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Citation: Asimit, A.V., Chi, Y. and Hu, J. (2015). Optimal non-life reinsurance under Solvency II Regime. *Insurance: Mathematics and Economics*, 65, pp. 627-637. doi: 10.1016/j.insmatheco.2015.09.006

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Optimal Non-life Reinsurance under Solvency II Regime

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October 2, 2015

Abstract: The optimal reinsurance contract is investigated from the perspective of an insurer who would like to minimise its risk exposure under Solvency II. Under this regulatory framework, the insurer is exposed to the retained risk, reinsurance premium and change in the risk margin requirement as a result of reinsurance. Depending on how the risk margin corresponding to the reserve risk is calculated, two optimal reinsurance problems are formulated. We show that the optimal reinsurance policy can be in the form of two layers. Further, numerical examples illustrate that the optimal two-layer reinsurance contracts are only slightly different under these two methodologies.

Keywords and phrases: Optimal Reinsurance; Risk Margin; General Premium Principle; Solvency II; Technical Provision.

1. INTRODUCTION

A standard reinsurance contract is usually reached between two parties: the *insurer*, *cedent*, *insurance buyer*, or even simpler, *buyer*, who has an interest in transferring part of its risk to the *reinsurer*, also known as *insurance seller*, or even simpler, *seller*. Mathematically, let $X \geq 0$ be the total risk that the insurer faces during a fixed period, with distribution function denoted by $F(\cdot)$ and survival function $\bar{F}(\cdot) = 1 - F(\cdot)$. Moreover, the right end-point of F is denoted by $x_F := \inf\{z \in \mathfrak{R} : F(z) = 1\}$, where $\inf \phi = +\infty$ by convention. The reinsurance seller agrees to pay, $R[X]$, the amount by which the entire loss exceeds the insurer's amount, $I[X]$, and therefore $I[X] + R[X] = X$. Two most common reinsurance contracts are the *Quota-share* and *Stop-loss*, where $I[X] = cX$ (with $0 \leq c \leq 1$) and

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$I[X] = X \wedge M := \min\{X, M\}$ (with $0 \leq M \leq x_F$), respectively. In order to avoid potential moral hazard issues arising from the reinsurance arrangement, the set of feasible contracts is usually given by

$$\mathcal{F} := \{0 \leq R[x] \leq x : R[x] \text{ and } x - R[x] \text{ are non-decreasing functions}\}. \quad (1.1)$$

There is a vast academic literature on identifying the optimal risk transfer for an insurance company within a one-period setting. The first attempts are attributed to Borch (1960) and Arrow (1963) who consider to minimise the variance of the insurer's retained risk and maximise the expected utility of the insurer's final wealth, respectively. Further extensions have been developed for various decision criteria (see for example, Van Heerwaarden *et al.*, 1989, Young, 1999, Kaluska, 2001, 2005, Verlaak and Beirlant, 2003, Kaluszka and Okolewski, 2008, Ludkovski and Young, 2009 etc). Two risk measures commonly used in practice, *Value-at-risk (VaR)* and *Conditional Value-at-risk (CVaR)*, are considered by Cai *et al.* (2008), Cheung (2010), Chi and Tan (2011) and Lu *et al.* (2014). The classical risk model setting has been successfully studied in the literature by Centeno and Guerra (2010) and Guerra and Centeno (2008, 2010), where a natural choice for optimisation is the maximisation of the adjustment coefficient. An optimal reinsurance contract where the counterparty default risk is incorporated into the model has been investigated by Biffis and Millossovich (2012), Bernard and Ludkovski (2012) and Asimit *et al.* (2013). Other extensions and variations of optimal reinsurance design studied in the literature include, for example, the model with multiple reinsurers by Chi and Meng (2014) and the model with risk margins determined via expectile risk measure by Cai and Weng (2015). A recent paper of Tan and Weng (2014) investigates the optimal reinsurance design using an empirical approach.

A relatively recent project, namely *Solvency II*, has been developing in order to harmonise the regulatory environment within the European Union (EU) insurance industry. This unified methodology applies to all insurance/reinsurance companies that operate in the EU insurance market and its legal framework is specified in European Commission (2009). The actual implementation of Solvency II is expected to be put in place 2016, and in the meantime, various *Quantitative Impact Studies (QIS)* have been performed. The main purpose of these QIS's has been to collect feedback from many insurance and reinsurance companies related to the constantly augmented Solvency II specifications. The most recent one, also known as *QIS 5* (see European Commission, 2010), defines the most probable recommendations that will later lead to the implementation of Solvency II. Thus, it is very interesting to investigate the optimal reinsurance policy for an insurance company under the Solvency II Regime, which is the aim of the paper.

In this paper, we only deal with a non-life insurer, since the life reinsurance contracts have different characteristics. In addition, the buyer is regulated with the help of the Solvency II current recommendations. The aim of this paper is to identify the optimal reinsurance arrangement where the buyer's risk is quantified by the VaR plus the change in the risk margin requirement as a result of reinsurance. Traditionally, the cost of reinsurance has been considered to be solely the reinsurance premium. It would be more realistic to also include the change in the cost of capital as a result of reinsurance, which is given by the *technical provisions (TP's)* under the Solvency II jurisdiction. More specifically, the TP's are composed of *best estimate (BE)* and *risk margin (RM)*, but only the RM component is incorporated in our optimisation objective as the BE's are calculated gross of reinsurance. Moreover, the RM's can

be calculated via the existing proxy model in QIS5 with Log-Normal approximation or by taking into account the risk profile of the underwriting risk. Depending on how the RM corresponding to the reserve risk is calculated, two optimal reinsurance models are formulated. We show that any admissible reinsurance policy defined in (1.1) is dominated by a two-layer reinsurance contract. Further, it is very interesting to point out that our numerical analysis shows similar optimal reinsurance contracts under the two different calculation methodologies of the RM's. The rest of the paper is organised as follows: Section 2 defines our model, while Section 3 provides us with the main results. Finally, Section 4 gives some numerical examples.

2. MODEL FORMULATION

This section provides the details of optimal reinsurance models that are investigated in the current paper. Let us denote P and $\pi(R[X])$ as the gross premium charged by the buyer and the seller, respectively, where $\pi(\cdot)$ represents the premium principle chosen by the reinsurer. To make our results accessible to many reinsurance premium principles, we follow the assumption of Chi (2012), that $\pi(\cdot)$ preserves the convex order, i.e. $\pi(X) \leq \pi(Y)$ if $X \leq_{cx} Y$ which is equivalent to

$$EX = EY \text{ and } E(X - d)_+ \leq E(Y - d)_+, \text{ for all } d \in \mathfrak{R},$$

where $(x)_+ := \max\{0, x\}$. The buyer's profit is the difference between $P - \pi(R[X]) - I[X]$ and the cost of setting the TP's as defined in Solvency II. The latter is defined and detailed in European Commission (2009, 2010), but a succinct description can be found in Asimit *et al.* (2015), which is further used as a baseline reference in our commentary on TP's. As discussed in Section 1, for each line of business, each TP consists of BE of the liabilities and its RM, where the BE is defined as an expectation of future liabilities that are evaluated gross of any risk transfer. Thus, our optimisation objective will not take this component into account. The RM's are usually evaluated by considering four sources of risk: *underwriting (UwR)*, *unavoidable market (UMR)*, *counterparty default (CDR)* and *operational (OpR)*.

Firstly, the OpR (see for example, Asimit *et al.*, 2015) is independent of any risk transfer, reinsurance transaction or coinsurance, and thus, we can remove this component from our analysis. In addition, the UwR includes the *premium (PR)* and *reserve (RR)* risks and its corresponding RM is calculated as follows

$$RM_{UwR}(R[X]) := g\left(a_1\left(P - \pi(R[X])\right), b_1EI[X]\right),$$

where $g(x, y) := \sqrt{x^2 + y^2 + xy}$ and parameters a_1 and b_1 are given by

$$a_1 = \lambda \left(\frac{\exp\left\{\Phi^{-1}(p)\sqrt{\log(1 + \sigma_{PR}^2)}\right\}}{\sqrt{1 + \sigma_{PR}^2}} - 1 \right), \quad b_1 = \lambda \left(\frac{\exp\left\{\Phi^{-1}(p)\sqrt{\log(1 + \sigma_{RR}^2)}\right\}}{\sqrt{1 + \sigma_{RR}^2}} - 1 \right).$$

Here, $p = 99.5\%$, $\Phi^{-1}(\cdot)$ is the quantile function of the standard Normal distribution and λ represents the adjusted Cost-of-Capital rate. The parameters σ_{PR} and σ_{RR} are the coefficients of variation for PR and RR, respectively, as calibrated in QIS5 under the Log-Normal assumption.

Secondly, the RM for the UMR component is evaluated via the following formula

$$RM_{UMR}(R[X]) := cEI[X], \text{ with } c = CoC \times (d - n)(d - n + 1)\Delta_n,$$

where CoC and d are the Cost-of-Capital rate and modified duration of the insurer's net liability $I[X]$, respectively. In addition, n is the longest duration of available risk-free financial instruments to cover $I[X]$. Moreover, Δ_n represents the absolute decrease of the risk-free interest for maturity n under the downward stress scenario of the interest rate risk sub-module (as defined in the SCR section of European Commission, 2009).

