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# Exponential Smoothing Methods in Pension Funding

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“Smoothed-market” methods are used by actuaries, when they value pension plan assets, in order to dampen the volatility in contribution rates recommended to plan sponsors. A method involving exponential smoothing is considered. The dynamics of the pension funding process is investigated in the context of a simple model where asset gains and losses emerge as a result of random rates of investment return and where the gains and losses are spread. It is shown that smoothing market values up to a point does improve the stability of contributions but excessive smoothing is inefficient. It is also shown that consideration should be given to the combined effect of the asset valuation and gain and loss adjustment methods. Practical and efficient combinations of gain/loss spreading periods and asset value smoothing parameters are suggested.

*Keywords:* actuarial valuation; pension funding; asset value; smoothing.

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## 1 Introduction

Actuaries carry out regular valuations of defined benefit pension plans. One of their aims is to advise on suitable contribution rates in order that pensions are funded over time. They also carry out other types of valuations, for example for solvency or accounting purposes. A basic premise of funding pensions in advance is that contributions towards a pension can be systematically planned, spread out and invested in the capital markets over time. The main aim of a funding valuation is therefore to compare the assets and liabilities of a pension plan and to recommend contribution rates from a going-concern perspective. Actuarial funding methods seek to budget these contributions in an organised and stable manner over time. An implicit feature of these methods is that they incorporate devices for smoothing contribution rates. By contrast, no such smoothing is involved in accounting valuations (which aim to measure the economic cost to plan sponsors of pension provision) and in solvency valuations (which aim to establish that pensions are payable in the event of a wind-up of the pension plan). This paper is concerned with actuarial methods in the context of funding valuations only.

One of the ways in which contribution rates are smoothed is through the use of an actuarial value of pension plan assets. Actuarial asset valuation methods are described in a survey carried out by the Committee on Retirement Systems Research (1998). Permissible methods are also mentioned in the Standard of Practice for Valuation of Pension Plans of the Canadian Institute of Actuaries (1994), in Actuarial Standard of Practice No. 4 of the Actuarial Standards Board (1993) in the United States, and in relevant parts of the U.S. Employee Retirement Income Security Act (ERISA) and the U.S. Internal Revenue Service code (see McGill *et al.*, 1996, p. 678). It is important to note that, for funding purposes, an actuarial asset value is not an estimator of the fundamental worth of pension plan assets but is used to moderate volatility in the market values of these assets and thereby generate

a stable and smooth pattern of contribution rates: see Ezra (1979, p. 40), Anderson (1992, p. 108) and Winklevoss (1993, p. 171).

This paper investigates the commonly used “smoothed-market value” method, which calculates an average of market values of assets while allowing for the time value of money and for cash flows (section 3). To this end, a simple model of a defined benefit pension plan is described in section 2. The first and second moments of the contribution rate and other variables in the pension fund are derived, in section 4, when asset gains and losses emerge as a result of random rates of investment return. Finally, it is shown that excessive smoothing is counterproductive and efficient smoothing parameters are suggested (section 5). Proofs of all results can be found in Owadally and Haberman (2003).

## 2 A Simple Model of the Pension Fund

The smoothed-market asset valuation method employing exponential smoothing is considered in the following. Certain simplifying assumptions are required to study the effect of valuing the assets of pension plans according to this method during funding valuations. A simple but mathematically tractable model of a defined benefit plan is used here. For more details, refer to Dufresne (1988) or Haberman (1992). The plan provides a pension based on final salary upon retirement at a normal retirement age.

The plan is valued at the beginning of every year. A contribution rate  $C_t$  is determined at the start of year  $(t, t + 1)$ . Plan assets are directly marketable, their market value  $F_t$  is instantly obtainable and an actuarial asset value  $AV_t$  is calculated. A set of valuation assumptions (known as the valuation basis) concerning mortality rates, early retirement rates, withdrawal rates, inflation, promotional salary scale etc. is used at each valuation. It is assumed that the actuarial valuation basis is constant. It is also assumed that the funding method or actuarial cost method is not changed. See Aitken (1994), McGill *et*

*al.* (1996) or Turner (1984) for descriptions of actuarial cost methods. The actuarial cost method generates an actuarial liability (also known as standard fund)  $AL$  and a normal cost (also known as standard contribution)  $NC$ . The valuation discount rate  $i$  is chosen such that the actuarial liability  $AL$  is a practical approximation to the fair value of pension liabilities. The actuarial assumption as to the projected long-term rate of return on plan assets is also  $i$ . In the rest of this paper, the following usual notation is employed:

$$u = 1 + i, \quad v = (1 + i)^{-1}. \quad (2.1)$$

Actual experience does not generally unfold according to valuation assumptions. Actuarial gains and losses arise when experience deviates from valuation assumptions. McGill *et al.* (1996, p. 522) and Aitken (1994, p. 149) discuss the relevance and calculation of various types of gains and losses. Favourable experience, such as higher investment returns or heavier post-retirement mortality than anticipated, result in gains. Conversely, unfavourable experience results in losses.

A simple projection of the experience of the plan is made here. The size and age profile of the membership of the pension plan is projected to be constant and to evolve exactly according to the life table used for valuation purposes. No gain or loss due to mortality arises. For simplicity, neither salaries nor benefits are subject to economic inflation. The actuarial liability  $AL$ , normal cost  $NC$  and yearly benefit outgo  $B$  are constant as a result of the assumptions made about pension liabilities and the funding method. (Alternatively, all salaries and benefits (including pensions in payment) may be assumed to increase at the *same* rate of inflation, and all monetary quantities (including  $i$ ,  $r_t$ ,  $F_t$  etc.) are then deflated.  $AL$ ,  $NC$ ,  $B$  are then constant in *real* terms.)

The economic experience of the plan is such that a variable rate of return  $r_t$  is earned in year  $(t - 1, t)$ . The only source of unpredictable experience in the plan is through volatile investment returns. That is, only asset gains and losses emerge.

