Solution of the Determinantal Assignment Problem using the Grassmann Matrices

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Abstract

The paper provides a direct solution to the Determinantal Assignment Problem (DAP) which unifies all frequency assignment problems of Linear Control Theory. The current approach is based on the solvability of the exterior equation $\underline{v}_1 \wedge \underline{v}_2 \wedge \ldots \wedge \underline{v}_m = \underline{z}$, where $\underline{z} \in \wedge^m \mathcal{U}$, $\underline{v}_i \in \mathcal{U}, \mathcal{U}$ is an $n-$dimensional vector space over $\mathcal{F}$ which is an integral part of the solution of DAP. New criteria for existence of solution and their computation based on the properties of structured matrices referred to as Grassmann matrices. The solvability of this exterior equation is referred to as decomposability of $\underline{z}$, and it is in turn characterised by the set of Quadratic Plücker Relations (QPR) describing the Grassmann variety of the corresponding projective space. Alternative new tests for decomposability of the multi-vector $\underline{z}$ are given in terms of the rank properties of the Grassmann matrix, $\Phi_n^m(\underline{z})$ of the vector $\underline{z}$, which is constructed by the coordinates of $\underline{z} \in \wedge^m \mathcal{U}$. It is shown that the exterior equation is solvable ($\underline{z}$ is decomposable), if and only if $dim\mathcal{N}_n^m(\underline{z}) = m$, where $\mathcal{N}_n^m(\underline{z}) = \mathcal{N}_r\{\Phi_n^m(\underline{z})\}$; the solution space for a decomposable $\underline{z}$, is the space $\mathcal{N}_n^m(\underline{z}) = \mathcal{V}_{\underline{z}} = sp\{\underline{v}_i,\ i \in \tilde{m}\}$. This provides an alternative linear algebra characterisation of the decomposability problem and of the Grassmann variety to that defined by the QPRs. Further properties of the Grassmann matrices are explored by defining the Hodge-Grassmann matrix as the dual of the Grassmann matrix. The connections of the Hodge-Grassmann matrix to the solution of exterior equations is examined, and an alternative new characterisation of decomposability is given in terms of the dimension of its image space. The framework based on the Grassmann matrices provides the means for the development of a new computational method for the solutions of the exact DAP, (when such solutions exist), as well as computing approximate solutions, when exact solutions do not exist.

1. Introduction

The Determinantal Assignment Problem (DAP) has emerged as the abstract problem to which the study of pole, zero assignment of linear systems may be reduced [4], [5], [6], [8]. The multilinear nature of DAP suggests that the natural framework for its study is that of exterior algebra [1]. The study of DAP [4] may be reduced to a linear problem of zero assignment of polynomial combinants [17] and a standard problem of multilinear algebra, that is the decomposability of multivectors [1]. The solution of the linear subproblem, whenever it exists, defines a linear space in a projective space $\mathcal{P}^t$, whereas decomposability is characterised by the set of Quadratic Plücker Relations (QPR), which define the Grassmann variety of $\mathcal{P}^t$ [2]. Thus, solvability of DAP is reduced to a problem of finding real intersections between the linear variety and the Grassmann variety of $\mathcal{P}^t$.

The importance of tools and techniques of algebraic geometry for control theory problems has been demonstrated by the work in [9], [10], [4] etc. The approach adopted in [4], [5], [6], [8] differs from that in [9], [10] in the sense that the problem is studied in a projective, rather than an affine space setting; the former approach relies on exterior algebra and on the explicit description of the Grassmann variety, in terms of the QPRs, and has the advantage of being computational. The multilinear nature of DAP has been recently handled by a "blow up" type methodology, using the notion of degenerate solution and known as "Global Linearisation" [8]. Under certain conditions, this methodology allows the computation of solutions of the DAP problem.

This paper introduces a new approach for the computation of exact solutions of DAP, whenever such solutions exist, as well as approximate solutions, when exact solutions do not exist based on some new results for the solution of exterior equations. This new approach is based on an alternative, linear algebra type, criterion for decomposability of multivectors to that defined by the QPRs [1], in terms of the properties of structured matrices, referred to as Grassmann matrices. Such matrices provide a new explicit matrix representation of abstract results on skew symmetric tensors [12], [13] relating to decomposability of multivectors [1]. The decomposability of the multivector $\underline{z} \in \wedge^m \mathcal{U}$, where $\mathcal{U}$ is a vector space, is equivalent to the solvability of the exterior equation $\underline{v}_1 \wedge \underline{v}_2 \wedge \ldots \wedge \underline{v}_m = \underline{z}$, with $\underline{v}_i \in \mathcal{U}$. The conditions for decomposability are given by the set of QPRs [1],[2] and the solution space $\mathcal{V}_{\underline{z}} = sp\{\underline{v}_i, i \in \tilde{m}\}$ may be constructed as shown in [3]. The present approach handles simultaneously the question of decomposability and the reconstruction of $\mathcal{V}_{\underline{z}}$. For every $\underline{z} \in \wedge^m \mathcal{U}$ with Plücker coordinates $\{a_\omega, \omega \in Q_{m,n}\}$, the Grassmann matrix $\Phi_n^m(\underline{z})$ of $\underline{z}$ has been introduced in [14] as a structured matix based on the Plücker coordinates . The study of the properties of $\Phi_n^m(\underline{z})$ is the subject of this paper; it is shown, that $rank\{\Phi_n^m(\underline{z})\} \geq n-m$, for all $\underline{z} \neq \underline{0}$, and $\underline{z}$ is decomposable, if and only if, the equality sign holds. If $rank\{\Phi_n^m(\underline{z})\} = n-m$, then the solution space $\mathcal{V}_{\underline{z}}$ is defined by $\mathcal{V}_{\underline{z}} = \mathcal{N}_r\{\Phi_n^m(\underline{z})\}$. The rank based test for decomposability is easier to handle than the QPRs and provides a simple method for the computation of $\mathcal{V}_{\underline{z}}$. This provides an alternative characterization of the Grassmann variety of a projective space in terms of the Grassmann matrices, which are structured matrices defined for every point of the projective apace, which have a fixed rank *n-m*.

The development of the new computational framework requires the development of the properties of Grassmann matrices. These are further developed by using the Hodge duality [1] that leads to the definition of the Hodge-Grassmann matrix $\Phi_n^{n-m}(\underline{z}^*)$, which is defined as the Grassmann matrix of the Hodge dual of the multivector $\underline{z}$, that is $\underline{z}$*. The properties of $\Phi_n^{n-m}(\underline{z}^*)$ are dual to those of the Grassmann matrix $\Phi_n^m(\underline{z})$. In fact decomposability turns out to be an image problem for the transpose of the Hodge-Grassmann matrix and the Quadratic Plucker Relations can be concretely written in terms of the Grassmann and Hodge-Grassmann matrices. It is shown that the kernel of Grassmann matrix and the image of the transpose of the Hodge Grassmann matrix of a multivector define two fundamental spaces that determine a canonical representation of multivectors. The relation between those two spaces are established which leads to new criteria for decomposability, as well as introducing a new metric for distance from decomposability, which provides new ways to compute approximate solutions . A number of interesting relationships between the singular values of $\Phi_n^m(\underline{z})$ and $\Phi_n^{n-m}(\underline{z}^*)$ are established.It is shown that the two matrices have the same right singular vectors and the sum of squares of the corresponding singular values is equal to the squared norm of $\underline{z}$. The approximate DAP is addressed is formulated as a distance problem from decomposability, when the exact problem is not solvable. This is expressed as minimization of the the distance between the linear variety associated with the linear sub-problem of DAP and the Grassmann variety, characterizing the set of all decomposable vectors. The results on decomposability based on the Grassmann matrices provides an appropriate framework for computing solutions of the approximate DAP based on an optimization problem.

The paper is organised as follows: Section 2 provides a brief review of DAP motivating the significance of the exterior equation in control problems, whereas Section 3 summarises known results on decomposability. The results on the properties of Grassmann matrices are given in Section 4. In Section 5 the Hodge-Grassmann matrix is defined and some results related to this operator are reported. In Section 6 some properties of the kernel of Grassmann matrix and the image of the transpose of the Hodge Grassmann matrix of a multivector are presented in relation to the decomposability problem. Finally, in Section 7 we use the Grassmann matrices framework to develop the computation of exact and approximate DAP as an optimizatrion problem.

Throughout the paper the following notation is adopted: If $\mathcal{F}$ is a field, or ring then $\mathcal{F}^{m\times n}$ denotes the set of $m\times n$ matrices over $\mathcal{F}$. If H is a map, then $\mathcal{R}(H)$, $\mathcal{N}_r(H)$, $\mathcal{N}_l(H)$ denote the range, right, left nullspaces respectively. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from the set $\tilde{n}=\{1,2,\dots,n\}$. If $\mathcal{V}$ is a vector space and $\{\underline{v}_{i_1},\dots,\underline{v}_{i_k}\}$ are vectors of $\mathcal{V}$ then $\underline{v}_{i1}\wedge\dots\wedge\underline{v}_{ik}=\underline{v}_{\omega}\wedge$, $\omega=(i_1,\dots,i_k)$ denotes their exterior product and $\wedge^r\mathcal{V}$ the $r-$th exterior power of $\mathcal{V}$ [1]. If $H\in\mathcal{F}^{m\times n}$ and $r\le\min\{m,n\}$, then $C_r(H)$ denotes the $r-$th compound matrix of H [11]. In most of the following, we will assume that $\mathcal{F}=\mathbb{R}$.

2. The General Determinantal Assignment Problem

Let $M(s)\in\mathbb{R}^{m\times l}[s]$, $m>l$, $rank_{\mathbb{R}[s]}\{M(s)\}=l$ and consider the set of matrices $\mathcal{H}=\{H(s):H(s)\in\mathbb{R}^{l\times m}[s],\ rank_{\mathbb{R}[s]}\{H(s)\}=l\}$; the subset of $\mathcal{H}$ defined by all $H\in\mathbb{R}^{l\times m}$ will be denoted by $\mathcal{H}_{\mathbb{R}}$. Finding $H\in\mathcal{H}_{\mathbb{R}}$ such that the polynomial

$$f_M(s,H)=det\{H(s)M(s)\} \tag{1}$$

has assigned zeros, is defined as the *Determinantal Assignment Problem (DAP)* [4]; if $H\in\mathcal{H}_{\mathbb{R}}$, then the corresponding problem is defined as the *constant DAP* ($\mathbb{R}-$DAP) [4]. By considering subsets of $\mathcal{H}$ made up from matrices with block diagonal structure such as $\mathcal{H}_v=bl-diag\{H_i(s),\ i\in\tilde{v}\}$, $\mathcal{H}_v=[I_p;bl-diag\{H_i(s),\ i\in\tilde{v}\}]$, the *Decentralised-DAP (D-DAP)* versions are defined in [5].