Thirdly, the RM for CDR is given by

$$RM_{CDR}(R[X]) := g\left(a_2\pi(R[X]), b_2ER[X]\right),$$

with parameters a_2 and b_2 given by

$$\begin{aligned} a_2 &= \lambda(1 - RecR) l\sqrt{q(1-q)} \times \\ &\quad \left(\frac{\Phi\left(\sqrt{\log(1 + \sigma_{PR}^2)} - \Phi^{-1}(p)\right)}{1-p} - \frac{\exp\left\{\Phi^{-1}(p)\sqrt{\log(1 + \sigma_{PR}^2)}\right\}}{\sqrt{1 + \sigma_{PR}^2}} \right), \\ b_2 &= \lambda(1 - RecR) l\sqrt{q(1-q)} \times \\ &\quad \left(\frac{\Phi\left(\sqrt{\log(1 + \sigma_{RR}^2)} - \Phi^{-1}(p)\right)}{1-p} - \frac{\exp\left\{\Phi^{-1}(p)\sqrt{\log(1 + \sigma_{RR}^2)}\right\}}{\sqrt{1 + \sigma_{RR}^2}} \right), \end{aligned}$$

where l is a constant (usually, $3 \leq l \leq 5$), while q and $RecR$ represent the reinsurer one-year default probability and recovery rate, respectively.

To quantify the insurer's risk exposure, we also need to introduce VaR, which is a well-known risk measure in finance and insurance. The VaR of a generic loss variable Z at a confidence level a , $VaR_a(Z)$, represents the minimum amount of capital that makes the insurance company to be solvent at least $a\%$ of the time, i.e.

$$VaR_a(Z) := \inf\{z \geq z_0 : \Pr(Z \leq z) \geq a\},$$

where $z_0 := \sup\{z \in \mathfrak{R} : \Pr(Z \leq z) = 0\}$ represents the left-end point of the distribution function of Z . Recall that $VaR_p(I[X]) = VaR_p(X) - VaR_p(R[X])$ is true as a result of $R[X] \in \mathcal{F}$.

We are now able to introduce our optimisation problem. The classical approach in the existing literature, since the pioneering work of Cai and Tan (2007), has been to minimise the insurer's retained risk and reinsurance premium. In the Solvency II framework, it is translated to $VaR_p(I[X]) + \pi(R[X])$, since $VaR_p(\cdot)$ is the standard risk measure used under this regulatory framework. In addition, it also includes the additional cost of capital as a result of buying reinsurance. That is, the difference between the RM's after and before the risk transfer is given by

$$\begin{aligned} &RM_{UwR}(R[X]) + RM_{UMR}(R[X]) + RM_{CDR}(R[X]) \\ &\quad - RM_{UwR}(X) - RM_{UMR}(X) - RM_{CDR}(X). \end{aligned}$$

Recall that the BE's are calculated gross of reinsurance, and thus, our objective functions may not take the BE's into account. To simplify the representation, we further remove the last three terms in the above equation from our objective function, since they are not sensitive to the ceded loss function $R[X]$.

Assuming that the buyer reinsures only one risk, the optimisation problem, formulated with Log-Normal approximation, becomes:

$$\min_{R \in \mathcal{F}} H_1(R[X]), \quad (2.1)$$

where

$$\begin{aligned} H_1(R[X]) : &= RM_{UwR}(R[X]) + RM_{UMR}(R[X]) + RM_{CDR}(R[X]) \\ &\quad + \pi(R[X]) + VaR_p(X) - VaR_p(R[X]) \\ &= g\left(a_1\left(P - \pi(R[X])\right), b_1 EI[X]\right) + g\left(a_2 \pi(R[X]), b_2 ER[X]\right) \\ &\quad + c(EX - ER[X]) + \pi(R[X]) + VaR_p(X) - VaR_p(R[X]). \end{aligned}$$

One may wonder why RR is calculated via a Log-Normal approximation, while the reinsurance premium calculation takes into account the risk profile of the underwriting policies. The insurer may choose the Standard Formula or adjust this formula by using its belief on the distribution of X , and the most advantageous one would be its choice.

In order to rewrite (2.1) in the absence of the Log-Normal approximation, we need to introduce another well-known risk measure, $CVaR$. This risk measure has various representations in the literature (see Acerbi and Tasche, 2002), and one of them is given by

$$CVaR_a(Z) := \frac{1}{1-a} \int_a^1 VaR_s(Z) ds = VaR_a(Z) + \frac{1}{1-a} \mathbf{E}(Z - VaR_a(Z))_+. \quad (2.2)$$

Interestingly, this risk measure is a special case of the Haezendonck-Goovaerts class, which was introduced many years ago by Haezendonck and Goovaerts (1982). Further details can be found in Bellini and Rosazza Gianin (2012), Goovaerts *et al.* (2004 and 2012) and the references therein.

Recall that all previous RM's have been calculated based on the Standard Formula from Solvency II, where the Log-Normal approximation has been used in the definition of RM's for UwR and CDR. An alternative calculation of the RR contributions is to take into account the actual variation within the risk itself, i.e. $I[X]$ and $R[X]$ for the UwR and CDR, respectively. Concretely, the Log-Normal approximation for the RM's of UwR and CDR showed that

$$\lambda\left(VaR_p(I[X]) - EI[X]\right) \approx b_1 EI[X]$$

and

$$\vartheta\left(CVaR_p(R[X]) - VaR_p(R[X])\right) \approx b_2 ER[X],$$

where $\vartheta = \lambda(1 - RecR) l\sqrt{q(1-q)}$. It has been implicitly assumed in the last two equations that $I[X]$ and $R[X]$ are Log-Normal distributed with the same coefficient of variation. This standard assumption is acceptable in the Solvency II framework, and the values of σ_{RR} for the recognised nine lines of business can be found in European Commission (2010) or Asimit *et al.* (2015). By the above approximations, we should guarantee that $VaR_p(I[X]) \geq EI[X]$, and the RM's for UwR and CDR can be rewritten by

$$RM_{UwR}(R[X]) = g\left(a_1\left(P - \pi(R[X])\right), \lambda\left(VaR_p(I[X]) - EI[X]\right)\right)$$

and

$$RM_{CDR}(R[X]) = g\left(a_2\pi(R[X]), \vartheta\left(CVaR_p(R[X]) - VaR_p(R[X])\right)\right)$$

respectively. Thus, if we refrain ourselves from using the Log-Normal approximation in the RM calculation corresponding to RR, the optimisation problem (2.1) can be rewritten as follows:

$$\min_{R \in \mathcal{F}} H_2(R[X]), \quad (2.3)$$

where

$$\begin{aligned} H_2(R[X]) : &= g\left(a_1\left(P - \pi(R[X])\right), \lambda\left(VaR_p(I[X]) - EI[X]\right)\right) + c(EX - ER[X]) \\ &+ g\left(a_2\pi(R[X]), \vartheta\left(CVaR_p(R[X]) - VaR_p(R[X])\right)\right) + \pi(R[X]) \\ &+ VaR_p(X) - VaR_p(R[X]). \end{aligned}$$

3. OPTIMAL REINSURANCE DESIGN

In this section, we proceed with solving the optimal reinsurance problems (2.1) and (2.3). Note that $a_1, a_2, b_1, b_2 \in (0, 1)$, since $CoC, \lambda \in (0, 1)$ and the market wide estimates for σ_{PR} and σ_{RR} showed to be less than 22%. Therefore, it is not difficult to see that the objective function from (2.1) is increasing in $\pi(R[X])$. Consequently, the first step in solving (2.1) is to find the solutions to the following optimisation problem

$$\begin{aligned} &\arg \min_{R \in \mathcal{F}} \pi(R[X]), \\ &\text{subject to } ER[X] = \mu \in [0, EX], \\ &VaR_p(R[X]) = \nu, \\ &\pi(R[X]) \leq P. \end{aligned}$$

Clearly, the last inequality constraint ensures that the reinsurance premium could not exceed the insurer's gross premium.

Before solving the above minimisation problem, we first define a layer $(a, b]$ of a risk Z by

$$\mathcal{L}_{(a,b]}(Z) := (Z - a)_+ \wedge (b - a) = (Z - a)_+ - (Z - b)_+, \quad 0 \leq a \leq b.$$

Next, the analysis of the above infinite-dimensional optimisation problem can be reduced to studying a certain finite-dimensional one, as shown in the following theorem.