This simple model resembles the model of Trowbridge (1952), who shows that when the liability structure of the pension plan is in equilibrium, the following equation holds:

$$AL = (1 + i)(AL + NC - B). \quad (2.2)$$

All cash flows occur at the start of the year. An asset recurrence relation may be written: for  $t \geq 0$ ,

$$F_{t+1} = (1 + r_{t+1})(F_t + C_t - B). \quad (2.3)$$

If the experience of the plan is favourable relative to valuation assumptions, successive actuarial gains will emerge and will tend to reduce any deficit (or increase any surplus) in the pension fund. Conversely, losses increase deficits. Gains and losses are paid off by adjusting pension contributions to restore financial balance to the pension fund. The excess of actuarial liability over the actuarial asset value is  $AL - AV_t$  and represents a notional actuarial deficit in the pension fund. This deficit is paid off by paying a total contribution rate  $C_t$  equal to the normal cost  $NC$  plus a supplementary contribution  $SC_t$ ,

$$C_t = NC + SC_t. \quad (2.4)$$

The supplementary contribution is equal to an amortization payment over a term of  $m$  years for the deficit, that is,  $SC_t = (AL - AV_t)/\ddot{a}_{\overline{m}|}$ , where  $\ddot{a}_{\overline{m}|} = (1 - v^m)/(1 - v)$  represents the present value of an annuity-certain payable in advance over  $m$  years at the valuation rate of interest  $i$ . Gains and losses are said to be spread over  $m$  years. This method of calculating the supplementary contribution is very common (particularly in the United Kingdom) and is discussed by Turner (1984), Dufresne (1988), McGill *et al.* (1996), and Owadally and Haberman (1999) among others.

It is convenient to define the following:

$$K = 1 - 1/\ddot{a}_{\overline{m}|}, \quad (2.5)$$

so that the supplementary contribution is a proportion  $(1 - K)$  of the actuarial deficit:  
 $SC_t = (1 - K)(AL - AV_t)$ .

The excess of actuarial liability over the market value of plan assets is called the unfunded liability of the plan:

$$UL_t = AL - F_t. \quad (2.6)$$

A pension plan may have an initial unfunded liability  $UL_0$ , arising at plan inception or from amendments to plan benefits or valuation methods. The initial unfunded liability may be explicitly amortized over  $n$  years (say) by payments

$$P_t = \begin{cases} UL_0/\ddot{a}_{\overline{n}|}, & 0 \leq t \leq n-1, \\ 0, & t \geq n. \end{cases} \quad (2.7)$$

$\ddot{a}_{\overline{n}|} = (1 - v^n)/(1 - v)$  represents the present value of an annuity-certain payable in advance over  $n$  years at the valuation rate of interest  $i$ . The unamortized part of  $UL_0$  is

$$U_t = \begin{cases} UL_0 \ddot{a}_{\overline{n-t}|}/\ddot{a}_{\overline{n}|}, & 0 \leq t \leq n-1, \\ 0, & t \geq n. \end{cases} \quad (2.8)$$

Note that

$$P_t = U_t - vU_{t+1}, \quad (2.9)$$

where we define  $U_n = U_{n+1} = \dots = 0$ .

If an initial unfunded liability is separately amortized, then gains and losses are spread by calculating the supplementary contribution as follows (Owadally and Haberman, 1999):

$$SC_t = (1 - K)(AL - AV_t - U_t) + P_t. \quad (2.10)$$

$(AL - AV_t - U_t)$  represents the portion of the actuarial deficit in excess of the unamortized part of the initial unfunded liability.

Finally, note that the recurrence relation (2.3) may be written in terms of the unfunded liability using equations (2.2), (2.6) and (2.10):

$$UL_{t+1} = AL + (1 + r_{t+1})(UL_t - SC_t - v AL). \quad (2.11)$$

### 3 Exponential Smoothing of Market Values

A smoothed-market actuarial asset value based on an exponential smoothing of market values is commonly used and is considered here. A simple average of the market values at different points in time cannot be used of course. The market values must be adjusted by allowing for both the time value of money and cash flows (Anderson, 1992, p. 110).

The actuarial asset value  $AV_t$  at time  $t$  is a weighted average of the market value  $F_t$  of the fund at time  $t$  and the actuarial value of the fund at time  $t$  as anticipated at time  $t - 1$  based on the valuation assumptions at time  $t - 1$ :

$$AV_t = \lambda u(AV_{t-1} + C_{t-1} - B) + (1 - \lambda)F_t, \quad (3.1)$$

where  $\lambda$  is a smoothing parameter such that  $0 \leq \lambda < v$ . A larger value of  $\lambda$  means that more weight is placed on the past market values and more smoothing is applied. It is easily verified that  $AV_t$  may be expressed as an infinite exponentially weighted average allowing for interest and cash flows (provided  $0 < \lambda < v$ ):

$$AV_t = \sum_{j=0}^{\infty} (1 - \lambda)(\lambda u)^j F_{t-j} + \sum_{j=1}^{\infty} (\lambda u)^j (C_{t-j} - B). \quad (3.2)$$

If assets are valued at market only, that is  $\lambda = 0$  and  $AV_t = F_t \forall t$ , then the model reduces to the one investigated by Dufresne (1988) (with the minor exception that he does not consider the separate amortization of the initial unfunded liability in equations (2.7)–(2.9)).

