Dynamic stiffness formulation for composite Mindlin plates for exact modal analysis of structures. Part I: Theory

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Abstract

The dynamic stiffness formulation for both inplane and bending free vibration based on the first order shear deformation theory for composite plates is presented. The explicit terms of the dynamic stiffness matrices are also given. Plates with different boundary conditions are considered. Rotation and offset matrices for the element are developed and an assembly technique given. The Wittrick and Williams algorithm is modified to avoid the troublesome computation of the clamped-clamped natural frequencies when solving the free vibration problem. The validation of the theory and its application to real structures are illustrated in the second part of this paper.

Keywords: Dynamic stiffness method, thin-walled structures, free vibration analysis, plates, composites.

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1. INTRODUCTION

Composite materials are increasingly being used in structural design, particularly in the aerospace industry. This is mainly due to the benefits derivable from their high specific strength and from their directional properties. The former enables design of structures with minimum weight and maximum strength whereas as the latter can be taken advantage of to produce desirable aeroelastic or other dynamic effects. The use of composite materials in aeronautical design has thus led to much lighter aircraft. However, for an efficient and optimum design of composite structures an accurate knowledge of their static and dynamic behaviour is important. In particular, the free vibration analysis of composite structures is an important consideration in design. The results from free vibration analysis are generally used to characterise aeroelastic behaviour, dynamic response, acoustic performance, and also to avoid possible resonance. The free vibration analysis has always been a fundamental prerequisite for aeroelastic analysis of aircraft structures particulary when the normal mode method is used.

In order to model various parts of aircraft structures such as wings, fuselage, tailplane, fin and rudder when using the finite element method (FEM) [1], it is a standard practice to use plate elements based on assumed shape functions. Components such skins, ribs and spars are generally modelled as plate elements to provide sufficient accuracy. However, it should be recognised that although the FEM is a versatile tool that can be used to analyse structures with complex geometry, it is, nevertheless, an approximate method which by its very nature, requires high computational resources and time, particularly in optimisation studies. Naturally, the excessive demand on computational resources and time can be avoided if more accurate methods of analysis and solution techniques are available. For free vibration analysis of structures, a more accurate and reliable method than the usually adopted FEM is indeed, available which is that of the dynamic stiffness method (DSM). The method has been quite extensively developed for beam elements [2–9] but relatively much less efforts have been expended with some limited, but noteworthy, success for the corresponding developments of plate elements [10–13]. This may probably be due to increased difficulty in formulating the DSM for a two dimensional plate element unlike the relatively simple case for one-dimensional beam element. The limitation in symbolic computation that existed in the past could also be another reason for the lack of progress in the dynamic stiffness developments for plate elements. Understandably, the DSM is appealing in dynamic analysis because unlike the FEM, it provides exact solution of the equation of motion of a structure once the initial assumptions on the displacement field have been made (e.g. Euler−Bernoulli, Timoshenko theories for beams or Kirchhoff, Mindlin or higher order theories for plates). No further approximation is required in the analysis and any number of natural frequencies can be computed using the DSM with as few as a single element which, of course, is impossible in the FEM. The DSM can be very effectively used to study the free vibration behaviour of complex structures because once the dynamic stiffness (DS) matrix of a structural element has been developed, it can be rotated, offset and assembled in a similar way to that of the FEM, to build the global dynamic stiffness matrix of the final structure. Thus by using the DSM, any number of exact natural frequencies and mode shapes of a complex structure can be computed without unnecessarily compromising the accuracy.

DS beam elements have already been implemented and validated in programs such as BUNVIS-RG [14] and PFVIBAT [15]. These programs have clearly demonstrated the efficiency and potential of the DSM to analyse frameworks. On the other hand, DS plate elements based on the classical plate theory (CPT) have been developed for simply support boundary conditions mainly due to research by Wittrick and Williams, which began in the early seventies [10–13]. They implemented their dynamic stiffness theories into a program called VIPASA [13, 16, 17]. In the engineering literature, this program made considerable impact at the time and it was subsequently developed further. Foremost amongst these developments are VICON [16], PASCO [18,19] and VICONOPT [9, 20] which are all well documented.

Wittrick and Williams' DS formulation for CPT based elements [10–13] has been enhanced by the present authors with particular reference to isotropic plates. In this respect, the authors made two principal contributions [21, 22] to the literature. In [21], the DSM for isotropic plates undergoing out of plane free vibration, was extended to include for the first time the first order shear deformation theory (FSDT, also known as Mindlin plate theory [23]). By contrast in [22], the DSM was developed for inplane free vibration of plates in a much simpler and straightforward way than the one published by Wittrick and Williams [13], but importantly, a missing set of solution, not accounted for in the earlier works [9, 13, 16–20], was identified and further developed.

The inadequacy of CPT when studying thick plates is well known [24–26]. This has recently been highlighted by the present authors [21]. Furthermore, it is well recognised that for composite plates the effect of shear deformation can be significant even when the plate is thin because fibre reinforced composites in general have low shear modulii.

Against the above background, in Part I of this two-part paper, the work of Wittrick and Williams [10–13] based on CPT and the previous works of the authors [21, 22] for isotropic plates based on FSDT have been extended to cover DS theories for composite laminates. First, some essential preliminaries such as assumptions on displacement field, derivation of equations of motion and natural boundary conditions for FSDT applications as well as some essential features of classical lamination theory (CLT) are reported briefly in Section 2. Subsequent to this, the DS matrices for bending (Section 3.1) and inplane (Section 3.2) free vibration analysis are developed. The complete DS matrix of the laminated plate element is then formulated in Section 3.3. Despite the complexity of the problem due to the inclusion of the effects of shear deformation and rotatory inertia, it has still been possible to generate explicit expressions for the DS elements by using symbolic computation (Mathematica, [27]). As necessitated by the analysis of any complex structure, rotation and offset transformation matrices that form essential parts of modelling and problem formulation, are described in Section 3.4. Finally, the procedures to assemble and constrain degrees of freedoms, i.e. to impose boundary conditions, and the application of the Wittrick-Williams algorithm [28] to compute natural frequencies and mode shapes are described in Sections 3.5, 3.6, and 3.7, respectively. In this way, the subject matter in Part I concludes (Section 4) with the theory, method of analysis and its description, whilst the numerical results and their validation together with the computational efficiency and accuracy of the proposed DSM and its applications to practical structures are reported in Part II [29] of this paper.

2. PRELIMINARIES

The displacement field for a plate based on Mindlin formulation [23] is assumed as:

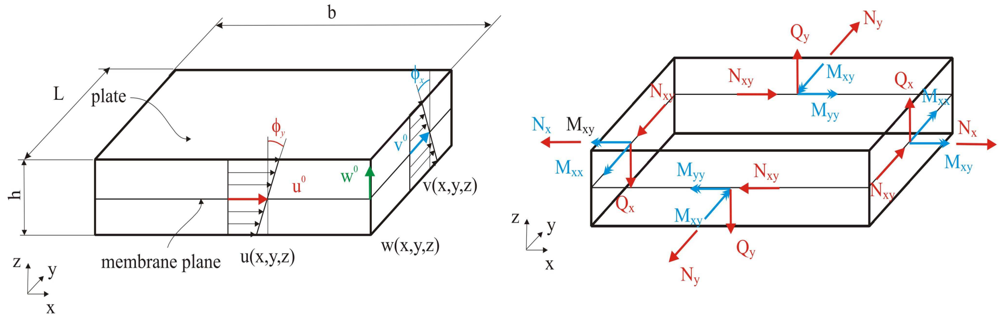
$$u(x,y,z,t) = u^0(x,y,t) + z\phi_y(x,y,t)\ ,\ v(x,y,z,t) = v^0(x,y,t) - z\phi_x(x,y,t)$$
$$w(x,y,z,t) = w^0(x,y,t) \tag{1}$$

where u^0, v^0, w^0 are the membrane displacements along x, y and z directions respectively and ϕ_x, ϕ_y the bending rotations (Fig. 1(a)). Although a composite plate is made of many layers of different materials, the displacement is assumed to be linear though the thickness, and the plate is considered to be an equivalent plate with equivalent properties (classical lamination theory [30, 31]). The geometric relations and constitutive laws used in the formulation are presented in Appendix A. Hamilton's principle is preferred to derive the equations of motion because it routinely provides the natural boundary conditions which are necessary for the dynamic stiffness formulation. Hamilton's principle in the usual notation states:

$$\delta \int_{t_1}^{t_2} (T - U)dt = 0 \tag{2}$$

where the kinetic energy T for the plate is given by:

$$T = \frac{1}{2} \int_A \sum_{k=1}^{N_l} \int_{z_{k-1}}^{z_k} \rho_k \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right) dzdA \tag{3}$$



(a) Displacement field (b) Forces

Figure 1: Coordinate system and notations for displacements and forces for a plate.

with ρ the density, k is the layer reference, and N_l is the number of layers of the composite plate.

Similarly, the potential energy U can be written as:

$$U = \frac{1}{2} \int_A \sum_{k=1}^{N_l} \int_{z_{k-1}}^{z_k} \boldsymbol{\sigma_k}^{\boldsymbol{T}} \boldsymbol{\varepsilon_k} dz dA \tag{4}$$

where:

$$\boldsymbol{\sigma}^T = [\sigma_{xx} \; \sigma_{yy} \; \sigma_{xy} \; \sigma_{yz} \; \sigma_{xz}] \quad \text{and} \quad \boldsymbol{\varepsilon}^T = [\varepsilon_{xx} \; \varepsilon_{yy} \; \varepsilon_{xy} \; \varepsilon_{yz} \; \varepsilon_{xz}] \tag{5}$$

By substituting the geometric and constitutive equations (see Appendix A) into Eqs. (3) and (4) and applying Hamilton's principle (Eq. (2)) the following equations of motion in free vibration, and the natural boundary conditions are obtained:

$$\begin{aligned} \delta u^0 : \quad & + A_{11} u^0_{,xx} + 2A_{16} u^0_{,xy} + A_{66} u^0_{,yy} + A_{16} v^0_{,xx} + (A_{66} + A_{12}) v^0_{,xy} \\ & + A_{26} v^0_{,yy} - B_{16} \phi_{x,xx} - (B_{66} + B_{12}) \phi_{x,xy} - B_{26} \phi_{x,yy} + B_{11} \phi_{y,xx} \\ & + 2B_{16} \phi_{y,xy} + B_{66} \phi_{y,yy} = I_0 \ddot{u}^0 + I_1 \ddot{\phi}_y \end{aligned} \tag{6}$$

$$
\begin{aligned}
\delta v^0 : \quad & + A_{16}u^0_{,xx} + (A_{66} + A_{12})u^0_{,xy} + A_{26}u^0_{,yy} + A_{66}v^0_{,xx} + 2A_{26}v^0_{,xy} \\
& + A_{22}v^0_{,yy} - B_{66}\phi_{x,xx} - 2B_{26}\phi_{x,xy} - B_{22}\phi_{x,yy} + B_{16}\phi_{y,xx} \\
& + (B_{66} + B_{12})\phi_{y,xy} + B_{26}\phi_{y,yy} = I_0\ddot{u}^0 - I_1\ddot{\phi}_x \qquad (7)\\
\delta w^0 : \quad & + kA_{55}w^0_{,xx} + 2kA_{45}w^0_{,xy} + kA_{44}w^0_{,yy} - kA_{45}\phi_{x,x} - kA_{44}\phi_{x,y} \\
& + kA_{55}\phi_{y,x} + kA_{45}\phi_{y,y} = I_0\ddot{w}^0 \qquad (8)\\
\delta \phi_y : \quad & + B_{11}u^0_{,xx} + 2B_{16}u^0_{,xy} + B_{66}u^0_{,yy} + B_{16}v^0_{,xx} + (B_{66} + B_{12})v^0_{,xy} \\
& + B_{26}v^0_{,yy} - D_{16}\phi_{x,xx} - (D_{66} + D_{12})\phi_{x,xy} - D_{26}\phi_{x,yy} + D_{11}\phi_{y,xx} \\
& + 2D_{16}\phi_{y,xy} + D_{66}\phi_{y,yy} - kA_{55}w^0_{,x} - kA_{45}w^0_{,y} + kA_{45}\phi_x - kA_{55}\phi_y \\
& = I_1\ddot{u}^0 + I_2\ddot{\phi}_y \qquad (9)\\
\delta \phi_x : \quad & - B_{16}u^0_{,xx} - (B_{66} + B_{12})u^0_{,xy} - B_{26}u^0_{,yy} - B_{66}v^0_{,xx} - 2B_{26}v^0_{,xy} \\
& - B_{22}v^0_{,yy} + D_{66}\phi_{x,xx} + 2D_{26}\phi_{x,xy} + D_{22}\phi_{x,yy} - D_{16}\phi_{y,xx} \\
& - (D_{66} + D_{12})\phi_{y,xy} - D_{26}\phi_{y,yy} + kA_{45}w^0_{,x} + kA_{44}w^0_{,y} - kA_{44}\phi_x \\
& + kA_{45}\phi_y = -I_1\ddot{v}^0 + I_2\ddot{\phi}_x \qquad (10)
\end{aligned}
$$

The natural boundary conditions with the sign conventions of Figure (1(b)) are:

$$
\begin{aligned}
\delta u^0 : \quad & N_{xx} = +A_{11}u^0_{,x} + A_{16}u^0_{,y} + A_{16}v^0_{,x} + A_{12}v^0_{,y} - B_{16}\phi_{x,x} - B_{12}\phi_{x,y} \\
& + B_{11}\phi_{y,x} + B_{16}\phi_{y,y} \qquad (11)
\end{aligned}
$$

$$\delta v^0: \quad N_{xy} = +A_{16}u^0_{,x} + A_{66}u^0_{,y} + A_{66}v^0_{,x} + A_{26}v^0_{,y} - B_{66}\phi_{x,x} - B_{26}\phi_{x,y} + B_{16}\phi_{y,x} + B_{66}\phi_{y,y} \tag{12}$$

$$\delta w^0: \quad Q_x = +kA_{55}w^0_{,x} + kA_{45}w^0_{,y} - kA_{45}\phi_x + kA_{55}\phi_y \tag{13}$$

$$\delta \phi_y: \quad M_{xx} = +B_{11}u^0_{,x} + B_{16}u^0_{,y} + B_{16}v^0_{,x} + B_{12}v^0_{,y} - D_{16}\phi_{x,x} - D_{12}\phi_{x,y} + D_{11}\phi_{y,x} + D_{16}\phi_{y,y} \tag{14}$$

$$\delta \phi_x: \quad M_{xy} = -B_{16}u^0_{,x} - B_{66}u^0_{,y} - B_{66}v^0_{,x} - B_{26}v^0_{,y} + D_{66}\phi_{x,x} + D_{26}\phi_{x,y} - D_{16}\phi_{y,x} - D_{66}\phi_{y,y} \tag{15}$$

where the suffix after the comma denotes the derivatives, k the shear correction factor ($\pi^2/12$ used by Mindlin [23], 5/6 used by Reissner [32]) and the matrix $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ and the inertia parameters I_0, I_1, and I_2 are given in the usual notation:

$$[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}] = \sum_{k=1}^{N_l} \overline{\boldsymbol{C}}_k \; [(z_k - z_{k-1}), 1/2({z_k}^2 - {z_{k-1}}^2), 1/3({z_k}^3 - {z_{k-1}}^3)] \tag{16}$$

$$[I_0, I_1, I_2] = \sum_{k=1}^{N_l} \rho_k \; [(z_k - z_{k-1}), 1/2({z_k}^2 - {z_{k-1}}^2), 1/3({z_k}^3 - {z_{k-1}}^3)] \tag{17}$$

where ρ_k is the mass density and $\overline{\boldsymbol{C}}_k$ the material property matrix in the laminate coordinate system of the $k-th$ layer which is defined in Appendix A.

