Optimal investment strategies and risk measures in defined contribution pension schemes.

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Abstract

In this paper, we derive a formula for the optimal investment allocation (derived from a dynamic programming approach) in a defined contribution (DC) pension scheme whose fund is invested in n assets. We then analyse the particular case of n = 2 (where we consider the presence in the market of a high-risk and a low-risk asset whose returns are correlated) and study the investment allocation and the downside risk faced by the retiring member of the DC scheme, where optimal investment strategies have been adopted. The behaviour of the optimal investment strategy is analysed when changing the disutility function and the correlation between the assets. Three different risk measures are considered in analysing the final net replacement ratios achieved by the member: the probability of failing the target, the mean shortfall and a Value at Risk measure. The replacement ratios encompass the financial and annuitization risks faced by the retiree. We consider the relationship between the risk aversion of the member and these different risk measures in order to understand better the choices confronting different categories of scheme member. We also consider the sensitivity of the results to the level of the correlation coefficient.

Key words: defined contribution pension scheme; optimal investment; downside risk.

1. INTRODUCTION

In this paper we first derive and analyse the optimal investment strategy for a defined contribution pension scheme whose fund is invested in n assets, and then consider the special case of 2 assets and study the optimal investment strategy behaviour and the downside risk (in terms of the net replacement ratio achieved at retirement) faced by the member of the scheme, thereby extending the model introduced in Vigna and Haberman (2001).

The extensions introduced are threefold:
1. we consider n assets instead of 2;
2. we now consider assets which are correlated with each other;
3. we generalise the disutility function in such a way that deviations of the fund above the target are not penalised to the same degree as deviations below and the risk profile of the individual is taken into consideration

For the case of the n=2, the downside risk has been studied by examining three risk measures: the probability of failing the target, the mean shortfall and the Value at Risk measure (VaR) at three different confidence levels (1%, 5% and 10%).

The annuity risk faced by the member has been analysed through these risk measures by comparing the results relative to the net replacement ratio both in the case of a fixed conversion factor and in the case of a random conversion factor, which depends on the prevailing yield on the low risk asset.

2. THE MODEL

We consider a defined contribution pension scheme with n-asset portfolio. The forces of interest corresponding to the investment returns of the n assets are assumed to be normally distributed and correlated at any time with a given variance-covariance matrix.

Contributions are paid in advance every year as a fixed proportion of the salary of the scheme's member. Taxation, expenses and decrements other than retirement are not taken into consideration. The scheme member is assumed to join the scheme at time $t = 0$ and contribute until retirement at time $t = N$, which is a time point that is fixed in advance.

The model is presented in discrete time and we assume that the portfolio is reallocated every year between the n assets, depending on the past history of the market returns and on the current size of the fund, which is compared to an a priori target. We then find the optimal investment allocation every year that minimises the deviations of the fund from these corresponding targets. We assume that there are no real salary increases and that for simplicity the salary is 1 each year.

The fund at time t+1 is given by the following equation:

(2.1)
$$f_{t+1} = (f_t + c)[\sum_{i=1}^{n-1} y_t^i (e^{X_t^i} - e^{X_t^n}) + e^{X_t^n}]$$
t=0, 1,…, N-1

where:
f_t: fund level at time t
c: contribution rate

y_t^i: proportion of the portfolio invested in the ith asset during year [t, t+1], i = 1,2,…,n-1, so that the proportion invested in the nth asset is $1 - \sum_{t=1}^{n-1} y_t^i$

X_t^i: force of interest of ith asset in year [t, t+1], assumed to be constant over year [t, t+1], i=1,2…n

For fixed i, the sequences $\{X_t^i\}_{t=0,1,…N-1}$ are assumed to be IID with a normal distribution, while the correlation structure for the annual forces of interest X_t^i and X_t^j is given by the variance-covariance matrix, which is assumed to be constant for any t.
Therefore:

$$X_t^i \approx N (\mu_i, \sigma_i^2) \qquad\qquad \text{for t} = 0, 1,…, \text{N-1}$$

where we assume, without loss of generality, that:

$$\mu_1 > \mu_2 > …> \mu_n \qquad \text{and} \qquad \sigma_1^2 > \sigma_2^2 > …> \sigma_n^2.$$

3. THE PROBLEM
3.1 FORMULATION OF THE PROBLEM
We define the "cost" incurred by the fund at time t as follows:

(3.1) $\qquad\qquad C_t = (F_t - f_t)^2 + \alpha(F_t - f_t) \qquad\qquad$ t = 0, 1,…,N-1

(3.2) $\qquad\qquad C_N = \theta \, [(F_N - f_N)^2 + \alpha(F_N - f_N)] \qquad\qquad$ t = N

with $\alpha \geq 0$ and $\theta \geq 1$, where F_t is the annual target for the fund at time t. The targets are assumed to be given a priori (for example, by the investment manager or trustees of the pension scheme) and, on grounds of simplicity, are assumed to be deterministic. In the specific case with 2 assets, which we will deal with in a later section, we will give a particular specification of the targets.

The use of target values in the cost function is supported by the analysis of Kahneman and Tversky (1979). The target-based approach in decision making under uncertainty is investigated and supported also by Bordley and Li Calzi (2000), although they present a more general model in which the targets are stochastic and the utility function is the probability of matching the target. The use of stochastic targets could describe the real situation faced by the investment manager of a pension fund, who would be likely to change the targets every year in response to actual experience, but this would increase considerably the complexity of the mathematical model underlying the problem. We have chosen deterministic targets so that the model is mathematically tractable.

It may be argued that the inclusion of yearly targets between joining the scheme and retirement is not practical, since it may not be possible for the scheme member to change the annual contribution to the fund by either withdrawing money from the fund or paying additional money to the fund. A similar argument regarding decrements other than retirement, which are excluded from our model, could be advanced. In this case, the cost function should be defined only at retirement, and the actual fund compared with the final target only. We think that this different formulation of the problem is interesting and we will consider it in further research. The choice of having a target every year is adopted for reasons of mathematical tractability of the model as well as reasons of

cautiousness (since in practice it would probably be easier to meet a final target if the path of the fund's growth were monitored at periodic intervals)[1].

The cost defined by (3.1) and (3.2) is positive when the fund is below the target and above a certain level, which is equal to the target plus α, and negative between the target and this level. Intuitively, this means that the deviations below the target are penalised, while the deviations above the target are rewarded, until a certain level (linked to the target), after which they are penalised again. The economic interpretation of this choice is that the possibility of gaining from high market returns is incentivised, but, when the fund becomes too large in relation to the target, the trade-off between risk and return means that the portfolio is cautiously invested.

Indeed, equation (3.1) can be rewritten as

$$(3.3) \qquad C_t = (F_t + \frac{1}{2}\alpha - f_t)^2 - \frac{1}{4}\alpha^2$$

so that it is clear that the "real" target being pursued by the model is $F_t + \frac{1}{2}\alpha$. With this formulation, the f_t values are pulled back towards the "real" targets so that the generation of very high values of f_t is penalized.

By varying the parameter α we are actually considering different disutility functions with different risk aversion factors, so that we are considering individuals with different risk profiles. In fact, it can be shown (see Pratt 1964, Owadally 1998) that an individual's risk aversion can be measured either by $A(x) = -\dfrac{u''(x)}{u'(x)}$ if u(x) is the individual's utility function, or by $A(z) = \dfrac{l''(z)}{l'(z)}$ if l(z) is the individual's disutility (or loss) function, where the relationship between utility and disutility functions is: u(x)= −al(z)+b (a>0) and the relationship between loss and gain is: x+z=constant.

The disutility function considered here is $l(z) = z^2 + \alpha z$ (with the loss being z = F_t–f_t), and therefore the resulting risk aversion is:

$$(3.4) \qquad A(z) = \frac{l''(z)}{l'(z)} = \frac{2}{2z + \alpha} .$$

We observe not only that the risk aversion depends on the value of α, but also that it is decreasing when α is increasing, which means that lower values of α represent more risk averse individuals and vice versa.

We also observe that, in our model if $\alpha \to 0$, i.e. for very risk averse individuals, the target pursued tends to F_t. Instead, if $\alpha \to \infty$, i.e. for risk neutral individuals (A(z) = 0 means a null risk aversion, which is equivalent to risk neutrality), the target pursued tends to infinity, which means that the individual wants to make as much money as possible and always gain from higher than usual rates of return. Thus, F_t and infinity are the lower and upper bounds for our real targets, when we consider the dependence between them and the risk aversion.

The cost at time N has a weight θ which can be greater than 1. When θ is greater than 1 more importance is given to the final target than to the yearly ones, and the rationale for this choice is that

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[1] It should be noticed, however, that the different weight given to F_t and F_N (1 and $\theta \geq 1$) reflects the greater importance of the final target in comparison with the yearly ones.

decrements other than retirement are not considered in our model, and therefore the achievement of the final target - at retirement – can be considered more important than the achievement of the annual ones – before retirement.

The total future cost at time t is obtained by discounting the future costs until N, using a subjective inter-temporal discount factor β as in Bellman and Kalaba (1965):

(3.5)
$$G_t = \sum_{s=t}^{N} \beta^{s-t} C_s$$

We define $\Im_t$ the σ–field generated by all information available at time t:

(3.6)
$$\Im_t = \sigma(f_0, f_1, ..., f_t, \{y_0^i\}_{i=1...n-1}, \{y_1^i\}_{i}, ... \{y_{t-1}^i\}_i) \qquad t = 0,1,...,N$$

with $\Im_0 = \sigma(f_0)$, f_0 being the size of the fund when the member joins the scheme, that can be either 0 or greater than 0, if there is a transfer value.

The value function at time t is defined as:

(3.7)
$$J(\Im_t) = \min_{\pi_t} E[G_t \mid \Im_t] \qquad\qquad t=0, 1,..., N-1$$

where π_τ is the set of the future investment allocations, i.e.:

$$\pi_t = \{\{y_s^i\}_{s=t, t+1,...,N-1; i=1,...,n-1}\} = \{\{y_t^i\}_{i=1...n-1}, \{y_{t+1}^i\}_{i=1...n-1}, ... \{y_{N-1}^i\}_{i=1,...n-1})$$

We now find the future portfolio allocations that minimise the discounted future cost incurred by the fund.

3.2 BELLMAN'S OPTIMALITY PRINCIPLE
By applying Bellman's optimality principle we find:

(3.8)
$$J(\Im_t) = \min_{\pi_t} E[\sum_{s=t}^{N} \beta^{s-t} C_s \mid \Im_t] = \min_{\{y_t^i\}_{i=1,2,...n-1}} [C_t + \beta E[J(\Im_{t+1}) \mid \Im_t]]$$

Since the sequences $\{X_t^i\}_{t=0,...,N-1}$ are assumed to be independent for any t, $\{f_t\}$ is a Markov chain and:

$Pr[f_{t+1} \mid \Im_t] = Pr[f_{t+1} \mid f_t]$, and also: $Pr[f_{t+1}, f_{t+2},..., f_N \mid \Im_t] = Pr[f_{t+1}, f_{t+2},..., f_N \mid f_t]$

so that:

$Pr[G_t \mid \Im_t] = Pr[G_t \mid f_t]$

and:

(3.9)
$$J(\mathfrak{I}_t) = \min_{\pi_t} E[G_t \mid \mathfrak{I}_t] = \min_{\{y_t^i\}_{i=1,2,\ldots n-1}} E[G_t \mid f_t] = J(f_t, t)$$

The dynamic programming problem, which we have defined, now becomes:

(3.10)
$$J(f_t, t) = \min_{\{y_t^i\}_{i=1,\ldots,n-1}} [(F_t - f_t)^2 + \alpha(F_t - f_t) + \beta E[J(f_{t+1}, t+1) \mid f_t]]$$

with boundary condition:

(3.11)
$$J(f_N, N) = C_N = \theta \, [(F_N - f_N)^2 + \alpha(F_N - f_N)]$$

with $\{F_t\}_{t=1,\ldots,N}$ given.

3.3 SOLUTION OF THE DYNAMIC PROGRAMMING PROBLEM
It can be proved by mathematical induction that (a sketch of the proof is in the appendix):

(3.12)
$$J(f_t, t) = P_t f_t^2 - 2Q_t f_t + R_t \qquad\qquad t = 0, 1,\ldots,N$$

where the sequences $\{P_t\}$ and $\{Q_t\}$ and $\{R_t\}$ are given by the recursive relationship:

(3.13)
$$P_t = 1 + \beta P_{t+1} \left[\sum_{i=1}^{n-1} a_{ii} \frac{D_i^2}{|A|^2} + \sum_{j<i}^{n-1}\sum_{i=1}^{n-1} a_{ij} \frac{D_i D_j}{|A|^2} - 2\sum_{i=1}^{n-1} d_i \frac{D_i}{|A|} + g_{X_n}(2) \right]$$

(3.14)
$$Q_t = F_t + \frac{\alpha}{2} + \beta Q_{t+1}\left[\sum_{i=1}^{n-1} a_{ii} \frac{B_i D_i}{|A|^2} + \sum_{j<i}^{n-1}\sum_{i=1}^{n-1} a_{ij}\left(\frac{B_i D_j + B_j D_i}{|A|^2} \right) - \sum_{i=1}^{n-1}\left(\frac{d_i B_i + b_i D_i}{|A|} \right) + g_{X_n}(1) \right] -$$
$$- \beta c P_{t+1}\left[\sum_{i=1}^{n-1} a_{ii} \frac{D_i^2}{|A|^2} + \sum_{j<i}^{n-1}\sum_{i=1}^{n-1} a_{ij} \frac{D_j D_i}{|A|^2} - 2\sum_{i=1}^{n-1} d_i \frac{D_i}{|A|} + g_{X_n}(2) \right]$$

(3.15)
$$R_t = F_t + \alpha F_t + \beta c^2 P_{t+1}\left[\sum_{i=1}^{n-1} a_{ii} \frac{D_i^2}{|A|^2} + \sum_{j<i}^{n-1}\sum_{i=1}^{n-1} a_{ij} \frac{D_j D_i}{|A|^2} - 2\sum_{i=1}^{n-1} d_i \frac{D_i}{|A|} + g_{X_n}(2) \right] -$$
$$- 2\beta c Q_{t+1}\left[\sum_{i=1}^{n-1} a_{ii} \frac{B_i D_i}{|A|^2} + \sum_{j<i}^{n-1}\sum_{i=1}^{n-1} a_{ij}\left(\frac{B_i D_j + B_j D_i}{|A|^2} \right) - \sum_{i=1}^{n-1} \frac{b_i D_i}{|A|} + g_{X_n}(1) \right] +$$
$$+ \beta \frac{Q_{t+1}^2}{P_{t+1}}\left[\sum_{i=1}^{n-1} a_{ii} \frac{B_i^2}{|A|^2} + \sum_{j<i}^{n-1}\sum_{i=1}^{n-1} a_{ij} \frac{B_i B_j}{|A|^2} - 2\sum_{i=1}^{n-1} b_i \frac{B_i}{|A|} \right] + R_{t+1}$$

starting from: $P_N = \theta$, $Q_N = \theta \, (F_N + \alpha/2)$ and $R_N = \theta \, F_N^2 + \alpha \, F_N$

with (when possible, we suppress the time index t, noting that the distributions of the asset returns do not change over time, in order to simplify the notation):

$W_i = e^{X_i}$

$a_{ij} = E[(W_i - W_n)(W_j - W_n)]$

$b_i = E[(W_i - W_n)]$

$d_i = E[(W_i - W_n)] - E[(W_n^2)]$

$A := (a_{ij})$ matrix

B_i = determinant of the matrix obtained by A, replacing the ith column with the vector $\mathbf{b} = (b_i)_{i=1,\ldots,n-1}$

D_i = determinant of the matrix obtained by A, replacing the ith column with the vector $\mathbf{d} = (d_i)_{i=1,\ldots,n-1}$

$g_{X_n}(s)$: moment generating function of X_n (therefore, $g_{X_n}(s) = E[(W_n^s)]$)

In addition, we find that the optimal investment strategy is given by the following optimal investment allocation at time t:

(3.16)
$$\mathbf{y}_t^{i*} = \frac{k_t B_i - D_i}{|A|}$$

with:

$$k_t = \frac{Q_{t+1}}{P_{t+1}(f_t + c)}$$

and the other notation as before.