We may use equations (2.2), (2.4) and (2.6) to rewrite equation (3.1) in terms of the unfunded liability and supplementary contribution:

$$AL - AV_t = \lambda u[AL - AV_{t-1} - SC_{t-1}] + (1 - \lambda)UL_t. \quad (3.3)$$

Equations (2.9) and (2.10) may also be used:

$$AL - AV_t - U_t = \lambda Ku(AL - AV_{t-1} - U_{t-1}) + (1 - \lambda)(UL_t - U_t). \quad (3.4)$$

Equation (3.4) may be written as:

$$AL - AV_t - U_t = (1 - \lambda) \sum_{j=0}^t (\lambda Ku)^{t-j} (UL_j - U_j). \quad (3.5)$$

Hence, the supplementary contribution rate when asset values are smoothed is given by (substituting equation (3.5) in equation (2.10)):

$$SC_t = (1 - K)(1 - \lambda) \sum_{j=0}^t (\lambda Ku)^{t-j} (UL_j - U_j) + P_t. \quad (3.6)$$

Finally, we may replace  $SC_t$  from equation (3.6) in equation (2.11) to yield an equation for the unfunded liability of the pension plan:

$$\begin{aligned} & (UL_{t+1} - U_{t+1}) - (AL - U_{t+1}) \\ &= (1 + r_{t+1}) \left[ (UL_t - U_t) - (1 - K)(1 - \lambda) \sum_{j=0}^t (\lambda Ku)^{t-j} (UL_j - U_j) \right. \\ & \qquad \qquad \qquad \left. - v(AL - U_{t+1}) \right]. \quad (3.7) \end{aligned}$$

The initial unfunded liability may be large and its treatment is important in practice. Nevertheless, it has only a transient effect since it is paid off in  $n$  years and  $U_t = 0$  for  $t \geq n$  (equation (2.8)). If the initial unfunded liability is disregarded (assumed to be zero or to be paid off from a separate fund), equation (3.7) has a somewhat simpler structure:

$$UL_{t+1} - AL = (1 + r_{t+1}) \left[ UL_t - (1 - K)(1 - \lambda) \sum_{j=0}^t (\lambda Ku)^{t-j} UL_j - vAL \right]. \quad (3.8)$$

Equation (3.8) reveals that the asset valuation method (through the parameter  $\lambda$ ) and the gain/loss spreading method (through the parameter  $K$ ) provide an exponential smoothing mechanism in the pension funding process. Note also that equations (3.6)–(3.8)

are symmetrical in  $K$  and  $\lambda$ : the values of  $K$  and  $\lambda$  could be interchanged without affecting the dynamics of the pension funding process. The perfect symmetry above is a consequence of the simplistic assumptions of the model. In practice, the actuarial asset value smoothes only the volatility in asset returns whereas the supplementary contribution smoothes all gains and losses, including demographic ones.

#### 4 Moments of the Pension Funding Process

Suppose now that the sequence  $\{r_t\}$  of rates of return on pension plan assets is a sequence of independent and identically distributed random variables, with mean  $r$  and variance  $\sigma^2$ . Such a projection assumption simplifies reality but does introduce volatility and reflect market efficiency. It is convenient to define  $d = i/(1 + i)$ ,  $d_r = r/(1 + r)$ , as well as

$$\theta = (1 - K)(1 - \lambda)/(1 - \lambda Ku). \quad (4.1)$$

The long-term expected values of various variables in the pension fund are shown in Proposition 1, which is proven in Owadally and Haberman (2003).

**PROPOSITION 1** *If  $i > -1$ ,  $r > -1$ ,  $\lambda K(1 + i)(1 + r) < 1$  and  $\theta > d_r$ , then*

$$\lim_{t \rightarrow \infty} EUL_t = AL(d_r - d)/(d_r - \theta), \quad (4.2)$$

$$\lim_{t \rightarrow \infty} EF_t = AL(d - \theta)/(d_r - \theta), \quad (4.3)$$

$$\lim_{t \rightarrow \infty} EAV_t = AL - AL\theta(d_r - d)/(1 - K)(d_r - \theta), \quad (4.4)$$

$$\lim_{t \rightarrow \infty} EC_t = NC + AL\theta(d_r - d)/(d_r - \theta). \quad (4.5)$$

The symmetry between  $K$  and  $\lambda$  in the first moments (except in that of the actuarial value  $AV_t$  which involves only smoothing through asset valuation and not through the supplementary contribution) is again evident. When unsmoothed market values of plan assets are used ( $\lambda = 0$ ) then  $\theta = 1 - K$ , in which case the results reduce exactly to those

obtained by Dufresne (1988). When asset gains and losses are not spread but are paid off immediately ( $m = 1$ ,  $K = 0$ ) then  $\theta = 1 - \lambda$ , in which case the results mirror those of Dufresne (1988) with  $\lambda$  exactly replacing  $K$ .

Simpler results follow if the actuarial assumption  $i$  as to the rate of return on plan assets is unbiased and equals the mean rate of return  $r$ .

COROLLARY 1 *Suppose that  $r = i$ . Then:*

$$EUL_t = \begin{cases} UL_0 \ddot{a}_{\overline{n-t}|} / \ddot{a}_{\overline{n}|}, & 0 \leq t \leq n-1, \\ 0, & t \geq n, \end{cases} \quad (4.6)$$

$$EF_t = EAV_t = \begin{cases} AL - UL_0 \ddot{a}_{\overline{n-t}|} / \ddot{a}_{\overline{n}|}, & 0 \leq t \leq n-1, \\ AL, & t \geq n, \end{cases} \quad (4.7)$$

$$EC_t = \begin{cases} NC + UL_0 / \ddot{a}_{\overline{n}|}, & 0 \leq t \leq n-1, \\ NC, & t \geq n. \end{cases} \quad (4.8)$$

See Owadally and Haberman (2003) for a proof. If the actuarial assumption as to returns on plan assets is a best estimate and is borne out by experience on average, then no asset gain or loss is expected to emerge. After the initial unfunded liability is defrayed, the plan is expected to remain fully funded and no supplementary contribution beyond the normal cost is paid on average. The Standard of Practice for Valuation of Pension Plans of the Canadian Institute of Actuaries (1994, para. 5.01) requires that an asset valuation method be consistent with liability valuation and that systematic gains or losses do not emerge. The smoothed-market method described in section 3 therefore satisfies this criterion for consistency.