The use of Hamilton's principle, as opposed to Newton's second law has the added advantage to give the natural boundary conditions. This is important because the connections between forces and displacements are essential when deriving the dynamic stiffness method.

3. DYNAMIC STIFFNESS FORMULATION

Once the equations of motion (Eqs. 6-10) and the general boundary conditions (Eqs. 11-15) are obtained, the classical method to carry out exact free vibration analysis of a plate consists of solving the system of differential equation in Navier's or Levi's form and applying particular boundary conditions to derive the frequency equation by eliminating the integration constants [26, 33–38]. This method, although extremely useful in studying a single plate, lacks generality and cannot be easily applied to complex structures that are often solved by approximate methods. On the contrary, the dynamic stiffness method retains the exactness of the solution whilst being applied to complex structures. Once the dynamic stiffness matrix of an element is obtained, it can be offset and/or rotated and finally assembled in a global DS matrix of a complex structure. This global DS matrix contains implicity all the exact natural frequencies of the structure which can be computed by using the Wittrick and Williams algorithm [28].

A general procedure to develop the dynamic stiffness matrix of a structural element can be summarised as follows:

(i) Seek a closed form solution of the governing differential equations of motion for free vibration.

(ii) Apply a number of general boundary conditions equal to twice the number of integration constants in algebraic form; these are usually the nodal displacements and forces.

(iii) Eliminate the constants by relating the harmonically varying nodal forces to the corresponding displacements which generates the frequency dependent dynamic stiffness matrix connecting the nodal forces to the nodal displacements.

Referring to the equations of motion (Eqs. 6-10), an exact solution can be found in Levi's form for symmetric and balanced cross ply laminates. For these laminates $\boldsymbol{B} = A_{16} = A_{26} = D_{16} = D_{26} = A_{45} = 0$ and the out of plane motions are decoupled from the inplane ones. The two cases will be studied separately in Section (3.1) and (3.2) respectively, and finally they will be combined in Section (3.3).

3.1. Out of plane formulation

The solution of Eqs. (8-10) is sought in the form:

$$\begin{aligned} w^0(x,y,t) &= \sum_{m=1}^{\infty} W_m(x)\mathrm{e}^{i\omega t}\sin(\alpha_m y) \\ \phi_y(x,y,t) &= \sum_{m=1}^{\infty} \Phi_{y_m}(x)\mathrm{e}^{i\omega t}\sin(\alpha_m y) \\ \phi_x(x,y,t) &= \sum_{m=1}^{\infty} \Phi_{x_m}(x)\mathrm{e}^{i\omega t}\cos(\alpha_m y) \end{aligned} \tag{18}$$

where ω is an arbitrary circular frequency, $\alpha_m = \frac{m\pi}{L}$ and $m = 1, 2, \ldots, \infty$. This is also called Levi's solution which assumes that two opposite sides of the plate are simply supported (SS), i.e. $w = \phi_y = 0$ at $y = 0$ and $y = L$. Substituting Eq. (18) into Eqs. (8-10), a set of three coupled ordinary differential equations is obtained which can be written in matrix form as:

$$\begin{bmatrix} I_0\omega^2 - kA_{44}\alpha_m^2 + kA_{55}\mathcal{D}^2 & kA_{55}\mathcal{D} & kA_{44}\alpha_m \\ -kA_{55}\mathcal{D} & D_{11}\mathcal{D}^2 - D_{66}\alpha_m^2 - kA_{55} + I_2\omega^2 & (D_{66}+D_{12})\alpha_m\mathcal{D} \\ kA_{44}\alpha_m & -(D_{66}+D_{12})\alpha_m\mathcal{D} & D_{66}\mathcal{D}^2 - D_{22}\alpha_m^2 - kA_{44} + I_2\omega^2 \end{bmatrix} \begin{bmatrix} W_m \\ \Phi_{y_m} \\ \Phi_{x_m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where $\mathcal{D}$ is the differential operator d/dx. The determinant of the matrix in Eq. (19) gives the following ordinary differential equation:

$$(\mathcal{D}^6 + a_1\mathcal{D}^4 + a_2\mathcal{D}^2 + a_3\mathcal{D})\Psi = 0 \tag{20}$$

where

$$\Psi = W_m \quad \text{or} \quad \Phi_{y_m} \quad \text{or} \quad \Phi_{x_m} \tag{21}$$

and a_1, a_2, and a_3 are

$$\begin{aligned}
a_1 = &\Big(D_{11}D_{66}(I_0\omega^2 - kA_{44}\alpha^2) + A_{55}k(\alpha^2(D_{12}^2 - D_{11}D_{22} + 2D_{12}D_{66}) - A_{44}D_{11}k \\
&+ (D_{11} + D_{66})I_2\omega^2)\Big)/\Big(kA_{55}D_{11}D_{66}\Big) \\
a_2 = &\Big(\alpha^2k(\alpha^2(A_{55}D_{22}D_{66} - A_{44}(D_{12}^2 - D_{11}D_{22} + 2D_{12}D_{66})) + 2kA_{44}A_{55}(D_{12} + 2D_{66})) \\
&+ A_{55}D_{66}I_0 + A_{44}A_{55}I_2k) + \alpha^2((D_{12}^2 - D_{11}D_{22} + 2D_{12}D_{66})I_0 - (A_{44}(D_{11} + D_{66}) \\
&+ (-k(A_{44}D_{11}I_0 + A_{55}(D_{22} + D_{66}))I_2k))\omega^2 + I_2\omega^4((D_{11} + D_{66})I_0 \\
&+ A_{55}I_2k)\Big)/\Big(kA_{55}D_{11}D_{66}\Big) \\
a_3 = -&\Big((\alpha^2D_{66} + A_{55}k - I_2\omega^2)(A_{44}\alpha^4D_{22}k - (A_{44}I_0k + \alpha^2(D_{22}I_0 + A_{44}I_2k))\omega^2 \\
&+ I_0I_2\omega^4\Big)/\Big(kA_{55}D_{11}D_{66}\Big)
\end{aligned} \tag{22}$$

Using a trial solution e^{λ} in Eq. (20) yields the following auxiliary equation:

$$\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3 = 0 \tag{23}$$

Substituting $\mu = \lambda^2$, the 6^{th} order Eq. (23) becomes:

$$\mu^3 + a_1\mu^2 + a_2\mu + a_3 = 0 \tag{24}$$

The three roots (μ_1, μ_2, μ_3) are given by:

$$\begin{aligned}
\mu_1 &= -\frac{1}{3}\left(a_1 + \sqrt[3]{\frac{p+\sqrt{q}}{2}} + \sqrt[3]{\frac{p-\sqrt{q}}{2}}\right) \\
\mu_2 &= -\frac{1}{3}\left(a_1 + \beta_2\sqrt[3]{\frac{p+\sqrt{q}}{2}} + \beta_1\sqrt[3]{\frac{p-\sqrt{q}}{2}}\right) \\
\mu_3 &= -\frac{1}{3}\left(a_1 + \beta_1\sqrt[3]{\frac{p+\sqrt{q}}{2}} + \beta_2\sqrt[3]{\frac{p-\sqrt{q}}{2}}\right)
\end{aligned} \tag{25}$$

where the above parameters are defined as:

$$\begin{aligned}
&\beta_1 = \frac{i\sqrt{3}-1}{2} \quad , \quad \beta_2 = -\frac{i\sqrt{3}+1}{2} \\
&p = 2a_1^3 - 9a_1a_2 + 27a_3 \quad , \quad q = p^2 - 4l^3 \quad , \quad l = a_1^2 - 3a_2
\end{aligned} \tag{26}$$

The discriminant Δ of Eq. (24) can be written as:

$$\Delta = 18a_1a_2a_3 - 4a_1^3a_3 + a_1^2a_2^2 - 4a_2^2 - 27a_3^2 \tag{27}$$

The sign of the discriminant Δ gives information about the nature of the roots (i.e. if any of them is complex) and by using Descartes' rule [39], the signs of the roots can also be determined. This method was applied earlier by the authors [21, 22] successfully to reduce the number of cases to be investigated. For the present case, the sign of the discriminant can not be generally determined a priori for any material and any trial frequency and can be either positive or negative. Following a parametric study and noting that the trial frequency and material properties are always positive, it was observed that no natural frequency could be found from the solutions coming from complex roots of Eq. (24). Therefore, only real roots of Eq. (24) which are μ_1, μ_2, μ_3 will be presented. If the three roots of Eq. (24) μ_1, μ_2, μ_3 are real, there are no more than four possible solutions, i.e. (i) all three roots are positive, (ii) one negative and two positive, (iii) two negative and one positive and (iv) all three roots are negative. These possibilities together with the associated solutions are elaborated as follows:

- Case 1: μ_1, μ_2, $\mu_3 > 0$. Six real roots r_1, $-r_1$, r_2, $-r_2$, r_3, $-r_3$ where $r_1 = \sqrt{\mu_1}$, $r_2 = \sqrt{\mu_2}$ and $r_3 = \sqrt{\mu_3}$. Thus:

$$\begin{aligned}
W_m(x) =& A_{1_m}\cosh(r_{1_m}x) + A_{2_m}\sinh(r_{1_m}x) + A_{3_m}\cosh(r_{2_m}x) \\
& + A_{4_m}\sinh(r_{2_m}x) + A_{5_m}\cosh(r_{3_m}x) + A_{6_m}\sinh(r_{3_m}x) \\
\Phi_{y_m}(x) =& B_{1_m}\cosh(r_{1_m}x) + B_{2_m}\sinh(r_{1_m}x) + B_{3_m}\cosh(r_{2_m}x) \\
& + B_{4_m}\sinh(r_{2_m}x) + B_{5_m}\cosh(r_{3_m}x) + B_{6_m}\sinh(r_{3_m}x) \\
\Phi_{x_m}(x) =& C_{1_m}\cosh(r_{1_m}x) + C_{2_m}\sinh(r_{1_m}x) + C_{3_m}\cosh(r_{2_m}x) \\
& + C_{4_m}\sinh(r_{2_m}x) + C_{5_m}\cosh(r_{3_m}x) + C_{6_m}\sinh(r_{3_m}x)
\end{aligned} \tag{28}$$

- Case 2: μ_1, $\mu_2 > 0$ and $\mu_3 < 0$. The roots are rearranged so that the negative root is μ_3, thus r_1, $-r_1$, r_2, $-r_2$, ir_3, $-ir_3$ where $r_1 = \sqrt{\mu_1}$, $r_2 = \sqrt{\mu_2}$ and $r_3 = \sqrt{-\mu_3}$. Thus the solutions can be written as:

$$\begin{aligned}
W_m(x) =& A_{1_m}\cosh(r_{1_m}x) + A_{2_m}\sinh(r_{1_m}x) + A_{3_m}\cosh(r_{2_m}x) \\
& + A_{4_m}\sinh(r_{2_m}x) + A_{5_m}\cos(r_{3_m}x) + A_{6_m}\sin(r_{3_m}x) \\
\Phi_{y_m}(x) =& B_{1_m}\cosh(r_{1_m}x) + B_{2_m}\sinh(r_{1_m}x) + B_{3_m}\cosh(r_{2_m}x) \\
& + B_{4_m}\sinh(r_{2_m}x) + B_{5_m}\cos(r_{3_m}x) + B_{6_m}\sin(r_{3_m}x) \\
\Phi_{x_m}(x) =& C_{1_m}\cosh(r_{1_m}x) + C_{2_m}\sinh(r_{1_m}x) + C_{3_m}\cosh(r_{2_m}x) \\
& + C_{4_m}\sinh(r_{2_m}x) + C_{5_m}\cos(r_{3_m}x) + C_{6_m}\sin(r_{3_m}x)
\end{aligned} \tag{29}$$

- Case 3. $\mu_1 > 0$ and μ_2, $\mu_3 < 0$. The roots are rearranged so that the negative roots are μ_2 and μ_3, thus r_1, $-r_1$, ir_2, $-ir_2$, ir_3, $-ir_3$ where

$r_1 = \sqrt{\mu_1}$, $r_2 = \sqrt{-\mu_2}$ and $r_3 = \sqrt{-\mu_3}$. Thus the solutions can be written as:

$$\begin{aligned}
W_m(x) =& A_{1_m}\cosh(r_{1_m}x) + A_{2_m}\sinh(r_{1_m}x) + A_{3_m}\cos(r_{2_m}x) \\
&+ A_{4_m}\sin(r_{2_m}x) + A_{5_m}\cos(r_{3_m}x) + A_{6_m}\sin(r_{3_m}x) \\
\Phi_{y_m}(x) =& B_{1_m}\cosh(r_{1_m}x) + B_{2_m}\sinh(r_{1_m}x) + B_{3_m}\cos(r_{2_m}x) \\
&+ B_{4_m}\sin(r_{2_m}x) + B_{5_m}\cos(r_{3_m}x) + B_{6_m}\sin(r_{3_m}x) \\
\Phi_{x_m}(x) =& C_{1_m}\cosh(r_{1_m}x) + C_{2_m}\sinh(r_{1_m}x) + C_{3_m}\cos(r_{2_m}x) \\
&+ C_{4_m}\sin(r_{2_m}x) + C_{5_m}\cos(r_{3_m}x) + C_{6_m}\sin(r_{3_m}x)
\end{aligned} \tag{30}$$

- Case 4. μ_1, μ_2, $\mu_3 < 0$. The six roots are imaginary ir_1, $-ir_1$, ir_2, $-ir_2$, ir_3, $-ir_3$ where $r_1 = \sqrt{-\mu_1}$, $r_2 = \sqrt{-\mu_2}$ and $r_3 = \sqrt{-\mu_3}$. Thus the solutions can be written as:

$$\begin{aligned}
W_m(x) =& A_{1_m}\cos(r_{1_m}x) + A_{2_m}\sin(r_{1_m}x) + A_{3_m}\cos(r_{2_m}x) \\
&+ A_{4_m}\sin(r_{2_m}x) + A_{5_m}\cos(r_{3_m}x) + A_{6_m}\sin(r_{3_m}x) \\
\Phi_{y_m}(x) =& B_{1_m}\cos(r_{1_m}x) + B_{2_m}\sin(r_{1_m}x) + B_{3_m}\cos(r_{2_m}x) \\
&+ B_{4_m}\sin(r_{2_m}x) + B_{5_m}\cos(r_{3_m}x) + B_{6_m}\sin(r_{3_m}x) \\
\Phi_{x_m}(x) =& C_{1_m}\cos(r_{1_m}x) + C_{2_m}\sin(r_{1_m}x) + C_{3_m}\cos(r_{2_m}x) \\
&+ C_{4_m}\sin(r_{2_m}x) + C_{5_m}\cos(r_{3_m}x) + C_{6_m}\sin(r_{3_m}x)
\end{aligned} \tag{31}$$

In this paper only the details regarding the first case will be given for conciseness. The other cases can be solved in a similar manner.