3.4 THE TWO-ASSET CASE
We have analysed the particular case of n = 2, considering the case of a pension fund invested in a high-risk and a low-risk asset.

The fund at time t+1 is given by the following equation:

(3.17)
$$f_{t+1} = (f_t + c)[y_t e^{\lambda_t} + (1 - y_t)e^{\mu_t}]$$

where:
f_t: fund level at time t
c: contribution rate
y_t: proportion of fund invested in the high-risk asset during year [t, t+1]

λ_t: real force of interest for the high-risk asset in year [t, t+1], assumed to be constant over the year [t, t+1]

μ_t: real force of interest for the low-risk asset in year [t, t+1], assumed to be constant over the year [t, t+1]

The sequences $\{\mu_t\}$ and $\{\lambda_t\}$ are assumed to be IID with normal distribution, while the annual forces of interest μ_t and λ_t are correlated with correlation factor ρ, assumed to be constant for any t.

Therefore:

$$\lambda_t \approx N(\lambda, \sigma_1^2) \qquad \text{and} \qquad \mu_t \approx N(\mu, \sigma_2^2) \qquad \text{for } t = 0, 1, \ldots, N-1$$

where:

$$\lambda > \mu \qquad \text{and} \qquad \sigma_1^2 > \sigma_2^2$$

It follows that:

(3.18)
$$J(f_t, t) = P_t f_t^2 - 2Q_t f_t + R_t$$

where the sequences $\{P_t\}$ and $\{Q_t\}$ and $\{R_t\}$ are given by the recursive relationship:

(3.19)
$$P_t = 1 + \beta\, P_{t+1}\, \Lambda$$

(3.20)
$$Q_t = F_t + \alpha/2 + \beta[Q_{t+1}\Gamma - cP_{t+1}\Lambda]$$

(3.21)
$$R_t = F_t + \alpha\, F_t + \beta[c^2\, P_{t+1}\, \Lambda - 2cQ_{t+1}\, g_{\mu_t}(1) + \frac{Q_{t+1}^2}{P_{t+1}}\, \Omega + R_{t+1}]$$

starting from: $\qquad P_N = \theta, Q_N = \theta\,(F_N + \alpha/2)$ and $R_N = \theta\, F_N^2 + \alpha\, F_N$, where:

$$\Lambda = \frac{E(W_1^2)E(W_2^2) - (E(W_1 W_2))^2}{E[(W_1 - W_2)^2]}$$

$$\Gamma = \frac{E(W_1^2)E(W_2) + E(W_1)E(W_2^2) - E(W_1 W_2)[E(W_1) + E(W_2)]}{E[(W_1 - W_2)^2]}$$

$$\Omega = \frac{E^2(W_1 - W_2)}{E[(W_1 - W_2)^2]}$$

$g_{\mu_t}(s)$ is the moment generating function of μ_t

and with:

$$W_1 = e^{\lambda_t} \qquad \text{and} \qquad W_2 = e^{\mu_t}.$$

The optimal investment strategy in the 2-asset case is given by:

$$(3.22) \qquad y_t^* = \frac{Q_{t+1}E(W_1 - W_2)}{P_{t+1}(f_t + c)E[(W_1 - W_2)^2]} - \frac{E(W_1 W_2) - E(W_2^2)}{E[(W_1 - W_2)^2]}$$

3.5 RELATIONSHIP BETWEEN α AND THE OPTIMAL INVESTMENT STRATEGY

Looking at formula (3.22) above, we observe that α affects the sequence $\{Q_t\}$ only (observing that the coefficients R_t do not appear in the formula of y_t^*), which increases in value as α increases, leading to higher values of y_t^*, everything else being equal. Thus, we conclude that y_t^* increases as α increases.

This is a reasonable result as we observe that α measures the risk aversion of the individual: a higher value of α corresponds to a lower risk aversion and leads to a higher fraction of the portfolio being invested in the riskier asset. This result is consistent also with intuition: by increasing α we are increasing both the penalisation of deviations below the target and the reward of deviations above, pushing the optimal portfolio to be invested more in riskier assets.

In particular, considering the behaviour of the risk neutral investor (α → ∞), we see that, in our model, the risk neutral individual would short sell as much as possible low-risk assets in order to buy as much as possible high-risk assets. If short selling were not allowed (a hypothesis that we will make next in the simulation), the risk neutral investor would invest the whole fund in the high risk asset and never switch into the low risk asset[2]. This asset allocation is supported by Blake et al (2001), who argue that there is no evidence for the appropriateness of switching the fund into lower risk assets prior to retirement for the risk neutral individual, except for prudential reasons. However, in their analysis they do not consider individuals with different levels of risk aversion.

The meaning of α will be considered again later, when we discuss the simulation results.

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[2] In this case the optimal investment strategy would be: $y_t^* = 1$ at any time $0 \leq t \leq N-1$.

4. THE SIMULATIONS
4.1 SIMULATIONS FRAMEWORK

We have investigated the solution presented in section 3.4 (2-asset case) for many scenarios, and have studied the sensitivity of the results to changes in the values of F_t, α and ρ by carrying out simulations.

We have also considered different generations of members by changing the value of N: 10, 20, 30 and 40 years to retirement. We assume that the member joins the scheme without a transfer value, which means that $f_0 = 0$. The contribution rate has been taken equal to 12%, the weight θ given to the final target has been chosen equal to 2 and the subjective inter-temporal discount factor β has been taken equal to 0.95.

The parameters of the asset returns chosen are:
$\lambda = 10\%$ $\qquad\qquad$ $\mu = 4\%$ $\qquad\qquad\qquad\qquad$ $\sigma_1 = 15\%$ $\qquad\qquad$ $\sigma_2 = 5\%$.

The correlation factor ρ takes the values -1, -1/2, 0, 1/2, 1; α takes many values, depending on the time to retirement.

The choice of different values of α for different durations is not arbitrary and is due to the fact that, since short selling is not allowed in our simulations (see later in this paragraph), the value of y_t^* stabilises at 1 after a certain value of α (we recall that y_t^* increases as α increases), which depends on the time to retirement (see section 6.3). Thus, for N = 10, α takes the following values: 0, 0.25, 0.5, 0.75, 1, 2, 3, 4, 5, 6; for N = 20, α takes the following values: 0, 1, 2, 3, 4, 5, 6, 7, 10, 15, 20, 30; for N = 30, α takes the following values: 0, 1, 2, 3, 4, 5, 7.5, 10, 12.5, 15, 20, 25, 30, 40, 50, 60; for N = 40, α takes the following values: 0, 1, 2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20, 22.5, 25, 30, 35, 40, 50, 75, 100, 150, 200.

For each scenario, 1000 simulations have been carried out by generating, for each simulation, the asset returns for N years (with N = 10, 20, 30, 40), and deriving, for each year, the optimal portfolio allocation derived by the model. In all the scenarios investigated, we use the same 1000 paths of returns and just change the other parameters like F_t, α and ρ. The rationale is that we wish to study the effect of changing the parameters and we do not wish the results to be affected (and confounded) by differences in the simulated paths of returns.

As mentioned above, in our simulations we have imposed the constraint that short selling is not allowed; therefore we have constrained the true value of y_t^* and set it equal to 0 when we had a negative value, and set it equal to 1 when we had a value greater than 1. We have then used the truncated value as the adopted investment strategy in the growth of the fund. We are aware of the fact that the simulations do not correspond exactly to the underlying model, due to this additional constraint, and the investment strategies adopted could be called "sub-optimal". The rationale for this choice is on the one hand to give results which can be easily compared with each other, the range being [0, 1] in all cases; on the other hand, to have a mathematical model which is tractable: adding the constraint $0 \leq y_t^* \leq 1$ would complicate further the model; however, we think that this could be a useful improvement of the model and defer this to further research.

At the time of retirement, N, we have followed Vigna and Haberman (2001) and analysed the behaviour of the optimal investment strategy and studied how it changes if we change the targets, the disutility function, and the correlation factor ρ. Furthermore, we have calculated the fund accumulated, and also the net replacement ratio achieved by the member using two different methods, in order to measure the effect of the annuity risk on the retiree income. For the first

method, we have converted the accumulated fund into an annuity using an actuarial value, a_x, based on discounting at the expected return of the low risk asset (i.e. a fixed rate[3]). For the second method, we have used a variable annuity value, $\tilde{a}_x$, with discounting based on the average of the realised returns by the low risk asset in the last 5 years before retirement (setting a minimum of 2% in order to avoid unreasonable values in the case of a very poor performance of the low risk asset prior to retirement) and the variance of the low risk asset experienced during the N years of membership[4]. In both cases, the annuity values depend on the returns on the low risk asset in order to represent the pricing behaviour of an insurance company, which would utilise the returns on matching fixed interest bonds in its calculations. For both methods, the mortality table used to calculate the actuarial value is the Italian projected mortality table (RGS48) and the retirement age has been chosen throughout to be $x = 62$.

Considering that the salary is 1 at any time, the net replacement ratio achieved using the first method of calculation is:

$$b_N = \frac{f_N}{a_x},$$

whereas the net replacement ratio achieved using the second method of calculation is:

$$\tilde{b}_N = \frac{f_N}{\tilde{a}_x}.$$

The net replacement ratio achieved using the first method, b_N, takes into account only the investment risk faced by the member[5], as the conversion rate used is fixed. The net replacement ratio achieved using the second method, $\tilde{b}_N$ ("bNt" or "bNtilde" in the Figures that follow), takes into account also the annuity risk faced by the member[6], as the conversion rate is linked to the simulated returns on the low risk asset in the years before retirement, and therefore it is variable, reflecting the simulated behaviour of the low risk asset.

We illustrate only the results for the 30 years' case (we note that the other durations, N=10, 20, 40, give similar results). Similarly, we illustrate only the results for the VaR at 5% confidence level (we note that the VaR measures at 1% and 10% confidence levels give similar results).

4.2 TARGETS
Since the fund and contributions are invested in a 2-asset portfolio, we have chosen the yearly targets as the accumulated fund using a particular average of the rates of return of the 2 assets. We have considered three different cases: (a) the rate of return equal to μ, as though the fund were fully

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[3] The discount factor used to calculate the annuity value a_x is $v= E[e^{-\mu_t}] = e^{-\mu+0.5\sigma_2^2}$, with $\mu = 4\%$ and $\sigma_2 = 5\%$.

[4] The discount factor used to calculate the annuity value $\tilde{a}_x$ is $\tilde{v} = e^{-\tilde{\mu}+0.5\tilde{\sigma}_2^2}$, with

$\tilde{\mu} = \max\{2\%, \text{average}(\mu_{N-1},....,\mu_{N-5})\}$, and $\tilde{\sigma}_2^2 = \text{variance}\{\mu_1,...,\mu_{N-1}\}$

[5] By "investment risk" we mean the risk that the returns experienced during the membership have been too low leading to a low final fund. This risk is borne during the accumulation period.

[6] By "annuity risk" we mean the risk that the rate used in the conversion of the capital into annuity is too low, leading to a low pension rate (the actual conversion rate used to calculate the annuity is directly linked to the current market yields, and so the perceived pension will strongly depend on the level of the markets rates at retirement). This risk is borne at retirement, when the annuity is purchased.

invested in the low risk asset; (b) the rate of return equal to the Chisini average of μ_t and λ_t relative to the expected return over one year of a portfolio invested equally in the 2 assets, as though the fund were invested half in the low risk and half in the high risk asset; (c) the rate of return equal to λ, as though the fund were fully invested in the high risk asset.

Therefore, the yearly target at time t is:

$$(4.1) \qquad\qquad F_t = f_0 e^{tj} + c\ddot{s}_{\overline{t}|j} \qquad\qquad t=1,\dots,N$$

with i such that:

(a) $j = \mu$; (b) $j = r^*$; (c) $j = \lambda$,

with[7]

$$(4.2) \qquad\qquad r^* = \frac{1}{2}(\mu + \lambda) + \frac{1}{8}(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)$$

Therefore, the value of j for determining the target values takes the three different values: 4%, r* as defined by (4.2) above (we observe that this value depends on the value of ρ, so it varies with the different values of the correlation factor) and 10%.

We set $F_0 = f_0$, where f_0 is the fund held by the member when he/she joins the scheme (i.e. the transfer value, which in our model is zero).

4.3 MEASURES OF RISK

The downside risk faced by the member of the pension scheme is analysed by comparing the net replacement ratios achieved, b_N and $\tilde{b}_N$, with the target ratio B_N, which is defined to be the fund target F_N divided by an actuarial annuity value a_x. In other words:

$$B_N = \frac{F_N}{a_x}.$$

The idea of comparing the net replacement ratio achieved with the target pursued is consistent with the analysis of Kahneman and Tversky (1979), who observe that individuals perceive the outcomes as gains and losses relative to some neutral "reference point" (which, in our case, is the target).

The risk has been measured in the three different ways:

a) Probability of failing the target.
This is defined as the proportion of outcomes where b_N (or $\tilde{b}_N$) is less than the target B_N. Thus $Pr(b_N < B_N) = \frac{k}{1000}$, where k represents the number of failures out of 1000 simulations. We note that the amount by which b_N (or $\tilde{b}_N$) falls below B_N is not taken into account by this risk measure.

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[7] We found r* by solving $E[e^{r^*}] = E[e^{\frac{\mu_t + \lambda_t}{2}}]$, following the definition given above.

In the same way we define $\Pr(\tilde{b}_N < B_N)$.

b) Mean shortfall.

The mean shortfall is the conditional mean of shortfall below the target, conditional on $(b_N - B_N) < 0$ (or $(\tilde{b}_N - B_N) < 0$). Thus, using the upper suffix j to refer to a simulation, we have:

$$\text{mean shortfall } (b_N) = \left| \frac{1}{k} \sum_{j=1}^{k} (b_N^j - B_N) \right| \text{ where } (b_N^j - B_N) < 0 \text{ for } j = 1, 2, \ldots, k.$$

In the same way we define the mean shortfall $(\tilde{b}_N)$.

As demonstrated by Artzner et al (1999) and Artzner (2000), risk measures based on the mean shortfall have more desirable properties than the commonly used Value-at-Risk (VaR) measures. In particular, these are coherent risk measures (as introduced by Artzner et al (1998)), whereas a VaR measure is not, as it fails to satisfy the sub-additive property (as many examples can show), which is required of any coherent risk measure. As a result, mean shortfall risk measures have become widely used in the recent actuarial literature: for example, see Albrecht et al (2001) for a discussion of equity risk and Hardy (2001) for an application to segregated funds.

c) Value at Risk (VaR).

The VaR measure at confidence level ε is defined to be the 100εth lowest percentile of the simulated distribution of b_N (or $\tilde{b}_N$).

5. RESULTS: THE OPTIMAL INVESTMENT STRATEGY

5.1 OPTIMAL INVESTMENT STRATEGY

We observe, as in Vigna and Haberman (2001), that the optimal investment allocation y_t^* decreases on average with the time, which indicates the suitability of the lifestyle policy[8] for defined contribution pension schemes. This is illustrated by the results in Figures 1-3 (and by other detailed results not shown here but available on request from the authors).

The only exception to the suitability of the lifestyle strategy is when the individual is risk neutral, as mentioned above. In this case, the scheme member will invest the whole fund in the high risk asset at any time between joining the scheme and retirement.

5.2 EFFECT OF CHANGING THE TARGETS

We have studied the behaviour of the optimal investment strategy when the targets change. The three graphs of Figure 1 report some percentiles (5th, 25th, 50th, 75th and 95th), the minimum and maximum of the distribution of y_t^* in the cases of μ-based targets, r*-based targets and λ-based targets, when $\alpha=1$.

