ANCHORS OF IRREDUCIBLE CHARACTERS

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Abstract. Given a prime number p, every irreducible character χ of a finite group G determines a unique conjugacy class of p-subgroups of G which we will call the *anchors* of χ. This invariant has been considered by Barker in the context of finite p-solvable groups. Besides proving some basic properties of these anchors, we investigate the relation to other p-groups which can be attached to irreducible characters, such as defect groups, vertices in the sense of J. A. Green and vertices in the sense of G. Navarro.

Dedicated to the memory of J. A. Green

November 2, 2015

1. Introduction

Let p be a prime number and $\mathcal{O}$ a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p and field of fractions K of characteristic 0. For G a finite group, we denote by $\mathrm{Irr}(G)$ the set of characters of the simple KG-modules. For $\chi \in \mathrm{Irr}(G)$, we denote by e_χ the unique primitive idempotent in $Z(KG)$ satisfying $\chi(e_\chi) \neq 0$. The $\mathcal{O}$-order $\mathcal{O}Ge_\chi$ in the simple K-algebra KGe_χ is a G-interior $\mathcal{O}$-algebra, via the group homomorphism $G \to (\mathcal{O}Ge_\chi)^\times$ sending $g \in G$ to ge_χ. Since $(\mathcal{O}Ge_\chi)^G = Z(\mathcal{O}Ge_\chi)$ is a subring of the field $Z(KGe_\chi)$, it follows that $\mathcal{O}Ge_\chi$ is a primitive G-interior $\mathcal{O}$-algebra. In particular, $\mathcal{O}Ge_\chi$ is a primitive G-algebra. By the fundamental work of J. A. Green [7], it has a defect group. This is used in work of Barker [1] to prove a part of a conjecture of Robinson (cf. [24, 4.1, 5.1]) for blocks of finite p-solvable groups. In order to distinguish this invariant from defect groups of blocks and from vertices of modules, we introduce the following terminology.

Definition 1.1. Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. An *anchor of* χ is a defect group of the primitive G-interior $\mathcal{O}$-algebra $\mathcal{O}Ge_\chi$.

By the definition of defect groups, an anchor of an irreducible character χ of G is a subgroup P of G which is minimal with respect to $e_\chi \in (\mathcal{O}Ge_\chi)^G_P$, where $(\mathcal{O}Ge_\chi)^G_P$ denotes the image of the relative trace map $\mathrm{Tr}^G_P : (\mathcal{O}Ge_\chi)^P \to (\mathcal{O}Ge_\chi)^G$. Green's general theory in [7, §5] implies that the anchors of χ form a conjugacy class of p-subgroups of G.

For the remainder of the paper we make the blanket assumption that K and k are splitting fields for the finite groups arising in the statements below. In a few places, this assumption is not necessary; see the Remark 1.7 below.

The second author gratefully acknowledges support by the DFG (SPP 1388).

Theorem 1.2. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. Let B be the block of $\mathcal{O}G$ containing χ and let L be an $\mathcal{O}G$-lattice affording χ. Let P be an anchor of χ and denote by ΔP the image $\{(x,x)\,|\,x \in P\}$ of P under the diagonal embedding of G in $G \times G$. The following hold.*

(a) *P is contained in a defect group of B.*
(b) *P contains a vertex of L.*
(c) *We have $O_p(G) \le P$.*
(d) *The suborder $\mathcal{O}Pe_\chi$ of $\mathcal{O}Ge_\chi$ is local, and $\mathcal{O}Ge_\chi$ is a separable extension of $\mathcal{O}Pe_\chi$.*
(e) *ΔP is contained in a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$ and $P \times P$ contains a vertex of $\mathcal{O}Ge_\chi$. Moreover, ΔP is a vertex of $\mathcal{O}Ge_\chi$ if and only if χ is of defect zero.*

For G a finite group, we denote by $\mathrm{IBr}(G)$ the set of $\mathcal{O}$-valued Brauer characters of the simple kG-modules. We denote by G° the set of p'-elements in G, and for χ a K-valued class function on G, we denote by χ° the restriction of χ to G°.

Theorem 1.3. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let B be the block of $\mathcal{O}G$ containing χ and let L be an $\mathcal{O}G$-lattice affording χ. Let P be an anchor of χ. The following hold.*

(a) *If $\chi^\circ \in \mathrm{IBr}(G)$, then L is unique up to isomorphism, P is a vertex of L, and $P \times P$ is a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$.*
(b) *Let τ be a local point of P on $\mathcal{O}Ge_\chi$. Then the multiplicity module of τ is simple. In particular, $O_p(N_G(P_\tau)) = P$ and P is centric in a fusion system of B.*
(c) *If B has an abelian defect group D, then D is an anchor of χ.*
(d) *If χ has height zero, then P is a defect group of B, and $P \times P$ is a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$.*

The hypothesis $\chi^\circ \in \mathrm{IBr}(G)$ in the first statement of Theorem 1.3 holds if χ is a height zero character of a nilpotent block. If G is p-solvable, then by the Fong-Swan theorem [4, §22], for any $\varphi \in \mathrm{IBr}(G)$ there is $\chi \in \mathrm{Irr}(G)$ such that $\chi^\circ = \varphi$. The fact that anchors are centric is essentially proved in the proof of [1, Theorem] as an immediate consequence of results of Knörr [14], Picaronny-Puig [20], and Thévenaz [26]; see the proof of 3.7 below for details.

The next result shows that anchors are invariant under Morita equivalences given by a bimodule with endopermutation source, hence in particular under source algebra equivalences.

Theorem 1.4. *Let G, G' be finite groups. Let B, B' be blocks of $\mathcal{O}G$, $\mathcal{O}G'$, with defect groups D, D', respectively. Suppose that B and B' are Morita equivalent via a B-B'-bimodule M which has an endopermutation source, when viewed as an $\mathcal{O}(G \times G')$-module. Let $\chi \in \mathrm{Irr}(B)$ and $\chi' \in \mathrm{Irr}(B')$ such that χ and χ' correspond to each other under the Morita equivalence determined by M. Then there is an isomorphism $D \cong D'$ sending an anchor of χ to an anchor of χ'. In particular, if B and B' are source algebra equivalent, then χ and χ' have isomorphic anchors.*

The last statement in Theorem 1.4 can be made more precise: if B, B' are source algebra equivalent, then the isomorphism $D \cong D'$ can be chosen to have an extension to a source algebra isomorphism; see Theorem 4.1 below.

In [17] Navarro associated, via the theory of special characters, to each ordinary irreducible character χ of a p-solvable group G, a G-conjugacy class of pairs (Q,δ), where Q is a p-subgroup of G and δ is an ordinary irreducible character of Q, which behave in certain ways as the Green vertices of indecomposable modules (see also [5],[2] and [3]). We call such a pair (Q,χ) a *Navarro vertex* of χ. We prove the following two results relating Navarro vertices and anchors (see Section 5 for the definitions).

Theorem 1.5. *Let G be a finite p-solvable group and let $\chi\in\mathrm{Irr}(G)$ such that $\chi^\circ\in\mathrm{IBr}(G)$. Let (Q,δ) be a Navarro vertex of χ. Then Q contains an anchor of χ. Further, if p is odd or δ is the trivial character, then Q is an anchor of χ.*

Theorem 1.6. *Let G be a finite group of odd order, and let $\chi\in\mathrm{Irr}(G)$ have Navarro vertex (Q,δ). Then Q is contained in an anchor of G.*

We give examples which show that equality does not always hold in the above theorem. We also give examples which show that if $|G|$ is even, then a Navarro vertex need not be contained in any anchor of χ.

Section 2 of the paper contains some basic properties of quotient orders of finite group algebras. In Section 3, we prove the theorems 1.2 and 1.3. In section 4 we characterise anchors at the source algebra level, and use this to prove Theorem 1.4. In Section 5 we prove some properties of anchors with respect to normal subgroups. Section 6 contains the proofs of Theorems 1.5 and 1.6. In Section 7 we compare anchors of $\mathcal{O}Ge_\chi$ to the defect groups of $k\otimes_{\mathcal{O}}\mathcal{O}Ge_\chi$.

Remark 1.7. The splitting field hypothesis on K and k is not needed in Theorem 1.2 and the Propositions 3.1, 3.2, 3.3, and 3.5, on which the proof of Theorem 1.2 is based. This hypothesis is also not needed in Theorem 4.1, stating that anchors can be read off the source algebras of a block.

Remark 1.8. Let G be a finite group, $\chi\in\mathrm{Irr}(G)$, and b the block idempotent of the block of $\mathcal{O}G$ to which χ belongs; that is, b is the primitive idempotent in $Z(\mathcal{O}G)$ satisfying $be_\chi=e_\chi$. Let P be an anchor of χ; that is, P is a minimal subgroup of G such that there exists an element $c\in(\mathcal{O}Ge_\chi)^P$ satisfying $e_\chi=\mathrm{Tr}_P^G(c)$. We clearly have $(\mathcal{O}G)^Pe_\chi\subseteq(\mathcal{O}Ge_\chi)^P$, but this inclusion need not be an equality, and this is one of the main issues for calculating anchors. The spaces $(\mathcal{O}G)^Pe_\chi$ and $(\mathcal{O}Ge_\chi)^P$ have the same $\mathcal{O}$-rank, since $(KG)^Pe_\chi=(KGe_\chi)^P$. An equality $(\mathcal{O}G)^Pe_\chi=(\mathcal{O}Ge_\chi)^P$ implies that P is a defect group of the block of $\mathcal{O}G$ to which χ belongs; see Proposition 3.9 below.

