On Hochschild cohomology and Modular representation theory

Lleonard Rubio y Degrassi

Doctor of Philosophy

City, University of London
Department of Mathematics

December 2016

Contents

Acknowledgments

I would like to extend my thanks to:

First and foremost my supervisor Markus Linckelmann for his intellectual support, for his enthusiasm, for his many valuable comments and his incredible patience.
City, University of London for the financial support.
The staff of the Maths department: in particular Chris Bowman, Radha Kessar and Maud De Visscher for helpful advice.
My office mates: in particular Karan for the many hours of life conversations in the department and in the pub, Niamh for sharing hard moments and victories, maths problems and solutions, Laura for the nice motivational coffee breaks and Malte for the awesome moments and discussions.

I am also deeply grateful to:

My dear friends: Lisa, Fabienne, Myriam and Damien for the epic adventures in Fairbridge Road.
My dear old best friends: Riccardo, Agustin, Pancho and Helena for being always present.
My girlfriend Gloria for her love and immense support in every moment.
My brothers: Jordi for the short but intense moments we spend each time together and Gustav for his support as an older brother.

This work is based on my passion for mathematics and would have never been possible without the incredible support of my parents: Marina and Eduardo.

Declaration

I grant powers of discretion to the University Librarian to allow this thesis to be copied in whole or in part without further reference to me. This permission covers only single copies made for study purpose, subject to normal conditions of acknowledgement.

ABSTRACT

The aim of this thesis is to study local and global invariants in representation theory of finite groups using the (restricted) Lie algebra structure of the first degree of Hochschild cohomology of a block algebra B as a main tool. This lead to two directions:
In the first part we investigate the global approach. In particular, we prove the compatibility of the p-power map under stable equivalence of Morita type of subclasses of the first Hochschild cohomology represented by integrable derivations. Further results in this aspect include an example showing that the p-power map cannot generally be expressed in terms of the BV operator. We also study some properties of r-integrable derivations and we provide a family of examples given by the quantum complete intersections where all the derivations are r-integrable.
In the second part our attention is focused on the local invariants. More precisely, we fully characterise blocks B with unique isomorphism class of simple modules such that the first degree Hochschild cohomology $\mathrm{HH}^1(B)$ is a simple as Lie algebra. In this case we prove that B is a nilpotent block with an elementary abelian defect group P of order at least 3 and $\mathrm{HH}^1(B)$ is isomorphic to the Witt algebra $\mathrm{HH}^1(kP)$.

Chapter 1

Introduction

Modular representation theory seeks to understand the representations of a finite group when the group algebra is not semisimple. The group algebra can be decomposed into indecomposable algebras called blocks such that for every simple module there is exactly one block that does not annihilate it. This leads to the strategy of investigating the structure of the blocks and the interaction between them.

There are two types of invariants associated with block algebras: those related to the algebra structure and those related to the group structure. The first family includes module categories, stable categories, numbers of isomorphism classes of simple modules and the Hochschild cohomology, for instance. The second family contains defect groups, fusion systems and some of their cohomological invariants.

In this context Hochschild cohomology plays a crucial role. It is shown in [26] that the Hochschild cohomology $\mathrm{HH}^*(B)$ of a block algebra B of a finite group G over an algebraically closed field k of prime characteristic p and the corresponding block cohomology $\mathrm{H}^*(B)$ are isomorphic modulo nilpotent ideals. Hence Hochschild cohomology can be seen as a bridge between the local and the global worlds - and there are not many of these bridges.

With this in mind, my thesis aims to analyse the interplay between the local and the global invariants with a focus on the first Hochschild cohomology $\mathrm{HH}^1(B)$ of a block algebra B. It is worth noting that $\mathrm{HH}^1(B)$ is a restricted Lie algebra. This rich algebraic structure points to analysing $\mathrm{HH}^1(B)$ in two directions: first, we investigate the extent to which the restricted Lie algebra structure of $\mathrm{HH}^1(B)$ is preserved under stable equivalences of Morita type; and, secondly, we investigate the connections between the local structure of a block B and the Lie algebra structure of $\mathrm{HH}^1(B)$.

Chapter 2 contains the background material and some results for the global approach. In particular, we give an example showing that the p-power map cannot generally be expressed in terms of the BV operator. Furthermore, we provide an example showing that p-power maps do not necessarily commute with transfer maps.

Chapter 3 continues with global invariants. In particular, it is devoted to the study of the invariance of the properties of p-power maps under stable equivalences. We define the notion of r-integrable derivation which are derivations that are induced by automorphisms on $A[[t]]$ such that induce the identity on $A[[t]]/t^rA[[t]]$. Using transfer maps as main tool, we show in Theorem 3.5.2 that the p-power map, restricted to the classes of r-integrable derivations, commutes with stable equivalences of Morita type between finite-dimensional selfinjective algebras. We also study some properties of r-integrable derivations and we provide a family of examples given by the quantum complete intersections where all the derivations are r-integrable.

In the last chapter we consider the second aspect. We prove that, if $\mathrm{HH}^1(B)$ is a simple Lie algebra such that B has a unique isomorphism class of simple modules, B is nilpotent with an elementary abelian defect group P of order at least 3 and $\mathrm{HH}^1(B)$ is isomorphic to the Witt algebra $\mathrm{HH}^1(kP)$. In particular, no other simple modular Lie algebras arise as $\mathrm{HH}^1(B)$ of a block B with a single isomorphism class of simple modules.

Chapter 2

Background and first results

2.1 Basics facts on algebras

The background for this section can be found in [2] and [4].

2.1.1 Idempotents and Blocks of an Algebra

Let k be a field. Let A be a finite dimensional associative unitary k-algebra. An *idempotent* i of A is a nonzero element of A such that $i^2 = i$. Two idempotents i, j of A are said to be *orthogonal* if $ij = ji = 0$. A *decomposition* of an idempotent i of A is a finite set J of pairwise orthogonal idempotents of A such that $i = \sum_{j \in J} j$. An idempotent i of A is called *primitive* if the only decomposition of i is given by i. A decomposition of an idempotent $i \in A$ consisting of primitive idempotents is called a *primitive decomposition* of i.

Let A be a finite-dimensional k-algebra. Recall that, if I is a primitive decomposition of 1 in A, we have a direct sum decomposition

$$A = \oplus_{i \in I} Ai.$$

Two summands $Ai, Ai^{'}$ in this sum are isomorphic if and only if the corresponding primitive idempotents $i, i^{'}$ are conjugate. Thus, if we choose a set of representatives J of the conjugacy classes of elements in I, then

$$A = \oplus_{i \in J} (Ai)^{n_i}$$

for some positive integers n_i, equal to the number of idempotents in I which are conjugate to i. Dividing by the radical yields a direct sum

$$A/J(A) = \oplus_{i \in J} (Ai/J(A)i)^{n_i}$$

and this is a direct sum of simple modules. More precisely, if we set $S_i = Ai/J(A)i$ then $\{S_i\}_{i \in J}$ is a set of representatives of the isomorphism classes of simple A-modules. The proof of Wedderburn's theorem yields an algebra isomorphism

$$A/J(A) \cong \prod_{i \in J} M_{n_i}(D_i)$$

where $D_i = \mathrm{End}_A(S_i)^{op}$. If k is algebraically closed, then

$$A/J(A) \cong \prod M_{n_i}(k) \cong \prod \mathrm{End}_k(S_i).$$

This shows that if k is algebraically closed, then the integer n_i is both equal to $\dim_k(S_i)$ and to the multiplicity of Ai as a direct summand of the regular A-module A.

Definition 2.1.1. Let A be a finite-dimensional k-algebra and let I be a set of representatives of the conjugacy classes of primitive idempotents in A. The Cartan matrix of A is the square matrix of non negative integers $C = (c_{ij})_{i,j \in I}$ where c_{ij} is the number of composition factors isomorphic to the simple A-module $S_i = Ai/J(A)i$ in a composition series of the projective indecomposable A-module A_j.

An idempotent i of A which lies in the center, $Z(A)$, of A is called a *central idempotent* of A. A primitive idempotent i of $Z(A)$ is called a *primitive central idempotent*, or a *block* of A. If 1_A has a primitive decomposition $\mathcal{B}$ in $Z(A)$, we have a decomposition of algebras

$$A = \prod_{j \in \mathcal{B}} Aj$$

where each Aj is an indecomposable algebra. Such Aj is called the block algebra of the block j.

Proposition 2.1.2. *Let A be a finite dimensional algebra over a field k. Then A has only finitely many blocks.*

Let U be a finite dimensional A-module. If $\mathcal{B}$ is a primitive decomposition of 1_A in Z(A), then $U = \oplus_{b \in \mathcal{B}} bU$ not only as vector spaces but as A-modules:

Proposition 2.1.3. *Let A be a finite-dimensional k-algebra, let B be the set of block idempotents of A, and let U be an A-module. We have a direct sum decomposition of U as an A-module of the form*

$$U = \oplus_{b \in \mathcal{B}} bU.$$

In particular, if U is an indecomposable or simple A-module, then there is a unique $b \in \mathcal{B}$ such that $bU = U$ and such that $b^{'}U = 0$ for all $b^{'} \in \mathcal{B}$ for all $b^{'} \neq b$. We then say that M *belongs to the block* b or to the block algebra Ab.

The next propositions will be useful later:

Proposition 2.1.4 (Rosenberg's Lemma)**.** *Let A be a finite dimensional algebra over a field k. Let i be a primitive idempotent of A. If $i \in \sum_{I \in \Delta} I$ where Δ is a set of ideals of A, then $i \in I$ for some $I \in \Delta$.*

Theorem 2.1.5 (Wedderburn-Malcev). *Let k be an algebraically closed field and let A be a finite dimensional k-algebra. Then there is a semisimple subalgebra $S \cong A/J(A)$ of A such that $A \cong S \oplus J(A)$, where the direct sum is a direct sum of vector spaces, and where S is a subalgebra of A.*

Proposition 2.1.6 (Idempotent Lifting Theorem). *Let k be a field. Let A, B be finite dimensional k-algebras with ideals I, J, respectively. Let $f : A \to B$ be a k-algebra homomorphism such that $f(I) = J$. Then*

- *If i is a primitive idempotent of A contained in I, then either $f(i) = 0$ or $f(i)$ is a primitive idempotent of B.*
- *If j is a primitive idempotent of B contained in J, then there exists a primitive idempotent i of A contained in I such that $f(i) = j$.*
- *Let $i, i^{'}$ be primitive idempotents of A contained in I such that $f(i) \neq 0 \neq f(i^{'})$. Then i and $i^{'}$ are conjugate in A if and only if $f(i)$ and $f(i^{'})$ are conjugate in B.*

Blocks are particular examples of symmetric algebras. In the next section introduce them.

2.1.2 Symmetric algebras

During this section we denote by k a unitary commutative ring unless otherwise specified. Let A, B be two k-algebras and let M be an A-B-bimodule. Then the k-dual $M^\vee = \mathrm{Hom}_k(M, k)$ has a structure of a B-A-bimodule given by $(b \cdot \alpha \cdot a)(m) = \alpha(amb)$ for all $a \in A$, $b \in B$, $m \in M$ and $\alpha \in \mathrm{Hom}_k(M, k)$. In particular $A^\vee$ is again an A-A-bimodule. An element $t \in A^\vee$ is called *symmetric* if $t(ab) = t(ba)$ for all $a, b \in A$.

Definition 2.1.7. Let A be a k-algebra. We say that A is *symmetric* if A is finitely generated projective as a k-module and if $A \cong A^\vee$ as A-A-bimodules. The image of $1 \in A$ under such isomorphism is called a *symmetrising form* and denoted by $s : A \to k$.

Equivalently we can define a symmetric algebra A by requiring there is a non-degenerate bilinear form $\big\langle -, - \big\rangle : A \times A \to k$. In this case the isomorphism is defined as $\phi : A \to A^\vee$ which sends a to $\big\langle a, - \big\rangle$. Conversely, given the isomorphism, hence the symmetrising form, the bilinear form is obtained by letting $\big\langle a, b \big\rangle = s(a \cdot b)$.

A symmetrising form s of a symmetric k-algebra A is automatically symmetric, in fact, if $\phi : A \cong A^\vee$ is an A-A-bimodule isomorphism such that $s = \phi(1_A)$, then for any $a \in A$, we have $a \cdot 1_A = a = 1_A \cdot a$. Applying ϕ on both sides, yields $a \cdot s = s \cdot a$, which is equivalent to $s(ab) = s(ba)$ for all $a, b \in A$.

Note that if k is a field and A is symmetric, then this implies that A is finite-dimensional.

Definition 2.1.8. Let k be a field and let A be a finite dimensional k-algebra. Let $\{u_i\}$ be a basis for A. We define $\{v_j\}$ to be the *dual basis* with respect to the bilinear form $\langle -, - \rangle$ if $\langle u_i, v_j \rangle = \delta_{ij}$.

Example 2.1.9. Let G be a finite group. We have an isomorphism of kG-kG-bimodules $(kG)^\vee \cong kG$ sending any k-linear map $\mu : kG \to k$ to $\mu_0 = \sum_{x \in G} \mu(x^{-1})x$ in kG. The symmetrising form $s : kG \to k$ is given by $s\big(\sum_{x \in G} \lambda_x x\big) = \lambda_1$ where $\lambda_x \in k$ for $x \in G$. Let $\{g\}_{g \in G}$ be the group basis of kG. Then $\{g^{-1}\}_{g \in G}$ is the dual basis.

Example 2.1.10. Let n be a positive integer. The matrix algebra $M_n(k)$ is symmetric with symmetrising form the trace map $tr : M_n(k) \to k$. In fact for $1 \leq i, j \leq n$ denote by $E_{i,j}$ the matrix whose coefficient at (i, j) is equal to 1 and zero otherwise. The set$\{E_{i,j}\}_{1 \leq i,j \leq n}$

is a k-basis of $M_n(k)$; in particular, $M_n(k)$ is finitely generated projective as a k-module. The isomorphism is constructed by sending $E_{i,j}$ to $E^{i,j}$ where $E^{i,j}$ is the dual basis element in $M_n(k)^\vee$ sending $E_{i,j}$ to 1 and $E_{i',j'}$ to 0 for $(i,j) \neq (i',j')$. Under this isomorphism the identity matrix is mapped to the trace map.

The following proposition will be useful in the last chapter:

Proposition 2.1.11. *Let A be a symmetric k-algebra with symmetrising forms $s \in A^\vee$. For any idempotent $e \in A$, the algebra eAe is symmetric with symmetrising form $s|_{eAe}$.*

Proof. Clearly eAe is finitely generated projective as k-module because it is a direct summand of A as k-module. Any bimodule isomorphism $A \cong A^\vee$ restricts to a bimodule isomorphism $eAe \cong e \cdot A^\vee \cdot e$, and any element in $e \cdot A^\vee \cdot e$ can be identified with an element in $(eAe)^\vee$, whence the statement. □

2.2 Some concepts from homological algebra

In the following two sections we follow [39] and [4].

2.2.1 Homotopy

In this section we recall some classical results on chain homotopy categories.

Let $\mathcal{C}$ be an abelian category. We denote by $\mathrm{Ch}(\mathcal{C})$ the category of chain complexes over $\mathcal{C}$. We denote by $\mathrm{Gr}(\mathcal{C})$ the category of graded objects over $\mathcal{C}$ with graded morphisms of degree zero.

Definition 2.2.1. Let $\mathcal{C}$ be an abelian category and let $X \in \mathrm{Ch}(\mathcal{C})$. Then X is *acyclic* if $\mathrm{H}_*(X) = \{0\}$.

Definition 2.2.2. Let $\mathcal{C}$ be an abelian category and let $X, Y \in \mathrm{Ch}(\mathcal{C})$. A chain map $f : X \to Y$ is a *quasi-isomorphism* if the induced map on homology, say $\mathrm{H}_*(f) : \mathrm{H}_*(X) \to \mathrm{H}_*(Y)$, is an isomorphism in $\mathrm{Gr}(\mathcal{C})$.

Definition 2.2.3. Let $\mathcal{C}$ be an additive category and let $X, Y \in \mathrm{Ch}(\mathcal{C})$ with differentials ϵ, δ respectively. A (*chain*) *homotopy* from X to Y is a family of morphisms $h_n : X_n \to Y_{n+1}$ in $\mathcal{C}$, for any $n \in \mathbb{Z}$. Two chain maps $f, f' : X \to Y$ are called *homotopic* , written $f \sim f'$, if there is a homotopy $h : X \to Y$ such that

$$f - f' = h \circ \delta + \epsilon \circ h,$$

or equivalently if $f_n - f_n' = h_{n-1} \circ \delta_n + \epsilon_{n+1} \circ h_n$ for any $n \in \mathbb{Z}$. Similarly, we define a *cochain homotopy* for cochain complexes X, Y but in this case $h^n : X^n \to Y^{n-1}$ for any $n \in \mathbb{Z}$.

Definition 2.2.4. Let $\mathcal{C}$ be an additive category and let $X, Y \in \mathrm{Ch}(\mathcal{C})$. A chain map $f : X \to Y$ is a *homotopy equivalence* if there is a chain map $g : Y \to X$ such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$; in that case, g is called a *homotopy inverse of* f, and the complexes X, Y are said to be *homotopy equivalent*, written $X \simeq Y$. If $X \simeq 0$, then X is called *chain contractible* or simply *contractible*.

Proposition 2.2.5. *Let $\mathcal{C}$ be an additive category and let $X \in \mathrm{Ch}(\mathcal{C})$. Then X is contractible if and only if the identity on X is homotopic to the zero chain map on X that is, $\mathrm{Id}_X \sim 0$.*

Let A be an algebra over a commutative ring k. During this section we let $\mathcal{C} = \mathrm{Mod}(A)$ denote the category of left A-modules. We denote by $\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)$ the k-module of chain maps from X to Y in $\mathrm{Ch}(\mathrm{Mod}(A))$.