We observe that moving from μ-based targets to λ–based targets, the optimal proportion of the portfolio to be invested in the high risk asset increases. This is evident if we look at formula (3.22) for y_t^*: it can be easily seen that the value of y_t^* increases as the target F_t increases, everything else being equal. This is also an intuitive result, since one must increase the aggressiveness of the strategy if the target to be attained increases.

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[8] We recall that by "lifestyle strategy" we mean the investment strategy largely adopted in defined contribution pension schemes in UK, which consists in investing at the beginning the whole fund in equities, switching it into bonds and cash as retirement approaches (usually 3-5 years before retirement).

FIGURE 1: CHANGING THE TARGETS (THE VALUE OF j)

Y*t with μ targets (α=1)


Y*t with r* targets (α=1)

Y*t with λ targets (α=1)
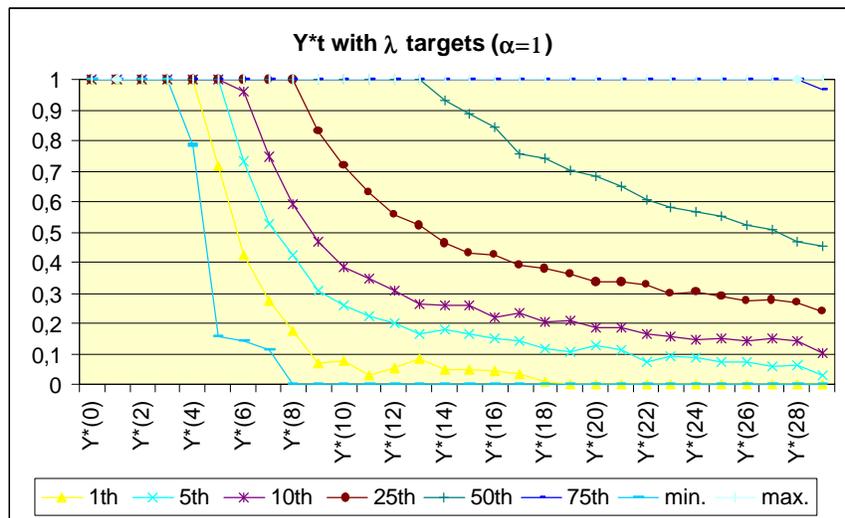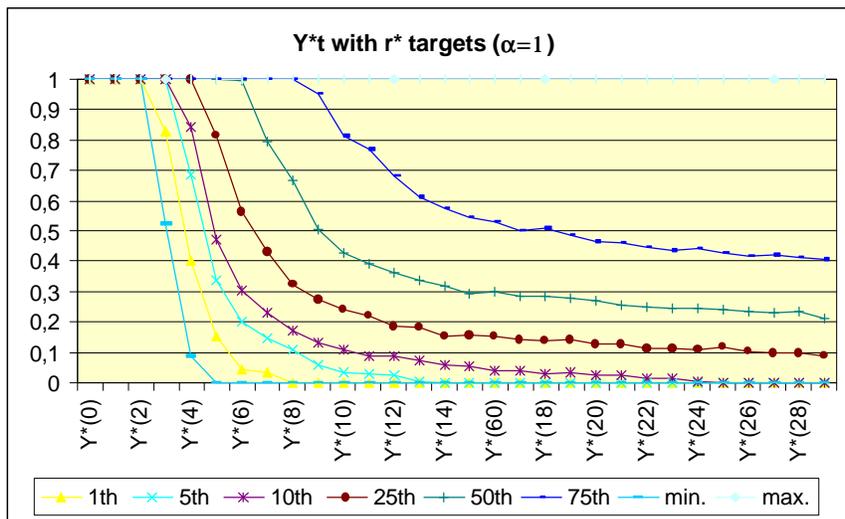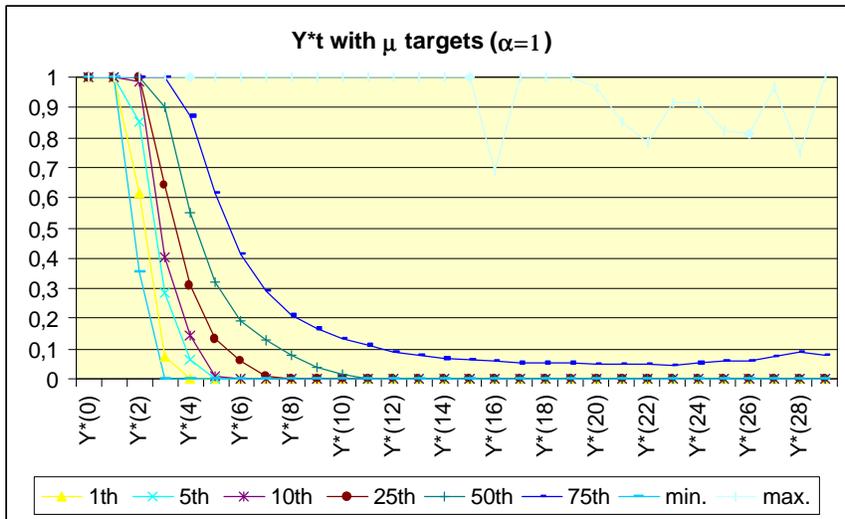
5.3 EFFECT OF CHANGING THE DISUTILITY FUNCTION

We have studied the effect of changing the disutility function by observing the behaviour of y_t^* when we change the value of α and leave unchanged the value of ρ.

In all cases (i.e. for any N and any ρ) we have found that y_t^* increases on average as α increases. This is consistent with the fact that, the higher is α, the lower is the risk aversion of the individual, hence the riskier the investment strategy adopted.

The graphs reported in Figure 2 show how the level of the optimal investment strategy increases with α in the 2 cases $\rho = -\frac{1}{2}$ (on the left) and $\rho = \frac{1}{2}$ (on the right) with the r*-based targets. The graphs report the mean of y_t^* (for t = 0, 1,…, 29) over the 1000 simulations that have been carried out.

FIGURE 2: CHANGING DISUTILITY FUNCTION (THE VALUE OF α)
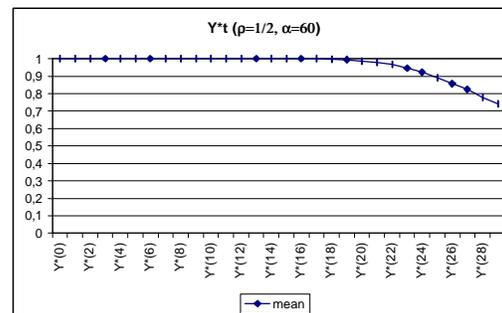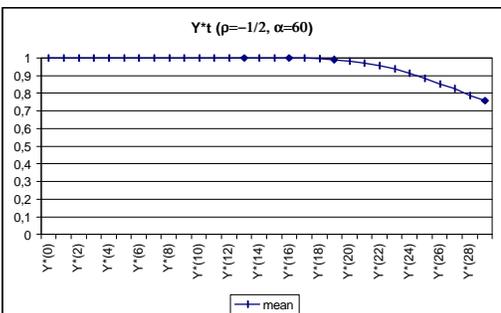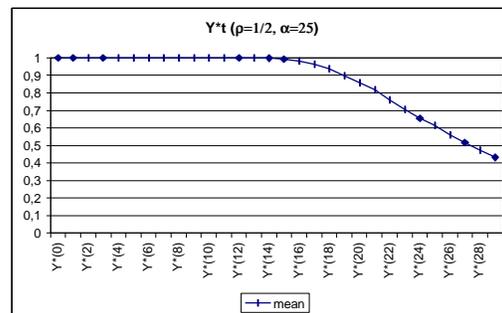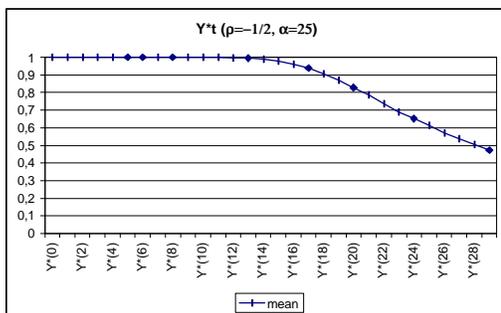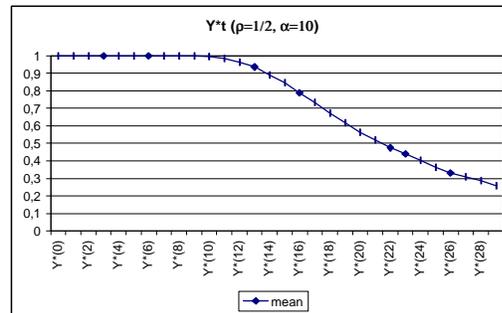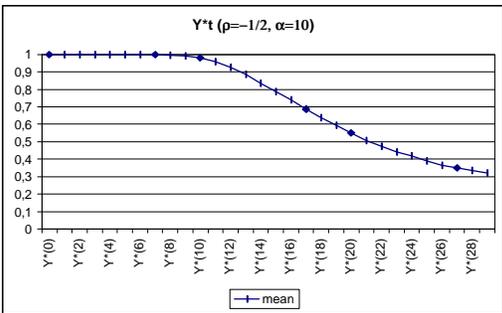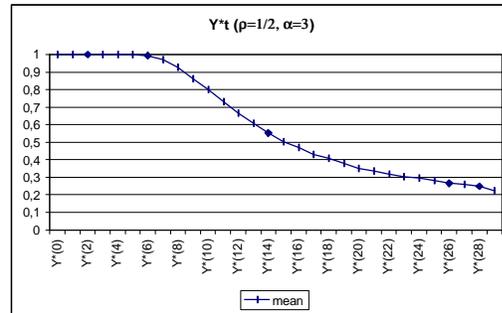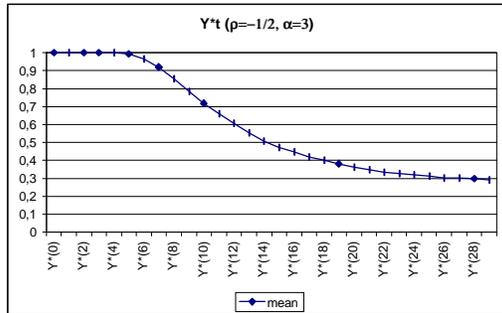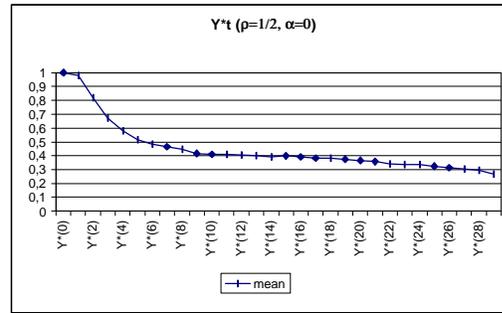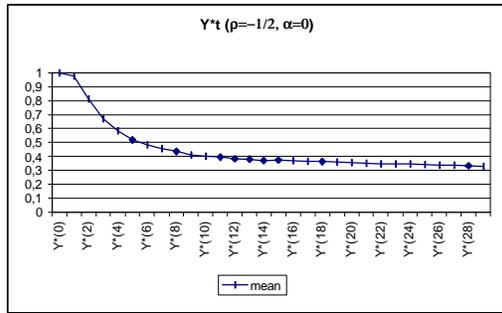
5.4 EFFECT OF CHANGING THE CORRELATION BETWEEN THE ASSET RETURNS

We have studied the effect of changing the correlation factor ρ on the optimal investment strategy by comparing the behaviour of y_t^* when N and α are fixed and ρ changes.

We have discovered two interesting trends:

1 – in the early years of membership, the optimal allocation y_t^* increases as ρ increases from negative to positive values, while in the last years of membership, towards retirement, it decreases as ρ increases;

2 – apart from the early years of membership, the standard deviation of y_t^* increases as ρ increases.

The graphs in Figure 3 show the optimal investment strategy for the different values of ρ (-1, -½, 0, ½, 1) when $\alpha = 0$ (on the left) and when $\alpha = 10$ (on the right). The graphs report certain percentiles (5th, 25th, 50th, 75th and 95th), the minimum and the maximum of the distribution of y_t^* over the 1000 simulations carried out. We can see that, when ρ increases from –1 to +1, the curves reporting the percentiles become steeper in their descent from 1 to 0, confirming the first trend above explained (higher values of y_t^* at the beginning and lower at the end when ρ increases).

An economic explanation of this feature is as follows. We observe that a strategy is well diversified if the curve y_t^* decreases gradually towards zero, less diversified if the curve decreases steeply towards zero. When ρ is low, in the range (-1, -½), there is negative correlation between the asset returns and it is convenient to diversify the portfolio between the assets and, therefore, the optimal investment allocation leads to a fraction to be invested in the riskier asset which decreases very gradually towards zero. When ρ is high, in the range (½, 1), there is positive correlation between the asset returns and the diversification effect is not so rewarding, so the portfolio can be invested more heavily either in the riskier or in the less risky asset, leading the optimal investment allocation to decrease steeply towards zero.

The graphs in Figure 4 show how the standard deviation of y_t^* changes in value when ρ increases. When $\alpha = 0$, the standard deviation increases as ρ increases. This feature is also observed in Figure 3: with negative values of ρ, the percentiles tend to stabilise around a certain percentage (30-40%), while with positive values of ρ the percentiles are more spread between 0 and 1.

An economic explanation of this feature is the following. When ρ increases from –1 to +1, the benefits of diversification decrease, pushing y_t^* upwards. On the other hand, with high positive values of ρ, the investor needs more hedging in order to achieve the target at time N and this factor pushes y_t^* downwards. The value of the optimal investment allocation will be high or low depending on which of these two effects is dominating (and this depends only on the returns experienced in the past), leading the values to be more spread and the standard deviation to be higher. For higher values of α, for example $\alpha = 10$, the optimal asset allocation y_t^* increases and the balance between these two effects is different.

This phenomenon seems to indicate that, with negative correlation between the asset returns, the investment strategy is more stable on average than for the case of positive correlation, where it remains on almost the same level regardless of the past experience of asset returns. This feature may be interesting from the pension scheme investment manager's point of view in two respects: firstly, when he/she considers projections and plans regarding the investment strategy to be adopted over a long period in the future, and, secondly, considering the fact that, in the real world, portfolios tend to be invested in assets with returns that are negatively correlated.

The detailed results for y_t^* indicate that they are relatively insensitive to small changes in ρ. This corresponds to the results of Chopra and Ziemba (1993), who find that optimal asset allocation results are much more sensitive to errors in means than to errors in variances and much more sensitive to errors in variances than to errors in covariances. This finding is particularly apparent as the risk aversion of the investor reduces – this corresponds here to increases in the value of α.