The solutions in Eq. (28) have 18 constants ($A_{1_m} - A_{6_m}$, $B_{1_m} - B_{6_m}$ and $C_{1_m} - C_{6_m}$) which are not all independent. By substituting Eqs. (28) into the last two equations of Eq. (19) and putting each of the terms to zero, 12 equations in 18 unknowns are obtained. By simultaneously solving the 12 equations in terms of one of the set of constants, i.e. $B_{1_m} - B_{6_m}$, a connection between the constants can be established and thus only 6 independent integration constants are now left. As a consequence, the following relationships are obtained:

$$\begin{array}{llllllll}
A_{1_m} = \delta_1 B_{2_m} & , & C_{1_m} = \gamma_1 B_{2_m} & , & A_{2_m} = \delta_1 B_{1_m} & , & C_{2_m} = \gamma_1 B_{1_m} \\
A_{3_m} = \delta_2 B_{4_m} & , & C_{3_m} = \gamma_2 B_{4_m} & , & A_{4_m} = \delta_2 B_{3_m} & , & C_{4_m} = \gamma_2 B_{3_m} \\
A_{5_m} = \delta_3 B_{6_m} & , & C_{5_m} = \gamma_3 B_{6_m} & , & A_{6_m} = \delta_3 B_{5_m} & , & C_{6_m} = \gamma_3 B_{5_m}
\end{array} \tag{32}$$

where:

$$\begin{aligned}
\delta_i =& \Big((\alpha^2 D_{22} + A_{44} k - I_2 \omega^2)(\alpha^2 D_{66} + A_{55} k - I_2 \omega^2) \\
&+ (\alpha^2 (D_{12}^2 - D_{11} D_{22} + 2 D_{12} D_{66}) - (A_{44} D_{11} + A_{55} D_{66}) k \\
&+ (D_{11} + D_{66}) I_2 \omega^2) r_i^2 + D_{11} D_{66} r_1^4 \Big) / \\
&\Big(k r_i (A_{44} \alpha^2 (D_{12} + D_{66}) - A_{44} A_{55} k + A_{55} (I_2 \omega^2 - \alpha^2 D_{22} + D_{66} r_i^2)) \Big)
\end{aligned} \tag{33}$$

$$\gamma_i = \frac{\alpha (A_{55} (D_{12} + D_{66}) r_i^2 + A_{44} (\alpha^2 D_{66} + A_{55} k - I_2 \omega^2 - D_{11} r_i^2))}{r_i (A_{44} \alpha^2 (D_{12} + D_{66}) - A_{44} A_{55} k + A_{55} (I_2 \omega^2 - \alpha^2 D_{22} + D_{66} r_i^2))} \tag{34}$$

with $i = 1, 2, 3$.

The above procedure must be completed with care. If a wrong set of equations is chosen from Eq. (19) to standardise the constants or if a wrong set of constants is assumed to be independent, numerical instabilities can occur. The authors have found that the above choice of constants leads to the exact solutions for all of the 4 cases.

If Eqs. (32) are substituted into Eqs. (28) a solution in terms of only 6 integration constants can be expressed as

$$
\begin{aligned}
W_m(x) =& + B_{2_m}\delta_1\cosh(r_{1_m}x) + B_{1_m}\delta_1\sinh(r_{1_m}x) + B_{4_m}\delta_2\cosh(r_{2_m}x) \\
& + B_{3_m}\delta_2\sinh(r_{2_m}x) + B_{6_m}\delta_3\cosh(r_{3_m}x) + B_{5_m}\delta_3\sinh(r_{3_m}x) \\
\Phi_{y_m}(x) =& + B_{1_m}\cosh(r_{1_m}x) + B_{2_m}\sinh(r_{1_m}x) + B_{3_m}\cosh(r_{2_m}x) \qquad (35) \\
& + B_{4_m}\sinh(r_{2_m}x) + B_{5_m}\cosh(r_{3_m}x) + B_{6_m}\sinh(r_{3_m}x) \\
\Phi_{x_m}(x) =& + B_{2_m}\gamma_1\cosh(r_{1_m}x) + B_{1_m}\gamma_1\sinh(r_{1_m}x) + B_{4_m}\gamma_2\cosh(r_{2_m}x) \\
& + B_{3_m}\gamma_2\sinh(r_{2_m}x) + B_{6_m}\gamma_3\cosh(r_{3_m}x) + B_{5_m}\gamma_3\sinh(r_{3_m}x)
\end{aligned}
$$

The expressions for forces and moments can be found in the same way by substituting Eqs. (35) into Eqs. (13-15). Thus

$$
\begin{aligned}
Q_{x_m}(x,y) =& \mathcal{Q}_{x_m}(x)\sin(\alpha_m y) = \\
=& \; A_{55}k\Big(+ B_{1_m}(1+\delta_1 r_{1_m})\cosh(r_{1_m}x) \\
& + B_{2_m}(1+\delta_1 r_{1_m})\sinh(r_{1_m}x) \\
& + B_{3_m}(1+\delta_2 r_{2_m})\cosh(r_{2_m}x) \\
& + B_{4_m}(1+\delta_2 r_{2_m})\sinh(r_{2_m}x) \\
& + B_{5_m}(1+\delta_3 r_{3_m})\cosh(r_{3_m}x) \\
& + B_{6_m}(1+\delta_3 r_{3_m})\sinh(r_{3_m}x)\Big)\sin(\alpha_m y) \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
M_{xx_m}(x,y) =& \mathcal{M}_{xx_m}(x)\sin(\alpha_m y) = \\
=& \Big(+ B_{2_m}(D_{11}r_{1_m} + \alpha D_{12}\gamma_1)\cosh(r_{1_m}x) \\
& + B_{1_m}(D_{11}r_{1_m} + \alpha D_{12}\gamma_1)\sinh(r_{1_m}x) \\
& + B_{4_m}(D_{11}r_{2_m} + \alpha D_{12}\gamma_2)\cosh(r_{2_m}x) \\
& + B_{3_m}(D_{11}r_{2_m} + \alpha D_{12}\gamma_2)\sinh(r_{2_m}x) \\
& + B_{6_m}(D_{11}r_{3_m} + \alpha D_{12}\gamma_3)\cosh(r_{3_m}x) \\
& + B_{5_m}(D_{11}r_{3_m} + \alpha D_{12}\gamma_3)\sinh(r_{3_m}x)\Big)\sin(\alpha_m y)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
M_{xy_m}(x,y) =& \mathcal{M}_{xy_m}(x)\cos(\alpha_m y) = \\
=& \; -D_{66}\Big(+ B_{1_m}(\alpha - \gamma_1 r_{1_m})\cosh(r_{1_m}x) \\
& + B_{2_m}(\alpha - \gamma_1 r_{1_m})\sinh(r_{1_m}x) \\
& + B_{3_m}(\alpha - \gamma_2 r_{2_m})\cosh(r_{2_m}x) \\
& + B_{4_m}(\alpha - \gamma_2 r_{2_m})\sinh(r_{2_m}x) \\
& + B_{5_m}(\alpha - \gamma_3 r_{3_m})\cosh(r_{3_m}x) \\
& + B_{6_m}(\alpha - \gamma_3 r_{3_m})\sinh(r_{3_m}x)\Big)\cos(\alpha_m y)
\end{aligned} \tag{38}
$$

At this point, zero boundary conditions are generally used to eliminate the constants in the classical method and establish the frequency equation. By contrast, in order to develop the dynamic stiffness matrix, general boundary conditions in algebraic form are used. These boundary conditions can be seen in Fig. (2) and formulated as

$$
\begin{aligned}
&\text{At} \quad x = 0: \qquad W_m = W_1\,,\; \Phi_{y_m} = \Phi_{y_1}\,,\; \Phi_{x_m} = \Phi_{x_1} \\
&\text{At} \quad x = b: \qquad W_m = W_2\,,\; \Phi_{y_m} = \Phi_{y_2}\,,\; \Phi_{x_m} = \Phi_{x_2}
\end{aligned} \tag{39}
$$

$$
\begin{array}{llll}
\text{At} & x = 0 : & \mathcal{Q}_{x_m} = -\mathcal{Q}_1 \,,\, \mathcal{M}_{xx_m} = -\mathcal{M}_{xx_1} \,,\, \mathcal{M}_{xy_m} = -\mathcal{M}_{xy_1} \\
\text{At} & x = b : & \mathcal{Q}_{x_m} = \quad \mathcal{Q}_2 \,,\, \mathcal{M}_{xx_m} = \quad \mathcal{M}_{xx_2} \,,\, \mathcal{M}_{xy_m} = \quad \mathcal{M}_{xy_2}
\end{array} \tag{40}
$$

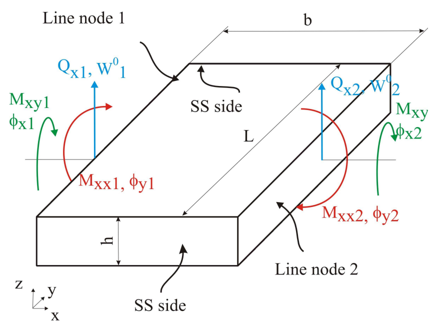


Figure 2: Edge conditions of the plate element and sign conventions

By substituting Eqs. (39) into Eqs. (35), the following matrix relation for the displacements is obtained:

$$
\begin{bmatrix} W_1 \\ \Phi_{y_1} \\ \Phi_{x_1} \\ W_2 \\ \Phi_{y_2} \\ \Phi_{x_2} \end{bmatrix} = \begin{bmatrix} 0 & \delta_1 & 0 & \delta_2 & 0 & \delta_3 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \delta_1 \mathcal{S}_{h_1} & \delta_1 \mathcal{C}_{h_1} & \delta_2 \mathcal{S}_{h_2} & \delta_2 \mathcal{C}_{h_2} & \delta_3 \mathcal{S}_{h_3} & \delta_3 \mathcal{C}_{h_3} \\ \mathcal{C}_{h_1} & \mathcal{S}_{h_1} & \mathcal{C}_{h_2} & \mathcal{S}_{h_2} & \mathcal{C}_{h_3} & \mathcal{S}_{h_3} \\ \gamma_1 \mathcal{S}_{h_1} & \gamma_1 \mathcal{C}_{h_1} & \gamma_2 \mathcal{S}_{h_2} & \gamma_2 \mathcal{C}_{h_2} & \gamma_3 \mathcal{S}_{h_3} & \gamma_3 \mathcal{C}_{h_3} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix} \tag{41}
$$

i.e.

$$
\boldsymbol{\delta} = \boldsymbol{AC} \tag{42}
$$

By applying the same procedure for forces and moments, i.e. substituting Eqs. (40) into Eqs. (36-38) the following matrix relationship is obtained:

$$\begin{bmatrix} \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \end{bmatrix} = \begin{bmatrix} -\mathcal{L}_1 & 0 & -\mathcal{L}_2 & 0 & -\mathcal{L}_3 & 0 \\ 0 & -\mathcal{R}_1 & 0 & -\mathcal{R}_2 & 0 & -\mathcal{R}_3 \\ -\mathcal{T}_1 & 0 & -\mathcal{T}_2 & 0 & -\mathcal{T}_3 & 0 \\ \mathcal{L}_1\mathcal{C}_{h_1} & \mathcal{L}_1\mathcal{S}_{h_1} & \mathcal{L}_2\mathcal{C}_{h_2} & \mathcal{L}_2\mathcal{S}_{h_2} & \mathcal{L}_3\mathcal{C}_{h_3} & \mathcal{L}_3\mathcal{S}_{h_3} \\ \mathcal{R}_1\mathcal{S}_{h_1} & \mathcal{R}_1\mathcal{C}_{h_1} & \mathcal{R}_2\mathcal{S}_{h_2} & \mathcal{R}_2\mathcal{C}_{h_2} & \mathcal{R}_3\mathcal{S}_{h_3} & \mathcal{R}_3\mathcal{C}_{h_3} \\ \mathcal{T}_1\mathcal{C}_{h_1} & \mathcal{T}_1\mathcal{S}_{h_1} & \mathcal{T}_2\mathcal{C}_{h_2} & \mathcal{T}_2\mathcal{S}_{h_2} & \mathcal{T}_3\mathcal{C}_{h_3} & \mathcal{T}_3\mathcal{S}_{h_3} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix} \tag{43}$$

i.e.

$$\boldsymbol{F} = \boldsymbol{RC} \tag{44}$$

where

$$\mathcal{L}_i = A_{55}k(1+\delta_i r_i) \quad , \quad \mathcal{R}_i = \alpha D_{12}\gamma_i + D_{11}r_i \quad , \quad \mathcal{T}_i = D_{66}(\alpha - \gamma_i r_i) \tag{45}$$

with $i = 1, 2, 3$.

Now the constant vector $\boldsymbol{C}$ from Eqs. (42) and (44) can be eliminated to form the dynamic stiffness matrix as follows:

$$\boldsymbol{F} = \boldsymbol{K\delta} \tag{46}$$

where

$$\boldsymbol{K} = \boldsymbol{RA}^{-1} \tag{47}$$

$\boldsymbol{K}$ can be written as:

$$\boldsymbol{K} = \begin{bmatrix} s_{qq} & s_{qm} & s_{qt} & f_{qq} & f_{qm} & f_{qt} \\ & s_{mm} & s_{mt} & -f_{qm} & f_{mm} & f_{mt} \\ & & s_{tt} & f_{qt} & -f_{mt} & f_{tt} \\ & \text{Sym} & & s_{qq} & -s_{qm} & s_{qt} \\ & & & & s_{mm} & -s_{mt} \\ & & & & & s_{tt} \end{bmatrix} \tag{48}$$

where there are only 12 independent, frequency dependent terms s_{qq}, s_{qm}, s_{qt}, s_{mm}, s_{mt}, s_{tt}, f_{qq}, f_{qm}, f_{qt}, f_{mm}, f_{mt}, f_{tt} given in explicit algebraic form in Appendix B.

