Non-overshooting stabilization via state and output feedback

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Abstract: The concept of "strong stability" of LTI systems has been introduced in a recent paper [KHP2]. This is a stronger notion of stability compared to alternative definitions (e.g. stability in the sense of Lyapunov, asymptotic stability), which allows the analysis and design of control systems with non-overshooting response in the state-space for arbitrary initial conditions. The paper reviews the notion of "strong stability" [KHP2] and introduces the problem of non-overshooting stabilization. It is shown that non-overshooting stabilization under dynamic and static output feedback are, in a certain sense, equivalent problems. Thus, we turn our attention to static non-overshooting stabilization problems under state-feedback, output injection and output feedback. After developing a number of preliminary results, we give a geometric interpretation to the problem in terms of the intersection of an affine hyperplane and the interior of an open convex cone. A solution to the problem is finally obtained via Linear Matrix Inequalities, along with the complete parametrization of the optimal solution set.

Keywords: Non-overshooting stabilization, state-feedback, output feedback, convex programming, Linear Matrix Inequalities (LMI)

1. Introduction

The concept of "strong stability" for autonomous internal LTI system descriptions was introduced in [KHP2]. This is a stronger version of stability compared to the standard definitions of asymptotic and Lyapunov stability [B], [HJ], [K], [MM], [HP], [H]. These two notions of stability are clearly necessary for bounding these variables in some sense, but do not guarantee that these physical variables do not overshoot. In contrast, strong stability characterizes the case where there is non-overshooting transient response for arbitrary initial conditions taken from a given hyper-sphere in the phase-space. Non-overshooting behavior is a desirable property in certain applications and can be considered as a special case of constrained control.

In [KHP2], three different notions of strong stability were introduced for the linear time-invariant autonomous system

$$\mathcal{S}(A) : \dot{x}(t) = Ax(t), \quad x(t_0) = x_0 \tag{1}$$

and these were related to properties of the state matrix A. (The relevant definitions and properties are briefly reviewed in section 2 below). In addition, the dependence of the strong stability property on general coordinate transformations was examined and the existence of special coordinate systems incompatible with strong stability was established. It was further shown that the strong-stability property is invariant under orthogonal transformations, which led to the use of the Schur canonical form as the basis for investigating further the parametrization of strongly stable state matrices. Finally, it was shown that the skewness of the eigen-frame of A is an important indicator of the violation of the

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strong stability property. Bounds on the eigen-frame skewness were derived (both for diagonalizable and non-diagonalizable matrices), for which asymptotic stability and strong stability are equivalent properties. The results indicate that there is a link between loss of strong stability due to eigen-frame skewness and reduced robustness of stability to parameter variations.

In this work we address non-overshooting stabilization problems under various feedback regimes (state-feedback, output injection and output feedback) for state space system descriptions defined in terms of physical state variables. Thus, the main problem addressed in the paper is the following: Given an LTI system of the form (1), does there exist an appropriate state or output feedback matrix for which the resulting ("closed-loop") state matrix is strongly stable?

The layout of the paper is as follows. Section 2 introduces the notation and section 3 reviews the notion of strong stability introduced in [KHP] and [KHP2]. In section 4 the non-overshooting stabilization problem is defined for general MIMO systems. It is first shown that a linear system is *non-overshooting stabilizable)* by output feedback if and only if it is *non-overshooting stabilizable* by static output feedback. Moreover, if a static output feedback can be found for which the closed-loop system is strongly stable, then dynamic non-overshooting stabilizing compensation schemes of arbitrary state-dimension can also be obtained. Thus non-overshooting stabilization is essentially a static feedback problem.

In section 5 general geometric conditions for z.o. stabilizability are derived under static state and output feedback, in terms of the intersection of an affine hyperplane and a convex cone. A solution to the problem is obtained in section 6 via convex programming/Linear Matrix Inequalities (LMI). Connections with asymptotic stabilization and the Kalman decomposition are also established. Section 7 gives a complete parametrization of the families of state and output feedback matrices which solve the z.o. stabilization problem. Finally, section 9 contains the main conclusions of the paper and discusses future research directions related to this work.

2. Notation and Background Results

The notation is mostly standard and is included here for completeness. $\mathcal{R}^{n\times m}$ denotes the space of all $n \times m$ matrices over the field $\mathcal{R}$. For a set $\Omega \subseteq \mathcal{R}^{n\times m}$, $\bar{\Omega}$ denotes its closure in $\mathcal{R}^{n\times m}$ (with respect to a suitable norm) and $\partial\Omega = \bar{\Omega}\backslash\Omega$. The interior of a set Ω in denoted by $\mathrm{int}(\Omega)$. The distance of $A \in \mathcal{R}^{n\times m}$ to Ω is defined as $\mathrm{dist}(A,\Omega) = \inf_{X\in\Omega} \|A - X\|$ where $\|\cdot\|$ denotes a suitably defined norm. A set $\Omega \subseteq \mathcal{R}^{n\times m}$ is called convex if whenever $\omega_1 \in \Omega$ and $\omega_2 \in \Omega$, $\lambda\omega_1 + (1-\lambda)\omega_2 \in \Omega$ for every $\lambda \in [0,1]$. A set $\Omega \subseteq \mathcal{R}^{n\times m}$ is said to be a cone if whenever $\omega \in \Omega$, $\lambda\omega \in \Omega$ for every $\lambda > 0$. The cone generated by a set $\Omega \subseteq \mathcal{R}^{n\times n}$ is defined as $\mathrm{cone}[\Omega] = \{x \in \mathcal{R}^{n\times n} : x = \lambda\omega, \omega \in \Omega, \lambda > 0\}$. A set $\Omega \subseteq \mathcal{R}^{n\times n}$ is called a convex invertible cone (cic) if it is a convex cone and $\omega \in \Omega \Rightarrow \omega^{-1} \in \Omega$.

The spectrum (set of eigenvalues) of a matrix $A \in \mathcal{R}^{n\times n}$ is the set of eigenvalues $\lambda(A) = \{\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\}$. $\rho(A) := \max\{|\lambda_1(A)|, |\lambda_2(A)|, \ldots, |\lambda_n(A)|\}$ is the spectral radius of A. The (column) range and (right) null-space of $A \in \mathcal{R}^{m\times n}$ are denoted as $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. The trace of A, $\mathrm{trace}(A)$, is defined as the sum of the diagonal elements of A. The set of all real $n \times n$ real symmetric matrices ($A = A'$) is denoted as $\mathcal{S}^n$ and the set of all $n \times n$ real skew-symmetric matrices ($A = -A'$) is denoted as $\mathcal{A}^n$. If $A \in \mathcal{S}^n$ the eigenvalues of A are denoted as

$\lambda_i(A)$ indexed in non-increasing order of magnitude. In this case, we define the inertia of A as the triplet $\text{In}(A) = (\pi(A), \delta(A), \nu(A))$ of positive, zero, and negative eigenvalues of A, respectively, counted according to their algebraic multiplicity. The spectral decomposition of a matrix $A \in \mathcal{S}^n$ is given by $A = U\Lambda U'$ where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the eigenvalue matrix of A and U is the eigenvector matrix of A which satisfies $UU' = U'U = I_n$. For $A \in \mathcal{S}^n$ we denote by $[A]_+$ ($[A]_-$) the matrix that results by setting all negative (positive) eigenvalues in the spectral decomposition of A to zero. An $n \times n$ symmetric positive-definite (positive semi-definite) matrix A is denoted by $A > 0$ ($A \geq 0$), while a negative-definite (negative semi-definite) matrix A is denoted as $A < 0$ ($A \leq 0$). The set of all $n \times n$ positive-definite (positive semi-definite) matrices is denoted by $\mathcal{S}_+^n$ ($\bar{\mathcal{S}}_+^n$) while $\mathcal{S}_-^n$ ($\bar{\mathcal{S}}_-^n$) denotes the set of all $n \times n$ negative-definite (negative semi-definite) symmetric matrices. It follows easily that the sets $\mathcal{S}_+^n$ (and $\mathcal{S}_-^n$) are convex invertible cones.

The spectral norm of $A \in \mathcal{R}^{n\times n}$ is denoted as $\|A\|$ or $\bar{\sigma}(A)$, where $\bar{\sigma}$ is the largest singular value of a matrix. The Frobenius norm of A is defined as $\|A\|_F^2 = \text{trace}(AA') = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2$. In matrix distance problems the convenience of using the Frobenius norm arises from the fact that it is induced by an inner product in $\mathcal{R}^{n\times n}$, $\langle A, B\rangle = \text{trace}\{B'A\}$, with $\|A\|_F^2 = \langle A, A\rangle$. Thus the space $(\mathcal{R}^{n\times n}, \mathcal{R})$ equipped with the Frobenius norm becomes an inner-product space (actually a Hilbert space due to completeness). Since any $A \in \mathcal{R}^{n\times n}$ can be written (uniquely) as the sum of a symmetric matrix $\frac{1}{2}(A + A') \in \mathcal{S}^n$ and a skew-symmetric matrix $\frac{1}{2}(A - A') \in \mathcal{A}^n$, $\mathcal{R}^{n\times n}$ can be written as the direct sum of the two subspaces $\mathcal{R}^{n\times n} = \mathcal{S}^n \oplus \mathcal{A}^n$ of dimensions $n(n+1)/2$ and $n(n-1)/2$, respectively. It can be easily seen that these two subspaces are orthogonal.