The different versions of DAP have been introduced as the abstract unifying descriptions of frequency assignment problems (pole, zero) that arise in linear systems theory. Thus pole assignment by state, constant output feedback [4], [6] and zero assignment by constant squaring down [4], [7] may be studied within the $\mathbb{R}-$DAP framework, whereas the corresponding problems of decentralised control belong to the $\mathbb{R}$-D-DAP class [5]. The general case, DAP, covers the dynamic version of frequency assignment problems. If we require that $f_M(s,H)$ is an arbitrary Hurwitz polynomial, then different classes of *Determinantal Stabilisation Problems (DSP)* are defined. DAP is clearly multilinear, as far as the parameters in H, and thus the natural setting for its study is that of exterior algebra [1]. Let $\underline{h}_i^t,\ \underline{m}_i(s),\ i \in \tilde{l}$ be the rows of $H \in \mathcal{H}$, columns of $M(s)$. Then,

$$C_l(H) = \underline{h}_1^t \wedge \ldots \wedge \underline{h}_l^t = \underline{h}^t \wedge \in \mathbb{R}^{1\times q}[s],\ q = \begin{pmatrix} m \\ l \end{pmatrix}, C_l(M(s)) = \underline{m}_1(s) \wedge \ldots \wedge \underline{m}_l(s) = \underline{m}(s) \wedge \in \mathbb{R}^q[s]$$

and by the Binet-Cauchy Theorem [11] we have

$$f_M(s,H) = C_l(H)C_l(M(s)) = <\underline{h}\wedge, \underline{m}(s)\wedge> = \sum_{\omega \in Q_{m,n}} h_\omega m_\omega(s) \qquad (2)$$

where $<\cdot,\cdot>$ denotes scalar product, $\omega = (\underline{i}_1, \ldots, \underline{i}_l) \in Q_{l,m}$ and $h_\omega, m_\omega(s)$ are the entries in $\underline{h}\wedge,\ \underline{m}(s)\wedge$ respectively. Note that h_ω is the $l \times l$ minor of H, which corresponds to the ω set of rows of H and thus is a multilinear alternating function of the h_{ij} entries of H. DAP may be reduced to a linear and a standard multilinear subproblem as shown below [4]:

- ***Linear Subproblem of DAP:*** Let $\underline{m}(s)\wedge = \underline{p}(s) \in \mathbb{R}^q[s]$. Investigate the existence of $\underline{k}(s) \in \mathbb{R}^q[s]$ such that for some given $\alpha(s) \in \mathbb{R}[s], d = deg\alpha(s)$,

$$f_{\underline{p}}(s,\underline{k}) = \underline{k}(s)^t \underline{p}(s) = \sum_{i=1}^{q} k_i(s) p_i(s) = \alpha(s) = \underline{\alpha}^t \underline{e}_d(s) \qquad (3)$$

  where $\underline{e}_d(s) = [1, s, \ldots, s^d]^t$.

- ***Multilinear subproblem of DAP:*** Assume that for the given $\alpha(s)$ part *(i)* is solvable and let $\mathcal{K}(\underline{\alpha})$ be the family of solutions. Determine whether there exists $H \in \mathcal{H},\ H^t = [\underline{h}_1, \ldots, \underline{h}_l]$ such that

$$\underline{h}_1 \wedge \ldots \wedge \underline{h}_l = \underline{k},\ \underline{k} \in \mathcal{K}(\underline{\alpha}) \qquad (4)$$

□

$f_{\underline{p}}(s,\underline{k})$, as defined by (3) for a given $\underline{p}(s)$, is called an $\mathbb{R}[s]-$*polynomial combinant*, [4], [17] if $k_i \in \mathbb{R}[s]$, and as $\mathbb{R}-$*polynomial combinant*, if $k_i \in \mathbb{R}$ [4]. The solution of the exterior equation (4) is a standard problem of exterior algebra, known as *decomposability of multivectors* [1]. Multilinear algebra also plays an important role in the linear subproblem since $f_{\underline{p}}(s,\underline{k})$ is generated by the decomposable multivector $\underline{m}(s)\wedge = \underline{p}(s)$. The solvability of the linear subproblem is a standard problem of linear algebra; in fact, if $k_i \in \mathbb{R}[s]$, is equivalent to solving a Diofantine equation over $\mathbb{R}[s]$, whereas if $k_i \in \mathbb{R}$, it is reduced to the solution of a system of linear equations [4]. In the latter case, the solution of (3) defines a linear space $\Psi(\underline{\alpha})$ of the projective space $\mathcal{P}^{q-1}$ [6]. The exterior equation (4) is central to the DAP approach and its solvability is characterised by the set of *Quadratic Plücker Relations (QPR)* [1], [2], which in turn describe the Grassmann variety $\Omega(l,m)$ of $\mathcal{P}^{q-1}$ [2]. Thus, solvability of $\mathbb{R}-$DAP is equivalent to finding real intersections between $\Psi(\underline{\alpha})$ and $\Omega(l,m)$; this clearly demonstrates the algebraic geometry context of DAP. The aim of this paper is to provide alternative criteria for solvability of (4), to those defined by the QPRs, as well as a simple procedure for reconstructing H. A summary of key notions and results from exterior algebra are summarised first.

3. Decomposability of Multi-vectors: Background Results

Let $\mathcal{U}$ be a vector space over a field $\mathcal{F}$ and let $\mathcal{G}(m,\mathcal{U})$ be the Grassmannian (set of all $m-$ dimensional subspaces of $\mathcal{U}$). For every $\mathcal{V}\in\mathcal{G}(m,\mathcal{U})$ the injection map $\wedge^p f: \wedge^p\mathcal{V}\to\wedge^p\mathcal{U}$ is well defined and if $p=m,$ then $\wedge^m\mathcal{V}$ is a 1-dimensional subspace of $\wedge^m\mathcal{U}$ if $\{\underline{v}_i, i\in\tilde{m}\}$ is a basis of $\mathcal{V}$, then $\wedge^m\mathcal{V}$ is spanned by $\underline{v}_1\wedge\ldots\wedge\underline{v}_m$. Let $B_{\mathcal{U}}=\{\underline{u}_i, i\in\tilde{n}\}$, $B_{\mathcal{U}}^m=\{\underline{u}_\omega\wedge:\ \underline{u}_\omega\wedge=\underline{u}_{i_1}\wedge\ldots\wedge\underline{u}_{i_m},\ \omega=(i_1,\ldots,i_m)\in Q_{m,n}\}$ be a basis of $\mathcal{U}$ and $\wedge^m\mathcal{U}$ spaces respectively. The general vector $\underline{z}\in\wedge^m\mathcal{U}$ may be expressed as

$$\underline{z}=\sum_{\omega\in Q_{m,n}} a_\omega\underline{u}_\omega\wedge \tag{5}$$

where $\{a_\omega,\omega\in Q_{m,n}\}$ are the coordinates of $\underline{z}$ with respect to $B_{\mathcal{U}}^m$. A vector $\underline{z}\in\wedge^m\mathcal{U}$ is called *decomposable,* if there exist $\underline{v}_i\in\mathcal{U},\ i\in\tilde{m}$ such that

$$\underline{v}_1\wedge\ldots\wedge\underline{v}_m=\underline{z} \tag{6}$$

The vector space $\mathcal{V}_{\underline{z}}=span_{\mathcal{F}}\{\underline{v}_i, i\in\tilde{m}\}$ is called the *generating space* of $\underline{z}$. It is known that if $\underline{z},\ \underline{z}\in\wedge^m\mathcal{U}$ are nonzero and decomposable, then $\underline{z}=c\cdot\underline{z}(c\neq 0)$ is equivalent to $\mathcal{V}_{\underline{z}}=\mathcal{V}_{\underline{z}}\in\mathcal{G}(m,\mathcal{U})$ and $\underline{z}$ is called a *Grassmann Representative (GR)* of $\mathcal{V}_{\underline{z}}$. All GRs of $\mathcal{V}\in\mathcal{G}(m,\mathcal{U})$ differ by a $c\in\mathcal{F}, c\neq 0$ and are denoted by $\underline{g}(\mathcal{V})$. The coordinates of a decomposable vector $\underline{z}\in\wedge^m\mathcal{U},\ \{a_\omega,\omega\in Q_{m,n}\}$ are known as the *Plücker coordinates (PC)* of $\mathcal{V}_{\underline{z}}$. The lexicographically ordered set of PCs is completely determined by $\mathcal{V}$ to within $c\in\mathcal{F}$. Note, that not every $\underline{z}\in\wedge^m\mathcal{U}$ is necessarily decomposable; if $\{a_\omega,\omega\in Q_{m,n}\}$ are the coordinates of $\underline{z}\in\wedge^m\mathcal{U}$ then $\underline{z}$ is decomposable if and only if the following conditions hold true [2]:

$$\sum_{k=1}^{m+1}(-1)^k a_{i_1,\ldots,i_{m-1},j_k}a_{j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{m+1}}=0 \tag{7}$$

where $1\le i_1<\cdots<i_{m-1}\le n$ and $1\le j_1<j_2<\cdots<j_{m+1}\le n.$ The set of quadratics defined by (7) are known as *Quadratic Plücker Relations (QPR)* and describe an $(n-m)m-$dimensional algebraic variety, $\Omega(m,n),$ of the projective space $\mathcal{P}^{\sigma-1},\sigma=\begin{pmatrix} m\\ n\end{pmatrix},$ known as *Grassmann variety* [2]. The map defined by $\nu:\ \mathcal{V}\in\mathcal{G}(m,\mathcal{U})\to\wedge^m\mathcal{V}$ in $\wedge^m\mathcal{U}$ expresses a natural injective correspondence between $\mathcal{G}(m,\boldsymbol{\mathcal{U}})$ and 1-dimensional subspaces of $\wedge^m\mathcal{U}$. By associating to every $\wedge^m\mathcal{V}$ the PCs $\{a_\omega,\omega\in Q_{m,n}\},$ the map $\rho:\mathcal{G}(m,\mathcal{U})\to\mathcal{P}^{\sigma-1}$ is defined, and it is known as the *Plücker embedding* [2] of $\mathcal{G}(m,\mathcal{U})$ in $\mathcal{P}^{\sigma-1}$; the image of $\mathcal{G}(m,\mathcal{U})$ under ρ is $\Omega(m,n).$ The term decomposability of a multi-vector and the solution of the exterior equation (6) are equivalent terms.

The notion of the GR is central in the study of DAP. For the rational vector space over $\mathbb{R}(s)$, $\mathcal{X}_M=\mathcal{R}(M(s)),$ a canonical polynomial ($\mathbb{R}[s]$) GR may be defined and through that a basis free invariant of $\mathcal{X}_M$ the Plücker matrix P_M [4]; the rank properties of P_M define the solvability conditions of the linear subproblem of $\mathbb{R}-$DAP. Using the set of QPRs for computation of solutions of $\mathbb{R}-\mathrm{DAP}$ is difficult. An alternative test for decomposability that also allows a more convenient framework for computations is considered next.

4. The Grassmann Matrix and Decomposability of Multivectors

The Grassmann matrix of $\underline{z} \in \wedge^m \mathcal{U}$ [14] is introduced in this section and a number of its properties are examined. This matrix provides an alternative test for decomposability of $\underline{z}$, which also allows the computation of the $\mathcal{V}_{\underline{z}}$ solution space in an easy manner. We state first the following result.

Proposition (1) [1]: Let $\mathcal{U}$ be an $n-$dimensional vector space over $\mathcal{F}$ and let $\underline{0} \neq \underline{z} \in \wedge^m \mathcal{U}$. Then, $\underline{z}$ is decomposable, if and only if, there exists a set of linearly independent vectors $\{\underline{v}_i, i \in \tilde{m}\}$ in $\mathcal{U}$ such that

$$\underline{v}_i \wedge \underline{z} = \underline{0}, \forall i \in \tilde{m} \tag{8}$$

□

This result is central in deriving the set of QPRs [1], as well as in deriving the alternative test that will be developed here . The coordinates of $\underline{v} \wedge \underline{z}$ in (8) may be computed as follows.

Lemma (1): Let $B_{\mathcal{U}} = \{\underline{u}_i,\ i \in \tilde{n}\}$ be a basis of $\mathcal{U}$, $B_{\mathcal{U}}{}^m = \{\underline{u}_\omega \wedge,\ \omega \in Q_{m,n}\}$ the corresponding basis of $\wedge^m \mathcal{U}$ and let $\underline{v} = \sum_{t=1}^m c_t \underline{u}_t$, $\underline{z} = \sum_{\omega \in Q_{m,n}} a_\omega \underline{u}_\omega \wedge$. Then,

$$\underline{v} \wedge \underline{z} = \sum_{\gamma \in Q_{m+1,n}} b_\gamma \underline{u}_\gamma \wedge, \quad b_\gamma = \sum_{k=1}^{m+1} (-1)^{k-1} c_{\gamma(k)} a_{\gamma(\hat{k})} \tag{9}$$

where $\gamma(k)$ denotes the $k-$th element of $\gamma \in Q_{m+1,n}$ and $\gamma(\hat{k})$ is the sequence $(\gamma(1), \ldots, \gamma(k-1), \gamma(k+1), \ldots, \gamma(m+1)) \in Q_{m,n}$.