Theorem 3.1. *For any admissible ceded loss function $R[X]$, denote $\mu = ER[X]$ and $\nu = VaR_p(R[X])$, then there must exist an admissible ceded loss function $\tilde{R}[X]$ in the form of*

$$\tilde{R}[X] = \begin{cases} X \wedge \nu + \mathcal{L}_{(VaR_p(X), t]}(X), & E(X \wedge \nu) \leq \mu; \\ X \wedge z + \mathcal{L}_{(VaR_p(X) - \nu + z, VaR_p(X)]}(X), & \text{otherwise,} \end{cases} \quad (3.1)$$

for some $z \in [0, \nu]$ and $t \geq VaR_p(X)$ such that

$$CVaR_p(R[X]) \geq CVaR_p(\tilde{R}[X]), \quad VaR_p(\tilde{R}[X]) = VaR_p(R[X]) \quad \text{and} \quad \tilde{R}[X] \leq_{cx} R[X].$$

Moreover, we have

$$CVaR_p(\tilde{R}[X]) - VaR_p(\tilde{R}[X]) = \frac{1}{1-p}(\mu - E(X \wedge \nu))_+. \quad (3.2)$$

Proof. Using a proof similar to that of Theorem 3.2 in Chi (2012), we get that there exists an admissible ceded loss function in the form of

$$\hat{R}[X] = \mathcal{L}_{(0,d_1]}(X) + \mathcal{L}_{(VaR_p(X)+d_1-\nu,d_2]}(X),$$

for some $d_1 \in [0, \nu]$ and $d_2 \in [VaR_p(X), x_F]$ such that

$$\hat{R}[X] \leq_{cx} R[X], \quad VaR_p(\hat{R}[X]) = \nu \quad \text{and} \quad CVaR_p(\hat{R}[X]) = CVaR_p(R[X]).$$

Now, building upon $\hat{R}[X]$, we can construct an admissible ceded loss function $\tilde{R}[X]$ as in (3.1), where t and z are determined by $E\tilde{R}[X] = E\hat{R}[X] = \mu$. More specifically, if $E[X \wedge \nu] \leq \mu$, it is not difficult to find that $\mathcal{H}(y) := E(X \wedge \nu + \mathcal{L}_{(VaR_p(X),y]}(X))$, $y \geq VaR_p(X)$, is a non-decreasing continuous function with

$$\mathcal{H}(VaR_p(X)) = E(X \wedge \nu) \leq \mu \quad \text{and} \quad \mathcal{H}(d_2) \geq E\hat{R}[X] = \mu.$$

Thus, there must exist a $t \in [VaR_p(X), d_2]$ such that $E\tilde{R}(X) = \mathcal{H}(t) = \mu$. Now, if $E(X \wedge \nu) > \mu$, we have $E\tilde{\mathcal{H}}[X; d_1] \leq E\hat{R}[X] = \mu$ and $E\tilde{\mathcal{H}}[X; \nu] = E(X \wedge \nu) > \mu$, where

$$\tilde{\mathcal{H}}[X; y] := X \wedge y + \mathcal{L}_{(VaR_p(X)-\nu+y, VaR_p(X))}[X], \quad d_1 \leq y \leq \nu.$$

Moreover, $E\tilde{\mathcal{H}}[X; y]$ is continuous and non-decreasing in y , and therefore, there must exist a $z \in [d_1, \nu]$ such that $E\tilde{R}[X] = E\tilde{\mathcal{H}}[X; z] = \mu$.

Further, we can see from the definition of $\tilde{R}[X]$ that $VaR_p(\tilde{R}[X]) = \nu$ and (3.2) are true. By comparing $\tilde{R}(X)$ with $\hat{R}(X)$, we note that $\hat{R}(X) \geq \tilde{R}(X)$ and $\hat{R}(X) \leq \tilde{R}(X)$ are true for any $X \geq VaR_p(X)$ and $0 \leq X \leq VaR_p(X)$, respectively. In other words, \hat{R} up-crosses \tilde{R} , and in turn, it follows from Lemma 3 of Ohlin (1969) that $\tilde{R}[X] \leq_{cx} R[X]$. It is well-known that the CVaR risk measure preserves the convex order, and thus, we have $CVaR_p(R[X]) \geq CVaR_p(\tilde{R}[X])$. The proof is now complete. \square

By the above theorem, we know that any admissible ceded loss function R is dominated by \tilde{R} (as defined in (3.1)). According to the definition of \tilde{R} , one can find that the parameters z and t when μ and ν are known. Moreover, as $\tilde{R}[X]$ is a function of μ and ν , we rewrite it as $\tilde{R}[X; \mu, \nu]$ in order to emphasise this dependence. As a result, the study of optimal reinsurance problem (2.1) can be simplified to solving a minimisation problem of two decision variables μ and ν . Before solving this minimisation problem, it is necessary to elaborate the relationship between μ and ν . Specifically, for any $R \in \mathcal{F}$ with $VaR_p(R[X]) = \nu$, we have

$$\mathcal{L}_{(VaR_p(X)-\nu, VaR_p(X))}(X) \leq R[X] \leq X \wedge \nu + (X - VaR_p(X))_+.$$

Consequently, (μ, ν) should satisfy

$$\int_{VaR_p(X)-\nu}^{VaR_p(X)} \bar{F}(x) dx \leq \mu \leq EX - \int_{\nu}^{VaR_p(X)} \bar{F}(x) dx \quad \text{and} \quad \nu \leq VaR_p(X). \quad (3.3)$$

Thus, optimisation problem (2.1) is equivalent to

$$\min_{\{(\mu, \nu) \in \mathbb{R}_+^2 : (3.3) \text{ is satisfied}, \pi(\tilde{R}[X; \mu, \nu]) \leq P\}} h_1(\mu, \nu), \quad \text{where } h_1(\mu, \nu) := H_1(\tilde{R}[X; \mu, \nu]). \quad (3.4)$$

The analysis of the above optimisation problem can be further simplified by using the following proposition.

Proposition 3.1. *For any $0 \leq \mu \leq EX$, we have*

$$h_1(\mu, \nu) \geq h_1(\mu, \nu(\mu)), \forall 0 \leq \nu \leq \nu(\mu),$$

where

$$\nu(\mu) := \sup \{0 \leq t \leq VaR_p(X) : E(X \wedge t) \leq \mu\}. \quad (3.5)$$

Proof. For any $0 \leq \nu^1 < \nu^2 \leq \nu(\mu)$, we can show that

$$\tilde{R}[X; \mu, \nu^2] \leq_{cx} \tilde{R}[X; \mu, \nu^1]. \quad (3.6)$$

Specifically, noting that $E(X \wedge \nu^i) \leq \mu$ for $i = 1, 2$, it can be seen from (3.1) that $\tilde{R}[X; \mu, \nu^1]$ up-crosses $\tilde{R}[X; \mu, \nu^2]$ and that $E\tilde{R}[X; \mu, \nu^i] = \mu$ for $i = 1, 2$. Then, (3.6) could be obtained by using Lemma 3 from Ohlin (1969).

Further, $\pi(\tilde{R}[X; \mu, \nu^2]) \leq \pi(\tilde{R}[X; \mu, \nu^1])$ follows from the assumption that $\pi(\cdot)$ preserves the convex order. Recall that $h_1(\mu, \nu)$ is non-increasing in ν and is non-decreasing in $\pi(\tilde{R}[X; \mu, \nu])$. Hence, the final result is obtained, which concludes our proof. \square

We now turn to study optimal reinsurance model (2.3). Recall that $a_1, a_2, b_1, b_2 \in (0, 1)$. It is not difficult to find that the objective function $H_2(R[X])$ is increasing in $\pi(R[X])$ and $CVaR_p(R[X])$. Further, using Theorem 3.1, we know that any admissible ceded loss function $R[X]$ is sub-optimal to $\tilde{R}[X; \mu, \nu]$ where $\mu = ER[X]$ and $\nu = VaR_p(R[X])$. As a result, the analysis of optimal reinsurance model (2.3) can be simplified to solving an optimisation problem of two decision variables μ and ν . However, in contrast to the optimisation problem (3.4), (μ, ν) is subject to more constraints. Specifically, in addition to (3.3), another constraint, $\nu \leq VaR_p(X) + \mu - EX$ (in order to guarantee the feasibility constraint $VaR_p(I[X]) \geq EI[X]$), is needed as a consequence of removing the Log-Normal approximation. Thus, the optimisation problem (2.3) is equivalent to

$$\min_{\substack{(\mu, \nu) \in \mathbb{R}_+^2 : \nu \leq VaR_p(X) + \mu - EX \\ \pi(\tilde{R}[X; \mu, \nu]) \leq P \text{ and (3.3) is satisfied}}} h_2(\mu, \nu), \quad \text{where } h_2(\mu, \nu) := H_2(\tilde{R}[X; \mu, \nu]). \quad (3.7)$$

We try to further reduce the dimension of the above minimisation problem, and define

$$\phi(\mu) =: \mu - E(X \wedge (VaR_p(X) + \mu - EX)), \quad 0 \leq \mu \leq EX.$$

Noting that $p = 99.5\%$, we assume that $VaR_p(X) \geq EX$ in the remainder of the paper. Thus, $\phi(\mu)$ is a non-decreasing and continuous function with $\phi(0) \leq 0$ and $\phi(EX) \geq 0$. Consequently,

$$\mu_0 := \inf \{0 \leq \mu \leq EX : \phi(\mu) = 0\}$$

is well-defined.

Proposition 3.2. *If $\mu \geq \mu_0$, we get*

$$h_2(\mu, \nu) \geq h_2(\mu, VaR_p(X) + \mu - EX), \quad \forall 0 \leq \nu \leq VaR_p(X) + \mu - EX.$$

Whenever $0 \leq \mu < \mu_0$, we have

$$h_2(\mu, \nu) \geq h_2(\mu, \nu(\mu)), \forall 0 \leq \nu \leq \nu(\mu),$$

where $\nu(\mu) \in [0, VaR_p(X))$ is defined in (3.5).