2. Orders of characters

Let G be a finite group and $\chi\in\mathrm{Irr}(G)$. Then $\mathcal{O}Ge_\chi$ is an $\mathcal{O}$-order in the simple K-algebra KGe_χ, called the $\mathcal{O}$*-order* of χ. In general, $\mathcal{O}Ge_\chi$ is not a subalgebra of $\mathcal{O}G$, but it is an $\mathcal{O}$-free quotient of $\mathcal{O}G$. The map $f:\mathcal{O}G\to\mathcal{O}Ge_\chi$ sending $x\in\mathcal{O}G$ to xe_χ is an epimorphism of $\mathcal{O}$-orders; in particular, the $\mathcal{O}$-orders $\mathcal{O}Ge_\chi$ and $\mathcal{O}G/\mathrm{Ker}(f)$ are isomorphic. Since we assume that K and k are splitting fields for all finite groups arising in this paper, we have $e_\chi=\frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g$, and as a K-algebra, KGe_χ is isomorphic to the matrix algebra $K^{\chi(1)\times\chi(1)}$. In particular, KGe_χ has up to isomorphism a unique simple left module M_χ, and we have $\dim_K(M_\chi)=\chi(1)$. Since the $\mathcal{O}$-rank of $\mathcal{O}Ge_\chi$ is $\chi(1)^2$, it follows that $k\otimes_{\mathcal{O}}\mathcal{O}Ge_\chi$ is a k-algebra of dimension $\chi(1)^2$ whose isomorphism classes of simple modules are in bijection with the set

$$\mathrm{IBr}(G|\chi) = \{\varphi \in \mathrm{IBr}(G) : d_{\chi,\varphi} \neq 0\}$$

where $d_{\chi,\varphi}$ denotes the decomposition number attached to χ and φ, defined by the equation $\chi^\circ = \sum_{\varphi\in\mathrm{IBr}(G)} d_{\chi,\varphi}\varphi$. It is well-known that $d_{\chi,\varphi} = \chi(i)$, where i is a primitive idempotent in $\mathcal{O}G$ such that $\mathcal{O}Gi$ is a projective cover of a simple module with Brauer character φ. In particular, $d_{\chi,\varphi} \neq 0$ if and only if $ie_\chi \neq 0$. The simple $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$-module N_φ corresponding to $\varphi \in \mathrm{IBr}(G|\chi)$ has dimension $\varphi(1)$. Thus we have an isomorphism of k-algebras

$$k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi / J(k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi) \cong \prod_{\varphi\in\mathrm{IBr}(G|\chi)} k^{\varphi(1)\times\varphi(1)}.$$

Let $P(N_\varphi)$ denote an $\mathcal{O}Ge_\chi$-lattice which is a projective cover of N_φ. Then the KGe_χ-module $K \otimes_{\mathcal{O}} P(N_\varphi)$ is isomorphic to $M^{d_{\chi,\varphi}}$; in particular, we have $\mathrm{rk}_{\mathcal{O}}(P(N_\varphi)) = d_{\chi,\varphi}\chi(1)$. Setting $\ell_\chi = |\mathrm{IBr}(G|\chi)|$, the decomposition matrix of the $\mathcal{O}$-order $\mathcal{O}Ge_\chi$ is the $1 \times \ell_\chi$-matrix

$$\Delta_\chi = (d_{\chi,\varphi} : \varphi \in \mathrm{IBr}(G|\chi)) \ .$$

Hence the Cartan matrix of $\mathcal{O}Ge_\chi$ is the $\ell_\chi \times \ell_\chi$-matrix

$$C_\chi = \Delta_\chi^\top \Delta_\chi = (d_{\chi,\varphi}d_{\chi,\psi} : \varphi, \psi \in \mathrm{IBr}(G|\chi)).$$

By definition, C_χ has rank 1. The only non-zero invariant factor of C_χ is

$$\gcd(d_{\chi,\varphi}d_{\chi,\psi} : \varphi, \psi \in \mathrm{IBr}(G|\chi)) = \gcd(d_{\chi,\varphi} : \varphi \in \mathrm{IBr}(G|\chi))^2.$$

If $\chi^\circ \in \mathrm{IBr}(G)$, then it is well-known that the $\mathcal{O}$-order $\mathcal{O}Ge_\chi$ is isomorphic to $\mathcal{O}^{\chi(1)\times\chi(1)}$ (see e. g. [13, Prop. 4.1]), and thus the k-algebra $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ is isomorphic to $k^{\chi(1)\times\chi(1)}$. In this case the decomposition matrix Δ_χ is the 1×1-matrix (1), and so is the Cartan matrix C_χ.

Since $Z(KGe_\chi) = Ke_\chi \cong K$, we have $Z(\mathcal{O}Ge_\chi) = \mathcal{O}e_\chi \cong \mathcal{O}$; in particular, $Z(\mathcal{O}Ge_\chi)$ is a local $\mathcal{O}$-order, and hence $Z(k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi)$ is a local k-algebra, by standard lifting theorems for central idempotents. It is obvious that

$$Z(k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi) \supseteq k \otimes_{\mathcal{O}} Z(\mathcal{O}Ge_\chi) = k \otimes e_\chi.$$

This inclusion can be proper, or equivalently, the canonical map $Z(\mathcal{O}Ge_\chi) \to Z(k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi)$ need not be surjective. The following example illustrates this.

Example 2.1. Let G be the dihedral group of order 8, let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 2$, and let $p = 2$. We represent G in the form $G = \langle a, b\rangle$ where

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathcal{O}Ge_\chi$ is isomorphic to the subalgebra Λ of $\mathcal{O}^{2\times 2}$ generated by a and b. Note that $ba = -ab$, so that $(1 \otimes b)(1 \otimes a) = (1 \otimes a)(1 \otimes b)$ in $k \otimes_{\mathcal{O}} \Lambda$. This shows that $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi \cong k \otimes_{\mathcal{O}} \Lambda$ is commutative and of dimension 4.

3. Proofs of the theorems 1.2 and 1.3

Proposition 3.1. *Let G be a finite group, $\chi \in \mathrm{Irr}(G)$, and let P be an anchor of χ. Let B be the block of $\mathcal{O}G$ containing χ, and let L be an $\mathcal{O}G$-lattice affording χ. The following hold.*

(i) P is contained in a defect group of B.

(ii) P contains a vertex of L.

Proof. Let D be a defect group of B. Then there exists an element $x \in (\mathcal{O}G)^D$ such that $\mathrm{Tr}_D^G(x) = 1_B$. Then $xe_\chi \in (\mathcal{O}Ge_\chi)^D$, and $\mathrm{Tr}_D^G(xe_\chi) = \mathrm{Tr}_D^G(x)e_\chi = 1_B e_\chi = e_\chi$. Thus D contains an anchor of χ, and (i) follows. Let $y \in (\mathcal{O}Ge_\chi)^P$ such that $\mathrm{Tr}_P^G(y) = e_\chi$. Then the map $\eta : L \to L$ sending z to yz, is an element in $\mathrm{End}_{\mathcal{O}P}(L)$ such that $\mathrm{Tr}_P^G(\eta) = \mathrm{id}_L$. By Higman's criterion, P contains a vertex of L, whence the result. □

Proposition 3.2. *Let G be a finite group, $\chi \in \mathrm{Irr}(G)$, and let P be an anchor of χ. Then $O_p(G) \leq P$.*

Proof. Set $N = O_p(G)$. Arguing by contradiction, suppose that P does not contain N. Then P is a proper subgroup of PN. For $g \in N$, we have $g - 1 \in J(\mathcal{O}N) \subseteq J(\mathcal{O}G)$. It follows that $gd - d$, $dg^{-1} - d$, and $gdg^{-1} - d = gdg^{-1} - dg^{-1} + dg^{-1} - d$ are contained in $J(\mathcal{O}Ge_\chi)$ for all $g \in N$ and all $d \in \mathcal{O}Ge_\chi$. Let $d \in (\mathcal{O}Ge_\chi)^P$ such that $\mathrm{Tr}_P^G(d) = e_\chi$. By the above , we have $\mathrm{Tr}_P^{PN}(d) - |PN : P|d \in J(\mathcal{O}Ge_\chi)$. Since p divides $|PN : P|$, it follows that $x = \mathrm{Tr}_P^{PN}(d) \in J(\mathcal{O}Ge_\chi)$. Applying Tr_{PN}^G to this element shows that $e_\chi = \mathrm{Tr}_P^G(d) = \mathrm{Tr}_{PN}^G(x) \in J(\mathcal{O}Ge_\chi)$, a contradiction. □

Let R be an $\mathcal{O}$-order with unitary suborder S. We recall that R is called a *separable extension* of S if the multiplication map $\mu : R \otimes_S R \to R$ sending $x \otimes y$ to xy for all x, $y \in R$ splits as a map of R-R-bimodules. This is equivalent to the condition that $1_R = \mu(z)$ for some $z \in (R \otimes_S R)^R$.