The second moments of the pension funding process are considered in Proposition 2. The simplifying assumption is made henceforth that the assumed rate of return is unbiased ( $r = i$ ). The variance of the random rate of return on plan assets is  $\sigma^2 = \text{Varr}_t$ .

Furthermore, define  $q = E(1 + r_t)^2 = u^2 + \sigma^2$  and  $V_\infty = \sigma^2 v^2 AL^2 / Q$ , where

$$Q = (1 - qK^2)(1 - \lambda^2 u^2)(1 - \lambda K u^2) - \lambda(1 - K)\sigma^2[2K(1 - \lambda^2 u^2) + \lambda(1 - K)(1 + \lambda K u^2)] \quad (4.9)$$

$$= (1 - q\lambda^2)(1 - K^2 u^2)(1 - \lambda K u^2) - K(1 - \lambda)\sigma^2[2\lambda(1 - K^2 u^2) + K(1 - \lambda)(1 + \lambda K u^2)]. \quad (4.10)$$

PROPOSITION 2 *Provided that  $r = i > -$ ,  $0 \leq K < v$ ,  $0 \leq \lambda < v$ ,  $Q > 0$  and*

$$(1 + \lambda^2 K^2 q u^2)(1 + \lambda^3 K^3 \sigma^2 u^2 - \lambda^4 K^4 q u^6) > 2\lambda^4 K^4 (\lambda + K) q \sigma^2 u^4 + \lambda K (\lambda + K)^2 q u^2 (1 - \lambda^2 K^2 q u^2), \quad (4.11)$$

then

$$\lim_{t \rightarrow \infty} \text{Var} F_t = V_\infty [(1 - \lambda K u^2)(1 - \lambda^2 K^2 u^2) + 2\lambda K (1 - \lambda)(1 - K) u^2], \quad (4.12)$$

$$\lim_{t \rightarrow \infty} \text{Var} AV_t = V_\infty (1 - \lambda)^2 (1 + \lambda K u^2), \quad (4.13)$$

$$\lim_{t \rightarrow \infty} \text{Var} C_t = V_\infty (1 - K)^2 (1 - \lambda)^2 (1 + \lambda K u^2), \quad (4.14)$$

$$\lim_{t \rightarrow \infty} \text{Cov}[F_t, AV_t] = V_\infty (1 - \lambda) [1 + \lambda K (1 - K - \lambda) u^2], \quad (4.15)$$

$$\lim_{t \rightarrow \infty} \text{Cov}[F_t, C_t] = -V_\infty (1 - K)(1 - \lambda) [1 + \lambda K (1 - K - \lambda) u^2], \quad (4.16)$$

$$\lim_{t \rightarrow \infty} \text{Cov}[C_t, AV_t] = -V_\infty (1 - K)(1 - \lambda)^2 (1 + \lambda K u^2). \quad (4.17)$$

Refer to Owadally and Haberman (2003) for a proof. Again, the moments (except those involving the smoothed actuarial asset value  $AV_t$ ) exhibit symmetry and the smoothing parameters  $K$  and  $\lambda$  can be interchanged. When pure market values of assets are used ( $\lambda = 0$ ) then  $Q = 1 - qK^2$  and the second moments are identical to those obtained by Dufresne (1988) ( $\lim_{t \rightarrow \infty} \text{Var} F_t = \lim_{t \rightarrow \infty} \text{Var} AV_t$ ). When asset gains and losses are not spread but are paid off immediately ( $K = 0$ ) then  $Q = 1 - q\lambda^2$  and Dufresne's (1988) results are again obtained but with  $\lambda$  exactly replacing  $K$ .

## 5 Effects of Smoothing Asset Values

### 5.1 Stability (Finite Variance)

An important property of an asset valuation method is that it should lead to stable funding for pension obligations: the funding process should at least exhibit finite variance. The convergence conditions of Proposition 1 and 2 are *sufficient* for finite variance. These conditions are realistic in normal economic circumstances and the condition that most constrains the choice of gain/loss spreading period  $m$  (or spreading parameter  $K$ ) and of asset valuation parameter  $\lambda$  is  $Q > 0$ . ( $K$  and  $m$  are in a direct one-to-one relationship: see equation (2.5).) It is necessary, but not sufficient, for stability that  $K < 1/\sqrt{q}$  and  $\lambda < 1/\sqrt{q}$ .

Table 1 exhibits the stability constraints in terms of maximum allowable spread periods for various choices of  $\{i, \sigma, \lambda\}$ . Table 2 shows maximum allowable smoothing parameters for various choices of  $\{i, \sigma, m\}$ . Both tables are based on the stability conditions of Proposition 2. It is easily verified in Tables 1 and 2 that inequalities  $K < 1/\sqrt{q}$  and  $\lambda < 1/\sqrt{q}$  hold.

It is clear from Table 1 that gains and losses should not be spread over very long periods as this could result in an unstable funding process. This conclusion is also emphasized by Dufresne (1988) who considers only pure market values of assets. Spreading periods should be even shorter if asset values are being smoothed.

Table 2 shows that excessive smoothing of asset values must be avoided, especially if gains and losses are being spread over long periods. Asset valuation and gain/loss adjustment perform a complementary actuarial smoothing function and there is a finite limit to the cumulative amount of smoothing that may be applied.

## 5.2 Effect on the Smoothed Actuarial Asset Value

A suitable asset valuation method should generate an asset value that is realistic in the sense that it remains fairly close to market values. Furthermore, the asset value should be more stable or less variable than the market value (Berin, 1989, p. 29). Since pension funds enjoy favourable tax treatment, the Internal Revenue Service in the United States imposes a maximum funding limit and for this purpose it requires that the smoothed asset value be within a corridor of 20% of the market value of assets (McGill *et al.*, 1996). Proposition 3 (proven in Owadally and Haberman, 2003) states that these properties do indeed hold.