Proposition 2.2.6. *Let $X, Y \in \mathrm{Mod}(A)$. The relation $\sim$ on the set of* $\mathrm{Hom}_{\mathrm{ChMod}(A)}(X, Y)$ *is an equivalence relation, compatible with sums and compositions of chain maps.*

Let us denote by $\mathrm{Hom}^0_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)$ the k-submodule of all chain maps $f : X \to Y$ satisfying $f \sim 0$, or equivalently, $f = h \circ \delta + \epsilon \circ h$ for some homotopy h from X to Y. Then Proposition 2.2.6 leads to the following definition:

Definition 2.2.7. The *homotopy category of complexes over* $\mathrm{Mod}(A)$ is the category $\mathrm{K}(\mathrm{Mod}(A))$ whose objects are the same as in $\mathrm{Ch}(\mathrm{Mod}(A))$ and whose morphisms are the homotopy equivalence classes of chain maps; that is,

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)/\mathrm{Hom}^0_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y).$$

Proposition 2.2.8. *Let $f : X \to Y$ be in* $\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)$. *If f is a homotopy equivalence, then f is a quasi-isomorphism.*

Corollary 2.2.9. *Let X be a complex of A-modules. If X is contractable then X is acyclic.*

Proof. If X is contractable, then X is quasi-isomorphic to zero by Proposition 2.2.8, which is equivalent to $\mathrm{H}_*(X) = 0$. □

There is a deep connection between the homology and cohomology of complexes and homotopy classes of chain maps. This is given by the following two propositions:

Proposition 2.2.10. *Let X be a complex of A-modules and let n be an integer. Let $A[n]$ be the complex equal to A in degree n and zero in all other degrees. Then there is a natural isomorphism*

$$\mathrm{H}_n(X) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(A[n], X).$$

Let X in $\mathrm{Ch}(\mathrm{Mod}(A))$ and let V be an A-module. Applying the contravariant functor $\mathrm{Hom}_A(-, V)$ to X yields a cochain complex $\mathrm{Hom}_A(X, V)$ of k-modules defined by

$$\mathrm{Hom}_A(X, V)^n = \mathrm{Hom}_A(X_n, V)$$

with differential

$$\delta^n : \mathrm{Hom}_A(X_n, V) \to \mathrm{Hom}_A(X_{n+1}, V)$$

given by $\delta^n(\alpha) = \alpha \circ \delta^{n+1}$ for any $\alpha \in \mathrm{Hom}_A(X_n, V)$. The cohomology of this cochain complex can be expressed as a family of morphisms in the homotopy category:

Proposition 2.2.11. *Let X be a chain complex of A-modules, and let V be an A-module. For any integer n we have a natural isomorphism of k-modules*

$$\mathrm{H}^n(\mathrm{Hom}_A(X, V)) \cong \mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(X, V[n]).$$

The right hand side of the isomorphism in Proposition 2.2.11 depends only on the homotopy category $\mathrm{K}(\mathrm{Mod}(A))$.Therefore any homotopy equivalence preserves the left side, that is

Corollary 2.2.12. *Let A be an algebra over a commutative ring k. Let $n \in \mathbb{Z}$ and let V be an A-module. Let $f : X \to Y$ be a homotopy equivalence of chain complexes of A-modules. Then f induces an isomorphism:*

$$\mathrm{H}^n(\mathrm{Hom}_A(Y, V)) \cong \mathrm{H}^n(\mathrm{Hom}_A(X, V)).$$

This interpretation in terms of the homotopy category will allow us to introduce extra structure on cohomology.

2.2.2 Ext functor

Definition 2.2.13. Let A be an algebra over a commutative ring k. A *projective resolution* of an A-module U is a pair (P, μ) consisting of a complex P of projective A-modules such that $P_i = 0$ for $i < 0$ and a quasi-isomorphism $\mu : P \to U$. If we view U as a chain complex concentrated in degree zero then a projective resolution of U can be written as:

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r, "\delta_2"] \arrow[d] & P_1 \arrow[r, "\delta_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\mu"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & U \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Equivalently, we define a projective resolution (P, δ) of an A-module U as an exact bounded below chain complex of the form:

$$\dots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\mu} U \longrightarrow 0$$

where all P_i are projective. The exactness of the above sequence is equivalent to the chain map being a quasi-isomorphism. In fact, the homology of both rows of the previous diagram is concentrated in degree zero, which is isomorphic to $P_0/\mathrm{Im}(\delta_1) = P_0/\ker(\mu) \cong U$. If μ is clear from the context, then we simply denote the projective resolution by P.

Definition 2.2.14. Let A be an algebra over a commutative ring k and let U, V be A-modules. Let (P, δ) be a projective resolution of U with differentials denoted by δ_n. Then the Ext functor is defined by:

$$\mathrm{Ext}^n_A(U, V) = \mathrm{H}^n(\mathrm{Hom}_A(P, V))$$

where $\mathrm{Hom}_A(P, V)$ is the cochain complex :

$$\mathrm{Hom}_A(P_0, V) \xrightarrow{\delta^0} \mathrm{Hom}_A(P_1, V) \xrightarrow{\delta^1} \mathrm{Hom}_A(P_2, V) \longrightarrow \dots$$

with differentials $\delta^n : \mathrm{Hom}_A(P_n, V) \to \mathrm{Hom}_A(P_{n+1}, V)$ given by $\delta^n(\alpha) = \alpha \circ \delta_{n+1}$. By Proposition 2.2.11 we have an interpretation of the Ext functor as

$$\mathrm{Ext}^n_A(U, V) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(P, V[n]).$$

Proposition 2.2.15. *Let A be an algebra over a commutative ring k and let V be an A-module. Then $\operatorname{Ext}_A^n(U,V)$ does not depend on the choice of the projective resolution, that is, for any two projective resolutions $(P,\mu),(P^{'},\mu^{'})$ of U we have*

$$\mathrm{H}^n(\operatorname{Hom}_A(P,V)) \cong \mathrm{H}^n(\operatorname{Hom}_A(P^{'},V)).$$

In order to prove this we recall the following two propositions:

Proposition 2.2.16. *Let A be an algebra over a commutative ring k. Let P be a complex of projective A-modules, and let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of complexes of A-modules. Suppose that g is a quasi-isomorphism and that one of P, Y is bounded below. The map

$$\operatorname{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P,Y) \to \operatorname{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(P,Z)$$

given by composition with g induces an isomorphism

$$\operatorname{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P,Y) \cong \operatorname{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P,Z).$$

Proposition 2.2.17. *Let A be an algebra over a commutative ring k. Let (P,μ), $(Q,\mu^{'})$ be projective resolutions of A-modules U, V, respectively. We have canonical isomorphisms*

$$\operatorname{Hom}_A(U,V) \cong \operatorname{Ext}_A^0(U,V) \cong \operatorname{Hom}_{K(\mathrm{Mod}(A))}(P,Q).$$

The isomorphism sends $\alpha : U \to V$ to the homotopy class of a chain map $\phi : P \to Q$ such that $\alpha \circ \mu \sim \mu^{'} \circ \phi$ as chain maps from P to V.

Proof. With the notation of Definition 2.2.14 we have $\operatorname{Ext}_A^0(U,V) = \ker(\delta^0)$. This is the family of all A-homomorphisms $\alpha : P_0 \to V$ such that $\alpha \circ \delta_1 = 0$, that is, all A-homomorphisms α such that $\operatorname{Im}(\delta_1) \subseteq \ker(\alpha)$. Any such homomorphism factors uniquely through the canonical surjection $P_0 \to P_0/\mathrm{Im}(\delta_1)$. Since $\mu : P \to U$ is quasi-isomorphism, this implies μ is determined by a surjective A-homomorphism, still denoted μ, from P_0 to V such that $\ker(\mu) = \mathrm{Im}(\delta_1)$. Thus $\ker(\delta^0)$ can be identified with the space of A-homomorphisms from $P/\ker(\mu) = U$ to V . This shows the first isomorphism. To show the second isomorphism we use the fact that the projective resolution Q comes with a quasi-isomorphism $\mu^{'} : Q \to V$. By Theorem 2.2.16 composition with $\mu^{'}$ is an isomorphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P, Q) \cong \mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P, V) = \mathrm{Ext}^0_A(U, V).$$

The compatibility with μ and $\mu^{'}$ follows from the explicit descriptions of these two isomorphisms. □

Proof of Proposition 2.2.15. We claim that for any two projective resolutions (P, μ), $(P^{'}, \mu^{'})$ of an A-module U there is a homotopy equivalence $\beta : P \to P^{'}$ such that $\mu^{'} \circ \beta = \mu$. By Corollary 2.2.12 we deduce the result.

We now prove the claim. Applying Proposition 2.2.17 with $U = V$ and $P^{'} = Q$ shows that Id_U corresponds to the homotopy class of a chain map $\phi : P \to P^{'}$ satisfying $\mu^{'} \circ \phi \sim \mathrm{Id}_U \circ \mu = \mu$. Exchanging P and $P^{'}$ yields a chain map $\psi : P^{'} \to P$ satisfying $\mu \circ \psi = \mu^{'}$. Thus $\mu \circ \psi \circ \phi = \mu$. But also $\mu \circ \mathrm{Id}_P = \mu$. Now, by Proposition 2.2.16, composition with μ induces an isomorphism $\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P, P) \cong \mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P, U)$,. Therefore, $\psi \circ \phi$ and Id_P should be the equal in the homotopy category, that is $\psi \circ \phi \sim \mathrm{Id}_P$. A similar argument shows that $\phi \circ \psi \sim \mathrm{Id}_{P'}$, and hence that $P \cong P^{'}$ as stated. □

Theorem 2.2.18. *Let A be an algebra over a commutative ring k. Let U, V be A-modules with projective resolutions P, Q, respectively, and let $n \geq 0$ be an integer. We have a natural k-linear isomorphism*

$$\mathrm{Ext}^n_A(U, V) \cong \mathrm{Hom}_{K(Mod(A))}(P, Q[n])$$

Proof. Let $\mu^{'} : Q \to V$ be the quasi-isomorphism associated with Q. Then $\mu^{'}[n] : Q[n] \to V[n]$ is a quasi-isomorphism and hence, by Proposition 2.2.16, induces an isomorphism $\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P, Q[n]) \cong \mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(A))}(P, V[n]) = \mathrm{Ext}^A_n(U, V)$ as stated. The naturality is an easy verification. □

2.3 Hochschild cohomology

In this section we let A be an algebra over a commutative ring k which is projective over k. We denote by A^{op} the opposite algebra, that is, $A^{op} = A$ as k-module but the product is defined as $a \cdot b = ba$ where the first product is in A^{op} and the second in A. The tensor product $A \otimes_k A^{op}$ over k is simply denoted by $A \otimes A^{op}$. We will work in the category of A-A-bimodules or equivalently in the $A \otimes A^{op}$-module category. We can regard A as an $A \otimes A^{op}$-module by left and right multiplication on itself.

Definition 2.3.1. Let M be a A-A-bimodule. The *Hochschild cohomology* of A with coefficients in M is

$$\mathrm{HH}^*(A; M) = \mathrm{Ext}^*_{A \otimes A^{op}}(A, M)$$

and the *Hochschild cohomology* of A is

$$\mathrm{HH}^*(A) = \mathrm{HH}^*(A; A) = \mathrm{Ext}^*_{A \otimes A^{op}}(A, A).$$

A fundamental feature of Hochschild cohomology is that there is a canonical projective resolution which is constructed as follows:

We denote by $A^0 = k$. We regard $A^{\otimes n}$ as an $A \otimes A^{op}$-module where A acts on the left on the first copy of A and A acts on the right on the last copy of A. Note that $A^{\otimes n+2}$ is a projective $A \otimes A^{op}$-module for every $n \geq 1$. Here the assumption of A being projective as a k-module is crucial. In fact, since A is projective, $A^{\otimes n}$ is projective as a k-module for any $n \geq 1$. If $n \geq 0$, then $A^{\otimes n+2} = A \otimes A^{\otimes n} \otimes A$ is a summand of a direct sum of copies of k, i.e $A^{\otimes n+2}$ is projective as an $A \otimes A^{op}$-module. Hence a projective resolution of A as $A \otimes A^{op}$-bimodules is given by

$$\dots \xrightarrow{d_2} A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \longrightarrow 0$$

where $d_n : A^{\otimes n+2} \to A^{\otimes n+1}$ is defined as

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \tag{2.3.1}$$

We denote by X the complex above, that is, for $n \geq -1$ set $X_n = A^{\otimes n+2}$ and for $n \geq 0$ denote by $d_n : X_n \to X_{n-1}$ the $A \otimes_k A^{op}$-homomorphism given by Equation 2.3.1 and set $X_n = 0$ for $n \leq -2$ and $d_n = 0$ for $n \leq -1$. The last step left in order to prove that X is a projective resolution is to show that it is exact. Hence it is enough to show that it is contractible as a complex of right and left A-modules (by Corollary 2.2.9). In the first case, using Proposition 2.2.5, it is equivalent to show that there exists an homotopy $h : X \to X$ such that $\mathrm{Id}_{X_n} = d_{n+1} \circ h_n + h_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$. The homotopy h is defined as $h_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}$for all $n \in \mathbb{Z}$. Similarly for the second case.

This resolution is called the *bar resolution* of A and we denote it by P_A. It can be used to explicitly calculate the Hochschild cohomology of A with coefficients in any $A \otimes_k A^{op}$-module M as

$$\mathrm{HH}^n(A; M) = \mathrm{H}^n(\mathrm{Hom}_{A \otimes_k A^{op}}(P_A, M)).$$

The differentials are given by $\delta^n(f) = f \circ d_{n+1}$. In particular, the Hochschild cohomology of A can be written as:

$$\mathrm{HH}^n(A) = \mathrm{H}^n(\mathrm{Hom}_{A \otimes A^{op}}(P_A, A)) = \frac{\mathrm{Ker}(\delta^n)}{\mathrm{Im}(\delta^{n-1})} \tag{2.3.2}$$

In order to work more explicitly with this cohomology, especially in lower degrees, we use the following canonical isomorphism:

Proposition 2.3.2. *There is a canonical isomorphism*

$$\mathrm{Hom}_k(A^{\otimes n}, M) \cong \mathrm{Hom}_{A\otimes A^{op}}(A^{\otimes n+2}, M)$$

for any $n \geq 0$. The isomorphism sends a k-linear map $\psi : A^{\otimes n} \to M$ to a unique $A \otimes A^{op}$-homomorphism $\phi : A \otimes A^{\otimes n} \otimes A \to M$ such that $\phi(a_0 \otimes c \otimes a_{n+1}) = a_0\psi(c)a_{n+1}$ for any $a_0, a_{n+1} \in A$ and $c \in A^{\otimes n}$.

Proof. To show that this is an isomorphism, we explicitly construct the inverse map: an $A \otimes A^{op}$-homomorphism $\phi : A \otimes A^{\otimes n} \otimes A \to M$ is sent to a unique k-linear map $\psi : A^{\otimes n} \to M$ by $\psi(c) = \phi(1 \otimes c \otimes 1)$. It is easy to check that the given maps are inverse to each other. □

Hence we define

Definition 2.3.3. Let M be an $A \otimes A^{op}$-module. We define k-modules $C^n(A, M)$ for $n \geq 0$ by

$$C^n(A, M) = \mathrm{Hom}_k(A^{\otimes n}, M).$$

We define the cochain maps $\delta^n : \mathrm{Hom}_k(A^{\otimes n}; M) \to \mathrm{Hom}_k(A^{\otimes n+1}; M)$ by

$$\begin{aligned} \delta^n(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 f(a_1 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n) \\ &\quad + (-1)^{n+1} f(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})a_n \end{aligned} \tag{2.3.3}$$

for any $f \in \mathrm{Hom}_k(A^{\otimes n}, M)$.

It is easy to prove that $C^*(A; M) = (C^n(A, M), \delta^n)$ is a cochain complex. By Proposition 2.3.2 we can relate the cohomology of $C^*(A; M)$ with the Hochschild cohomology.

Theorem 2.3.4. *Let M be an $A \otimes A^{op}$-module. The cochain complex $C^*(A;M)$ is isomorphic to $\mathrm{Hom}_{A\otimes A^{op}}(P_A, M)$. In particular the cohomology of this cochain complex is the Hochschild cohomology of A with coefficients in M.*

As we are mostly interested in the Hochschild cohomology of A from now on we will focus on $C^n(A,A) = \mathrm{Hom}_k(A^{\otimes n}, A)$. However some of these results can be generalised to $C^n(A,M)$. Let us study the degree zero of the Hochschild cohomology of A using Theorem 2.3.4.

Proposition 2.3.5. *The degree zero term of the Hochschild complex can be identified as*

$$\mathrm{Hom}_{A\otimes A^{op}}(A \otimes A, A) \ \cong C^0(k;A) = \mathrm{Hom}_k(k,A) \cong A$$

where the first isomorphism is from Propositon 2.3.2 and the second sends the linear map $\sigma : k \to A$ to $\sigma(1)$. The composition of these two isomorphisms send a bimodule homomorphism $\zeta : A \otimes A \to A$ to the element $\zeta(1 \otimes 1)$ in A. The differential $\delta^0 : A \to \mathrm{Hom}_k(A,A)$ sends an element $b \in A$ to the map $\delta^0(b) : A \to A$ given by $\delta^0(b)(a) = ab - ba$.

Consequently we have:

Proposition 2.3.6. *We have a canonical isomorphism*

$$\mathrm{HH}^0(A) \cong Z(A).$$

Proof. By definition we have

$$\mathrm{HH}^0(A;A) = \mathrm{Ker}(\delta^0) = \{b \in A \ st \ ab - ba = 0 \ for \ all \ a \in A\} \tag{2.3.4}$$

Hence the result. □

The following definitions allow us to provide an explicit interpretation of the degree 1 of the Hochschild cohomology:

Definition 2.3.7. The k-linear map $f : A \to A$ is called a *derivation* if $0 = af(b) + f(a)b - f(ab)$ for all $a, b \in A$. The set of all derivations is denoted by $\mathrm{Der}(A)$.

Definition 2.3.8. Let $[-,-] : A \otimes A \to A$ denote the additive commutator that is, $[a, b] = ab - ba$ for every a, b in A. Then $[-, b]$ is a derivation, called an *inner derivation*, and the set of them is denoted by $\mathrm{IDer}(A)$.

Proposition 2.3.9. *We have a canonical isomorphism*

$$\mathrm{HH}^1(A; A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A).$$

Proof. By definition we have $\mathrm{HH}^1(A) = \ker(\delta^1)/\mathrm{Im}(\delta^0)$. Consequently we need to show that $\ker(\delta^1) = \mathrm{Der}(A)$ and $\mathrm{Im}(\delta^0) = \mathrm{IDer}(A)$. Let $f \in \mathrm{Hom}_k(A; A)$. Then $\delta^1(f) \in \mathrm{Hom}_k(A \otimes_k A; A)$ is defined by $\delta^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$. Hence $\mathrm{Ker}(\delta^1) = \{f : A \to A \mid f(ab) = af(b) + f(a)b\} = \mathrm{Der}(A)$. We have $f \in \mathrm{Im}(\delta^0)$ if and only if there exists $b \in A$ such that $f = \delta^0(b)$, that is, if and only if $f(a) = ab - ba$ for all $a \in A$. Or equivalently, $\mathrm{Im}(\delta^0) = \{f : A \to A \mid f(a) = ab - ba\}$. Hence the result. □

In the few next sections we analyse some of the rich algebraic structure of Hochschild cohomology. We begin with a recollection of gradings in different algebraic structures:

2.3.1 Grading

Definition 2.3.10. A *graded algebra* A over a commutative ring k is a k-algebra which is a direct sum of k-modules A_i such that $A_iA_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}$. An element $f \in A_i$ is called homogeneous of degree i.

Definition 2.3.11. A *graded-commutative algebra* A over a commutative ring k is a graded algebra such that for f, g homogeneous elements of degree m, n respectively, we have $fg = (-1)^{n+m}gf$.

Similarly, we can define a graded version of a Lie algebra:

Definition 2.3.12. A *graded Lie algebra of degree* -1 is a Lie algebra $\mathcal{L}$ over a field k endowed with a grading which is compatible with the Lie bracket. That is, as a graded vector space $\mathcal{L} = \oplus_{i\in\mathbb{Z}}\mathcal{L}_i$ we have:

$$[\mathcal{L}^m, \mathcal{L}^n] \subseteq \mathcal{L}^{m+n-1}.$$

For every $f, g \in \mathcal{L}$ of degree m, n respectively, we have

$$[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$$

and for every $f, g, h \in \mathcal{L}$ of degree m, n, l respectively, we have

$$(-1)^{(m-1)(l-1)}[[f, g], h] + (-1)^{(n-1)(m-1)}[[g, h], f] + (-1)^{(l-1)(m-1)}[[h, f], g] = 0.$$

2.3.2 Gerstenhaber and BV algebras

Using Proposition 2.2.18 we have

Proposition 2.3.13. *Let A be an associative unital algebra over a commutative ring k. Let U, V be A-modules. Then the graded k-module*

$$\mathrm{Ext}^*_A(U, U) = \oplus_{n\geq 0}\mathrm{Ext}^n_A(U, U)$$

is a graded unital associative k-algebra through composition of chain maps.