Proposition 3.3. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let P be an anchor of χ. Then the $\mathcal{O}$-order $\mathcal{O}Ge_\chi$ is a separable extension of its local suborder $\mathcal{O}Pe_\chi$.*

Proof. The $\mathcal{O}$-order $\mathcal{O}P$ is local, and the map $\mathcal{O}P \to \mathcal{O}Ge_\chi$ induced by multiplication with e_χ is a homomorphism of $\mathcal{O}$-algebras with image $\mathcal{O}Pe_\chi$. Thus $\mathcal{O}Pe_\chi$ is a local suborder of $\mathcal{O}Ge_\chi$. Let T be a transversal for G/P, and let $d \in (\mathcal{O}Ge_\chi)^P$ such that

$$e_\chi = \mathrm{Tr}_P^G(d) = \sum_{g \in T} gdg^{-1}.$$

Then the element $x = \sum_{g \in T} gde_\chi \otimes g^{-1}e_\chi \in \mathcal{O}Ge_\chi \otimes_{\mathcal{O}Pe_\chi} \mathcal{O}Ge_\chi$ is independent of the choice of T since

$$gude_\chi \otimes u^{-1}g^{-1}e_\chi = gde_\chi ue_\chi \otimes u^{-1}e_\chi g^{-1}e_\chi = gde_\chi \otimes g^{-1}e_\chi$$

for $g \in T$ and $u \in P$. But, for $h \in G$, hT is another transversal for G/P. Thus

$$x = \sum_{g \in T} hgde_\chi \otimes g^{-1}h^{-1}e_\chi = hxh^{-1}.$$

This shows that $x \in (\mathcal{O}Ge_\chi \otimes_{\mathcal{O}Pe_\chi} \mathcal{O}Ge_\chi)^{\mathcal{O}Ge_\chi}$, and

$$\mu(x) = \sum_{g \in T} gde_\chi g^{-1}e_\chi = \mathrm{Tr}_P^G(d)e_\chi^2 = e_\chi^3 = e_\chi$$

where $\mu : \mathcal{O}Ge_\chi \otimes_{\mathcal{O}Pe_\chi} \mathcal{O}Ge_\chi \to \mathcal{O}Ge_\chi$, denotes the multiplication map sending $a \otimes b$ to ab. The result follows. $\square$

Remark 3.4. The proposition above implies that $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ is also a separable extension of $k \otimes_{\mathcal{O}} \mathcal{O}Pe_\chi$. Note that $k \otimes_{\mathcal{O}} \mathcal{O}Pe_\chi$ is a homomorphic image of the group algebra kP. Thus, if P is cyclic, then $k \otimes_{\mathcal{O}} \mathcal{O}Pe_\chi$ has finite representation type.

Proposition 3.5. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let P be an anchor of χ. Then ΔP is contained in a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$ and $P \times P$ contains a vertex of $\mathcal{O}Ge_\chi$. Moreover, ΔP is a vertex of $\mathcal{O}Ge_\chi$ if and only if χ is of defect zero.*

Proof. Viewing $\mathcal{O}Ge_\chi$ as an $\mathcal{O}(G \times G)$-module, we have $\mathcal{O}Ge_\chi(\Delta P) \neq 0$, where $\mathcal{O}Ge_\chi(\Delta P)$ is the Brauer quotient of $\mathcal{O}Ge_\chi$ (cf. [27, §11]). The first assertion follows by [27, Exercise 27.2 (a)]. Next, we claim that $\mathcal{O}Ge_\chi$ is relatively $P \times G$-projective. Indeed, let T be a transversal for G/P, and let $d \in (\mathcal{O}Ge_\chi)^P$ such that

$$e_\chi = \mathrm{Tr}_P^G(d) = \sum_{g \in T} gdg^{-1}.$$

The map

$$\mathcal{O}Ge_\chi \to \mathcal{O}(G \times G) \otimes_{\mathcal{O}(P \times G)} \mathcal{O}Ge_\chi, \quad (x \to \sum_{u \in T} (g,1) \otimes dg^{-1}x), \quad x \in \mathcal{O}Ge_\chi$$

is an $\mathcal{O}(G \times G)$-module splitting of the surjective $\mathcal{O}(G \times G)$-module homomorphism

$$\mathcal{O}(G \times G) \otimes_{\mathcal{O}(P \times G)} \mathcal{O}Ge_\chi, \quad (y \otimes y' \to yy'), \quad y \in \mathcal{O}(G \times G), y' \in \mathcal{O}Ge_\chi$$

proving the claim. Similarly, $\mathcal{O}Ge_\chi$ is relatively $G \times P$-projective. Let R_1 be a vertex of $\mathcal{O}Ge_\chi$ contained in $G \times P$ and let R_2 be a vertex of $\mathcal{O}Ge_\chi$ contained in $P \times G$. Since R_1 and R_2 are $G \times G$-conjugate, it follows that $R_1 \leq {}^xP \times P$ for some $x \in G$ and hence that $\mathcal{O}Ge_\chi$ is relatively $P \times P = {}^{(x^{-1},1)}({}^xP \times P)$-projective. This proves the second assertion.

Finally, if ΔP is a vertex of $\mathcal{O}(G \times G)$, then $\mathrm{Res}^{\mathcal{O}(G \times G)}_{\mathcal{O}(G \times 1)} \mathcal{O}Ge_\chi$ is projective. In particular, the character of $\mathcal{O}Ge_\chi$ as a left $\mathcal{O}G$-module vanishes on the p-singular elements of G. Since the character of $\mathcal{O}Ge_\chi$ is a multiple of χ, it follows that χ is of p-defect zero. Conversely, if χ is of p-defect zero, then by Proposition 3.1 we have $P = 1$, hence $1 = P \times P = \Delta P$ is a vertex of $\mathcal{O}Ge_\chi$. $\square$

Proof of Theorem 1.2. Proposition 3.1 implies (a) and (b) of the theorem. Proposition 3.2 proves (c). Part (d) follows from Proposition 3.3 and Part (e) is proved in Proposition 3.5. $\square$

Proposition 3.6. *Let G be a finite group, and let $\chi \in \mathrm{Irr}(G)$ with anchor P. Suppose that $\chi^\circ \in \mathrm{IBr}(G)$. Let L be an $\mathcal{O}G$-lattice affording χ. Then L is unique up to isomorphism, P is a vertex of L, and $P \times P$ is a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$.*

Proof. The hypotheses imply that the $\mathcal{O}$-orders $\mathcal{O}Ge_\chi$ and $\mathcal{O}^{\chi(1) \times \chi(1)}$ are isomorphic. Since $\mathcal{O}^{\chi(1)}$ is the only indecomposable $\mathcal{O}^{\chi(1) \times \chi(1)}$-lattice, up to isomorphism, L is the only indecomposable $\mathcal{O}Ge_\chi$-lattice, up to isomorphism. Thus the canonical map

$$\mathcal{O}Ge_\chi \longrightarrow \mathrm{End}_{\mathcal{O}}(L)$$

is an isomorphism of G-interior $\mathcal{O}$-algebras, and hence these two primitive G-interior $\mathcal{O}$-algebras have the same defect groups. Higman's criterion implies that P is a vertex of L. Moreover, we have a canonical $\mathcal{O}(G \times G)$-module isomorphism $\mathrm{End}_{\mathcal{O}}(L) \cong L \otimes_{\mathcal{O}} L^*$, where L^* is the $\mathcal{O}$-dual of L. This implies that $P \times P$ is a vertex of $\mathcal{O}Ge_\chi$. □

We observe next that the multiplicity modules of the primitive G-interior $\mathcal{O}$-algebras $\mathcal{O}Ge_\chi$ are simple. Background material on multiplicity modules can be found in [26, §9 Appendix].