PROPOSITION 3 *Provided that the stability conditions of Proposition 2 hold,*

$$\lim_{t \rightarrow \infty} E[F_t - AV_t]^2 < \infty, \quad (5.1)$$

$$\lim_{t \rightarrow \infty} \text{Var}AV_t \leq \lim_{t \rightarrow \infty} \text{Var}F_t. \quad (5.2)$$

Inequality (5.2) shows that the smoothed asset value is less variable than market value, provided the given conditions hold. Inequality (5.1) shows that the deviation between the smoothed actuarial asset value and the market value of plan assets remains bounded in the mean-square, provided that the amount of smoothing in the asset valuation and gain/loss adjustment methods are constrained as discussed in section 5.1. Excessive averaging of market values (as well as spreading of gains/losses over very long periods) must therefore be avoided. Note in particular that if  $\lambda = 1$ , the actuarial asset value  $AV_t$  in equation (3.1) does not revert towards the market value  $F_t$  and unless the fund is marked-to-market regularly the actuarial asset value will diverge from the market value of pension plan assets.

### 5.3 *Effect on the Fund Level*

Dufresne (1988) and Owadally and Haberman (1999) consider only pure market values of assets ( $\lambda = 0$ ) but show that spreading or amortizing gains and losses over longer periods lead to more variable fund levels. This is reasonable. As gains and losses are deferred for longer periods, fast enough action is not taken to defray them and the level of funding becomes more volatile. Likewise, one anticipates that heavier smoothing of asset values, which delays the recognition of asset gains and losses, should also adversely affect the security of pension benefits. This is encapsulated in Proposition 4, proven in Owadally and Haberman (2003).

*PROPOSITION 4* *Provided that the stability conditions of Proposition 2 hold,  $\lim \text{Var}F_t$  increases monotonically with both  $m$  and  $\lambda$ .*

This result is illustrated in the first contour plot in Figure 1. The symmetry between asset valuation and asset gain/loss spreading is clearly exhibited: the contour plot is symmetrical in the plane  $K = \lambda$ .

### 5.4 *Effect on the Contribution Rate*

Slower recognition and amortization of gains and losses should result in smoother and more stable contribution rates. In the context of pure market values of assets ( $\lambda = 0$ ), Dufresne (1988) shows that spreading gains and losses over longer periods does initially stabilize contributions, but beyond a certain critical period contributions become more variable:  $\lim \text{Var}C_t$  against  $m$  has a minimum at  $m^*$  corresponding to  $K^* = 1/q$ . (Owadally and Haberman (1999) also assume that assets are valued at market prices and prove a similar result when gains and losses are directly amortized rather than indirectly spread.)

An immediate consequence of the symmetry between gain/loss spreading and the ex-

ponential smoothing asset valuation methods used here is that, if gains and losses are immediately paid off and not spread forward ( $m = 1$  or  $K = 0$ ), then  $\lim \text{Var}C_t$  against  $\lambda$  has a minimum at  $\lambda^* = 1/q$ . Therefore, smoothing beyond a certain amount (weighting the current market value of assets by less than  $1 - \lambda^*$ ) is countereffective, as contributions become more variable. (The proof is obtained, as a matter of course, by repeating Dufresne's (1988) proof and replacing all  $K$  by  $\lambda$ .)

The combined effect of asset valuation and gain/loss spreading on the stability of contribution rates is investigated in Proposition 5 (proof in Owadally and Haberman, 2003).

**PROPOSITION 5** *Suppose  $m > 1$  and  $\lambda > 0$ . Provided that the stability conditions of Proposition 2 hold,*

1. *as  $m$  increases,*

*$\lim \text{Var}C_t$  has at least one minimum at some  $m < m^*$ , provided  $0 < \lambda < \lambda^*$ ;*

*$\lim \text{Var}C_t$  increases monotonically, provided either  $\lambda \geq \lambda^*$  or  $m \geq m^*$ ;*

2. *as  $\lambda$  increases,*

*$\lim \text{Var}C_t$  has at least one minimum at some  $\lambda < \lambda^*$ , provided  $1 < m < m^*$ ;*

*$\lim \text{Var}C_t$  increases monotonically, provided either  $m \geq m^*$  or  $\lambda \geq \lambda^*$ .*

The variation of  $\lim \text{Var}C_t$  with  $K$  and  $\lambda$  is illustrated in the second contour plot in Figure 1 and in Figure 2. The two parts of Proposition 5 are identical except that  $K$  and  $\lambda$  are interchanged. The variation of  $\lim \text{Var}C_t$  with  $K$  is similar to its variation with  $\lambda$ . The boomerang-shaped contours of Figure 1 are a further indication of the complementary function of gain/loss adjustment and asset valuation: the same contribution or fund level variability may be achieved by trading off  $\lambda$  and  $K$ .

Proposition 5 does not state whether no more than one minimum occurs but numerical work, as illustrated by Figure 2, does indicate at most one minimum. This suggests that

- $\lim \text{Var}C_t$  against  $K$  exhibits a minimum, except for large enough  $\lambda$  when  $\lim \text{Var}C_t$  increases monotonically;
- $\lim \text{Var}C_t$  against  $\lambda$  exhibits a minimum, except for large enough  $K$  when  $\lim \text{Var}C_t$  increases monotonically.

Thus, in Figure 2,  $\lim \text{Var}C_t$  versus  $K$  exhibits a minimum when  $\lambda < \lambda^* = 0.82$ . The minimum for  $\lambda = 0$  is seen to occur at  $K = K^* = 0.82$ . The minima for  $0 < \lambda < \lambda^*$  clearly occur at some  $K < 0.82$ . But when  $\lambda \geq \lambda^* = 0.82$ ,  $\lim \text{Var}C_t$  versus  $K$  has no minimum and increases monotonically. In other words, if asset values are being heavily smoothed ( $\lambda \geq \lambda^*$ ), it is counterproductive to spread gains and losses in an effort to smooth contribution rates further. Likewise, if gains/losses are being spread over long periods ( $K \geq K^*$  or  $m \geq m^*$ ), averaging market values of plan assets in an effort to generate smoother contribution rates is counterproductive.