Proof. Let P be a projective resolution of U. Given

$$\begin{aligned} \zeta &\in \mathrm{Ext}^n_A(U, U) = \mathrm{Hom}_{K(Mod(A))}(P, P[n]) \\ \tau &\in \mathrm{Ext}^m_A(U, U) = \mathrm{Hom}_{K(Mod(A))}(P, P[m]) \end{aligned} \tag{2.3.5}$$

define the product $\zeta \cdot \tau$ in $\mathrm{Ext}^{m+n}_A(U, U)$ by

$$\zeta \cdot \tau = \zeta[m] \circ \tau \in \mathrm{Ext}^{n+m}(U, U) = \mathrm{Hom}_{K(Mod(A))}(P, P[n + m]).$$

This defines a graded product on $\mathrm{Ext}^*_A(U, U)$ which is associative because it is induced by composition in the homotopy category. The unit element of this multiplication is given by Id_U, viewed as an element of $\mathrm{Ext}^0_A(U, U)$. □

Corollary 2.3.14. *The Hochschild cohomology of A is a graded unital associative k-algebra.*

Since $\mathrm{HH}^*(A) \cong \mathrm{Ext}^*_{A\otimes_k A^{op}}(A, A)$, we can regard the Hochschild cohomology of degree n as being naturally isomorphic to the abelian group of equivalence classes of extensions of A by A of length n. Consequently we have a cup product on $\mathrm{HH}^*(A)$. This is given by:

Definition 2.3.15. Let A be an associative unital k-algebra over a commutative ring. Let $f \in C^n(A, A) = \mathrm{Hom}_k(A^{\otimes n}, A)$ and $g \in C^m(A, A)$. Then the *cup-product* $f \smile g \in C^{n+m}(A, A)$ is given by:

$$(f \smile g)(a_1, \ldots, a_{n+m}) = f(a_1, \ldots, a_n) \cdot g(a_{n+1}, \ldots, a_{n+m}).$$

Under the isomorphism in Theorem 2.3.4, the cup product on Hochschild cohomology corresponds to graded composition in the homotopy category (defined in Proposition 2.3.13).

The following proposition is due to Gerstenhaber [12]:

Proposition 2.3.16. *Let A be a k-algebra. Then the Hochschild cohomology of A is a graded-commutative algebra with respect to the cup product.*

We recall the definitions of Gerstenhaber and Batalin-Vilkovisky algebras.

Definition 2.3.17. Let k be a commutative ring. A *Gerstenhaber algebra* is a graded k-module $A = \oplus_{i\in\mathbb{Z}} A_i$ such that:

- A is a graded-commutative algebra with the operation denoted by $\smile$.

- A is a graded Lie algebra of degree -1
- The Lie bracket and $\smile$ satisfy the Poisson rule, that is, for any $c \in A_l$ the map $[-, c] : A_i \to A_{i+l-1}$ satisfies

$$[a \smile b, c] = [a, c] \smile b + (-1)^{m(l-1)} a \smile [b, c] \tag{2.3.6}$$

  where m, l are respectively the degrees of a and c.

Gerstenhaber showed in [12] that the Hochschild cohomology of an algebra A is a Gerstenhaber algebra. We recall the construction:

Let A be a k-algebra. Let $f \in C^n(A) = \mathrm{Hom}_k(A^n, A)$ and $g \in C^m(A)$ with $n, m \geq 0$. If $n, m \geq 1$, then for $1 \leq i \leq n$, define

$$(f \circ_i g)(a_1, \ldots, a_{n+m-1}) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1}) \tag{2.3.7}$$

If we let $n \geq 1$ and $m = 0$, then g can be identified as an element of A. For $1 \leq i \leq n$, define

$$(f \circ_i g)(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_{i-1}, g, a_i, \ldots, a_{n-1})$$

for any other case, define $f \circ_i g$ to be zero. Let

$$f \circ g = \sum_{i=1}^{n} (-1)^{(n-1)(i-1)} f \circ_i g \tag{2.3.8}$$

then we define the Gerstenhaber Lie bracket as

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f \tag{2.3.9}$$

This Lie bracket with the cup product makes Hochschild cohomology into a Gerstenhaber algebra.

Remark 2.3.18. If we let $f, g \in C^1(A) = \mathrm{Hom}_k(A, A)$, then the Gerstenhaber Lie bracket becomes the usual Lie bracket $[f, g] = f \circ g - g \circ f$ which induces a Lie algebra structure on $\mathrm{HH}^1(A)$.

Definition 2.3.19. A *Batalin-Vilkovisky (BV) algebra* is a Gerstenhaber algebra $A = \oplus_{n\in\mathbb{Z}} A_n$ together with a degree -1 operator $\Delta : A_n \to A_{n-1}$, called the BV-operator, such that $\Delta \circ \Delta = 0$ and such that we have

$$[a, b] = -(-1)^{(m-1)n}(\Delta(a \smile b) - (\Delta a) \smile b - (-1)^m a \smile (\Delta b)). \tag{2.3.10}$$

for any two homogeneous elements a, b of degrees n, m, respectively. In other words, the deviation of Δ from being a derivation is the bracket.

In [38] Tradler noticed that the Hochschild cohomology of a symmetric algebra is a BV algebra. In order to state this result, we need the following definition:

Definition 2.3.20. Let k be a unital commutative ring and let A be symmetric k-algebra with a non-degenerate bilinear form $\langle -, - \rangle$ induced by a bimodule isomorphism between A and its k-linear dual $A^\vee$. Let $n \geq 1$ and $f \in C^n(A, A)$. For $i \in \{1, \ldots, n\}$ define $\Delta_i f \in C^{n-1}(A, A)$ by the equation

$$\langle \Delta_i f(a_1, \ldots, a_{n-1}), a_n \rangle = \langle f(a_i, \ldots, a_{n-1}, a_n, a_1, \ldots, a_{i-1}), 1 \rangle \tag{2.3.11}$$

with this we define

$$\Delta = \sum_{i=1}^{n} (-1)^{i(n-1)} \Delta_i$$

Theorem 2.3.21 ([38, Theorem 1])**.** *Let A be a finite dimensional symmetric k-algebra with non-degenerate bilinear form $\langle -, - \rangle : A \times A \to k$. For $f \in C^n(A, A)$, let $\Delta f \in C^{n-1}(A, A)$ be given by the Equation* 2.3.11*. Then Hochschild cohomology is a BV algebra with BV operator Δ defined above.*

2.3.3 Hochschild cohomology of the tensor product of two algebras

In [21], Le and Zhou define the tensor product of Gerstenhaber algebras:

Definition 2.3.22. Let $(A, \smile_A, [-,-]_A)$ and $(B, \smile_B, [-,-]_B)$ be two Gerstenhaber algebras over k. Then there is a new Gerstenhaber algebra $(L, \smile, [-,-])$ over k given as follows:

(i) $L_n = \oplus_{i+j=n} A_i \otimes B_j$ as a k-vector space for each $n \in \mathbb{Z}$;

(ii) $(a \otimes b) \smile (a' \otimes b') = (-1)^{|a'||b|}(a \smile_A a') \otimes (b \smile_B b')$;

(iii) $[a \otimes b, a' \otimes b'] = (-1)^{(|a|+|b|-1)|b'|}[a, a']_A \otimes (b \smile_B b') + (-1)^{|a|(|a'|+|b'|-1)}(a \smile_A a') \otimes [b, b']_B$

where $a, a' \in A$ and $b, b' \in B$ are homogeneous elements. We call $(L, \smile, [-,-])$ the tensor product of the two Gerstenhaber algebras A and B, and denote it by $A \otimes B$.

The following result will be useful in Chapter 4.

Theorem 2.3.23 ([21, Theorem 3.3])**.** *Let A and B be two k-algebras such that one of them is finite dimensional. Then there is an isomorphism of Gerstenhaber algebras:*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B).$$

Corollary 2.3.24 ([21, Theorem 3.4])**.** *Let A and B be two k-algebras such that one of them is finite dimensional. Then there is an isomorphism of Lie algebras*

$$\mathrm{HH}^1(A \otimes B) \cong \mathrm{HH}^1(A) \otimes \mathrm{HH}^0(B) \oplus \mathrm{HH}^0(A) \otimes \mathrm{HH}^1(B).$$

Let n be a positive integer. In the following corollary we denote by $\prod_{i=1}^n A_i$ the n-fold tensor products $A_1 \otimes A_2 \otimes \cdots \otimes A_n$.

Corollary 2.3.25. *Let n be a positive integer. Let A_i be a finite dimensional k-algebras* *for* $1 \leq i \leq n$. *Then there is an isomorphism of Lie algebras*

$$\mathrm{HH}^1\Big(\prod_{i=1}^{n} A_i\Big) \cong \sum_{\substack{i_1,\dots,i_n \geq 0,\\ i_1+\cdots+i_n=1}} \prod_{j=1}^{n} \mathrm{HH}^{i_j}(A_j).$$

Proof. We prove it by induction. For $n = 2$ it holds by Corollary 2.3.24. Let assume that it holds for n, then we have:

$$\begin{aligned}
\mathrm{HH}^1\Big(\prod_{i=1}^{n+1} A_i\Big) &\cong \mathrm{HH}^1\Big(\prod_{i=1}^{n} A_i\Big) \otimes \mathrm{HH}^0(A_{n+1}) \oplus \mathrm{HH}^0\Big(\prod_{i=1}^{n} A_i\Big) \otimes \mathrm{HH}^1(A_{n+1}) \\
&= \sum_{\substack{i_1,\dots,i_n \geq 0,\\ i_1+\cdots+i_n=1}} \prod_{j=1}^{n} \mathrm{HH}^{i_j}(A_j) \otimes \mathrm{HH}^0(A_{n+1}) \oplus \prod_{i=1}^{n} \mathrm{HH}^0(A_i) \otimes \mathrm{HH}^1(A_{n+1}) \\
&= \sum_{\substack{i_1,\dots,i_n \geq 0,\\ i_1+\cdots+i_{n+1}=1}} \prod_{j=1}^{n} \mathrm{HH}^{i_j}(A_j).
\end{aligned}$$

□

2.3.4 Restricted Lie algebra structure on Hochschild cohomology

Definition 2.3.26. Let $\mathcal{L}$ be a Lie algebra over a field k and let $x, y \in \mathcal{L}$. We denote by

$$\begin{aligned}
\mathrm{Ad} : \mathcal{L} &\to \mathrm{End}_k(\mathcal{L}) \\
x &\mapsto Ad(x)(y) = [x, y]
\end{aligned} \tag{2.3.12}$$

the adjoint representation of $\mathcal{L}$.

We introduce the last structure on Hochschild cohomology

Definition 2.3.27. Let $\mathcal{L}$ be a Lie algebra over a field k of positive characteristic p. We say that $\mathcal{L}$ is a *restricted Lie algebra* if there exists a map $[p] : \mathcal{L} \to \mathcal{L}$, called p-power or p-operator, such that:

- $\mathrm{Ad}(x^{[p]})(y) = Ad(x)^p(y)$
- $(\lambda x)^{[p]} = \lambda^p x^{[p]}$

- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$

for all $x, y \in \mathcal{L}$ and $\lambda \in k$. Here, the element $is_i(x,y)$ is the coefficient of t^{i-1} in

$$\mathrm{Ad}^{p-1}(ta+b)((a)) = [\dots[a, ta+b], \dots, ta+b].$$

The following theorem is due to Zimmermann [40]

Theorem 2.3.28. *For any field k of characteristic $p \geq 2$ the sum of the odd degree of the Hochschild cohomology, $\oplus_{n\in\mathbb{N}}\mathrm{HH}^{2n+1}(A,A)$, is a restricted Lie algebra under the Gerstenhaber bracket.*

In particular $\mathrm{HH}^1(A)$ is a restricted Lie algebra with the p-power map given by composing f with itself p times, that is $f^{[p]} = f \circ \cdots \circ f = f^p$.

We end the section with two definitions that will be useful later.

Definition 2.3.29. Let $\mathcal{L}$ be a restricted Lie algebra. An element x of $\mathcal{L}$ is *p-nilpotent* if $x^{[p]} = 0$.

Definition 2.3.30. Let $\mathcal{L}$ be a restricted Lie algebra. An element x of $\mathcal{L}$ is *p-idempotent* if $x^{[p]} = x$.

2.4 Transfer map and Hochschild cohomology

One of the fundamental tools in this thesis are transfer maps in Hochschild cohomology. In order to introduce them, we need first some background covered by the following section.

2.4.1 Adjoint functors

Definition 2.4.1. Let $F : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be two covariant functors of two categories $\mathcal{C}, \mathcal{D}$. We say $\mathcal{G}$ is the *left adjoint* to $\mathcal{F}$, or equivalently, $\mathcal{F}$ is *right adjoint* to $\mathcal{G}$, if there is an isomorphism of bifunctors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-),-) \cong \mathrm{Hom}_{\mathcal{D}}(-,\mathcal{F}(-))$, that is, a family of isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V),U) \cong \mathrm{Hom}_{\mathcal{D}}(V,\mathcal{F}(U))$$

where U and V are objects of $\mathcal{C}$ and $\mathcal{D}$ respectively such that

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-),U) \cong \mathrm{Hom}_{\mathcal{D}}(-,\mathcal{F}(U))$$

and

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V),-) \cong \mathrm{Hom}_{\mathcal{D}}(V,\mathcal{F}(-))$$

are isomorphisms of contravariant and covariant functors respectively.

Such an isomorphism of bifunctors need not be unique. If $\mathcal{C},\mathcal{D}$ are k-linear categories, for some commutative ring k, we always require such an isomorphism of bifunctors to be k-linear.

Given an adjunction isomorphism $\phi : \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(-),-) \cong \mathrm{Hom}_{\mathcal{D}}(-,\mathcal{F}(-))$, we can evaluate ϕ at an object V in $\mathcal{D}$ and $\mathcal{G}(V)$ and hence obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(V,\mathcal{F}(\mathcal{G}(V))) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(V),\mathcal{G}(V)).$$

We denote by $f(V) : V \to \mathcal{F}\mathcal{G}(V)$ the morphism corresponding to $\mathrm{Id}_{\mathcal{G}(V)}$ through this isomorphism, that is $f(V) = \phi(V,\mathcal{G}(V))(\mathrm{Id}_{\mathcal{G}(V)})$. It is easy to check that the family of morphism $f(V)$ defined in this way is a natural transformation

$$f : \mathrm{Id}_{\mathcal{D}} \to \mathcal{F} \circ \mathcal{G} \tag{2.4.1}$$

called the *unit* of the adjuction isomorphism ϕ. Similarly if we evaluate ϕ at an object U in $\mathcal{C}$ and $\mathcal{F}(U)$ we get an isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(U),U)) \cong \mathrm{Hom}_{D}(\mathcal{F}(U),\mathcal{F}(U))$. We denote by $g(U) : \mathcal{G}(\mathcal{F}(U)) \to U$ the morphism corresponding to $\mathrm{Id}_{\mathcal{F}(U)}$ through the isomorphism

$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(U),U)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(U),\mathcal{F}(U))$ i.e $g(U) = \phi(\mathcal{F}(U),U)^{-1}(\mathrm{Id}_{\mathcal{F}(U)})$. Again this is a natural transformation

$$g : \mathcal{G} \circ \mathcal{F} \to \mathrm{Id}_{\mathcal{C}} \tag{2.4.2}$$

called the *counit* of the adjuction isomorphism ϕ.

Example 2.4.2. Let A, B be algebras and let M be an A-B-bimodule. For any A-module U and any B-module V we have natural inverse isomorphisms of k-modules

$$\begin{aligned} \mathrm{Hom}_A(M \otimes_B V, U) &\cong \mathrm{Hom}_B(V, \mathrm{Hom}_A(M,U)) \\ \varphi &\to (v \to (m \to \varphi(m \otimes v))) \\ (m \otimes v \to \psi(v)(m)) &\leftarrow \psi \end{aligned} \tag{2.4.3}$$

Hence $M \otimes_B -$ is left adjoint to $\mathrm{Hom}_A(M,-)$.

Proposition 2.4.3. *Let A, B be two finite dimensional k-algebras and let ${}_AM_B$ be an A-B-bimodule. Furthermore, if A is symmetric, then $\mathrm{Hom}_A(M,A)$ is isomorphic to $M^\vee = \mathrm{Hom}_k(M,k)$ as B-A-modules. The isomorphism sends $\psi \in \mathrm{Hom}_A(M,A)$ to $s \circ \psi$.*

Proof. In order to define the inverse map we take $\{u_i\}$ to be a basis of A and let $\{v_i\}$ be the dual basis respect the bilinear form of A denoted by $\langle -,- \rangle_A$. Then if we consider $\theta \in M^\vee$ and $x \in M$ the image of θ under the inverse map is given by $\sum_i \theta(v_i x)u_i$. □

Proposition 2.4.4 ([41, Lemma 4.2.5])**.** *Let A, B be two k-algebras. Let M be an A-B-bimodule. Then there is a natural transformation*

$$\eta : \mathrm{Hom}_A(M,A) \otimes_A - \to \mathrm{Hom}_A(M,-) \tag{2.4.4}$$

of functors from the category of A-modules to the category of B-modules. In addition if M is finitely generated and projective as an A-module, then $\mathrm{Hom}_A(M,A)$ is a finitely generated projective right A-module and η is an isomorphism of functors.

Let A, B be two symmetric finite dimensional k-algebras. Using Proposition 2.4.3 and Proposition 2.4.4 and Example 2.4.2 we have the adjoint pair $(M \otimes_B -, M^\vee \otimes_A -)$. Let $\{u_i\}$ be a basis of A and let $\{v_i\}$ be the dual basis with respect to the bilinear form of A. The counit morphism $\epsilon_M : M \otimes_B M^\vee \to A$ is explicitly given by $\epsilon_M(x \otimes_A \theta) = \sum_i \theta(v_i x) u_i$ for $x \in M$ and $\theta \in M^\vee$.

Similarly, if B has a symmetrising form $t \in B^\vee$ and if M is finitely generated and projective as a right B-module, then $M^\vee$ is finitely generated and projective as a left B-module and we have an isomorphism of functors

$$\mathrm{Hom}_B(M^\vee, -) \cong \mathrm{Hom}_B(M^\vee, B) \otimes_B - \cong M^{\vee\vee} \otimes_B - \cong M \otimes_B -. \tag{2.4.5}$$

Consequently we have another adjoint pair $(M^\vee \otimes_A -, M \otimes_B, -)$ with unit morphism $\mu_{M^\vee} : A \to M \otimes_B M^\vee$ which can be calculated as follows. Since M is finitely generated and projective as a right B-module, there exist a positive integer s and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \leq i \leq s$ such that for any $x \in M$, $x = \sum_i x_i \varphi_i(x)$. Hence $\mu_{M^\vee}$ sends $a \in A$ to $\sum a x_i \otimes_B t \circ \varphi_i$.