FIGURE 3: CHANGING CORRELATION BETWEEN ASSETS (VALUE OF ρ)
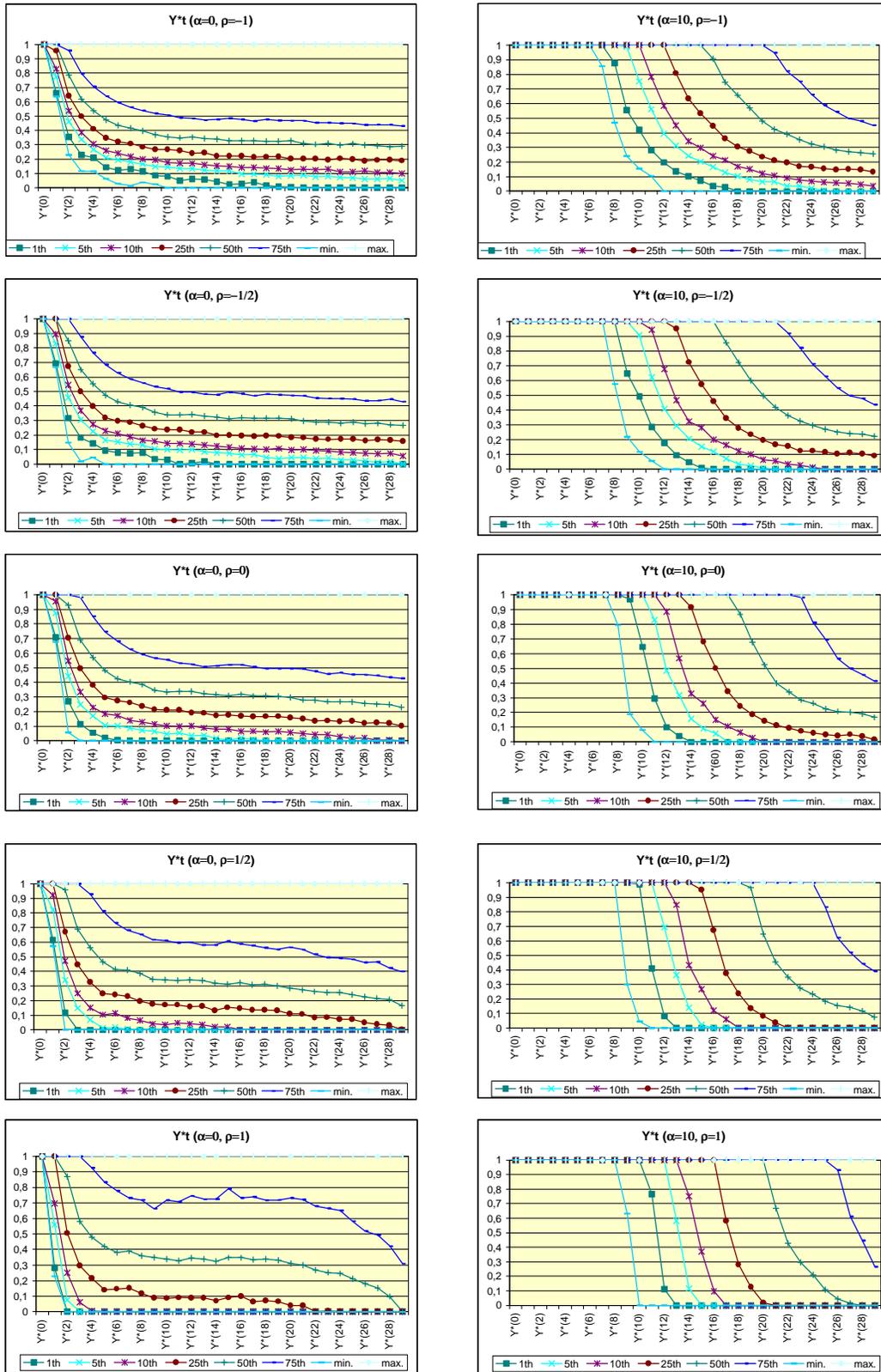
FIGURE 4: STANDARD DEVIATION OF Y_t^* WHEN ρ CHANGES
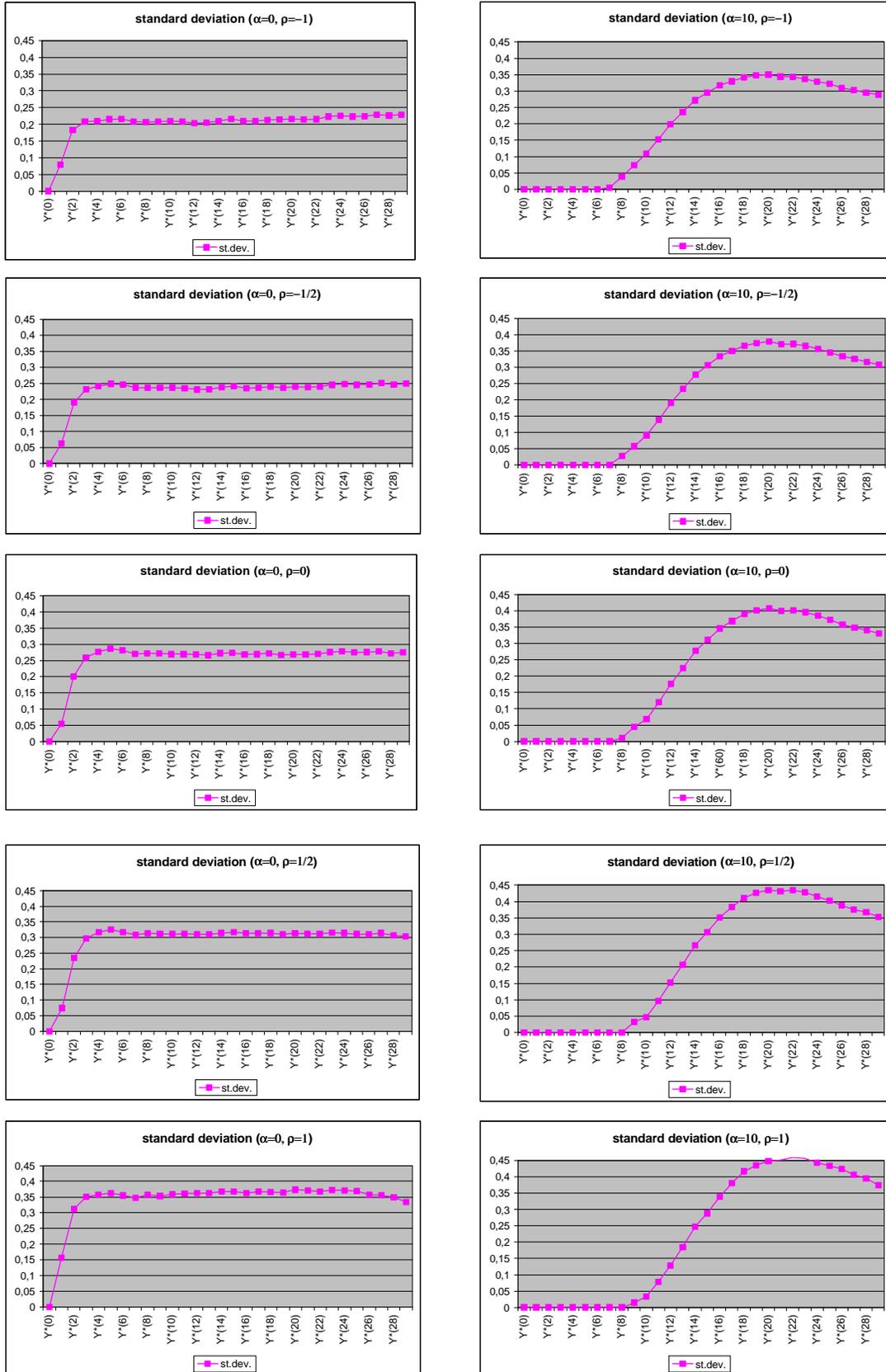
6. RESULTS: THE DOWNSIDE RISK

6.1 THE MEANING OF α

Before consideration of the results, it is worth recalling that α can be given two meanings in our model. On the one hand, as we have already seen, it is a measure of the risk profile of the individual with the disutility function C_t: the higher is its value, the less risk averse is the individual. On the other hand, it is a measure of the aggressiveness of the optimal investment strategy. In fact, if we look at the formula that determines y^*_t above, (3.22), we can easily observe that it depends directly on α, so that we have an increasing y^*_t with α increasing. Therefore, a low value of α indicates a cautious investment strategy, while a higher value of α indicates a riskier investment strategy. It is clear that these two viewpoints are consistent with each other.

6.2 THE DIFFERENT RISK MEASURES

In Figure 5, we have plotted the three different risk measures considered against the different values of α in the case of 30 years, r*-targets and $\rho = 0$.

We observe the following results:
a) the probability of failing the target decreases as α increases;
b) the mean shortfall increases slightly as α increases;
c) the VaR at 5% level is relatively stable as α increases.

The explanation for a) and b) is the following. Lower values of α lead to more cautious strategies, which lead to a greater number of failures but to more limited "losses" when a failure occurs. In contrast, higher values of α lead to riskier strategies and to a higher mean and a higher standard deviation of the distribution of both b_N and $\tilde{b}_N$, and this leads to a smaller number of failures (due to a higher mean) but slightly greater deviations from target when a failure occurs. This arises because the higher mean and standard deviation lead to a much longer right tail of the distribution of the net replacement ratio and to a slightly longer left tail of the distribution, with the lowest percentiles being smaller, so that a slightly greater mean shortfall results (as α increases).

The explanation for c) is the following: the VaR at 5% level remains stable because of the combined and opposing effects of the higher mean and standard deviation of the distribution of b_N and $\tilde{b}_N$, as α increases, and the net effect is to lead to the lower percentiles remaining fairly stable.

FIGURE 5: RISK MEASURES AGAINST α

Probability of failing the target (r*)

Mean shortfall (r*)

VaR at 5% confidence (r*)
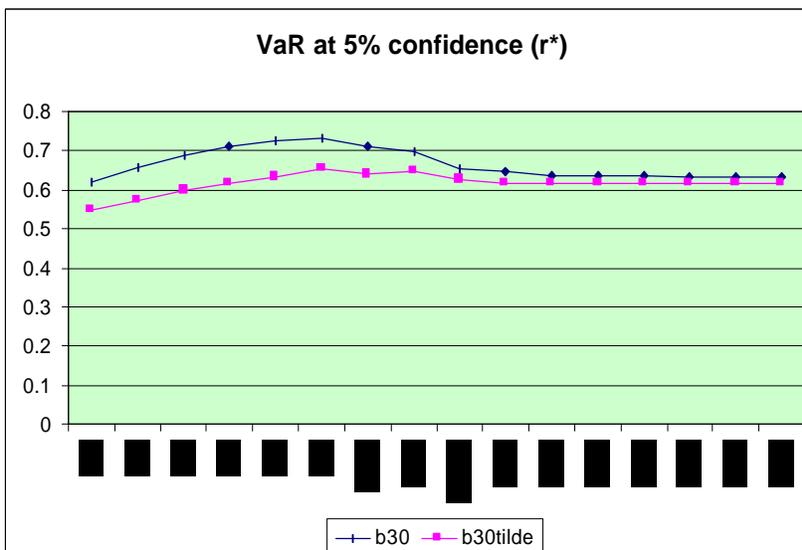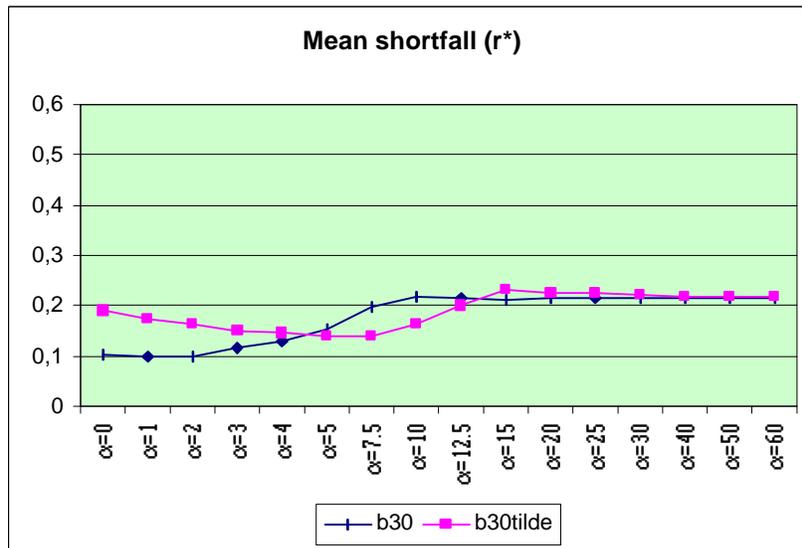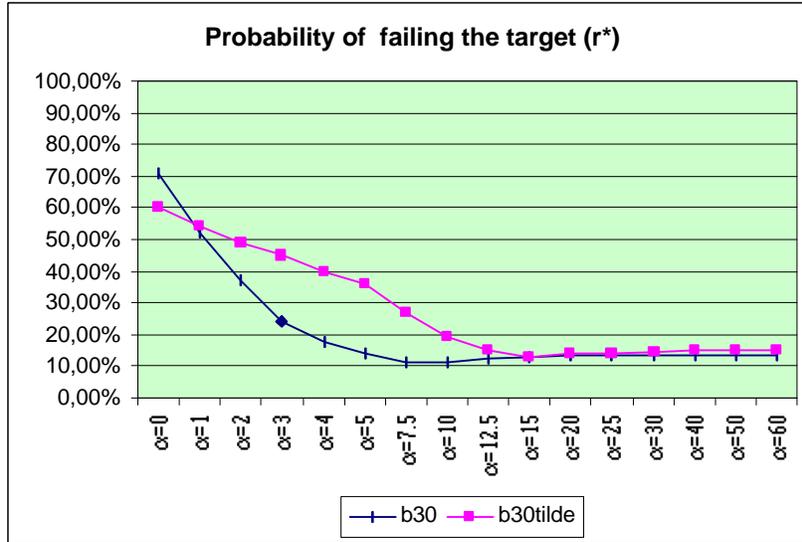
6.3 THE DIFFERENT TARGETS AND THE RISK MEASURES

In Figures 6, 7 and 8, we show the results for $b_{30}t$ and b_{30} as α varies, for the three different risk measures with each of the different targets chosen, and for the case of $\rho=0$. We consider the effect of changes in ρ in a later section.

A general result that comes out of the graphs is that, by increasing the targets (ie moving down the page from μ-based targets to λ-based targets), the levels of all of the risk measures increase. This is an intuitive result, as it is more difficult to reach higher targets than lower ones. Furthermore, as we move down the page, the optimal investment strategy becomes riskier, since y^*_t increases as the target values of F_t increase (see section 5.2).

Another general result is that, for very high values of α, in all of the graphs, the risk measures tend to stabilise themselves, leading to the curves becoming approximately horizontal. As mentioned before, this is a feature of the model: since short selling is not allowed and since increasing the value of α will increase the riskiness of the strategy, after a certain value of α the value of y^*_t will be always 1. This leads to the same strategy regardless of the rates of return that have been simulated (in which all the fund is invested in the high risk asset at any time), and hence to very small differences in the results, and therefore to the flat curves displayed in Figures 6-8 for the risk measures at extreme values of α.

We also find (in results not shown here) that the value of α after which y^*_t reaches 1 increases with the time to retirement N, and this suggests that the choice of α for different durations should be different. This also reflects the intuitive fact that, with a short time to retirement, an individual is more risk averse so that α takes low values, while, with a long time to retirement, an individual is less risk averse so that α takes high values.

We now analyse the different figures separately.

Probability of failing the target.