3.2. Inplane formulation

As was the case with out of plane free vibration, the solutions of the equations of motion (6) and (7) are again sought in Levi's form, i.e two opposite sides are simply supported (S). The S assumption for inplane displacements can be of two types [36, 37], namely S1 (for $y = 0$ and $y = L \Rightarrow u = 0$ and $v \neq 0$; for $x = 0$ and $x = b \Rightarrow v = 0$ and $u \neq 0$) and S2 (for $y = 0$ and $y = L \Rightarrow v = 0$ and $u \neq 0$; for $x = 0$ and $x = b \Rightarrow u = 0$ and $v \neq 0$). In order to allow compatibility of inplane and out of plane displacements for the general case[1], only S1 on $y = 0$ and $y = L$ needs to be considered. Therefore, the solution for S1 boundary conditions is sought in the following form:

$$\begin{aligned} u^0(x,y,t) &= \sum_{m=0}^{\infty} U_m(x)\mathrm{e}^{i\omega t}\sin(\alpha_m y) \\ v^0(x,y,t) &= \sum_{m=0}^{\infty} V_m(x)\mathrm{e}^{i\omega t}\cos(\alpha_m y) \end{aligned} \tag{49}$$

Substituting Eq. (49) into Eqs. (6) and (7) two coupled ordinary differential equations are obtained. These can be written in matrix form as:

$$\begin{bmatrix} A_{11}\mathcal{D}^2 - A_{66}\alpha_m^2 + I_0\omega^2 & -(A_{66}+A_{12})\alpha_m\mathcal{D} \\ (A_{66}+A_{12})\alpha_m\mathcal{D} & A_{66}\mathcal{D}^2 - A_{22}\alpha_m^2 + I_0\omega^2 \end{bmatrix} \begin{bmatrix} U_m \\ V_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$

[1]For example, after a 90^o rotation, local displacement w becomes local u thus u and w must have the same distributions on y

where $\mathcal{D}$ is the differential operator d/dx. It is necessary to split the solution in two independent cases. The first case is for $m \neq 0$ and the second case is for $m = 0$. When $m = 0$, the nature of the differential equation (50) changes and the solution must be sought separately. (The case $m = 0$ was missed in previous formulations for the inplane dynamic stiffness of plates [13] based on the classical plate theory.)

3.2.1. General case ($m \neq 0$) for inplane vibration

For $m \neq 0$ the determinant of the matrix in Eq. (50) gives the following governing differential equation:

$$(\mathcal{D}^4 + b_1\mathcal{D}^2 + b_2)\Psi = 0 \tag{51}$$

where:

$$\Psi = U_m \quad \text{or} \quad V_m \tag{52}$$

Using a trial solution e^{λ} in Eq. (51) yields the following auxiliary equation:

$$\lambda^4 + b_1\lambda^2 + b_2 = 0 \tag{53}$$

where:

$$\begin{aligned} b_1 &= \frac{(A_{12}^2 - A_{11}A_{22} + 2A_{12}A_{66})\alpha^2 + (A_{11} + A_{66})I_0\omega^2}{A_{11}A_{66}} \\ b_2 &= \frac{(A_{22}\alpha^2 - I_0\omega^2)^2}{A_{11}A_{66}} \end{aligned} \tag{54}$$

Substituting $\mu = \lambda^2$, the 4^{th} order Eq. (53) becomes:

$$\mu^2 + b_1\mu + b_2 = 0 \tag{55}$$

the two roots (μ_1, μ_2) are given by:

$$\mu_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} \tag{56}$$

By analysing the discriminant of Eq. (55) it can be concluded that the two roots μ_1 and μ_2 are always real. Thus only three solution cases are possible. Case 1: μ_1, $\mu_2 > 0$. Four real roots r_1, $-r_1$, r_2 and $-r_2$ where $r_1 = \sqrt{\mu_1}$ and $r_2 = \sqrt{\mu_2}$. Thus:

$$\begin{aligned} U_m(x) =& A_{1_m}\cosh(r_{1_m}x) + A_{2_m}\sinh(r_{1_m}x) + A_{3_m}\cosh(r_{2_m}x) + A_{4_m}\sinh(r_{2_m}x) \\ V_m(x) =& B_{1_m}\cosh(r_{1_m}x) + B_{2_m}\sinh(r_{1_m}x) + B_{3_m}\cosh(r_{2_m}x) + B_{4_m}\sinh(r_{2_m}x) \end{aligned} \tag{57}$$

Case 2: $\mu_1 > 0$ and $\mu_2 < 0$. The roots are rearranged so that the negative root is μ_2, thus r_1, $-r_1$, ir_2, $-ir_2$ where $r_1 = \sqrt{\mu_1}$ and $r_2 = \sqrt{\mu_2}$. Thus the solutions can be written as:

$$\begin{aligned} U_m(x) =& A_{1_m}\cosh(r_{1_m}x) + A_{2_m}\sinh(r_{1_m}x) + A_{3_m}\cos(r_{2_m}x) + A_{4_m}\sin(r_{2_m}x) \\ V_m(x) =& B_{1_m}\cosh(r_{1_m}x) + B_{2_m}\sinh(r_{1_m}x) + B_{3_m}\cos(r_{2_m}x) + B_{4_m}\sin(r_{2_m}x) \end{aligned} \tag{58}$$

Case 3: μ_1, $\mu_2 < 0$. All four roots are imaginary ir_1, $-ir_1$, ir_2 and $-ir_2$ where $r_1 = \sqrt{\mu_1}$ and $r_2 = \sqrt{\mu_2}$. Thus the solution can be written as:

$$\begin{aligned} U_m(x) =& A_{1_m}\cos(r_{1_m}x) + A_{2_m}\sin(r_{1_m}x) + A_{3_m}\cos(r_{2_m}x) + A_{4_m}\sin(r_{2_m}x) \\ V_m(x) =& B_{1_m}\cos(r_{1_m}x) + B_{2_m}\sin(r_{1_m}x) + B_{3_m}\cos(r_{2_m}x) + B_{4_m}\sin(r_{2_m}x) \end{aligned} \tag{59}$$

In this paper only the solution for the first case is given, the other cases can be solved by using the same procedure, but for sake of brevity, they are not reported here.

The solutions of Eq. (57) have two sets of four constants $A_{1_m} - A_{4_m}$ and $B_{1_m} - B_{4_m}$ which are not all independent. By substituting Eqs. (57) into the first equation of Eq. (50) and putting each of the terms to zero, 4 equations in 8 unknowns are obtained. By simultaneously solving the 4 equations in terms of one of the set of constants, i.e. $A_{1_m} - A_{4_m}$, a relationship between the constants can be established with only 4 independent integration constants. In this way

$$\begin{aligned} B_{1_m} &= \beta_1 A_{2_m} \quad , \quad B_{2_m} = \beta_1 A_{1_m} \\ B_{3_m} &= \beta_2 A_{4_m} \quad , \quad B_{4_m} = \beta_2 A_{3_m} \end{aligned} \tag{60}$$

where

$$\beta_i = \frac{I_0\omega^2 - A_{66}\alpha^2 + A_{11}r_i^2}{(A_{12} + A_{66})\alpha r_i} \quad \text{with} \quad i = 1, 2 \tag{61}$$

Likewise, Eqs (60) are substituted into Eqs. (57) a solution in terms of only 4 integration constants can be found as to give

$$\begin{aligned} U_m(x) =& A_{1_m}\cosh(r_{1_m}x) + A_{2_m}\sinh(r_{1_m}x) + A_{3_m}\cosh(r_{2_m}x) + A_{4_m}\sinh(r_{2_m}x) \\ V_m(x) =& A_{2_m}\beta_1\cosh(r_{1_m}x) + A_{1_m}\beta_1\sinh(r_{1_m}x) + A_{4_m}\beta_2\cosh(r_{2_m}x) \\ &+ A_{3_m}\beta_2\sinh(r_{2_m}x) \end{aligned} \tag{62}$$

The equations for the inplane forces can be found in the same way by substituting Eqs. (62) into Eqs. (11) and (12):

$$
\begin{aligned}
N_{xx_m}(x,y) =&\mathcal{N}_{xx_m}(x)\cos(\alpha_m y) = \Big(A_{1_m}(A_{11}r_{1_m} - A_{12}\alpha_m\beta_1)\sinh(r_{1_m}x)+\\
&A_{2_m}(A_{11}r_{1_m} - A_{12}\alpha_m\beta_1)\cosh(r_{1_m}x)+\\
&A_{3_m}(A_{11}r_{2_m} - A_{12}\alpha_m\beta_2)\sinh(r_{2_m}x)+\\
&A_{4_m}(A_{11}r_{2_m} - A_{12}\alpha_m\beta_2)\cosh(r_{2_m}x)\Big)\cos(\alpha_m y)
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
N_{xy_m}(x,y) =&\mathcal{N}_{xy_m}(x)\sin(\alpha_m y) = A_{66}\Big(A_{1_m}(\alpha_m + \beta_1 r_{1_m})\cosh(r_{1_m}x)+\\
&A_{2_m}(\alpha_m + \beta_1 r_{1_m})\sinh(r_{1_m}x) + A_{3_m}(\alpha_m + \beta_2 r_{2_m})\cosh(r_{2_m}x)+\\
&A_{4_m}(\alpha_m + \beta_2 r_{2_m})\sinh(r_{2_m}x)\Big)\sin(\alpha_m y)
\end{aligned}
\tag{64}
$$

To establish the dynamic stiffness matrix, general boundary conditions are imposed. These boundary conditions can be seen in Fig. (3) and formulated as

$$
\begin{aligned}
&\text{At } x=0: \quad && U=U_1\,,\ V=V_1\,,\ \mathcal{N}_{xx}=-\mathcal{N}_{xx_1}\,,\ \mathcal{N}_{xy}=-\mathcal{N}_{xy_1}\\
&\text{At } x=b: \quad && U=U_2\,,\ V=V_2\,,\ \mathcal{N}_{xx}=\ \ \mathcal{N}_{xx_2}\,,\ \mathcal{N}_{xy}=\ \ \mathcal{N}_{xy_2}
\end{aligned}
\tag{65}
$$

By substituting Eqs. (65) into Eqs. (62), the following matrix relationship for the displacements is obtained:

$$
\begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{bmatrix} =
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & \beta_1 & 0 & \beta_2 \\
\mathcal{C}_{h_1} & \mathcal{S}_{h_1} & \mathcal{C}_{h_2} & \mathcal{S}_{h_2} \\
\beta_1\mathcal{S}_{h_1} & \beta_1\mathcal{C}_{h_1} & \beta_2\mathcal{S}_{h_2} & \beta_2\mathcal{C}_{h_2}
\end{bmatrix}
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}
\tag{66}
$$

i.e.

$$
\boldsymbol{\delta} = \boldsymbol{AC} \tag{67}
$$

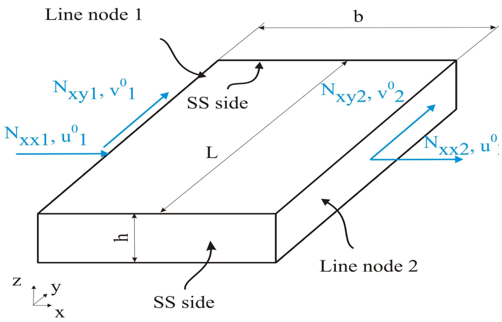


Figure 3: Boundary conditions of the plate element and positive sign conventions

By applying the same procedure for the inplane forces, i.e. substituting Eqs. (65) into Eqs. (63) and (64) the following matrix relation is obtained:

$$\begin{bmatrix} \mathcal{N}_{xx_1} \\ \mathcal{N}_{xy_1} \\ \mathcal{N}_{xx_2} \\ \mathcal{N}_{xy_2} \end{bmatrix} = \begin{bmatrix} 0 & -\mathcal{H}_1 & 0 & -\mathcal{H}_2 \\ -\mathcal{P}_1 & 0 & -\mathcal{P}_2 & 0 \\ \mathcal{H}_1\mathcal{S}_{h_1} & \mathcal{H}_1\mathcal{C}_{h_1} & \mathcal{H}_2\mathcal{S}_{h_2} & \mathcal{H}_2\mathcal{C}_{h_2} \\ \mathcal{P}_1\mathcal{C}_{h_1} & \mathcal{P}_1\mathcal{S}_{h_1} & \mathcal{P}_2\mathcal{C}_{h_2} & \mathcal{P}_2\mathcal{S}_{h_2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \tag{68}$$

i.e.

$$\boldsymbol{F} = \boldsymbol{RC} \tag{69}$$

where

$$\mathcal{H}_i = A_{11} r_{i_m} - A_{12}\alpha_m\beta_i \quad , \quad \mathcal{P}_i = A_{66}(\alpha_m + \beta_i r_{i_m}) \tag{70}$$

with $i = 1, 2$.

By eliminating the constants from Eqs. (67) and (69) the inplane dynamic stiffness matrix is given by:

$$\boldsymbol{F} = \boldsymbol{K\delta} \tag{71}$$

where

$$\boldsymbol{K} = \boldsymbol{RA}^{-1} \tag{72}$$

$\boldsymbol{K}$ can be written as:

$$\boldsymbol{K} = \begin{bmatrix} s_{nn} & s_{nl} & f_{nn} & f_{nl} \\ & s_{ll} & -f_{nl} & f_{ll} \\ & & s_{nn} & -s_{nl} \\ & \text{Sym} & & s_{tl} \end{bmatrix} \tag{73}$$

It should be noted that the dynamic stiffness matrix has only 6 independent terms s_{nn}, s_{nl}, s_{ll}, f_{nn}, f_{nl} and f_{ll}. Explicit expressions for these terms are given in Appendix C for all three cases.

3.2.2. Particular case for inplane vibration when $m = 0$

When $m = 0$ in Eq. (49), $u^0(x, y, t)$ is zero while $v^0(x, y, t) = V(x)e^{i\omega t}$. Thus, the equation of motion Eq. (50) becomes:

$$A_{66}\frac{d^2V}{dx^2} + I_0\omega^2 V = 0 \tag{74}$$

The solution of Eq. (74) can be written as:

$$V = A_1\cos(rx) + A_2\sin(rx) \quad \text{with} \quad r = \omega\sqrt{\frac{I_0}{A_66}} \tag{75}$$

Substituting Eqs. (75) into Eqs. (11) and (12), $N_{xx}(x, y)$ becomes zero and

$$N_{xy}(x, y) = \mathcal{N}_{xy}(x) = A_{66}r\left(-A_1\sin(rx) + A_2\cos(rx)\right) \tag{76}$$

The BC for the displacement and force (Fig. 3) are

$$\begin{aligned} &\text{At } x = 0 : \qquad V = V_1\,,\ \mathcal{N}_{xy} = -\mathcal{N}_{xy_1}\,; \\ &\text{At } x = b : \qquad V = V_2\,,\ \mathcal{N}_{xy} = +\mathcal{N}_{xy_2} \end{aligned} \tag{77}$$

By applying BCs for displacements, i.e. substituting Eq. (77) into Eq. (75), the following relationship is obtained:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathcal{C} & \mathcal{S} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{78}$$

where $\mathcal{C} = \cos(rb)$ and $\mathcal{S} = \sin(rb)$.