2.4.2 Transfer maps in Hochschild cohomology

We recall that the Hochschild cohomology of a k-algebra A is the cohomology of the Hochschild complex $\mathrm{Hom}_{A \otimes A^{op}}(P_A, A) \cong C^*(A)$. By Theorem 2.2.18 we have another characterisation of the Hochschild cohomology which is given in terms of the homotopy category that is,

$$\mathrm{HH}^n(A) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A \otimes A^{op}))}(P_A, P_A[n]).$$

In [24] Linckelmann introduced the transfer maps in Hochschild cohomology for symmetric algebras, say A, B, as follows: Let ${}_A M_B$ be an A-B-bimodule such that ${}_A M$ and M_B are finitely generated and projective. Let P_A (resp. P_B) be a projective resolution of A (resp.

of B) as bimodules. Suppose we are given $\zeta \in \mathrm{HH}^n(B) \cong \mathrm{Hom}_{K(B\otimes B^{op})}(P_B, P_B[n])$. Then we define $t^M(\zeta) \in \mathrm{HH}^n(A) \cong \mathrm{Hom}_{K(A\otimes A^{op})}(P_A, P_A[n])$ to be the class of the composition

$$P_A \xrightarrow{\tau} M \otimes_B P_B \otimes_B M^\vee \xrightarrow{\mathrm{Id}_M\otimes\zeta\otimes\mathrm{Id}_{M^\vee}} M \otimes_B P_B[n] \otimes_B M^\vee \xrightarrow{\sigma} P_A[n] \qquad (2.4.6)$$

where τ lifts the unit morphism and σ lifts a translation of the counit morphism ϵ_M : $M \otimes_B M^\vee \to A$.

An explicit construction of the transfer map is due to Koenig, Liu, and Zhou [20]. They choose P_A, respectively P_B, to be the bar resolution P_A, respectively P_B and they explicitly construct the first lift. This is given in the following proposition:

Proposition 2.4.5. *Let A, B and M be as before. Then for $n \geq 0$ we let*

$$\theta_n : A^{\otimes(n+2)} \to M \otimes_B B^{\otimes(n+2)} \otimes_B M^\vee \qquad (2.4.7)$$

denote the map which sends $a_0 \otimes \cdots \otimes a_{n+1} \in A^{\otimes n+2}$ to

$$\sum_{i_0,\ldots,i_n} a_0x_{i_1} \otimes \varphi_{i_1}(a_1x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_nx_{i_0}) \otimes (t \circ \varphi_{i_0})a_{n+1}. \qquad (2.4.8)$$

The map θ_n commutes with the differential, that is, $\theta_{n-1}d_i = (\mathrm{Id}_M\otimes d_i\otimes\mathrm{Id}_{M^\vee})\theta_n$ for each i, where $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \leq i \leq s$ such that any $x \in M, x = \sum_i x_i\varphi_i(x)$. Moreover θ lifts the unit morphism that is, $\eta_{M^\vee}\mu_A = (\mathrm{Id}_M \otimes \mu_B \otimes \mathrm{Id}_{M^\vee})\theta_0$.

Proof. We just prove that $\theta^\vee$ lifts the unit morphism $\eta_{M^\vee} : A \to M \otimes_B M^\vee$, that is, $\eta_{M^\vee}\nu_A = (\mathrm{Id}_M \otimes \nu_B \otimes \mathrm{Id}_{M^\vee})\theta_0$, where $\nu_A : A \otimes_k A \to A$ and $\nu_B : B \otimes_k B \to B$ are the multiplication maps. For $a_0, a_1 \in A$ we have $(\mathrm{Id}_M \otimes \nu_B \otimes \mathrm{Id}_{M^\vee})\theta_0(a_0 \otimes a_1) = (\mathrm{Id}_M\otimes\nu_B\otimes\mathrm{Id}_{M^\vee})(a_0x_i\otimes t\circ\phi_ia_1) = a_0x_i\otimes_B t\circ\phi_ia_1$, and on the other hand, $\nu_{M^*}\nu_A(a_0\otimes a_1) = \eta_{M^\vee}(a_0a_1) = a_0x_i \otimes_B t \circ \phi_ia_1$ since $\eta_\vee$ is an A-A-bimodule homomorphism. ☐

Hence if we let $f \in C^n(B)$, then $\mathrm{Tr}^M(f)$ is given by:

$$A^{\otimes(n+2)} \xrightarrow{\theta_n} M \otimes_B B^{\otimes(n+2)} \otimes_B M^\vee \xrightarrow{\mathrm{Id}_M\otimes f\otimes\mathrm{Id}_{M^\vee}} M \otimes_B B \otimes_B M^\vee \xrightarrow{\epsilon_M} A \qquad (2.4.9)$$

Explicitly we have:

Proposition 2.4.6. *For $f \in C^n(B) = \mathrm{Hom}_k(B^{\otimes n}, B)$ with $n \geq 0$, the map $\mathrm{Tr}^M(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to*

$$\sum_{i_0,\ldots,i_n,j} \Big\langle \varphi_{i_0}(v_j x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0})) \Big\rangle_B u_j. \tag{2.4.10}$$

Here $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \leq i \leq s$ such that for any $x \in M, x = \sum_i x_i \varphi_i(x)$, where $\langle -, - \rangle_B$ is the bilinear form over B and where $\{u_j\}, \{v_j\}$ are the dual bases in A. In addition $\mathrm{Tr}^M : C^n(B) \to C^n(A)$ is a chain map.

Proof. The proof of the first statement follows from the explicit construction of θ_n and ϵ_M. The second part is also direct since θ^* is a chain map by Proposition 2.4.5 .

□

Remark 2.4.7. Let $f \in C^1(B)$. Using notation from Proposition 2.4.5, the map $\mathrm{Tr}^M(f)$ sends $a_1 \in A$ to :

$$\sum_{i_0,i_1,j} \Big\langle \varphi_{i_0}(v_j x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_0})) \Big\rangle_B u_j. \tag{2.4.11}$$

2.4.3 Compatibility between transfer and p-power map

Let k be a field of positive characteristic p. Let A, B be two finite dimensional symmetric k-algebras. Since the first Hochschild cohomology group is a restricted Lie algebra, we can ask about the compatibility between the p-power and the transfer map. That is, we can ask if the following diagram commutes:

$$\begin{tikzcd}
\mathrm{HH}^1(B) \arrow[r, "\mathrm{Tr}^M"] \arrow[d, "{[p]}"] & \mathrm{HH}^1(A) \arrow[d, "{[p]}"] \\
\mathrm{HH}^1(B) \arrow[r, "\mathrm{Tr}^M"] & \mathrm{HH}^1(A)
\end{tikzcd}$$

Example 2.4.8. Let k be a field of characteristic 2. Let $H = \{1, (12)\} \cong C_2 \leq S_3$ and $M = kS_3$ considered as a kS_3-kC_2 bimodule. By $\langle -, - \rangle$ we mean the standard bilinear form for the group algebra kH. We choose $R = \{1, (13), (23)\}$ as set of representatives of S_3/C_2. We note that M is finitely generated and projective as a right kC_2-module, since $[S_3 : H] = 3$, so there exist $\varphi_i \in \mathrm{Hom}_{kC_2}(kS_3, kC_2)$ with $1 \leq i \leq 2$ such that for any $x \in M$, $x = \sum_i x_i \varphi_i(x)$. Explicitly:

$$\varphi_1(1) = 1,\ \varphi_1((12)) = (12),\ \varphi_1(g) = 0 \tag{2.4.12}$$

for every other $g \in S_3$. Similarly we define φ_2 and φ_3, which are denoted by $\varphi_{(13)}$ and $\varphi_{(23)}$ respectively, as follows:

$$\begin{aligned} &\varphi_{(13)}((13)) = 1,\ \varphi_{(13)}((123)) = (12),\ \varphi_{(13)}(g) = 0 \\ &\varphi_{(23)}((23)) = 1,\ \varphi_{(23)}((132)) = (12),\ \varphi_{(23)}(g) = 0 \end{aligned} \tag{2.4.13}$$

for every other $g \in S_3$.

It is easy to check that these maps satisfy $x = \sum_i x_i \varphi_i(x)$ for every $x \in M$. Indeed, let $g \in S_3$, $h \in H$ such that $g = x^{'}h$ and let $\varphi_x(xh) = h$, where $x, x^{'} \in R$. Then we have:

$$\sum_{x \in R} x\varphi_x(g) = \sum_{x \in R} x\varphi_x(x^{'}h) = x^{'}\varphi_{x^{'}}(x^{'}h) = x^{'}h = g. \tag{2.4.14}$$

Since C_2 is commutative, $\mathrm{HH}^1(kC_2) = \mathrm{Der}_k(kC_2)$ is generated by $\{f_0, f_1\}$ such that $f_0((12)) = 1,\ f_1((12)) = (12)$. In this case if we let $f \in \mathrm{Der}(kC_2)$, then the transfer map can be expressed as:

$$\mathrm{Tr}^M(f)(a) = \sum_{x,x^{'} \in R, g \in G} \left\langle \varphi_{x^{'}}(g^{-1}x), f(\varphi_x(ax^{'})) \right\rangle_B g \tag{2.4.15}$$

where $x, x^{'}$ are coset representatives.

For $a = (13)$ we get:

$$\sum_{g\in G}\Big\langle \varphi_1(g^{-1}(13)), f(\varphi_{(13)}((13)))\Big\rangle_B g + \sum_{g\in G}\Big\langle \varphi_{(13)}(g^{-1}), f(\varphi_1(1))\Big\rangle_B g +$$

$$\sum_{g\in G}\Big\langle \varphi_{(23)}(g^{-1}(23)), f(\varphi_{(23)}((23)))\Big\rangle_B g = \sum_{g\in G}\Big\langle \varphi_{(23)}(g^{-1}(23)), f((12))\Big\rangle_B g$$

$$= \Big\langle 1, f((12))\Big\rangle 1 + \Big\langle (12), f(12)\Big\rangle (13)$$

In particular for f_0 and f_1 we have:

$$\mathrm{Tr}^M(f_0)(13) = 1,\ \mathrm{Tr}^M(f_1)(13) = (13).$$

By a similar calculation we have:

$$\mathrm{Tr}^M(f_0)(23) = 1,\ \mathrm{Tr}^M(f_1)(23) = (23).$$

In order to compute $\mathrm{Tr}^M(f_0)(123)$ and $\mathrm{Tr}^M(f_0)(132)$, we use the fact that $(13)(23) = (132)$, $(23)(13) = (123)$ and the property that the transfer map of a derivation is itself a derivation. Hence:

$$\mathrm{Tr}^M(f_0)(123) = \mathrm{Tr}^M(f_0)((23)(13)) = \mathrm{Tr}^M(f_0)(23)(13) + (23)\mathrm{Tr}^M(f_0)(13) = (13) + (23)$$

Similarly:

$$\mathrm{Tr}^M(f_0)(132) = (13) + (23),\ \mathrm{Tr}^M(f_1)(123) = \mathrm{Tr}^M(f_1)(132) = 0.$$

Finally writing $a = (12) = (132)(13)$:

$$\mathrm{Tr}^M(f_0)(12) = 1 + (123) + (132),\ \mathrm{Tr}^M(f_1)(12) = (12).$$

It is straightforward to check the following conditions on element of S_3:

$$\mathrm{Tr}^M(f_0)^{[2]} = \mathrm{Tr}^M(f_0^{[2]}) = 0$$

$$\mathrm{Tr}^M(f_1)^{[2]} = \mathrm{Tr}^M(f_1^{[2]}) = \mathrm{Tr}^M(f_1)$$

since f_0 is p-nilpotent and f_1 is p-idempotent. Let us now consider an element $f \in \mathrm{Der}(kC_2)$. Then $f = \lambda f_0 + \mu f_1$ where $\lambda_0, \lambda_1 \in k$. Hence

$$f^{[2]} = (\lambda f_0 + \mu f_1)^{[2]} = \lambda^2 f_0 + \lambda\mu(f_0 f_1 + f_1 f_0) + \mu^2 f_1^2$$
$$= \lambda^2 f_0^2 + \lambda\mu f_0 + \mu^2 f_1^2 = \lambda\mu f_0 + \mu^2 f_1$$

since $f_0 f_1 = f_0$ and $f_1 f_0 = 0$. So

$$\mathrm{Tr}^M(f^2) = \lambda\mu \mathrm{Tr}^M(f_0) + \mu^2 \mathrm{Tr}^M(f_1).$$

On the other hand

$$\begin{aligned}\mathrm{Tr}^M(\lambda f_0 + \mu f_1)^2 &= (\mathrm{Tr}^M(\lambda f_0) + \mathrm{Tr}^M(\mu f_1))^2 \\ &= \lambda^2(\mathrm{Tr}^M(f_0))^2 + \lambda\mu(\mathrm{Tr}^M(f_0)\mathrm{Tr}^M(f_1) + \mathrm{Tr}^M(f_1)\mathrm{Tr}^M(f_0)) + \mu^2(\mathrm{Tr}^M(f_1))^2 \\ &= \mu^2 \mathrm{Tr}^M(f_1) + \lambda\mu \mathrm{Tr}^M(f_0)\end{aligned}$$

since $\mathrm{Tr}^M(f_0)$ is 2-nilpotent, $\mathrm{Tr}^M(f_1)$ is 2-idempotent and $\mathrm{Tr}^M(f_0)\mathrm{Tr}^M(f_1) + \mathrm{Tr}^M(f_1)\mathrm{Tr}^M(f_0) = \mathrm{Tr}^M(f_0)$. Hence the diagram commutes.

Despite what we have seen in this example, the p-power and transfer maps do not commute in general. In fact we give a negative example in the next section.

There are many open problems regarding the compatibility between the transfer and the p-power maps. For example, it is not known if the transfer maps send p-nilpotent elements to p-nilpotent elements. The main problem lies in the explicit calculation of the composition of the transfer with itself p times. Even if in the next chapter we show that p-power maps commute with stable equivalences of Morita type on the subgroup of classes represented by integrable derivations, this does not give a complete answer. In fact, not all p-nilpotent derivations are integrable. For example, if we let k be a field of characteristic 2 and $A = kC_2$ as in the previous example, then $\mathrm{HH}^1(kC_2) = \mathrm{Der}_k(kC_2)$ is generated by $\{f_0, f_1\}$. It is easy to prove that f_0 is not integrable but it is 2-nilpotent.

2.4.4 Transfer and p-power maps need not commute

In the previous section we provided an example of the compatibility between transfer and p-power maps for the algebras $A = kS_3$ and $B = kC_2$. In Chapter 3 we will show that this compatibility holds for any finite-dimensional selfinjective k-algebras A, B, with separable semisimple quotients, in a certain subgroup of HH^1 when A and B are stably equivalent of Morita type. A natural question emerges: can we have the compatibility between these maps without the assumption of stable equivalence of Morita type? The following example gives a negative answer to this question.

Example 2.4.9. In this example we will show that Tr^M does not always commute with $[p]$. Let k be a field of characteristic 3. Let $H = \{1, (123), (132)\} \cong C_3 \leq S_3$ and $M = kS_3$ considered as a kS_3-kC_3 bimodule. By $\langle -, - \rangle$ we mean the standard bilinear form for the group algebra kH. We choose $R = \{1, t = (12)\}$ as set of representatives of S_3/H. We note that M is finitely generated and projective as a right kC_3-module, since $[G : H] = 2$, so there exist $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_{kC_3}(kS_3, kC_3)$ with $i = 1$ or 2 such that for any $x \in M$, $x = \sum_i x_i \varphi_i(x)$. Explicitly:

$$\begin{aligned} &\varphi_1(1) = 1, \varphi_1((123)) = (123), \\ &\varphi_1((132)) = (132), \varphi_1(g) = 0 \end{aligned} \tag{2.4.16}$$

for every other $g \in G$. Similarly we define φ_2, denoted by φ_t, as follows:

$$\varphi_t((12)) = 1, \varphi_t((13)) = (132), \varphi_t((23)) = (123), \varphi_t(g) = 0 \tag{2.4.17}$$

for every other $g \in G$. Since C_3 is commutative, $\mathrm{HH}^1(kC_3) = \mathrm{Der}_k(kC_3)$ is generated by $\{f_0, f_1, f_2\}$ such that $f_0((123)) = 1$, $f_1((123)) = (123)$ and $f_2((123)) = (132)$. Using Equation 2.4.15, if we let $f \in Der_k(kC_3)$ and $g = xh$ for some $x \in R$ and $h \in H$ then we have :

$$\begin{aligned}\mathrm{Tr}^M(f)(a) &= \sum_{x,x'\in R, g\in G} \Big\langle \varphi_{x'}(g^{-1}x), f(\varphi_x(ax'))\Big\rangle_B g \\ &= \sum_{h\in H} \Big\langle h^{-1}, f(\varphi_1(a))\Big\rangle h + \Big\langle h^{-1}, f(\varphi_t(a))\Big\rangle th + \\ &\Big\langle h^{-1}, f(\varphi_1(at))\Big\rangle ht + \Big\langle\ h^{-1}, f(\varphi_t(at))\Big\rangle tht\end{aligned} \tag{2.4.18}$$

where in the second equation we use the identity $tht = h^{-1}$. In particular for $a = (123)$ we have:

$$\begin{aligned}\mathrm{Tr}^M(f_0)((123)) &= \sum_{h\in H} \Big(\Big\langle h^{-1}, f_0((123))\Big\rangle + \Big\langle h, f_0((132))\Big\rangle\Big) h \\ &= \sum_{h\in H} \Big(\Big\langle h^{-1}, 1\Big\rangle + \Big\langle h, -(123)\Big\rangle\Big) h = 1-(132)\end{aligned} \tag{2.4.19}$$

similarly we have:

$$\mathrm{Tr}^M(f_0)(132) = 1-(123). \tag{2.4.20}$$

We can note that $\mathrm{Tr}^M(f_0^{[3]}) = 0$ since f_0 is p-nilpotent. Therefore $\mathrm{Tr}^M(f_0^{[3]})(132) = 0$. On the other hand $\mathrm{Tr}^M(f_0)^{[3]}((132)) = \mathrm{Tr}^M(f_0) \circ \mathrm{Tr}^M(f_0)(1-(123)) = \mathrm{tr}^M(f_0)(-1+(132)) = 1-(123)$. Now, the transfer maps send elements of $\mathrm{HH}^1(B)$ to elements $\mathrm{HH}^1(A)$. Therefore there should exists a inner derivation in S_3 which sends (132) to $1-(123)$ if we require the commutativity of the diagram. But there is no element in $a \in kS_3$ such that $[a,(132)] = 1$. Hence in this case the p-power map does not commute with the transfer map.

2.4.5 The p-power map and BV-operator

In the previous section, we noticed that the stable equivalence of Morita type is a fundamental condition for the compatibility between the p-power and transfer maps. We will show this for the spaces in HH^1 given by integrable derivations. It is natural to ask if this compatibility holds for the entire Hochschild cohomology of degree 1. Koenig et al. in [20] prove that if A and B are two symmetric k-algebras that are related under a stable equivalence of Morita type then their Hochschild cohomology are isomorphic as BV algebras. As a corollary we have that the Gerstenhaber bracket is invariant under stable equivalences. This is because the bracket can be written in terms of the BV-operator and the cup product (as shown by Tradler in [38]). More precisely, Tradler expressed the Gerstenhaber bracket of two elements $f \in \mathrm{HH}^n(A)$, $g \in \mathrm{HH}^m(A)$, in the following way:

$$[f, g] = \Delta(f \smile g) - \Delta(f) \smile g - (-1)^n f \smile \Delta(g).$$

The aim of this section is to prove that the BV-operator and the cup product do not determine the p-power map on HH^1.