Attention must therefore be paid to the *combined* smoothing effect of gain/loss adjustment and asset valuation.

### 5.5 Efficient Asset Valuation and Gain/Loss Spreading

It is argued by Dufresne (1988) that maximizing the security of plan members' benefits (by minimizing  $\lim \text{Var}F_t$ ) and maximizing the stability of contributions required from the plan sponsor (by minimizing  $\lim \text{Var}C_t$ ) are rational actuarial objectives in pension funding in the long term. Given such objectives, it is possible to go further than in section 5.4 and state that:

PROPOSITION 6 *Under the objectives of minimizing  $\lim \text{Var}F_t$  and  $\lim \text{Var}C_t$ ,*

1. *it is not efficient to smooth asset values by weighting current market value by less than  $1 - \lambda^*$ ;*

2. *it is not efficient to adjust gains/losses by spreading them over periods exceeding  $m^*$ .*

Proposition 4 states that increasing  $\lambda$  causes  $\lim \text{Var}F_t$  to increase. Proposition 5 states that increasing  $\lambda$  initially causes  $\lim \text{Var}C_t$  to decrease but eventually increasing  $\lambda$  beyond  $\lambda^*$  causes  $\lim \text{Var}C_t$  to increase. Hence, it is inefficient to smooth asset values using  $\lambda > \lambda^*$  as there is some other choice of  $\lambda$  for which both  $\lim \text{Var}F_t$  and  $\lim \text{Var}C_t$  may be reduced. By symmetry, the second part of Proposition 6 is also proven.

The second part of Proposition 6 encompasses the conclusions of Dufresne (1988) who investigates the choice of  $m$  when pure market values are used ( $\lambda = 0$ ).

Numerical work indicates that  $\lim \text{Var}C_t$  against  $\lambda$  or  $K$  has at most one minimum, as discussed in section 5.4. For any given gain/loss spreading period  $m$ , it is inefficient to smooth asset values by more than the  $[\lim \text{Var}C_t]$ -minimizing value of  $\lambda$  as a lower  $\lambda$  will reduce both  $\lim \text{Var}F_t$  and  $\lim \text{Var}C_t$ . If  $m$  is long enough and  $\lim \text{Var}C_t$  is strictly increasing with  $\lambda$ , then pure market values should be used. Table 4 lists the  $[\lim \text{Var}C_t]$ -minimizing values of  $\lambda$  for various choices of  $\{i, \sigma, m\}$ . It is efficient to smooth asset values using a value of  $\lambda$  between 0 and the  $[\lim \text{Var}C_t]$ -minimizing value in Table 4. The first column of Table 4 ( $m = 1$  or  $K = 0$ ) contains the upper bound  $\lambda^* = 1/q$ . The values in Table 4 are of course lower than the corresponding maximum allowable values of  $\lambda$  for stability in Table 2.

By symmetry, Table 3 shows the longest periods over which gains and losses can be efficiently spread for various choices of  $\{i, \sigma, \lambda\}$ .

Tables 3 and 4 suggest that pension benefits would be efficiently funded if gains and losses are spread over terms of 1–5 years with a weight of 20–100% placed on the current market value of assets ( $\lambda$  should be at most 80%). Spreading asset gains and losses over up to 10 years requires the current market value to be weighted by at least 60% for efficiency (assuming real rates of return averaging up to 5% with standard deviations of up to 15%). This analysis is limited by the fact that only asset gains and losses were allowed.

Uncertainty in mortality, early retirement and other factors was assumed to be negligible compared with investment uncertainty.

## 6 Conclusion

The motivation for the use of special actuarial methods to value the assets of defined benefit pension plans was discussed in the context of funding valuations: an actuarial asset value is employed to moderate volatility in market values. A simple pension plan model was described where experience unfolds deterministically except for random investment returns. Asset gains and losses emerge and supplementary contributions are paid so as to spread the gains and losses. A smoothed-market asset value incorporating exponential smoothing was described. Symmetry between asset gain/loss adjustment and smoothed-market asset valuation was demonstrated and it was shown that they have a similar smoothing function in the pension funding process.

The first two moments of several variables (the level of contribution required, and the market and smoothed actuarial values of pension plan assets) were obtained. An important result is that asset valuation and gain/loss adjustment techniques have a complementary function in achieving smoothness in the pension funding process and their combined effect must be considered. Conditions for the funding process to be stable in the mean-square were obtained, restricting the total amount of smoothing through both techniques. The actuarial asset value does not diverge from, and is more stable than, the market value of plan assets if the conditions for stability hold.

It was also shown that the total amount of smoothing is further constrained if funding is to be efficient and the long-term variability of both contribution and funding levels is to be minimized. Numerical work appears to indicate that a combination of a gain/loss spreading period of 1–5 years and a 20–100% weighting on the current market value of

assets is efficient, as is a combination of 1–10 years and 60–100% respectively.

These results are mitigated by the fact that only asset gains and losses were considered in the model. This is a reasonable approximation to reality if volatility in mortality and other factors is small compared to volatility in investment returns. Further research is required to establish the effects of more general economic and demographic uncertainty in pension funding.

## References

ACTUARIAL STANDARDS BOARD (1993) *Actuarial Standard of Practice No. 4: Measuring Pension Obligations*. Pensions Committee of the Actuarial Standards Board, American Academy of Actuaries, Washington, D.C.

AITKEN, W.H. (1994) *A Problem-Solving Approach to Pension Funding and Valuation*. Winsted, Connecticut: ACTEX Publications.