Proof:

$$\underline{v} \wedge \underline{z} = (\sum_{t=1}^n c_t \underline{u}_t) \wedge (\sum_{\omega \in Q_{m,n}} a_\omega \underline{u}_\omega \wedge) = \sum_{t=1}^n \sum_{\omega \in Q_{m,n}} c_t a_\omega \underline{u}_t \wedge \underline{u}_\omega \wedge \tag{10}$$

To compute b_γ for a fixed $\gamma \in Q_{m+1,n}$ in $\underline{v} \wedge \underline{z}$ we argue as follows: A pair t, ω produces $e_\gamma \wedge$, if and only if $\{t\} \cup I_m\{\omega\} = I_m\{\gamma\}$, where $I_m\{\omega\}$ denotes the set of indices in ω (not necessarily ordered). In other words, there exists $k \in \{1, \ldots, m+1\}$ for which $\gamma(\hat{k}) = \omega$ and $t = \gamma(k)$. Then,

$$\underline{u}_t \wedge \underline{u}_\omega \wedge = \underline{u}_{\gamma(k)} \wedge \underline{u}_{\gamma(1)} \wedge \ldots \wedge \underline{u}_{\gamma(k-1)} \wedge \underline{u}_{\gamma(k+1)} \wedge \ldots \wedge \underline{u}_{\gamma(m+1)} = (-1)^{k-1} \underline{u}_\gamma \wedge \tag{11}$$

If $\{t\} \cup I_m\{\omega\} = I_m\{\omega\}$, then clearly $b_\gamma = 0$. By (10), (11) and the previous arguments the expression for b_γ in (9) readily follows.

□

Notation: Let $\gamma = (j_1, j_2, \ldots, j_k, j_{m+1}) \in Q_{m+1,n}, m+1 \leq n$. We denote by $Q^\gamma_{m,m+1}$ the subset of $Q_{m,n}$ sequences with elements taken from the γ set of integers. $Q^\gamma_{m,m+1}$ has $m+1$ elements and the sequences in it are defined from γ by deleting an index in γ. Thus, we may write:

$$Q^\gamma_{m,m+1} = \{\rho_\gamma[\hat{j}_k] = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{m+1}), k \in m+1\} \tag{12}$$

Definition (1): Let $\{a_\omega, \omega \in Q_{m,n}\}$ be the coordinates of $\underline{z} \in \wedge^m \mathcal{U}$ with respect to a basis $B_{\mathcal{U}}^m$ of $\wedge^m \mathcal{U}$, $m+1 \le n$, $\gamma = (j_1, \ldots, j_k, j_{m+1}) \in Q_{m+1,n}$. We may define the function $\phi : \{i : i = 1, \ldots, n\} \times \{\gamma, \gamma \in Q_{m+1,n}\} \to \mathcal{F}$ with $\rho_\gamma[\hat{j}_k] = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{m+1}) \in Q_{m,m+1}^\gamma$ by:

$$\begin{cases} \phi_\gamma^i = \phi_\gamma(i) = 0, \text{ if } i \neq \gamma \\ \phi_\gamma^i = \phi_\gamma(i) = sign(j_k : \rho_\gamma[\hat{j}_k]) a_{\rho_\gamma[\hat{j}_k]}, \text{ if } i = j_k \in \gamma \end{cases} \tag{13}$$

where and $sign(j_k : \rho_\gamma[\hat{j}_k]) = sign(j_k, j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{m+1})$.

□

With the above notation we may state the following result:

Proposition (2): Let $B_{\mathcal{U}} = \{\underline{u}_i, \ i \in \tilde{n}\}$, $B_{\mathcal{U}}^m = \{\underline{u}_\omega \wedge, \omega \in Q_{m,n}\}$ be bases of $\mathcal{U}$, $\wedge^m \mathcal{U}$, $\underline{v} = \sum_{i=1}^n c_i \underline{u}_i \in U, \underline{v} \neq \underline{0}$, and $\underline{z} = \sum_{\omega \in Q_{m,n}} a_\omega \underline{u}_\omega \wedge \ \in \wedge^m \mathcal{U}$, $\underline{z} \neq \underline{0}$. $\underline{v} \wedge \underline{z} = \underline{0}$, if and only if

$$\sum_{i=1}^n \phi_\gamma^i c_i = 0, \text{ for all } \gamma \in Q_{m+1,n} \tag{14}$$

Proof: By Lemma (1), $\underline{v} \wedge \underline{u}$ is expressed as in (9). Given that the set $\{\underline{u}_\gamma \wedge, \ \gamma \in Q_{m+1,n}\}$ is a basis for $\bigwedge^{m+1} \mathcal{U}$, then $\underline{u} \wedge \underline{z} = 0$ and (9) imply

$$\sum_{k=1}^{m+1} (-1)^{k-1} c_{\gamma(k)} a_{\gamma(\hat{k})} = 0, \text{for all} \gamma \in Q_{m+1,n} \tag{15}$$

For every $\gamma \in Q_{m+1,n}$, the above summation may be extended to a summation from 1 to n by using the ϕ function. In fact, if $i = j_k \in \gamma$, then $\phi_\gamma(j_k) = (-1)^{k-1} a_{\gamma(\hat{k})}$ and $c_i = c_{j_k} = c_{\gamma(k)}$, whereas, if $i \notin \gamma$, then $\phi_\gamma(i) c_i = 0 \cdot c_i = 0$. The sufficiency is obvious.

□

If we denote by γ_t the elements of $Q_{m+1,n}$ (assumed to be lexicographically ordered), $t = 1, 2, \ldots, \binom{n}{m+1} = \tau$, then (14) may be expressed in a matrix form as

$$\underbrace{\begin{bmatrix} \phi_{\gamma_1}^1, & \phi_{\gamma_1}^2, & \ldots, & \phi_{\gamma_1}^i, & \ldots, & \phi_{\gamma_1}^n \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi_{\gamma_t}^1, & \phi_{\gamma_t}^2, & \ldots, & \phi_{\gamma_t}^i, & \ldots, & \phi_{\gamma_t}^n \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi_{\gamma_\tau}^1, & \phi_{\gamma_\tau}^2, & \ldots, & \phi_{\gamma_\tau}^i, & \ldots, & \phi_{\gamma_\tau}^n \end{bmatrix}}_{\Phi_n^m(\underline{z})} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{bmatrix}}_{\underline{c}} = \underline{0} \tag{16}$$

The matrix $\Phi_n^m(\underline{z}) \in \mathcal{F}^{\tau \times n}$ is a structured matrix (has zeros in fixed positions), it is defined by the pair (m,n) and the coordinates $\{a_\omega, \omega \in Q_{m,n}\}$ of $\underline{z} \in \wedge^m \mathcal{U}$ and will be called the *Grassmann matrix*

(GM) of $\underline{z}$ and it was originaly defined in [14]. We illustrate the canonical structure of GM by two examples.

Example (1): Let $m=2, n=4$ and $\{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}\}$ be the coordinates of $\underline{z} \in \wedge^2 \mathcal{U}$, $dim\mathcal{U}=4$, with respect to some basis. Then,

$$\Phi_4^2(\underline{z}) = \begin{array}{cl} \begin{bmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{bmatrix} & \begin{array}{l} \leftarrow (1,2,3)=\gamma_1 \\ \leftarrow (1,2,4)=\gamma_2 \\ \leftarrow (1,3,4)=\gamma_3 \\ \leftarrow (2,3,4)=\gamma_4 \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 \end{array} & \end{array} \tag{17}$$

□

Example (2): Let $m=2, n=5$ and $\{a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}, a_{35}, a_{45}\}$ be the coordinates of $\underline{z} \in \wedge^2 \mathcal{U}$, $dim\mathcal{U}=5$, with respect to some basis. Then,

$$\Phi_5^2(\underline{z}) = \begin{array}{cl} \begin{bmatrix} a_{23} & -a_{13} & a_{12} & 0 & 0 \\ a_{24} & -a_{14} & 0 & a_{12} & 0 \\ a_{25} & -a_{15} & 0 & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} & 0 \\ a_{35} & 0 & -a_{15} & 0 & a_{13} \\ a_{45} & 0 & 0 & -a_{15} & a_{14} \\ 0 & a_{34} & -a_{24} & a_{23} & 0 \\ 0 & a_{35} & -a_{25} & 0 & a_{23} \\ 0 & a_{45} & 0 & -a_{25} & a_{24} \\ 0 & 0 & a_{45} & -a_{35} & a_{34} \end{bmatrix} & \begin{array}{l} \leftarrow (1,2,3)=\gamma_1 \\ \leftarrow (1,2,4)=\gamma_2 \\ \leftarrow (1,2,5)=\gamma_3 \\ \leftarrow (1,3,4)=\gamma_4 \\ \leftarrow (1,3,5)=\gamma_5 \\ \leftarrow (1,4,5)=\gamma_6 \\ \leftarrow (2,3,4)=\gamma_7 \\ \leftarrow (2,3,5)=\gamma_8 \\ \leftarrow (2,4,5)=\gamma_9 \\ \leftarrow (3,4,5)=\gamma_{10} \end{array} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 & 5 \end{array} & \end{array} \tag{18}$$

□

The matrix $\Phi_n^m(\underline{z})$ is defined for every $\underline{z} \in \wedge^m \mathcal{U}$ and the decomposability property of $\underline{z}$ is expressed by the following result.

Theorem (1): Let $\mathcal{U}$ be an $n-$dimensional vector space over $\mathcal{F}$, $B_{\mathcal{U}}$ a basis of $\mathcal{U}$, $\underline{0} \neq \underline{z} \in \wedge^m \mathcal{U}$, $\Phi_n^m(\underline{z})$ the GM of $\underline{z}$ with respect to $B_{\mathcal{U}}$ and let $\mathcal{N}_n^m(\underline{z}) = \mathcal{N}_r\{\Phi_n^m(\underline{z})\}$, Then,

(i) $dim\mathcal{N}_n^m(\underline{z}) \leq m$ and equality holds, if and only if $\underline{z}$ is decomposable.

(ii) If $dim\mathcal{N}_n^m(\underline{z}) = m$, then a representation of the solution space, $\mathcal{V}_{\underline{z}}$, of $\underline{v}_1 \wedge \ldots \wedge \underline{v}_m = \underline{z}$ with respect to $B_{\mathcal{U}}$ is given by $\mathcal{N}_n^m(\underline{z})$.

Proof: By Proposition (1), $\underline{0} \neq \underline{z} \in \wedge^m \mathcal{U}$ is decomposable, if and only if there exists an independent set of vectors $\{\underline{v}_i, i \in \tilde{m}\}$ in $\mathcal{U}$, such that $\underline{v}_i \wedge \underline{z} = 0$, for all $i \in \tilde{m}$. By Proposition (2) and Eq. (16), it follows that such vectors $\underline{v}_i$ may be found, if

$$\Phi_n^m(\underline{z})\underline{c} = \underline{0} \tag{19}$$

has at least m independent solutions, or equivalently $dim\mathcal{N}_n^m(\underline{z}) \geq m$. If $dim\mathcal{N}_n^m(\underline{z}) = p > n$, then (19) defines p independent vectors $\underline{c}_i$ and thus p independent vectors $\underline{v}_i$ for which $\underline{v}_i \wedge \underline{z} = \underline{0}$. By Proposition (1), $\underline{z}$ is decomposable and thus we may write $\underline{z} = \underline{v}_1 \wedge ... \wedge \underline{v}_m$. However, since $\underline{v}_j \wedge \underline{z} = 0$, $j = m+1,..., p$, it follows that $\underline{v}_1 \wedge ... \wedge \underline{v}_m \wedge \underline{v}_j = \underline{0}$ and thus the set $\{\underline{v}_1,..., \underline{v}_m, \underline{v}_j\}$, $j = m+1,..., p$ is linearly dependent, which is a contradiction. Thus, $dim\mathcal{N}_n^m(\underline{z}) \leq m$ and decomposability holds when equality holds. The sufficiency of part (i) follows by reversing the steps. Note, that if $\{\underline{c}_i, i \in \tilde{m}\}$ is a basis of $\mathcal{N}_n^m(\underline{z})$, when $dim\mathcal{N}_n^m(\underline{z}) = m$, then m independent vectors $\underline{v}_i$ may be defined by $\underline{v}_j = \sum_{i=1}^{n} c_{ji}\underline{u}_i$, where $B_{\mathcal{U}} = \{\underline{u}_i, i \in \tilde{n}\}$ is a basis of $\mathcal{U}$; clearly, $\mathcal{V}_{\underline{z}} = span_{\mathcal{F}}\{\underline{v}_i, i \in \tilde{m}\}$ and this establishes part (ii).