Proposition 3.7. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let P_τ be a defect pointed group of the primitive G-interior $\mathcal{O}$-algebra $\mathcal{O}Ge_\chi$. Then P is an anchor of χ and the multiplicity module of τ is simple. In particular, we have $O_p(N_G(P_\tau)) = P$, and P is centric in a fusion system of the block containing χ.*

Proof. The fact that P is an anchor of χ is a standard property of local pointed groups on primitive G-algebras (see e. g. [27, (18.3)]). As noted earlier, we have $(\mathcal{O}Ge_\chi)^G \cong \mathcal{O}$. Set $\bar{N} = N_G(P_\tau)/P)$. It follows from [26, 9.1.(c), 9.3.(b)] that the multiplicity module V_τ of τ is simple. The well-known Lemma 3.8 below implies that $O_p(N_G(P_\tau)) = P$. By results of Knörr in [14], every vertex of a lattice with irreducible character χ is centric in a fusion system of the block containing χ. Centric subgroups in a fusion system are upwardly closed. Since the anchor P of χ contains a vertex of every lattice with character χ, the last statement follows. One can prove this also by applying the results of [20] directly to the G-interior $\mathcal{O}$-algebra $\mathcal{O}Ge_\chi$. □

Lemma 3.8. *Let G be a finite group, A a primitive G-algebra, and let P_τ be a defect pointed group on A. If the multiplicity module of τ is simple, then $O_p(N_G(P_\tau)) = P$.*

Proof. Set $\bar{N} = N_G(P_\tau)/P$. The multiplicity module V_τ of τ is a module over a twisted group algebra $k_\alpha\bar{N}$ for some $\alpha \in H^2(\bar{N}; k^\times)$. Since P_τ is maximal, V_τ is projective (cf. [26, 9.1]). Since V_τ is also simple by the assumptions, it follows that $k_\alpha\bar{N}$ has a block which is a matrix algebra over k. By [27, (10.5)] we have $k_\alpha\bar{N} \cong kN'e$ for some central p'-extension N' of $\bar{N}$ and some idempotent $e \in Z(kN')$. Thus the multiplicity module corresponds to a defect zero block of kN'. Since $O_p(N')$ is contained in all defect groups of all blocks of kN', it follows that $O_p(N')$ is trivial. By elementary group theory, the canonical map $N' \to \bar{N}$ sends $O_p(N')$ onto $O_p(\bar{N})$, and hence $O_p(\bar{N})$ is trivial, or equivalently, $O_p(N_G(P_\tau)) = P$. □

Proof of Theorem 1.3. Part (a) is proved in Proposition 3.6. Part (b) follows from Proposition 3.7, and (c) is an immediate consequence of (b). If χ has height zero then $\chi(1)_p = |G : D|_p$ where D is a defect group of B. Then D contains a vertex Q of L, and $|G : Q|_p$ divides $(\mathrm{rk}_{\mathcal{O}}L)_p = \chi(1)_p = |G : D|_p$. This implies $Q = D$. Hence D is an anchor of χ in this case. A similar argument shows that $D \times D$ is a vertex of the $\mathcal{O}(G \times G)$-module $\mathcal{O}Ge_\chi$. This proves (d). □

Proposition 3.9. *Let G be a finite group, $\chi \in \mathrm{Irr}(G)$, and let P be an anchor of χ. If $(\mathcal{O}G)^P e_\chi = (\mathcal{O}Ge_\chi)^P$, then P is a defect group of the block of $\mathcal{O}G$ to which χ belongs.*

Proof. Denote by b the primitive idempotent in $Z(\mathcal{O}G)$ such that $be_\chi = e_\chi$. Suppose that $(\mathcal{O}G)^P e_\chi = (\mathcal{O}Ge_\chi)^P$. Then $e_\chi = \mathrm{Tr}_P^G(ce_\chi) = \mathrm{Tr}_P^G(c)e_\chi$ for some $c \in (\mathcal{O}Gb)^P$.

Thus $w = \mathrm{Tr}_P^G(c)$ is not contained in $J(Z(\mathcal{O}Gb))$, hence invertible in $Z(\mathcal{O}Gb)$. Therefore $b = w^{-1}w = \mathrm{Tr}_P^G(w^{-1}c)$, which implies that P contains a defect group of b, hence is equal to a defect group of b by Theorem 1.2 (a). □

4. Anchors and source algebras

We show that anchors of characters in a block can be read off the source algebras of that block, and use this to prove Theorem 1.4. As before, we refer to [27, §11] for the Brauer quotient and Brauer homomorphism.

Let G be a finite group, B a block of $\mathcal{O}G$, and D a defect group of B. We denote by $\mathrm{Irr}(B)$ the subset of all $\chi \in \mathrm{Irr}(G)$ satisfying $\chi(1_B) = \chi(1)$. Let i be a source idempotent in B^D; that is, i is a primitive idempotent in B^D satisfying $\mathrm{Br}_D(i) \neq 0$. Then $A = iBi = i\mathcal{O}Gi$ is a *source algebra of* B. We view A as a D-interior $\mathcal{O}$-algebra with the embedding of $D \to A^\times$ induced by multiplication with i. By [21, 3.5], the A-B-bimodule $iB = i\mathcal{O}G$ and the B-A-bimodule $Bi = \mathcal{O}Gi$ induce a Morita equivalence between A and B. In particular, if X is a simple $K \otimes_{\mathcal{O}} B$-module, then iX is a simple $K \otimes_{\mathcal{O}} A$-module, and this correspondence induces a bijection between $\mathrm{Irr}(B)$ and the set of isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules. Equivalently, the map $e_\chi \mapsto ie_\chi$ is a bijection between primitive idempotents in $Z(K \otimes_{\mathcal{O}} B)$ and $Z(K \otimes_{\mathcal{O}} A)$. If U is a B-lattice with character $\chi \in \mathrm{Irr}(B)$, then iU is an A-lattice such that $K \otimes_{\mathcal{O}} iU$ is a simple $K \otimes_{\mathcal{O}} A$-module corresponding to χ. This Morita equivalence induces a bijection between the $\mathcal{O}$-free quotients of B and of A. If $\chi \in \mathrm{Irr}(B)$, then the $\mathcal{O}$-free quotient $\mathcal{O}Ge_\chi = Be_\chi$ corresponds to the $\mathcal{O}$-free quotient $i\mathcal{O}Gie_\chi = Aie_\chi = Ae_\chi$. Note that Ae_χ is again a D-interior $\mathcal{O}$-algebra, via the canonical surjection $A \to Ae_\chi$. Note further that Ae_χ is a direct summand of Be_χ as an $\mathcal{O}D$-$\mathcal{O}D$-bimodule, since it is obtained from multiplying Be_χ on the left and on the right by the idempotent ie_χ in $(Be_\chi)^D$. The next result shows that anchors of χ can be characterised in terms of the order Ae_χ. This is based on a variation of standard arguments, similar to those used in [15, §6], identifying vertices of modules at the source algebra level.

Theorem 4.1. *Let G be a finite group, B a block of $\mathcal{O}G$, D a defect group of B, and $i \in B^D$ a source idempotent. Set $A = iBi$. Let $\chi \in \mathrm{Irr}(B)$.*

(i) Let Q be a p-subgroup of G such that $(Be_\chi)(Q) \neq 0$. Then there is $x \in G$ such that ${}^xQ \leq D$ and such that $(Ae_\chi)({}^xQ) \neq \{0\}$.

(ii) Let Q be a subgroup of D of maximal order subject to $(Ae_\chi)(Q) \neq \{0\}$. Then Q is an anchor of χ.

Proof. We use basic properties of local pointed groups; see e. g. [27, §18] for an expository account of this material. Let γ be the local point of D on B containing i. Let Q be a p-subgroup of G such that $(Be_\chi)(Q) \neq \{0\}$. Since e_χ is the unit element of Be_χ, this is equivalent to $\mathrm{Br}_Q^{Be_\chi}(e_\chi) \neq 0$. By considering a primitive decomposition of 1_B in B^Q it follows that there is a primitive idempotent $j \in B^Q$ such that $\mathrm{Br}_Q^{Be_\chi}(je_\chi) \neq 0$. Then necessarily also $\mathrm{Br}_Q^B(j) \neq 0$, because the canonical map $B \to Be_\chi$ sends $\ker(\mathrm{Br}_Q^B)$ to $\ker(\mathrm{Br}_Q^{Be_\chi})$. Thus j belongs to a local point δ of Q on B. Since the maximal local pointed groups on B are all G-conjugate, it follows that there is $x \in G$ such that ${}^xQ_\delta \leq D_\gamma$. In other words, after replacing Q_δ by a suitable G-conjugate, we may assume that $Q_\delta \leq D_\gamma$, and hence that $j \in A^Q$ for some choice of j in δ. For this choice of j, we have $je_\chi \in Ae_\chi$. Thus the condition

$\mathrm{Br}_Q^{Be_\chi}(je_\chi) \neq 0$ is equivalent to $\mathrm{Br}_Q^{Ae_\chi}(je_\chi) \neq 0$; we use here the fact, mentioned above, that Ae_χ is a direct summand of Be_χ as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule. In particular, we have $(Ae_\chi)(Q) \neq \{0\}$. This proves (i). For (ii), let Q be a subgroup of D such that $(Ae_\chi)(Q) \neq \{0\}$ and such that the order of Q is maximal with respect to this property. Then $(Be_\chi)(Q) \neq \{0\}$, and hence Q is contained in an anchor R of χ. By (i), there is $x \in G$ such that ${}^xR \leq D$ and such that $(Ae_\chi)({}^xR) \neq \{0\}$. The maximality of $|Q|$ forces $Q = R$, whence the result. □