The three graphs of Figure 6 report the probability of failing the target with the three different targets. We observe the following points:

1 – when we consider the initial values of the probability of failing the target in the two cases of b_{30} and $b_{30}t$, when $\alpha=0$, we see that this probability is higher for $b_{30}t$ than for b_{30} in the case of μ-based targets (with an approximate 10% gap), whereas it is lower for $b_{30}t$ than for b_{30} in the case of the r*-based (with an approximate 10% gap) and λ-based targets (with an approximate 20% gap). The fact that the probability of failing the target is higher for $b_{30}t$ than for b_{30} may not be a surprise, as it arises from the fact that the magnitude of b_{30} is affected by the investment risk only, whereas the magnitude of $b_{30}t$ is affected by both the investment and annuity risk. What may seem strange is the fact that the probability of failing the target is lower for $b_{30}t$ than for b_{30} with the r*- and λ-based targets. A possible explanation for this is given by looking at the aggressiveness of the strategy in the considered cases. In the case of the μ-based targets, the $\alpha=0$ strategies are very cautious, so the fund is likely to be invested more in the low risk asset during membership and entirely in the low risk asset just before retirement (because of the lifestyle profile shown by the optimal values of y^*_t), so that the reason for failing the target is likely to be mainly due to poor performance of the low risk asset, either during the period of membership and/or just before retirement. On the other hand, in the case of the r*- and λ-based targets, the strategies are riskier and the reason for failure may depend also on the adverse performance of the high risk asset. When this happens, there are cases in which the poor performance of the high risk asset leads to a failure regarding the achievement of b_{30}, but not to a failure regarding the achievement of $b_{30}t$ due to the good performance of the low

risk asset prior to retirement (leading to a more favourable annuity conversion rate than the one used in the b_{30} case). The gap between the probabilities values becomes larger when moving from the r*- to λ-based targets as the strategies become riskier and the weight attaching to the adverse performance of the high risk asset increases;

2 – the descent of the probability curve is steeper for the b_{30} case than for the $b_{30}t$ one. This is probably to be explained by referring to the annuity risk and the aggressiveness of the strategies. Moving from the left to the right in each of the graphs, the value of α increases and the strategy becomes more aggressive. The increased aggressiveness of the strategy affects the general level of the final fund f_N which increases, but does not affect the values of the simulated μ_t in the final years before retirement, which are used (by the insurance company) to price the annuity in the case of $b_{30}t$. We observe that, in the case of b_{30} , failures are only due to the low value of f_N in relation to the target F_N, whereas in the case of $b_{30}t$, failures are due also to the low value of the simulated μ_t in the final years before retirement. Therefore, acting on the general level of the final fund f_N will have a greater effect on the probability of failing the target in the b_{30} case, because we are reducing the impact of the only cause of failures in the b_{30} case (but only one of the 2 causes in the $b_{30}t$ case), and this leads to a steeper decrease in the probability curve.

These two features explain the shape of the curves in Figure 6 and the fact that in the last 2 graphs (r*- and λ-based targets) the b_{30} and the $b_{30}t$ curves exhibit a cross-over.

Mean shortfall.
The three graphs of Figure 7 report the mean shortfall with the three different targets. We observe the following points:

1 – when we consider the initial values of the mean shortfall, when α=0, in the two cases of b_{30} and $b_{30}t$ we see that this is always higher for $b_{30}t$ than for b_{30}. A failure in the case of b_{30} occurs when $f_{30} < F_{30}$, and the size of the failure is proportional to the difference $f_{30} - F_{30}$. But, in the case of $b_{30}t$, the size of the failure is also determined by the behaviour of the low-risk asset prior to retirement, with a poor performance leading to a bigger deviation in the case of $b_{30}t$ than in the case of b_{30}. We see that, in most cases of failure for both b_{30} and $b_{30}t$, the low risk asset has a return lower than 4% in the last 5 years before retirement (also noting that failure depends heavily on the return of the low risk asset, due to the lifestyle strategy adopted and cautiousness of the strategy), and this leads to bigger deviations for $b_{30}t$ and therefore a bigger mean shortfall;

2 – as before, the mean shortfall curve increases more smoothly for $b_{30}t$ than for b_{30} (Note that, in some places, the curve appears to be decreasing, but this is a small effect and may be due to the relatively small number of cases considered – recalling that the mean shortfall does not consider the deviations from the target of the 1000 simulations, but only the simulations in which the target is missed). This difference in smoothness can be explained in the following way. When we move from the left to the right of the graph, the strategy becomes more aggressive and this leads to results that are more spread out in terms of final fund achieved, affecting directly the net replacement achieved in the b_{30} case. In the $b_{30}t$ case, this effect is not so direct, as the final fund has still to be transformed into a net replacement ratio by applying the variable conversion rate; therefore, it may happen that, in some of the cases of failure, the effect of a very low final fund may be reduced by a lower than usual conversion rate (i.e. a lower $\tilde{a}_x$). Again, increasing the aggressiveness of the strategy will affect only the final fund achieved, not the distribution of the simulated rates of return of the low risk asset, and this has a more adverse effect on the b_{30} case than the $b_{30}t$ case.

These 2 features explain the shape of the curves and the fact that, in all of the figures, the b_{30} and the $b_{30}t$ curves cross over.

VaR at 5% confidence.

The three graphs of Figure 8 report the VaR at 5% confidence with the three different targets. We observe the following points:

1 – the VaR curves are fairly stable when α changes and they slightly increase with low – medium values of α in the case of μ-based targets;

2 – the VaR curve lies always at a higher level for the b_{30} case than for the $b_{30}t$ one and the two curves do not cross over. This underlines the annuity risk that the member has to face, as the VaR at 5% confidence is the 5th percentile of the distribution of the net replacement ratio and the fact that the VaR of the $b_{30}t$ is lower than the VaR of the b_{30} indicates that the poor outcomes are more adverse in the case of a variable annuity rate;

3 – for high values of α the VaR curves stay in the same range (60% - 65%), regardless of the target chosen.

The relative stability of the VaR values shown in Figure 8 is explained by the effect of increasing α on the underlying simulated distributions of b_{30} and $b_{30}t$. These are not shown here but examination of these distributions and the associated sample moments and quantiles shows that, as α increases, the means, medians and standard deviations all increase, as do the higher quantiles eg the 75th. However, the lower quantiles are relatively stable (and these, of course, directly affect the VaR estimates), representing the net effect of 2 conflicting influences – distributions with increasing means and increasing spreads about the means.

FIGURE 6: PROBABILITY OF FAILING THE TARGET

Probability of failing the target (mu)

Probability of failing the target (r*)

Probability of failing the target (lambda)

FIGURE 7: MEAN SHORTFALL

Mean shortfall (mu)
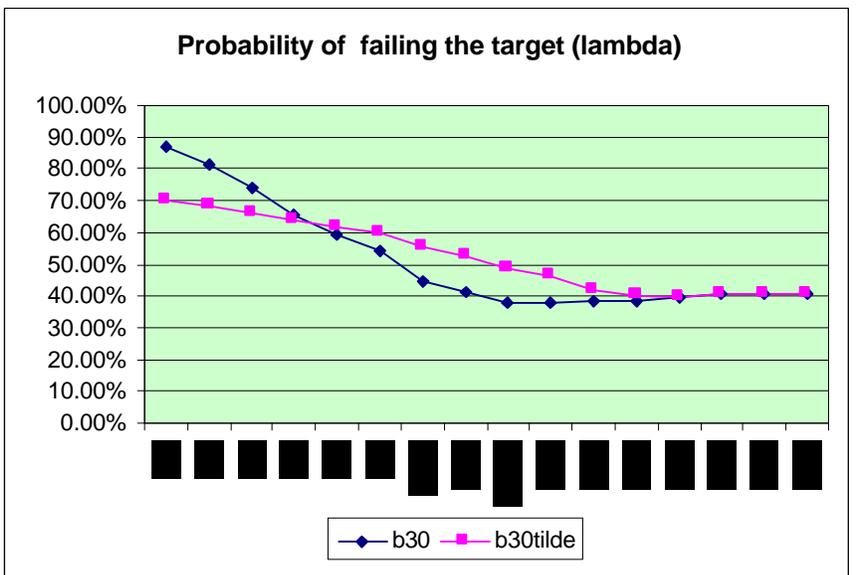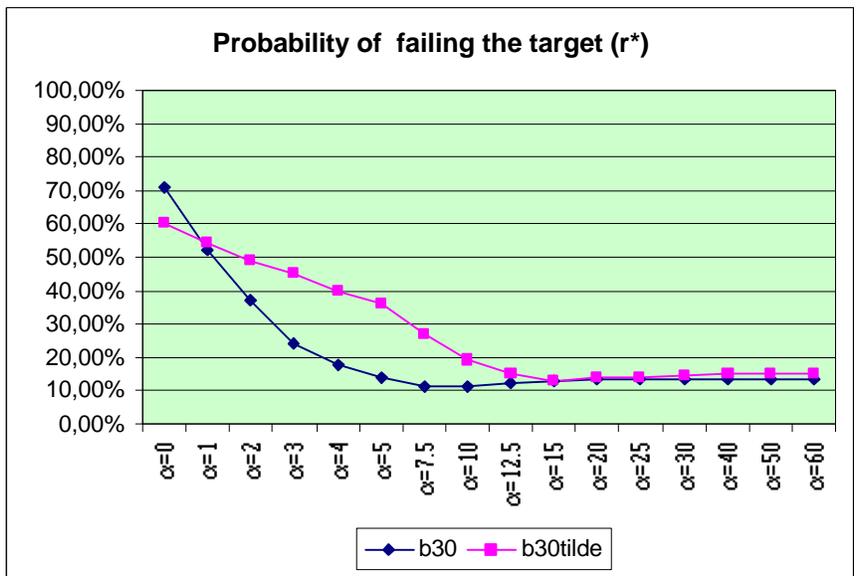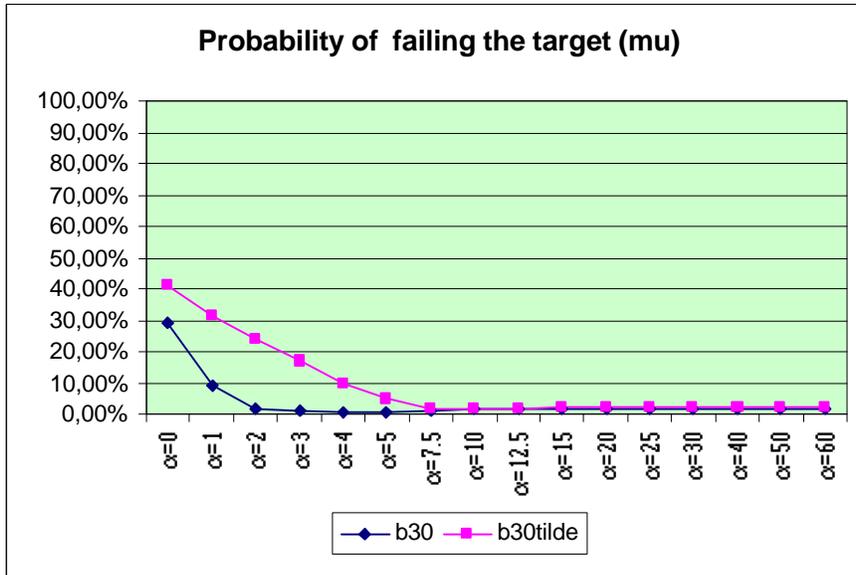
Mean shortfall (r*)

Mean shortfall (lambda)
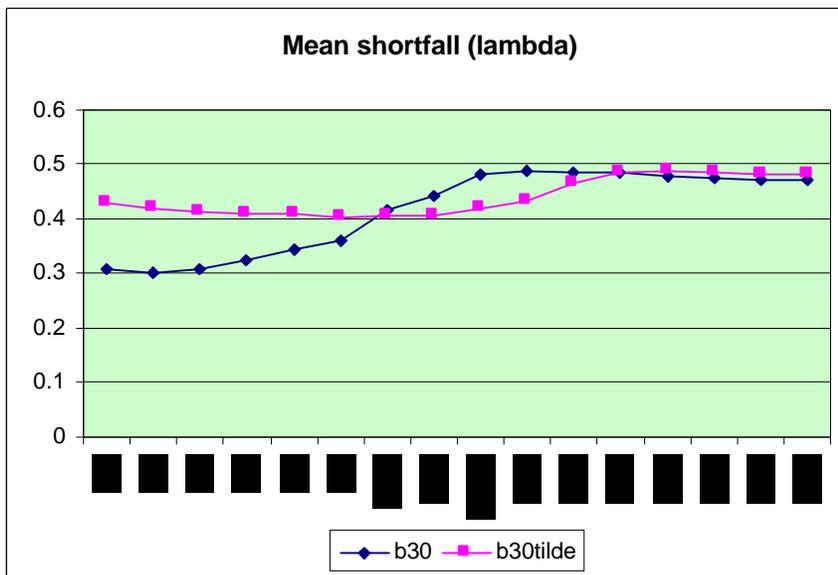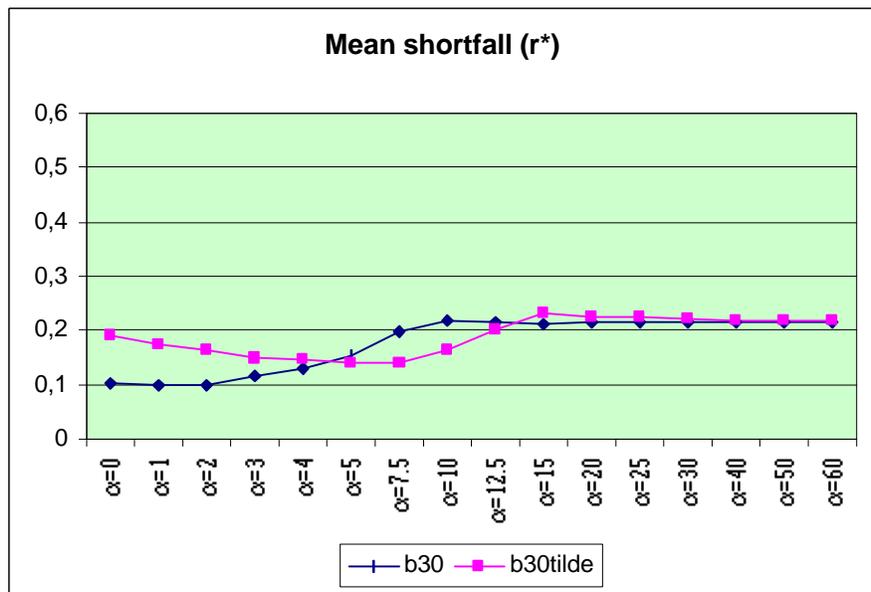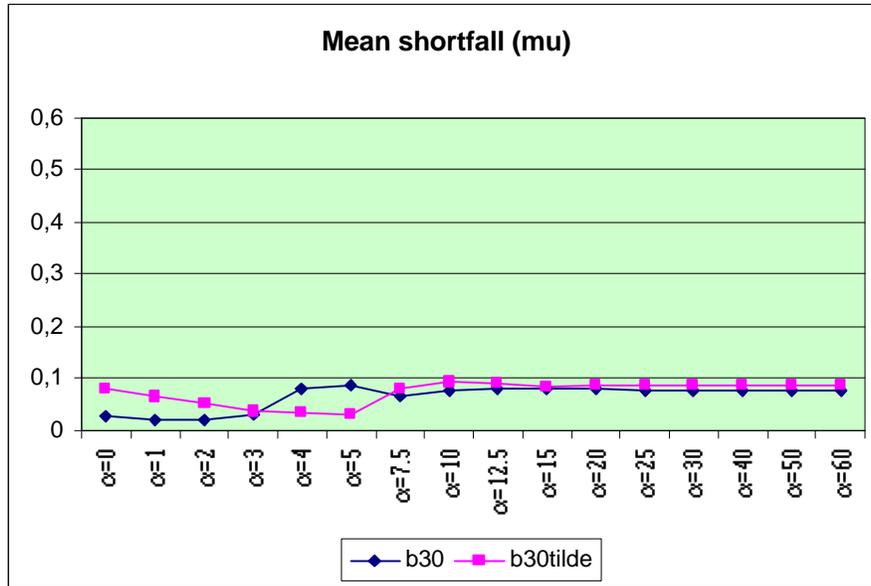
FIGURE 8: VaR AT 5% CONFIDENCE

VaR at 5% confidence (mu)

VaR at 5% confidence (r*)