By applying BC for the forces, i.e. substituting Eq. (77) into Eqs. (76), the following relationship is obtained:

$$\begin{bmatrix} \mathcal{N}_{xy_1} \\ \mathcal{N}_{xy_2} \end{bmatrix} = \begin{bmatrix} 0 & -A_{66}r \\ -A_{66}r\mathcal{S} & A_{66}r\mathcal{C} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{79}$$

From Eqs. (78) and (79), the vector of the integration constants can be eliminated and in this way, the dynamic stiffness matrix can be obtained as

$$\boldsymbol{K} = \begin{bmatrix} s_{nn} & f_{nn} \\ f_{nn} & s_{nn} \end{bmatrix} \tag{80}$$

where the 2 independent terms are:

$$s_{nn} = A_{66}r\cot(rb) \quad , \quad f_{nn} = -A_{66}r\,\mathrm{cosec}(rb) \tag{81}$$

3.2.3. Inplane free vibration with S2 boundary conditions

Although S2 boundary conditions on sides $y = 0$ and $y = L$ can not be used to study a structure with complex geometry since in order to have compatibility of the displacements after rotation, u and w must be in phase, the S2 boundary conditions can still be used to study inplane vibration of simple plates. The dynamic stiffness matrix for inplane free vibration based on S2 boundary conditions has never been developed before. Different from Equation (49), the displacement field is now assumed to be:

$$
\begin{aligned}
u^0(x,y,t) &= \sum_{m=0}^{\infty} U_m(x)\mathrm{e}^{i\omega t}\cos(\alpha_m y) \\
v^0(x,y,t) &= \sum_{m=0}^{\infty} V_m(x)\mathrm{e}^{i\omega t}\sin(\alpha_m y)
\end{aligned}
\tag{82}
$$

Following the same procedure presented in Section 3.2.1 the dynamic stiffness matrix is obtained. The matrix is of the same as the one reported in Equation (73) with the only differences in the following terms:

$$
s_{nl}^{SS2} = -s_{nl}^{SS1} \text{ and } f_{nl}^{SS2} = -f_{nl}^{SS1} \tag{83}
$$

The explicit expressions for S1 boundary condition are reported in Appendix C.

For what concerns the particular case for $m = 0$, the displacements become $u^0(x,y,t) = U(x)\mathrm{e}^{i\omega t}$ while $v^0(x,y,t)$ is zero. Following the same procedure presented in section 3.2.2, the dynamic stiffness matrix has the same expression reported in Equation (80) but the 2 independent terms are:

$$
s_{nn} = A_{11} r \cot(rb) \quad , \quad f_{nn} = -A_{11} r \operatorname{cosec}(rb) \tag{84}
$$

3.3. Complete dynamic stiffness matrix of a single element

The complete dynamic stiffness matrix of a composite plate element based on the FSDT can now be obtained by combining Eqs. (48) and (73) to give

$$
\begin{bmatrix} \mathcal{N}_{xx_1} \\ \mathcal{N}_{xy_1} \\ \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{N}_{xx_2} \\ \mathcal{N}_{xy_2} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \end{bmatrix}
=
\begin{bmatrix}
s_{nn} & s_{nl} & 0 & 0 & 0 & f_{nn} & f_{nl} & 0 & 0 & 0 \\
 & s_{ll} & 0 & 0 & 0 & -f_{nl} & f_{ll} & 0 & 0 & 0 \\
 & & s_{qq} & s_{qm} & s_{qt} & 0 & 0 & f_{qq} & f_{qm} & f_{qt} \\
 & & & s_{mm} & s_{mt} & 0 & 0 & -f_{qm} & f_{mm} & f_{mt} \\
 & & & & s_{tt} & 0 & 0 & f_{qt} & -f_{mt} & f_{tt} \\
 & & & & & s_{nn} & -s_{nl} & 0 & 0 & 0 \\
 & & & & & & s_{tl} & 0 & 0 & 0 \\
 & & & & & & & s_{qq} & -s_{qm} & s_{qt} \\
 & & & & & & & & s_{mm} & -s_{mt} \\
 & & & & & & & & & s_{tt}
\end{bmatrix}
\begin{bmatrix} U_1 \\ V_1 \\ W_1 \\ \Phi_{y_1} \\ \Phi_{x_1} \\ U_2 \\ V_3 \\ W_2 \\ \Phi_{y_2} \\ \Phi_{x_2} \end{bmatrix}
\tag{85}
$$

Forces and displacements sign conventions are shown in Fig. (4).

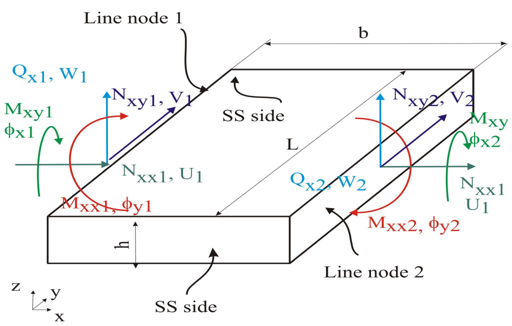


Figure 4: Complete dynamic stiffness FSDT plate element

3.4. Rotation and offset of dynamic stiffness element

The global dynamic stiffness matrix of an element may often need to be rotated and/or offset before being assembled to form the global dynamic stiffness matrix. Rotation and offset are applied by using standard transformation matrices.

Referring to Fig. (5), let the global reference system be x', y' and z' and local one x, y and z.

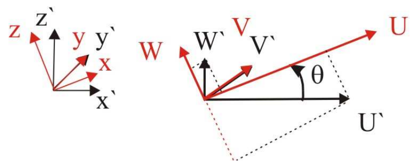


Figure 5: Displacement transformations from global to local reference system

The local displacements and forces on node 1 and 2 can be expressed as functions of global displacements and forces on node 1 and 2 as follow:

$$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \\ \Phi_{y_1} \\ \Phi_{x_1} \\ U_2 \\ V_2 \\ W_2 \\ \Phi_{y_2} \\ \Phi_{x_2} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos(\theta) & 0 & \sin(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin(\theta) & 0 & \cos(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U'_1 \\ V'_1 \\ W'_1 \\ \Phi'_{y_1} \\ \Phi'_{x_1} \\ U'_2 \\ V'_2 \\ W'_2 \\ \Phi'_{y_2} \\ \Phi'_{x_2} \end{bmatrix} \tag{86}$$

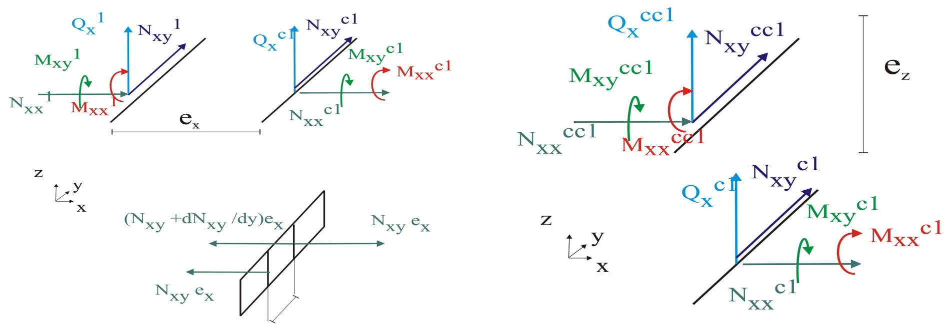
i.e.

$$\boldsymbol{\delta} = \boldsymbol{T_r \delta'} \quad \text{or} \quad \boldsymbol{F} = \boldsymbol{T_r F'} \tag{87}$$

where T_r is the rotation matrix.

Also eccentric connections need to be considered for certain problems when investigating the free vibration behaviour of plate assemblies. Let us consider eccentricities e_x and e_z at node 1 of the plate element (Fig. 6). The two eccentricities can be used separately for simplicity. The first eccentricity is e_x which moves node 1 to node C_1. The equivalent forces and moments on node C_1 can be written as (see Fig. 6(a)):

$$\begin{aligned} &M_{xxC1} = M_{xx_1} + V_1 e_x, \quad M_{xyC1} = M_{xy_1}, \quad M_{zzC1} = N_{xy_1} e_x \\ &Q_{xC1} = Q_{x_1}, \quad N_{xyC1} = N_{xy_1}, \quad N_{xxC1} = N_{xx_1} \end{aligned} \tag{88}$$



(a) Offset in the x direction. (b) Offset in the z direction.

Figure 6: Transformation of forces due to eccentricities, equivalent forces and moments.

It should be noted that as the moment M_{zzC1} does not exist, it should be balanced by some inplane forces. By referring to Fig. 6(a), it can be seen that the inplane component is N_{xxC1} which becomes:

$$N_{xyC1} = N_{xy_1} - \frac{dN_{xy_1}}{dy} e_x = N_{xy_1} + \alpha N_{xy_1} e_x \tag{89}$$

The transformation can now be written in matrix form as

$$\begin{bmatrix} \mathcal{N}_{xxC1} \\ \mathcal{N}_{xyC1} \\ \mathcal{Q}_{xC1} \\ \mathcal{M}_{xxC1} \\ \mathcal{M}_{xyC1} \end{bmatrix} = \begin{bmatrix} 1 & \alpha e_x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & e_x & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{N}_{xx_1} \\ \mathcal{N}_{xy_1} \\ \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \end{bmatrix} \tag{90}$$

The second transformation using e_z which moves node $C1$ to node $CC1$ (Fig. 6(b)) is now applied. In this case, the equivalent forces and moments are:

$$M_{xxCC1} = M_{xxC1} - N_{xxC1}e_z, \quad M_{xyCC1} = M_{xyC1} - N_{xyC1}e_z,$$
$$Q_{xCC1} = Q_{xC1}, \quad N_{xyCC1} = N_{xyC1}, \quad N_{xxCC1} = N_{xxC1} \tag{91}$$

The transformation can be written in matrix form as:

$$\begin{bmatrix} \mathcal{N}_{xxCC1} \\ \mathcal{N}_{xyCC1} \\ \mathcal{Q}_{xCC1} \\ \mathcal{M}_{xxCC1} \\ \mathcal{M}_{xyCC1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -e_z & 0 & 0 & 1 & 0 \\ 0 & -e_z & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{N}_{xxC1} \\ \mathcal{N}_{xyC1} \\ \mathcal{Q}_{xC1} \\ \mathcal{M}_{xxC1} \\ \mathcal{M}_{xyC1} \end{bmatrix} \tag{92}$$

By combining Eqs. (90) and (92) and considering eccentricity e_{x_1} and e_{z_1} at node 1 and e_{x_2} and e_{z_2} at node 2 the transpose of the total eccentricity matrix is obtained as

$$\boldsymbol{T_e}^T = \begin{bmatrix} 1 & \alpha e_{x_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e_{z_1} & 0 & e_{x_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e_{z_1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha e_{x_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e_{z_2} & 0 & e_{x_2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -e_{z_2} & 0 & 0 & 1 \end{bmatrix} \tag{93}$$

The dynamic stiffness matrix of an element, in the global coordinate system with eccentricity can now be computed by using the local stiffness matrix (Eq. 85) and the transformation matrices (Eqs. 86 and 93) to give

$$\boldsymbol{K}^{r,e} = \boldsymbol{T_r}^T \boldsymbol{T_e}^T \boldsymbol{K} \boldsymbol{T_e} \boldsymbol{T_r} \tag{94}$$

3.5. Assembly procedure, boundary conditions and similarities with FEM

Once the DS matrix of a laminate element has been computed, rotated and offset if required, it can be assembled in a global DS matrix of the whole structure as schematically shown in Fig. (7). The procedure is similar to that used in the FEM and the global matrix is banded as well. Although a mesh is also required in the DSM, it should be noted that the results are not mesh dependent and additional elements are required only when a change in the geometry or structural property occurs in the structure. A single DS laminate element is sufficient to compute any number of natural frequencies for an individual plate to any desired accuracy.

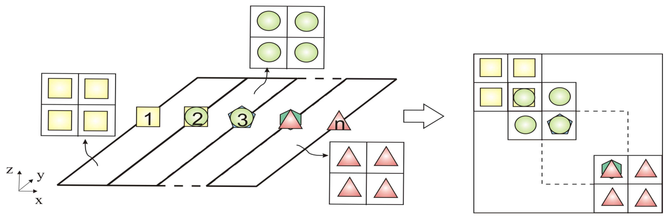


Figure 7: Assembly of dynamic stiffness matrices

Unlike the FEM, DS plate elements do not have point nodes but have line nodes for each strip. Furthermore, no change in geometry along the y-direction can be modelled and the two sides $y = 0$ and $y = b$ must be simply supported (SS1 as described in Section 3.2). The other two sides of the structure can have any boundary condition. Boundary conditions are applied to the global dynamic stiffness matrix using the penalty method. This consists of adding a large stiffness to the position on the leading diagonal term which corresponds to the degree of freedom of the node which needs to be constrained. Any of the following boundary conditions on the two sides $x = 0$ and $x = b$ can be applied: (i) Free (F): no penalty is applied, (ii) Simply supported (SS1): V_i, W_i and Φ_{x_i} are penalised, (iii) Simply supported (SS2): U_i, W_i and Φ_{x_i} are penalised, (iv) Clamped (C): U_i, V_i, W_i, Φ_{y_i}, Φ_{x_i} are penalised; i the node to be constrained.

Because of the similarities between DS and finite elements, DS elements can be implemented in FEM codes to increase the accuracy very considerably for free vibration analysis of structures. It should be emphasised that when analytical solutions are available, resorting to numerical techniques results in loss of accuracy and often excessive computational time.

3.6. The Wittrick-Williams algorithm

In order to compute the natural frequencies of a structure using the DSM, the most efficient method is to apply the Wittrick and Williams algorithm [28]. For clarity and completeness, the procedure is briefly summarised as follows.

The global dynamic stiffness matrix of the structure K^* is computed at a trial frequency ω^*. By applying Gauss elimination the global stiffness matrix is then triangulated in upper triangular $K^{*^{\triangle}}$ form. If the number of negative terms on the leading diagonal of $K^{*^{\triangle}}$ is defined as the sign count $s(K^*)$, the number (j) of natural frequencies (ω) which lie below the trial frequency (ω^*) is given by [28]:

$$j = j_0 + s(K^*) \tag{95}$$

where j_0 is the number of frequencies of all individual single strip elements in the structure when clamped on their opposite sides which are still lower than the trial frequency (ω^*). Note that the DSM allows for an infinite number of natural frequencies between nodes to be accounted for when all nodal displacements, i.e. the displacements components to which the overall DS matrix corresponds to are zero. When $s(K^*)$ and j_0 are known, bi-section method can be used to bracket any natural frequency up to the required accuracy.