□

The above result provides an alternative characterisation for decomposability of multivectors,as well as a simple procedure for reconstruction of the solution space of the exterior equation. The matrix $\Phi_n^m(\underline{z})$ that corresponds to a decomposable $\underline{z}$ will be referred to as *canonical*.

Remark (1): Let $\Phi_n^m(\underline{z})$ be the canonical GR of $\underline{z} \in \wedge^m \mathcal{U}$ which has been defined with respect to the $B_{\mathcal{U}} = \{\underline{u}_i, i \in \tilde{n}\}$ basis of $\mathcal{U}$. If $\{\underline{c}_j, j \in \tilde{m}\}$ is a basis for $\mathcal{N}_n^m(\underline{z})$, then the space $\mathcal{V}_{\underline{z}} \in \mathcal{G}(m, \mathcal{U})$ for which $\underline{g}(\mathcal{V}_{\underline{z}})$ is defined by $\mathcal{V}_{\underline{z}} = span_{\mathcal{F}}\{\underline{v}_j, j \in \tilde{m}\}$, where

$$\underline{v}_j = \sum_{i=1}^{n} c_{ij}\underline{u}_i, \quad \underline{c}_j = [c_{1j,..,c_{nj}}]^t, \quad \forall j \in \tilde{m} \tag{20}$$

□

Corollary (1): Let $\Phi_n^m(\underline{z})$ be the GR of $\underline{z} \in \wedge^m \mathcal{U}$, $\underline{z} \neq 0$. Then,

(i) If $m = 1$, then for all n, $\Phi_n^1(\underline{z})$ is always canonical; furthermore, if $n \geq 3$, $rank_F\{\Phi_n^1(\underline{z})\} = n-1$.

(ii) If $m = n-1$, then $\Phi_n^{n-1}(\underline{z}) \in F^{1\times n}$ and it is always canonical with $rank_F\{\Phi_n^{n-1}(\underline{z})\} = 1$.

(iii) If $m = n-\rho, m > 1$ and $\rho \geq 2$, then for all $\underline{z}$, $rank_F\{\Phi_n^m(\underline{z})\} \geq n-m$; equality holds, if and only if $\Phi_n^m(\underline{z})$ is canonical.

***Proof*:**

(i) If $m = 1$, $\wedge^1 \mathcal{U} = \mathcal{U}$ and every $\underline{z} = \underline{v} \in \mathcal{U}$ is decomposable. Given that $\Phi_n(\underline{z})$ has $\binom{n}{2} \times n$ dimensions and the only vectors $\underline{v}$ for which $\underline{v} \wedge \underline{z} = \underline{0}$ are those written as $\underline{v} = c\underline{z}$, it follows that $dim\mathcal{N}_n(\underline{z}) = 1$. For $n \geq 3$, $\binom{n}{2} > n$ and thus $rank_F\{\Phi_n^1(\underline{z})\} = \; = n-1$.

(ii) If $m+1 = n$, then $\Phi_n^{n-1}(\underline{z})$ is $1\times n$ and since $\underline{z} \neq \underline{0}$, $dim\mathcal{N}_n^{n-1}(\underline{z}) = n-1 = m$; thus, $\Phi_n^{n-1}(\underline{z})$ is always canonical with rank 1.

(iii) If $m=n-\rho$, $m>1$, $\rho\geq 2$, then $\binom{n}{m+1}\geq n$. In fact $\Phi_n^m(\underline{z})$ is $n\times n$, if $m+2=n$ and $\binom{n}{m+1}>n$ if $m+2<n$. The condition, $dim\mathcal{N}_n^m(\underline{z})=m$, for $\Phi_n^m(\underline{z})$ to be canonical clearly yields the result.

□

Note that parts $(i),(ii)$ of the above result express the well known result for decomposability of all vectors of $\wedge^1\mathcal{U}$, $\wedge^{n-1}\mathcal{U}$ [1]. From part (iii) we also have:

Corollary (2): Let $\underline{0}\neq\underline{z}\in\wedge^m\mathcal{U}$, $n-m\geq 2$, $m>1$. $\Phi_n^m(\underline{z})$ is canonical, if and only if $C_{n-m+1}\{\Phi_n^m(\underline{z})\}=0$.

□

This result establishes the links between the new decomposability result based on $\Phi_n^m(\underline{z})$ and the set of QPRs. It may be readily shown that the quadratics in the compound matrix $C_{n-m+1}\{\Phi_n^m(\underline{z})\}$ are dependent on the set of RQPRs. Finally, it is worth pointing out that the new decomposability test also provides an alternative characterisation of the Grassmann variety $\Omega(m,n)$ of $\mathscr{P}^{\sigma-1}, \sigma=\binom{n}{m}$.

Remark (2): Let $\underline{z}\in\wedge^m\mathcal{U}$, $\underline{z}\neq 0$, and let $P(\underline{z})$ be the point of $\mathscr{P}^{\sigma-1}$ defined by the coordinates $\{a_\omega,\omega\in Q_{m,n}\}$ of $\underline{z}$. $P(\underline{z})\in\Omega(m,n)$ if and only if the Grassmann matrix Φ_n^m is canonical.

□

We close this section by describing a systematic procedure for constructing $\Phi_n^m(\underline{z})$ and by making some final remarks on the relationship between $\Phi_n^m(\underline{z})$ and the QPRs.

Procedure for construction of $\Phi_n^m(\underline{z})$

Given (n,m), $\tau=\binom{n}{m+1}$ we form a $\tau\times n$ matrix, where the rows are indexed by the sequences $\gamma\in Q_{m+1,n}$ lexicographically ordered, and the columns by $i\in\tilde{n}$. The elements of the $\gamma=(j_1,j_2,\ldots,j_{m+1})\in Q_{m+1,n}$ indexed row are defined for every $i\in\tilde{n}$ as follows:

(a) If $i\notin\{j_1,\ldots,j_{m+1}\}$, then $\phi_\gamma^i=0$.

(b) If $i=j_k\in\{j_1,\ldots,j_{m+1}\}$, then we define as $\omega=\{j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{m+1}\}\in Q_{m,n}$ and $\phi_\gamma^i=sign(j_k;\omega)a_\omega$.

(c) The procedure is repeated for all $i\in\tilde{n}$ and for all $\gamma\in Q_{m+1,n}$ indexed rows.

□

Some interesting observations on the structure of $\Phi_n^m(\underline{z})$ are summarised below.

Remark (3): For every $\omega\in Q_{m,n}$ the coordinate a_ω appears only in $n-m$ rows with indices $I_\omega=(i_1,i_2,\ldots,i_{n-m})$ and in $n-m$ columns with indices $J_\omega=(j_1,j_2,\ldots,j_{n-m})$ of $\Phi_n^m(\underline{z})$. The I_ω, J_ω sets of indices are distinct and have the following properties: (i) $i_k\in I_\omega$ is the index of $\gamma_{i_k}\in Q_{m+1,n}$ row for which all indices in ω are contained in γ_{i_k}; (ii) $j_k\in J_\omega$ is the index of the column which is not contained in ω.

□

The above observations, together with the assumption that $\underline{z} \neq 0$ (and thus at least one $a_\omega \neq 0$), verify the property that $rank_F\{\Phi_n^m(\underline{z})\} \geq n-m$ and suggest an alternative procedure for deriving the set of QPRs from the $\Phi_n^m(\underline{z})$ matrix.

Grassmann matrix procedure for deriving the QPRs

Let $\omega \in Q_{m,n}$ such that $a_\omega \neq 0$ and denote by $J_\omega = (j_1, j_2, \ldots, j_{n-m})$ the column index of ω (columns containing a_ω) and $\hat{J}_\omega = (k_1, \ldots, k_m)$ the complementary set of J_ω with respect to $\tilde{n} = (1, \ldots, n)$ (k_i are the indices of columns not containing a_ω). If $\underline{\phi}_i$ denote the columns of $\Phi_n^m(\underline{z})$ and $\underline{\phi}_\omega \wedge = \underline{\phi}_{j_1} \wedge \ldots \wedge \underline{\phi}_{j_{n-m}}$ ($\underline{\phi}_\omega \wedge$ by Remark 3), then the set of QPRs are defined by the nontrivial relations derived from

$$\underline{\phi}_{k_i} \wedge \underline{\phi}_\omega \wedge = \underline{0}, \quad \forall \ i \in \tilde{m} \tag{21}$$

If $\mathcal{R}_\omega = span_{\mathcal{F}}\{\underline{\phi}_{j_i}, \ldots, \underline{\phi}_{j_{n-m}}\}$, equations (21) are equivalent to $\underline{\phi}_{k_i} \in \mathcal{R}_\omega, \ \forall i \in \tilde{m}.$

□

Remark(4): For $\underline{z} \in \wedge^m \mathcal{U}$, $\underline{z} \neq 0$, $\Phi_n^m(\underline{z})$ may be interpreted as the matrix representation of the right multiplication operator $\wedge_{\underline{z}}^R : \mathcal{U} \to \wedge^{m+1}\mathcal{U}$ defined as: $\wedge_{\underline{z}}^R(\underline{u}) = \underline{u} \wedge \underline{z}$, for $\forall \underline{u} \in \mathcal{U}$.

□

5. The Hodge-Grassmann Matrix and the Decomposability of Multivectors

The Hodge-Grassmann matrix is the Grassmann matrix of the Hodge dual of the multivector $\underline{z}$ and its properties are dual to those of the Grassmann matrix. In fact decomposability turns out to be an image problem for the transpose of the Hodge-Grassmann matrix and the Quadratic Plucker Relations can be expressed in terms of the Grassmann and Hodge-Grassmann matrices.This will provide additional criteria for decomposabilty that can be used for development of a new algorithm for the computation of solutions of DAP. We give first some background definitions.

Definition(2) [1]**:** The Hodge *-operator, for a oriented n-dimensional vector space $\mathcal{U}$ equipped with an inner product <.,.>, is an operator defined as: $*: \wedge^m \mathcal{U} \to \ \wedge^{n-m}\mathcal{U}$ such that $\underline{a} \wedge (\underline{b*})$=<$\underline{a}$,$\underline{b}$> $\underline{w}$ where $\underline{a}$, $\underline{b}$, $\underline{w} \in \wedge^n \mathcal{U}$ defines the orientation on $\mathcal{U}$ and *m*<*n*.

□

To compute the Hodge star of a multivector in $\wedge^m \mathcal{U}$ we follow the procedure: Let $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ be an orthonormal basis for $\mathcal{U}$ then an element of $\underline{z} \in \wedge^m \mathcal{U}$ can be written as $\underline{z} = \sum_{\omega \in Q_{m,n}} a_\omega \underline{u}_\omega \wedge$ and the Hodge star of $\underline{z}$ may be calculated as:

$$\underline{z}^* = \sum_{\omega \in Q_{m,n}} a_\omega (\underline{u}_\omega \wedge)^*$$

Therefore it suffices to calculate the Hodge star of all the elements of the basis $\wedge^m \mathcal{U}$ ie of the set $B^m{}_{\mathcal{U}} = \{\underline{u}_\omega \wedge\}_{\omega \in Q_{m,n}}$. Let $\underline{u}_{i_1} \wedge \underline{u}_{i_2} \wedge \ldots \wedge \underline{u}_{i_m} \in B^m{}_{\mathcal{U}}$ where $1 \leq i_1 < i_2 < \ldots < i_m \leq n$. Then:

$$(\underline{u}_{i_1} \wedge \underline{u}_{i_2} \wedge \ldots \wedge \underline{u}_{i_m})^* = sign(\sigma) \underline{u}_{j_1} \wedge \underline{u}_{j_2} \wedge \ldots \wedge \underline{u}_{j_{n-m}}$$

Where j_k are the *n-m* complementary to the i_k indices considered in ascending order and σ is the permutation: $\sigma = (i_1, i_2, ..., i_m, j_1, j_2, ..., j_{n-m})$.