Given $A \in \mathcal{R}^{n\times m}$ define $\text{vec}(A) : \mathcal{R}^{n\times m} \to \mathcal{R}^{nm}$ as the column vector:

$$\text{vec}(A) = \left(\begin{array}{cccc|cccc|c|cccc} a_{11} & a_{21} & \ldots & a_{n1} & a_{12} & a_{22} & \ldots & a_{n2} & \ldots & a_{1m} & a_{2m} & \ldots & a_{nm} \end{array} \right)'$$

It is straightforward to show that $\text{vec}(\cdot)$ defines an isometric isomorphism between the spaces $\mathcal{R}^{n\times n}$ and $\mathcal{R}^{n^2}$, so that $\|A\|_F = \|\text{vec}(A)\|$ for every $A \in \mathcal{R}^{n\times n}$, where $\|\cdot\|$ denotes the usual Euclidian norm. Note also that, $\text{vec}(\mathcal{S}^n) = \{\text{vec}(A) : A \in \mathcal{S}^n\} \subseteq \mathcal{R}^{n^2}$ is a linear subspace of $\mathcal{R}^{n^2}$ of dimension $r = n(n+1)/2$. Let $\{w_1, w_2, \ldots, w_r\}$, be an orthonormal basis set for $\text{vec}(S^n)$ and define $W_{\mathcal{S}} = [w_1\ w_2\ \ldots w_r]$. For each $A \in \mathcal{S}^n$ the column vector of co-ordinates of $\text{vec}(A)$ with respect to $\{w_1, w_2, \ldots, w_r\}$ is denoted by $\overline{\text{vec}}_{\mathcal{S}}(A)$. Clearly, we have that: $\text{vec}(A) = W_{\mathcal{S}}\overline{\text{vec}}_{\mathcal{S}}(A) \Rightarrow \overline{\text{vec}}_{\mathcal{S}}(A) = W_{\mathcal{S}}'\text{vec}(A)$ where also, $W_{\mathcal{S}}'W_{\mathcal{S}} = I_r$, $\mathcal{R}[W_{\mathcal{S}}'] = \mathcal{R}^r$ and $\mathcal{R}[W_{\mathcal{S}}] = \text{vec}(\mathcal{S}^n)$.

The characterization of positive semi-definiteness in [All] is based on the fact that $A \in \bar{\mathcal{S}}_+^n$ can be written (e.g. via its spectral decomposition) as $A = \alpha B^2$ for some $B = B'$ and $\alpha \geq 0$. Let:

$$\mathcal{U}_{\mathcal{S}} := \{B \in \mathcal{R}^{n\times n} : B = B' \text{ and } \|B\|_F = 1\} \subseteq \mathcal{S}^n$$

Also define:

$$\Psi_{\mathcal{S}} := \{\text{vec}(B^2) : \ B \in \mathcal{U}_{\mathcal{S}}\} \subseteq \mathcal{R}^{n^2} \text{ and } \Omega_{\mathcal{S}} = \text{conv}[\Psi_{\mathcal{S}}]$$

Then the following result is proved in [All]:

Lemma 2.1 [All]:

(i) $\text{vec}(\bar{\mathcal{S}}_+^n) = \text{cone}[\Omega_{\mathcal{S}}]$ with $\text{vec}(\mathcal{S}_+^n) = \text{int cone}[\Omega_{\mathcal{S}}]$.

(ii) $\overline{\text{vec}}_{\mathcal{S}}(\bar{\mathcal{S}}_+^n) = \text{cone}[W'_{\mathcal{S}}\Omega_{\mathcal{S}}]$ with $\overline{\text{vec}}_{\mathcal{S}}(\mathcal{S}_+^n) = \text{int cone}[W'_{\mathcal{S}}\Omega_{\mathcal{S}}]$.

(iii) $\Psi_{\mathcal{S}}$ is a compact set, $\Omega_{\mathcal{S}}$ is a non empty convex compact set with $\text{dist}(0, \Omega_{\mathcal{S}}) = 1/\sqrt{n}$ and cone$[\Omega_{\mathcal{S}}]$ is a nonempty closed convex cone. □

The Kronecker product of two matrices $A \in \mathcal{R}^{m\times n}$ and $B \in \mathcal{R}^{p\times q}$ is denoted as $A \otimes B \in \mathcal{R}^{mp\times nq}$. A useful identity involving the vectorization of three matrices of compatible dimensions is $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$ (see [HJ]). We will also make use of the following result:

Lemma 2.2 [HJ]: Let m, n be given positive integers. There is a unique matrix $P(m, n) \in \mathcal{R}^{m\times n}$ such that:

$$\text{vec}(X') = P(m, n)\text{vec}(X) \ \text{ for all } X \in \mathcal{R}^{m\times n}$$

This matrix $P(m, n)$ depends only on the dimensions m and n and is given by

$$P(m, n) = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E'_{ij} = [E_{ij}]_{i=1,\ldots,m}^{j=1,\ldots,n}$$

where each $E_{ij} \in \mathcal{R}^{m\times n}$ has entry 1 in position (i, j) and all other entries are zero. Moreover $P(m, n)$ is a permutation matrix and $P(m, n) = P'(n, m) = P(n, m)^{-1}$. □

A matrix $A \in \mathcal{R}^{n\times n}$ is said to be strongly stable iff $A + A' \in \mathcal{S}_-^n$ (see [KHP], [KHP2] and section 3 below). The set of all strongly-stable matrices of dimension $n \times n$ is denoted by $\mathcal{K}^n$ and is a convex invertible cone in $\mathcal{R}^{n\times n}$ [CL], [L]. Given $A \in \mathcal{R}^{n\times n}$ we define the Lyapunov cone of A as the set $\mathcal{P}_A = \{P \in \mathcal{S}_+^n : -AP - PA' \in \mathcal{S}_+^n\}$. Lyapunov's stability theorem for LTI systems states that A is asymptotically stable if and only if $\mathcal{P}_A$ is non-empty [B], [BS], and that A is strongly stable if and only if $I_n \in \mathcal{P}_A$. It is straightforward to verify that $\mathcal{P}_A$ is also a convex invertible cone (cic) in $\mathcal{R}^{n\times n}$.

3. Strong Stability: Definitions and basic results

We start by giving the two standard definitions of Lyapunov and asymptotic stability [B], [K]:

Definition 3.1: For a linear system: $\mathcal{S}(A)$ we define:

1. $\mathcal{S}(A)$ is *Lyapunov stable* iff for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|x(t_0)\| < \delta(\epsilon)$ implies that $\|x(t)\| < \epsilon$ for all $t \geq t_0$.

2. $\mathcal{S}(A)$ is *asymptotically stable* iff it is Lyapunov stable and $\delta(\epsilon)$ in part (1) of the definition can be selected so that $\|x(t)\| \to 0$ as $t \to \infty$. □

Remark 3.1: Note that asymptotic stability is here taken to mean that the origin is the unique equilibrium point and that it is asymptotically stable (in the sense of Definition 3.1 part 2). □

In the paper we use a refined version of stability which characterizes systems with non-overshooting behavior in the (Euclidian) norm of their state trajectories for arbitrary initial conditions in phase-space (with the exception of the origin). We refine this notion by introducing the following definitions (see [KHP2] for details):

Definition 3.2: For the LTI system $S(A)$ we define:

1. The system $\mathcal{S}(A)$ is *strongly Lyapunov stable* iff $\|x(t)\| \leq \|x(t_0)\|$, $\forall t > t_0$ and $\forall\, x(t_0) \in \mathcal{R}^n$.
2. The system $\mathcal{S}(A)$ is *strongly asymptotically stable w.s.* (in the wide sense), iff $\|x(t)\| < \|x(t_0)\|$, $\forall t > t_0$ and $\forall\, x(t_0) \neq 0$.
3. The system $\mathcal{S}(A)$ is *strongly asymptotically stable s.s.* (in the strict sense, or simply strongly asymptotically stable) iff $\frac{d\|x(t)\|}{dt} < 0$, $\forall t \geq t_0$ and $\forall\, x(t_0) \neq 0$. □

Remark 3.2: The three definitions of strong stability introduced above make precise the notion of non-overshooting responses. Thus, strong Lyapunov stability does not allow state trajectories to exit (at any time) the (closed) hyper-sphere with center the origin and radius the norm of the initial state vector $r_0 = \|x(t_0)\|$ (although motion on the boundary of the sphere $\|x(t)\| = r_0$ is allowed, e.g. an oscillator's trajectory). Strong asymptotic stability (strict sense) requires that all state trajectories enter each hyper-sphere $\|x(t)\| = r \leq r_0$ from a non-tangential direction, whereas for systems which are strongly asymptotically stable (wide-sense), tangential entry is allowed. □

Remark 3.3: Strong Lyapunov stability implies Lyapunov stability and strong asymptotic stability (in either sense) implies asymptotic stability. Moreover, strong asymptotic stability (s.s.) implies strong asymptotic stability (w.s.) which in turn implies strong Lyapunov stability. For concrete examples of each type of strong stability see [KHP] and [KHP2]. □

The characterization of the properties of LTI systems for which we may have, or can avoid, overshoots is a property dependent entirely on the matrix A. Necessary and sufficient conditions for each type of strong stability are stated below:

Theorem 3.1 [KHP2]: For the system $\mathcal{S}(A)$, the following properties hold true:

(i) $\mathcal{S}(A)$ is strongly asymptotically stable (s.s.) if and only if $A + A' < 0$. □

(ii) $\mathcal{S}(A)$ is strongly asymptotically stable (w.s.) if and only if one of the following two equivalant conditions hold:

  (a) $A + A' \leq 0$ and A is asymptotically stable.

  (b) $A + A' \leq 0$ and the pair $(A, A + A')$ is observable.

(iii) $\mathcal{S}(A)$ is strongly Lyapunov stable, if and only if $A + A' \leq 0$.

Remark 3.4: In the remaining parts of the paper we consider only strong asymptotic stability in the strict sense (s.s.), which will be simply referred to as "non-overshooting", or in simpler terms as "strong stability". □

4. Non-overshooting Stabilization: Problem definition and preliminary results

In this section we consider the general non-overshooting stabilization problem. We first consider the general dynamic output feedback case and show that, in a certain sense that is made precise subsequently, dynamic compensation does not offer additional flexibility to static stabilization.

It should be stressed at this point that the problem of strong stability (and non-overshooting stabilization) does not have any meaning under general co-ordinate transformations, since the states of the underlying system realization are assumed to represent physical variables. Further note that, even for the state-feedback case, the problem of non-overshooting stabilization is qualitatively different from the corresponding asymptotic stabilization problem. In the later case, a simple necessary and sufficient condition for state-feedback stabilization of a pair (A, B) is the pair' stabilizability, i.e. that all $\overline{\mathcal{C}}_+$ eigenvalues of A are controllable. To see that this does not apply for non-overshooting stabilizability, consider a pair (A, b) in controllable-canonical form. Then we have the following result:

Proposition 4.1: If $A \in \mathcal{R}^{n\times n}$ is in companion form, then it is not strongly stable. Hence no pair (A, b) in controllable-canonical form can be non-overshooting stabilizable by state (or output) feedback.

Proof: State, or output feedback leaves the companion form invariant, i.e. it produces a closed-loop system in companion form. It has been proved [KHP2] that no companion form can be strongly stable and this completes the proof. □

It is shown next, that dynamic output feedback does not offer any additional flexibility to the problem of non-overshooting stabilization. Thus, consider the following feedback configuration shown in Figure 1, which is used for the study of dynamic stabilization problems.