Computing j_0 can sometimes be cumbersome, but can be avoided if a sufficiently fine mesh is used although this will increase the computational time. The value of j_0 can be computed for each trial frequency if the C-C frequencies of the elements within the structure are know beforehand. These C-C frequencies can be computed by splitting each element in smaller sub-strip elements and then computing the natural frequencies of the global structure up to the first C-C natural frequency of the largest sub-strip. This is the upper limit of the trial frequency in the analysis analysis, if higher frequencies needs to be computed, smaller strips will need to be used. Once these C-C frequencies are know, j_0 can be computed, thus W-W algorithm can be applied without any increase in the computational time due to a fine mesh.

3.7. Mode shape computation

The mode shapes are routinely computed by using the global dynamic stiffness matrix of the structure, and setting the force vector to zero. A carefully chosen nodal displacement is given an arbitrary value and then determining the rest of the nodal displacements in terms of the chosen one. The best degree of freedom to choose for normalising the mode shapes is the one which causes the sign count of K to increase when applying the Wittrick and Williams algorithm (see section 3.6). In fact, if this procedure is not adopted and a displacement is chosen randomly, inplane modes could be erroneously picked up as the out of plane modes. It seems that the degree of freedom which is associated with the increase of the sign count of K is also the one which triggers the mode. This observation has never been made before because thick plates with both inplane and out of plane modes have apparently not been analysed using the DSM before.

Once the correct degree of freedom has been chosen to normalise the mode, solving the system of algebraic equations gives the nodal displacements of the strips. In order to have an accurate plot of the modes either a finer mesh or further post processing may be required. A finer mesh increases the computational time considerably in which case further post-processing is preferred. This consists of computing the integration constants for each element from Eqs. (42) and (67) and subsequently computing the displacements of each strip by using Eqs. (35) and (62). In this way the modes can be plotted as accurately as required.

4. CONCLUDING REMARKS

In Part I of this two-part paper the complete dynamic stiffness matrix of a laminated composite element based on the first order shear deformation theory has been formulated. This is a new development and in sharp contrast to previous work reported in the literature, the effects of shear deformation and rotatory inertia have been considered and exact explicit expressions for the elements of the dynamic stiffness matrix have been presented. The transformation matrices for rotation and offset connections have been developed and the assembly procedure to generate the global dynamic stiffness matrix of the complete structure has been fully described. The Wittrick-Williams algorithm which is essential to solve the free vibration problem has been

illustrated along with a method to avoid computation of the otherwise required clamped-clamped natural frequencies. Once the natural frequencies are computed, the procedure to obtain the mode shapes has been explained with particular attention to a method for correctly distinguishing between inplane and out of plane modes. The dynamic stiffness laminate elements, although similar to finite elements, provide an exact solution for free vibration analysis of complex structures.
In the second part of this paper [29], the method and theory presented herein is first validated against exact results in the literature for simple plates, and then the free vibration analysis of typical aeronautical structures such as stringer panels are investigated. In the sequel, the results obtained by the dynamic stiffness method are compared with results from finite element analysis using NASTRAN in order to show the superiority of the present method in terms of both accuracy and computational efficiency.

Acknowledgements

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APPENDIX A: LAMINATE GEOMETRIC AND CONSTITUTIVE EQUATIONS

The geometric relation for a lamina in the local or lamina reference system can be written as:

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & 0 & z\mathcal{D}_x & 0 \\ 0 & \mathcal{D}_y & 0 & 0 & -z\mathcal{D}_y \\ \mathcal{D}_y & \mathcal{D}_x & 0 & z\mathcal{D}_y & -z\mathcal{D}_x \\ 0 & 0 & \mathcal{D}_y & 0 & -1 \\ 0 & 0 & \mathcal{D}_x & 1 & 0 \end{bmatrix} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \\ \phi_y \\ \phi_x \end{bmatrix} \tag{96}
$$

where $\mathcal{D}_x$ and $\mathcal{D}_y$ are the derivatives in x and y respectively. The constitutive equations in the lamina reference system can be written as:

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix} \tag{97}
$$

where

$$
\begin{aligned} & C_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}; \quad C_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}; \quad C_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}; \\ & C_{66} = G_{12}; \quad C_{44} = G_{23}; \quad C_{55} = G_{13} \end{aligned} \tag{98}
$$

where E_1 is the elastic modulus in the fibre direction, E_2 the elastic modulus in perpendicular to the fibre, ν_{12} and $\nu_{21} = \nu_{12}E_2/E_1$ the Poisson's ratios, $G_{12} = G_{13}$ and G_{23} the shear modulus of each single orthotropic lamina. If the lamina is placed at an angle θ in the laminate or global reference system, the equation need to be transformed as follows:

$$
\overline{C}_{11} = C_{11}\mathcal{C}^4 + 2(C_{12} + 2C_{66})\mathcal{S}^2\mathcal{C}^2 + C_{22}\mathcal{S}^4 \tag{99}
$$

$$\overline{C}_{12} =(C_{11} + C_{22} - 4C_{66})\mathcal{S}^2\mathcal{C}^2 + C_{12}(\mathcal{S}^4 + \mathcal{C}^4) \tag{100}$$

$$\overline{C}_{16} =(C_{11} - C_{12} - 2C_{66})\mathcal{S}\mathcal{C}^3 + (C_{12} - C_{22} + 2C_{66})\mathcal{S}^3\mathcal{C} \tag{101}$$

$$\overline{C}_{22} =C_{11}\mathcal{S}^4 + 2(C_{12} + 2C_{66})\mathcal{S}^2\mathcal{C}^{\in} + C_{22}\mathcal{C}^4 \tag{102}$$

$$\overline{C}_{26} =(C_{11} - C_{12} - 2C_{66})\mathcal{S}^3\mathcal{C} + (C_{12} - C_{22} + 2C_{66})\mathcal{S}\mathcal{C}^3 \tag{103}$$

$$\overline{C}_{66} =(C_{11} + C_{22} - 2C_{12} - 2C_{66})\mathcal{S}^2\mathcal{C}^2 + C_{66}(\mathcal{S}^4 + \mathcal{C}^4) \tag{104}$$

$$\overline{C}_{44} =C_{44}\mathcal{C}^2 + C_{55}\mathcal{S}^2 \quad \overline{C}_{55} = C_{44}\mathcal{S}^2 + C_{55}\mathcal{C}^2, \quad \overline{C}_{45} = (C_{55} - C_{44})\mathcal{C}\mathcal{S} \tag{105}$$

where $\mathcal{C} = cos\theta$ and $\mathcal{S} = sin\theta$. This leads to the constitutive equation for the k-th lamina in the laminate or global reference system:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & 0 & 0 \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & 0 & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix} \tag{106}$$

that in compact form can be written for each k-th lamina as:

$$\boldsymbol{\sigma}_k = \overline{\boldsymbol{C}}_k \boldsymbol{\varepsilon}_k \tag{107}$$

APPENDIX B: EXPLICIT EXPRESSIONS FOR THE DYNAMIC STIFFNESS ELEMENTS FOR OUT OF PLANE MOTION

Explicit expressions for the coefficients of the DS matrix are given only for case 1 for brevity. The coefficients for the other 3 cases can be obtained by following the procedure reported in section 3.1. Given the complexity of these coefficient, the use of a symbolic computation program such as Mathematica [27] is essential. In order to avoid numerical instabilities and overflows, the expressions need to be simplified and carefully constructed. Full simplification is achieved by using the explicit expressions of the terms appearing in the matrix $\boldsymbol{A}$ and $\boldsymbol{R}$ (Eqs. 41 and 43). Numerical inversion of matrix $\boldsymbol{A}$

should be avoided because it may cause ill-conditioning.

$$
\begin{aligned}
s_{qq} =& \frac{1}{\Delta}\Big(k\, A_{55}\, (-(\mathcal{C}_{h_3}\, (\gamma_1\, \delta_3\, (r_2\, \gamma_2\, \delta_2 + r_3\, \gamma_3\, \delta_2 - 2\, r_3\, \gamma_2\, \delta_3) + \delta_1\, (-((r_1\, \gamma_1 + r_2\, \gamma_2)\, \gamma_3\, \delta_2) \\
&+ \gamma_2\, (r_1\, \gamma_1 + r_3\, \gamma_3)\, \delta_3) + \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, (\delta_3\, ((r_2\, {\gamma_1}^2 + r_3\, \gamma_2\, \gamma_3)\, \delta_2 - r_3\, ({\gamma_1}^2 + {\gamma_2}^2)\, \delta_3) \\
&+ \delta_1\, (-((r_2\, \gamma_1 + r_1\, \gamma_2)\, \gamma_3\, \delta_2) + (r_1\, {\gamma_2}^2 + r_3\, \gamma_1\, \gamma_3)\, \delta_3)))) + \mathcal{C}_{h_2}\, (\delta_1\, (-((r_1\, \gamma_1 + r_2\, \gamma_2)\, \gamma_3\, \delta_2) \\
&+ \gamma_2\, (r_1\, \gamma_1 + r_3\, \gamma_3)\, \delta_3) + \gamma_1\, \delta_2\, (2\, r_2\, \gamma_3\, \delta_2 - (r_2\, \gamma_2 + r_3\, \gamma_3)\, \delta_3) + \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, (\delta_1\, (-((r_2\, \gamma_1\, \gamma_2 + r_1\, {\gamma_3}^2)\, \delta_2) \\
&+ \gamma_2\, (r_3\, \gamma_1 + r_1\, \gamma_3)\, \delta_3) + \delta_2\, (r_2\, ({\gamma_1}^2 + {\gamma_3}^2)\, \delta_2 - (r_3\, {\gamma_1}^2 + r_2\, \gamma_2\, \gamma_3)\, \delta_3))) + \mathcal{C}_{h_1}\, (2\, r_1\, \gamma_2\, \gamma_3\, {\delta_1}^2 \\
&- r_1\, \gamma_1\, \gamma_3\, \delta_1\, \delta_2 - r_2\, \gamma_2\, \gamma_3\, \delta_1\, \delta_2 - r_1\, \gamma_1\, \gamma_2\, \delta_1\, \delta_3 - r_3\, \gamma_2\, \gamma_3\, \delta_1\, \delta_3 + r_2\, \gamma_1\, \gamma_2\, \delta_2\, \delta_3 + r_3\, \gamma_1\, \gamma_3\, \delta_2\, \delta_3 \\
&+ \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, (r_1\, ({\gamma_2}^2 + {\gamma_3}^2)\, {\delta_1}^2 + \gamma_1\, (r_3\, \gamma_2 + r_2\, \gamma_3)\, \delta_2\, \delta_3 - \delta_1\, (r_1\, \gamma_1\, \gamma_2\, \delta_2 + r_2\, {\gamma_3}^2\, \delta_2 + r_3\, {\gamma_2}^2\, \delta_3 \\
&+ r_1\, \gamma_1\, \gamma_3\, \delta_3)) - \mathcal{C}_{h_2}\, \mathcal{C}_{h_3}\, (2\, r_1\, \gamma_2\, \gamma_3\, {\delta_1}^2 - \delta_1\, ((r_1\, \gamma_1 + r_2\, \gamma_2)\, \gamma_3\, \delta_2 \\
&+ \gamma_2\, (r_1\, \gamma_1 + r_3\, \gamma_3)\, \delta_3) + \gamma_1\, (2\, r_2\, \gamma_3\, {\delta_2}^2 - (r_2\, \gamma_2 + r_3\, \gamma_3)\, \delta_2\, \delta_3 + 2\, r_3\, \gamma_2\, {\delta_3}^2)))))\Big)
\end{aligned} \tag{B.1}
$$