Example(3): Let

$$\underline{z} = a_{12}\underline{u}_1 \wedge \underline{u}_2 + a_{13}\underline{u}_1 \wedge \underline{u}_3 + a_{14}\underline{u}_1 \wedge \underline{u}_4 + a_{23}\underline{u}_2 \wedge \underline{u}_3 + a_{24}\underline{u}_2 \wedge \underline{u}_4 + a_{34}\underline{u}_3 \wedge \underline{u}_4 \in \wedge^2 \mathbb{R}^4$$

Then applying the previously mentioned computational procedure we get:

$$\underline{z}^* = a_{12}(\underline{u}_1 \wedge \underline{u}_2)^* + a_{13}(\underline{u}_1 \wedge \underline{u}_3)^* + a_{14}(\underline{u}_1 \wedge \underline{u}_4)^* + a_{23}(\underline{u}_2 \wedge \underline{u}_3)^* + a_{24}(\underline{u}_2 \wedge \underline{u}_4)^* + a_{34}(\underline{u}_3 \wedge \underline{u}_4)^* =$$

$$= a_{12}\underline{u}_3 \wedge \underline{u}_4 - a_{13}\underline{u}_2 \wedge \underline{u}_4 + a_{14}\underline{u}_2 \wedge \underline{u}_3 + a_{23}\underline{u}_1 \wedge \underline{u}_4 - a_{24}\underline{u}_1 \wedge \underline{u}_3 + a_{34}\underline{u}_1 \wedge \underline{u}_2$$

Which in terms of coordinates it is:

$$\left(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \right)^* = \left(a_{34}, -a_{24}, a_{23}, a_{14}, -a_{13}, a_{12} \right)$$

□

Remark(5): The relation of the * operator with the inner product that demonstrates the involutive nature of the operator is: $<\underline{a},\ \underline{b}> = (\underline{b} \wedge\ (\underline{a}*))^* = (\underline{a} \wedge\ (\underline{b}*))^*,\ \underline{a}** = (-1)^{m(n-m)}\underline{a}$

□

Definition(3): The Hodge-Grassmann matrix of a multivector $\underline{z}$, $\underline{z} \in \wedge^m \mathcal{U}$, $\underline{z} \neq 0$, is defined as the Grassmann matrix of the Hodge dual of $\underline{z}$, $\underline{z}^*$, ie it is the matrix $\Phi_n^{n-m}(\underline{z}^*)$ representing the linear map $\wedge_{\underline{z}^*}^R : \mathcal{U} \to \wedge^{n-m+1}\mathcal{U}$ defined as the representation of:

$$\wedge_{\underline{z}^*}^R(\underline{u}) = \underline{u} \wedge \underline{z}^*,\ \text{for}\ \forall \underline{u} \in \mathcal{U}$$

□

A number of properties of the Hodge-Grassmann matrix of a multivector $\underline{z}$ are considered next.

Proposition(3): For any $\underline{z} \in \wedge^m \mathcal{U}$ the following are equivalent:

1. $\underline{z}$ is decomposable
2. $\underline{z}^*$ is decomposable

Proof:

(1→2) Let $\underline{z} \in \wedge^m \mathcal{U}$ be decomposable, then $\underline{z}$ can be written as $\underline{z} = \lambda \underline{u}_1 \wedge ... \wedge \underline{u}_m$ where the vectors $\underline{u}_1, ..., \underline{u}_m$ are orthonormal. We extend this set to a positively oriented orthonormal basis [1], $\underline{u}_1, ..., \underline{u}_m, \underline{u}_{m+1}, ..., \underline{u}_n$, of $\mathcal{U}$. Then

$$\underline{z}^* = \lambda \underline{u}_{m+1} \wedge ... \wedge \underline{u}_n$$

which establishes that $\underline{z}^*$ is decomposable.

(2→1) Immediate from the previous part of the proof and the fact that $\underline{z}** = (-1)^{m(n-m)}\underline{z}$

□

Proposition(4): The following statements hold true:

a) $dim \mathcal{N}_r\{\Phi_n^{n-m}(\underline{z}^*)\} \leq n - m$ equality holding, iff $\underline{z}$ is decomposable.

b) $dim\, rowspan\{\Phi_n^{n-m}(\underline{z}^*)\} \geq m$ equality holding, iff $\underline{z}$ is decomposable.

Proof

a) immediate from theorem(1) and Proposition(1)
b) immediate from a)

□

Proposition(5): For $\underline{z} \in \wedge^m \mathcal{U}$, $\underline{z} \neq 0$, the matrix $\Phi_n^m(\underline{z})^T$ is the representation of the map ${}^T\wedge_{\underline{z}}^R : \wedge^{m+1}\mathcal{U} \to \mathcal{U}$ given by:

$$ {}^T\wedge_{\underline{z}}^R(\underline{\mathrm{y}}) = (-1)^{n-1}(\underline{z} \wedge \underline{\mathrm{y}}^*)^*, \text{ where } \underline{\mathrm{y}} \in \wedge^{m+1}\mathcal{U} $$

Proof:

Assuming $\underline{u} \in \mathcal{U}$, $\underline{y} \in \wedge^{m+1}\mathcal{U}$

$$ \underline{u}^T \Phi_n^m(\underline{z})^T \underline{y} = <\underline{y}, \Phi_n^m(\underline{z})\underline{u}> = <\underline{y}, \wedge_{\underline{z}}^R(\underline{u})> = <\underline{y}, \underline{u} \wedge \underline{z}> = $$
$$ = (\underline{u} \wedge \underline{z} \wedge \underline{y}^*)^* = (-1)^{n-1}(\underline{u} \wedge (\underline{z} \wedge \underline{y}^*)^{**})^* = <\underline{u}, (-1)^{n-1}(\underline{z} \wedge \underline{y}^*)^*> $$

Proving that $\Phi_n^m(\underline{z})^T$ is the matrix representation of ${}^T\wedge_{\underline{z}}^R(\underline{y})$

□

Corollary(3): The matrix $\Phi_n^{n-m}(\underline{z}^*)^T$ is the representation of the map ${}^T\wedge_{\underline{z}^*}^R : \wedge^{n-m+1}\mathcal{U} \to \mathcal{U}$ given by:

$$ {}^T\wedge_{\underline{z}^*}^R(\underline{\mathrm{y}})) = (-1)^{n-1}(\underline{z}^* \wedge \underline{\mathrm{y}}^*)^* \text{ , where } \underline{\mathrm{y}} \in \wedge^{n-m+1}\mathcal{U} $$

□

The above results lead to a new test for decomposability in terms of relations based on the Grassmann and Hodge-Grassmann matrices (for an abstract formulation see also [12,13]):

Theorem(2): For any $\underline{z} \in \wedge^m \mathcal{U}$ the following are equivalent:

1. $\underline{z}$ is decomposable

2. $\Phi_n^m(\underline{z})(\Phi_n^{n-m}(\underline{z}^*))^T = \underline{0} \in \mathbb{R}^{\binom{n}{m+1} \times \binom{n}{n-m+1}}$

Proof

(1→2) Let $\underline{z} \in \wedge^m \mathcal{U}$ be decomposable, then $\underline{z}$ can be written as $\underline{z} = \lambda \underline{u}_1 \wedge \ldots \wedge \underline{u}_m$ where the vectors $\underline{u}_1, \ldots, \underline{u}_m$ are orthonormal. We extend this set to a positively oriented orthonormal basis, $\underline{u}_1, \ldots, \underline{u}_m, \underline{u}_{m+1}, \ldots, \underline{u}_n$, of $\mathcal{U}$. Then to prove 2 is equivalent to proving that

$$ \wedge_{\underline{z}}^R({}^T\wedge_{\underline{z}^*}^R(\underline{\mathrm{y}})) = (-1)^{n-1}(\underline{z}^* \wedge \underline{\mathrm{y}}^*)^* \wedge \underline{z} = 0, \quad \forall \underline{\mathrm{y}} \in \wedge^{n-m+1}\mathcal{U} $$

Let $\underline{\mathrm{y}}^* = \sum_{\omega \in Q_{m-1,n}} a_\omega \underline{u}_\omega \wedge$ then

$$ (\underline{z}^* \wedge \underline{\mathrm{y}}^*)^* = \lambda(\underline{u}_{m+1} \wedge \ldots \wedge \underline{u}_n \wedge \sum_{\omega \in Q_{m-1,n}} a_\omega \underline{u}_\omega \wedge)^* = \sum_{i=1}^m r_i \underline{u}_i $$

Implying that:

$$ (\underline{z}^* \wedge \underline{\mathrm{y}}^*)^* \wedge \underline{z} = (\sum_{i=1}^m r_i \underline{u}_i) \wedge (\lambda \underline{u}_1 \wedge \ldots \wedge \underline{u}_m) = 0 $$

(2→1) Assume that $\underline{z}$ is not decomposable and 2 holds. Then

$$ m > \dim \mathcal{N}_r\{\Phi_n^m(\underline{z})\} \geq \dim \mathrm{row}\mathit{span}\{\Phi_n^{n-m}(\underline{z}^*)\} > m $$

which is a contradiction. Note that the matrices

$$\Phi_n^m(\underline{z}) \ , \ (\Phi_n^{n-m}(\underline{z}^*))^T$$

are linear in $\underline{z}$, and making their product equal to zero leads to the quadratic relations defining decomposability.

□

Example(4): Here we will derive the Quadratic Plucker relations for multivectors $\underline{z}$ in $\wedge^2 \mathbb{R}^4$. For this case we calculate the Grassmann and Hodge Grassmann Matrices for this space. Thus, we have:

$$\Phi_4^2(\underline{z}) = \begin{bmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{bmatrix}, \quad \Phi_4^2(\underline{z}^*) = \begin{bmatrix} a_{14} & a_{24} & a_{34} & 0 \\ -a_{13} & -a_{23} & 0 & a_{34} \\ a_{12} & 0 & -a_{23} & -a_{24} \\ 0 & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

According to Theorem(2) the Quadratic Relations defining decomposability are given by the product:

$$\Phi_4^2(\underline{z})(\Phi_4^2(\underline{z}^*))^T = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}).\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which is the single quadratic Plucker relation defining the decomposable vectors in $\wedge^2 R^4$

□

Example(5): Next we will derive the Quadratic Plucker relations for multivectors $\underline{z}$ (2,5) in $\wedge^2 \mathbb{R}^5$. For this case, the QPRs are given by:

$$q_1 = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}, \quad q_2 = a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23}, \quad q_3 = a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24},$$
$$q_4 = a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34}, \quad q_5 = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$$

We may verify the derivation of QPRs using Theorem(2). In fact it may be verified that using the above result we have that the Grassmann and Hodge Grassmann matrices for this space are:

$$\Phi_5^2(\underline{z}) = \begin{bmatrix} a_{23} & -a_{13} & a_{12} & 0 & 0 \\ a_{24} & -a_{14} & 0 & a_{12} & 0 \\ a_{25} & -a_{15} & 0 & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} & 0 \\ a_{35} & 0 & -a_{15} & 0 & a_{13} \\ a_{45} & 0 & 0 & -a_{15} & a_{14} \\ 0 & a_{34} & -a_{24} & a_{23} & 0 \\ 0 & a_{35} & -a_{25} & 0 & a_{23} \\ 0 & a_{45} & 0 & -a_{25} & a_{24} \\ 0 & 0 & a_{45} & -a_{35} & a_{34} \end{bmatrix}, \ \Phi_5^3(\underline{z}^*) = \begin{bmatrix} -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \\ a_{14} & a_{24} & a_{34} & 0 & -a_{45} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ a_{12} & 0 & -a_{23} & -a_{24} & -a_{25} \\ 0 & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix}$$

We calculate

$$\Phi_5^2(\underline{z})(\Phi_5^3(\underline{z}^*))^T = q_1 \cdot \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + q_2 \cdot \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + q_3 \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+ q_4 \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + q_5 \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which provides an alternative way for computing the five quadratic Plucker relations defining decomposable vectors in $\wedge^2 \mathbb{R}^5$.