Proposition 2.4.10. *Let A be symmetric k-algebra with symmetrising form $s : A \to k$ and let $f : A \to A$ be a derivation. Then $s \circ f : A \to k$ is a symmetric map.*

Proof. We need to prove that $s \circ f$ is symmetric. We have:

$$\begin{aligned}(s \circ f)(ab) &= s(f(ab)) = s(f(a)b + af(b)) = s(f(a)b) + s(af(b)) \\ &= s(f(b)a) + s(bf(a)) = (s \circ f)(ba)\end{aligned} \tag{2.4.21}$$

since s is symmetric. □

Remark 2.4.11. Let A be a symmetric algebra with a symmetrising form $s : A \mapsto k$ or equivalently with a non-degenerate bilinear form $< -, - >$. Let $f \in C^1(A, A)$. Using Equation 2.3.11 the BV operator in degree 1, $\Delta f \in C^0(A, A) = A$, is given by

$$< \Delta f(1), a >=< f(a), 1 >= (s \circ f)(a) \tag{2.4.22}$$

Since Δ is of degree -1, this implies Δ sends $\mathrm{HH}^1(A)$ to $Z(A)$. Alternatively this can be proven using Proposition 2.4.10. In fact, if we let $f \in \mathrm{Der}(A)$, then we have to show that $(\Delta f)(1) \cdot a = a \cdot (\Delta f)$ for every $a \in A$ where we denote by $\cdot$ the multiplication in A. It is enough to prove that $< (\Delta f)(1) \cdot a, b >=< a \cdot (\Delta f), b >$ for every $a, b \in A$ since

$< -, - >$ is not degenerate. Using Proposition 2.4.10 and the properties of bilinear forms we have that

$$\begin{aligned} < (\Delta f)(1) \cdot a, b > &= < (\Delta f)(1), ab) > = < f(ab), 1 > = (s \circ f)(ab) \\ &= (s \circ f)(ba) = < (\Delta f)(1), ba > \\ &= < ba, (\Delta f)(1) > = < a \cdot (\Delta f)(1), b > \end{aligned} \tag{2.4.23}$$

Finally if $g \in \mathrm{Inn}(A)$, then $< \Delta g, a > = (s \circ g)(a) = s(ab - ba) = 0$ for every $a \in A$. So $\Delta g = 0$.

Note that the p-power map can be defined as a map, only on the diagonal entries, from the p-ary cartesian product $\mathrm{HH}^1(A) \times \cdots \times \mathrm{HH}^1(A)$ to $\mathrm{HH}^1(A)$ sending $(f, \ldots, f)$ to f^p. Our question is the following: is it possible to express the p-power map as k-linear combination of compositions of the cup product and the BV-operator having as domain the p-ary cartesian product $\mathrm{HH}^1(A) \times \cdots \times \mathrm{HH}^1(A)$ and codomain $\mathrm{HH}^1(A)$?

The first step is to write down all the possible ways to compose the BV-operator and the cup product. For the sake of simplicity, we denote by the p-tuple $(n_1, \ldots, n_p)$ the p-ary cartesian product $\mathrm{HH}^{n_1}(A) \times \cdots \times \mathrm{HH}^{n_p}(A)$ where the entries denote the degrees of Hochschild cohomology. With this notation, the BV-operator sends (n_1) to $(n_1 - 1)$ and the cup product sends the couple (n_1, n_2) to $(n_1 + n_2)$ for non-negative integers n_1 and n_2. Another example is the map $\smile \otimes \mathrm{Id}$ which sends $(1, 1, 1)$ to $(2, 1)$.

Theorem 2.4.12. *Let A be symmetric algebra over a field k of positive characteristic p. If $p = 2$ let $n = 2$, otherwise let $n \geq 2$. Let $f \in \mathrm{Der}(A)$ and let $z_1, \ldots, z_n \in Z(A)$. Every composition of the BV-operator and the cup product sends an element of the form $(z_1 \cdot f, \ldots, z_n \cdot f) \in \mathrm{HH}^1(A) \times \cdots \times \mathrm{HH}^1(A)$ to $z' \cdot f \in \mathrm{HH}^1(A)$ where $z' \in Z(A)$.*

We need some technical lemmas in order to prove Theorem 2.4.12:

Lemma 2.4.13. *Let A be symmetric algebra over a field k of positive characteristic p and let $f : A \to A$ be a derivation. For $p > 2$ then $f \smile f$ is zero.*

Proof. Let $p > 2$ and let $f \in \mathrm{Der}(A)$. Since Hochschild cohomology has the structure of a graded-commutative algebra and since f has odd degree we have that $f \smile f = (-1)^{|f|^2}(f \smile f) = (-1)f \smile f$. Hence $f \smile f$ should be zero. In particular this implies that for every $p > 2$ the cup product of f with itself is zero. Hence the statement. □

For $p = 2$ we have the following:

Lemma 2.4.14. *Let A be symmetric algebra over a field of characteristic 2, let $f : A \to A$ be a derivation and let Δ denote the BV operator. Then $\Delta(f \smile f) = 0$.*

Proof. Let $f \in C^1(A, A)$. Then $f \smile f \in C^2(A, A)$. Using the Equation 2.3.11 we have $\Delta = -\Delta_1 + \Delta_2$ where:

$$< \Delta_1(f \smile f)(a_1), a_2 >=< (f \smile f)(a_1, a_2), 1 >= s \circ (f(a_1)f(a_2)) = s \circ (f(a_2)f(a_1)) \tag{2.4.24}$$

since s is symmetric. Similarly for Δ_2. Hence we have:

$$\begin{aligned} < \Delta(f \smile f)(a_1), a_2 > &= - < f(a_1)f(a_2), 1 > + < f(a_2)f(a_1), 1 > \\ &= -s(f(a_1)f(a_2)) + s(f(a_2)f(a_1)) = 0 \end{aligned} \tag{2.4.25}$$

for every a_1, a_2. Thus $\Delta(f \smile f) = 0$. □

Proof of Theorem 2.4.12. The proof is by induction. Let $n = 2$ then the product $\mathrm{HH}^1(A) \times \mathrm{HH}^1(A)$ is represented by the couple $(1, 1)$. The cup product sends $(1, 1)$ to (2). If $\mathrm{char}(k) > 2$ we get zero by Lemma 2.4.13. In fact if we let $z_1, z_2 \in Z(A)$ then

$$(z_1 \cdot f \smile z_2 \cdot f)(a_1, a_2) = z_1 \cdot f(a_1) \cdot z_2 \cdot f(a_2) = (z_1 \cdot z_2) \cdot (f \smile f)(a_1, a_2) = 0$$

for all $a_1, a_2 \in A$. In characteristic 2 the only possible composition at this point is to apply Δ which send (2) to (1). By Lemma 2.4.14 we get zero. Starting again from $(1,1)$, the other possibility is to apply either $\Delta \otimes \mathrm{Id}$ or $\mathrm{Id} \otimes \Delta$ which send $(1,1)$ to $(0,1)$ or $(1,0)$, respectively. In both cases we can only compose with the cup product which sends $(1,0)$ or $(0,1)$ to (1), that is, to $\mathrm{HH}^1(A)$. At this point we cannot neither compose with the cup product nor with Δ (since $\Delta^2 = 0$). It is easy to check that an element $(z_1 \cdot f, z_2 \cdot f) \in \mathrm{HH}^1(A) \times \mathrm{HH}^1(A)$ is sent through these maps to $z^{'} \cdot f$ where $z^{'} \in Z(A)$. For example, the map $\smile \circ(\Delta \otimes \mathrm{Id})$ sends $(z_1 \cdot f, z_2 \cdot f)$ to $\Delta(z_1 \cdot f) \cdot z_2 \cdot f$.

Assume it holds for $n-1$, then we show it holds for n. The n-ary product $\mathrm{HH}^1(A) \times \cdots \times \mathrm{HH}^1(A)$ is represented by the n-tuple $(1, \ldots, 1)$. If we apply the cup product to any two components it will give us the zero by Lemma 2.4.13. The only possibility is to apply the BV-operator to one of the entries, that is, $\mathrm{Id} \otimes \cdots \otimes \Delta \otimes \cdots \otimes \mathrm{Id}$ which sends the n-tuple $(1, \ldots, 1)$ to the n-tuple which has as entries $n-1$ ones and 1 zero. The cup product can only be applied on the entries of the form $1, 0$ or $0, 1$ otherwise we get zero. In this cases, the cup product sends an n-tuple to an $n-1$ tuple with one in all entries. By induction hypothesis the result follows. Considering again the n-tuple which has $n-1$ ones and 1 zero, we can apply the BV-operator to a non-zero entry which has as codomain a n-tuple with $n-2$ ones and 2 zeros. If we keep applying the BV-operator to non-zero entries, we get a n-tuple with all zero entries except one. In this case we can only apply the cup product until we have as codomain $\mathrm{HH}^1(A)$. It is easy to check that an element of the form $(z_1 \cdot f, \ldots, z_n \cdot f)$ is sent through these compositions to $z^{'} \cdot f$ for some $z^{'} \in Z(A)$. Otherwise at certain point we get an m-tuple, where m is a positive integer less than n, with one in all entries. By induction hypothesis the result follows.

□

Corollary 2.4.15. *Let A be a symmetric k-algebra over a field of positive characteristic p. Let $\underline{f} = (f, \dots, , f)$ be a p-tuple in $\mathrm{HH}^1(A) \times \cdots \times \mathrm{HH}^1(A)$. Then all possible compositions of the cup product and the BV-operator send $\underline{f}$ to $z \cdot f$ for some $z \in Z(A)$. In particular every k-linear combination of compositions of $\smile$ and Δ sends $\underline{f}$ to $z \cdot f$.*

Theorem 2.4.16. *Let A be symmetric k-algebra over a field of positive characteristic p and let $f : A \to A$ be a derivation. Then $f^{[p]}$ cannot always be expressed k-linear combination of compositions of the BV-operator and the cup product.*

Proof. For any fixed p, we need to find a derivation f such that $f^{[p]}$ cannot be expressed as $z \cdot f$ for some $z \in Z(A)$. We provide a family of examples with these properties. Consider the algebra $A = k[x, y]/ < x^p, y^p >$. Let $\{f_{a,b}\}, \{g_{c,d}\} \subseteq \mathrm{Der(A)}$ be such that $f_{a,b}(x) = x^a y^b$, $f_{a,b}(y) = 0$, $g_{c,d}(x) = 0$ and $g_{c,d}(y) = x^c y^d$ where $0 \leq a, b, c, d \leq p - 1$. Then a basis for $\mathrm{Der}(A)$ is given by $\{f_{a,b}\} \cup \{g_{c,d}\}$. Since A is commutative this implies $\mathrm{Inn}(A) = 0$. An element of the algebra A acts on $\{f_{i,j}\}$ and $\{g_{i,j}\}$ in the following way:

$$x^k y^l \cdot f_{i,j} = \begin{cases} f_{i+k,j+l} & \text{for } 0 \leq i+k, j+l \leq p-1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for $g_{i,j}$. We can also note that $f_{1,0} \circ g_{0,1} = g_{0,1} \circ f_{1,0} = 0$ and $f_{1,0}, g_{0,1}$ are p-idempotents. Let $f = \lambda f_{1,0} + \mu g_{0,1}$ for $\lambda, \mu \in k$. Since $\mathrm{Der}(A)$ is a restricted Lie algebra we have:

$$\begin{aligned} f^{[p]} = (\lambda f_{1,0} + \mu g_{0,1})^{[p]} &= \lambda^p f_{1,0} + \mu^p g_{0,1} + \sum_i s_i \\ &= \lambda^p f_{1,0} + \mu^p g_{0,1} \end{aligned} \tag{2.4.26}$$

where s_i are zero since they are consecutive compositions of $f_{1,0}$ and $g_{0,1}$. An element of the centre $z \in Z(A)$ can be written as $z = \sum_{i,j} \lambda_{ij} x^i y^j$. Consequently

$$z \cdot f = (\sum_{i,j} \lambda_{ij} x^i y^j)(\lambda f_{1,0} + \mu g_{0,1}) = \sum \lambda_{ij}(\lambda f_{i+1,j} + \mu g_{i,j+1}) \tag{2.4.27}$$

for $0 \leq i,j \leq p-1$. Since $\{f_{i,j}\}$, $\{g_{i,j}\}$ are linearly independent, in order to have $f^{[p]} = z \cdot f$, we should impose $\lambda_{0,0} = \lambda^{p-1}$ and $\lambda_{0,0} = \mu^{p-1}$. For a field k large enough we have a contradiction. $\square$

Chapter 3

Invariance and properties of r-integrable derivations

Sections $3.2, 3.3, 3, 4$ and 3.5 of this chapter are based on the paper [34].

3.1 Introduction

Since the pioneering work of Happel [15, 16], Rickard [31], [32], [33] and Keller [19], derived categories have played an increasing role in the representation theory of groups and algebras. Broué's abelian defect group conjecture [7] has been the starting point of a major development in block theory using derived equivalences as main tool. Broué [8] has introduced stable equivalences of Morita type, which are implied by derived equivalences between symmetric algebras (as a consequence of a theorem of Rickard [32]). This puts the focus on understanding invariants under derived and stable equivalences. Derived equivalences have been shown to preserve the Hochschild cohomology. However stable equivalences only preserve the positive part of the Hochschild cohomology. Keller [19] has shown that a derived equivalence preserves the Gerstenhaber algebra structure. Koenig et al. [20] have shown that a derived equivalence between symmetric algebras preserves the BV algebra structure using the transfer maps introduced in [24]. Apart from a paper of Zimmerman [40], little has been done with the Lie restricted p-power map. In addition, there are also evidences from calculations in examples in [6] that suggest there is a strong relation between the algebra structure of A and the restricted Lie algebra structure of $\mathrm{HH}^1(A)$. In this chapter we address this compatibility problem for a subfamily of $\mathrm{HH}^1(A)$ called integrable derivations.

Integrable derivations have been introduced by Gerstenhaber in [13] and then studied by Matsumura [28] and others. The main aim of this chapter is threefold. Following Farkas, Geiss and Marcos (who have proved in [11] that integrable derivations are invariant under Morita equivalences) we generalise the concept of integrable derivations, called r-integrable derivations. In a similar fashion to [26] we prove they are invariant under stable equivalences of Morita type. On the other hand, let k be a field of prime characteristic p, for symmetric k-algebras, Zimmermann proved in [40] that the p-power map on (the positive part of) Hochschild cohomology commutes with derived equivalences. We show that the p-power map, restricted to the classes of r-integrable derivations, commutes with stable equivalences of Morita type between finite-dimensional selfinjective algebras (See Theorem 3.5.1 and Theorem 3.5.2 below). Finally we provide some properties of r-integrable derivations and a family of examples when all derivations are integrable.

The chapter is divided into the following sections: in 3.2 we introduce the concept of r-integrable derivations and some of their properties. Section 3.3 explains how to endow the set of integrable derivations with a p-power map. Section 3.4 gives a cohomological interpretation of integrable derivation whilst in 3.5 we prove the two main results. In section

3.6 we study different properties of the p-power maps and integrable derivations. Then we prove that the family of the r-integrable derivations form a poset and the subfamily of p^a-integrable derivations, where p is a prime number and a a positive integer, form a vector space. We also establish the invariance of the Jacobson radical under r-integrable derivations. In the last section we provide a family of examples given by quantum complete intersection in which all the derivations are integrable.

3.2 Integrable derivations of degree r

We recall some background lemmas that will be helpful during the chapter. Let A be a finite-dimensional k-algebra over a field k. Then we denote by $A[[t]]$ the formal power series with coefficients in A and by $Z(A[[t]])$ the centre of $A[[t]]$. In the following, all the tensor products are over a field k unless otherwise specified.

Lemma 3.2.1. *Let A be a finite-dimensional algebra over k. Then the multiplication in $A[[t]]$ induces a $k[[t]]$-algebra isomorphism $k[[t]] \otimes_k A \cong A[[t]]$.*

Proof. The given map sends $\sum_{i\geq 0} \lambda_i t^i \otimes a$ to $\sum_{i\geq 0} \lambda_i a t^i$ where $\lambda_i \in k$ and $a \in A$. In order to show that this is an isomorphism, we construct its inverse as follows: let $\sum_{i\geq 0} a_i t^i \in A[[t]]$ and let $\{e_j\}_{1\leq j\leq n}$ be a k-basis of A. Write $a_i = \sum_{j=1}^{n} \mu_{ij} e_j$ for every non-negative integer i where $\mu_{ij} \in k$. The inverse map sends $\sum_{i\geq 0} a_i t^i$ to $\sum_{j=1}^{n} \Big(\sum_{i\geq 0} \mu_{ij} t^i \otimes e_j \Big)$. □

Lemma 3.2.2. *Let A be a finite-dimensional algebra over k and let r be a positive integer. Then the canonical map $Z(A[[t]]) \to Z(A[[t]]/t^r A[[t]])$ is surjective.*

Proposition 3.2.3. *(cf. [26, 2.1]) Let $A[[t]]$ be the formal power series with coefficients in A. Then the canonical map $A[[t]] \to A[[t]]/t^r A[[t]]$ induces an isomorphism*

$$\mathrm{HH}^n(A[[t]]; A[[t]]/t^r A[[t]]) \cong \mathrm{HH}^n(A[[t]]/t^r A[[t]])$$

for all $n \geq 0$ and $r > 0$.

We denote by $\mathrm{Aut}(A[[t]])$ the group of $k[[t]]$-algebra automomorphisms of $A[[t]]$ and by $\mathrm{Out}(A[[t]])$ the group of outer $k[[t]]$-algebra automorphisms of $A[[t]]$. We introduce now a subgroup of $\mathrm{Aut}(A[[t]])$ related to the notion of r-integrable derivations.

For a fixed positive integer r, we denote by $\mathrm{Aut_r}(A[[t]])$ the group of all $k[[t]]$-algebra automorphisms of $A[[t]]$ which induce the identity on $A[[t]]/t^rA[[t]]$. Clearly we have an inclusion $\mathrm{Aut}_r(A[[t]]) \subseteq \mathrm{Aut}_1(A[[t]])$ for every $r \geq 1$. We denote by $\mathrm{Out_r}(A[[t]])$ the image of the canonical map $\varphi : \mathrm{Aut_r}(A[[t]]) \to \mathrm{Out}(A[[t]])$.

Lemma 3.2.4. *Let A be a finite-dimensional k-algebra. Let r be a positive integer. Then* $\mathrm{Out_r}(A[[t]])$ *is the kernel of the canonical group homomorphism*

$$\psi : \mathrm{Out}(A[[t]]) \to \mathrm{Out}(A[[t]]/(t^rA[[t]])). \tag{3.2.1}$$

Proof. Clearly $\mathrm{Out}_r(A[[t]]) \subseteq \mathrm{Ker}(\psi)$. Let α be a representative of an element in the kernel of ψ. Then $\psi(\alpha)$ is given by conjugation with an invertible element $\bar{u} = u + t^rA[[t]]$ in $A[[t]]/t^rA[[t]]$ where $u \in A[[t]]$. We denote by $\overline{A[[t]]} = A[[t]]/t^rA[[t]]$. Now, $\bar{u}$ is invertible in $\overline{A[[t]]}$ and so $\overline{A[[t]]} = \overline{A[[t]]}\bar{u}$. We lift u to $A[[t]] = A[[t]]u + t^rA[[t]] \subseteq A[[t]]u + J(A[[t]])A[[t]]$. By Nakayama's Lemma we have $A[[t]] = A[[t]]u$ hence u is invertible. Consequently if we replace α by $u^{-1}\alpha u$, then the resulting automorphism is in the same class as α and it induces the identity on $A[[t]]/t^rA[[t]]$. □

We recall a definition from [28] which is connected with $\mathrm{Aut}_1(A[[t]])$:

Definition 3.2.5. (cf. [28, 1.1]) Let A be a finite-dimensional k-algebra. A *higher derivation* $\underline{D}$ of A is a sequence $\underline{D} = (D_i)_{i \geq 0}$ of k-linear endomorphisms $D_i : A \to A$ such that $D_0 = \mathrm{Id}$ and $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$ for all $n \geq 1$ and all $a, b \in A$.