ANDERSON, A.W. (1992) *Pension Mathematics for Actuaries*, 2nd ed. Winsted, Connecticut: ACTEX Publications.

BERIN, B.N. (1989) *The Fundamentals of Pension Mathematics*. Schaumburg, Illinois: Society of Actuaries.

BRAND, L. (1966) *Differential and Difference Equations*. New York: Wiley.

CANADIAN INSTITUTE OF ACTUARIES (1994) *Standard of Practice for Valuation of Pension Plans*. Ottawa, Ontario.

COMMITTEE ON RETIREMENT SYSTEMS RESEARCH (1998) *Survey of Asset Valuation Methods for Defined Benefit Pension Plans*. Schaumburg, Illinois: Society of Actuaries.

DUFRESNE, D. (1988) Moments of pension contributions and fund levels when rates of return are random. *Journal of the Institute of Actuaries*, **115**, 535–544.

- EZRA, D.D. (1979) *Understanding Pension Fund Finance and Investment*. Toronto, Canada: Pagurian Press.
- HABERMAN, S. (1992) Pension funding with time delays—a stochastic approach. *Insurance: Mathematics and Economics*, **11**, 179–189.
- JURY, E.I. (1964) *Theory and Applications of the z-Transform Method*. New York: John Wiley & Sons.
- MARDEN, M. (1966) *The Geometry of Polynomials*, 2nd ed. Providence, Rhode Island: American Mathematical Society.
- MCGILL, D.M., BROWN, K.N., HALEY, J.J. AND SCHIEBER, S.J. (1996) *Fundamentals of Private Pensions*, 7th ed. Philadelphia, Pennsylvania: University of Pennsylvania Press.
- OWADALLY, M.I. AND HABERMAN, S. (1999) Pension fund dynamics and gains/losses due to random rates of investment return. *North American Actuarial Journal*, **3:3**, 105–117.
- OWADALLY, M.I. AND HABERMAN, S. (2003) Exponential Smoothing Methods in Pension Funding, mimeo, Faculty of Actuarial Science and Statistics, Cass Business School.
- TROWBRIDGE, C.L. (1952) Fundamentals of pension funding. *Transactions of the Society of Actuaries*, **4**, 17–43.
- TURNER, M.J. ET AL. (1984) Report of a Working Party of the Pensions Standards Joint Committee on Terminology of Pension Funding Methods. London: Institute and Faculty of Actuaries.
- WINKLEVOSS, H.E. (1993) *Pension Mathematics with Numerical Illustrations*, 2nd ed. Philadelphia, Pennsylvania: University of Pennsylvania Press.

Table 1: *Maximum allowable  $m$  for various choices of  $\{i, \sigma, \lambda\}$ . Blanks indicate that stability conditions do not hold.*

$\sigma$	$i$	$\lambda = 0$	$\lambda = 20\%$	$\lambda = 40\%$	$\lambda = 60\%$	$\lambda = 80\%$	$\lambda = 90\%$
0.1	1%	112	111	110	109	104	94
	3%	67	67	66	64	59	47
	5%	51	50	49	47	42	29
	10%	33	33	32	30	23	5
	15%	25	25	25	22	14	
0.2	1%	42	41	41	39	34	26
	3%	32	32	31	28	24	20
	5%	27	27	24	22	19	9
	10%	20	20	17	15	11	
	15%	17	16	15	13	7	

Table 2: Maximum allowable  $\lambda$  (%) for various choices of  $\{i, \sigma, m\}$ . Blanks indicate that stability conditions do not hold.

$\sigma$	$i$	$m = 1$	3	5	10	15	20	25	30	40	50
0.1	1%	98.5	98.5	98.4	98.4	98.3	98.2	98.0	97.8	97.5	97.0
	3%	96.6	96.6	96.5	96.2	95.9	95.5	94.9	94.2	92.4	88.9
	5%	94.8	94.7	94.6	94.1	93.4	92.5	91.2	89.6	82.9	34.7
	10%	90.5	90.4	90.1	88.8	86.7	83.6	78.0	63.2		
	15%	86.6	86.4	85.9	83.6	79.5	71.4	40.8			
0.2	1%	97.1	96.9	96.6	95.8	94.5	92.9	90.6	86.8	54.9	
	3%	95.3	94.9	94.5	92.8	90.4	86.6	79.4	58.0		
	5%	93.6	93.0	92.4	89.8	85.7	78.2	55.0			
	10%	89.4	88.6	87.4	82.1	71.5	25.2				
	15%	85.7	84.6	82.7	74.2	49.7					

Table 3:  $\lim \text{Var}C_t$ -minimizing values of  $m$  for various choices of  $\{i, \sigma, \lambda\}$ . † indicates that  $\lim \text{Var}C_t$  increases monotonically with  $m$  with smallest value at  $m = 1$ . Blanks indicate instability.

$\sigma$	$i$	$\lambda = 0$	$\lambda = 20\%$	$\lambda = 40\%$	$\lambda = 60\%$	$\lambda = 80\%$	$\lambda = 90\%$
0.1	1%	42	41	41	39	36	28
	3%	20	19	19	17	14	3
	5%	13	13	12	11	6	1
	10%	7	7	6	5	1	†
	15%	5	5	4	2	†	
0.2	1%	19	19	18	17	13	5
	3%	13	13	12	11	6	1
	5%	9	9	9	7	3	†
	10%	6	6	5	4	1	
	15%	5	4	4	2	†	

Table 4:  $\lim \text{Var}C_t$ -minimizing values of  $\lambda$  (%) for various choices of  $\{i, \sigma, m\}$ . † indicates that  $\lim \text{Var}C_t$  increases monotonically with  $\lambda$  with smallest value at  $\lambda = 0$ . Blanks indicate instability.