□

6. The Grassmann and Hodge-Grassmann Matrices and the Canonical Representation of Multivectors

The kernel of the Grassmann matrix and the image of the transpose of the Hodge Grassmann matrix of a multivector define two fundamental spaces that determine a canonical representation of multivectors. The relation between those two spaces is demonstrated by the following result.

Theorem(3): Let $\underline{z} \in \wedge^m \mathcal{U}$ then the following holds true:

$$\mathcal{N}_r\{\Phi_n^m(\underline{z})\} \subseteq RowSpan\{\Phi_n^{n-m}(\underline{z}^*)\} = \mathcal{R}\{\Phi_n^{n-m}(\underline{z}^*)^T\}$$

Proof:

Let us consider $\underline{u}_1 \in \mathcal{N}_r\{\Phi_n^m(\underline{z})\}$ with $\|\underline{u}_1\| = 1$, then $\underline{u}_1 \wedge \underline{z} = 0$ and thus $\underline{z} = \lambda \underline{u}_1 \wedge \underline{z}_1$ for some λ>0 and $\underline{z}_1 \in \wedge^{m-1}\mathcal{U}$, $\|\underline{z}_1\| = 1$. We will prove that such a $\underline{u}_1$ belongs to the image of the transpose of the Hodge Grassmann matrix $(\Phi_n^{n-m}(\underline{z}^*))^T$. To establish this we will calculate first the expression $((\underline{u}_1 \wedge \underline{z}_1)^* \wedge \underline{z}_1)^*$. First we consider $\underline{u}_1, \dots, \underline{u}_n$ an oriented orthonormal basis of $\mathcal{U}$ with $\underline{u}_1$ as its first element. Then

$$\underline{z}_1 = \sum_{\omega \in Q_{m-1,n}, 1 \notin \omega} z_\omega \underline{u}_\omega \wedge$$

Hence

$$((\underline{u}_1 \wedge \underline{z}_1)^* \wedge \underline{z}_1)^* = ((\underline{u}_1 \wedge \sum_{\omega \in Q_{m-1,n}, 1 \notin \omega} z_\omega \underline{u}_\omega \wedge)^* \wedge \sum_{\omega_1 \in Q_{m-1,n}, 1 \notin \omega_1} z_{\omega_1} \underline{u}_{\omega_1} \wedge)^* =$$
$$= \sum_{\omega \in Q_{m-1,n}, 1 \notin \omega} \sum_{\omega_1 \in Q_{m-1,n}, 1 \notin \omega_1} z_\omega z_{\omega_1} ((\underline{u}_1 \wedge \underline{u}_\omega \wedge)^* \wedge (\underline{u}_{\omega_1} \wedge))^*$$

The only nonzero terms of the above expression are those that $\omega = \omega_1$ hence:

$$((\underline{u}_1 \wedge \underline{z}_1)^* \wedge \underline{z}_1)^* = \sum_{\omega \in Q_{m-1,n}, 1 \notin \omega} z_\omega{}^2 ((\underline{u}_1 \wedge \underline{u}_\omega \wedge)^* \wedge (\underline{u}_\omega \wedge))^* \tag{22}$$

Now we also have that:

$$< \underline{u}_i, ((\underline{u}_1 \wedge \underline{u}_\omega \wedge)^* \wedge (\underline{u}_\omega \wedge))^* >= (-1)^{n-1} (\underline{u}_i \wedge (\underline{u}_1 \wedge \underline{u}_\omega \wedge)^* \wedge (\underline{u}_\omega \wedge))^* =$$
$$= (-1)^{n-1}(-1)^{nm}((\underline{u}_i \wedge \underline{u}_\omega \wedge) \wedge (\underline{u}_1 \wedge \underline{u}_\omega \wedge)^*)^* = (-1)^{n-1}(-1)^{nm} < \underline{u}_i \wedge \underline{u}_\omega \wedge), (\underline{u}_1 \wedge \underline{u}_\omega \wedge) >= (-1)^{n-1}(-1)^{nm} \delta_{1i} \underline{u}_1$$

Proving that:

$$((\underline{u}_1 \wedge \underline{u}_\omega \wedge)^* \wedge (\underline{u}_\omega \wedge))^* = (-1)^{n-1}(-1)^{nm} \underline{u}_1$$

Which combined with (22) implies:

$$((\underline{u}_1 \wedge \underline{z}_1)^* \wedge \underline{z}_1)^* = (-1)^{n-`}(-1)^{nm} (\sum_{\omega \in Q_{m-1,n}, 1 \notin \omega} z_\omega{}^2) \underline{u}_1 = (-1)^{n-1}(-1)^{nm} \underline{u}_1 \tag{23}$$

Equation (23) can be rewritten as:

$$(-1)^{n-1}(\underline{z}^* \wedge (\frac{(-1)^{nm}}{\lambda} \underline{z}_1))^* = \underline{u}_1$$

Which by Corollary(3) implies that $\underline{u}_1 \in \mathrm{Im}(\Phi_n^{n-m}(\underline{z}^*)^T)$ proving the result.

□

We consider now the two fundamental spaces associated with the multivector $\underline{z}$:

$$\mathcal{D}_1(\underline{z}) = \mathcal{N}_r(\Phi_n^m(\underline{z})) \ \text{ with } \ d_1(\underline{z}) = \dim \mathcal{N}_r(\Phi_n^m(\underline{z}))$$
$$\mathcal{D}_2(\underline{z}) = \mathcal{R}(\Phi_n^{n-m}(\underline{z}^*)^T) \ \text{ with } \ d_2(\underline{z}) = \dim \mathcal{R}(\Phi_n^{n-m}(\underline{z}^*)^T)$$
$$\{0\} \subseteq \mathcal{D}_1(\underline{z}) \subseteq \mathcal{D}_2(\underline{z}) \subseteq \mathcal{U}, \ \text{ where } \ 0 \le d_1(\underline{z}) \le d_2(\underline{z}) \le m$$

We may now establish the following result:

Theorem(4): The following properties hold true for a $\underline{z} \in \wedge^m \mathcal{U}$

a. Let $\{\underline{u}_1, \ldots, \underline{u}_{d_1}\}$ be a basis for $\mathcal{D}_1(\underline{z})$ then $\underline{z}$ can be written as $\underline{z} = \underline{u}_1 \wedge \ldots \wedge \underline{u}_{d_1} \wedge \underline{z}_1$.

b. $\underline{z} \in \wedge^m \mathcal{D}_2(\underline{z})$

Proof:

a. This part of the proof follows from the fact that if $\underline{u}_i \wedge \underline{z} = 0$ then $\underline{u}_i$ is a factor of $\underline{z}$.

b. Consider now the orthogonal decompositions:

$$\mathcal{D}_2(\underline{z}) \oplus \mathcal{D}_2(\underline{z})^\perp = \mathcal{U}$$
$$\wedge^m \mathcal{D}_2(\underline{z}) \oplus (\wedge^m \mathcal{D}_2(\underline{z}))^\perp = \wedge^m \mathcal{U}$$

It is easy to see that the elements that span $(\wedge^m \mathcal{D}_2(\underline{z}))^\perp$ are of the form $\underline{w} = \underline{u} \wedge \underline{w}_1$ where $\underline{u} \in (\mathcal{D}_2(\underline{z}))^\perp$. It suffices to prove that $< \underline{z}, \underline{w} >= 0$ for all elements $\underline{w}$ spanning the space $(\wedge^m \mathcal{D}_2(\underline{z}))^\perp$. Indeed:

$$< \underline{z}, \underline{u} \wedge \underline{w}_1 >= (\underline{u} \wedge \underline{w}_1 \wedge \underline{z}^*)^* = (-1)^{n-1} < \underline{u}, (\underline{w}_1 \wedge \underline{z}^*)^* > \ = \ < \underline{u}, (-1)^{n-1}(\underline{w}_1 \wedge \underline{z}^*)^* > \ = 0$$

since $\underline{\mathrm{u}} \in (\mathcal{D}_2(\underline{z}))^\perp$ and $(-1)^{n-1}(\underline{w}_1 \wedge \underline{z}^*)^* \in \mathcal{D}_2(\underline{z})$ and this proves the result.

□

Corollary(4): If $\{\underline{u}_1, \ldots, \underline{u}_{d_1}\}$ a basis for $\mathcal{D}_1(\underline{z})$, then the multivector $\underline{z}$ can be represented as:

$$\underline{z} = \underline{u}_1 \wedge \ldots \wedge \underline{u}_{d_1} \wedge \underline{z}_1$$

where $\underline{z}_1 \in \wedge^{m-d_1} \mathcal{D}_3(\underline{z})$, where $\mathcal{D}_3(\underline{z})$ is the orthogonal complement of $\mathcal{D}_1(\underline{z})$ in $\mathcal{D}_2(\underline{z})$.

□

Example (6): Consider Multivectors $\underline{z}$ in $\wedge^3 \mathbb{R}^6$ for which $d_1=1$ and $d_2=4$ and let $\{\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4\}$ be a basis for $\mathcal{D}_2(\underline{z})$ by extending the basis $\{\underline{u}_1\}$ of $\mathcal{D}_1(\underline{z})$. Then, the canonical representation of $\underline{z}$ is:

$$\underline{z} = \underline{u}_1 \wedge (a\underline{u}_2 \wedge \underline{u}_3 + b\underline{u}_2 \wedge \underline{u}_4 + c\underline{u}_3 \wedge \underline{u}_4) = a\underline{u}_1 \wedge \underline{u}_2 \wedge \underline{u}_3 + b\underline{u}_1 \wedge \underline{u}_2 \wedge \underline{u}_4 + c\underline{u}_1 \wedge \underline{u}_3 \wedge \underline{u}_4$$

□

Finnaly, we present a result that establishes some fundamental relationships between the singular vectors and the singular values of the Grassmann and Hodge-Grassmann matrices. This is deduced by the following theorem that describes a relationship between these two matrices:

Theorem(5): For any $\underline{z} \in \wedge^m \mathbb{R}^n$ the following holds true:

$$\Phi_n^m(\underline{z})^T \Phi_n^m(\underline{z}) + \Phi_n^{n-m}(\underline{z}^*)^T \Phi_n^{n-m}(\underline{z}^*) = \|\underline{z}\|^2 I_n$$

Proof:

The above relation is equivalent to proving that the bilinear form

$$Q(\underline{u},\underline{w}) = <\underline{u}\wedge\underline{z},\underline{w}\wedge\underline{z}> + <\underline{u}\wedge\underline{z}^{*},\underline{w}\wedge\underline{z}^{*}> - \|z\|^2 <\underline{u},\underline{w}> \qquad (24)$$

is equal to zero $\forall \underline{u},\underline{w}\in\mathbb{R}^n$. To this end it is equivalent to ptove that a) $Q(\underline{u}_1,\underline{u}_1)=0,\ \forall\underline{u}_1:\|\underline{u}_1\|=1$ and b) $Q(\underline{u}_1,\underline{u}_2)=0,\ \forall\underline{u}_1,\underline{u}_2:\|\underline{u}_1\|=\|\underline{u}_1\|=1$ and $<\underline{u}_1,\underline{u}_2>=0$. To prove a), we consider $\underline{u}_1,\dots,\underline{u}_n$ an oriented orthonormal basis of $\mathbb{R}^n$ with $\underline{u}_1$ as its first element. Then

$$<\underline{u}_1\wedge\underline{z},\underline{u}_1\wedge\underline{z}> = <\sum_{\omega\in Q_{m,n},1\notin\omega} z_\omega\underline{u}_1\wedge\underline{u}_\omega\wedge, \sum_{\omega\in Q_{m,n},1\notin\omega} z_\omega\underline{u}_1\wedge\underline{u}_\omega\wedge> = \sum_{\omega\in Q_{m,n},1\notin\omega} z_\omega{}^2$$

$$<\underline{u}_1\wedge\underline{z}^{*},\underline{u}_1\wedge\underline{z}^{*}> = <\sum_{\omega\in Q_{m,n},1\in\omega} z_\omega\underline{u}_1\wedge(\underline{u}_\omega\wedge)^{*}, \sum_{\omega\in Q_{m,n},1\in\omega} z_\omega\underline{u}_1\wedge(\underline{u}_\omega\wedge)^{*}> = \sum_{\omega\in Q_{m,n},1\in\omega} z_\omega{}^2$$