Proof. It is first assumed that $\mu \geq \mu_0$. Now, $E(X \wedge \nu) \leq \mu$ holds as a result of $\nu \leq VaR_p(X) + \mu - EX$. We can see from the proof of Proposition 3.1 that

$$\pi(\tilde{R}[X; \mu, \nu^2]) \leq \pi(\tilde{R}[X; \mu, \nu^1]) \text{ for any } 0 \leq \nu^1 < \nu^2 \leq VaR_p(X) + \mu - EX.$$

Furthermore, relation (3.2) yields that $CVaR_p(\tilde{R}[X; \mu, \nu]) - VaR_p(\tilde{R}[X; \mu, \nu])$ is non-increasing in ν . Thus, $h_2(\mu, \nu)$ is non-increasing in ν by keeping in mind that $h_2(\cdot)$ is non-decreasing in $\pi(\cdot)$ and $CVaR_p(\cdot)$. Hence, our first claim is justified in full.

The case in which $0 \leq \mu < \mu_0$ can be dealt as above. The proof is finally completed. \square

Remark 3.1. Proposition 3.2 says that whenever $\mu \geq \mu_0$, any admissible ceded loss function $\tilde{R}[X; \mu, \nu]$ is dominated by $\tilde{R}[X; \mu, VaR_p(X) + \mu - EX]$, which is well-defined as

$$E\left(X \wedge (VaR_p(X) + \mu - EX)\right) + E(X - VaR_p(X))_+ = EX - \int_{VaR_p(X) + \mu - EX}^{VaR_p(X)} \bar{F}(t) dt \geq \mu.$$

Otherwise, for $0 \leq \mu < \mu_0$, Proposition 3.2 implies that the optimal ν must be attained over the interval $[\nu(\mu), VaR_p(X) + \mu - EX] \cap \mathfrak{N}_\mu$, where

$$\mathfrak{N}_\mu := \left\{ \nu \in [0, VaR_p(X) + \mu - EX] : (3.3) \text{ is satisfied.} \right\}, 0 \leq \mu \leq EX.$$

Now, for any $\nu \in [\nu(\mu), VaR_p(X) + \mu - EX] \cap \mathfrak{N}_\mu$, we have $E(X \wedge \nu) \geq E(X \wedge \nu(\mu)) = \mu$. This, together with (3.2), yields that

$$CVaR_p(\tilde{R}[X; \mu, \nu]) - VaR_p(\tilde{R}[X; \mu, \nu]) = 0,$$

which in turn implies that the third term of $h_2(\cdot)$ from (2.3) is reduced to

$$RM_{CDR}(\tilde{R}[X; \mu, \nu]) = a_2 \pi(\tilde{R}[X; \mu, \nu]).$$

Collecting the above arguments, the analysis of minimisation problem (3.7) can be reduced to solving two simpler optimisation problems:

- (i) $\min_{\substack{\mu_0 \leq \mu \\ \pi(\tilde{R}[X; \mu, VaR_p(X) + \mu - EX]) \leq P}} h_2(\mu, VaR_p(X) + \mu - EX);$
- (ii) $\min_{\substack{(\mu, \nu) \in \mathcal{R} \\ \pi(\tilde{R}[X; \mu, \nu]) \leq P}} h_2(\mu, \nu),$

where

$$\mathcal{R} := \left\{ (\mu, \nu) \in \mathfrak{R}_+^2 : 0 \leq \mu < \mu_0, \nu \in [\nu(\mu), VaR_p(X) + \mu - EX] \cap \mathfrak{N}_\mu \right\}.$$

It seems impossible to find closed-form solutions without specifying the premium principle $\pi(\cdot)$. Thus, some well-known premium principles are investigated in the next section in order to derive the optimal parameters of the two-layer reinsurance.

4. EXAMPLES AND NUMERICAL ANALYSIS

The final section of the paper provides the solutions to optimal reinsurance problems (2.1) and (2.3) for some well-known premium principles. Initially, closed-form solutions are found for expected value premium principle, which is formally defined as

$$\pi(\cdot) = (1 + \rho)E[\cdot], \quad (4.1)$$

where $\rho > 0$ represents the *safety loading coefficient*. Numerical solutions are then investigated for a large class of premium principles, known as Wang's class, for which we need to assume that the loss sample space is composed of a finite number of possible outcomes.

4.1. Expected value premium principle. As anticipated, we now find the closed-form solutions of optimisation problems (2.1) and (2.3) under the expected value premium principle assumption. By Theorem 3.1, the optimal ceded loss function can be in the form of (3.1) and the analysis reduces to solving the minimisation problems (3.4) and (3.7). A two-step procedure is used to solve these problems, where the first step derives the optimal ν for a fixed μ , while the second step provides the global optimal solution. Noting that μ is upper-bounded by $\frac{P}{1+\rho}$ since $\pi(\tilde{R}[X; \mu, \nu]) \leq P$, the value of P is critical to the set of feasible solutions. Thus, our analysis is divided accordingly. As reinsurance is expected to be more costly than insurance, the economic constraint, $P \leq (1 + \rho)EX$, is made in this subsection.

We first study the optimisation problem (2.1), where Log-Normal approximation is used. By some simple algebra, the objective function $h_1(\cdot)$ can be rewritten as follows:

$$h_1(\mu, \nu) = cEX + (C_2 - c + 1 + \rho)\mu + C_1 \sqrt{\left(\mu - \frac{B}{2C_1^2}\right)^2 + \frac{3a_1^2 b_1^2}{4C_1^4} (P - (1 + \rho)EX)^2 + VaR_p(X) - \nu}, \quad (4.2)$$

where $C_i := g(a_i(1 + \rho), b_i)$, $i = 1, 2$ and $B := 2a_1^2 P(1 + \rho) + 2b_1^2 EX + a_1 b_1 (P + (1 + \rho)EX)$.

In the first step, note that $h_1(\mu, \nu)$ is non-increasing in ν , and then we have $h_1(\mu, \nu) \geq h_1(\mu, \nu^*(\mu))$, where $\nu^*(\mu) = \sup \left\{ 0 \leq t \leq VaR_p(X) : \int_{VaR_p(X)-t}^{VaR_p(X)} \bar{F}(z) dz \leq \mu \right\}$. The second step is used to solve the following optimisation problem $\min_{\mu \in [0, P/(1+\rho)]} h_1(\mu, \nu^*(\mu))$.

1a) If $P/(1 + \rho) < E(X \wedge VaR_p(X))$, then for any $\mu \in [0, P/(1 + \rho)]$, we have $\nu^*(\mu) < VaR_p(X)$, and together with (4.2) we get that

$$\begin{aligned} h_1(\mu, \nu^*(\mu)) &= (C_2 - c + 1 + \rho)\mu + C_1 \sqrt{\left(\mu - \frac{B}{2C_1^2}\right)^2 + \frac{3a_1^2 b_1^2}{4C_1^4} (P - (1 + \rho)EX)^2} \\ &\quad + VaR_p(X) - \nu^*(\mu) + cEX. \end{aligned}$$

Differentiating $h_1(\mu, \nu^*(\mu))$ with respect to μ , we obtain

$$\begin{aligned} h_1'(\mu, \nu^*(\mu)) &= \frac{C_1 \left(\mu - \frac{B}{2C_1^2}\right)}{\sqrt{\left(\mu - \frac{B}{2C_1^2}\right)^2 + \frac{3a_1^2 b_1^2}{4C_1^4} (P - (1 + \rho)EX)^2}} - \frac{1}{\bar{F}(VaR_p(X) - \nu^*(\mu))} \\ &\quad + C_2 - c + 1 + \rho. \end{aligned} \quad (4.3)$$

One may find that $h_1'(\mu, \nu^*(\mu))$ is non-decreasing. Hence, the minimal value of $h_1(\mu, \nu^*(\mu))$ is obtained at

$$\mu_1^* = \left(\sup \{ 0 \leq \mu \leq P/(1 + \rho) : h_1'(\mu, \nu^*(\mu)) \leq 0 \} \right)_+,$$

where $\sup \emptyset = -\infty$ by convention.

- 1b)** Otherwise, if $P/(1 + \rho) \geq E(X \wedge VaR_p(X))$, we separate the discussion into two subcases: $\mu \in [0, E(X \wedge VaR_p(X))]$ and $\mu \in [E(X \wedge VaR_p(X)), P/(1 + \rho)]$.

For any $\mu \in [0, E(X \wedge VaR_p(X))]$, similar arguments to those used in case (1a), the minimal value of $h_1(\mu, \nu^*(\mu))$ is attainable at

$$\mu_2^* = \left(\sup \left\{ 0 \leq \mu < E(X \wedge VaR_p(X)) : h_1'(\mu, \nu^*(\mu)) \leq 0 \right\} \right)_+,$$

where $h_1'(\mu, \nu^*(\mu))$ is given in (4.3).