Theorem 4.1 implies that anchors are invariant under source algebra equivalences. By a result of Scott [25] and Puig [23, 7.5.1], an isomorphism between source algebras is equivalent to a Morita equivalence given by a bimodule with a trivial source (see also [16, §4] for an expository account). In order to extend the invariance of anchors to Morita equivalences given by bimodules with endopermutation source, we need to describe these Morita equivalences at the source algebra level. Let G, G' be finite groups, and let B, B' be blocks of $\mathcal{O}G$, $\mathcal{O}G'$ with defect groups D, D', respectively. By results of Puig in [23, §7], a Morita equivalence between B and B' given by a bimodule with endopermutation source implies an identification $D = D'$ such that for some choice of source idempotents $i \in B^D$, $i' \in (B')^D$, setting $A = iBi$ and $A' = i'B'i'$, we have D-interior $\mathcal{O}$-algebra isomorphisms

$$A' \cong e(S \otimes_{\mathcal{O}} A)e\ , \quad A \cong e'(S^{\mathrm{op}} \otimes_{\mathcal{O}} A')e'\ ,$$

where $S = \mathrm{End}_{\mathcal{O}}(V)$ for some indecomposable endopermutation $\mathcal{O}D$-module V with vertex D, and where e, e' are primitive idempotents in $(S \otimes_{\mathcal{O}} A)^D$, $(S^{\mathrm{op}} \otimes_{\mathcal{O}} A')^D$, respectively, satisfying $\mathrm{Br}_D(e) \neq 0$, $\mathrm{Br}_D(e') \neq 0$. These isomorphisms induce inverse equivalences between $\mathrm{mod}(A)$ and $\mathrm{mod}(A')$, sending an A-module U to the A'-module $e(V \otimes_{\mathcal{O}} U)$, and an A'-module U' to the A-module $e'(V^* \otimes_{\mathcal{O}} U')$. Here V^* is the $\mathcal{O}$-dual of V; note that $\mathrm{End}_{\mathcal{O}}(V^*) \cong S^{\mathrm{op}}$ as D-interior $\mathcal{O}$-algebras.

Proof of Theorem 1.4. We use the notation above. Let $\chi \in \mathrm{Irr}(B)$ and $\chi' \in \mathrm{Irr}(B')$ such that χ and χ' correspond to each other through the Morita equivalence $\mathrm{mod}(B) \cong \mathrm{mod}(A) \cong \mathrm{mod}(A') \cong \mathrm{mod}(B')$ described above. As mentioned at the beginning of this section, the primitive idempotent in $Z(K \otimes_{\mathcal{O}} A)$ corresponding to χ is ie_χ. Similarly, the primitive idempotent in $Z(K \otimes_{\mathcal{O}} A')$ is equal to $i'e_{\chi'}$. The explicit description of the Morita equivalence between A and A' above implies that we have

$$i'e_{\chi'} = e \cdot (1_S \otimes ie_\chi)$$

$$ie_\chi = e' \cdot (1_{S^{\mathrm{op}}} \otimes i'e_{\chi'})$$

where these equalities are understood in the algebras $K \otimes_{\mathcal{O}} A$ and $K \otimes_{\mathcal{O}} A'$. By Theorem 4.1, it suffices to show that for Q a subgroup of D, we have $(Ae_\chi)(Q) \neq \{0\}$ if and only if $(A'e_{\chi'})(Q) \neq \{0\}$. It suffices to show one implication, because the other follows then from exchanging the roles of A and A'. Thus it suffices to show that if $(Ae_\chi)(Q) = \{0\}$, then $(A'e_{\chi'})'(Q) = \{0\}$. Let Q be a subgroup of D such that $(Ae_\chi)(Q) = \{0\}$. We have $(S \otimes_{\mathcal{O}} A)(1_S \otimes ie_\chi) = S \otimes_{\mathcal{O}} Ae_\chi$. Since S has a D-stable basis, it follows from [22, 5.6] that $(S \otimes_{\mathcal{O}} Ae_\chi)(Q) = S(Q) \otimes_k (Ae_\chi)(Q) = \{0\}$. Since $A'e_{\chi'}$ is obtained from $S \otimes_{\mathcal{O}} Ae_\chi$ by left and right multiplication with the idempotent e, it follows that $(A'e_{\chi'})(Q) = \{0\}$ as required. □

5. Anchors and normal subgroups

We prove in this section some results on anchors of characters which are induced from a normal subgroup or inflated from quotients. Since an anchor of an irreducible character χ contains a vertex of any lattice affording χ, constructing suitable lattices is one of the tools for getting lower bounds on anchors. The following is well-known (we include a proof for the convenience of the reader).

Lemma 5.1. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let S be a simple kG-module with Brauer character φ such that $d_{\chi\varphi} \neq 0$. Then there exists an $\mathcal{O}G$-lattice L affording χ such that L has a unique maximal submodule M, and such that $L/M \cong S$.*

Proof. Let i be a primitive idempotent in $\mathcal{O}G$ such that $\mathcal{O}Gi$ is a projective cover of S. Since $\chi(i) = d_{\chi\varphi} \neq 0$, there is an $\mathcal{O}$-pure submodule L' of $\mathcal{O}Gi$ such that $L = \mathcal{O}Gi/L'$ affords χ. Since the projective indecomposable $\mathcal{O}G$-module $\mathcal{O}Gi$ has a unique maximal submodule and S is its unique simple quotient, it follows that the image, denoted M, in L of the unique maximal submodule of $\mathcal{O}Gi$ is the unique maximal submodule of L, and satisfies $L/M \cong S$. □

In [19], Plesken showed that if G is a p-group and χ is an irreducible character of G, then there exists an $\mathcal{O}G$-lattice affording χ whose vertex is G. Our next result is a slight variation on this theme.

Proposition 5.2. *Let G be a finite group, N a normal subgroup of G of p-power index, and $\chi \in \mathrm{Irr}(G)$. If there exists $\varphi \in \mathrm{IBr}(G)$ of degree not divisible by $|G : N|$ and such that $d_{\chi,\varphi} \neq 0$, then there exists an $\mathcal{O}G$-lattice L with character χ which is not relatively $\mathcal{O}N$-projective. In particular, in that case, N does not contain the anchors of χ.*

Proof. Let S be a simple kG-module with Brauer character φ of degree not divisible by $|G : N|$ and such that $d_{\chi,\varphi} \neq 0$. By 5.1 there exists an $\mathcal{O}G$-lattice L with a unique maximal submodule M such that χ is the character of L and such that $L/M \cong S$. Note that the character of M is also equal to χ. We will show that one of M or L is not relatively $\mathcal{O}N$-projective. Arguing by contradiction, suppose that L and M are relatively $\mathcal{O}N$-projective. By Green's indecomposability theorem, there are indecomposable $\mathcal{O}N$-modules Z and U such that $L \cong \mathrm{Ind}_N^G(Z)$ and $M \cong \mathrm{Ind}_N^G(U)$. Then $\chi = \mathrm{Ind}_N^G(\tau)$, where τ is the character of Z. Since $\chi = \mathrm{Ind}_N^G(\tau)$ is irreducible, it follows that the different G-conjugates ${}^x\tau$ of τ, with x running over a set of representatives $\mathcal{R}$ of G/N in G, are pairwise different. Similarly, $\chi = \mathrm{Ind}_N^G(\tau')$, where τ' is the character of U. After replacing U by ${}^xU$ for a suitable element $x \in G$, we may assume that $\tau' = \tau$. By the above, we have $\mathrm{Res}_N^G(L) \cong \bigoplus_{x\in\mathcal{R}} {}^xZ$, and the characters of these summands are the pairwise different conjugates ${}^x\tau$ of τ. In particular, $\mathrm{Res}_N^G(L)$ has a unique $\mathcal{O}$-pure summand with character τ, and this summand is isomorphic to Z. We denote this summand abusively again by Z. Similarly, $\mathrm{Res}_N^G(M)$ has a unique $\mathcal{O}$-pure summand, abusively again denoted by U, with character τ. Since $M \subseteq L$ induces an equality $K \otimes_{\mathcal{O}} M = K \otimes_{\mathcal{O}} L$, it follows that $K \otimes_{\mathcal{O}} U = K \otimes_{\mathcal{O}} Z$. Moreover, we have $U \subseteq K \otimes_{\mathcal{O}} Z \cap L = Z$, where the second equality holds as Z is $\mathcal{O}$-pure in L. Thus the inclusion $M \subseteq L$ is obtained from inducing the inclusion map $U \subseteq Z$ from N to G. By the construction of M, the inclusion $M \subseteq L$ induces a map $k \otimes_{\mathcal{O}} M \to k \otimes_{\mathcal{O}} L$ with cokernel S. Thus we have

$$\dim_k(S) = \mathrm{codim}(k \otimes_{\mathcal{O}} M \to k \otimes_{\mathcal{O}} L) = |G:N|\mathrm{codim}(k \otimes_{\mathcal{O}} U \to k \otimes_{\mathcal{O}} Z)\ .$$

This contradicts the assumption that $\varphi(1)$ is not divisible by $|G:N|$. Thus one of L or M is not relatively $\mathcal{O}N$-projective. □

Corollary 5.3. *Let G be a finite group, N a normal subgroup of p-power index, and let $\chi \in \mathrm{Irr}(G)$. Let P be an anchor of χ. If there exists $\varphi \in \mathrm{IBr}(G)$ of degree prime to p such that $d_{\chi\varphi} \neq 0$, then $G = PN$.*

Proof. Arguing by contradiction, suppose that PN is a proper subgroup of G. Since G/N is a p-group, it follows that PN is contained in a normal subgroup M of index p in G. Then M contains every anchor of χ, hence M contains the vertices of any $\mathcal{O}G$-lattice affording χ. Let $\varphi \in \mathrm{IBr}(G)$ such that $\varphi(1)$ is prime to p and such that $d_{\chi,\varphi} \neq 0$. Proposition 5.2 implies however that $|G:M| = p$ divides $\varphi(1)$, a contradiction. □

We record here an extension of [19, Lemma 3] which will be used in the next section.