$$
\begin{aligned}
s_{qm} =& \frac{1}{\Delta}\Big(k\, A_{55}\, (-((-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2)\, (1 + r_3\, \delta_3)\, ((-1 + \mathcal{C}_{h_1}\, \mathcal{C}_{h_3})\, \mathcal{S}_{h_2}\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3) \\
&+ \mathcal{S}_{h_1}\, (-(\gamma_3\, \delta_1) + \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, (-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2) + \gamma_1\, \delta_3 - \mathcal{C}_{h_2}\, \mathcal{C}_{h_3}\, (-(\gamma_3\, \delta_1) + \gamma_1\, \delta_3)))) \\
&- (1 + r_2\, \delta_2)\, (-(\gamma_3\, \delta_1) + \gamma_1\, \delta_3)\, ((-1 + \mathcal{C}_{h_1}\, \mathcal{C}_{h_2})\, \mathcal{S}_{h_3}\, (\gamma_3\, \delta_2 - \gamma_2\, \delta_3) + \mathcal{S}_{h_1}\, (-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2 \\
&- \mathcal{C}_{h_2}\, \mathcal{C}_{h_3}\, (-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2) + \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, (-(\gamma_3\, \delta_1) + \gamma_1\, \delta_3))) + (1 + r_1\, \delta_1)\, (\gamma_3\, \delta_2 \\
&- \gamma_2\, \delta_3)\, ((-1 + \mathcal{C}_{h_1}\, \mathcal{C}_{h_2})\, \mathcal{S}_{h_3}\, (\gamma_3\, \delta_1 - \gamma_1\, \delta_3) + \mathcal{S}_{h_2}\, (\gamma_2\, \delta_1 - \gamma_1\, \delta_2 + \mathcal{C}_{h_1}\, \mathcal{C}_{h_3}\, (-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2) \\
&+ \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3)))))\Big)
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
s_{qt} =& \frac{1}{\Delta}\Big(k\, A_{55}\, (\mathcal{C}_{h_3}\, (\gamma_1\, \delta_2\, \delta_3\, (r_2\, \delta_2 - r_3\, \delta_3) + r_1\, {\delta_1}^2\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3) - \delta_1\, (r_2\, \gamma_3\, {\delta_2}^2 \\
&- 2\, r_3\, \gamma_3\, \delta_2\, \delta_3 + r_3\, \gamma_2\, {\delta_3}^2) - \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, (\gamma_3\, {\delta_1}^2\, (r_2\, \delta_2 - r_3\, \delta_3) \\
&+ r_3\, \delta_2\, \delta_3\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3) + \delta_1\, (r_1\, \gamma_3\, {\delta_2}^2 - (r_2\, \gamma_1 + r_1\, \gamma_2)\, \delta_2\, \delta_3 + r_3\, \gamma_1\, {\delta_3}^2))) \\
&- \mathcal{C}_{h_2}\, (\gamma_1\, \delta_2\, \delta_3\, (r_2\, \delta_2 - r_3\, \delta_3) + r_1\, {\delta_1}^2\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3) + \delta_1\, (r_2\, \gamma_3\, {\delta_2}^2 - 2\, r_2\, \gamma_2\, \delta_2\, \delta_3 + r_3\, \gamma_2\, {\delta_3}^2) \\
&+ \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, (r_2\, \delta_2\, \delta_3\, (\gamma_3\, \delta_2 - \gamma_2\, \delta_3) + {\delta_1}^2\, (-(r_2\, \gamma_2\, \delta_2) + r_3\, \gamma_2\, \delta_3) + \delta_1\, (r_2\, \gamma_1\, {\delta_2}^2 \\
&- (r_3\, \gamma_1 + r_1\, \gamma_3)\, \delta_2\, \delta_3 + r_1\, \gamma_2\, {\delta_3}^2))) + \mathcal{C}_{h_1}\, (-(r_1\, \gamma_3\, {\delta_1}^2\, \delta_2) + r_2\, \gamma_3\, \delta_1\, {\delta_2}^2 - r_1\, \gamma_2\, {\delta_1}^2\, \delta_3 \\
&+ 2\, r_1\, \gamma_1\, \delta_1\, \delta_2\, \delta_3 - r_2\, \gamma_1\, {\delta_2}^2\, \delta_3 + r_3\, \gamma_2\, \delta_1\, {\delta_3}^2 - r_3\, \gamma_1\, \delta_2\, {\delta_3}^2 - \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, (\gamma_1\, \delta_2\, \delta_3\, (r_3\, \delta_2 + r_2\, \delta_3) \\
&+ r_1\, {\delta_1}^2\, (\gamma_2\, \delta_2 + \gamma_3\, \delta_3) - \delta_1\, (r_1\, \gamma_1\, {\delta_2}^2 + r_3\, \gamma_2\, \delta_2\, \delta_3 + r_2\, \gamma_3\, \delta_2\, \delta_3 + r_1\, \gamma_1\, {\delta_3}^2)) \\
&+ \mathcal{C}_{h_2}\, \mathcal{C}_{h_3}\, (\gamma_1\, \delta_2\, \delta_3\, (r_2\, \delta_2 + r_3\, \delta_3) + r_1\, {\delta_1}^2\, (\gamma_3\, \delta_2 + \gamma_2\, \delta_3) \\
&+ \delta_1\, (r_2\, \gamma_3\, {\delta_2}^2 - 2\, (r_1\, \gamma_1 + r_2\, \gamma_2 + r_3\, \gamma_3)\, \delta_2\, \delta_3 + r_3\, \gamma_2\, {\delta_3}^2))))\Big)
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
f_{qq} =& \frac{1}{\Delta}\Big(k\, A_{55}\, (-(r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, {\gamma_2}^2\, {\delta_1}^2) - 2\, r_1\, \gamma_2\, \gamma_3\, {\delta_1}^2 - r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, {\gamma_3}^2\, {\delta_1}^2 + r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_1\, \gamma_2\, \delta_1\, \delta_2 \\
&+ r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \gamma_2\, \delta_1\, \delta_2 + r_1\, \gamma_1\, \gamma_3\, \delta_1\, \delta_2 - r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_1\, \gamma_3\, \delta_1\, \delta_2 + r_2\, \gamma_2\, \gamma_3\, \delta_1\, \delta_2 \\
&- r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_2\, \gamma_3\, \delta_1\, \delta_2 + r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, {\gamma_3}^2\, \delta_1\, \delta_2 + r_2\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, {\gamma_3}^2\, \delta_1\, \delta_2 - r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, {\gamma_1}^2\, {\delta_2}^2 \\
&- 2\, r_2\, \gamma_1\, \gamma_3\, {\delta_2}^2 - r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, {\gamma_3}^2\, {\delta_2}^2 + r_1\, \gamma_1\, \gamma_2\, \delta_1\, \delta_3 - r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_1\, \gamma_2\, \delta_1\, \delta_3 \\
&+ r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, {\gamma_2}^2\, \delta_1\, \delta_3 + r_3\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, {\gamma_2}^2\, \delta_1\, \delta_3 + r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_1\, \gamma_3\, \delta_1\, \delta_3 + r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \gamma_3\, \delta_1\, \delta_3 \\
&+ r_3\, \gamma_2\, \gamma_3\, \delta_1\, \delta_3 - r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_2\, \gamma_3\, \delta_1\, \delta_3 + r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, {\gamma_1}^2\, \delta_2\, \delta_3 + r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, {\gamma_1}^2\, \delta_2\, \delta_3 + r_2\, \gamma_1\, \gamma_2\, \delta_2\, \delta_3 \\
&- r_3\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \gamma_2\, \delta_2\, \delta_3 + r_3\, \gamma_1\, \gamma_3\, \delta_2\, \delta_3 - r_2\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \gamma_3\, \delta_2\, \delta_3 + r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_2\, \gamma_3\, \delta_2\, \delta_3 \\
&+ r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_2\, \gamma_3\, \delta_2\, \delta_3 - r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, {\gamma_1}^2\, {\delta_3}^2 - 2\, r_3\, \gamma_1\, \gamma_2\, {\delta_3}^2 - r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, {\gamma_2}^2\, {\delta_3}^2 \\
&+ \mathcal{C}_{h_2}\, \mathcal{C}_{h_3}\, (2\, r_1\, \gamma_2\, \gamma_3\, {\delta_1}^2 + \gamma_1\, (r_2\, \gamma_2 + r_3\, \gamma_3)\, \delta_2\, \delta_3 - \delta_1\, (r_1\, \gamma_1\, \gamma_3\, \delta_2 + r_2\, \gamma_2\, \gamma_3\, \delta_2 + r_1\, \gamma_1\, \gamma_2\, \delta_3 \\
&+ r_3\, \gamma_2\, \gamma_3\, \delta_3)) - \mathcal{C}_{h_1}\, (\mathcal{C}_{h_3}\, (\gamma_1\, \delta_2\, (-2\, r_2\, \gamma_3\, \delta_2 + r_2\, \gamma_2\, \delta_3 + r_3\, \gamma_3\, \delta_3) + \delta_1\, ((r_1\, \gamma_1 + r_2\, \gamma_2)\, \gamma_3\, \delta_2 \\
&- \gamma_2\, (r_1\, \gamma_1 + r_3\, \gamma_3)\, \delta_3)) + \mathcal{C}_{h_2}\, (\gamma_1\, \delta_3\, (r_2\, \gamma_2\, \delta_2 + r_3\, \gamma_3\, \delta_2 - 2\, r_3\, \gamma_2\, \delta_3) + \delta_1\, (-((r_1\, \gamma_1 + r_2\, \gamma_2)\, \gamma_3\, \delta_2) \\
&+ \gamma_2\, (r_1\, \gamma_1 + r_3\, \gamma_3)\, \delta_3)))))\Big)
\end{aligned} \tag{B.4}
$$