We recall from [28, 1.5]:

Proposition 3.2.6. *Let A be a finite-dimensional k-algebra. The set of higher derivation of A is a group, called the Hasse-Schmidt group of A and denoted by* $\mathrm{HS}(A)$, *with the product defined by*

$$\underline{D} \circ \underline{D'} = \Big(\sum_{i=0}^{n} D_i \circ D'_{n-i}\Big)_{n\geq 0} \tag{3.2.2}$$

for any two higher derivations $\underline{D} = (D_i)_{i\geq 0}$ and $\underline{D'} = (D'_i)_{i\geq 0}$.

From [28] we have the following relation between $\mathrm{HS}(A)$ and $\mathrm{Aut}_1(A[[t]])$:

Proposition 3.2.7. *Let A be a finite-dimensional k-algebra. The Hasse-Schmidt group of A is isomorphic to* $\mathrm{Aut}_1(A[[t]])$. *The isomorphism sends a higher derivation $\underline{D} = (D_i)_{i\geq 0}$ to $\alpha \in \mathrm{Aut}_1(A[[t]])$ defined as $\alpha(a) = \sum_{i\geq 0} D_i(a)t^i$ for all $a \in A$.*

Proof. Note that any $k[[t]]$-ring endomorphism of $A[[t]]$ is determined by its restriction to A since it can be extended linearly. Let $\underline{D} = (D_i)_{i\geq 0}$ be an higher derivation. Then we can construct $\alpha \in \mathrm{Aut}_1(A[[t]])$ as follows: $\alpha(a) = \sum_{i\geq 0} D_i(a)t^i$ for all $a \in A$. It is easy to check that $\alpha \in \mathrm{Aut}_1(A[[t]])$ using Proposition 3.2.6. Conversely, let $\alpha \in \mathrm{Aut}_1(A)$ and let $d_i : A \to A$ such that $\alpha(a) = \sum_{i\geq 0} d_i(a)t^i$ for all $a \in A$. Using the fact that α is an automorphism which induce the identity on A, we have that the d_i satisfy 3.2.2 for every i. Hence $(d_i)_{i\geq 0}$ is a higher derivation. □

Definition 3.2.8. Let A be a finite-dimensional k-algebra. A higher derivation $\underline{D} = (D_i)_{i\geq 0}$ is of degree r if $D_0 = \mathrm{Id}$, $D_i = 0$ for $1 \leq i \leq r-1$ and $D_r \neq 0$. The set of higher derivations of degree r is denoted by HS_r (A).

We have the following:

Corollary 3.2.9. *Let A be a finite-dimensional k-algebra. The set* $\mathrm{HS}_r(A)$ *is a subgroup of* $\mathrm{HS}(A)$.

Proof. Let us consider $\mathrm{Aut_r}(A[[t]])$. Then the image of $\mathrm{Aut_r}(A[[t]])$ under the isomorphism in Proposition 3.2.7 is $\mathrm{HS}_r(A)$. $\square$

Remark 3.2.10. Let $(D_i)_{i\geq 0}$ be a higher derivation of degree r. Since $D_0 = \mathrm{Id}$, $D_i = 0$ for $1 \leq i \leq r-1$ then by definition of higher derivation we have that D_i are derivations for $r \leq i < 2r$. In fact for $r \leq i < 2r$:

$$D_i(ab) = \sum_{j+k=i} D_j(a)D_k(b) = aD_i(b) + D_i(a)b.$$

In particular D_r is a derivation.

Consequently, we can slightly extend Matsumura [28] in order to introduce the following terminology:

Definition 3.2.11. Let A be a finite-dimensional k-algebra and let r be a positive integer. A derivation $D \in \mathrm{Der}(A)$ is called *r-integrable* if there exists a higher derivation of degree r, say $\underline{D} = (D_i)_{i\geq 0}$, such that $D = D_r$. We denote by $\mathrm{Der}_r(A)$ the set of r-integrable derivations of A.

Note that for $r = 1$ we have the notion of integrable derivation defined in [13]; that is, integrable derivations are 1-integrable.

Remark 3.2.12. The notion of integrable derivation can be defined over more general rings (see e.g [26]) but we only work over a field k since the main results mostly require a field as a base ring.

Now using Proposition 3.2.6 we have that:

Corollary 3.2.13. *Let A be a finite-dimensional k-algebra and let r be a positive integer. The set* $\mathrm{Der}_r(A)$ *is a subgroup of the additive abelian group* $\mathrm{Der}(A)$.

Proof. If we let $D, D' \in \mathrm{Der}_r(A)$ with $\underline{D}, \underline{D'}$ the corresponding higher derivations, then the rth term of $\underline{D} \circ \underline{D'}$ is $D + D'$. Thus the collection of r-integrable derivations is closed under addition. Using Proposition 3.2.6, the rth term of the inverse of $\underline{D}$, say $\underline{D}^{-1}$, is $-D_r$. It is easy to check that the unit in the group of higher derivation is given by $D_i = 0$ for $i > 0$ and $D_0 = 1$. In particular $D_r = 0$. $\square$

It is helpful to have an equivalent characterisation of $\mathrm{Der}_r(A)$ in terms of $\mathrm{Aut}_r(A[[t]])$. Using Proposition 3.2.7 we can write $\alpha \in \mathrm{Aut}_r(A[[t]])$ as follows: let $a = \sum_{i=0}^{\infty} a_i t^i$. Then

$$\alpha(a) = \sum_{i,n \geq 0} D_n(a_i) t^{i+n} = a + t^r \sum_{k \geq r} \sum_{\substack{n \geq 1, \\ i \geq 0 \\ i+n=k}}^{k} D_n(a_i) t^{i+n-r}$$

since $D_i = 0$ for $1 \leq i \leq r-1$. Hence we can write α as $\alpha(a) = a + t^r \mu(a)$ where μ is a $k[[t]]$-linear endomorphism of $A[[t]]$. We need now the following:

Proposition 3.2.14. *Let A be a finite-dimensional k-algebra. Let r be a positive integer, let $\alpha \in \mathrm{Aut_r}(A[[t]])$. Let $\mu : A[[t]] \to A[[t]]$ be the unique $k[[t]]$-linear map such that $\alpha(a) = a + t^r \mu(a)$ for all $a \in A[[t]]$. Then the map $\bar{\mu} : A \cong A[[t]]/tA[[t]] \to A \cong A[[t]]/tA[[t]]$ induced by μ is a derivation.*

Proof. Let $a, b \in A[[t]]$, since α is an automorphism we have $\alpha(ab) = ab + t^r \mu(ab)$ is equal to $\alpha(a)\alpha(b) = ab + t^r \mu(a) + t^r \mu(b) + t^{2r} \mu(a)\mu(b)$ hence we obtain $\mu(ab) = a\mu(b) + \mu(a)b + t^r \mu(a)\mu(b)$. Reducing modulo t we have

$$\bar{\mu}(ab) = a\bar{\mu}(b) + \bar{\mu}(a)b \tag{3.2.3}$$

hence $\bar{\mu}$ is a derivation on A. $\square$

By Proposition 3.2.14 the map $\bar{\mu} : A \to A$ induced by μ is a derivation over A, in fact, $\bar{\mu}$ is exactly D_r. We can now introduce an equivalent definition of r-integrable derivation:

Remark 3.2.15. Let A be a finite-dimensional k-algebra. A derivation D is r-integrable if there is a $k[[t]]$-algebra automorphism of $A[[t]]$, say α, and a $k[[t]]$-linear endomorphism μ of $A[[t]]$ such that $\alpha(a) = a + t^r\mu(a)$ for all $a \in A[[t]]$ and such that D is equal to the map $\bar{\mu}$ induced by μ on $A \cong A[[t]]/tA[[t]]$.

Definition 3.2.16. Let A be a finite-dimensional k-algebra. Let r be a positive integer. We denote by $\mathrm{HH}^1_r(A)$ the image of $\mathrm{Der}_r(A)$ in $\mathrm{HH}^1(A)$.

We can now prove some results regarding r-integrable derivations:

Proposition 3.2.17. *Let A be a finite-dimensional k-algebra. Let r be a positive integer, let $\alpha \in \mathrm{Aut_r}(A[[t]])$. Let $\mu : A[[t]] \to A[[t]]$ be the unique $k[[t]]$-linear map such that $\alpha(a) = a + t^r\mu(a)$ for all $a \in A[[t]]$. Let $\bar{\mu}$ the map induced by μ on A. The following hold:*

(*a*) *Let α be an inner automorphism. Then $\bar{\mu} = [\bar{d}, -]$ for some $\bar{d} \in A$; that is $\bar{\mu}$ is a inner derivation.*

(*b*) *The class of $\bar{\mu} \in \mathrm{HH}^1_r(A)$ depends only on the class of $\alpha \in \mathrm{Out}_r(A[[t]])$.*

Proof. Let α be an inner automorphism induced by conjugation by an element $c \in (A[[t]])^\times$; that is $\alpha(a) = cac^{-1}$ for $a \in A[[t]]$. Since α induces the identity on $A[[t]]/t^rA[[t]]$ then taking the projection of α in $A[[t]]/t^rA[[t]]$ we have $\bar{c}\bar{a}\bar{c}^{-1} = \bar{a}$, that is $\bar{c}\bar{a} = \bar{a}\bar{c}$ hence $\bar{c} \in Z(A[[t]]/t^rA[[t]])^\times$. Since the map $Z(A[[t]]) \to Z(A[[t]]/t^rA[[t]])$ is surjective (Remark 3.2.1), there is an element $z \in Z(A[[t]])^\times$ such that $\bar{z} = \bar{c}$ hence such that $cz^{-1} \in 1+t^rA[[t]]$. Therefore if we replace c by cz^{-1} we have $c = 1+t^rd$ for some $d \in A[[t]]$. If we take $a \in A[[t]]$ we have $cac^{-1} = \alpha(a) = a + t^r\mu(a)$ and hence $ca = ac + t^r\mu(a)c$, that is $[c, a] = t^r\mu(a)c$. Now if we replace c by $1 + t^rd$ and we divide by t^r we obtain

$$[d, a] = \mu(a) + t^r\mu(a)d. \tag{3.2.4}$$

Consequently $[\bar{d}, \bar{a}] = \bar{\mu}(\bar{a})$, whence statement a).

For the second part we let α_1, α_2 be two representatives of the same class in $\mathrm{Out}_r(A[[t]])$ with induced derivations μ_1, μ_2. Since α_1, α_2 are in the same class, then $\alpha_1 \circ \alpha_2^{-1}$ is an inner automorphism of $A[[t]]$. By Proposition 3.2.6 the induced derivation of $\alpha_1 \circ \alpha_2^{-1}$ is $\mu_1 - \mu_2$. Then by statement a) we have that $\mu_1 - \mu_2$ is an inner derivation. Hence the result. $\square$

The following proposition is useful for Proposition 3.2.19 and for the next section:

Proposition 3.2.18 ([34, Theorem 3.6]). *Let A be a finite-dimensional k-algebra and let $\alpha \in \mathrm{Aut}_1(A[[t]])$. Let $(D_i)_{i\geq 0}$ be the higher derivation satisfying $\alpha(a) = \sum_{i\geq 0} D_i(a)t^i$ for $a \in A$. The map that sends α to $\sum_{i\geq 0} D_i t^i$ induces a group homomorphism ϕ : $\mathrm{Aut}_1(A[[t]]) \to (\mathrm{End}_k(A)[[t]])^{\times}$.*

Proof. Let $\beta \in \mathrm{Aut}_1(A[[t]])$. For $l \geq 0$ let $E_l \in \mathrm{End}_k(A)$ such that $\beta(a) = \sum_{l\geq 0} E_l(a)t^l$. For all $a \in A$ let $\{e_j\}_{1\leq j\leq n}$ be a k-basis of A. For every $i \geq 0$ define $\mu_{ij} : A \to k$ such that $D_i(a) = \sum_{j=1}^{n} \mu_{ij}(a)e_j$ where $a \in A$. On one side we have:

$$\begin{aligned}(\beta \circ \alpha)(a) &= \beta\Big(\sum_{i\geq 0} D_i(a)t^i\Big) = \sum_{j=1}^{n} \beta\Big(\sum_{i\geq 0} \mu_{ij}(a)t^i e_j\Big) \\ &= \sum_{j=1}^{n}\sum_{i\geq 0} \mu_{ij}(a)t^i\beta(e_j) = \sum_{l\geq 0}\sum_{i\geq 0}\sum_{j=1}^{n} \mu_{ij}(a)E_l(e_j)t^{i+l}\end{aligned} \tag{3.2.5}$$

where the third equation holds since β is an automorphism over $k[[t]]$. If we fix a degree $m \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{\substack{l,i \\ i+l=m}} \sum_{j=1}^{n} \mu_{ij}(a)E_l(e_j)t^{i+l} &= \sum_{\substack{l,i \\ i+l=m}} E_l(\sum_{j=1}^{n} \mu_{ij}(a)e_j)t^m \\ &= \sum_{\substack{l,i \\ i+l=m}} E_l(D_i(a))t^m\end{aligned} \tag{3.2.6}$$

Hence $\phi(\beta \circ \alpha)$ in degree m is equal to $\sum_{\substack{i,l \geq 0 \\ i+l=m}} E_l \circ D_i t^m$. This is clearly equal to the coefficient at t^m of $\phi(\beta)\phi(\alpha)$. $\square$

Proposition 3.2.19. *Let A be a finite-dimensional k-algebra. Let r be a positive integer and let $\alpha \in \mathrm{Aut_r}(A[[t]])$. Let μ be the unique $k[[t]]$-linear map on $A[[t]]$ such that $\alpha(a) = a + t^r\mu(a)$ for all $a \in A[[t]]$. We denote by $\bar{\mu}$ the derivation induced on A by μ as in Remark* 3.2.15.

(*a*) *The derivation $\bar{\mu}$ is inner if and only if α induces an inner automorphism in $A[[t]]/t^{r+1}A[[t]]$.*

(*b*) *We have the following short exact sequence of groups:*

$$1 \longrightarrow \mathrm{Out_{r+1}}(A[[t]]) \longrightarrow \mathrm{Out_r}(A[[t]]) \xrightarrow{\psi} \mathrm{HH}^1_r(A) \longrightarrow 1.$$

where ψ is defined as in Proposition 3.2.17

Proof. Let us assume that $\bar{\mu}$ is an inner derivation so $\bar{\mu} = [\bar{d}, -]$ for some $d \in A[[t]]$. Take $c = 1 + t^r d$ as in the proof of Proposition 3.2.17. By Equation 3.2.4 we can choose $\tau(a) = -\mu(a)d$ so that we have $[d, a] = \mu(a) - t^r\tau(a)$. Since $c = 1 + t^r d$ we get

$$[c, a] = [1 + t^r d, a] = t^r[d, a]. \tag{3.2.7}$$

So $[c, a] = t^r[d, a] = t^r\mu(a) - t^{2r}\tau(a)$. Hence $t^r\mu(a) = [c, a] + t^{2r}\tau(a)$. Consequently $cac^{-1} = a + t^r\mu(a)c^{-1} - t^{2r}\tau(a)c^{-1}$. Now, $\alpha(a) = a + t^r\mu(a)$ implies that $\alpha(a) - cac^{-1} = t^r\mu(a)(1 - c^{-1}) + t^{2r}\tau(a)c^{-1}$. Since c belongs to $1 + t^r A[[t]]$, we have $c^{-1} \in 1 + t^r A[[t]]$ and hence $1 - c^{-1} \in t^r A[[t]]$. This shows that $\alpha(a) - cac^{-1} \in t^{2r}A[[t]] \subset t^{r+1}A[[t]]$. Consequently α induces an inner automorphism on $A[[t]]/t^{r+1}A[[t]]$.

Conversely, suppose that α acts as an inner automorphism on $A[[t]]/t^{r+1}A[[t]]$. Using the same argument as in Lemma 3.2.4 we may assume that α acts as identity on $A[[t]]/t^{r+1}A[[t]]$ and hence it induces an inner derivation on $A[[t]]/t^{r+1}A[[t]]$. Hence we can assume $\alpha \in \mathrm{Aut}_{r+1}(A[[t]])$. Hence $\alpha(a) = a + t^{r+1}\mu'(a)$ for some $\mu'(a) \in A[[t]]$, which gives the equality $\mu(a) = t\mu'(a)$. Consequently we have that μ induces the zero map on A.

For the second part, we let $\beta \in \mathrm{Aut_r}(A)$. Let ν be the unique $k[[t]]$-linear map on $A[[t]]$ such that $\beta(a) = a + t^r\nu(a)$ for all $a \in A[[t]]$. For $i, l \geq 0$ let $D_i, E_l \in \mathrm{End}_k(A)$ be such that $\alpha(a) = \sum_{i\geq 0} D_i(a)t^i$ and $\beta(a) = \sum_{l\geq 0} E_l(a)t^l$ for all $a \in A[[t]]$. By Proposition 3.2.18, the coefficient of the term of degree r of $\beta\circ\alpha$ is $\sum_{\substack{l,i \\ i+l=r}} E_l(D_i(a)) = E_r(a)+D_r(a) = \bar{\mu}(a)+\bar{\nu}(a)$ since $E_i = D_j = 0$ for $1 \leq i, j \leq r-1$. Hence by Proposition 3.2.17 we have that the class determined by $\beta \circ \alpha$ in $\mathrm{HH}^1_\mathrm{r}(A)$ is the class determined by $\bar{\mu} + \bar{\nu}$. The exactness follows from part a). □

3.3 The p-power map on r-integrable derivations

In this section we denote by k be a field of positive characteristic p. A way to understand the action of the p-power map on the integrable derivations is by studying it on $\mathrm{Aut}_1(A[[t]])$ and then using the homomorphism in Proposition 3.2.18, $\phi : \mathrm{Aut}_1(A[[t]]) \to (\mathrm{End_k}(A)[[t]])^\times$. In the following proposition the expression $\prod_{j=1}^{c} D_{i_j}$ is to be understood as the composition $D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_c}$. Note that this product is not commutative.