Proposition 5.4. *Let G be a finite group, P a Sylow p-subgroup, and $\chi \in \mathrm{Irr}(G)$. Suppose that $\mathrm{Res}^G_P(\chi)$ is irreducible and that there exists an irreducible Brauer character φ of p'-degree of G such that $d_{\chi,\varphi} \neq 0$. Then there exists an $\mathcal{O}G$-lattice L affording χ with vertex P. In particular, the Sylow p-subgroups of G are the anchors of χ.*

Proof. Let π be a generator of $J(\mathcal{O})$ and let S be a simple kG-module with Brauer character φ. By 5.1, there is an $\mathcal{O}G$-lattice L affording χ such that $k \otimes_{\mathcal{O}} L$ has a simple head isomorphic to S. Let N be the maximal submodule of L. Then $\pi L \subset N$ and N is an $\mathcal{O}G$-lattice affording χ. The invariant factors of the $\mathcal{O}$-module L/N are either 1 or π and the number of non-trivial invariant factors of L/N equals $\dim_k(L/N) = \dim_k(S)$. Thus the product of the invariant factors of L/N equals $\pi^{\dim_k S}$. By hypothesis, $\mathrm{Res}^G_P(L)$ is irreducible. If $\mathrm{Res}^G_P(L)$ has vertex P, then L has vertex P. So, we may assume that $\mathrm{Res}^G_P(L)$ is relatively U-projective for some proper subgroup U of P. By Green's indecomposability theorem, $\mathrm{Res}^G_P(L) = \mathrm{Ind}^P_U(M)$ for some $\mathcal{O}U$-lattice M. By [19, Lemma 3], $\mathrm{Res}^G_P(N)$ has vertex P, whence N has vertex P. Note that [19, Lemma 3] is stated for $\mathcal{O}$ a localisation of the $|P|$-th cyclotomic integers, but as remarked in [19, Page 235], [19, Lemma 3] remains true in our setting. The second assertion of the proposition follows from Proposition 3.1. □

Proposition 5.5. *Let G be a finite group, N a normal subgroup, and let $\chi \in \mathrm{Irr}(G)$ such that $\chi = \mathrm{Ind}^G_N(\tau)$ for some $\tau \in \mathrm{Irr}(N)$. Let V be an $\mathcal{O}N$-lattice with character τ. Suppose that the composition series of the kN-modules $k \otimes_{\mathcal{O}} {}^xV$, with x running over a set of representatives of G/N in G, are pairwise disjoint. Then $\mathcal{O}Ge_\chi \cong \mathrm{Ind}^G_N(\mathcal{O}Ne_\tau)$ as G-interior $\mathcal{O}$-algebras. In particular, N contains the vertices of all lattices affording χ, and N contains the anchors of χ.*

Proof. It suffices to show that e_τ belongs to $\mathcal{O}Ge_\chi$. Indeed, if this is true, then the assumptions on χ and τ imply that $e_\chi = \mathrm{Tr}^G_N(e_\tau)$, and the different conjugates of e_τ appearing in $\mathrm{Tr}^G_N(e_\tau)$ are pairwise orthogonal idempotents in $\mathcal{O}Ge_\chi$. In particular, we have $e_\tau\mathcal{O}Ge_\tau = \mathcal{O}Ne_\tau$. It follows from [27, (16.6)] that $\mathcal{O}Ge_\chi \cong \mathrm{Ind}^G_N(\mathcal{O}Ne_\tau)$.

It remains to show that e_τ belongs to $\mathcal{O}Ge_\chi$. Let I be a primitive decomposition of 1 in $\mathcal{O}N$. Let $i \in I$ such that $e_\tau i \neq 0$. This condition is equivalent to $k \otimes_{\mathcal{O}} V$ having a composition factor isomorphic to the unique simple quotient T_i of the $\mathcal{O}N$-module $\mathcal{O}Ni$. Since the different G-conjugates of the kN-module $k \otimes_{\mathcal{O}} V$ have pairwise disjoint composition series, it follows that $e_{{}^x\tau} i = 0$ for $x \in G \setminus N$. Thus $e_\chi i = e_\tau i \in \mathcal{O}Ge_\chi$ for any $i \in I$ such that $e_\tau i \neq 0$. Taking the sum over all such i implies that $e_\tau \in \mathcal{O}Ge_\chi$. The last statement follows from the fact that $e_\chi = \mathrm{Tr}_N^G(e_\tau)$ and Higman's criterion, for instance, or directly from the fact that Ind_N^G induces a Morita equivalence between $\mathcal{O}Ne_\tau$ and $\mathcal{O}Ge_\chi$. □

Corollary 5.6. *Let G be a finite group, N a normal subgroup of p-power index, and let $\chi \in \mathrm{Irr}(G)$ such that $\chi = \mathrm{Ind}_N^G(\tau)$ for some $\tau \in \mathrm{Irr}(N)$. Suppose that $d_{\chi,\varphi}$ is either 1 or 0 for every $\varphi \in \mathrm{IBr}(G)$. Then N contains the anchors of χ.*

Proof. Let V be an $\mathcal{O}N$-lattice affording τ. Let I be a primitive decomposition of 1 in $\mathcal{O}N$. By Green's indecomposability theorem, I remains a primitive decomposition in $\mathcal{O}G$. Let $i \in I$. We have

$$\chi(i) = \sum_x {}^x\tau(i)$$

where x runs over a set of representatives $\mathcal{R}$ of G/N in G. By the assumptions on the decomposition numbers of χ, the left side is either 1 or 0. Thus either ${}^x\tau(i) = 0$ for all $x \in \mathcal{R}$, or there is exactly one $x = x(i) \in \mathcal{R}$ with ${}^x\tau(i) \neq 0$. This implies that the composition series of the different G-conjugates of $k \otimes_{\mathcal{O}} V$ are pairwise disjoint. The result follows from 5.5. □

Proposition 5.7. *Let G be a finite group, N a normal subgroup of G, and $\chi \in \mathrm{Irr}(G)$. Suppose that χ is the inflation to G of an irreducible character $\psi \in \mathrm{Irr}(G/N)$. Let P be an anchor of χ. Then PN/N is an anchor of ψ, and $P \cap N$ is a Sylow p-subgroup of N.*

Proof. Let $d \in (\mathcal{O}Ge_\chi)^P$ such that $\mathrm{Tr}_P^G(d) = e_\chi$. The assumptions imply that the canonical map $G \to G/N$ induces a G-algebra isomorphism $\mathcal{O}Ge_\chi \cong \mathcal{O}G/Ne_\psi$ such that N acts trivially on both algebras. Thus $e_\chi = \mathrm{Tr}_P^G(d) = |PN : P|\mathrm{Tr}_{PN}^G(d)$. This implies that P is a Sylow p-subgroup of PN, and hence that $P \cap N$ is a Sylow p-subgroup of N. Since the isomorphism $\mathcal{O}Ge_\chi \cong \mathcal{O}G/Ne_\psi$ sends $\mathrm{Tr}_{PN}^G(d)$ to $\mathrm{Tr}_{PN/N}^{G/N}(\bar{d})$, where $\bar{d}$ is the canonical image of d, it follows that PN/N contains an anchor of ψ. Using the fact that P is a Sylow p-subgroup of PN, one easily checks that any proper subgroup of PN/N is of the form QN/N for some proper subgroup Q of P containing $P \cap N$. The previous isomorphism implies that PN/N is an anchor of ψ. □

Example 5.8. (1) $p = 2$, $G = \mathfrak{S}_3$, $\chi(1) = 2$: Then χ lies in a defect zero block of G, hence by Theorem 1.2, the trivial group is the only anchor of χ.

(2) $p = 2$, $G = \mathfrak{S}_4$, $\chi(1) = 2$: Then χ is inflated from the character in part (1). In this case, by Proposition 5.7 the Klein four subgroup V_4 of $\mathfrak{S}_4$ is the only anchor of χ. However, the defect groups of the block containing χ (i.e. of the principal block of $\mathfrak{S}_4$) are the Sylow 2-subgroups of $\mathfrak{S}_4$ (i.e. dihedral groups of order 8).

(3) $p = 2$, $G = \mathfrak{S}_5$: All irreducible characters of G except the one of degree 6 are of height zero in their block. So their anchors coincide with their defect groups, by Theorem 1.3 (a) .