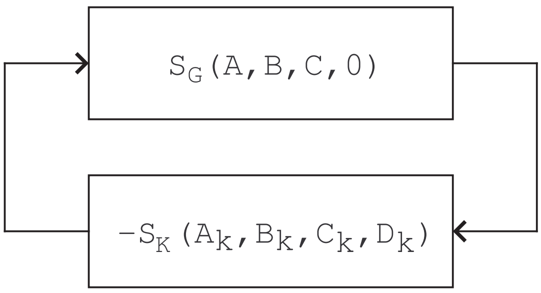

Figure 1: Feedback Configuration

Definition 4.1: Given a system $\Sigma_G(A, B, C, 0)$ and a dynamic compensator $\Sigma_K(A_k, B_k, C_k, D_k)$ in the feedback configuration of Figure 1, we say that Σ_K is a non-overshooting stabilizer of Σ_G if the natural state-space realization of the closed-loop system (Σ_G, Σ_k) is strongly stable. □

Remark 4.1: Note that strong stability of (Σ_G, Σ_k) also implies asymptotic stability and hence it is also an internal stability condition of the feedback system. □

Proposition 4.2: A system $\Sigma_G(A, B, C, 0)$ is non-overshooting stabilizable by output dynamic feedback if and only if it is non-overshooting stabilizable by static output feedback.

Proof: (a) Necessity is obvious since the set of static controllers is a subset of the set of dynamic controllers. (b) Assume that the dynamic controller $K(s)$ with state space realization Σ_K:

$$\dot{\xi}(t) = A_k\xi(t) + B_k y(t)$$
$$u(t) = -C_k\xi(t) - D_k y(t)$$

is a non-overshooting stabilizer of $\Sigma_G(A,B,C,0)$. Then the natural state-space realization of the closed-loop system is:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} = \begin{pmatrix} A - BD_kC & -BC_k \\ B_kC & A_k \end{pmatrix} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} := A_c \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}$$

Since by assumption Σ_K is a non-overshooting stabilizer, A_c is strongly stable, i.e. $A_c + A_c' < 0$. This implies that

$$A_c + A_c' = \begin{pmatrix} A - BD_kC + (A - BD_kC)' & -BC_k + C'B_k' \\ B_kC - C_k'B' & A_k + A_k' \end{pmatrix} < 0 \tag{2}$$

so that, in particular, $A - BD_kC + (A - BD_kC)' < 0$. Thus $A - BD_kC$ is strongly stable and D_k is a non-overshooting stabilizing static output feedback of $\Sigma_G(A,B,C,0)$. $\square$

The above result shows that a LTI system is non-overshooting stabilizable by static output feedback if and only if it is non-overshooting stabilizable by dynamic output feedback. Next we establish a slightly stronger result, i.e. if an LTI system is non-overshooting stabilizable by static output feedback, then it is also non-overshooting stabilizable by dynamic output feedback of arbitrary state-dimension. $\square$

Proposition 4.3: If a system $\Sigma_G(A,B,C,0)$ is non-overshooting stabilizable by output static feedback then it is also non-overshooting stabilizable by dynamic output feedback of arbitrary state dimension.

Proof: Suppose $\Sigma_G(A,B,C,0)$ is non-overshooting stabilizable by output feedback D_k so that $A - BD_kC + (A - BD_kC)' < 0$. To prove the result it suffices to construct a dynamic controller $\Sigma_k(A_k, B_k, C_k, D_k)$ of arbitrary state-dimension $r = \dim(A_k)$ such that equation (2) holds. The result follows immediately if $r = 0$. If $r > 0$, choose any A_k such that $\dim(A_k) = r$ and $A_k + A_k' < 0$. Then a Schur-type argument shows that $A_c + A_c' < 0$ if and only if:

$$A - BD_kC + (A - BD_kC)' - (-BC_k + C'B_k')(A_k + A_k')^{-1}(B_kC - C_k'B') < 0$$

or equivalently:

$$A - BD_kC + (A - BD_kC)' - \begin{pmatrix} C' & B \end{pmatrix} \begin{pmatrix} B_k' \\ -C_k \end{pmatrix} (A_k + A_k')^{-1} \begin{pmatrix} B_k & -C_k' \end{pmatrix} \begin{pmatrix} C \\ B' \end{pmatrix} < 0$$

Since $A - BD_kC + (A - BD_kC)' < 0$ and $(A_k + A_k')^{-1} < 0$, a continuity argument shows that the left-hand side of the above equation can be made negative definite by choosing

$$\left\| \begin{pmatrix} B_k & -C_k' \end{pmatrix} \right\| \leq \epsilon$$

for a sufficiently small $\epsilon > 0$. $\square$.

The two last Propositions show that the design of non-overshooting stabilizers (static or dynamic) can be reduced to a Linear Matrix Inequality (LMI) condition in terms of the controller parameters (D_k in the static case or (A_k, B_k, C_k, D_k) in the dynamic case). It is also clear that non-overshooting stabilization is a static feedback property and there is no need to consider dynamics. In the remaining parts of the paper we turn our attention to static non-overshooting stabilization problems. We distinguish three types of such problems:

P.1 *State-feedback non-overshooting stabilization:* Given a matrix pair (A, B) with $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$, find a state-feedback matrix $F \in \mathcal{R}^{m\times n}$ such that the matrix $A + BF$ is strongly stable.

P.2 *Output injection non-overshooting stabilization:* Given a matrix pair (A, C) with $A \in \mathcal{R}^{n\times n}$ and $C \in \mathcal{R}^{p\times n}$, find an output injection matrix $H \in \mathcal{R}^{n\times p}$ such that the matrix $A + HC$ is strongly stable.

P.3 *Output feedback non-overshooting stabilization:* Given a matrix triplet (A, B, C) with $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{p\times n}$ find an output feedback matrix $F \in \mathcal{R}^{m\times p}$ such that the matrix $A + BFC$ is strongly stable.

The main objective of the work is to establish necessary and sufficient conditions of non-overshooting stabilization (for each problem type) and parametrize the set of all non-overshooting stabilizing state-feedback (resp. output injection, output feedback) matrices.

Remark 4.2: Since a matrix A is strongly stable if and only if A' is strongly stable, the problem of non-overshooting stabilization by state feedback is *dual* to non-overshooting stabilization by output injection. □

It is shown in [KHP2] that strong stability is essentially equivalent to asymptotic stability along with a "small" degree of eigen-frame skewness. Thus the problem of non-overshooting stabilization is in principle related to the problem of robust eigen-structure assignment, i.e. assigning the eigenvalues of the closed loop matrix in the stable region of the complex plane together with the selection of an eigenvector matrix whose distance from orthogonality is minimal. A number of methods have been proposed in the literature for achieving this objective, e.g. minimization of the eigenvector's matrix condition number [AD], [Su]. Although the two problems are intimately related, in the next section we follow a direct approach for achieving strong stabilization which is independent of all techniques related to the robust eigen-structure assignment problem.

5. Non-overshooting Stabilization: Geometric conditions

In this section we consider the general non-overshooting stabilization problem under state or output feedback via convex optimization. Using a concrete representation of positive semi-definite matrices in terms of convex cones [All] we give a geometric interpretation of the problem in terms of "conic sections"; in particular it is shown that the problem is solvable if and only if the intersection of an affine hyperplane with the interior of a convex cone is non-empty. The technique leads to formulation of the non-overshooting stabilization problem as a convex feasibility problem which can be efficiently solved via Linear Matric Inequalities or alternative convex programming techniques [All2], [O], [SW], [SIG]. These are further developed in sections 6 and 7 to derive easily verifiable necessary and sufficient conditions for the solvability of the general state and output feedback non-overshooting stabilization problem and to derive a closed-form parametrization of the solution sets in each case.

We start by giving a geometric interpretation to the problem. This is based on the characterization of the cone of positive semi-definite matrices [All] summarized in section 2.

Theorem 5.1: There exists an output feedback matrix F such that $A+BFC$ is strongly stable if and only if

$$[\alpha+\mathcal{R}(D)]\cap \text{int cone}[\Omega_{\mathcal{S}}]\neq\emptyset \tag{3}$$

where $\alpha=-\text{vec}(A+A')$, $D=-(C'\otimes B)-(B\otimes C')P(m,p)$ and $P(m,p)$ is defined in Lemma 2.2.

Proof: The closed loop matrix $A_c=A+BFC$ is strongly stable if and only if:

$$A_c+A_c'<0\Leftrightarrow A+BFC+(A+BDC)'<0\Leftrightarrow -A-A'-BFC-C'F'B'<0$$

for some $F\in\mathcal{R}^{m\times p}$. Taking Kronecker products, this is equivalent to

$$[-\text{vec}(A+A')-(C'\otimes B)\text{vec}(F)-(B\otimes C')\text{vec}(F')]\in\text{vec}(\mathcal{S}_+^n)$$

for some $F\in\mathcal{R}^{m\times p}$, and using Lemma 2.1, this leads to:

$$[-\text{vec}(A+A')-(C'\otimes B)\text{vec}(F)-(B\otimes C')\text{vec}(F')]\in\text{int cone}(\Omega_{\mathcal{S}})$$

Now, let $P(m,p)$ be the unique permutation matrix such that $\text{vec}(F')=P(m,p)\text{vec}(F)$ (see Lemma 2.2). Using the definitions of vector $\alpha\in\mathcal{R}^{n^2}$ and matrix $D\in\mathcal{R}^{m\times n}$ we derive the stated equivalent condition, by noting that since F varies freely over $\mathcal{R}^{m\times p}$, $\text{vec}(F)$ varies freely over $\mathcal{R}^{mp}$. □

Remark 5.1: $\mathcal{R}(D)$ is a subspace in $\mathcal{R}^{n^2}$ and α is a fixed vector in $\mathcal{R}^{n^2}$. Hence the above Theorem states that the non-overshooting stabilization problem has a solution if and only if an affine hyperplane in $\mathcal{R}^{n^2}$ has a nonempty intersection with the interior of a convex cone. Thus, provided the intersection is non-empty, all solutions are geometrically described in terms of "conic sections" in n^2-dimensional space. □

An equivalent result to Theorem 5.1 using the $\overline{\text{vec}}(\cdot)$ operation is given next. This effectively reduces space-dimensionality (from n^2 to r), by taking into account the symmetry constraints of the problem.