$\sigma$	$i$	$m = 1$	3	5	10	15	20	25	30	40	50
0.1	1%	97.1	96.9	96.7	96.0	94.8	91.6	73.1	34.5	3.2	†
	3%	93.4	92.6	91.4	80.6	23.9	†	†	†	†	†
	5%	89.9	87.9	83.8	22.8	†	†	†	†	†	†
	10%	82.0	73.0	35.0	†	†	†	†	†		
	15%	75.0	50.4	4.0	†	†	†	†			
0.2	1%	94.3	93.4	92.0	81.5	27.1	†	†	†	†	
	3%	90.8	88.6	84.2	25.0	†	†	†	†		
	5%	87.5	83.2	70.7	†	†	†	†			
	10%	80.0	66.0	20.2	†	†	†				
	15%	73.4	42.0	†	†	†					

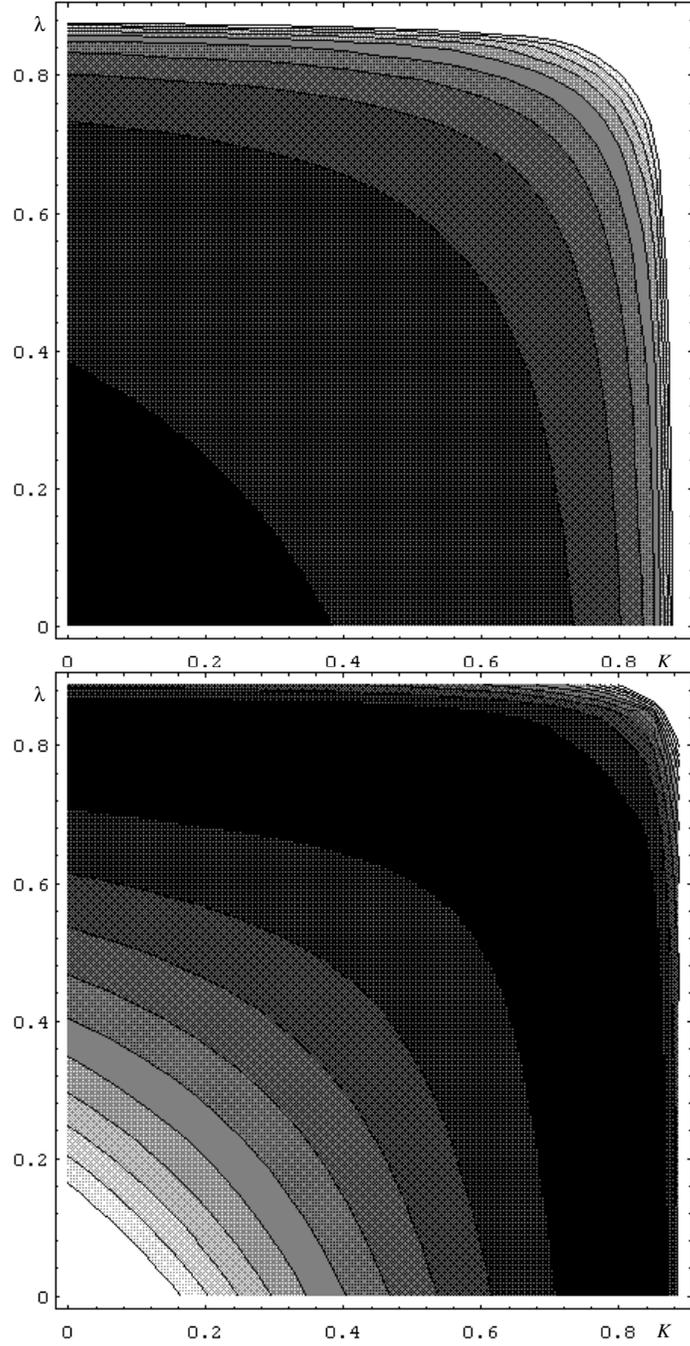


Figure 1: Contour plots of  $\lim \text{Var}F_t$  (above) and  $\lim \text{Var}C_t$  (below) against  $K$  and  $\lambda$ .  $i = 10\%$ ,  $\sigma = 5\%$ . Lighter shading represents higher values.

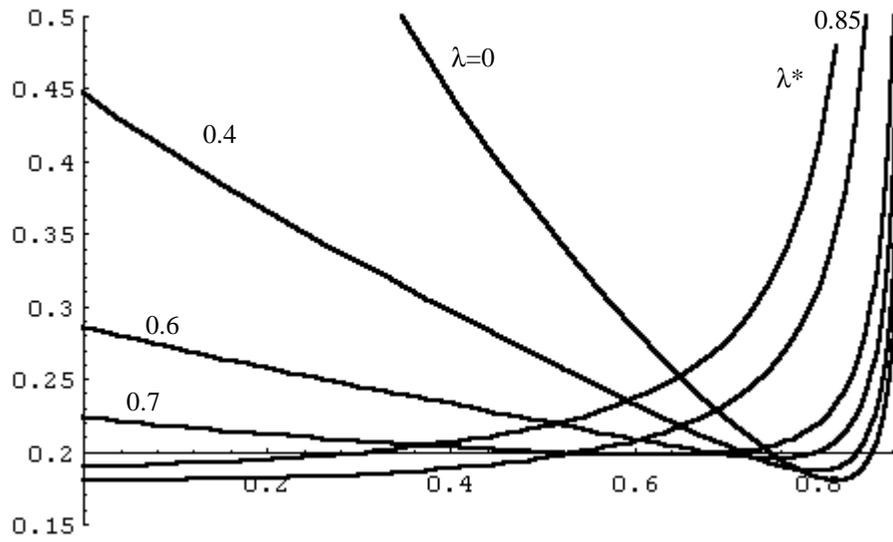


Figure 2:  $\lim\text{Var}C_t$  (scaled) against  $K$  for various  $\lambda$ .  $K$  and  $\lambda$  can be interposed.  $i = 10\%$ ,  $\sigma = 10\%$ ,  $\lambda^* = K^* = 0.82$ .