Therefore:

$$Q(\underline{u}_1,\underline{u}_1) = \sum_{\omega\in Q_{m,n},1\notin\omega} z_\omega{}^2 + \sum_{\omega\in Q_{m,n},1\in\omega} z_\omega{}^2 - \|\underline{z}\|^2 = \|\underline{z}\|^2 - \|\underline{z}\|^2 = 0$$

To prove b), we consider $\underline{u}_1,\dots,\underline{u}_n$ an oriented orthonormal basis of $\mathbb{R}^n$ with $\underline{u}_1,\underline{u}_2$ as its first two elements. Then

$$\begin{aligned}
&<\underline{u}_1\wedge\underline{z},\underline{u}_2\wedge\underline{z}> = <\sum_{\omega\in Q_{m,n},1\notin\omega} z_\omega\underline{u}_1\wedge\underline{u}_\omega\wedge, \sum_{\omega\in Q_{m,n},2\notin\omega} z_\omega\underline{u}_2\wedge\underline{u}_\omega\wedge> =\\
&<\sum_{\omega\in Q_{m,n},1\notin\omega,2\in\omega} z_\omega\underline{u}_1\wedge\underline{u}_\omega\wedge, \sum_{\omega\in Q_{m,n},2\notin\omega,1\in\omega} z_\omega\underline{u}_2\wedge\underline{u}_\omega\wedge> =\\
&<\sum_{\omega_1\in Q_{m-1,n},1,2\notin\omega_1} z_{2,\omega_1}\underline{u}_1\wedge\underline{u}_2\wedge\underline{u}_{\omega_1}\wedge, \sum_{\omega_1\in Q_{m-1,n},1,2\notin\omega_1} z_{1,\omega_1}\underline{u}_2\wedge\underline{u}_1\wedge\underline{u}_{\omega_1}\wedge> =\\
&= -\sum_{\omega_1\in Q_{m-1,n},1,2\notin\omega_1} z_{2,\omega_1}z_{1,\omega_1}
\end{aligned}$$

Also

$$\begin{aligned}
&<\underline{u}_1\wedge\underline{z}^{*},\underline{u}_2\wedge\underline{z}^{*}> = <\sum_{\omega\in Q_{m,n},1\in\omega} z_\omega\underline{u}_1\wedge(\underline{u}_\omega\wedge)^{*}, \sum_{\omega\in Q_{m,n},2\in\omega} z_\omega\underline{u}_2\wedge(\underline{u}_\omega\wedge)^{*}> =\\
&<\sum_{\omega\in Q_{m,n},1\in\omega,2\notin\omega} z_\omega\underline{u}_1\wedge(\underline{u}_\omega\wedge)^{*}, \sum_{\omega\in Q_{m,n},2\in\omega,1\notin\omega} z_\omega\underline{u}_2\wedge(\underline{u}_\omega\wedge)^{*}> =\\
&<(-1)^{m-1}\sum_{\omega_1\in Q_{m-1,n},1,2\in\omega_1} z_{1,\omega_1}\underline{u}_1\wedge\underline{u}_2\wedge(\underline{u}_{\omega_1}\wedge)^{*}, (-1)^{m}\sum_{\omega_1\in Q_{m-1,n},1,2\in\omega_1} z_{2,\omega_1}\underline{u}_2\wedge\underline{u}_1\wedge(\underline{u}_{\omega_1}\wedge)^{*}> =\\
&= \sum_{\omega_1\in Q_{m-1,n},1,2\notin\omega_1} z_{1,\omega_1}z_{2,\omega_1}
\end{aligned}$$

Therefore

$$Q(\underline{u}_1,\underline{u}_2) = -\sum_{\omega_1\in Q_{m-1,n},1,2\notin\omega_1} z_{2,\omega_1}z_{1,\omega_1} + \sum_{\omega_1\in Q_{m-1,n},1,2\notin\omega_1} z_{2,\omega_1}z_{1,\omega_1} - \|\underline{z}\|^2 <\underline{u}_1,\underline{u}_2> = 0$$

And this establishes the result.

□

Corollary(5): The matrices $\Phi_n^m(\underline{z})$ and $\Phi_n^{n-m}(\underline{z}^{*})$ have the same right singular vectors $\underline{x}_i$ and the corresponding singular values $\sigma_i,\bar{\sigma}_i$ obey the identity $\sigma_i{}^2+\bar{\sigma}_i{}^2=\|\underline{z}\|^2\ \ \forall i=1,\dots,n$.

□

The above, leads to a result demonstrating the relationship between decomposability and the singular values of the Grassmann matrix.

Corollary(6): The vector $\underline{z} \in \wedge^m \mathbb{R}^n$ is decomposable, iff the matrix $\Phi_n^m(\underline{z})$ has k singular values equal to 0 and n-k singular values equal to $\|\underline{z}\|$.

Proof

From Theorem 2 and Corollary 6 we have:

$$\sigma_i^2(\|\underline{z}\|^2 - \sigma_i^2) = 0 \ \forall i = 1, ..., n$$

Therefore all singular values of the Grassmann matrix are either 0 or $\|\underline{z}\|$. The proof then follows immediately by Proposition 4.

□

The dual result of the above is the following corollary:

Corollary(7): The vector $\underline{z} \in \wedge^m \mathbb{R}^n$ is decomposable iff the matrix $\Phi_n^{n-m}(\underline{z}^*)$ has n-k singular values equal to 0 and k singular values equal to $\|\underline{z}\|$.

The above lead to the following result:

Corollary(8): The vector $\underline{z} \in \wedge^m \mathbb{R}^n$ is decomposable iff

$$\mathcal{N}_r\{\Phi_n^m(\underline{z})\} = col\ span\{\Phi_n^{n-m}(\underline{z}^*)^T\} = span(\underline{x}_1, ..., \underline{x}_m)$$

Where $\underline{x}_1, ..., \underline{x}_m$ are the right singular vectors of the Grassmann matrix corresponding to its 0 singular value or the right singular vectors of the Hodge-Grassmann matrix corresponding to its singular value that equals to $\|\underline{z}\|$.

□

The properties of the Grassman matrices provide the means for developing a new approach for the direct computation of exact, or approximate solutions of DAP in a direct way, without resorting to the use of methods based on *Global Linearization* [8].

7. The Solution of the Exact and Approximate DAP

As described in section 2, the Determinantal Assignment Problem can be decomposed into a linear and a multi-linear problem defined as:

Linear problem: given by equations (3) which can be rewritten as

$$\underline{k}^t P = \underline{a}^t \qquad (26)$$

where $\underline{k}^t$ is an unknown $l-$vector, P is a $q \times (d+1)$ matrix, known as the Plucker matrix of the problem [4] and $\underline{a}$ is $d+1$ coefficient vector of a $d-$degree polynomial $a(s)$.

Multi-linear problem: given by equations (4) which express the fact that the $l-$vector $\underline{k}^t$ is decomposable.

The exact DAP is to find a decomposable $l-$vector $\underline{k}^t$ that satisfies (26) and is an intersection problem between a linear variety and the Grassmann variety. The approximate DAP is addressed when the exact problem is not solvable. In that case we try to minimize the distance between the linear variety given by (25) and the Grassmann variety of all decomposable vectors.

Since we are interested to place the roots of the polynomial $a(s)=\underline{a}^t\underline{e}(s)$, $\underline{e}(s)^t = [1, s,...., s^d]$, where d is the degree of the polynomial to be assigned. Let A be a right annihilator matrix of the vector $\underline{a}^t$ (i.e. $\underline{a}^t A=\underline{0}$), then (25) may be expressed as

$$\underline{k}^t PA=\underline{0}$$

If V is an orthonormal basis matrix for the left kernel of PA, then $\underline{k}^t$ equals to $\underline{k}^t = \underline{x}^t V$, $V \in \mathbb{R}^{p\times q}$ where p being the dimension of the left kernel of PA.

Thus for $\underline{k}^t$ to be decomposable, or to be the closest to decomposability, we require that either

(a) The QPRs are exactly zero, that is $\Phi_m^l(\underline{k})\cdot\Phi_{m-l}^l(\underline{k}^*)^T=\underline{0}$

or

(b) The square norm of the QPRs is minimum, that is minimize $\left\|\Phi_m^l(\underline{k})\cdot\Phi_{m-l}^l(\underline{k}^*)^T\right\|$

Therefore for both exact and approximate DAP we have to solve the following optimization problem

Problem: $\min\left\|\Phi_m^l(\underline{k})\cdot\Phi_{m-l}^l(\underline{k}^*)^T\right\|$ subject to $\underline{k}^t=\underline{x}^t V$ and $\|\underline{x}\|=1$

We may express the objective function of this problem as

$$\left\|\Phi_m^l(\underline{k})\cdot\Phi_m^l(\underline{k}^*)^T\right\|^2 = tr\left(\Phi_m^l(\underline{k})\cdot\Phi_m^l(\underline{k}^*)^T\cdot\Phi_m^l(\underline{k}^*)\cdot\Phi_m^l(\underline{k})^T\right)=tr\left(\Phi_m^l(\underline{k}^*)^T\cdot\Phi_m^l(\underline{k}^*)\cdot\Phi_m^l(\underline{k})^T\cdot\Phi_m^l(\underline{k})\right)$$

Substituting now, $\Phi_m^l(\underline{k}^*)^T\cdot\Phi_m^l(\underline{k}^*)$ by $\|k\|^2 I_m-\Phi_m^l(\underline{k})^T\Phi_m^l(\underline{k})$, we get

$$\left\|\Phi_m^l(\underline{k})\cdot\Phi_m^l(\underline{k}^*)^T\right\|^2 = tr\left(\|k\|^2\Phi_m^l(\underline{k})^T\Phi_m^l(\underline{k})-\left[\Phi_m^l(\underline{k})^T\Phi_m^l(\underline{k})\right]^2\right)=(m-l)\|k\|^4-tr\left(\Phi_m^l(\underline{k})^T\Phi_m^l(\underline{k})\right)^2$$

Hence, the optimization problem (26) may be written as

$$\max\ tr\left(\Phi_m^l(\underline{x}^t V)^T\cdot\Phi_m^l(\underline{x}^t V)\right)^2,\ \text{subject to } \|\underline{x}\|=1 \qquad (27)$$

The objective function of the new optimization problem is a homogeneous polynomial in p variables $\underline{x}=\left(x_1,x_2,\ldots,x_p\right)$ under the constraint $\|\underline{x}\|=1$. This is a nonlinear maximization problem which can be solved using usual optimization methods and algorithms.

Remark (6): It is known [8], [21] that this problem is similar to the zero assignment by squaring down and thus it has generically real solutions when *l(m-l)>d*. When *l(m-l)=d* the existence of real solutions depends on the degree of the corresponding Grassmannian [20]; in these cases the optimization algorithm may provide exact solutions. In the case where *l(m-l)<d* the problem of exact DAP is not generically solvable and then the algorithm provides approximate solutions.

□

Iterative Method for Computing Solutions: Here we will propose an iterative method resembling the power method [18], [19] for finding the largest modulus eigenvalue and its corresponding eigenvector of a matrix that solves the above problem.