Theorem 5.2: There exists an output feedback matrix F such that $A+BFC$ is strongly stable if and only if: $[\hat{\alpha}+\mathcal{R}(\hat{D})]\cap\text{int cone}[W_{\mathcal{S}}'\Omega_{\mathcal{S}}]\neq\emptyset$ where $\hat{\alpha}=-\overline{\text{vec}}(A+A')$ and

$$\hat{D}=\begin{pmatrix}\overline{\text{vec}}(D_{11}) & \overline{\text{vec}}(D_{21}) & \dots & \overline{\text{vec}}(D_{m1}) & \dots & \overline{\text{vec}}(D_{p1}) & \dots & \overline{\text{vec}}(D_{pm})\end{pmatrix}\in\mathcal{R}^{r\times pm}$$

where $r=n(n+1)/2$, the E_{ij} are defined in Lemma 2.2 and

$$D_{ij}=\left[-BE_{i1}E_{j1}'C-C'E_{j1}E_{i1}'B'\right].$$

Proof: Solvability of the strong stabilization problem by output feedback matrix F is equivalent to: $-(A+A')+\sum_{i=1}^m\sum_{j=1}^p f_{ij}\left[-BE_{i1}E_{j1}'C-C'E_{j1}E_{i1}'B'\right]>0$, or equivalently, $[-(A+A')+\sum_{i=1}^m\sum_{j=1}^p f_{ij}D_{ij}]>0$. Vectorizing, this leads to: $[\hat{\alpha}+\sum_{i=1}^m\sum_{j=1}^p f_{ij}\overline{\text{vec}}(D_{ij})]\in\text{int cone}[W_{\mathcal{S}}'\Omega_{\mathcal{S}}]$ using Lemma 2.1(ii) and the definition of the D_{ij}'s, or: $[\hat{\alpha}+\hat{D}\text{vec}(F)]\in\text{int cone}[W_{\mathcal{S}}'\Omega_{\mathcal{S}}]$. The result follows again on noting that $\mathcal{R}(\hat{D})=\{\hat{D}\text{vec}(F):F\in\mathcal{R}^{m\times p}\}$. □

In conclusion, the existence of a strongly stabilizing output feedback F can be expressed as follows:

Feasibility Linear Matrix Inequality (LMI) problem: This is defined as:

$$\text{Find } f_{ij} \text{ such that: } D_0+\sum_{i=1}^m\sum_{j=1}^p f_{ij}D_{ij}>0 \tag{4}$$

where $D_0 = -(A + A')$ and $D_{ij} = -BE_{i1}E'_{j1}C - C'E_{j1}E'_{i1}B'$. The feasibility problem is equivalent to:

$$\text{Find } f_{ij} \text{ such that: } \lambda_{\min}\left[D_0 + \sum_{i=1}^{m}\sum_{j=1}^{p} f_{ij}D_{ij}\right] > 0$$

The numerical verification of the above relations involves the maximization of the smallest eigenvalue of a linear combination of symmetric matrices [All2], [O] and is a standard convex feasibility programme. For example, if we wish to minimize $\|F\|$ subject to non-overshooting stabilization we can enforce the additional constraint:

$$\gamma^2 I - FF' \geq 0 \Leftrightarrow \begin{pmatrix} \gamma I_m & F \\ F' & \gamma I_p \end{pmatrix} \geq 0 \tag{5}$$

and solve the optimization problem: $\inf \gamma$ subject to the constrains (4) and (5), which is a standard LMI problem with variables $\gamma = \|F\|$ and $\{f_{ij}\}$. Additional LMI constraints can be added to enforce other design objectives, e.g. pole placement in a convex region of the open left half plane, special block-structure of F for decentralized problems, etc. Additional techniques for solving non-overshooting stabilization problems are described in sections 6 and 7 of the paper.

6. Non-overshooting stabilization: Convexity and LMI conditions

In this section we analyze the problem of non-overshooting stabilisation under state feedback, output injection and output feedback, using techniques based on convex programming and the theory of Linear Matrix Inequalities (LMI's). First, a number of links between non-overshooting and asymptotic stabilizability are established. The use of Finsler's lemma [SIG] and Schur-type arguments allows the development of solution of the non-overshooting stabilization problem, in the form of easily verifiable necessary and sufficient conditions and provide a complete parametrization of all strongly stabilizing solutions of the state feedback, output injection or output feedback type, respectively. A number of the results presented in this section are based on the theory of Linear Matrix Inequalities [SIG], [SW], which are reproduced here (with minor adaptations) for continuity of the main arguments.

Before stating the results of this section we make the following remarks:

Remark 6.1: Let (A, B, C) be a state-space realization of a dynamic LTI system. In this section we are concerned with the problem of non-overshooting stabilization of this system under state feedback, output feedback and output injection. It will be assumed throughout the section (and the next) that B has full (column) rank and that C has full (row) rank. These assumptions are standard for well-formed systems, and although not strictly necessary for our purposes, simplify the presentation considerably.

Remark 6.2: In the later part of the section we refer to the left and right annihilators, respectively, of the matrices B and C, corresponding to the input and output system matrices, respectively. If $B \in \mathcal{R}^{n\times m}$ (with $m \leq n$ - see Remark 6.1), we define its left annihilator $B^{\perp} \in \mathcal{R}^{n_l \times n}$ as *any* matrix with linearly independent rows such that $B^{\perp}B = 0$, where n_l is the dimension of the left null-space of B (so that $n_l = n - \rho$ with ρ the rank of B). Similarly, we define the right annihilator of $C \in \mathcal{R}^{p\times n}$ to be *any* matrix $C^{\perp} \in \mathcal{R}^{n\times n_r}$ with linear independent columns such that $CC^{\perp} = 0$. Here $n_r = n - \rho$ is the dimension of the right null space of C and ρ is the rank of C. In view of Remark 6.1 above, left and right annihilators of B and C, respectively, always exist unless $n = m$ or $n = p$. In such a case, we will assume that any statement involving the corresponding annihilator is vacuously satisfied.

Remark 6.3: In some of the proofs in this section, we refer to a "Schur-type argument". This is an argument based on the following well-known result (see, e.g. [HJ]): Let $A = A' \in \mathcal{R}^{n\times n}$, partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}' & A_{22} \end{pmatrix}$$

with A_{11} square. Then $A > 0$ if and only if $A_{22} > 0$ and $A_{11} - A_{12}A_{22}^{-1}A_{12}' > 0$. The last term will be referred to as a "Schur complement" of A. Variants of this result using permutations of the rows and columns of A will also be used.

Some preliminary results for non-overshooting stabilizability by state feedback are established next:

Lemma 6.1: A necessary condition for (A, B) to be non-overshooting stabilizable via state-feedback is that the pair (A, B) is stabilizable.

Proof: This is based on the fact that strong stability implies asymptotic stability (see [KHP2]). If (A, B) is not a stabilizable pair, then $A + BF$ has an eigenvalue in the closed right-half of the complex plane for every state feedback matrix $F \in \mathcal{R}^{m\times n}$. Thus $A + BF$ is not asymptotically stable for any $F \in \mathcal{R}^{p\times n}$ and thus (A, B) is not non-overshooting stabilizable by state-feedback. $\square$

Lemma 6.2: A necessary condition for (A, B) to be non-overshooting stabilizable by state feedback is that the pair $(A + A', B)$ is stabilizable.

Proof: Suppose that $(A + A', B)$ is not stabilizable; then $(A + A', B)$ has an uncontrollable mode $s_0 \geq 0$. Thus, $A + A' + BF$ has an eigenvalue $s_0 \geq 0$ for every $F \in \mathcal{R}^{m\times n}$ and hence $\frac{1}{2}(A + A') + \frac{1}{2}BF$ has an non-negative eigenvalue $\frac{s_0}{2} \geq 0$ for every $F \in \mathcal{R}^{m\times n}$. Defining $\hat{F} = \frac{1}{2}F$, this implies that $\frac{1}{2}(A + A') + B\hat{F}$ has a non-negative eigenvalue $\frac{s_0}{2}$ for every $\hat{F} \in \mathcal{R}^{m\times n}$ and thus

$$\left[\frac{1}{2}(A + A') + B\hat{F}\right] + \left[\frac{1}{2}(A + A') + B\hat{F}\right]' = (A + B\hat{F}) + (A + B\hat{F})'$$

is not negative definite for any $F \in \mathcal{R}^{m\times n}$, i.e. (A, B) is not non-overshooting stabilizable. $\square$

Remark 6.4: It can be shown by straightforward dual arguments that detectability of (A, C) and $(A + A', C)$ are (independently) necessary conditions for non-overshooting stabilizability of (A, C) under output injection. $\square$

We next investigate the effect of uncontrollable (unobservable) modes of the pair (A, B) ((A, C)) on non-overshooting stabilizability by state-feedback (output injection). We first state the following necessary and sufficient condition for non-overshooting stabilizability.

Lemma 6.3: Non-overshooting stabilizability under output feedback is invariant under orthogonal state-space transformations, i.e. for each orthogonal matrix U, (A, B, C) is non-overshooting stabilizable by output feedback if and only if $(U'AU, U'B, CU)$ is non-overshooting stabilizable by output feedback.

Proof: This follows from the fact that strong stability is invariant under orthogonal transformations [KHP2]. Suppose (A, B, C) is non-overshooting stabilizable by output feedback. Then there exists a matrix F such that $A+BFC$ is strongly stable. Hence for every orthogonal matrix U, $U'(A+BFC)U$ is strongly stable and hence $(U'AU, U'B, CU)$ is strongly stabilizable by output feedback. The reverse implication is immediate. □

Remark 6.5: The Lemma above implies that (A, B) is non-overshooting stabilizable by state feedback if and only if, for each orthogonal matrix U, $(U'AU, U'B)$ is non-overshooting stabilizable by state feedback. Dually, (A, C) is non-overshooting stabilizable by output injection if and only if $(U'AU, CU)$ is non-overshooting stabilizable by output injection. □

Proposition 6.1: Given a pair (A, B), the following properties hold true:

(i) There exists an orthogonal transformation V such that:

$$V'AV = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad V'B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \tag{6}$$

with (A_{11}, B_1) controllable.

(ii) A necessary condition for (A, B) to be non-overshooting stabilizable under state feedback is that: (i) A_{22} is strongly stable and (ii) (A_{11}, B_1) is non-overshooting stabilizable by state feedback.