VaR at 5% confidence (lambda)
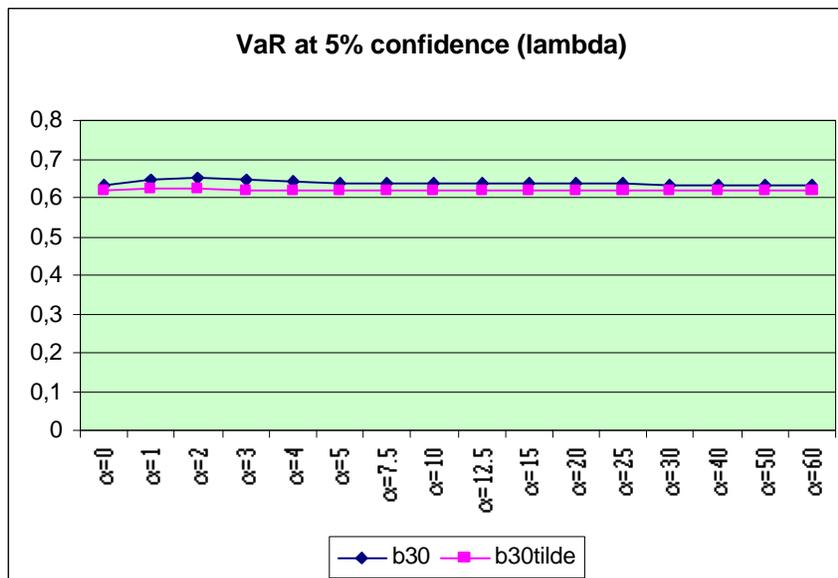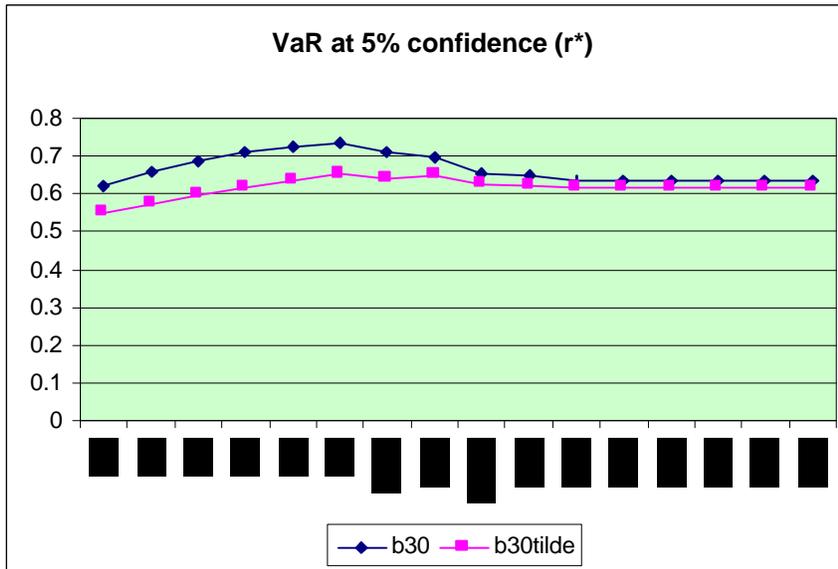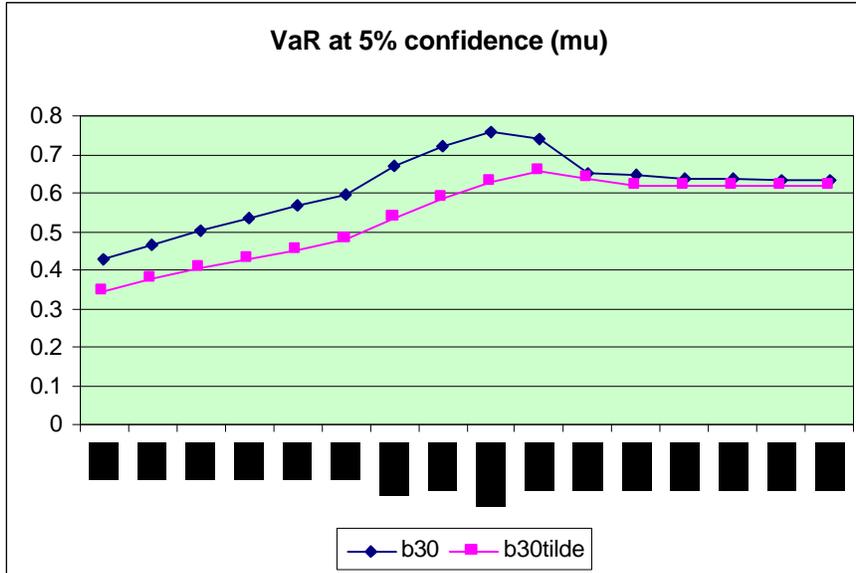
6.4 EFFECT OF CHANGING CORRELATION BETWEEN THE ASSET RETURNS

Up to now (in Figures 5-8) the downside risk borne by the member has been analysed in the case of uncorrelated assets, i.e. $\rho=0$. Tables 1, 2 and 3 and Figures 9 and 10 report the different results relative to the three risk measures when we let the correlation factor vary from $\rho = -1$ to $\rho = 1$ (for a term of 30 years and r*-based targets).

We observe that, when ρ changes, there are very small changes in the behaviour of the three risk measures. This phenomenon is to be explained by considering the behaviour of the optimal investment strategies: the optimal investment strategy changes very little when the correlation factor ρ varies, and this is due to the small influence of the coefficient ρ on y_t^* (consider formula (3.22) above), compared to the much larger influence of other factors like α and the targets F_t. As noted earlier, a similar result has been found by Chopra and Ziemba (1993) in respect of covariances.

TABLE 1: PROBABILITY OF FAILING THE TARGET WHEN ρ CHANGES

PROBABILITY OF FAILING THE TARGET										
	ρ=−1		ρ=−1/2		ρ=0		ρ=1/2		ρ=1	
	b30	b30t	b30	b30t	b30	b30t	b30	b30t	b30	b30t
α=0	60.00%	57.90%	65.80%	59.30%	71.20%	60.00%	75.50%	61.40%	73.30%	62.00%
α=1	44.00%	51.60%	47.90%	53.10%	52.20%	54.30%	57.40%	54.90%	57.50%	54.60%
α=2	31.60%	47.30%	34.00%	48.10%	36.90%	48.80%	39.00%	50.20%	40.00%	49.30%
α=3	22.90%	41.80%	23.30%	43.20%	24.40%	45.00%	25.10%	45.00%	23.90%	43.90%
α=4	17.60%	37.00%	18.10%	38.60%	18.00%	40.00%	17.90%	40.90%	16.40%	40.90%
α=5	15.00%	33.10%	14.90%	34.30%	14.10%	36.10%	14.20%	36.90%	13.60%	36.40%
α=7.5	10.80%	23.20%	11.00%	24.90%	11.30%	26.70%	11.70%	28.90%	11.90%	28.80%
α=10	10.70%	17.50%	11.10%	18.70%	11.20%	19.50%	11.80%	20.10%	12.70%	20.70%
α=12.5	11.30%	14.00%	11.90%	14.70%	12.30%	15.00%	12.60%	15.10%	13.40%	15.40%
α=15	12.30%	13.50%	12.60%	13.40%	12.90%	13.10%	13.20%	13.90%	13.50%	14.20%
α=20	13.00%	13.30%	13.00%	13.70%	13.40%	13.90%	13.50%	14.00%	13.60%	14.50%
α=25	13.30%	13.80%	13.30%	13.90%	13.40%	14.10%	13.50%	14.40%	13.70%	14.70%
α=30	13.30%	13.80%	13.30%	14.20%	13.50%	14.40%	13.70%	14.80%	13.80%	14.90%
α=40	13.50%	14.40%	13.50%	14.80%	13.60%	14.90%	13.80%	15.10%	13.90%	15.40%
α=50	13.60%	14.70%	13.60%	14.90%	13.70%	15.10%	13.80%	15.20%	13.90%	15.40%
α=60	13.60%	14.80%	13.60%	15.00%	13.70%	15.10%	13.80%	15.20%	13.90%	15.40%

TABLE 2: MEAN SHORTFALL WHEN ρ CHANGES

MEAN SHORTFALL										
	ρ=−1		ρ=−1/2		ρ=0		ρ=1/2		ρ=1	
	b30	b30t	b30	b30t	b30	b30t	b30	b30t	b30	b30t
α=0	10.93%	18.20%	10.57%	18.51%	10.39%	18.97%	10.35%	19.09%	10.95%	19.28%
α=1	10.55%	16.91%	10.16%	17.18%	9.77%	17.54%	9.26%	17.87%	9.57%	18.20%
α=2	10.89%	15.38%	10.42%	15.93%	9.87%	16.45%	9.51%	16.58%	9.43%	17.08%
α=3	12.04%	14.62%	11.98%	14.98%	11.52%	15.16%	11.27%	15.74%	11.82%	16.14%
α=4	13.38%	14.10%	13.03%	14.26%	13.09%	14.52%	13.49%	14.88%	15.16%	15.08%
α=5	14.20%	13.55%	14.29%	13.80%	15.48%	13.93%	16.15%	14.37%	17.90%	15.06%
α=7.5	18.71%	14.41%	19.17%	14.00%	19.72%	13.89%	20.69%	13.79%	22.41%	14.59%
α=10	20.32%	16.52%	20.69%	16.14%	21.79%	16.26%	22.26%	16.59%	22.14%	16.78%
α=12.5	20.94%	20.09%	20.86%	19.82%	21.32%	20.13%	21.89%	20.92%	21.65%	21.05%
α=15	20.52%	21.12%	20.83%	21.99%	21.23%	23.30%	21.60%	22.58%	22.02%	23.05%
α=20	20.83%	22.36%	21.44%	22.32%	21.41%	22.63%	21.84%	23.09%	22.10%	22.75%
α=25	20.88%	22.11%	21.37%	22.45%	21.62%	22.56%	21.86%	22.51%	21.95%	22.56%
α=30	21.00%	22.27%	21.40%	22.05%	21.48%	22.16%	21.67%	22.14%	22.00%	22.49%
α=40	20.84%	21.67%	21.26%	21.51%	21.52%	21.85%	21.62%	22.07%	21.87%	22.09%
α=50	20.76%	21.48%	21.15%	21.61%	21.39%	21.73%	21.63%	21.99%	21.87%	22.09%
α=60	20.76%	21.38%	21.15%	21.49%	21.39%	21.74%	21.63%	21.99%	21.87%	22.09%

TABLE 3: VaR AT 5% CONFIDENCE WHEN ρ CHANGES

VAR AT 5% CONFIDENCE										
	ρ=−1		ρ=−1/2		ρ=0		ρ=1/2		ρ=1	
	b30	b30t	b30	b30t	b30	b30t	b30	b30t	b30	b30t
α=0	61.80%	54.57%	62.40%	54.34%	62.04%	54.74%	59.81%	54.40%	58.32%	54.83%
α=1	65.65%	56.54%	65.12%	57.08%	65.67%	57.24%	66.06%	56.90%	63.32%	57.24%
α=2	67.88%	59.23%	68.21%	58.74%	68.83%	59.77%	67.87%	59.54%	66.85%	59.28%
α=3	70.15%	60.49%	70.69%	60.98%	71.14%	61.49%	71.47%	61.66%	70.79%	61.78%
α=4	71.98%	62.55%	72.85%	62.74%	72.62%	63.31%	72.69%	62.71%	71.71%	63.80%
α=5	73.63%	64.56%	73.67%	65.33%	73.15%	65.33%	71.76%	64.74%	70.52%	64.92%
α=7.5	73.76%	65.57%	71.98%	65.02%	71.03%	64.04%	68.10%	64.65%	64.71%	62.52%
α=10	70.60%	65.57%	70.49%	64.75%	69.68%	64.83%	65.22%	63.48%	64.71%	61.76%
α=12.5	69.65%	64.72%	67.05%	63.50%	65.22%	62.57%	64.71%	61.76%	63.76%	61.76%
α=15	65.22%	63.80%	65.22%	62.93%	64.71%	61.78%	63.76%	61.76%	63.60%	61.76%
α=20	63.80%	61.78%	63.76%	61.76%	63.60%	61.76%	63.60%	61.65%	63.60%	61.65%
α=25	63.60%	61.76%	63.60%	61.65%	63.60%	61.65%	63.60%	61.65%	63.60%	61.65%
α=30	63.60%	61.65%	63.60%	61.65%	63.60%	61.65%	63.60%	61.65%	63.36%	61.65%
α=40	63.60%	61.65%	63.60%	61.65%	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%
α=50	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%
α=60	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%	63.36%	61.65%

FIGURE 9: RISK MEASURES WHEN ρ CHANGES (b30)

Probability of failing target when ρ changes (b30)

Legend: ρ=−1, ρ=−1/2, ρ=0, ρ=1/2, ρ=1

Mean shortfall when ρ changes (b30)

Legend: ρ=−1, ρ=−1/2, ρ=0, ρ=1/2, ρ=1

VaR at 5% confidence when ρ changes (b30)

Legend: ρ=−1, ρ=−1/2, ρ=0, ρ=1/2, ρ=1
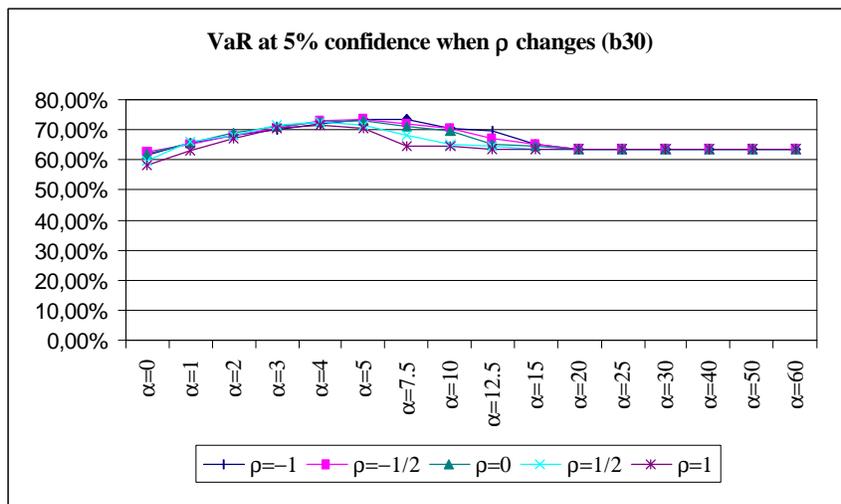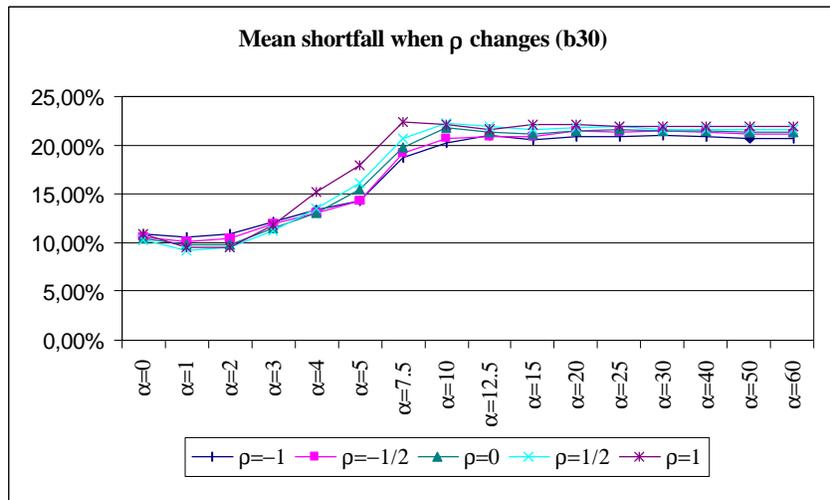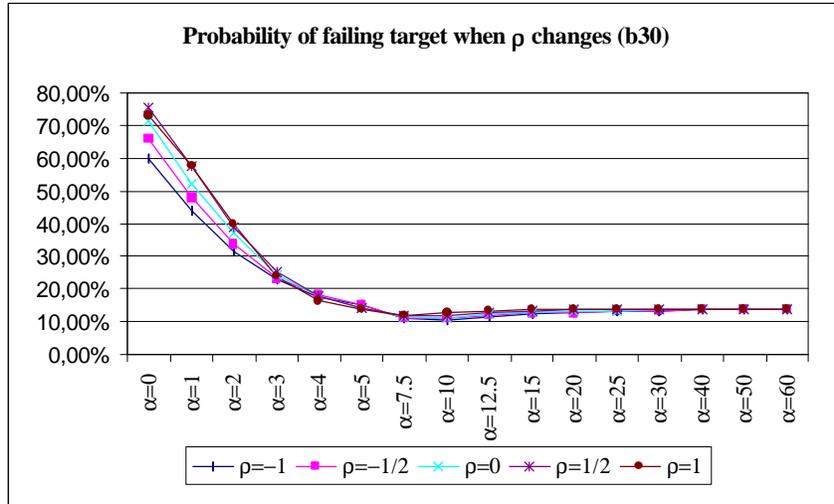
FIGURE 10: RISK MEASURES WHEN ρ CHANGES (b30t)