We define by Φ the matrix matrix

$$\Phi = \Phi_m^l(\underline{x}^t V)\cdot \Phi_m^l(\underline{x}^t V)^T = \begin{pmatrix} & \vdots & \\ \cdots & \varphi_{ij}(\underline{x}) & \cdots \\ & \vdots & \end{pmatrix}$$

where $\varphi_{ij}(\underline{x}) = \underline{x}^t A_{ij}\,\underline{x}$ a quadratic function in $\underline{x}$, then the objective function is $tr\left(\Phi\right)^2 = \sum_{i,j=1}^{m} \phi_{ij}^2(\underline{x})$ and the Lagrangian of the problem is given by

$$L(\underline{x},\lambda) = \sum_{i,j=1}^{m} \phi_{ij}^2(\underline{x}) - \lambda\left(\|x\|^2 - 1\right) \tag{28}$$

It is readily shown that the first order conditions are given by

$$4\sum_{i,j=1}^{m} \phi_{ij}(\underline{x}) A_{ij}\,\underline{x} - 2\lambda \underline{x} = \underline{0}$$

If we now define by $A(\underline{x})$ the $p \times p$ matrix defined by

$$A(\underline{x}) = \sum_{i,j=1}^{m} \phi_{ij}(\underline{x}) A_{ij}$$

the first order conditions can be rewritten as a nonlinear eigenvalue problem defined by

$$A(\underline{x})\underline{x} = \frac{\lambda}{2}\underline{x}$$

The solution of the problem is that $\underline{x}$ that correspond to the maximum eigenvalue of the above. Thus can be found by applying the iteration that resembles the power method:

$$\underline{x}_{n+1} = A(\underline{x}_n)\underline{x}_n \,/\, \|A(\underline{x}_n)\underline{x}_n\| \tag{29}$$

The stopping criteria can be of the form $\|\underline{x}_{n+1} - \underline{x}_n\| < \varepsilon$. We have an exact solution to DAP wherever the objective function takes the value $m-l$. The method can be summarized as follows:

Computational Procedure

1. Calculate the solution of the linear problem parametrised in the form $\underline{x}^t V$:
2. Calculate the parametrised Grassmann matrix $\Phi_m^l(\underline{x}^t V)$ and the matrix Φ.
3. Calculate hence the matrix $A(\underline{x}) = \sum_{i,j=1}^{m} \phi_{ij}(\underline{x}) A_{ij}$
4. Apply the iteration (29) until some stopping criteria are met. The vector $\underline{x}_n$ of the last iteration gives rise to the multivector $k^t = \underline{x}_n^t V$ which is closer to the Grassmannian representing the set of acceptable solutions for DAP.
5. Calculate the decomposable vector and hence a solution of approximate DAP, that best approximates k^t

Remark (7): Such multilinear eigenvalue eigenvector problems has been studied in the literature [18,19] for symmetric tensors and similar power methods are employed for their solution. The main problem for these methods is that convergence is not always guaranteed as in the static matrix case. For this, a shifted power methods has to be employed employed [18].

□

Remark (8):A major application of DAP is the pole placement problem where the polynomial matrix M(s) is the composite MFD of a linear system the degree of which equals to the number of states n of the system. The unknown matrix *[I_p,K]* has dimensions *p×(p+m)* where *p* is the number of inputs and *m* is the number of outputs of the system and finally the polynomial *a(s)* correspond to the closed loop pole polynomial which has degree *n*. The generic solvability conditions for real solutions now become *mp≥n* whereas when *mp<n* the problem cannot generically solved.

□

Here we present two examples for DAP corresponding to the pole placement problem one that we can find exact solutions and one for approximate solutions.

Example (7): Consider the system of 3 inputs, 3 outputs and 7 states with transfer function which has the following composite MFD:

$$M(s)^T = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}^T = \begin{bmatrix} s^3 & s^2 & s & s+1 & 1 & 1 \\ 0 & s^2+1 & s^2-s-2 & 2s+1 & s & 1 \\ 0 & 0 & s^2 & s-1 & s-3 & 1 \end{bmatrix}$$

The system has 5 poles at 0 and 2 poles at $\pm j$, it is therefore not BIBO stable. We would like to place its poles at positions $-1,-2,\ldots,-7$ and we are seeking an output feedback $K \in \mathbb{R}^{3\times 7}$ such that

$$\det\left(\begin{bmatrix} I, K \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \right) = (s+1)(s+2)\cdots(s+7) = a(s)$$

By applying the Binet-Cauchy theorem we get

$$k^t P = \underline{a}^t \text{, with } k^t \in \mathbb{R}^{20},\ P \in \mathbb{R}^{20\times 8}$$

where $\underline{a}^t$ is the coefficient vector of the polynomial $a(s)$. Note that for the exact problem to be solvable $\underline{k}^t$ has to be decomposable. The solution of the linear problem is of the form

$$\underline{k}^t = \underline{x}^t V \text{, where } \underline{x}^t \in \mathbb{R}^{13} \text{ and } V \in \mathbb{R}^{13\times 20}$$

The optimization problem (27) has as an objective function a 4th order homogeneous polynomial in 13 variables, i.e. $\underline{x} = (x_1, x_2, \ldots, x_{13})$. The matrix $A(\underline{x})$ is a 13×13 matrix of the form

$$A(\underline{x}) = \sum_{i=1,\, j=1}^{6} \phi_{ij}(\underline{x}) A_{ij}$$

where $\phi_{ij}(\underline{x})$ are 36 quadratics in 13 variables whose representation matrix is A_{ij}. Starting from an appropriate selected vector $\underline{x}_0 \in \mathbb{R}^{13}$, we apply the iteration

$$\underline{x}_{n+1} = A(\underline{x}_n)\underline{x}_n / \left\| A(\underline{x}_n)\underline{x}_n \right\|$$

and after a sufficiently large number of iterations we stopped when the value of the objective function becomes $m-l = 6-3 = 3$ in which case we have exact pole placement.

The solution $\underline{k}^t = \underline{x}_{n+1} V$ is given by

$\underline{k}^t$ = (-0.000345, -0.198335, 0.271311, -0.0244774, 0.0826322,-0.112460, 0.0100429, 0.327394,-0.0891579, 0.330639, 0.0815575,-0.451625, 0.0404264, 0.363256, -0.210901, 0.243668, 0.394281, -0.0607202, 0.0347231, 0.184841) this is a decomposable vector which gives rise to the feedback controller

$$K = \begin{bmatrix} -958.381 & 1309.17 & -117.214 \\ 239.588 & -326.091 & 29.119 \\ 576.064 & -786.652 & 70.971 \end{bmatrix}$$

□

The previous example demonstrated the case where an exact solution exists. We give now an example for the case where the generic solvability conditions are not satisfied, The proposed algorithm in that case provides an approximate solution.

Example(8): Let the system of 2-inputs, 4-outputs and 9-states given by the following MFD

$$M(s) = \begin{bmatrix} (s+3)^5 & s^4 & s^3 & s^2+1 & s-1 & 1 \\ 0 & s(s+2)^3 & s^3-s-2 & 2s^2+1 & s & 1 \end{bmatrix}^T$$

This system is unstable having one pole at $s=0$. We seek to place its poles at $s=-1,-2,-3,\ldots,-9$ by static output feedback and thus to stabilize it. We form the matrix $P=\left[C_2(M(-1)) \,\vdots\, C_2(M(-2)) \,\vdots\, \cdots \,\vdots\, C_2(M(-9))\right] \in \mathbb{R}^{15\times 9}$ and let $V \in \mathbb{R}^{6\times 15}$ an orthonormal basis matrix for the left kernel of P. Then, a representative z of the linear problem satisfies: $z = \underline{x}V, \quad \underline{x} \in \mathbb{R}^6$. To find the best decomposable vector we consider the matrix $\Phi(\underline{x}) = \Phi_6^2(\underline{x}V)$ and the 4th order homogeneous polynomial

$$p(\underline{x}) = \mathrm{tr}\left[\Phi^T(\underline{x})\Phi(\underline{x})\right]^2$$

We solve the maximization problem: $\max p(\underline{x}) \quad \text{s.t.} \quad \|\underline{x}\| = 1$ and hence we find,

$$\underline{x} = -0.0286776, \ -0.781733, \ -0.48096, \ -0.196208, \ -0.331037, \ 0.0930859$$

which gives rise to

$$\begin{aligned} z = \underline{x}V = (&0.0000703886, \ 0.00168933, \ 0.0111918, \ 0.125527, \ 0.0986488, \\ &0.00523094, \ 0.0146791, \ 0.150223, \ 0.118616, \ -0.0519013, \\ &-0.656751, \ -0.514066, \ -0.375565, \ -0.283407, \ 0.133626) \end{aligned}$$

the best decomposable approximation for z is

$$\begin{aligned} z' = (&0.00053719, \ 0.00190004, \ 0.0110096, \ 0.125537, \ 0.0986499, \\ &0.00508362, \ 0.014781, \ 0.150217, \ 0.118615, \ -0.0519128, \\ &-0.656751, \ -0.514066, \ -0.375565, \ -0.283407, \ 0.133626) \end{aligned}$$

which gives rise to the controller:

$$K = \begin{bmatrix} -9.46425 & -27.5179 & -279.661 & -220.828 \\ 3.53733 & 20.4968 & 233.715 & 183.658 \end{bmatrix}$$

Using this controller the closed loop pole polynomial is calculated via

$$\det\left[I_2, \ K\right] \cdot M(s) \ = p'(s)$$

and the roots of $\mathrm{p}'(\mathrm{s})$ are:

$$-6{,}12657, -4.01 \pm 2.17818\mathrm{j}, -2.60657 \pm 0.787584\mathrm{j}$$
$$-1.93878, -1.12736 \pm 4.99248\mathrm{j}, -0.984124$$

Clearly, all of them have negative real part and thus $\mathrm{p}'(\mathrm{s})$ is stable. Therefore, the solution of the approximate DAP of the above defined pole placement problem guarantees stability (but not exact pole placement).

□

8. Conclusions

A new method for computing solutions of the DAP problem has been presented based on some new criteria for the solution of exterior equations, or for the decomposability of multivectors $\underline{z} \in \wedge^m \mathcal{U}$. These new criteria have been given in terms of the rank properties of the structured Grassmann matrix $\Phi_n^m(\underline{z})$ and its Hodge dual $\Phi_n^{n-m}(\underline{z}^*)$ defined for every $\underline{z} \in \wedge^m \mathcal{U}$. The new tests are simpler in nature to that given by the QPRs and they have the extra advantage that allow the reconstruction of the solution space $\mathcal{V}_{\underline{z}}$ of the $\underline{v}_1 \wedge ... \wedge \underline{v}_m = \underline{z}$ equation, by computing the right null space of $\Phi_n^m(\underline{z})$. The new framework based on Grassman matrices provide an alternative formulation for investigation of existence, as well as computation of solutions of $\mathbb{R}-$DAP that is reduced to an optimization. It is known [7], that for a given assignable polynomial $\alpha(s)$ the solution space of the linear subproblem of $\mathbb{R}-$DAP may be parametrically expressed as $\mathcal{K}(\underline{\alpha}, \underline{t})$, where $\underline{t}$ is a free parameter vector; by substituting into the GM, the $\Phi_n^m(\underline{\alpha}, \underline{t})$ GM is obtained with its entries being linear functions of $(\underline{\alpha}, \underline{t})$ vectors. Solvability of $\mathbb{R}-$DAP is thus reduced to finding the $\underline{t}$ vectors such that the rank condition is satisfied. In comparison to the algebraic geometry framework (real intersections of $\Psi(\underline{\alpha})$ and $\Omega(m,n)$), this alternative formulation has the advantage that it may tackle nongeneric cases and whenever a solution exists, their computation is straightforward.

The nature of the control problem may impose restrictions on the matrix H of $\mathbb{R}-$DAP, which may be expressed either as fixed values, or as inequality constraints on certain entries of H. The algebraic geometry framework, although useful for establishing existence of solutions in generic cases, may be difficult, or almost impossible to use. The alternative approach, based on the structured Grassmann matrix, is more suitable; in fact, fewer free parameters in $\Phi_n^m(\underline{z})$ make the investigation of its rank properties simpler, rather than more difficult. By combining the power of algebraic geometry methods (in establishing conditions for generic solvability) with the concreteness of the GM framework (tackling specific cases, as well as computations), an integrated powerful approach will emerge for the study of DAP. This new method transforms the exact or approximate DAP to a nonlinear eigenvalue-eigenvector problem which can be solved efficiently using appropriate numerical methods. A main feature of this approach is the convergence of the method that it is apparent experimentally but it has to be rigorously proven. Similar power methods for symmetric tensors have been addressed in [18,19] and convergence has been proven for appropriate modifications called shifted power methods.

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