Proposition 3.3.1. *Let $\underline{D}$ be a higher derivation and let l, n be positive integers. The coefficient of the monomial t^l in $\left(\sum_{i\geq 0} D_i t^i\right)^n$ is equal to*

$$\sum_{c=1}^{l} \binom{n}{c} \sum_{\substack{i_1,\ldots i_c \geq 1 \\ i_1+\cdots+i_c=l}} \prod_{j=1}^{c} D_{i_j} \tag{3.3.1}$$

Proof. The term at t^l in $\left(\sum_{i\geq 0} D_i t^i\right)^n$ is given by

$$\sum_{\substack{i_1,\ldots,i_n \geq 0 \\ i_1+\cdots+i_n=l}} \prod_{j=1}^{n} D_{i_j}. \tag{3.3.2}$$

If $i_j = 0$, then $D_{i_j} = \mathrm{Id}$, so we regroup the sum $\sum_{\substack{i_1,\dots,i_n \geq 0 \\ i_1+\cdots+i_n=l}}$ over the n-tuples in terms of the number c of indices i_j which are strictly positive. Then for each c-tuple $(i'_1, i'_2, \dots, i'_c)$ which has non-zero components and such that $\sum_{j=1}^{c} i'_j = l$, there are $\binom{n}{c}$ different n-tuples $(i_1, i_2, \dots, i_n)$ which have the c non-zero components of the c-tuple $(i'_1, i'_2, \dots, i'_c)$ and rest equal to zero. Since $D_0 = \mathrm{Id}$, this implies $\prod_{j=1}^{n} D_{i_j} = \prod_{j=1}^{c} D_{i'_j}$. For a fixed c the Equation (3.3.2) is given by $\binom{n}{c} \sum_{\substack{i_1,\dots i_c \geq 1 \\ i_1+\cdots+i_c=l}} \prod_{j=1}^{c} D_{i_j}$. If we sum over all c we have the result. □

Proposition 3.3.2. *Let A be a finite-dimensional k-algebra and let $\alpha \in \mathrm{Aut_r}(A[[t]])$ for some positive integer r. Then $\alpha^p \in \mathrm{Aut}_{rp}(A[[t]])$. The p-power map sends $\mathrm{HH}^1_r(A)$ to $\mathrm{HH}^1_{rp}(A)$, and $\mathrm{Out}_r(A[[t]])$ to $\mathrm{Out}_{rp}(A[[t]])$ and we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Out}_r(A[[t]]) \arrow[r, "(\ )^p"] \arrow[d] & \mathrm{Out}_{rp}(A[[t]]) \arrow[d] \\
\mathrm{HH}^1_r(A) \arrow[r, "{[p]}"] & \mathrm{HH}^1_{rp}(A)
\end{tikzcd}$$

where the vertical maps are from Proposition 3.2.19 (b), $(\ )^p$ *is the p-fold composition and* $[p]$ *is the p-power map.*

Proof. Let $\alpha \in \mathrm{Aut_r}(A[[t]])$ and let D_r the derivation in $\mathrm{Der}_r(A)$. Let $\underline{D'}$ be the higher derivation associated to α^p. Using Proposition 3.3.1, in degree $l \leq p-1$ we have:

$$\sum_{c=1}^{l} \binom{p}{c} \sum_{\substack{i_1,\dots i_c \geq 1 \\ i_1+\cdots+i_c=l}} \prod_{j=1}^{c} D_{i_j} t^l = 0 \tag{3.3.3}$$

since the binomial coefficient are multiples of p. For $l \geq p$

$$\sum_{c=1}^{l} \binom{p}{c} \sum_{\substack{i_1,\dots i_c \geq 1 \\ i_1+\cdots+i_c=l}} \prod_{j=1}^{c} D_{i_j} t^l = \sum_{\substack{i_1,\dots i_p \geq 1 \\ i_1+\cdots+i_p=l}} \prod_{j=1}^{c} D_{i_j} t^l \tag{3.3.4}$$

Now we know that each D_i is zero for $i = 1, \ldots, r-1$, so in order to have an element different from zero we should impose that each i_j be at least r. Therefore $i_1 + \cdots + i_p = rp$, that is $l = rp$, hence the first non-zero coefficient is D_r^p. Consequently the diagram commutes. $\square$

3.4 A cohomological interpretation of r-integrable derivations

In this section we let k be an arbitrary field. Integrable derivation can also be interpreted using a cohomological point of view (See [26], [34]). We recall a standard theorem from cohomology which is fundamental for the following sections:

Proposition 3.4.1. *Let*

$$0 \longrightarrow X \xrightarrow{\tau} Y \xrightarrow{\sigma} Z \longrightarrow 0$$

be a short exact sequence of cochain complexes of modules over some k-algebra A with differentials δ, ϵ, ζ respectively. This induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n(X) \xrightarrow{\mathrm{H}^n(\tau)} \mathrm{H}^n(Y) \xrightarrow{\mathrm{H}^n(\sigma)} \mathrm{H}^n(Z) \xrightarrow{d^n} \mathrm{H}^{n+1}(X) \longrightarrow \cdots$$

depending functorially on the short exact sequence, where d^n is called the connecting homomorphism. The functoriality dependence means that given a commutative diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\tau"] \arrow[d, "f"] & Y \arrow[r, "\sigma"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\tau'"] & Y' \arrow[r, "\sigma'"] & Z' \arrow[r] & 0
\end{tikzcd}$$

we get a commutative ladder of long exact sequences

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^n(X) \arrow[r, "\mathrm{H}^n(\tau)"] \arrow[d, "\mathrm{H}^n(f)"] & \mathrm{H}^n(Y) \arrow[r, "\mathrm{H}^n(\sigma)"] \arrow[d, "\mathrm{H}^n(g)"] & \mathrm{H}^n(Z) \arrow[r, "d^n"] \arrow[d, "\mathrm{H}^n(h)"] & \mathrm{H}^{n+1}(X) \arrow[r] \arrow[d, "\mathrm{H}^{n+1}(f)"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^n(X') \arrow[r, "\mathrm{H}^n(\tau')"] & \mathrm{H}^n(Y') \arrow[r, "\mathrm{H}^n(\sigma')"] & \mathrm{H}^n(Z') \arrow[r, "d'^n"] & \mathrm{H}^{n+1}(X') \arrow[r] & \cdots
\end{tikzcd}
$$

Proof. We only prove how d^n is obtained. Let $\bar{z} = z + \mathrm{Im}(\zeta^{n-1}) \in \mathrm{H}^n(Z)$ for some $z \in \mathrm{Ker}(\zeta^n) \subseteq Z^n$. Since σ is surjective in each degree, there exists $y \in Y^n$ such that $\sigma^n(y) = z$. Then $\epsilon(y) \in Y^{n+1}$ satisfies

$$\sigma^{n+1}(\epsilon^n(y)) = \zeta^n(\sigma^n(y)) = \zeta^n(z) = 0 \tag{3.4.1}$$

Hence $\epsilon(y) \in \ker(\sigma^{n+1}) = \mathrm{Im}(\tau^{n+1})$. Thus the is an $x \in X^{n+1}$ such that $\tau^{n+1}(x) = \epsilon^n(y)$. In addition we have that

$$\tau^{n+2}(\delta^{n+1}(x)) = \epsilon^{n+2}(\tau^{n+1}(x)) = \epsilon^{n+1}(\epsilon^n(y)) = 0.$$

Since τ^{n+2} is a monomorphism, this shows that $\delta^{n+1}(x) = 0$. Consequently $x + \mathrm{Im}(\delta^n)$ is an element in $\mathrm{H}^{n+1}(X)$. One can verify that if $z \in \mathrm{Im}(\zeta^{n+1})$ then $x \in \mathrm{Im}(\delta^n)$. This implies that there is a well-defined map $d^n : \mathrm{H}^n(Z) \to \mathrm{H}^{n+11}(X)$ sending $z + \mathrm{Im}(\zeta^n)$ to $x + \mathrm{Im}(\delta^n)$. $\square$

Let $\alpha \in \mathrm{Aut}(A[[t]])$ and let U be a right, equivalently left, $A[[t]]$-module. We denote by U_α (or ${}_\alpha U$) the $A[[t]]$-module which is equal to U as a $k[[t]]$-module, with $a \in A[[t]]$ acting on the right, equivalently on left, as $\alpha(a)$ on U.

Let us consider the short exact sequence of $A[[t]]$-$A[[t]]$-bimodules:

$$0 \longrightarrow A[[t]]/tA[[t]] \xrightarrow{t^r} A[[t]]/t^{r+1}A[[t]] \longrightarrow A[[t]]/t^rA[[t]] \longrightarrow 0.$$

Twisting the exact sequence on the right by the automorphism $\alpha \in \mathrm{Aut}_r(A[[t]])$ does not affect the $A[[t]]$-$A[[t]]$-bimodules $A[[t]]/tA[[t]]$ and $A[[t]]/t^rA[[t]]$, since α induces the identity on $A[[t]]/t^r A[[t]]$, but only $A[[t]]/t^{r+1}A[[t]]$. The resulting sequence is also exact hence we have the following short exact sequence:

$$0 \longrightarrow A[[t]]/tA[[t]] \xrightarrow{t^r} (A[[t]]/t^{r+1}A[[t]])_\alpha \longrightarrow A[[t]]/t^r A[[t]] \longrightarrow 0.$$

Proposition 3.4.2. *Let A be a finite-dimensional algebra over k. Set $\hat{A} = A[[t]]$ and set $\hat{A}^e = \hat{A} \otimes_{k[[t]]} \hat{A}^{op}$. Let $\alpha \in \mathrm{Aut_r}(\hat{A})$. Let r a positive integer and let $\mu : \hat{A} \to \hat{A}$ be the unique $k[[t]]$-linear map satisfying $\alpha(a) = a + t^r \mu(a)$. Let P be a projective resolution of $\hat{A}$ as $\hat{A}^e$-module. Applying the functor* $\mathrm{Hom}_{\hat{A}^e}(P, -)$ *to the exact sequence of $\hat{A}^e$-modules*

$$0 \longrightarrow \hat{A}/t\hat{A} \xrightarrow{t^r} (\hat{A}/t^{r+1}\hat{A})_\alpha \longrightarrow \hat{A}/t^r\hat{A} \longrightarrow 0$$

yields a short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_{\hat{A}^e}(P, A) \xrightarrow{t^r} \mathrm{Hom}_{\hat{A}^e}(P, (\hat{A}/t^{r+1}\hat{A})_\alpha) \longrightarrow \mathrm{Hom}_{\hat{A}^e}(P, \hat{A}/t^r\hat{A}) \longrightarrow 0$$

The first non trivial connecting homomorphism can be identified with a map

$$\mathrm{End}_{\hat{A}^e}(\hat{A}/t^r\hat{A}) \to \mathrm{HH}^1(A) \tag{3.4.2}$$

and this map sends $\mathrm{Id}_{\hat{A}/t^r\hat{A}}$ *to the class of the derivation induced by μ on A.*

Proof. We take as a projective resolution the bar resolution P of $\hat{A}$ where the tensor products are over $k[[t]]$:

$$\dots \longrightarrow \hat{A}^{\otimes n+2} \xrightarrow{\delta_n} \hat{A}^{\otimes n+1} \longrightarrow \dots$$

which is given by $\delta_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$. The last non-zero differential is the map $\delta_1 : \hat{A}^{\otimes 3} \to \hat{A}^{\otimes 2}$ which sends $a \otimes b \otimes c$ to $ab \otimes c - a \otimes bc$ for $a, b, c \in \hat{A}$. We have the following identifications:

$$\begin{aligned} \mathrm{H}^0(\mathrm{Hom}_{\hat{A}^e}(P, \hat{A}/t^r\hat{A})) &= \mathrm{HH}^0(\hat{A}, \hat{A}/t^r\hat{A}) \\ &\cong \mathrm{HH}^0(\hat{A}/t^r\hat{A}) = \mathrm{End}_{\hat{A}^e}(\hat{A}/t^r\hat{A}) \end{aligned} \tag{3.4.3}$$

The identity map in $\mathrm{End}_{\hat{A}^e}(\hat{A}/t^r\hat{A})$ corresponds to the homomorphism

$$\begin{aligned} \zeta : \hat{A} \otimes_{k[[t]]} \hat{A} &\to \hat{A}/t^r\hat{A} \\ a \otimes b &\mapsto \zeta(a \otimes b) = ab + t^r\hat{A} \end{aligned} \tag{3.4.4}$$

for all $a, b \in A[[t]]$. This lifts to an $\hat{A}^e$-homomorphism

$$\begin{aligned} \bar{\zeta} : \hat{A} \otimes_{k[[t]]} \hat{A} &\to (\hat{A}/t^{r+1}\hat{A})_\alpha \\ a \otimes b &\mapsto \bar{\zeta}(a \otimes b) = a\alpha(b) + t^{r+1}\hat{A} \end{aligned} \tag{3.4.5}$$

for $a, b \in \hat{A}$ since α induces the identity on $\hat{A}/t^r\hat{A}$.

Following the proof of Proposition 3.4.1, since $\bar{\zeta} \in \mathrm{Hom}_{\hat{A}^e}(\hat{A} \otimes \hat{A}, (\hat{A}/t^{r+1}\hat{A})_\alpha)$, we need to apply the first non-zero differential

$$\epsilon : \mathrm{Hom}_{\hat{A}^e}(\hat{A}^{\otimes 2}, (\hat{A}/t^{r+1}\hat{A})_\alpha) \to \mathrm{Hom}_{\hat{A}^e}(\hat{A}^{\otimes 3}, (\hat{A}/t^{r+1}\hat{A})_\alpha) \tag{3.4.6}$$

which is given by composing with $-\delta_1$. Hence in $\hat{A}/t^{r+1}\hat{A}$ we have:

$$\begin{aligned} (-\bar{\zeta} \circ \delta_1)(a \otimes b \otimes c) &= -\bar{\zeta}(ab \otimes c - a \otimes bc) = -ab\alpha(c) + a\alpha(bc) \\ &= a(\alpha(b) - b)\alpha(c) = t^r a\mu(b)\alpha(c). \end{aligned} \tag{3.4.7}$$

for all $a, b, c \in \hat{A}$. In order to construct the first non trivial connection homomorphism, we observe that $t^r a\mu(b)\alpha(c) + t^{r+1}\hat{A} \in \hat{A}/t^{r+1}\hat{A}$ is the image, under $t^r : \hat{A}/t\hat{A} \to (\hat{A}/t^{r+1}\hat{A})_\alpha$, of the map $\psi : \hat{A}^{\otimes 3} \to \hat{A}/t\hat{A}$, that is we have the following commutative diagram:

$$\begin{tikzcd} & \hat{A}^{\otimes 3} \arrow[dl, "\psi"'] \arrow[d, "-\zeta\circ\delta"] \\ \hat{A}/t\hat{A} \arrow[r, "t^r"] & (\hat{A}/t^{r+1}\hat{A})_\alpha \end{tikzcd}$$

where ψ sends $a \otimes b \otimes c$ to $a\mu(b)\alpha(c) + t\hat{A}$ which is equal to $a\mu(b)c + t\hat{A}$ since $\alpha(c) - c \in t^r\hat{A} \subseteq t\hat{A}$. Consequently ψ induces a map $\bar{\psi} : \hat{A}^{\otimes 3} \to A$ which sends $a \otimes b \otimes c$ to $\bar{a}\bar{\mu}(\bar{b})\bar{c}$ that can be restricted to the map $\bar{\psi} : A \to A$ that sends $\bar{b}$ to $\mu(\bar{b})$. Using Proposition 3.2.3 the result follows.

$\square$

Remark 3.4.3. Let $\mathcal{O}$ be is a complete discrete valuation ring with maximal ideal $J(\mathcal{O}) = \pi\mathcal{O}$ for some nonzero element $\pi \in \mathcal{O}$. Let B be an $\mathcal{O}$-algebra such that B is free of finite rank as an $\mathcal{O}$-module. If we consider the exact sequence in [26, 4.1] with $\mathcal{O} = k[[t]]$ and $B = A \otimes_k k[[t]]$ for some finite-dimensional k-algebra A then we have:

$$0 \longrightarrow \hat{A}/t^r\hat{A} \xrightarrow{t^r} (\hat{A}/t^{2r}\hat{A})_\alpha \longrightarrow \hat{A}/t^r\hat{A} \longrightarrow 0.$$

The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \hat{A}/t^r\hat{A} \arrow[r, "t^r"] \arrow[d, two heads] & (\hat{A}/t^{2r}\hat{A})_\alpha \arrow[r] \arrow[d, two heads] & \hat{A}/t^r\hat{A} \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] & \hat{A}/t\hat{A} \arrow[r, "t^{2r}"] & (\hat{A}/t^{r+1}\hat{A})_\alpha \arrow[r] & \hat{A}/t^r\hat{A} \arrow[r] & 0
\end{tikzcd}$$

together with the naturality of the connecting homomorphism implies that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{End}_{\hat{A}^e}(\hat{A}/t^r\hat{A}) \arrow[r] \arrow[d, equal] & \mathrm{HH}^1(\hat{A}/t^r\hat{A}) \arrow[d, two heads] \\
\mathrm{End}_{\hat{A}^e}(\hat{A}/t^r\hat{A}) \arrow[r] & \mathrm{HH}^1(\hat{A}/t\hat{A})
\end{tikzcd}$$

Consequently Proposition 3.4.2 can be deduced from [26, 4.1].

3.5 Invariance theorems

Our first aim in this section is to prove the invariance of the r-integrable derivations under stable equivalences which is a variation of [26, 5.1]:

Theorem 3.5.1. *Let A, B be finite-dimensional selfinjective k-algebras indecomposable with separable semisimple quotients. Let r be a positive integer and let M, N be an*

A-B-bimodule, B-A bimodule, respectively, inducing a stable equivalence of Morita type between A and B. Then for any $\alpha \in \mathrm{Aut}_r(A[[t]])$ there is a $\beta \in \mathrm{Aut}_r(B[[t]])$ such that ${}_{\alpha^{-1}}M[[t]] \cong M[[t]]_\beta$ as $A[[t]]$-$B[[t]]$-bimodules. This correspondence induces a group isomorphism $\mathrm{Out}_r(A[[t]]) \cong \mathrm{Out}_r(B[[t]])$ making the following diagram commute:

$$\begin{tikzcd} \mathrm{Out}_r(A[[t]]) \arrow[r, "\cong"] \arrow[d] & \mathrm{Out}_r(B[[t]]) \arrow[d] \\ \mathrm{HH}^1_r(A) \arrow[r, "\cong"] & \mathrm{HH}^1_r(B) \end{tikzcd}$$

where the vertical maps are from Proposition 3.2.19 *and the lower horizontal isomorphism is induced by the functor $N \otimes_A - \otimes_A M$*

Proof. Let $\alpha \in \mathrm{Aut}_r(A)$. Then α induces the identity on $A[[t]]/tA[[t]] \cong A$, hence stabilises the isomorphism classes of all A-modules, hence in particular of all simple A-modules and all finitely generated projective $A[[t]]$-modules. By Theorem 4.2 in [23] there exists an automorphism $\beta \in \mathrm{Aut}_r(B[[t]])$, unique up to inner automorphisms, such that ${}_{\alpha^{-1}}M[[t]] \cong M[[t]]_\beta$ as $A[[t]]$-$B[[t]]$-bimodules. Then by [26, *Lemma* 5.2] we have that the upper horizontal map is a group isomorphism. Note that in [26, *Lemma* 5.2] A, B are nonsimple in order to exclude trivial cases, that is, when HH^1 is zero. The hypotheses of separable semisimple quotients and indecomposablity of A and B are needed in order to apply Theorem 4.2 in [23].