If $\mu \in [E(X \wedge VaR_p(X)), P/(1 + \rho)]$, then we have $\nu^*(\mu) = VaR_p(X)$ and

$$h_1(\mu, \nu^*(\mu)) = cEX + (C_2 - c + 1 + \rho)\mu + C_1 \sqrt{\left(\mu - \frac{B}{2C_1^2}\right)^2 + \frac{3a_1^2 b_1^2}{4C_1^4} (P - (1 + \rho)EX)^2}.$$

Clearly, $h_1(\mu, \nu^*(\mu))$ is a convex function and its minimum is attained at

$$\mu_3^* = \max \left\{ \sup \left\{ E[X \wedge VaR_p(X)] \leq \mu \leq \frac{P}{1 + \rho} : C_2 - c + 1 + \rho + \frac{C_1 \left(\mu - \frac{B}{2C_1^2}\right)}{\sqrt{\left(\mu - \frac{B}{2C_1^2}\right)^2 + \frac{3a_1^2 b_1^2}{4C_1^4} (P - (1 + \rho)EX)^2}} \leq 0 \right\}, E(X \wedge VaR_p(X)) \right\}.$$

Finally, the minimum between $h_1(\mu_2^*, \nu^*(\mu_2^*))$ and $h_1(\mu_3^*, \nu^*(\mu_3^*))$ provides the global optimal solution to (2.1).

Now, we turn our attention to solving the optimisation problem (2.3), where the Log-Normal approximation is not used to estimate the RR within the UwR and CDR . It has been found at the end of Remark 3.1 that for any $\mu \geq \mu_0$, we only need to solve a minimisation problem in one variable, namely μ . The same conclusion is drawn in the next proposition for the other case in which $0 \leq \mu < \mu_0$.

Proposition 4.1. *Assume that the reinsurance premium is calculated by the expected value principle as given in (4.1). Whenever $0 \leq \mu < \mu_0$, we have $h_2(\mu, \nu) \geq h_2(\mu, \nu^*(\mu))$ for all $\nu \in \mathfrak{N}_\mu$, where*

$$\nu^*(\mu) := \min \{ VaR_p(X) + \mu - EX, \mathfrak{M}(\mu) \}.$$

Here, $\mathfrak{M}(\mu) := \sup \left\{ s \in [0, VaR_p(X)] : \int_{VaR_p(X)-s}^{VaR_p(X)} \bar{F}(t) dt = \mu \right\}$.

Proof. For any $\nu \in [\nu(\mu), VaR_p(X) + \mu - EX] \cap \mathfrak{N}_\mu$, we have $\pi(\tilde{R}[X; \mu, \nu]) = (1 + \rho)\mu$ and

$$CVaR_p(\tilde{R}[X; \mu, \nu]) - VaR_p(\tilde{R}[X; \mu, \nu]) = 0.$$

Moreover, we get

$$VaR_p(\tilde{I}[X; \mu, \nu]) - E\tilde{I}[X; \mu, \nu] = VaR_p(X) + \mu - EX - \nu,$$

which is non-negative and non-increasing in ν . Therefore, $h_2(\mu, \nu)$ is non-increasing in ν , which completes the proof. \square

Remark 4.1. *It is necessary to determine whenever $\nu^*(\mu) = \mathfrak{M}(\mu)$. Let*

$$\varphi(\mu) := \int_{EX-\mu}^{VaR_p(X)} \bar{F}(t) dt - \mu, \quad 0 \leq \mu \leq \mu_0.$$

Clearly, $\varphi(\mu)$ is a decreasing function with $\varphi(0) = \int_{EX}^{VaR_p(X)} \bar{F}(t)dt \geq 0$ and

$$\varphi(\mu_0) = \int_{EX-\mu_0}^{VaR_p(X)} \bar{F}_X(t)dt - \mu_0 \leq E\left(X \wedge (VaR_p(X) + \mu_0 - EX)\right) - \mu_0 = 0.$$

Thus,

$$\mu_0^s =: \inf \{0 \leq \mu \leq \mu_0 : \varphi(\mu) \leq 0\}$$

is well-defined, and in turn, $\nu^*(\mu) = VaR_p(X) + \mu - EX$ if and only if $\mu \geq \mu_0^s$.

Now, one may find μ^* and solve minimisation problem (2.3), by assuming that μ falls into one of the following three intervals: $[0, \mu_0^s]$, $[\mu_0^s, \mu_0]$ and $[\mu_0, +\infty]$. As mentioned earlier, μ is upper-bounded by $\frac{P}{1+\rho}$, and therefore, the analysis is again divided according to the value of P .

- 2a)** If $P/(1+\rho) < \mu_0^s$, we have $\mu \in [0, P/(1+\rho)]$ and $\mu < \mu_0^s$. Thus, $\nu^*(\mu) = \mathfrak{M}(\mu)$ and $CVaR_p(\tilde{R}[X; \mu, \nu^*(\mu)]) - VaR_p(\tilde{R}[X; \mu, \nu^*(\mu)]) = 0$ are true according to Remarks 3.1 and 4.1. Consequently,

$$h_2(\mu, \nu^*(\mu)) = g\left(a_1(P - (1+\rho)\mu), \lambda K(\mu)\right) + ((1+a_2)(1+\rho) - c)\mu + cEX + VaR_p(X) - \mathfrak{M}(\mu),$$

where $K(\mu) =: VaR_p(X) - EX + \mu - \mathfrak{M}(\mu)$. Clearly,

$$K'(\mu) = 1 - \mathfrak{M}'(\mu) = 1 - \frac{1}{\bar{F}(VaR_p(X) - \mathfrak{M}(\mu))} \leq 0.$$

Thus, if $(1+a_2)(1+\rho) - c - \mathfrak{M}'(\mu) \leq 0$ is satisfied for any $\mu \in [0, P/(1+\rho)]$, then the minimum value of $h_2(\mu, \nu^*(\mu))$ is attained at $\hat{\mu}_1^* = \frac{P}{1+\rho}$. Otherwise, a numerical solution will be employed.

- 2b)** If $\frac{P}{1+\rho} \in [\mu_0^s, \mu_0]$, we give separate discussions for $\mu \in [0, \mu_0^s]$ and $\mu \in [\mu_0^s, P/(1+\rho)]$.

For any $\mu \in [0, \mu_0^s]$, we use similar arguments as used in case 2a). The minimum value of $h_2(\mu, \nu^*(\mu))$ is attained at $\hat{\mu}_2^* = \mu_0^s$, if $(1+a_2)(1+\rho) - c - \mathfrak{M}'(\mu) \leq 0$ is satisfied for any $\mu \in [0, \mu_0^s]$. Otherwise, a numerical solution may be employed.

For any $\mu \in [\mu_0^s, P/(1+\rho)]$, Remarks 3.1 and 4.1 yield that $\nu^*(\mu) = VaR_p(X) + \mu - EX$ and

$$CVaR_p(\tilde{R}[X; \mu, \nu^*(\mu)]) - VaR_p(\tilde{R}[X; \mu, \nu^*(\mu)]) = 0,$$

which in turn imply $VaR_p(\tilde{I}[X; \mu, \nu^*(\mu)]) - E\tilde{I}[X; \mu, \nu^*(\mu)] = 0$. Thus,

$$h_2(\mu, \nu^*(\mu)) = ((a_2 - a_1)(1+\rho) + \rho - c)\mu + a_1P + (1+c)EX,$$

and therefore, its minimum is attained at $\hat{\mu}_3^* = \mu_0^s$ if $(a_2 - a_1)(1+\rho) + \rho - c > 0$, or $\hat{\mu}_3^* = \frac{P}{1+\rho}$, otherwise.

Finally, the minimum between $h_2(\hat{\mu}_2^*, \nu^*(\hat{\mu}_2^*))$ and $h_2(\hat{\mu}_3^*, \nu^*(\hat{\mu}_3^*))$ provides the global optimal solution to (2.3).

- 2c)** If $\frac{P}{1+\rho} \in [\mu_0, EX]$, our analysis is divided into three cases: $\mu \in [0, \mu_0^s]$, $\mu \in [\mu_0^s, \mu_0]$ and $\mu \in [\mu_0, P/(1+\rho)]$.

The first subcase, $\mu \in [0, \mu_0^s]$, is similar to case 2a). The minimum value of $h_2(\mu, \nu^*(\mu))$ is attained at $\hat{\mu}_4^* = \mu_0^s$, if for any $\mu \in [0, \mu_0^s]$ we have $(1+a_2)(1+\rho) - c - \mathfrak{M}'(\mu) \leq 0$. Otherwise, a numerical solution is needed.

Whenever $\mu \in [\mu_0^s, \mu_0]$, similar arguments used to derive $\hat{\mu}_3^*$ show that the minimum value of $h_2(\mu, \nu^*(\mu))$ is attained at $\hat{\mu}_5^* = \mu_0^s$ if $(a_2 - a_1)(1+\rho) + \rho - c > 0$, or $\hat{\mu}_5^* = \mu_0$, otherwise.