Now let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 6$. Then χ is induced from an irreducible character of the alternating group $\mathfrak{A}_5$ of degree 3. Thus there exists an $\mathcal{O}G$-lattice affording χ with vertex V_4.
On the other hand, χ is labelled by the partition $\lambda = (3, 1, 1)$ of 5. By the remark on p. 511 of [28], the Specht module S^λ is indecomposable, and $\mathfrak{S}_2 \times \mathfrak{S}_2$ is a vertex of S^λ, by Theorem 2 in [28]. Thus the anchors of χ are Sylow 2-subgroups of G, by Theorem 1.2 (b).

6. Navarro vertices

We prove Theorems 1.5 and 1.6.

Theorem 6.1. *Let G be a finite p-solvable group. Let $\chi \in \mathrm{Irr}(G)$ and let (Q, δ) be a Navarro vertex of χ. Supppose that $\chi^\circ \in \mathrm{IBr}(G)$. Then Q contains an anchor of χ. Moreover, if $\delta = 1_Q$ or if p is odd, then Q is an anchor of χ.*

Proof. Since $\chi^\circ \in \mathrm{IBr}(G)$, there is a unique $\mathcal{O}G$-lattice L affording χ, up to isomorphism. Moreover, $k \otimes_{\mathcal{O}} L$ is the unique simple kG-module with Brauer character χ°, up to isomorphism. Recall that there is a nucleus (W, γ) of χ such that $\chi = \mathrm{Ind}_W^G(\gamma)$, and $Q \in \mathrm{Syl}_p(W)$ (cf. [17, p. 2763]). Further, $\gamma \in \mathrm{Irr}(W)$ has a unique factorization $\gamma = \alpha\beta$ where $\alpha \in \mathrm{Irr}(W)$ is p'-special and $\beta \in \mathrm{Irr}(W)$ is p-special. Going over to Brauer characters, we have $\chi^\circ = \mathrm{Ind}_W^G(\gamma^\circ)$ and $\gamma^\circ = \alpha^\circ\beta^\circ$; in particular, $\gamma^\circ, \alpha^\circ, \beta^\circ \in \mathrm{IBr}(W)$. Let R be a vertex of the unique $\mathcal{O}W$-lattice affording γ and let R_0 be a vertex of the unique kW-module affording γ°. Then, up to conjugation in W, $R_0 \leq R \leq Q$. Since $\chi = \mathrm{Ind}_W^G(\gamma)$, R is also a vertex of the $\mathcal{O}G$-lattice affording χ and hence by Proposition 3.1 (iii), R is an anchor of χ. This proves the first assertion.

Since α is p'-special, the p-part of the degree of γ equals the p-part of the degree of β. Since G is p-solvable, it follows that

$$|R_0| = \frac{|W|_p}{\beta(1)_p}.$$

Now suppose that p is odd. Since β° is irreducible, by [18, Lemma 2.1], β is linear. It follows from the above that $R_0 = Q = R$, proving the second assertion when p is odd. Since $\delta = \mathrm{Res}_Q^W(\beta)$, a similar argument works when $\delta = 1_Q$. $\square$

Lemma 6.2. *Let G be a finite p-solvable group and $\chi \in \mathrm{Irr}(G)$. Suppose that χ is p-special and that there exists $\varphi \in \mathrm{IBr}(G)$ of p'-degree such that $d_{\chi\varphi} \neq 0$. Then there exists an $\mathcal{O}G$-lattice affording χ with vertex a Sylow p-subgroup of G.*

Proof. This is immediate from Proposition 5.4 and the fact that the restriction of a p-special character of G to a Sylow p-subgroup of G is irreducible (cf. [6, Prop. 6.1]). $\square$

The following is due to G. Navarro.

Lemma 6.3. *Let G be a finite group of odd order and $\chi \in \mathrm{Irr}(G)$. Suppose that χ is p-special. Then the trivial Brauer character of G is a constituent of χ°.*

Proof. By the Feit-Thompson theorem, G is solvable and hence p-solvable. Let H be a p-complement of G and let ζ be a primitive $|G|_p$-th root of unity. By [6, Theorem 6.5], $\mathbb{Q}[\zeta]$ is a splitting field of χ. Thus $\mathrm{Res}_H^G(\chi)$ is a rational valued character of odd degree. Hence, $\mathrm{Res}_H^G(\chi)$ contains a real valued irreducible constituent, say

α. By Brauer's permutation lemma, the number of real-valued irreducible characters of H equals the number of real conjugacy classes of H. Since $|H|$ is odd, α is the trivial character of H. By Frobenius reciprocity, χ is an irreducible constituent of $\mathrm{Ind}_H^G(\alpha)$. On the other hand, since H is a p-complement of G, $\mathrm{Ind}_H^G(\alpha)$ is the character of the projective indecomposable $\mathcal{O}G$-module corresponding to the trivial kG-module. $\square$

Combining the two results above yields the Theorem 1.6. In fact we prove more.

Theorem 6.4. *Let G be a finite group of odd order, let $\chi \in \mathrm{Irr}(G)$ and let (Q, δ) be a Navarro vertex of χ. Then there exists an $\mathcal{O}G$-lattice affording χ with vertex Q. In particular, Q is contained in an anchor of χ.*

Proof. Let (W, γ) be a nucleus of χ such that Q is a Sylow p-subgroup of W and $\delta = \mathrm{Res}_Q^W(\alpha)$, where $\gamma = \alpha\beta$, with α a p'-special character and β a p-special character of W (cf. [17, Sections 2,3]). By Lemma 6.3, the trivial Brauer character is a constituent of β°. By Lemma 6.2, there exists an $\mathcal{O}W$-lattice X affording β and with vertex Q. Let Y be an $\mathcal{O}W$-lattice affording α. Then $V = Y \otimes X$ is an $\mathcal{O}W$-lattice affording γ. We claim that V has vertex Q. Indeed if V is relatively R-projective, then every indecomposable sumand of $Y^* \otimes V$ is relatively R-projective. On the other hand, since α has p'-degree, the $\mathcal{O}W$-lattice $Y^* \otimes Y$ has a direct summand isomorphic to the trivial $\mathcal{O}W$-module. Thus, $Y^* \otimes Y \otimes X$ has a direct summand isomorphic to X. Since Q is a vertex of X, and Q is a Sylow p-subgroup of W, it follows that R is a Sylow p-subgroup of W. Then $\mathrm{Ind}_W^G(V)$ is an $\mathcal{O}G$-lattice with character $\chi = \mathrm{Ind}_W^G(\gamma)$. Clearly, $\mathrm{Ind}_W^G(V)$ is relatively $\mathcal{O}Q$-projective. Suppose if possible that $\mathrm{Ind}_W^G(V)$ is relatively $\mathcal{O}R$-projective for some proper subgroup R of Q, say $\mathrm{Ind}_W^G(V)$ is a summand of $\mathrm{Ind}_R^G(X)$ for some R properly contained in Q and for some $\mathcal{O}R$-lattice X. By the Mackey formula, V is a summand of $\mathrm{Ind}_{W\cap {}^xR}^W(\mathrm{Res}_{W\cap {}^xR}^{{}^xR}\, {}^xX)$ for some $x \in G$. This is a contradiction as $|{}^xR| < |Q|$. Thus Q is a vertex of $\mathrm{Ind}_W^G(V)$ proving the first assertion. The second is immediate from the first and Proposition 5.4. $\square$

I. M. Isaacs and G. Navarro provided us with an example of a p-special character of a p-solvable group none of whose irreducible Brauer constituents have degree prime to p. Proposition 5.5 can be used to prove that the anchors of the Isaacs-Navarro example, which we give below, are strictly contained in the Sylow p-subgroups of the ambient group (so in particular, these characters are not afforded by any lattice with full vertex).

Example 6.5. [Isaacs -Navarro] Let $p = 5$ and let M be the semidirect product of an extraspecial group of order 5^3 and of exponent 5, acted on faithfully by Q_8 where the action is trivial on the center. Let $G = M \wr C_5$ be the wreath product of M by a cyclic group of order 5. In G, there is the normal subgroup $N = M_1 \times \cdots \times M_5$, with each M_i isomorphic to M. Also, there is a cyclic subgroup C of order 5 that permutes the M_i transitively. Note that M_1 has a 5-special character α of degree 5.

Let $\theta \in \mathrm{Irr}(N)$ be the product of α with trivial characters of M_2, M_3, M_4 and M_5. Then θ has degree 5 and $\chi = \theta^G$ is 5-special of degree 25.

There is a Sylow 2-subgroup S of G with the form $Q_1 \times Q_2 \times ... \times Q_5$, where Q_i is a Sylow 2-subgroup of M_i and the Q_i are permuted transitively by C. Now θ_S is the product of α_{Q_1} with trivial characters on the other Q_i.

Also, α_{Q_1} is the sum of the irreducible character of degree 2 and the three nontrivial linear characters, so there is no trivial constituent. It follows that θ_S has no C-invariant irreducible constituent. The same is therefore true about χ_S. Then each 5-Brauer irreducible constituent of χ has degree divisible by 5.

The construction also shows that if $x \in G \setminus N$, then θ and ${}^x\theta$ have no irreducible Brauer constituents in common. So by Proposition 5.5, the anchors of χ are contained in N.