Probability of failing target when ρ changes (b30t)

Mean shortfall when ρ changes (b30t)

VaR at 5% confidence when ρ changes (b30t)
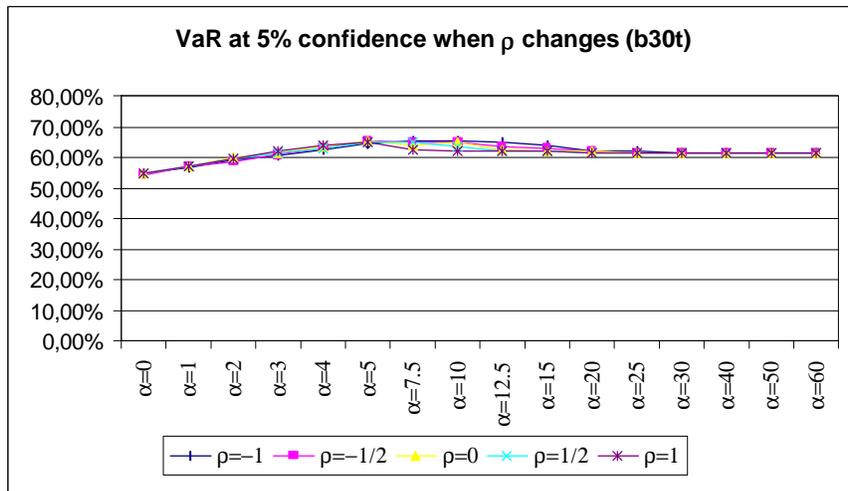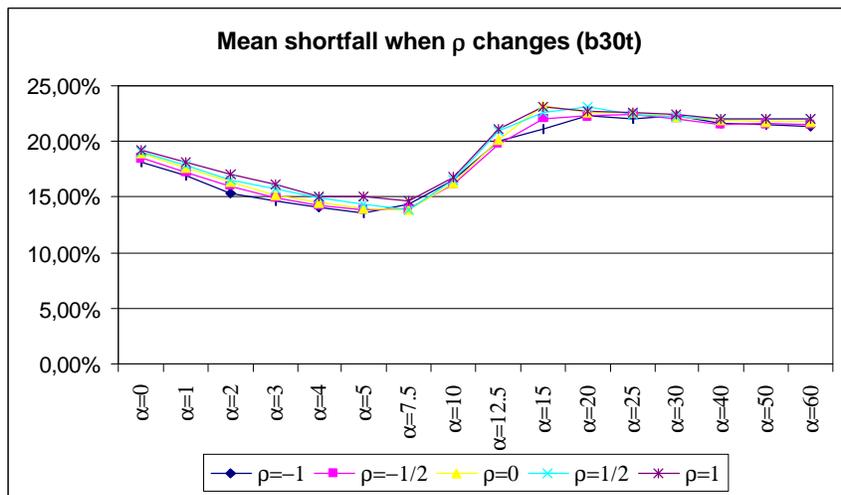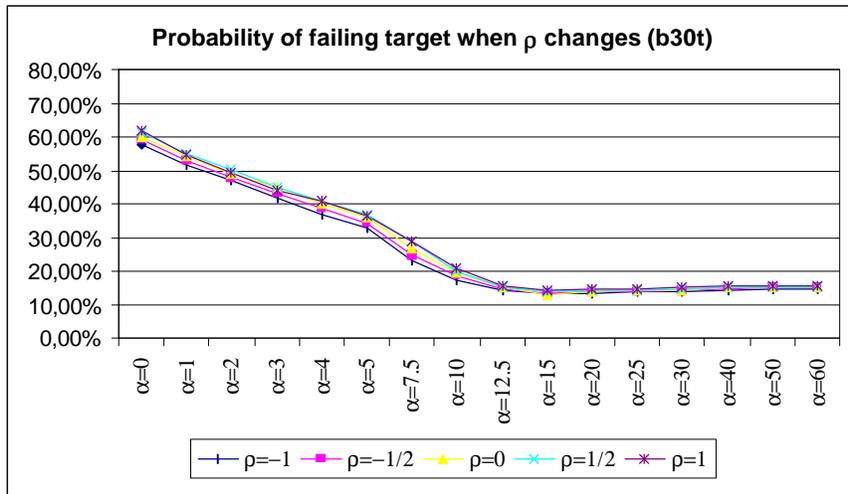
6.5 DISTRIBUTIONS OF b30 AND b30t WITH THE DIFFERENT TARGETS

Figures 11 and 12 report the observed distributions of b30 and b30t over the 1000 simulations, with the different targets (for the case where $\alpha = 0$ and $\rho = 0$)[9]. The final target in terms of net replacement ratio (B30) is also indicated, in order to facilitate comparisons.

First, we notice that a simple comparison of the distributions of the net replacement ratio with different targets is not feasible, the ranges of the distributions being different (we recall that the targets affect considerably the optimal investment strategy, and therefore the outcomes). However, we notice that the different targets affect the distributions of the net replacement ratio achieved in a similar way in the cases b30 and b30t: moving down the page, from λ-based targets to μ-based targets, we see that the mode moves towards the left tail of the distribution and, furthermore, the right tail becomes longer, and the left tail shorter.

The comparison between b30 and b30t is even more striking. Whereas in the b30 case we notice that the final target B30 lies next to the mode (sometimes it is located to the right of the mode, but with a small gap between them), in the b30t case B30 is located to the right of the mode, with a considerable gap between the two values. This underlines again the annuity risk borne by the member of a DC scheme: when the annuity risk is added to the investment risk, there are fewer chances for the target to be achieved.

On the other hand, we notice also that the results, as we expected from our previous discussion, are much more spread out in the b30t case (for example, they range between 0.27 and 2.04 in the r*-target case, against the range [0.33, 1.1] for the b30 case). This means that, in the case of higher than expected investment returns, the variable annuity factor turns to be advantageous to the member, leading to a higher net replacement ratio than with a fixed annuity factor.

---

[9] Note the different scales used for the horizontal axes in Figures 11 and 12.

FIGURE 11: DISTRIBUTIONS OF b30

**Frequency of b30,   -targets (B_{30}=1.49)**

Frequency of b30, =0, r*-targets (B_{30}=0.88)

Frequency of b30, =0,   -targets (B_{30}=0.48)
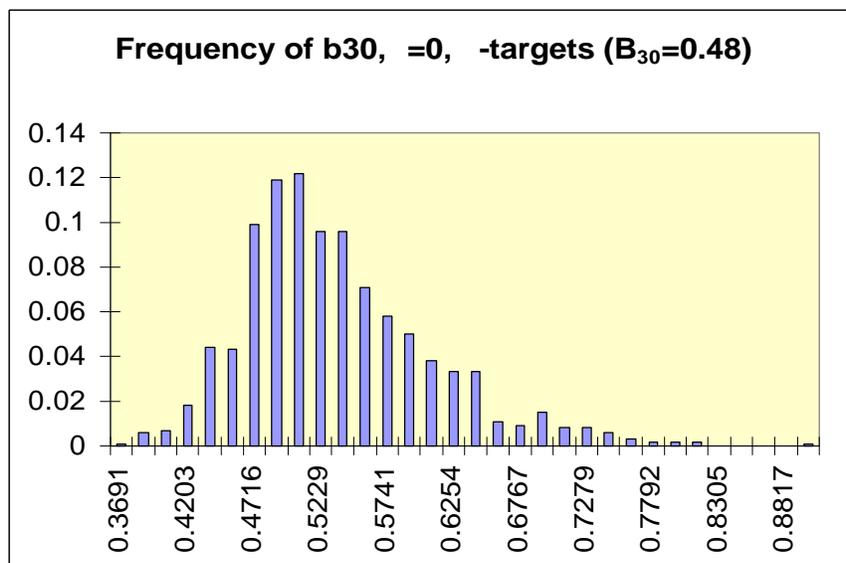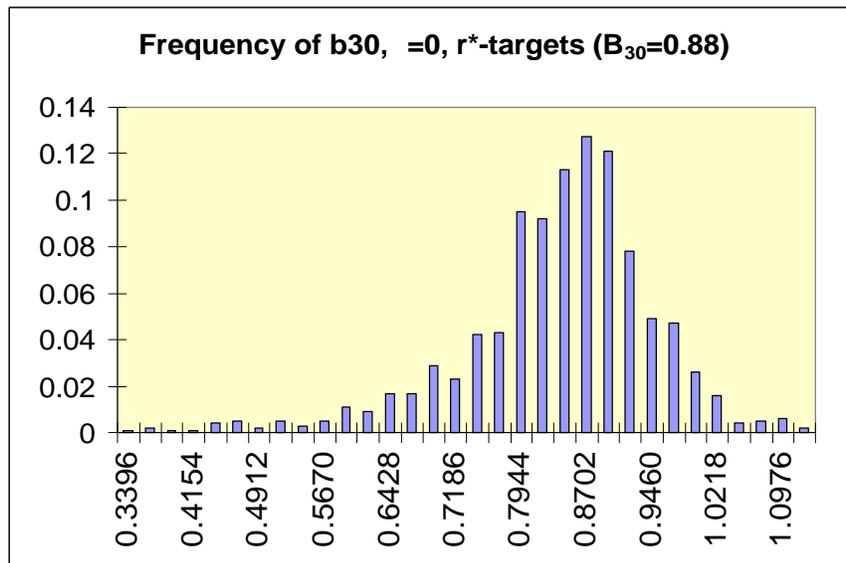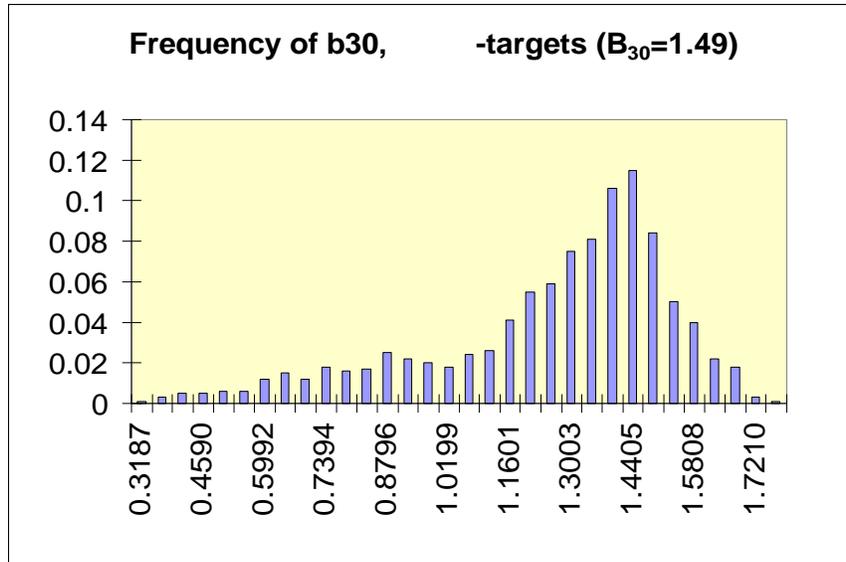
FIGURE 12: DISTRIBUTIONS OF b30t

Frequency of b30t, =0, -targets (B$_{30}$=1.49)

Frequency of b30t, =0, r*-targets (B$_{30}$=0.88)

Frequency of b30t, =0, -targets (B$_{30}$=0.48)
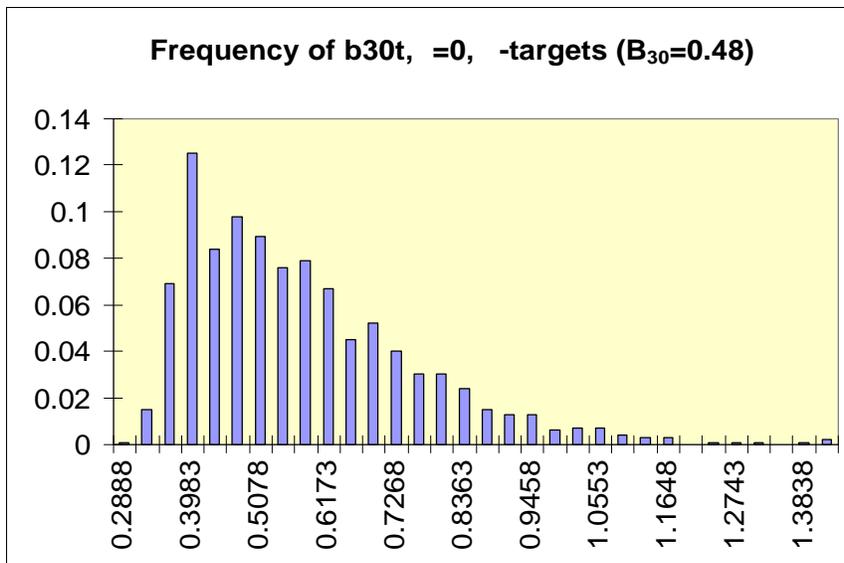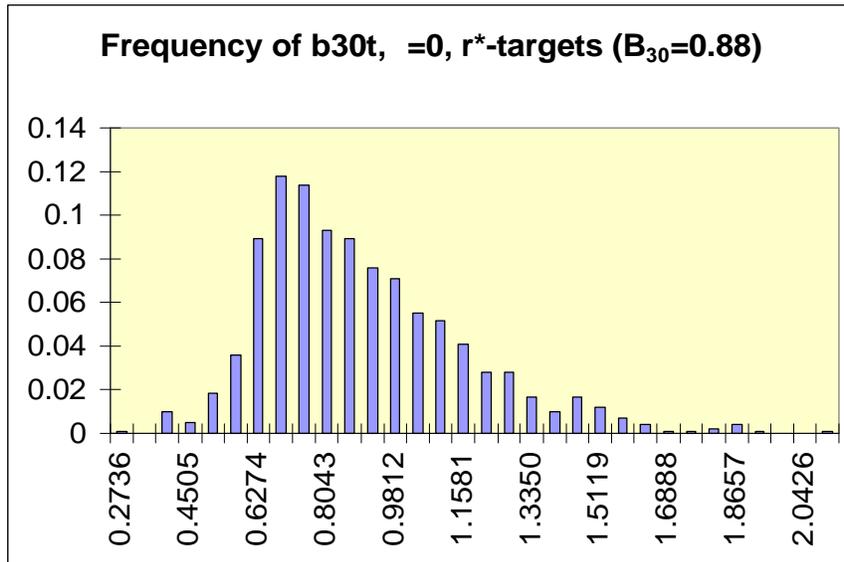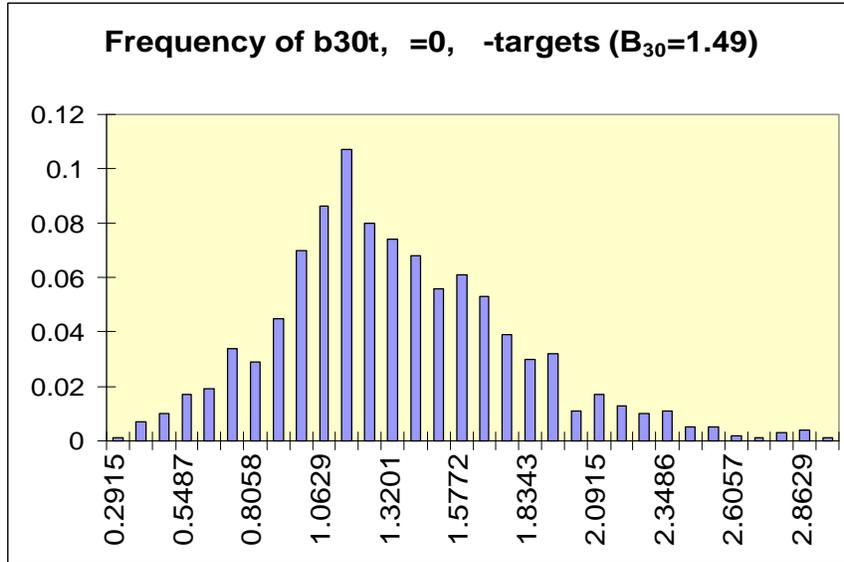
7. CONCLUDING COMMENTS

In our paper we have derived a formula for the optimal investment allocation (using a dynamic programming approach as in Vigna and Haberman (2001)) in a defined contribution pension scheme whose fund is invested in n assets.