(iii) If A_{22} is strongly stable then the existence of matrices F_1 and F_2 such that

$$A_{11} + A'_{11} + B_1F_1 + F'_1B'_1 - (A_{12} + B_1F_2)(A_{22} + A'_{22})^{-1}(A_{12} + B_1F_2)' < 0 \tag{7}$$

is necessary and sufficient for the non-overshooting stabilizability of the pair (A, B) by state feedback.

Proof:

(i) The indicated realization is Kalman's decomposition into the controllable and uncontrollable parts of the system. It is well known [B], [HP] that V can be chosen orthogonal.

(ii) Since non-overshooting stabilisation is invariant under orthogonal transformations (see Lemma 6.3 and Remark 6.5), it follows that (A, B) is non-overshooting stabilizable if and only if the pair

$$\left(\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right) \tag{8}$$

is non-overshooting stabilizable, i.e. if and only if there exists a matrix $F = (F_1 \;\; F_2)$ such that:

$$A_c + A'_c = \begin{pmatrix} A_{11} + B_1F_1 + A'_{11} + F'_1B'_1 & A_{12} + B_1F_2 \\ A'_{12} + F'_2B_1 & A_{22} + A'_{22} \end{pmatrix} < 0 \tag{9}$$

Thus, a necessary condition for $A_c + A'_c < 0$ is that (A_{11}, B_1) is non-overshooting stabilizable and A_{22} is strongly stable.

(iii) A Schur argument (see Remark 6.3) establishes the necessary and sufficient conditions given by (iii).

$\square$

Note that the necessary conditions for non-overshooting stabilizability (under state) feedback of the pair (A, B) given in Proposition 6.1 part (iii) above are not sufficient in general. Necessary and sufficient conditions for non-overshooting stabilizability are considered next.

Remark 6.6: Alternative necessary conditions for non-overshooting stabilizability under state feedback can be obtained as follows: Let $B_1^\perp$ be a left annihilator of B_1. Multiplying from left and right equation (7) by $B_1^\perp$ and $(B_1^\perp)'$, respectively, shows that an alternative set of necessary conditions for strong stabilizability of (A, B) by state feedback are that: (i) A_{22} is strongly stable, and (ii) $B_1^\perp[A_{11}+A_{11}'-A_{12}(A_{22}+A_{22}')^{-1}A_{12}'](B_1^\perp)' < 0$. It is shown later in the section (Theorem 6.3) that this set of conditions is actually both sufficient and necessary for the non-overshooting stabilizability of (A, B). $\square$

Remark 6.7: Left and right annihilators may be easily constructed, e.g. via the singular value decomposition: Let $B \in \mathcal{R}^{n\times m}$ be full column rank (see Remark 6.2) so that $\text{Rank}(B) = m \leq n$. Then B has a singular value decomposition:

$$B = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma \\ 0_{n-m,m} \end{pmatrix} V' \tag{10}$$

where $\Sigma = \text{diag}(\Sigma) \in \mathcal{R}^{m\times m}$ is positive definite and matrices $[U_1\ U_2]$ and V are orthogonal. Then, all left annihilators of B are given as $B^\perp = \Theta U_2'$ where Θ is an arbitrary $(n-m)\times(n-m)$ non-singular matrix. A dual construction may be followed to generate all right annihilators of matrix C. $\square$

Non-overshooting stabilizability of a pair (A, C) by output injection follows by duality:

Proposition 6.2: Given a pair (A, C), the following properties hold true:

(i) There exists an orthogonal transformation V such that:

$$V'AV = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad V'B = \begin{pmatrix} C_1 & 0 \end{pmatrix} \tag{11}$$

with (A_{11}, C_1) observable.

(ii) A necessary condition for (A, C) to be non-overshooting stabilizable by output injection is that: (i) A_{22} is strongly stable and (ii) (A_{11}, C_1) is non-overshooting stabilizable by output injection.

(iii) If A_{22} is strongly stable then the existence of matrices H_1 and H_2 such that

$$A_{11} + A_{11}' + H_1C_1 + C_1'H_1' - (A_{21} + H_2C_1)'(A_{22} + A_{22}')^{-1}(A_{21} + H_2C_1) < 0 \tag{12}$$

is necessary and sufficient for the non-overshooting stabilizability of (A, C) by output injection.

Proof: Dual to the proof of Proposition 6.1. $\square$

Remark 6.8: Let $C_1^\perp$ be a right annihilator of C_1. Then multiplying from left and right condition (12) by $(C_1^\perp)'$ and $C_1^\perp$, respectively, shows immediately that an alternative set of necessary conditions for the non-overshooting stabilizability of (A, C) by output injection is that: (i) A_{22} is strongly stable, and (ii) $(C_1^\perp)'[A_{11} + A_{11}' - A_{21}'(A_{22} + A_{22}')^{-1}A_{21}]C_1^\perp < 0$. It is shown in the sequel (Theorem 6.3) that this set of conditions is actually both sufficient and necessary. □

A necessary and sufficient condition for the solution of the non-overshooting stabilization problem (under state-feedback) is given in Theorem 6.1 below. This follows from a standard result in Linear Algebra and the theory of Linear Matrix Inequalities [SIG]. Before stating this theorem, we give two standard preliminary results.

Lemma 6.4 [SIG],[SW]: There exists a symmetric matrix X such that

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}' & P_{22} + X & P_{23} \\ P_{13}' & P_{23}' & P_{33} \end{pmatrix} < 0 \tag{13}$$

if and only if

$$\begin{pmatrix} P_{11} & P_{13} \\ P_{13}' & P_{33} \end{pmatrix} < 0 \tag{14}$$

Lemma 6.5: (Projection Lemma [SW]). Let P be a symmetric matrix partitioned in three block rows and columns and consider the Linear Matrix Inequality (LMI):

$$\begin{pmatrix} P_{11} & P_{12} + X' & P_{13} \\ P_{21} + X & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} < 0 \tag{15}$$

Then, there exists a matrix X satisfying this LMI if and only if:

$$\begin{pmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{pmatrix} < 0 \quad \text{and} \quad \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix} < 0 \tag{16}$$

In this case, one particular solution of the LMI is $X = P_{32}' P_{33}^{-1} P_{31} - P_{21}$.

The results below give necessary and sufficient conditions for non-overshooting stabilization by output feedback and (dually) by output injection. First, we consider the necessary and sufficient conditions of non-overshooting stabilization under state feedback.

Theorem 6.1: Let $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$ be two given matrices with Rank$(B) = m$. Then, the following two statements are equivalent:

(i) There exists a matrix F such that:

$$A + A' + BF + F'B' < 0 \tag{17}$$

(ii) $B^\perp(A + A')(B^\perp)' < 0$, where $B^\perp$ is any left annihilator of B.

Proof: The Theorem is a special case of a more general result which is fully proved below - see Theorem 6.2. □

The more general result involving non-overshooting stabilization via output feedback follows:

Theorem 6.2: Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{p\times n}$ be given matrices with Rank$(B) = m$ and Rank$(C) = p$. Then, the following two statements are equivalent:

1. There exists a matrix F such that:

$$A + A' + BFC + C'F'B' < 0 \tag{18}$$

2. The following two conditions hold:

   (i) $B^{\perp}(A + A')(B^{\perp})' < 0$.

   (ii) $(C')^{\perp}(A + A')((C')^{\perp})' < 0$.

   where $B^{\perp}$ is any left annihilator of B and $C^{\perp}$ is any right annihilator of C.

Proof: The Theorem is a generalization of Theorem 6.1 and its proof is adapted from a parallel result in [SIG]: Let $B^{\perp}$ and $C^{\perp}$ be left and right annihilators of B and C, respectively. Multiplying equation (18) by $B^{\perp}$ from the left and by $(B^{\perp})'$ from the right gives the first necessary condition. Similarly, multiplying by (18) by $(C^{\perp})'$ from the left and $C^{\perp}$ from the right gives the second necessary condition.

For proving the reverse implication let $S = (S_1\ S_2\ S_3\ S_4)$ be a nonsingular matrix such that the columns of S_3 span $\mathcal{N}_l(B) \cap \mathcal{N}_r(C)$, the columns of $(S_1\ S_3)$ span $\mathcal{N}_l(B)$ and the columns of $(S_2\ S_3)$ span $\mathcal{N}_r(C)$. Instead of (18) consider the equivalent inequality:

$$S'(A + A')S + (S'B)F(CS) + (CS)'F'(S'B)' < 0 \tag{19}$$

It will be shown that provided the two conditions given in part 2 of the Theorem hold, the above LMI is satisfied for some matrix F. Note that $S'B$ and CS have the structure

$$S'B = \begin{pmatrix} S_1' \\ S_2' \\ S_3' \\ S_4' \end{pmatrix} B = \begin{pmatrix} 0 \\ B_2 \\ 0 \\ B_4 \end{pmatrix} \quad \text{and} \quad CS = C\begin{pmatrix} S_1 & S_2 & S_3 & S4 \end{pmatrix} = \begin{pmatrix} C_1 & 0 & 0 & C_4 \end{pmatrix} \tag{20}$$

where $(B_2'\ B_4')$ and $(C_1\ C_4)$ have full column rank. Thus $(S'B)F(CS)$ has the structure

$$(S'B)F(CS) = \begin{pmatrix} 0 \\ B_2 \\ 0 \\ B_4 \end{pmatrix} F \begin{pmatrix} C_1 & 0 & 0 & C_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Z_{21} & 0 & 0 & Z_{24} \\ 0 & 0 & 0 & 0 \\ Z_{41} & 0 & 0 & Z_{44} \end{pmatrix} \tag{21}$$