$$
\begin{aligned}
f_{qm} =& \frac{1}{\Delta}\Big(k\, A_{55}\, (\mathcal{C}_{h_1}\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3)\, (\mathcal{S}_{h_2}\, (-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2)\, (r_1\, \delta_1 - r_3\, \delta_3) \\
&+ \mathcal{S}_{h_3}\, (r_1\, \delta_1 - r_2\, \delta_2)\, (\gamma_3\, \delta_1 - \gamma_1\, \delta_3)) + \mathcal{C}_{h_2}\, (-(\gamma_3\, \delta_1) + \gamma_1\, \delta_3)\, (\mathcal{S}_{h_1}\, (\gamma_2\, \delta_1 - \gamma_1\, \delta_2)\, (r_2\, \delta_2 - r_3\, \delta_3) \\
&+ \mathcal{S}_{h_3}\, (r_1\, \delta_1 - r_2\, \delta_2)\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3)) + \mathcal{C}_{h_3}\, (-(\gamma_2\, \delta_1) + \gamma_1\, \delta_2)\, (\mathcal{S}_{h_1}\, (r_2\, \delta_2 - r_3\, \delta_3)\, (-(\gamma_3\, \delta_1) \\
&+ \gamma_1\, \delta_3) + \mathcal{S}_{h_2}\, (-(r_1\, \delta_1) + r_3\, \delta_3)\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3)))\Big)
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
f_{qt} =& \frac{1}{\Delta}\Big(k\, A_{55}\, (-(r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_2\, {\delta_1}^2\, \delta_2) + r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_2\, {\delta_1}^2\, \delta_2 + r_1\, \gamma_3\, {\delta_1}^2\, \delta_2 + r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_3\, {\delta_1}^2\, \delta_2 \\
&+ r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_1\, \delta_1\, {\delta_2}^2 - r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \delta_1\, {\delta_2}^2 + r_2\, \gamma_3\, \delta_1\, {\delta_2}^2 + r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_3\, \delta_1\, {\delta_2}^2 \\
&+ r_1\, \gamma_2\, {\delta_1}^2\, \delta_3 + r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_2\, {\delta_1}^2\, \delta_3 - r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_3\, {\delta_1}^2\, \delta_3 + r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_3\, {\delta_1}^2\, \delta_3 - 2\, r_1\, \gamma_1\, \delta_1\, \delta_2\, \delta_3 \\
&- r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_1\, \delta_1\, \delta_2\, \delta_3 - r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_1\, \delta_1\, \delta_2\, \delta_3 - 2\, r_2\, \gamma_2\, \delta_1\, \delta_2\, \delta_3 - r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_2\, \delta_1\, \delta_2\, \delta_3 \\
&- r_3\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_2\, \delta_1\, \delta_2\, \delta_3 - 2\, r_3\, \gamma_3\, \delta_1\, \delta_2\, \delta_3 - r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_3\, \delta_1\, \delta_2\, \delta_3 - r_2\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_3\, \delta_1\, \delta_2\, \delta_3 \\
&+ r_2\, \gamma_1\, {\delta_2}^2\, \delta_3 + r_3\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, {\delta_2}^2\, \delta_3 - r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_3\, {\delta_2}^2\, \delta_3 + r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_3\, {\delta_2}^2\, \delta_3 \\
&+ r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_1\, \delta_1\, {\delta_3}^2 - r_1\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \delta_1\, {\delta_3}^2 + r_3\, \gamma_2\, \delta_1\, {\delta_3}^2 + r_1\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_2\, \delta_1\, {\delta_3}^2 \\
&+ r_3\, \gamma_1\, \delta_2\, {\delta_3}^2 + r_2\, \mathcal{S}_{h_2}\, \mathcal{S}_{h_3}\, \gamma_1\, \delta_2\, {\delta_3}^2 + r_3\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_2}\, \gamma_2\, \delta_2\, {\delta_3}^2 - r_2\, \mathcal{S}_{h_1}\, \mathcal{S}_{h_3}\, \gamma_2\, \delta_2\, {\delta_3}^2 \\
&- \mathcal{C}_{h_2}\, \mathcal{C}_{h_3}\, (\gamma_1\, \delta_2\, \delta_3\, (r_2\, \delta_2 + r_3\, \delta_3) + r_1\, {\delta_1}^2\, (\gamma_3\, \delta_2 + \gamma_2\, \delta_3) - \delta_1\, (r_2\, \gamma_3\, {\delta_2}^2 + 2\, r_1\, \gamma_1\, \delta_2\, \delta_3 + r_3\, \gamma_2\, {\delta_3}^2)) \\
&+ \mathcal{C}_{h_1}\, (-(\mathcal{C}_{h_3}\, (\gamma_1\, \delta_2\, \delta_3\, (r_2\, \delta_2 - r_3\, \delta_3) + r_1\, {\delta_1}^2\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3) + \delta_1\, (r_2\, \gamma_3\, {\delta_2}^2 - 2\, r_2\, \gamma_2\, \delta_2\, \delta_3 \\
&+ r_3\, \gamma_2\, {\delta_3}^2))) + \mathcal{C}_{h_2}\, (\gamma_1\, \delta_2\, \delta_3\, (r_2\, \delta_2 - r_3\, \delta_3) + r_1\, {\delta_1}^2\, (-(\gamma_3\, \delta_2) + \gamma_2\, \delta_3) - \delta_1\, (r_2\, \gamma_3\, {\delta_2}^2 \\
&- 2\, r_3\, \gamma_3\, \delta_2\, \delta_3 + r_3\, \gamma_2\, {\delta_3}^2)))))\Big)
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
s_{mm} = & \frac{1}{\Delta}\Big(D_{11}\,(-(r_3\,\gamma_2\,\delta_1) + r_2\,\gamma_3\,\delta_1 + r_3\,\gamma_1\,\delta_2 - r_1\,\gamma_3\,\delta_2 - r_2\,\gamma_1\,\delta_3 + r_1\,\gamma_2\,\delta_3)\,(\mathcal{C}_{h_3}\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,(-(\gamma_2\,\delta_1) \\
& + \gamma_1\,\delta_2) + \mathcal{S}_{h_3}\,(\mathcal{C}_{h_2}\,\mathcal{S}_{h_1}\,(\gamma_3\,\delta_1 - \gamma_1\,\delta_3) + \mathcal{C}_{h_1}\,\mathcal{S}_{h_2}\,(-(\gamma_3\,\delta_2) + \gamma_2\,\delta_3)))\Big)
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
s_{mt} = & \frac{1}{\Delta}\Big((-(D_{11}\,r_3) - \alpha\,D_{12}\,\gamma_3)\,(-((-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_3})\,\mathcal{S}_{h_2}\,\delta_2\,(-(\gamma_2\,\delta_1) + \gamma_1\,\delta_2)) \\
& + (-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_2})\,\mathcal{S}_{h_3}\,(-2\,\gamma_3\,\delta_1\,\delta_2 + \gamma_2\,\delta_1\,\delta_3 + \gamma_1\,\delta_2\,\delta_3) + \mathcal{S}_{h_1}\,(\delta_1\,(\gamma_2\,\delta_1 - \gamma_1\,\delta_2) - \mathcal{C}_{h_2}\,\mathcal{C}_{h_3}\,\delta_1\,(\gamma_2\,\delta_1 \\
& - \gamma_1\,\delta_2) + \mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,(\gamma_3\,{\delta_1}^2 + \gamma_3\,{\delta_2}^2 - \gamma_1\,\delta_1\,\delta_3 - \gamma_2\,\delta_2\,\delta_3))) + (-(D_{11}\,r_1) - \alpha\,D_{12}\,\gamma_1)\,((\mathcal{S}_{h_2}\,(\delta_2 \\
& - \mathcal{C}_{h_1}\,\mathcal{C}_{h_3}\,\delta_2) + (-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_2})\,\mathcal{S}_{h_3}\,\delta_3)\,(\gamma_3\,\delta_2 - \gamma_2\,\delta_3) + \mathcal{S}_{h_1}\,(-(\gamma_3\,\delta_1\,\delta_2) \\
& - \gamma_2\,\delta_1\,\delta_3 + 2\,\gamma_1\,\delta_2\,\delta_3 - \mathcal{C}_{h_2}\,\mathcal{C}_{h_3}\,(-(\gamma_3\,\delta_1\,\delta_2) - \gamma_2\,\delta_1\,\delta_3 + 2\,\gamma_1\,\delta_2\,\delta_3) \\
& + \mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,(-(\gamma_2\,\delta_1\,\delta_2) + \gamma_1\,{\delta_2}^2 - \gamma_3\,\delta_1\,\delta_3 + \gamma_1\,{\delta_3}^2))) + (-(D_{11}\,r_2) \\
& - \alpha\,D_{12}\,\gamma_2)\,(-((-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_2})\,\mathcal{S}_{h_3}\,\delta_3\,(-(\gamma_3\,\delta_1) + \gamma_1\,\delta_3)) + (-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_3})\,\mathcal{S}_{h_2}\,(\gamma_3\,\delta_1\,\delta_2 \\
& - 2\,\gamma_2\,\delta_1\,\delta_3 + \gamma_1\,\delta_2\,\delta_3) + \mathcal{S}_{h_1}\,(\delta_1\,(\gamma_3\,\delta_1 - \gamma_1\,\delta_3) - \mathcal{C}_{h_2}\,\mathcal{C}_{h_3}\,\delta_1\,(\gamma_3\,\delta_1 - \gamma_1\,\delta_3) + \mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,(\gamma_2\,{\delta_1}^2 \\
& - \gamma_1\,\delta_1\,\delta_2 - \gamma_3\,\delta_2\,\delta_3 + \gamma_2\,{\delta_3}^2))))\Big)
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
f_{mm} = & \frac{1}{\Delta}\Big(D_{11}\,(-(r_3\,\gamma_2\,\delta_1) + r_2\,\gamma_3\,\delta_1 + r_3\,\gamma_1\,\delta_2 - r_1\,\gamma_3\,\delta_2 - r_2\,\gamma_1\,\delta_3 + r_1\,\gamma_2\,\delta_3)\,(\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,(\gamma_3\,\delta_2 \\
& - \gamma_2\,\delta_3) + \mathcal{S}_{h_1}\,(\mathcal{S}_{h_2}\,(\gamma_2\,\delta_1 - \gamma_1\,\delta_2) + \mathcal{S}_{h_3}\,(-(\gamma_3\,\delta_1) + \gamma_1\,\delta_3)))\Big)
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
f_{mt} = & \frac{1}{\Delta}\Big(D_{11}\,(r_3\,\gamma_2\,\delta_1 - r_2\,\gamma_3\,\delta_1 - r_3\,\gamma_1\,\delta_2 + r_1\,\gamma_3\,\delta_2 + r_2\,\gamma_1\,\delta_3 - r_1\,\gamma_2\,\delta_3)\,(\mathcal{C}_{h_3}\,(-(\mathcal{S}_{h_1}\,\delta_1) + \mathcal{S}_{h_2}\,\delta_2) \\
& + \mathcal{C}_{h_2}\,(\mathcal{S}_{h_1}\,\delta_1 - \mathcal{S}_{h_3}\,\delta_3) + \mathcal{C}_{h_1}\,(-(\mathcal{S}_{h_2}\,\delta_2) + \mathcal{S}_{h_3}\,\delta_3))\Big)
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
s_{tt} = & \frac{1}{\Delta}\Big(D_{66}\,(\mathcal{C}_{h_2}\,(-(\gamma_1\,(r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_2\,\delta_3) - \delta_1\,((-(r_1\,\gamma_1) + r_2\,\gamma_2)\,\gamma_3\,\delta_2 + \gamma_2\,(r_1\,\gamma_1 - 2\,r_2\,\gamma_2 \\
& + r_3\,\gamma_3)\,\delta_3) - \mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,((r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_1\,(-(\gamma_2\,\delta_1) + \gamma_1\,\delta_2) + (-(r_1\,\gamma_1) + r_2\,\gamma_2)\,\gamma_3\,\delta_2\,\delta_3 \\
& + \gamma_2\,(r_1\,\gamma_1 - r_2\,\gamma_2)\,{\delta_3}^2)) - \mathcal{C}_{h_3}\,(\gamma_1\,(-(r_2\,\gamma_2) + r_3\,\gamma_3)\,\delta_2\,\delta_3 + \delta_1\,(\gamma_3\,(r_1\,\gamma_1 + r_2\,\gamma_2 - 2\,r_3\,\gamma_3)\,\delta_2 \\
& + (-(r_1\,\gamma_1\,\gamma_2) + r_3\,\gamma_2\,\gamma_3)\,\delta_3) + \mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,(\gamma_3\,(r_1\,\gamma_1 - r_3\,\gamma_3)\,{\delta_2}^2 + \gamma_2\,(-(r_1\,\gamma_1) + r_3\,\gamma_3)\,\delta_2\,\delta_3 \\
& + (-(r_2\,\gamma_2) + r_3\,\gamma_3)\,\delta_1\,(-(\gamma_3\,\delta_1) + \gamma_1\,\delta_3))) + \mathcal{C}_{h_1}\,(-(r_1\,\gamma_1\,\gamma_3\,\delta_1\,\delta_2) + r_2\,\gamma_2\,\gamma_3\,\delta_1\,\delta_2 \\
& - r_1\,\gamma_1\,\gamma_2\,\delta_1\,\delta_3 + r_3\,\gamma_2\,\gamma_3\,\delta_1\,\delta_3 + 2\,r_1\,{\gamma_1}^2\,\delta_2\,\delta_3 - r_2\,\gamma_1\,\gamma_2\,\delta_2\,\delta_3 - r_3\,\gamma_1\,\gamma_3\,\delta_2\,\delta_3 \\
& + \mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,(\gamma_2\,(-(r_1\,\gamma_1) + r_3\,\gamma_3)\,\delta_1\,\delta_2 + \gamma_1\,(r_1\,\gamma_1 - r_3\,\gamma_3)\,{\delta_2}^2 + (r_1\,\gamma_1 - r_2\,\gamma_2)\,\delta_3\,(-(\gamma_3\,\delta_1) \\
& + \gamma_1\,\delta_3)) - \mathcal{C}_{h_2}\,\mathcal{C}_{h_3}\,(\gamma_2\,(-(r_1\,\gamma_1) + 2\,r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_1\,\delta_3 + \delta_2\,(\gamma_3\,(-(r_1\,\gamma_1) - r_2\,\gamma_2 \\
& + 2\,r_3\,\gamma_3)\,\delta_1 + \gamma_1\,(2\,r_1\,\gamma_1 - r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_3)))))\Big)
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
f_{tt} =& \frac{1}{\Delta}\Big(D_{66}\,(-(r_2\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_2{}^2\,\delta_1{}^2) + r_2\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_2\,\gamma_3\,\delta_1{}^2 + r_3\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_2\,\gamma_3\,\delta_1{}^2 \\
&- r_3\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_3{}^2\,\delta_1{}^2 + r_2\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_2\,\delta_1\,\delta_2 + r_1\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_2\,\delta_1\,\delta_2 + r_1\,\gamma_1\,\gamma_3\,\delta_1\,\delta_2 \\
&- r_3\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_3\,\delta_1\,\delta_2 + r_2\,\gamma_2\,\gamma_3\,\delta_1\,\delta_2 - r_3\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_2\,\gamma_3\,\delta_1\,\delta_2 - 2\,r_3\,\gamma_3{}^2\,\delta_1\,\delta_2 \\
&- r_1\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_1{}^2\,\delta_2{}^2 + r_1\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_1\,\gamma_3\,\delta_2{}^2 + r_3\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_3\,\delta_2{}^2 - r_3\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_3{}^2\,\delta_2{}^2 \\
&+ r_1\,\gamma_1\,\gamma_2\,\delta_1\,\delta_3 - r_2\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_1\,\gamma_2\,\delta_1\,\delta_3 - 2\,r_2\,\gamma_2{}^2\,\delta_1\,\delta_3 + r_3\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_1\,\gamma_3\,\delta_1\,\delta_3 \\
&+ r_1\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_3\,\delta_1\,\delta_3 + r_3\,\gamma_2\,\gamma_3\,\delta_1\,\delta_3 - r_2\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_2\,\gamma_3\,\delta_1\,\delta_3 - 2\,r_1\,\gamma_1{}^2\,\delta_2\,\delta_3 + r_2\,\gamma_1\,\gamma_2\,\delta_2\,\delta_3 \\
&- r_1\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_1\,\gamma_2\,\delta_2\,\delta_3 + r_3\,\gamma_1\,\gamma_3\,\delta_2\,\delta_3 - r_1\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_3\,\delta_2\,\delta_3 + r_3\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\gamma_2\,\gamma_3\,\delta_2\,\delta_3 \\
&+ r_2\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_2\,\gamma_3\,\delta_2\,\delta_3 - r_1\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_1{}^2\,\delta_3{}^2 + r_1\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_2\,\delta_3{}^2 + r_2\,\mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,\gamma_1\,\gamma_2\,\delta_3{}^2 \\
&- r_2\,\mathcal{S}_{h_1}\,\mathcal{S}_{h_3}\,\gamma_2{}^2\,\delta_3{}^2 + \mathcal{C}_{h_2}\,\mathcal{C}_{h_3}\,(\gamma_2\,(-(r_1\,\gamma_1) + r_3\,\gamma_3)\,\delta_1\,\delta_3 + \delta_2\,((-(r_1\,\gamma_1) + r_2\,\gamma_2)\,\gamma_3\,\delta_1 \\
&+ \gamma_1\,(2\,r_1\,\gamma_1 - r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_3)) + \mathcal{C}_{h_1}\,(\mathcal{C}_{h_2}\,(\gamma_1\,(r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_2\,\delta_3 + \delta_1\,(\gamma_3\,(-(r_1\,\gamma_1) \\
&- r_2\,\gamma_2 + 2\,r_3\,\gamma_3)\,\delta_2 + \gamma_2\,(r_1\,\gamma_1 - r_3\,\gamma_3)\,\delta_3)) - \mathcal{C}_{h_3}\,(\gamma_1\,(r_2\,\gamma_2 - r_3\,\gamma_3)\,\delta_2\,\delta_3 \\
&+ \delta_1\,((-(r_1\,\gamma_1) + r_2\,\gamma_2)\,\gamma_3\,\delta_2 + \gamma_2\,(r_1\,\gamma_1 - 2\,r_2\,\gamma_2 + r_3\,\gamma_3)\,\delta_3))))\Big)
\end{aligned}
\tag{B.12}
$$

where

$$
\begin{aligned}
\Delta =& -2\,(-(\gamma_3\,\delta_2) + \gamma_2\,\delta_3)\,((-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_3})\,\mathcal{S}_{h_2}\,(\gamma_2\,\delta_1 - \gamma_1\,\delta_2) + (-1 + \mathcal{C}_{h_1}\,\mathcal{C}_{h_2})\,\mathcal{S}_{h_3}\,(-(\gamma_3\,\delta_1) \\
&+ \gamma_1\,\delta_3)) + \mathcal{S}_{h_1}\,(2\,(-(\gamma_2\,\delta_1) + \gamma_1\,\delta_2)\,(-(\gamma_3\,\delta_1) + \gamma_1\,\delta_3) - 2\,\mathcal{C}_{h_2}\,\mathcal{C}_{h_3}\,(-(\gamma_2\,\delta_1) + \gamma_1\,\delta_2)\,(-(\gamma_3\,\delta_1) \\
&+ \gamma_1\,\delta_3) + \mathcal{S}_{h_2}\,\mathcal{S}_{h_3}\,((\gamma_2{}^2 + \gamma_3{}^2)\,\delta_1{}^2 + (\gamma_1{}^2 + \gamma_3{}^2)\,\delta_2{}^2 - 2\,\gamma_1\,\gamma_3\,\delta_1\,\delta_3 + (\gamma_1{}^2 + \gamma_2{}^2)\,\delta_3{}^2 \\
&- 2\,\gamma_2\,\delta_2\,(\gamma_1\,\delta_1 + \gamma_3\,\delta_3)))
\end{aligned}
\tag{B.13}
$$

APPENDIX C: EXPLICIT EXPRESSIONS FOR THE DYNAMIC STIFFNESS ELEMENTS FOR INPLANE MOTION

Explicit expressions of the elements of the dynamic stiffness matrix are given only for case 1 for brevity. The coefficients for the other 2 cases can be obtained by following the procedure reported in section 3.2.1.:

$$s_{nn} = \frac{A_{11}\,(r_1\,\delta_2 - r_2\,\delta_1)\,(\mathcal{C}_{h_1}\,\mathcal{S}_{h_2}\,\delta_2 - \mathcal{C}_{h_2}\,\mathcal{S}_{h_1}\,\delta_1)}{\Lambda} \tag{C.1}$$

$$
\begin{aligned}
s_{nl} =& \frac{1}{\Lambda}\Big(- A_{11}\,r_2\,\delta_1 - A_{11}\,r_1\,\delta_2 + 2\,\alpha\,A_{12}\,\delta_1\,\delta_2 - \mathcal{C}_{h_1}\,\mathcal{C}_{h_2}\,(2\,\alpha\,A_{12}\,\delta_1\,\delta_2 - A_{11}\,(r_2\,\delta_1 + r_1\,\delta_2)) \\
&+ \mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,(\alpha\,A_{12}\,(\delta_1{}^2 + \delta_2{}^2) - A_{11}\,(r_1\,\delta_1 + r_2\,\delta_2))\Big)
\end{aligned}
\tag{C.2}
$$

$$f_{nn} = \frac{A_{11}\,(r_1\,\delta_2 - r_2\,\delta_1)\,(\mathcal{S}_{h_1}\,\delta_1 - \mathcal{S}_{h_2}\,\delta_2)}{\Lambda} \tag{C.3}$$

$$f_{nl} = \frac{A_{11}\,(\mathcal{C}_{h_1} - \mathcal{C}_{h_2})\,(r_2\,\delta_1 - r_1\,\delta_2)}{\Lambda} \tag{C.4}$$

$$s_{ll} = \frac{A_{66}\,(r_1\,\delta_1 - r_2\,\delta_2)\,(\mathcal{C}_{h_1}\,\mathcal{S}_{h_2}\,\delta_1 - \mathcal{C}_{h_2}\,\mathcal{S}_{h_1}\,\delta_2)}{\Lambda} \tag{C.5}$$

$$f_{ll} = \frac{A_{66}\,(r_2\,\delta_2 - r_1\,\delta_1)\,(\mathcal{S}_{h_2}\,\delta_1 - \mathcal{S}_{h_1}\,\delta_2)}{\Lambda} \tag{C.6}$$

where

$$\Lambda = 2\,\delta_1\,\delta_2 - 2\,\mathcal{C}_{h_1}\,\mathcal{C}_{h_2}\,\delta_1\,\delta_2 + \mathcal{S}_{h_1}\,\mathcal{S}_{h_2}\,\left({\delta_1}^2 + {\delta_2}^2\right) \tag{C.7}$$

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