Finally, for any $\mu \in [\mu_0, \frac{P}{1+\rho}]$, Proposition 3.2 suggests that the local solution is found via the minimisation of $h_2(\mu, VaR_p(X) + \mu - EX)$. Specifically, we have

$$VaR_p\left(\tilde{I}[X; \mu, VaR_p(X) + \mu - EX]\right) - E\tilde{I}[X; \mu, VaR_p(X) + \mu - EX] = 0,$$

and hence

$$h_2(\mu, VaR_p(X) + \mu - EX) = a_1P + (1+c)EX + ((1-a_1)\rho - a_1 - c)\mu + g(a_2(1+\rho)\mu, \vartheta\mathcal{H}(\mu))$$

where

$$\mathcal{H}(\mu) := \frac{1}{1-p} \left(\mu - E\left(X \wedge (VaR_p(X) + \mu - EX)\right) \right).$$

If $(1-a_1)\rho - a_1 \geq c$ (which is expected to hold under a realistic parametrisation), then $\hat{\mu}_6^* = \mu_0$. Otherwise, a numerical solution is employed.

We now give a numerical illustration of the above derivations, and in fact we solve optimisation problems (2.1) and (2.3) under a realistic parametrisation as suggested in QIS5. Alternatively, a short description is provided in Asimit *et al.* (2015). We also compare our results with the optimal contract given by Proposition 4.2 of Chi and Tan (2013), where the RM's have not been included in the objective function.

Example 4.1. *In this example, we consider two different distributions for X . The first distribution is assumed to be Pareto with shape parameter $\alpha = 3$ and scale parameter $\theta = 1,000$, and therefore, $\bar{F}(x) = (1,000/(x + 1,000))^3$. The second one is a Log-Normal distribution with location parameter $\mu_{LN} = 5.786$ and scale parameter $\sigma_{LN} = 0.926$. The parameter values are chosen such that the two distributions have the same value for EX and $VaR_{0.8}(X)$ respectively. The choice of a confidence level of 80% is made, since anecdotically, this level represents a rough estimate of the minimal capital that an insurer needs to hold to keep its business running. We also choose $\lambda = 6\%/1.04$, $CoC = 6\%$, $\sigma_{PR} = 0.1$ and $\sigma_{RR} = 0.11$ as suggested in QIS5 for Fire and Other Property Damage, while $d = 1.56$, $n = 1$ and $\Delta_n = 3\%$ are suggested in Asimit *et al.* (2015). We set $RecR = 50\%$, which is the benchmark value used in Solvency II, and $q = 6.04\%$ that represents the one-year default probability of a B-rating reinsurer calibrated in QIS5. Finally, it is also assumed that $\rho = 0.5$ and $l = 3$.*

Assuming that the reinsurer adopts an expected value premium principle as defined in (4.1), the feasible region of μ is upper-bounded by $\frac{P}{1+\rho}$ as the premium charged by the reinsurer should not exceed the gross premium received by the insurer. Thus, we now turn to the discussion of choosing a proper value for P . As analysed in the derivation of closed-form solutions to the optimisation problems (2.1) and (2.3), the feasible region of P , i.e. $[0, (1+\rho)EX]$, is further divided into two sub-intervals, (1a) $[0, (1+\rho)E(X \wedge VaR_p(X))]$ and (1b) $[(1+\rho)E(X \wedge VaR_p(X)), (1+\rho)EX]$, for problem (2.1), while in problem (2.3) three sub-intervals of feasible value of P are considered, (2a) $[0, (1+\rho)\mu_0^s]$, (2b) $[(1+\rho)\mu_0^s, (1+\rho)\mu_0]$ and (2c) $[(1+\rho)\mu_0, (1+\rho)EX]$. Thus, our numerical illustration is divided according to these five intervals of P . Although the numerical boundaries of these five intervals are different under the two distributions of X considered in this example, we try our best to employ the same value of P for both distributions when each interval is considered, and therefore our optimal solutions may be compared across the two distributions. In particular, Table 4.1 summarises the numerical intervals of P

under Pareto and Log-Normal distributions respectively, together with the chosen value of P from each interval.

Model	Interval of P	Pareto	Log-Normal	P
(2.1)	(1a)	[0, 728.07]	[0, 739.51]	700
	(1b)	[728.07, 750]	[739.51, 750]	740
(2.3)	(2a)	[0, 591.76]	[0, 555.46]	555
	(2b)	[591.76, 727.96]	[555.46, 739.46]	700
	(2c)	[727.96, 750]	[739.46, 750]	740

TABLE 4.1. Parametrisation of P .

For each of the five cases described above, Figure 4.1 contains the plots of the optimal ceded risk when X is Pareto and Log-Normal distributed. We find that, under the same optimisation model and parametrisation, the insurer always retains more risk by itself when X is Log-Normal distributed as compared to the Pareto case. This is not surprising, since Pareto has a heavier tail than Log-Normal.

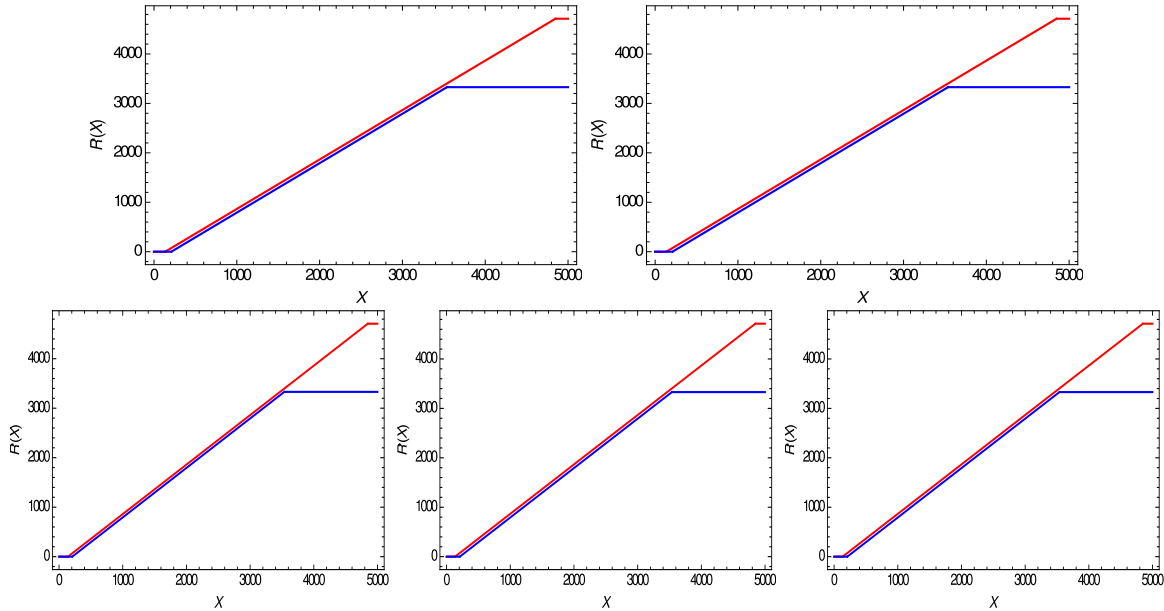


FIGURE 4.1. Closed-form solution under expected value principle with a Pareto distribution (red line) and a Log-Normal distribution (blue line), where the top row displays (1a) and (1b) and the bottom row shows (2a), (2b) and (2c).

Table 4.2 provides the solutions across the two optimisation models (2.1) and (2.3). In all of the ten examples investigated, the global optimal reinsurance contracts have the same two-layer form

$$R^*[X] = \mathcal{L}_{(VaR_p(X) - \nu^*, VaR_p(X))}(X), \quad (4.4)$$

where the values of ν^* are summarised in Table 4.2 together with the corresponding expected value of the reinsurer's risk μ^* . Also, under our parametrisation, $VaR_p(X) = 4,848.04$ in the Pareto distribution

case, while $VaR_p(X) = 3,537.65$ in the Log-Normal distribution setting. The conclusion drawn from Figure 4.1, that the insurer always retains more risk under Log-Normal distribution assumption than Pareto, is affirmed by studying Table 4.2. If we compare the values of ν^* and μ^* across optimisation models, we find that there is only negligible differences between the optimal contracts solved under the two methodologies where the RM is calculated either with or without applying the Log-Normal approximation. Thus, we can conclude that the optimal reinsurance contract is quite robust with respect to the two methodologies of the RM evaluation within Solvency II.

Model	P	Distribution	ν^*	μ^*
(2.1)	700	Pareto	4,712.81	373.36
		Log-Normal	3,328.37	309.84
	740	Pareto	4,712.84	373.38
		Log-Normal	3,328.38	309.85
(2.3)	555	Pareto	4,707.87	370
		Log-Normal	3,330.21	311.10
	700	Pareto	4,713.97	374.15
		Log-Normal	3,329.68	310.74
	740	Pareto	4,713.83	374.06
		Log-Normal	3,329.58	310.67

TABLE 4.2. Numerical illustration of the closed-form solution under the expected value premium principle.

Finally, it would be interesting to compare our results with the optimal contract when the objective function does not take into account the change in the RM's, which can be found in Proposition 4.2 in Chi and Tan (2013), and it is

$$R^*[X] = \mathcal{L}_{(VaR_{\beta^*}(X), VaR_p(X))}(X), \quad \text{with } \beta^* = 1/3.$$

It is found that $VaR_{\beta^*}(X) = 144.71$ in the Pareto distribution setting, while $VaR_{\beta^*}(X) = 218.58$ in the Log-Normal distribution case. Thus, the insurer always retains less risk if the RM's are included in the optimisation problem. In other words, our optimal contracts are more conservative than the one from Chi and Tan (2013).