Note that the rank properties of $(B_2'\ B_4')$ and $(C_1\ C_4)$ imply that the map

$$F \in \mathcal{R}^{m\times p} \to \begin{pmatrix} B_2 \\ B_4 \end{pmatrix} F \begin{pmatrix} C_1 & C_4 \end{pmatrix} \in \mathcal{R}^{\text{Rank}(B)\times\text{Rank}(C)} \tag{22}$$

is surjective (onto), since given any

$$Z = \begin{pmatrix} Z_{21} & Z_{24} \\ Z_{41} & Z_{44} \end{pmatrix} \in \mathcal{R}^{\text{Rank}(B)\times\text{Rank}(C)}$$

we have

$$\begin{pmatrix} B_2 \\ B_4 \end{pmatrix} (B^r Z C^l) \begin{pmatrix} C_1 & C_4 \end{pmatrix} = Z$$

where B^r and C^l denote two arbitrary right and left inverses of $(B_2' \; B_4')'$ and $(C_1 \; C_4)$, respectively. Thus as F varies over $\mathcal{R}^{m \times p}$, Z_{21}, Z_{24}, Z_{41} and Z_{44} are *arbitrary* matrices. In the new coordinate system the LMI given in equation (19) can be written as:

$$\begin{pmatrix} Q_{11} & Q_{12} + Z_{21}' & Q_{13} & Q_{14} + Z_{41}' \\ Q_{21} + Z_{21} & Q_{22} & Q_{23} & Q_{24} + Z_{24}' \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} + Z_{41} & Q_{32} + Z_{24}' & Q_{43} & Q_{44} + Z_{44} + Z_{44}' \end{pmatrix} < 0 \tag{23}$$

where $Q_{ij} = S_i'(A+A')S_j = Q_{ji}'$. Thus, we need to show that this LMI is satisfied for a suitable choice of Z_{21}, Z_{24}, Z_{41} and Z_{44} (which can be chosen freely). On noting that

$$(SB)^{\perp} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \quad \text{and} \quad (CS)^{\perp} = \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \tag{24}$$

are full rank left and right annihilators of SB and CS, respectively, the two conditions given in part 2 of the Theorem can be written as:

$$\begin{pmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q_{33} \end{pmatrix} < 0 \quad \text{and} \quad \begin{pmatrix} Q_{22} & Q_{23} \\ Q_{32} & Q_{33} \end{pmatrix} < 0 \tag{25}$$

Thus, by Lemma 6.5 ("Projection Lemma") we can find Z_{21} such that the sub-matrix of (23) consisting of the first three block rows and columns is negative definite. Fix Z_{41} and Z_{24} to arbitrary matrices (of appropriate dimensions). With Z_{21}, Z_{41} and Z_{24} fixed, Z_{44} can be determined according to Lemma 6.4 so that the left hand side of the inequality in (23) is negative definite. $\square$

Using the Theorem above, the following Corollary readily follows:

Corollary 6.1: The linear system $\mathcal{S}(A, B, C)$ with B full column rank and C full row rank is:

(i) Non-overshooting stabilizable by output feedback if and only if conditions (i) and (ii) of Theorem 6.2 part 2 hold.

(ii) Non-overshooting stabilizable by state-feedback if and only if $B^{\perp}(A + A')(B^{\perp})' < 0$.

(iii) Non-overshooting stabilizable by output injection if and only if $(C')^{\perp}(A + A')((C')^{\perp})' < 0$.

Proof: Part (i) follows immediately from Theorem 6.2 above. Parts (ii) and (iii) follow from part (i) by setting $C = I_n$ and $B = I_n$, respectively. $\square$

Corollary 6.2: The system (A, B, C) is non-overshooting stabilizable by output feedback if and only if the following conditions hold true:

(i) (A, B) is non-overshooting stabilizable by state feedback.

(ii) (A, C) is non-overshooting stabilizable by output injection.

Proof: Sufficiency is immediate. Necessity also follows immediately from Corollary 6.1. □

We can also state the following necessary and sufficient conditions for non-overshooting stabilizability under either state-feedback or output injection:

Theorem 6.3: (i) Given a pair (A, B), there exists an orthogonal transformation V such that:

$$V'AV = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad V'B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

with (A_{11}, B_1) controllable. Then, if B has full column rank, a set of necessary and sufficient conditions for (A, B) to be non-overshooting stabilizable under state feedback is that: (a) A_{22} is strongly stable, and (b)

$$B_1^{\perp}[A_{11} + A_{11}' - A_{12}(A_{22} + A_{22}')^{-1}A_{12}'](B_1^{\perp})' < 0 \tag{26}$$

where $B_1^{\perp}$ is any left annihilator of B_1. (ii) Given a pair (A, C), there exists an orthogonal transformation V such that:

$$V'AV = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad V'B = \begin{pmatrix} C_1 & 0 \end{pmatrix}$$

with (A_{11}, C_1) observable. Then, if C has full row rank, a set of necessary and sufficient conditions for (A, C) to be non-overshooting stabilizable under output injection is that: (a) A_{22} is strongly stable and, (b)

$$(C_1^{\perp})'[A_{11} + A_{11}' - A_{21}'(A_{22} + A_{22}')^{-1}A_{21}]C_1^{\perp} < 0 \tag{27}$$

where $C_1^{\perp}$ is any right annihilator of C_1.

Proof: Part (i) is proved only; part (ii) follows by duality. First note that since the non-overshooting stabilizability property is invariant under orthogonal state-space transformations, the pair (A, B) is non-overshooting stabilizable if and only if

$$\left(\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right)$$

is non-overshooting stabilizable. Let $B_1^{\perp}$ be any left annihilator of B_1; then $\text{diag}(B_1^{\perp}, I)$ is a left annihilator of $(B_1' \ 0)'$ and hence according to Theorem 6.2, (A, B) is non-overshooting stabilizable if and only if

$$\begin{pmatrix} B_1^{\perp}(A_{11} + A_{11}')(B_1^{\perp})' & B_1^{\perp}A_{12} \\ A_{12}'(B_1^{\perp})' & A_{22} + A_{22}' \end{pmatrix} < 0$$

Using a Schur argument (see Remark 6.3), the last LMI is satisfied if and only if: (a) $A_{22} + A_{22}' < 0$ (i.e. A_{22} is strongly stable) and, (b) $B_1^{\perp}[A_{11} + A_{11}' - A_{12}(A_{22} + A_{22}')^{-1}A_{12}'](B_1^{\perp})' < 0$ as required. □

Remark 6.9: Proposition 6.1 states that conditions, (a) A_{22} strongly stable, and (b) (A_{11}, B_1) non-overshooting stabilizable (by state feedback) are together necessary for non-overshooting stabilizability of (A, B) (by state feedback). In the light of Corollary 6.1, condition (ii) is equivalent to $B_1^{\perp}(A_{11} + A_{11}')(B_1^{\perp})' < 0$. Clearly, this is a weaker condition, in general, than the condition $B_1^{\perp}[A_{11} + A_{11}' - A_{12}(A_{22} + A_{22}')^{-1}A_{12}'](B_1^{\perp})' < 0$ given in Theorem 6.3 above, which along with condition (a), is both necessary and sufficient for the non-overshooting stabilizability of (A, B). □

The following is also an immediate consequence of Theorem 6.2:

Corollary 6.3: The following properties hold true:

(i) Suppose that $\mathcal{R}(B) \subseteq \mathcal{R}(C')$. Then the system (A, B, C) is non-overshooting stabilizable by output feedback if and only if (A, B) is non-overshooting stabilizable by state feedback.

(ii) Let $\mathcal{R}(C') \subseteq \mathcal{R}(B)$. Then (A, B, C) is non-overshooting stabilizable by output feedback if and only if (A, C) is non-overshooting stabilizable by output injection.

Proof: (i) If $\mathcal{R}(B) \subseteq \mathcal{R}(C')$, then $\mathcal{N}_r(C) = \mathcal{N}_l(C') \subseteq \mathcal{N}_l(B)$ and condition 2(ii) of Theorem 6.2 is redundant. Thus output-feedback stabilizability is equivalent to state-feedback stabilizability in this case. Part (ii) follows similarly. □

7. Parametrization of all non-overshooting stabilizing solutions

We now assume that the necessary and sufficient conditions for non-overshooting state-feedback stabilizability (resp. non-overshooting output-injection stabilizability, non-overshooting output-feedback stabilizability) are satisfied and give a parametrization of all non-overshooting stabilizing state feedback (resp. output-injection, output feedback) matrices. In general, two separate techniques are reported in the literature for parametrizing the solutions of LMI's with simple structure. The first approach proceeds via Finsler's theorem [SIG] (see next) and the second via Parrot's theorem [LGA]. Here we follow the first approach. We first need two preliminary results related to LMI's.

Lemma 7.1 [SIG]: Suppose that P_{11} and P_{22} are two symmetric positive-definite matrices. Then, there exists a matrix X such that

$$\begin{pmatrix} -P_{11} & X \\ X' & -P_{22} \end{pmatrix} < 0 \tag{28}$$

if and only if $\|P_{11}^{-1/2} X P_{22}^{-1/2}\| < 1$, where $\|\cdot\|$ denotes the largest singular value of a matrix. Further, all solutions of the matrix inequality (28) are parametrized as $X = P_{11}^{1/2} Z P_{22}^{1/2}$ where Z is an arbitrary matrix satisfying $\|Z\| < 1$ and $P^{1/2}$ denotes a symmetric square root of the symmetric positive-definite matrix P.

Theorem 7.1 (Finsler's Theorem [SIG]): Let matrices $B \in \mathcal{R}^{n \times m}$ and $Q \in \mathcal{R}^{n \times n}$ be given. Suppose further that $\text{Rank}(B) = m$ and $Q = Q'$. Then the following statements are equivalent.

(i) There exists a scalar μ such that

$$\mu B B' - Q > 0 \tag{29}$$

(ii) The following condition holds: $P := B^{\perp} Q (B^{\perp})' < 0$ where $B^{\perp}$ is any left annihilator of B.

If the above two statements hold, then all scalars μ satisfying $\mu BB' - Q > 0$ are given by:

$$\mu > \mu_{min} := \lambda_{max}[B_l^{\dagger}(Q - Q(B^{\perp})' P^{-1} B^{\perp} Q)(B_l^{\dagger})'] \tag{30}$$

Further, $\mu_{min} \leq 0$ if and only if $Q \leq 0$.