In conjunction with Proposition 5.2, the following example provides characters whose anchors are not contained in Navarro vertices. The construction is similar to that in the Isaacs-Navarro example above.

Example 6.6. Suppose that $M = O_{p,p'}(M)$, and α is an irreducible p-special character of M such that $\mathrm{Res}^M_{O_p(M)}\alpha$ is irreducible. Suppose further that there exists a nontrivial irreducible Brauer character φ of M such that $d_{\alpha,\varphi} \neq 0$. Let β be the irreducible character of M with $O_p(M)$ in the kernel of β and such that β° equals the dual φ^* of φ. Then β is p'-special.

Let $G = M \wr C_p$. In G, there is the normal subgroup $N = M_1 \times \cdots \times M_p$ with each M_i isomorphic to M. Let $\tilde{\alpha} \in \mathrm{Irr}(N)$ be the product of α and the trivial characters of $M_2, \cdots, M_p$, let $\tilde{\beta} \in \mathrm{Irr}(N)$ be the product of β and the trivial characters of $M_2, \cdots, M_p$ and let $\tilde{\varphi}$ be the product of φ with the trivial Brauer characters of $M_2, \cdots, M_p$. Let $\chi = \mathrm{Ind}^G_N(\tilde{\alpha}\tilde{\beta})$.

Since $\tilde{\alpha}$ is p-special and $\tilde{\beta}$ is p'-special, by results of [6], $\tilde{\alpha}\tilde{\beta}$ is an irreducible character of N. By construction, neither $\tilde{\alpha}$ nor $\tilde{\beta}$ is G-stable. Hence, also by general results on p-factorable characters, $\tilde{\alpha}\tilde{\beta}$ is not G-stable. Since $|G/N| = p$, it follows that χ is an irreducible character of G. Now, since $\tilde{\beta}$ is not G-stable, it is easy to see that χ is not p-factorable. On the other hand, N is a maximal normal subgroup of G. Thus $(N, \tilde{\alpha}\tilde{\beta})$ is a nucleus of G in the sense of [17], and the Sylow p-subgroups of N are the first components of the Navarro vertices of χ.

We have $(\tilde{\alpha}\tilde{\beta})^\circ = \tilde{\alpha}^\circ\tilde{\beta}^\circ$, and $\tilde{\varphi}$ is an irreducible Brauer constituent of $\tilde{\alpha}$ and $\tilde{\beta}^\circ = \tilde{\varphi}^*$. Since $\tilde{\varphi}(1) = \varphi(1)$ is relatively prime to p, it follows that the trivial Brauer character of N is a constituent of $(\tilde{\alpha}\tilde{\beta})^\circ$. Consequently, the trivial Brauer character of G occurs as a constituent of χ. Thus, by Proposition 5.2, the anchors of χ are not contained in N.

7. Lifting

Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Let P be an anchor of χ. Then $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ is a G-interior k-algebra. Since

$$(k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi)^G = Z(k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi)$$

is a local k-algebra, it follows that $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ is a primitive G-interior k-algebra. Since

$$k \otimes_{\mathcal{O}} (\mathcal{O}Ge_\chi)^P \subseteq (k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi)^P,$$

$k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ has a defect group Q contained in P. We will see below that we often (but not always) have equality here. If $\chi^\circ \in \mathrm{IBr}(G)$, then there is, up to isomorphism, a unique $\mathcal{O}G$-lattice L affording χ, and $k \otimes_{\mathcal{O}} L$ is the unique simple kG-module with Brauer character χ°, up to isomorphism. We have seen above that in that case the G-interior $\mathcal{O}$-algebra $\mathcal{O}Ge_\chi$ is isomorphic to $\mathrm{End}_{\mathcal{O}}(L)$. This implies that the G-interior k-algebra $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ is isomorphic to $\mathrm{End}_k(k \otimes_{\mathcal{O}} L)$. Thus the anchor P of χ is a vertex of L, and the defect group Q of $k \otimes_{\mathcal{O}} \mathcal{O}Ge_\chi$ is a vertex of $k \otimes_{\mathcal{O}} L$. The examples 7.1 and 7.2 below illustrate the cases where $Q = P$ and $Q < P$, respectively.

Example 7.1. Let G be the symmetric group $\mathfrak{S}_n$, for a positive integer n. Let $\chi \in \mathrm{Irr}(G)$ such that $\chi^\circ \in \mathrm{IBr}(G)$, and let L be an $\mathcal{O}G$-lattice affording χ. We claim that L and $k \otimes_{\mathcal{O}} L$ have the same vertices.

Indeed, let λ be the partition of n labelling χ. Since the Specht lattice $S^\lambda_{\mathcal{O}}$ is an $\mathcal{O}G$-lattice affording χ, the uniqueness of L implies that $S^\lambda_{\mathcal{O}} \cong L$. Thus the kG-module $S^\lambda_k \cong k \otimes_{\mathcal{O}} S^\lambda_{\mathcal{O}} \cong k \otimes_{\mathcal{O}} L$ has Brauer character χ° and is therefore simple.

A result by Hemmer (cf. [8]) implies that S^λ_k lifts to a p-permutation $\mathcal{O}G$-lattice M. Then $K \otimes_{\mathcal{O}} M$ is a simple KG-module; that is, $K \otimes_{\mathcal{O}} M \cong S^\mu_K \cong K \otimes_{\mathcal{O}} S^\mu_{\mathcal{O}}$ for some partition μ of n. Moreover, S^μ_k is isomorphic to $k \otimes_{\mathcal{O}} M \cong S^\lambda_k$; in particular, we have

$$\mathrm{Hom}_{kG}(S^\lambda_k, S^\mu_k) \neq 0 \neq \mathrm{Hom}_{kG}(S^\mu_k, S^\lambda_k).$$

Suppose first that $p > 2$. Then [11, Proposition 13.17] implies that $\lambda \geq \mu$ and $\mu \geq \lambda$, hence $\mu = \lambda$. The uniqueness of L implies that $S^\lambda_{\mathcal{O}} \cong L \cong M$; in particular, L is a p-permutation $\mathcal{O}G$-lattice. Hence L and $k \otimes_{\mathcal{O}} L$ have the same vertices.

It remains to consider the case $p = 2$. In this case a theorem by James and Mathas (cf. [12]) implies that either λ is 2-regular, or the conjugate partition λ' is 2-regular, or $n = 4$ and $\lambda = (2, 2)$. The last alternative is trivial. Multiplying by the sign character, if necessary, we may therefore assume that λ is 2-regular. If μ is also 2-regular then we certainly have $\lambda = \mu$. If μ' is 2-regular then we have $\lambda = \mu'$, in a similar way. Now, arguing as in the case $p > 2$, we conclude that L and $k \otimes_{\mathcal{O}} L$ have the same vertices.

Example 7.2. Let $p = 2$, $G = GL(2, 3)$ and $N = SL(2, 3)$. Let R be the unique Sylow 2-subgroup of N and H a complement of R in N. Let τ be the 2-dimensional irreducible character of R and let η be the unique extension of τ to N with determinantal order a power of 2 (cf. Corollary (6.28) in [10]). Let χ be an extension of η to G. Then χ is 2-special, by [9, Proposition 40.5]. Further, χ° is irreducible and equals $\mathrm{Ind}^G_N(\psi)$, where ψ is a linear Brauer character of N. (Note that the restriction of χ° to H is a sum of two distinct irreducible Brauer characters).

Thus, R is a vertex of the unique kG-module affording χ° and R is contained in some (and hence every) vertex of the $\mathcal{O}G$-lattice affording χ. Since χ is not induced from any character of N, and G/N is a 2-group, Green's indecomposability theorem implies that the $\mathcal{O}G$-lattice affording χ is not relatively N-projective. Hence R is properly contained in a vertex of the $\mathcal{O}G$-lattice affording χ, which is consequently a Sylow 2-subgroup of G.

Remark 7.3. Let G be a finite p-solvable group and $\chi \in \mathrm{Irr}(G)$ such that $\chi^\circ \in \mathrm{IBr}(G)$. Let L be an $\mathcal{O}G$-lattice affording χ. Suppose, as in the above example that a vertex P of L strictly contains a vertex R of $k \otimes_{\mathcal{O}} L$. Let S be an $\mathcal{O}P$-lattice source of L. We claim that S is not an endopermutation module. Indeed, assume the contrary. Since P is a vertex of S and since S is endopermutation, $k \otimes_{\mathcal{O}} S$ is an indecomposable endopermutation kP-module with vertex P. On the other hand, $k \otimes_{\mathcal{O}} S$ is a direct summand of $k \otimes_{\mathcal{O}} L$, $k \otimes_{\mathcal{O}} L$ has vertex R, and R is strictly contained in P, a contradiction.

Acknowledgements. The authors are grateful to Susanne Danz for her help with the example on symmetric groups. They would also like to thank Gabriel Navarro for providing us with Lemma 6.3 and Marty Isaacs and Gabriel Navarro for providing us with Example 6.5. Work on this paper started when the second author visited the City University of London in 2013, with the kind support of the DFG, SPP 1388. He is grateful for the hospitality received at the City University.

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