Then, considering the particular case of a 2-asset portfolio, we have investigated the financial risks of a DC scheme, considering the investment risk borne by the member during the accumulation period up to retirement and the annuity risk arising when the fund is converted into an annuity at retirement.

We have analysed numerically the investment allocation and the downside risk faced by the retiring member, where approximately optimal investment strategies have been adopted (or "sub-optimal" investment strategies, due to the fact that we have added the constraint that short selling is not allowed). The behaviour of the optimal investment strategy has been analysed allowing for changes in the disutility function (via the parameter α). Three different risk measures have been considered in analysing the final net replacement ratios achieved by the member: the probability of failing the target, the mean shortfall and a Value at Risk measure. We have considered the relationship between the risk aversion of the member and these different risk measures in order to understand better the choices confronting different categories of scheme member. We have also considered the sensitivity of the results to the level of the correlation coefficient ρ between the two asset returns.

The main results of our investigation are the following:

- The optimal investment strategy to be adopted by a risk averse member of a defined contribution pension scheme is the so-called lifestyle strategy, which consists in investing the whole fund in high risk assets at the beginning of the membership, and then switching into low risk assets some years prior to retirement. The point in time when the switch occurs depends on both the risk aversion of the individual (the more risk averse, the sooner the switch) and the time to retirement (the longer the accumulation period, the later the switch).
- The optimal investment strategy for a risk neutral member of a defined contribution pension scheme is to invest the whole fund in high risk assets for the whole period of membership, and never switch into low risk assets.
- The different risk measures of the downside risk faced by the member of a defined contribution pension scheme give different and contradictory indications.
- Looking at the results for the probability of failing the target, the conclusion seems to be that increasing the risk aversion of the individual or (which is the same) adopting more cautious strategies leads to a greater number of failures relative to a target, chosen a priori.
- Looking at the results for the mean shortfall, the conclusion seems to be that increasing the risk aversion of the individual or (which is the same) adopting more cautious strategies leads to slightly lower mean shortfall, which means more limited reductions in pensioner income when a failure occurs.
- Looking at the VaR results, we note that the VaR at 1%, 5% and 10% level does not change very much when changing the risk aversion of the individual.
- The effect of changing the correlation factor ρ between the assets is very small both on the optimal investment strategy and on the downside risk borne by the member of the scheme.
- The annuity risk borne by the member is underlined both by the behaviour of the VaR (with VaR values of bn being always higher than VaR values of bnt), which shows that poor outcomes are more adverse when a variable conversion factor is used to buy the annuity, and by the observed distributions of bn and bnt (for example, the mode of the distribution being higher in the bn case than in the bnt case).

We suggest that the risk profile of the individual and the trade-off between different risk measures of the downside risk borne by the member (for example, the number of failures and size of failures in respect of a certain target), are important factors to be taken into consideration when determining the choice of investment strategies in defined contribution pension schemes.

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APPENDIX

SKETCH OF THE PROOF

We want to prove that there are some sequences of coefficients $\{P_t\}_t$, $\{Q_t\}_t$ and $\{R_t\}_t$ such that:

(A1) $$J(f_t, t) = P_t f_t^2 - 2Q_t f_t + R_t$$

for t = 0, 1 , 2,…N.

We show this by induction.

INDUCTION BASIS

For t=N (A1) is true. In fact,

(A2) $$J(f_N, N) = C_N = \theta \,[(F_N - f_N)^2 + \alpha(F_N - f_N)]$$

And therefore:

$P_N = \theta$
$Q_N = \theta\, F_N - \alpha/2$
$R_N = \theta\, F_N^2 + \alpha\, F_N$

INDUCTION STEP

Let us assume that (A1) is true for t+1, i.e.:

(A3) $$J(f_{t+1}, t+1) = P_{t+1} f_{t+1}^2 - 2Q_{t+1} f_{t+1} + R_{t+1}$$

for some coefficients P_{t+1}, Q_{t+1} and R_{t+1}.

We will now show that (A1) is then true also for t.

By application of Bellmann's principle (see 3.10 above), we obtain:

(A4) $$J(f_t, t) = \min_{\{y^i_t\}_{i=1,\dots,n-1}} [(F_t - f_t)^2 + \alpha(F_t - f_t) + \beta E[J(f_{t+1}, t+1) \mid f_t]]$$

i.e.:

(A5) $$J(f_t, t) = (F_t - f_t)^2 + \alpha(F_t - f_t) + \beta \min_{\{y^i_t\}_{i=1,\dots,n-1}} E[J(f_{t+1}, t+1) \mid f_t]$$

Applying the induction step, we have:

(A6) $$E[J(f_{t+1}, t+1)\big| f_t] = P_{t+1} E[f_{t+1}^2 \big| f_t] - 2Q_{t+1} E[f_{t+1} \big| f_t] + R_{t+1}$$

Considering now the expression for f_t (see 2.1 above), we have:

(A7) $\quad E[f_{t+1}|f_t] = (f_t + c)\left[\sum_{i=1}^{n-1} E(W_i - W_n)y_i + E(W_n)\right]$

(A8) $\quad E[f_{t+1}^2|f_t] = (f_t + c)^2 \times \left\{\sum_{i=1}^{n-1} E(W_i - W_n)^2 y_i^2 + \right.$

$$+ 2\sum_{j<i}\sum_{i=1}^{n-1} E[(W_i - W_n)(W_j - W_n)]y_i y_j + 2\sum_{i=1}^{n-1}[E(W_i W_n) - E(W_n^2)]y_i + 2E(W_n^2)\left.\right\}.$$

By replacing terms in (A6), we get:

(A9) $\quad E[J(f_{t+1}, t+1)|f_t] = \Psi(y_1, y_2, ..., y_{n-1})$

(we omit the index t for convenience of notation).

We now solve the optimisation problem:

(A10) $\quad \min_{y_1,...,y_{n-1}} \Psi(y_1, y_2, ..., y_{n-1})$

and we find the gradient of Ψ, $\underline{\nabla}\Psi$, and set it equal to $\underline{0}$:

$$\frac{\partial \Psi}{\partial y_k} = P_{t+1}(f_t + c)^2\left\{2E[(W_k - W_n)^2]y_k + 2\sum_{i\neq k} E[(W_i - W_n)(W_k - W_n)]y_i + 2[E(W_k W_n) - E(W_n^2)]\right\} -$$
$$-2Q_{t+1}(f_t + c)[E(W_k - W_n)], \qquad \text{for k = 1, 2, ..., n-1.} \qquad (A11)$$

Setting $\dfrac{\partial \Psi}{\partial y_k} = 0$, for k = 1, 2, ..., n-1 yields:

$$E[(W_k - W_n)^2]y_k + \sum_{i\neq k} E[(W_i - W_n)(W_k - W_n)]y_i = \frac{Q_{t+1}}{P_{t+1}(f_t + c)}[E(W_k - W_n)] - [E(W_k W_n) - E(W_n^2)]$$

for k = 1, 2, ..., n-1. $\qquad\qquad$ (A12)

We have now to solve the linear system with n-1 unknowns and n-1 equations:

(A13) $\qquad\qquad\qquad\qquad \mathbf{A}\,\underline{y} = \underline{h}$

with:

(A14) $\qquad\qquad \mathbf{A} = (a_{ij})_{i,j=1,...,n-1}$

(A15) $\qquad\qquad a_{ij} = E[(W_i - W_n)(W_j - W_n)]$

(A16) $\qquad\qquad \underline{y} = (y_1, y_2, ..., y_{n-1})^T$

(A17) $$\underline{h} = (h_1, h_2, \ldots, h_{n-1})^T$$

(A18) $$h_i = \frac{Q_{t+1}}{P_{t+1}(f_t + c)} b_i - d_i$$

(A19) $$b_i = E[(W_i - W_n)] \qquad \text{and} \qquad \underline{b} = (b_1, b_2, \ldots, b_{n-1})^T$$

(A20) $$d_i = E(W_i \, W_n) - E(W_n^2) \qquad \text{and} \qquad \underline{d} = (d_1, d_2, \ldots, d_{n-1})^T$$

We note that the Hessian matrix of Ψ is the matrix $\mathbf{A}$, so we have now to prove the following lemma:

LEMMA

The matrix $\mathbf{A}$, as defined by (A14) and (A15), is positive definite.

Proof.

$\mathbf{A}$ is positive definite if and only if:

$\underline{v}^T \mathbf{A} \, \underline{v} > 0$ for any vector $\underline{v} = (v_1, v_2, \ldots v_{n-1})^T$.

Let us define:

$Z_i := W_i - W_n$ for $i = 1,\ldots,n-1$, then:

(A21) $$\underline{v}^T \mathbf{A} \, \underline{v} =$$

$$\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} a_{ij} v_i v_j = \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} E(Z_i Z_j) v_i v_j = E\left(\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} Z_i Z_j v_i v_j\right) = E\left(\sum_{j=1}^{n-1} Z_j v_j \sum_{i=1}^{n-1} Z_i v_i\right) = E\left[\left(\sum_{i=1}^{n-1} Z_i v_i\right)^2\right] \geq 0$$

.

$\underline{v}^T \mathbf{A} \, \underline{v} = 0$ if and only if the random variable $\sum_{i=1}^{n-1} Z_i v_i$ is identically zero, i.e. if and only if:

(A22) $$\sum_{i=1}^{n-1} Z_i v_i \equiv 0.$$

The proof proceeds by contradiction.

If (A22) is true, then:

(A23) $$\sum_{i=1}^{n-1} W_i v_i \equiv W_n \left(\sum_{i=1}^{n-1} v_i\right)$$

which means:

$$
(A24) \qquad W_n \equiv \frac{\displaystyle\sum_{i=1}^{n-1} v_i W_i}{\left(\displaystyle\sum_{i=1}^{n-1} v_i\right)}
$$

so, W_n would be the weighted average of n-1 lognormal random variable, which is impossible, as W_n itself is a lognormal random variable. Hence, we have a contradiction.

Therefore, **A** is definite positive.

The system (A13) has a unique solution given by:

$$
(A25) \qquad y_i^* = \frac{A_i}{|A|} \qquad\qquad i = 1, 2, \ldots, n\text{-}1
$$

where A_i is the determinant of the matrix obtained by **A**, replacing the ith column with the vector **h**.

Then, the following holds:

$$
(A26) \qquad \mathbf{y_t^{i*}} = \frac{k_t \mathbf{B_i} - \mathbf{D_i}}{|\mathbf{A}|} \qquad\qquad i = 1, 2, \ldots, n\text{-}1
$$

with $k_t = \dfrac{Q_{t+1}}{P_{t+1}(f_t + c)}$, B_i is the determinant of the matrix obtained by **A**, replacing the ith column with the vector **b**, and D_i is the determinant of the matrix obtained by **A**, replacing the ith column with the vector **d.**

Since **A** is definite positive, $\underline{\mathbf{y}}_t^* = (\mathbf{y_t^{1^*}}, \mathbf{y_t^{2^*}}, \ldots \mathbf{y_t^{n-1^*}})$ is the minimum of ψ:

$$
(A27) \qquad \min_{y_1, \ldots, y_{n-1}} \Psi\,(y_1, y_2, \ldots, y_{n-1}) = \psi\,(y_1^*, y_2^*, \ldots, y_{n-1}^*)
$$

(where we omit the index t for convenience of notation).

By replacing the values of y_1^*, y_2^*, ..., y_{n-1}^* obtained, we get:

$$
(A28) \qquad \psi\,(y_1^*, y_2^*, \ldots, y_{n-1}^*) = L_t\, f_t^2 + M_t\, f_t + N_t
$$

where:

$$\text{(A29)} \qquad L_t = P_{t+1}\left[\sum_{i=1}^{n-1} a_{ii}\frac{D_i^2}{|A|^2} + \sum_{j<i}\sum_{i=1}^{n-1} a_{ij}\frac{D_iD_j}{|A|^2} - 2\sum_{i=1}^{n-1} d_i\frac{D_i}{|A|} + E(W_n^2)\right]$$

$$\text{(A30)} \qquad M_t = 2cP_{t+1}\left[\sum_{i=1}^{n-1} a_{ii}\frac{D_i^2}{|A|^2} + \sum_{j<i}\sum_{i=1}^{n-1} a_{ij}\frac{D_jD_i}{|A|^2} - 2\sum_{i=1}^{n-1} d_i\frac{D_i}{|A|} + E(W_n^2)\right] -$$
$$- 2\,Q_{t+1}\left[\sum_{i=1}^{n-1} a_{ii}\frac{B_iD_i}{|A|^2} + \sum_{j<i}\sum_{i=1}^{n-1} a_{ij}\left(\frac{B_iD_j + B_jD_i}{|A|^2}\right) - \sum_{i=1}^{n-1}\left(\frac{d_iB_i + b_iD_i}{|A|}\right) + E(W_n)\right]$$

$$N_t = c^2 P_{t+1}\left[\sum_{i=1}^{n-1} a_{ii}\frac{D_i^2}{|A|^2} + \sum_{j<i}\sum_{i=1}^{n-1} a_{ij}\frac{D_jD_i}{|A|^2} - 2\sum_{i=1}^{n-1} d_i\frac{D_i}{|A|} + g_{X_n}(2)\right] -$$
$$\text{(A31)} \qquad - 2cQ_{t+1}\left[\sum_{i=1}^{n-1} a_{ii}\frac{B_iD_i}{|A|^2} + \sum_{j<i}\sum_{i=1}^{n-1} a_{ij}\left(\frac{B_iD_j + B_jD_i}{|A|^2}\right) - \sum_{i=1}^{n-1}\frac{b_iD_i}{|A|} + g_{X_n}(1)\right] +$$
$$+ \frac{Q_{t+1}^2}{P_{t+1}}\left[\sum_{i=1}^{n-1} a_{ii}\frac{B_i^2}{|A|^2} + \sum_{j<i}\sum_{i=1}^{n-1} a_{ij}\frac{B_iB_j}{|A|^2} - 2\sum_{i=1}^{n-1} b_i\frac{B_i}{|A|}\right] + R_{t+1}$$

Therefore:

$$\text{(A32)} \qquad J(f_t, t) = (F_t - f_t)^2 + \alpha(F_t - f_t) + \beta\min_{\{y^i_t\}_{i=1,\dots,n-1}} E[J(f_{t+1}, t+1) \mid f_t] =$$
$$= (F_t - f_t)^2 + \alpha(F_t - f_t) + \beta\,\Psi(y_t^{1^*}, y_t^{2^*}, \dots y_t^{n-1^*}) = (F_t - f_t)^2 + \alpha(F_t - f_t) + \beta(L_t f_t^2 + M_t f_t + N_t) =$$
$$= P_t f_t^2 - 2Q_t f_t + R_t$$

with:

$$\text{(A33)} \qquad \begin{cases} P_t = 1 + \beta L_t \\ Q_t = F_t + \dfrac{\alpha}{2} - \dfrac{\beta}{2}M_t \\ R_t = F_t + \alpha F_t + \beta N_t \end{cases}$$

which is (A1).