Proof: The proof is adapted from [SIG] using the simplifying assumption that B has full column rank. Let T be a square non-singular matrix:

$$T = \begin{pmatrix} B_l^\dagger \\ B^\perp \end{pmatrix}$$

By a congruence transformation with T, (29) is equivalent to:

$$\begin{pmatrix} \mu I - B_l^\dagger Q(B_l^\dagger)' & -B_l^\dagger Q(B^\perp)' \\ -B^\perp Q(B_l^\dagger)' & -B^\perp Q(B^\perp)' \end{pmatrix} > 0$$

or equivalently,

$$P := B^\perp Q(B^\perp)' < 0 \tag{31}$$

and

$$\mu I - B_l^\dagger Q(B_l^\dagger)' + B_l^\dagger Q(B^\perp)' P^{-1} B^\perp Q(B_l^\dagger)' > 0 \tag{32}$$

which proves the necessity of equation (29). Now, to prove sufficiency, suppose that $P < 0$. Clearly there exists a μ satisfying equation (32) and all such μ are given by equation (30). Finally to show that $\mu_{min} \leq 0$ if and only if $Q \leq 0$ note that $\mu_{min} \leq 0$ if and only if $B_l^\dagger(Q - Q(B^\perp)'P^{-1}B^\perp Q)(B_l^\dagger)' \leq 0$ which is equivalent to $TQT' \leq 0$ since $P < 0$. □

The following Theorem gives a complete parametrization of all non-overshooting stabilizing state-feedback matrices F. It is assumed that B has full-column rank. The assumption can be removed, if required at the expense of a significantly more complex parametrization.

Theorem 7.2: Suppose that (A, B) is non-overshooting stabizable by state-feedback and that B has full column rank. Then all F such that $A + A' + BF + F'B' < 0$ are given as:

$$F = -\rho B' + \sqrt{\rho} L \Omega^{1/2} \tag{33}$$

where, L is an arbitrary contraction (i.e. any matrix such that $\|L\| < 1$) and $\rho > 0$ is any scalar such that $\Omega_\rho = \rho BB' - A - A' > 0$; in particular all such ρ are given as $\rho > \max(\rho_{min}, 0)$ where:

$$\rho_{min} := \lambda_{max}\{B^\dagger[A + A' - (A + A')(B^\perp)'(B^\perp(A + A')(B^\perp)')^{-1}B^\perp(A + A')](B^\dagger)'\} \tag{34}$$

Proof: Since Theorem 7.2 is a special case of Theorem 7.4 below, we do not supply a separate proof. □

Two properties of the strongly stable "closed-loop" matrix $A_c = A + BF$ corresponding to a non-overshooting stabilizing state-feedback matrix F are given in the following Theorem. The first property makes the strong stability of A_c more transparent, while the second property shows that the symmetric part of all (strongly-stable) matrices A_c which arise from the parametrization of Theorem 7.2 is constant, when restricted to the range of $(B^\perp)'$ and projected onto the range of $B^\perp$.

Theorem 7.3 Suppose that (A, B) is non-overshooting stabilizable by state-feedback and consider any strongly stabilizing state-feedback matrix F as given in Theorem 7.2. Denote by A_c the "closed-loop" A-matrix of the system, i.e. $A_c = A + BF$. Then:

(i) $A_c + A_c' = -(\Omega_\rho^{1/2} - \sqrt{\rho}BL)(\Omega_\rho^{1/2} - \sqrt{\rho}BL)' - \rho B(I - LL')B' < 0$ for all $\rho > \rho_{min}$, and

(ii) $B^{\perp}(A_c + A_c')(B^{\perp})' = B^{\perp}(A + A')(B^{\perp})' < 0$ independent of $\rho > \rho_{min}$ and L.

Proof: To show (i) note that:

$$\begin{aligned} A_c + A_c' &= (A + BF) + (A + BF)' = -\rho BB' - \Omega_\rho + \sqrt{\rho}\Omega_\rho^{1/2} L'B' + \sqrt{\rho} BL\Omega_\rho^{1/2} \\ &= -(\Omega_\rho^{1/2} - \sqrt{\rho}BL)(\Omega_\rho^{1/2} - \sqrt{\rho}BL)' - \rho B(I - LL')B' < 0 \end{aligned}$$

since $\rho > 0$ and $\|L\| < 1$. For part (ii) note that

$$B^{\perp}(A_c + A_c')(B^{\perp})' = -B^{\perp}\Omega_\rho(B^{\perp})' = -B^{\perp}(\rho BB' - A - A')(B^{\perp})' = B^{\perp}(A + A')(B^{\perp})' < 0$$

□

Our next result illustrates the parametrization of all non-overshooting stabilizing state feedback matrices given in Theorem 7.2.

Proposition 7.1: Assume that (A, B) is non-overshooting stabilizable by state feedback. Let all variables be defined as in Theorem 7.2, and denote by

$$E_\rho = \{F \in \mathcal{R}^{m\times n} : \|(F + \rho B')(\rho\Omega_\rho)^{-1/2}\| < 1\}$$

the set of non-overshooting stabilizing state feedback matrices (for fixed ρ). Then $E_{\rho_1} \subseteq E_{\rho_2}$ whenever $\rho_2 \geq \rho_1 > \rho_{\min}$. Thus the set of all non-overshooting stabilizing state feedback matrices $\mathcal{F}_s \subseteq \mathcal{R}^{m\times n}$ is given as $\cup_{\rho>\rho_{\min}} E_\rho = \lim_{\rho\to\infty} E_\rho$.

Proof: First note that the set of all non-overshooting stabilizing state-feedback matrices can be expressed as

$$\begin{aligned} \mathcal{F}_s &= \{-\rho B' + \sqrt{\rho}L\Omega^{1/2} : \rho > \rho_{\min}, \|L\| < 1\} \\ &= \{F \in \mathcal{R}^{m\times n} : \rho > \rho_{\min}, \|(F + \rho B')(\rho\Omega_\rho)^{-1/2}\| < 1\} = \cup_{\rho>\rho_{\min}} E_\rho \end{aligned}$$

Take any pair (ρ_1, ρ_2) such that $\rho_2 \geq \rho_1 > \rho_{\min}$. Since (A, B) is assumed non-overshooting stabilizable and $\rho_2 \geq \rho_1 > \rho_{\min}$, E_{ρ_1} and E_{ρ_2} are non-empty. Let $F \in E_{\rho_1}$ be a non-overshooting stabilizing state feedback matrix. We need to show that $F \in E_{\rho_2}$. First note that:

$$\rho_1 > \rho_{\min} \Rightarrow \Omega_{\rho_1} > 0 \tag{35}$$

Further,

$$F \in E_{\rho_1} \Rightarrow \|(F + \rho_1 B')(\rho_1\Omega_{\rho_1})^{-1/2}\| < 1 \Rightarrow \rho_1 I - (F + \rho_1 B')\Omega_{\rho_1}^{-1}(F' + \rho_1 B) > 0 \tag{36}$$

Equations (35) and (36) taken together imply that

$$\begin{pmatrix} \rho_1 I & \rho_1 B' + F \\ \rho_1 B + F' & \Omega_{\rho_1} \end{pmatrix} > 0 \Rightarrow \rho_1\Omega_{\rho_1} - (\rho_1 B + F')(\rho_1 B' + F) > 0 \tag{37}$$

which may be written, using the definition of Ω_{ρ_1}, as:

$$A + A' + BF + F'B' + \rho_1^{-1}F'F < 0 \Rightarrow A + A' + BF + F'B' + \rho_2^{-1}F'F < 0 \tag{38}$$

since $\rho_1 \leq \rho_2$. Equivalently,

$$\rho_2^{-1}(\rho_2 B + F')(\rho_2 B' + F) < \rho_2 BB' - A - A' = \Omega_{\rho_2} \Leftrightarrow \begin{pmatrix} -\rho_2 I & \rho_2 B' + F \\ \rho_2 B + F' & -\Omega_{\rho_2} \end{pmatrix} < 0$$

From Lemma 7.1, F must be of the form: $F = -\rho_2 B' + \sqrt{\rho_2} L \Omega_{\rho_2}^{1/2}$ for some $\|L\| < 1$. Thus $\|(F + \rho_2 B')(\rho_2 \Omega_{\rho_2})^{-1/2}\| < 1$ and hence $F \in E_{\rho_2}$, so that $E_{\rho_1} \subseteq E_{\rho_2}$ as required. We also conclude that the set of all non-overshooting stabilizing state feedback matrices, $\mathcal{F}_s$, is equal to the limit set $\lim_{\rho \to \infty} E_\rho$. $\square$

Remark 7.1: In the single input case $m = 1$, $B := b$ is a column vector and $F := f'$ is a row vector. In this case, the regions E_ρ defined in Proposition 7.1 above can be written as

$$E_\rho = \{f \in \mathcal{R}^n : (f + \rho b)'(\rho\Omega_\rho)^{-1}(f + \rho b) \leq 1\}$$

Since $\rho\Omega_\rho > 0$ for every $\rho > \rho_{\min}$, each set E_ρ corresponds to the region inside an ellipsoid in $\mathcal{R}^n$. $\square$

The following example illustrates the parametrization given in Theorem 7.2 in the light of Proposition 7.1 and Remark 7.1.

Example 7.1: Consider the matrix $A = \text{diag}(1, -2)$ and let $B = \begin{pmatrix} 1 & 0 \end{pmatrix}'$. Then

$$B^\perp(A + A')(B^\perp)' = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -4 < 0$$

and hence A is non-overshooting stabilizable by Theorem 6.1. Let $F = (f_1 \ f_2)$. Since

$$A + BF + (A + BF)' = \begin{pmatrix} 2(1 + f_1) & f_2 \\ f_2 & -4 \end{pmatrix}$$

necessary and sufficient conditions for non-overshooting stabilizability are obtained as $f_1 < -\frac{1}{8}f_2^2 - 1$ and $f_1 < 1$. In view of the first condition, the second condition is clearly redundant. Thus, all non-overshooting stabilizing state feedback matrices are specified by the condition $f_1 < -\frac{1}{8}f_2^2 - 1$ which corresponds to the area below a parabola in the (f_2, f_1) space shown in Figure 2.