4.2. Wang's premium principle. The expected value premium principle allows us to provide closed-form solutions to our optimisation problems. Next, we numerically find the optimal contracts for a large family of premium principles, namely Wang's premium principle, for which closed-form solutions usually seem difficult to be found. There is a rich literature on insurance pricing that recognises the advantages of this pricing method (see for example, Wang, 2000, Wang and Young, 1998, Wang *et al.*, 1997, Yaari, 1987). Under this premium principle, we have

$$\pi(X) := (1 + \rho) \int_0^\infty w(\bar{F}(x)) dx, \quad (4.5)$$

where the distortion function $w : [0, 1] \mapsto [0, 1]$ is a non-decreasing and concave function such that $w(0) = 0$ and $w(1) = 1$. The safety loading coefficient, ρ , is employed as in the expected value premium principle. Thus, (4.5) also includes the expected value principle as defined in (4.1), and therefore, allows us to conveniently compare our numerical results with the closed-form solutions. Jones and Zitikis (2003) investigates the empirical estimator for (4.5), which is:

$$\hat{\pi}(X) = (1 + \rho) \sum_{i=1}^n \phi_i x_{i,n},$$

where $\phi_i = w\left(1 - \frac{i-1}{n}\right) - w\left(1 - \frac{i}{n}\right)$ and $x_{1,n} \leq x_{2,n} \leq \dots \leq x_{n,n}$ are the sample ordered statistics.

Recall that we deal with the following family of admissible ceded loss functions:

$$R[X; d_1, d_2, \mu, \nu] := \mathcal{L}_{(0, d_1]}(X) + \mathcal{L}_{(VaR_p(X) + d_1 - \nu, d_2]}(X)$$

s.t. $ER[X; d_1, d_2, \mu, \nu] = \mu, 0 \leq d_1 \leq \nu \leq VaR_p(X) \leq d_2 \leq x_F,$ (4.6)

$$\int_{VaR_p(X) - \nu}^{VaR_p(X)} \bar{F}(x) dx \leq \mu \leq EX - \int_{\nu}^{VaR_p(X)} \bar{F}(x) dx, \quad (4.7)$$

which is discussed in the proof of Theorem 3.1. We further assume that the loss distribution is discrete in order to use the sample estimates for the reinsurance premium. This is not a restrictive assumption from the practical point of view, since this sample represents the claim history. If there is no reliable claim history but a proxy model is socially accepted, then one may draw a sample from this model. That is, we observe a sample of size n , which without loss of generality is assumed to be increasingly ordered, e.g. $x_1 \leq x_2 \leq \dots \leq x_n$. Therefore, the reinsurance premium under Wang's premium principle becomes

$$\pi(R[X; d_1, d_2, \mu, \nu]) = (1 + \rho) \sum_{i=1}^n \phi_i \times (x_i \wedge d_1 + (x_i - x_{[np]} - d_1 + \nu)_+ - (x_i - d_2)_+), \quad (4.8)$$

where $x_{[np]}$ is the sample estimation of $VaR_p(X)$. The feasibility conditions for the optimisation problem from (2.1) are

$$\pi(R[X; d_1, d_2, \mu, \nu]) \leq P, \quad (4.9)$$

while for (2.3) they are

$$\pi(R[X; d_1, d_2, \mu, \nu]) \leq P, \nu \leq VaR_p(X) + \mu - EX. \quad (4.10)$$

Note that $\int_{VaR_p(X) - \nu}^{VaR_p(X)} \bar{F}(x) dx$ and $\int_{\nu}^{VaR_p(X)} \bar{F}(x) dx$ can be replaced by

$$E(X - VaR_p(X) + \nu)_+ - E(X - VaR_p(X))_+ = \frac{1}{n} \sum_{i=1}^n ((x_i - x_{[np]} + \nu)_+ - (x_i - x_{[np]})_+),$$

and

$$E(X - \nu)_+ - E(X - VaR_p(X))_+ = \frac{1}{n} \sum_{i=1}^n ((x_i - \nu)_+ - (x_i - x_{[np]})_+),$$

respectively. Therefore, we deal with two non-convex optimisation problems that need to be transformed into implementable formulations in some commercial optimisation software. The next two propositions show how the optimisation problems (2.1) and (2.3) can be reformulated as Mixed Integer Nonlinear Programming (*MINLP*) with linear constraints that can be efficiently solved. The idea behind implementing (2.1) and (2.3) is the same, and thus, we only show Proposition 4.2.

Proposition 4.2. Denote $\mathcal{A} = \{0, 1\}$, and let M be such that $M \geq 2x_n$. Solving

$$\min_{\mu, \nu, d_1, d_2} H_1(R[X; d_1, d_2, \mu, \nu]) \quad \text{subject to (4.6), (4.7) and (4.9)}$$

is equivalent to

$$\begin{aligned} & \min_{\substack{(d_1, d_2, \mu, \nu, \mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\ (\mathbf{s}, \mathbf{t}, \boldsymbol{\eta}, \boldsymbol{\gamma}, \boldsymbol{\xi}, \boldsymbol{\psi}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A}^n \times \mathcal{A}^n \times \mathcal{A}^n \times \mathcal{A}^n \times \mathcal{A}^n}} \left\{ -\nu - c\mu + (1 + \rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}) \right. \\ & \left. + g(a_2(1 + \rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}), b_2\mu) + g(a_1P - a_1(1 + \rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}), b_1\bar{x} - b_1\mu) \right\} \\ & \text{s.t. } 0 \leq d_1 \leq \nu \leq x_{[np]} \leq d_2 \leq x_n \\ & \quad -\mathbf{x} + \mathbf{u} \leq \mathbf{0}, -d_1\mathbf{1} + \mathbf{u} \leq \mathbf{0}, \\ & \quad \mathbf{x} - \mathbf{u} - M\boldsymbol{\eta} \leq \mathbf{0}, -M\mathbf{1} + d_1\mathbf{1} - \mathbf{u} + M\boldsymbol{\eta} \leq \mathbf{0}, \\ & \quad \mathbf{x} - (x_{[np]} + d_1 - \nu)\mathbf{1} - \mathbf{v} \leq \mathbf{0}, -\mathbf{v} \leq \mathbf{0}, \\ & \quad -\mathbf{x} + (x_{[np]} + d_1 - \nu)\mathbf{1} + \mathbf{v} - M\boldsymbol{\theta} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{v} + M\boldsymbol{\theta} \leq \mathbf{0}, \\ & \quad \mathbf{x} - d_2\mathbf{1} - \mathbf{w} \leq \mathbf{0}, -\mathbf{w} \leq \mathbf{0}, \\ & \quad -\mathbf{x} + d_2\mathbf{1} + \mathbf{w} - M\boldsymbol{\gamma} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{w} + M\boldsymbol{\gamma} \leq \mathbf{0}, \\ & \quad \mathbf{x} - (x_{[np]} - \nu)\mathbf{1} - \mathbf{s} \leq \mathbf{0}, -\mathbf{s} \leq \mathbf{0}, \\ & \quad -\mathbf{x} + (x_{[np]} - \nu)\mathbf{1} + \mathbf{s} - M\boldsymbol{\xi} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{s} + M\boldsymbol{\xi} \leq \mathbf{0}, \\ & \quad \mathbf{x} - \nu\mathbf{1} - \mathbf{t} \leq \mathbf{0}, -\mathbf{t} \leq \mathbf{0}, \\ & \quad -\mathbf{x} + \nu\mathbf{1} + \mathbf{t} - M\boldsymbol{\psi} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{t} + M\boldsymbol{\psi} \leq \mathbf{0}, \\ & \quad \mathbf{1}^T \mathbf{s} - nK - n\mu \leq 0, \\ & \quad n\mu - n\bar{x} + \mathbf{1}^T \mathbf{t} - nK \leq 0, \\ & \quad -P + (1 + \rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}) \leq 0, \\ & \quad n\mu - \mathbf{1}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}) = 0, \end{aligned}$$

where \bar{x} represents the sample mean and $K = \frac{1}{n} \sum_{i=1}^n (x_i - x_{[np]})_+$.

Proof. The relevant terms of the objective function are

$$\begin{aligned} & -\nu - \frac{c}{n} \sum_{i=1}^n \left(x_i \wedge d_1 + (x_i - (x_{[np]} + d_1 - \nu))_+ - (x_i - d_2)_+ \right) \\ & + (1 + \rho) \sum_{i=1}^n \phi_i \left(x_i \wedge d_1 + (x_i - (x_{[np]} + d_1 - \nu))_+ - (x_i - d_2)_+ \right) \\ & + g \left(a_1P - a_1(1 + \rho) \sum_{i=1}^n \phi_i \left(x_i \wedge d_1 + (x_i - (x_{[np]} + d_1 - \nu))_+ - (x_i - d_2)_+ \right), b_1\bar{x} - b_1\mu \right) \\ & + g \left(a_2(1 + \rho) \sum_{i=1}^n \phi_i \left(x_i \wedge d_1 + (x_i - (x_{[np]} + d_1 - \nu))_+ - (x_i - d_2)_+ \right), b_2\mu \right), \end{aligned} \tag{4.11}$$

by taking into account (4.8) and the fact that $x_1 \leq x_2 \leq \dots \leq x_n$. Now, note that

$$x \wedge y = z \iff z \leq x, z \leq y, x - M\eta \leq z, y - M(1 - \eta) \leq z$$

and

$$\max\{x, y\} = z \iff z \geq x, z \geq y, x + M\eta \geq z, y + M(1 - \eta) \geq z,$$

where M is a large number and η is a binary variable, i.e. $\eta \in \mathcal{A}$. By multiple use of the latter in (4.11) one may conclude the first reformulation, where any $M \geq 2x_n$ satisfies all required inequality constraints. This completes the proof. \square

The next proposition gives the reformulation of optimisation problem (2.3) and is based on the objective function without Log-Normal approximation for the RM's. Before stating the next proposition, we note that (2.2) and (4.6) yield that

$$\begin{aligned} CVaR_p(R[X; \mu, \nu, d_1, d_2]) - VaR_p(R[X; \mu, \nu, d_1, d_2]) &= \frac{1}{1-p} \int_{VaR_p(X)}^{d_2} \bar{F}(x) dx \\ &= \frac{1}{1-p} \left(E(X - VaR_p(X))_+ - E(X - d_2)_+ \right) \\ &= \frac{1}{(1-p)n} \sum_{i=1}^n ((x_i - x_{[np]})_+ - (x_i - d_2)_+). \end{aligned}$$

Proposition 4.3. *Solving*

$$\min_{\mu, \nu, d_1, d_2} H_2(R[X; d_1, d_2, \mu, \nu]) \quad \text{subject to (4.6), (4.7) and (4.10)}$$

is equivalent to

$$\begin{aligned}
& \min_{\substack{(d_1, d_2, \mu, \nu, \mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\ (\mathbf{s}, \mathbf{t}, \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\gamma}, \boldsymbol{\xi}, \boldsymbol{\psi}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A}^n \times \mathcal{A}^n \times \mathcal{A}^n \times \mathcal{A}^n \times \mathcal{A}^n}} \left\{ -\nu - c\mu + (1+\rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}) \right. \\
& + g(a_2(1+\rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}), \frac{\vartheta}{(1-p)n}(nK - \mathbf{1}^T \mathbf{w})) \\
& \left. + g(a_1 P - a_1(1+\rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}), \lambda(x_{\lceil np \rceil} - \nu) - \lambda(\bar{x} - \mu)) \right\} \\
& \text{s.t. } 0 \leq d_1 \leq \nu \leq x_{\lceil np \rceil} \leq d_2 \leq x_n \\
& \quad -\mathbf{x} + \mathbf{u} \leq \mathbf{0}, -d_1 \mathbf{1} + \mathbf{u} \leq \mathbf{0}, \\
& \quad \mathbf{x} - \mathbf{u} - M\boldsymbol{\eta} \leq \mathbf{0}, -M\mathbf{1} + d_1 \mathbf{1} - \mathbf{u} + M\boldsymbol{\eta} \leq \mathbf{0}, \\
& \quad \mathbf{x} - (x_{\lceil np \rceil} + d_1 - \nu)\mathbf{1} - \mathbf{v} \leq \mathbf{0}, -\mathbf{v} \leq \mathbf{0}, \\
& \quad -\mathbf{x} + (x_{\lceil np \rceil} + d_1 - \nu)\mathbf{1} + \mathbf{v} - M\boldsymbol{\theta} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{v} + M\boldsymbol{\theta} \leq \mathbf{0}, \\
& \quad \mathbf{x} - d_2 \mathbf{1} - \mathbf{w} \leq \mathbf{0}, -\mathbf{w} \leq \mathbf{0}, \\
& \quad -\mathbf{x} + d_2 \mathbf{1} + \mathbf{w} - M\boldsymbol{\gamma} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{w} + M\boldsymbol{\gamma} \leq \mathbf{0}, \\
& \quad \mathbf{x} - (x_{\lceil np \rceil} - \nu)\mathbf{1} - \mathbf{s} \leq \mathbf{0}, -\mathbf{s} \leq \mathbf{0}, \\
& \quad -\mathbf{x} + (x_{\lceil np \rceil} - \nu)\mathbf{1} + \mathbf{s} - M\boldsymbol{\xi} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{s} + M\boldsymbol{\xi} \leq \mathbf{0}, \\
& \quad \mathbf{x} - \nu \mathbf{1} - \mathbf{t} \leq \mathbf{0}, -\mathbf{t} \leq \mathbf{0}, \\
& \quad -\mathbf{x} + \nu \mathbf{1} + \mathbf{t} - M\boldsymbol{\psi} \leq \mathbf{0}, -M\mathbf{1} + \mathbf{t} + M\boldsymbol{\psi} \leq \mathbf{0}, \\
& \quad \mathbf{1}^T \mathbf{s} - nK - n\mu \leq 0, \\
& \quad n\mu - n\bar{x} + \mathbf{1}^T \mathbf{t} - nK \leq 0, \\
& \quad \nu - \mu + \bar{x} - x_{\lceil np \rceil} \leq 0, \\
& \quad -P + (1+\rho)\boldsymbol{\phi}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}) \leq 0, \\
& \quad n\mu - \mathbf{1}^T(\mathbf{u} + \mathbf{v} - \mathbf{w}) = 0.
\end{aligned}$$

With the help of Propositions 4.2 and 4.3, numerical solutions can be obtained for a large class of premium principles. In fact, we produce a discrete approximation of the objective functions that can be solved in any commercial optimisation software such as MATLAB, if the underlying distribution of X is known. In particular, we illustrate further results for the expected value principle in Example 4.2, while results for the Wang's premium principle are presented in Example 4.3

Example 4.2. *In this example, we provide numerical approximations of optimal solutions under the expected value premium principle. One set of 100 samples of size 1,000 is simulated from each of the two parametric models used in Example 4.1, so that the numerical results given in the two examples can be compared. As it has been seen from Example 4.1, optimal contracts found in all of the ten examples have the same two-layer form regardless of the value of P chosen. Thus, we only provide examples for cases (1b) and (2c) with $P = 740$, since these cases provide a large set of feasible solutions.*

Loss Distribution	Model (2.1)	Model (2.3)
Pareto	4,597.5	4,596.9
Log-Normal	3,461.6	3,462.3

TABLE 4.3. Empirical solutions of ν^* to optimisation models (2.1) and (2.3) under expected value principle with $P = 740$.

Numerous examples have been employed and showed that the estimated value of d_2^* has negligible deviation from its closed-form solution value $\text{VaR}_p(X)$. However, variations exist in the estimated value of d_1^* , which is due to the lack of efficiency in using the MATLAB algorithm to solve MINLP. Nevertheless, we have seen cost-inefficiency in transferring very small liabilities when studying Example 4.1, and therefore, it is expected that the bottom-layer of the optimal contract to be strictly positive, i.e. $d_1^* = 0$. Thus, in order to speed up the computation process, we can explicitly set $d_1^* = 0$ and $d_2^* = x_{\lceil np \rceil}$ and only compute ν^* . Clearly, the optimal reinsurance contract is as given in equation (4.4), and the mean value of estimated ν^* is summarised in Table 4.3. The results are satisfactorily close to its corresponding closed-form value, since the relative errors are less than 2.5% in the Pareto distribution case and are less than 4% in the Log-Normal setting. Thus, it can be concluded that the numerical procedure is stable.

Example 4.3. The final example provides numerical illustrations to optimal reinsurance contracts under Wang's premium principle. In particular, we choose the Proportional Hazards Transform (PHT) principle, where $w(t) = t^{0.95}$. In addition, $\rho = 0.38$ is set, while all other parameters remain the same as in Example 4.1. The value of ρ is chosen in such manner that, under the PHT principle, the optimal contracts found in Example 4.1 are as expensive as priced under the expected value principle. Instead of simulating new samples, we use the same simulated data as in Example 4.2. Again, we obtain robust estimates for $d_2^* = \text{VaR}_p(X)$, while variation exists in the estimates of d_1^* . However, by a similar argument given in Example 4.2, it is unnatural for the insurer to cede very low layers from the total loss. Thus, it can be explicitly set $d_1^* = 0$ and $d_2^* = \text{VaR}_p(X)$. The optimal reinsurance contract has the form as in

Loss Distribution	Model (2.1)	Model (2.3)
Pareto	4,598.3	4,598.3
Log-Normal	3,462.4	3,460.9

TABLE 4.4. Empirical solutions of ν^* to optimisation models (2.1) and (2.3) under PHT premium principle with $P = 740$.

equation (4.4), and a summary of the estimates of ν^* based on our 100 samples are given in Table 4.4. It can be seen that the optimal contracts found under PHT principle are very close to those found under the expected value principle, which can be explained by our choice for ρ .

ACKNOWLEDGMENTS

Chi's research was supported by a grant from the National Natural Science Foundation of China (No. 11471345) and a MOE (China) Project of Key Research Institute of Humanities and Social Sciences

at Universities (13JJD790040). Junlei Hu would like to thank Cass Business School for funding her research.

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