In the notation of Theorem 7.2,

$$\Omega_\rho = \rho BB' - A - A' = \begin{pmatrix} \rho - 2 & 0 \\ 0 & 4 \end{pmatrix} > 0$$

which satisfies equation (34) with $\rho_{\min} = 2$. All non-overshooting stabilizing matrices F are generated (see Theorem 7.2) as

$$F = \begin{pmatrix} -\rho + l_1\sqrt{\rho(\rho - 2)} & 2l_2\sqrt{\rho} \end{pmatrix}, \ \rho > \rho_{\min} \tag{39}$$

where l_1 and l_2 are any two real numbers such that $l_1^2 + l_2^2 < 1$. For each (fixed) $\rho > \rho_{\min}$, the contour described by equation (39) can be written as

$$E_\rho = \left\{(f_2, f_1) : \left(\frac{f_2}{2\sqrt{\rho}}\right)^2 + \left(\frac{f_1 + \rho}{\sqrt{\rho(\rho - 2)}}\right)^2 < 1\right\}$$

and corresponds to the interior of an ellipse in the (f_2, f_1) space. Figure 2 plots a number of these ellipses for 20 equally spaced ρ-values between $\rho = 2.1$ and $\rho = 10$. It can be seen, in agreement with Proposition 7.1, that each ellipse corresponding to a specific ρ value contains all ellipses corresponding to lower ρ-values and that as $\rho \to \infty$ the area inside the ellipse tends to the region below the parabola, effectively covering the entire non-overshooting stabilization region. □

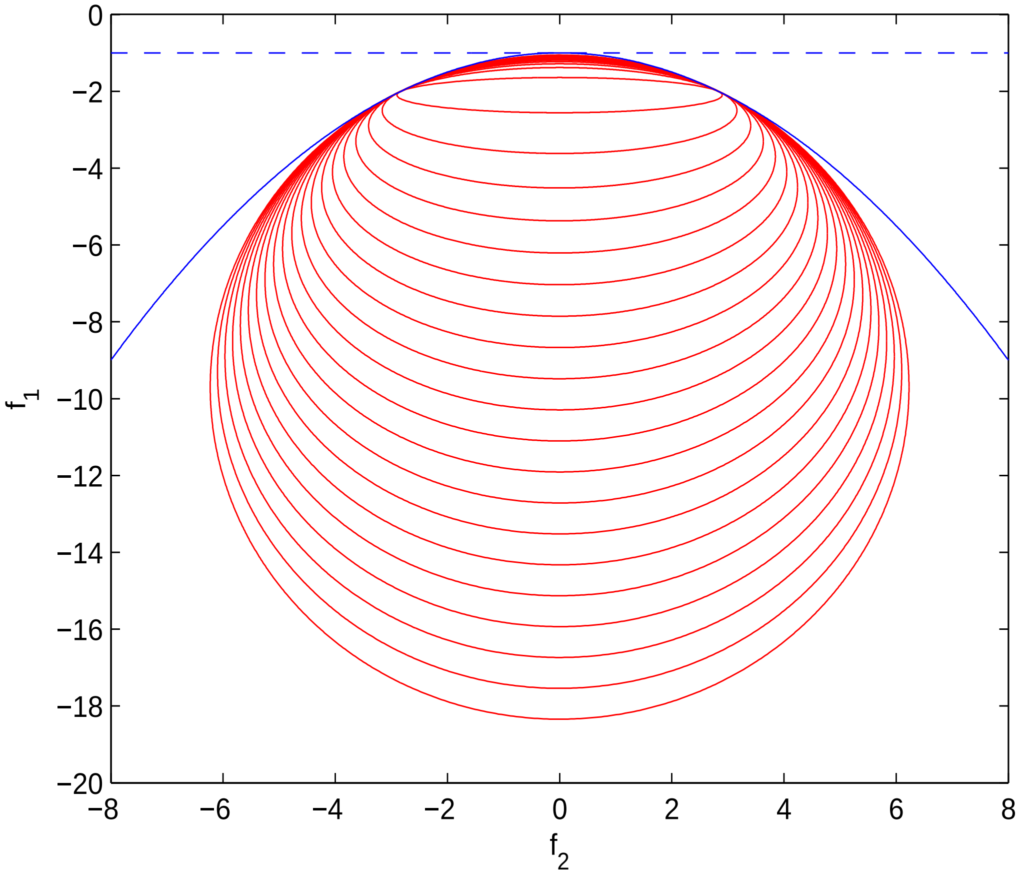

Figure 2: Non-overshooting stabilizing region given in Example 7.1.

A similar parametrization for non-overshooting stabilization by output injection follows by duality:

Theorem 7.4: Suppose that C has full row rank and that (A, C) is non-overshooting stabilizable by output injection. Then all H such that $A + A' + HC + C'H' < 0$ are given by:

$$H = -\varrho C' + \sqrt{\varrho}\Phi^{1/2} Z \tag{40}$$

where Z is an arbitrary matrix such that $\|Z\| < 1$ and ϱ is any positive scalar such that $\Phi = \varrho C'C - A - A' > 0$.

Proof: Follows immediately from Theorem 7.2 by identifying H with F' and C with B'. □

A parametrization of all non-overshooting stabilizing output feedback matrices is given next. This generalizes Theorem 7.2 and is again adapted from a result in [SIG].

Theorem 7.5: Suppose that (A, B, C) is non-overshooting stabizable by output-feedback and that matrices B and C have full column rank and full row rank, respectively. Then all matrices F such that $A + A' + BFC + C'F'B' < 0$ are given as:

$$F = -\rho B'\Phi C'(C\Phi C')^{-1} + \Psi^{1/2}L(C\Phi C')^{-1/2} \tag{41}$$

where, L is an arbitrary contraction (i.e. any matrix such that $\|L\| < 1$),

$$\Psi = \rho I - \rho^2 B'\Phi B + \rho^2 B'\Phi C'(C\Phi C')^{-1}C\Phi B$$

and $\rho > 0$ is any scalar such that $\Phi = (\rho BB' - A - A')^{-1} > 0$; in particular all such ρ are given as $\rho > \max(\rho_{min}, 0)$ where:

$$\rho_{min} := \lambda_{max}\{B^\dagger[A + A' - (A + A')(B^\perp)'(B^\perp(A + A')(B^\perp)')^{-1}B^\perp(A + A')](B^\dagger)'\} \tag{42}$$

Proof: First assume that F satisfies $A + A' + BFC + C'F'B' < 0$. We need to show that F can be written as $F = -\rho B'\Phi C'(C\Phi C')^{-1} + \Psi^{1/2}L(C\Phi C')^{-1/2}$ for some (ρ, Φ, Ψ, L) as defined above. First note that the existence of a $\rho > 0$ (as defined in equation (42)) such that $\Phi = \rho BB' - A - A' > 0$ follows from Finsler's theorem and the non-overshooting stabilizability of (A, B, C) (see Theorem 6.2). Now, since $A + A' + BFC + C'F'B' < 0$, there exists a positive scalar ρ (sufficiently large) such that:

$$A + A' + BFC + C'B'F' + \rho^{-1}C'F'FC < 0$$

or equivalently,

$$\rho^{-1}(\rho B + C'F')(\rho B' + FC) < \rho BB' - A - A' = \Phi^{-1}$$

or

$$(\rho B' + FC)\Phi(\rho B + C'F') < \rho I$$

This is further equivalent to:

$$(F + \rho B'\Phi C'\Phi_c^{-1})\Phi_c(F' + \rho\Phi_c^{-1}C\Phi B) < \rho I - \rho^2 B'\Phi B + \rho^2 B'\Phi C'(C\Phi C')^{-1}C\Phi B = \Psi$$

where, $\Phi_c = C\Phi C'$, and thus also to:

$$\begin{pmatrix} -\Psi & F + \rho B'\Phi C'\Phi_c^{-1} \\ F' + \rho\Phi_c^{-1}C\Phi B & -\Phi_c^{-1} \end{pmatrix} < 0$$

From Lemma 7.1 above, all F which satisfy the above LMI are given by:

$$F + \rho B'\Phi C'\Phi_c^{-1} = \Psi^{-1/2}L\Phi_c^{1/2} \Leftrightarrow F = -\rho B'\Phi C'(C\Phi C')^{-1} + \Psi^{1/2}L(C\Phi C')^{-1/2}$$

where, L is an arbitrary matrix such that $\|L\| < 1$) and $\Phi = (\rho BB' - A - A')^{-1} > 0$.

Conversely, it is shown that any F given by equation (33) satisfies $A + A' + BFC + C'F'B' < 0$. It suffices to show that:

$$A + A' + BFC + C'B'F' + \rho^{-1}C'F'FC < 0$$

or equivalently (for a sufficiently large ρ) that

$$\rho^{-1}(\rho B + C'F')(\rho B' + FC) < \rho BB' - A - A' = \Phi^{-1}$$

This is further equivalent to:

$$(\rho B' + FC)\Phi(\rho B + C'F') < \rho I$$

or, equivalently, to:

$$(F + \rho B'\Phi C'\Phi_c^{-1})\Phi_c(F' + \rho\Phi_c^{-1}C\Phi B) < \rho I - \rho^2 B'\Phi B + \rho^2 B'\Phi C'(C\Phi C')^{-1}C\Phi B = \Psi$$

for some $\rho > 0$. Using equation (41) shows that this is further equivalent to the condition:

$$(\Psi^{1/2}L(C\Phi C')^{-1/2}(C\Phi C')(C\Phi C')^{-1/2}L'\Psi^{1/2} < \Psi \Leftrightarrow \Psi^{1/2}LL'\Psi^{1/2} < \Psi \Leftrightarrow LL' < I$$

which is valid since $\|L\| < 1$. □

8. Conclusions

The paper extends our previous work [KHP], [KHP2] in the area of strong stability of internal system descriptions. This is a finer notion of stability compared to classical definitions (stability in the sense of Lyapunov, asymptotic stability) which is relevant for systems with physical variables that do not exhibit overshooting behavior in the phase-space for arbitrary initial conditions. It has been shown in [KHP2] that strong stability is intimately related to the skewness of the eigen-frame of the state matrix.

In this paper we address the problem of non-overshooting stabilization, i.e. designing a compensation scheme for which the closed-loop state matrix is strongly stable. It was shown that non-overshooting stabilization under dynamic and static feedback are, in a certain sense, equivalent problems. A number of results were developed for the static non-overshooting stabilization problem under state-feedback, output injection and output feedback, leading to the derivation of easily verifiable necessary and sufficient conditions and a complete parametrization of the non-overshooting stabilizing matrix feedback sets, in each case, using convex programming and LMI techniques. Geometric conditions were also derived, showing that the problem of non-overshooting stabilization is solvable if the intersection of an affine hyperplane with the interior of a convex cone is non-empty.

Our future work in this area will concentrate on the following topics: (a) Development of robust zero-overshoot stabilization methods, when the system state model is subjected to uncertainties; (b) Definition of metrics for characterizing the distance of a matrix from strong stability and their efficient computation; (c) Extension of the theory to the non-linear case; (d) Applications of strong stability in the area of switched systems; and, (e) Development of a novel methodology for achieving robust partial eigen-structure assignment using zero-overshooting stabilization ideas.

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