We also have that α is such that $\alpha(a) = a + t^r\mu(a)$ for all $a \in A[[t]]$ and β such that $\beta(b) = b + t^r\nu(b)$ for all $b \in B[[t]]$ for some $k[[t]]$-linear endomorphisms μ, ν. We denote by $\bar{\mu}$ and $\bar{\nu}$ the classes in $\mathrm{HH}^1_r(A)$ and $\mathrm{HH}^1_r(B)$ respectively determined by the canonical group homomorphism $\mathrm{Out}_r(A[[t]]) \to \mathrm{HH}^1(A)$ and $\mathrm{Out}_r(B[[t]]) \to \mathrm{HH}^1(B)$. Set $\hat{M} = M[[t]]$. By the assumptions, tensoring by M yields a stable equivalence of Morita type between A

and B. In particular we have:

$$\mathrm{HH}^1(A) \cong \mathrm{Ext}^1_{A\otimes_k B^{op}}(M,M) \cong \mathrm{HH}^1(B) \tag{3.5.1}$$

induced by the functors $-\otimes_A M$ and $M\otimes_B -$. In addition since $B[[t]]$ is isomorphic to $\hat{N}\otimes_{A[[t]]}\hat{M}$ in the relatively $k[[t]]$-stable category of $B[[t]]\otimes_{k[[t]]}B[[t]]^{op}$-modules, it follows that the isomorphism

$$\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B) \tag{3.5.2}$$

given by the composition of the two previous isomorphisms is induced by the functor $N\otimes_A - \otimes_A M$. The functors $M\otimes_B -$, $-\otimes_A M$ also induce algebra homomorphisms

$$\mathrm{End}_{A\otimes A^{op}}(A) \to \mathrm{End}_{A\otimes B^{op}}(M) \leftarrow \mathrm{End}_{B\otimes B^{op}}(B) \tag{3.5.3}$$

Tensoring the following two exact sequences

$$0 \longrightarrow A \longrightarrow (A[[t]]/t^{r+1}A[[t]])_\alpha \longrightarrow A[[t]]/t^rA[[t]] \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow (B[[t]]/t^{r+1}B[[t]])_\alpha \longrightarrow B[[t]]/t^rB[[t]] \longrightarrow 0$$

by $-\otimes_{A[[t]]}\hat{M}$ and $\hat{M}\otimes_{B[[t]]}-$ yields short exact sequences of the form

$$0 \longrightarrow M \longrightarrow {}_{\alpha^{-1}}(M[[t]]/t^{r+1}M[[t]]) \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow (M[[t]]/t^{r+1}M[[t]])_\beta \longrightarrow M \longrightarrow 0$$

By the naturality properties of the connecting homomorphism and the description of $\bar{\mu}$, $\bar{\nu}$ in Proposition 3.4.2 the image of $\bar{\mu}\otimes \mathrm{Id}_M$ and $\mathrm{Id}_M\otimes\bar{\nu}$ in $\mathrm{Ext}^1_{A\otimes_k B^{op}}(M,M)$ are equal to the images of $\mathrm{Id}_{\hat{M}}$ under the two connecting homomorphisms

$$\mathrm{End}_{A\otimes_k B^{op}}(\hat{M}) \to \mathrm{Ext}^1_{A\otimes_k B^{op}}(M,M) \tag{3.5.4}$$

obtained after applying the functor $\mathrm{Hom}_{A[[t]]\otimes B[[t]]^{op}}(\hat{M}, -)$ to the short exact sequences using the same identification used in Proposition 3.4.2. By Lemma [26, 4.3] the two exact sequences are equivalent, consequently the connecting homomorphisms are equal. Hence the two images of Id_M coincide. This shows that the group isomorphism $\mathrm{HH}^1_r(B) \cong \mathrm{HH}^1_r(A)$ induced by $\mathrm{Out}_r(B[[t]]) \cong \mathrm{Out}_r(A[[t]])$ is equal to the one determined by the functor $N \otimes_A - \otimes_A M$. Hence the result. $\square$

We want now to prove the compatibility between the transfer map and the p-power map under stable equivalences:

Theorem 3.5.2. *Let k be a field of positive characteristic p. Let A, B be finite-dimensional selfinjective indecomposable k-algebras with separable semisimple quotients, and let M, N be an A-B-bimodule, B-A-bimodule, respectively, inducing a stable equivalence of Morita type between A and B. For any positive integer r, the p-power map sends $\mathrm{HH}^1_r(A)$ to $\mathrm{HH}^1_{rp}(A)$, and we have a commutative diagram of maps*

$$\begin{tikzcd}
\mathrm{HH}^1_r(A) \arrow[r, "\cong"] \arrow[d, "{[p]}"] & \mathrm{HH}^1_r(B) \arrow[d, "{[p]}"] \\
\mathrm{HH}^1_{rp}(A) \arrow[r, "\cong"] & \mathrm{HH}^1_{rp}(B)
\end{tikzcd}$$

where the horizontal isomorphisms are induced by the functor $N \otimes_A - \otimes_A M$, and where the vertical maps are the p-power maps.

Proof. We show first that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Out}_r(A[[t]]) \arrow[r, "\cong"] \arrow[d, "(\ )^p"] & \mathrm{Out}_r(B[[t]]) \arrow[d, "(\ )^p"] \\
\mathrm{Out}_{rp}(A[[t]]) \arrow[r, "\cong"] & \mathrm{Out}_{rp}(B[[t]])
\end{tikzcd}$$

where the horizontal maps are from Theorem 3.5.1 and the vertical maps are p-fold compositions. Let $\alpha \in \mathrm{Aut}_r(A[[t]])$ and $\beta \in \mathrm{Aut}_r(B[[t]])$ such that ${}_{\alpha^{-1}}M[[t]] \cong M[[t]]_\beta$. Let μ, ν be the unique linear maps on $A[[t]]$ such that $\alpha(a) = a + t^r\mu(a)$ and $\beta(b) = b + t^r\nu(b)$ respectively. By Corollary 3.3.2 we have $\alpha^p \in \mathrm{Aut}_{rp}(A[[t]])$, $\beta^p \in \mathrm{Aut}_{rp}(A[[t]])$ and also that the maps $\bar{\mu}$, $\bar{\nu}$, induced by μ, ν on A, are sent under the p-power map to $\bar{\mu}^p$ and $\bar{\nu}^p$ respectively. Hence we have the commutativity of the diagram above since ${}_{\alpha^{-p}}M[[t]] \cong M[[t]]_{\beta^p}$. Using the commutative diagram above and Theorem 3.5.1 we have that the class of $\bar{\mu}^p$ is sent though the isomorphism defined in Theorem 3.5.2 to the class of $\bar{\nu}^p$. Hence we have the commutativity of the diagram in the beginning of the theorem. □

3.6 Structure on r-integrable derivations

Let p be a prime number and let a, r be two positive integers. The aim of this section is to study the structure of the r-integrable derivations. We divide this section in two parts: if $r = p^a$ and if r is arbitrary.

3.6.1 Vector space structure of p^a-integrable derivations

Matsumura in [28] already notes the following for 1-integrable derivations:

Proposition 3.6.1. *Let A be a k-algebra. Then the set of 1-integrable derivations of A form a $Z(A)$-submodule of $\mathrm{Der}(A)$. In particular $\mathrm{Der}_1(A)$ is a k-vector space.*

Proof. In Proposition 3.2.13 we prove that $\mathrm{Der}_1(A)$ is a subgroup of the additive abelian group $\mathrm{Der}(A)$. Let $\underline{D} = (D_i)_{i\geq 0}$ be a higher derivation of degree 1 . For any central element in $Z(A)$ the sequence

$$\lambda\underline{D} = (1, \lambda D_1, \ldots, \lambda^r D_r, \ldots)$$

is a higher derivation of degree 1. $\square$

Slightly generalising we have the following:

Theorem 3.6.2. *Let k be a perfect field of positive characteristic p and let A be a finite dimensional k-algebra. Let a be a non-negative integer. Then* $\mathrm{Der}_{p^a}(A)$ *is a k-vector subspace of* $\mathrm{Der}(A)$.

Proof. In Proposition 3.2.13 we prove for any positive integer r that $\mathrm{Der}_r(A)$ is a subgroup of the additive abelian group $\mathrm{Der}(A)$. Hence we just need to show that $\mathrm{Der}_{p^a}(A)$ is closed under scalar multiplication. Let $\lambda^{\frac{1}{p^a}}$ be the unique p^a-th root of λ. If we let n be a positive integer, we denote by $\lambda^{n/p^a} = (\lambda^{1/p^a})^n$. Let $\underline{D} = (D_i)_{i\geq 0}$ be a higher derivation of degree p^a. Then we define the scalar multiplication as follows:

$$\lambda \cdot \underline{D} = (1, 0, \ldots, \lambda D_{p^a}, \lambda^{\frac{p^a+1}{p^a}} D_{p^a+1}, \ldots) \tag{3.6.1}$$

It is easy to check that $\lambda \cdot \underline{D}$ is a higher derivation for every $\lambda \in k$, in fact for any positive integer r:

$$\begin{aligned} \sum_{i=0}^{p^a+r} \lambda^{\frac{i}{p^a}} D_i(a) \lambda^{\frac{p^a+r-i}{p^a}} D_{p^a+r-i}(b) &= \lambda^{\frac{p^a+r}{p^a}} \sum_{i=0}^{p^a+r} D_i(a) D_{p^a+r-i}(b) \\ &= \lambda^{\frac{p^a+r}{p^a}} D_{p^a+r}(ab) \end{aligned} \tag{3.6.2}$$

$\square$

Remark 3.6.3. It is not known if $\mathrm{Der}_{p^a}(A)$ has a $Z(A)$-module structure.

Theorem 3.6.4. *Let k be an algebraically closed field of positive characteristic p and let A be a finite dimensional k-algebra. Let n be a positive integer. Then* $\mathrm{Der}_n(A)$ *is a k-vector subspace of* $\mathrm{Der}(A)$.

Proof. Using Proposition 3.2.13 we just need to show that $\mathrm{Der}_n(A)$ is closed under scalar multiplication. Let $\lambda^{\frac{1}{n}}$ be a n-th root of λ. If we let m be a positive integer, we denote by $\lambda^{m/n} = (\lambda^{1/n})^m$. Let $\underline{D} = (D_i)_{i\geq 0}$ be a higher derivation of degree n. Then we define the scalar multiplication as follows:

$$\lambda \cdot \underline{D} = (1, 0, \dots, \lambda D_n, \lambda^{\frac{n+1}{n}} D_{n+1}, \dots) \tag{3.6.3}$$

Similarly to Theorem 3.6.2 it is easy to check that $\lambda \cdot \underline{D}$ is a higher derivation for every $\lambda \in k$. ☐

Remark 3.6.5. By Proposition 3.6.1 the sequence

$$\lambda\underline{D} = (1, \lambda D_1, \dots, \lambda^n D_n, \dots)$$

is higher derivation but it is not additive in λ, so does not yields a vectors space structure in $\mathrm{Der}_n(A)$.

3.6.2 Filtrations of r-integrable derivations

In this section we ask if there are inclusions among integrable derivations. The following theorem provides the answer:

Theorem 3.6.6. *Let A be a finite dimensional algebra over a field of positive characteristic p. Let n, k be two positive integers. Let $\{\mathrm{Der}_r(A)\}_{r\geq 1}$ be the set having elements r-integrable derivations for $r \geq 1$. If k divides n, then $\mathrm{Der}_k(A) \subseteq \mathrm{Der}_n(A)$. Hence, $\{\mathrm{Der}_r(A)\}_{r\geq 1}$ is a poset where the partial order is given by the inclusion.*

Proof. Let $\alpha \in \mathrm{Aut}_r(A[[t]])$ and let $\underline{D} = (D_k)_{k\geq 0}$ be the higher derivation of degree r associated to it. Clearly $D_r \in \mathrm{Der}_r(A)$. We need to construct $\beta \in \mathrm{Aut}_n(A[[t]])$ such that $D'_n = D_r$, where $\underline{D'} = (D'_k)_{k\geq 0}$ is the higher derivation associated to β. We divide the proof in different steps given by the homomorphisms that are defined below:

$$\begin{tikzcd}
\mathrm{Aut}_r(A[[t]]) \arrow[r, "\psi_1"] & \mathrm{Aut}_r(A[[t^n]]) \arrow[r, "\psi_2"] & \mathrm{Aut}_r(A\otimes k[[t^n]]) \arrow[d, "\psi_3"] \\
\mathrm{Aut}_n(A[[t]]) & \mathrm{Aut}_n(A\otimes k[[t]]) \arrow[l, "\psi_5"'] & \mathrm{Aut}_r(A\otimes k[[t^n]]\otimes_{k[[t^n]]} k[[t]]) \arrow[l, "\psi_4"']
\end{tikzcd}$$

Since all the maps are $k[[t]]$-linear, it is enough to evaluate them in A rather than in $A[[t]]$. Let $a \in A$ and let $\psi_1(a) = \sum_i at^{in}$. Set $\beta_1(a)$ to be equal to $(\psi_1 \circ \alpha)(a) = \sum_i D_i(a)t^{ni}$ for $a \in A$. Then the image of β_1 under ψ_2, say β_2, is defined as the unique automorphism of $A[[t]]$ such that the following diagram commutes:

$$\begin{tikzcd}
A[[t^n]] \arrow[r, "\beta_1"] \arrow[d, "\phi_1"] & A[[t^n]] \arrow[d, "\phi_1"] \\
A\otimes k[[t^n]] \arrow[r, "\beta_2"] & A\otimes k[[t^n]]
\end{tikzcd}$$

where ϕ_1 is the isomorphism from Lemma 3.2.1. Hence $\beta_2(a)$ is defined as

$$\phi_1(\beta_1(a)) = \sum D_i(a)\otimes t^{ni} \tag{3.6.4}$$

Since $\beta_2 \in \mathrm{Aut}_r(A\otimes k[[t^n]])$, it can be extended to the automorphism on $A\otimes k[[t^n]]\otimes_{k[[t^n]]} k[[t]]$, which is the image of ψ_3 of β_2, in the following way:

$$a \mapsto \sum_i D_i(a)\otimes t^{ni}\otimes 1 \tag{3.6.5}$$

We define $\beta_4 = \psi_4\circ\beta_3$ to be the unique automorphism of $A\otimes k[[t]]$ such that $\beta_4\circ\phi_2 = \phi_2\circ\beta_3$ where ϕ_2 is the isomorphism between $A\otimes k[[t^n]]\otimes_{k[[t^n]]} k[[t]]$ and $A\otimes k[[t]]$, hence :

$$\beta_4(a) = \sum_i D_i(a)\otimes t^{ni}. \tag{3.6.6}$$

Finally β_5 is given by $\beta_5(a) = \sum_i D_i(a)t^{ni}$. Hence β_5 is in $\mathrm{Aut}_n(A[[t]])$. □

As a consequence we have:

Corollary 3.6.7. *Let A be a finite dimensional k-algebra over a perfect field of positive characteristic. Let a, b be two positive integers such that $a \leq b$. Then* $\mathrm{Der}_{p^a}(A)$ *is a k-vector subspace of* $\mathrm{Der}_{p^b}(A)$.

3.7 Other properties of r-integrable derivations

In this section we collect some other properties of r-integrable derivations.

3.7.1 Invariance of the Jacobson radical under integrable derivations

We recall from Farkas, Geiss and Marcos in [11]:

Proposition 3.7.1. *Let A be a finite dimensional k-algebra. If $D : A \to A$ is a 1-integrable derivation. Then $D(J(A)) \subseteq J(A)$.*

The result is based on the following theorem:

Theorem 3.7.2 ([11, Theorem 2.1])**.** *Let A be a finite dimensional k-algebra. Let $(D_i)_{i \geq 0}$ be a higher derivation and let $\alpha \in \mathrm{Aut}_1(A[[t]])$ be the corresponding automorphism. Then $\alpha(J(A)) \subseteq (J(A))[[t]]$, that is $D_i(J(A)) \subseteq J(A)$ for all i.*

We generalise Theorem 3.7.1 for every r-integrable derivation.

Proposition 3.7.3. *Let A be a finite dimensional k-algebra and let $D : A \to A$ be a r-integrable derivation. Then $D(J(A)) \subseteq J(A)$.*

Proof. Let $\underline{D}$ be a higher derivation of degree r. Then using Theorem 3.7.2, we deduce that $D_m(J(A)) \subseteq J(A)$ for every positive integer m. Hence the result. □

3.8 Integrable derivations of quantum complete intersections

In [11] Farkas, Geiss and Marcos provide a family of examples given by commutative monomial algebras such that all their derivations are 1-integrable. The aim of this section is to provide another family given by certain quantum complete intersections.

Let k be a field of odd prime characteristic and let $q \in k^{\times}$ be an element of finite order $e \geq 2$ such that e divides $p-1$. Then

$$A = k\langle x, y|\ x^p = y^p = 0, yx = qxy\rangle \tag{3.8.1}$$

is a symmetric local k-algebra of dimension p^2. This k-algebra is called a quantum complete intersection. The set of monomials

$$V = \{x^i y^j |\ 0 \leq i, j \leq p-1\} \tag{3.8.2}$$

is a k-basis of A. The symmetrising form is given by the linear map that sends $x^{p-1}y^{p-1}$ to 1 and all other monomials in V to 0. The set

$$X^{'} = \{x^i y^j |\ 0 \leq i, j \leq p-1, \text{i and } j \text{ divisible by } e, \text{or } i = p-1, \text{ or } j = p-1\}$$

is a k-basis of $Z(A)$.

Remark 3.8.1. One can relax the condition that e divides $p-1$ in the definition of A. However, the resulting algebra A is not symmetric (although it is selfinjective [6]). Therefore A cannot be Morita equivalent to a block algebra of a finite group. To see this, assume that A is symmetric. Then any symmetrising form s of A is nonzero on the socle. In particular on $x^{p-1}y^{p-1}$. Thus

$$0 \neq s(x^{p-1}y^{p-1}) = s(x^{p-2}y^{p-1}x) = q^{p-1}s(x^{p-1}y^{p-1}), \tag{3.8.3}$$

and consequently $q^{p-1} = 1$.

Proposition 3.8.2 ([6, Lemma 5.2]). *Let a, b be integers such that $0 \leq a, b \leq p-1$.*

- *If e divides $a-1$, and b, or $b = p-1$ there is a derivation, say, $f_{a,b}$ on A satisfying*

$$f_{a,b}(x) = x^a y^b$$

  and

$$f_{a,b}(y) = 0.$$

- *If e divides $b-1$ and a, or $a = p-1$ then there is a derivation, say, $g_{a,b}$ on A satisfying*

$$g_{a,b}(x) = 0$$

  and

$$g_{a,b}(y) = x^a y^b.$$

We recall a result from [6] that allows us to determined the set of derivations and inner derivations in A.

Lemma 3.8.3 ([6, Lemma 5.3]). *Let a, b, c, d be integers such that $0 \leq a, b, c, d \leq p-1$. Let*

$$X_1 = \{f_{a,b} |\ 0 \leq a, b \leq p-1, e \text{ divides } a-1, \textit{ and } b, \textit{or } b = p-1\} \tag{3.8.4}$$

$$X_2 = \{g_{a,b} |\ 0 \leq a, b \leq p-1, e \text{ divides } a, \textit{ and } b-1, \textit{or } a = p-1\}. \tag{3.8.5}$$

Then $X_1 \cup X_2$ is the complement of $\mathrm{IDer}(A)$ in $\mathrm{Der}(A)$. Hence every element in $\mathrm{HH}^1(A)$ is represented by $X_1 \cup X_2$.

The following Lemma describes the $Z(A)$-module structure of $\mathrm{Der}(A)$.

Lemma 3.8.4. *Let i, j be two non-negative integers and let $x^i y^j$ be an element of the basis of $Z(A)$. Let a, b, c, d be integers such that $0 \leq a, b, c, d \leq p-1$ and let $f_{a,b} \in X_1$, $g_{c,d} \in X_2$. Then the action of the monomials on the derivations is described as follows:*

$$x^i y^j \cdot f_{a,b} = \begin{cases} f_{a+i,b+j} & \text{for } 0 \leq a+i-1, b+j \leq p-1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly:

$$x^i y^j \cdot g_{c,d} = \begin{cases} g_{c+i,d+j} & \text{for } 0 \leq c+i, d+j-1 \leq p-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is enough to prove the statement for the generators of the algebra A. Since $f_{a,b}(x) = x^a y^b$ then $x^i y^j f_{a,b}(x) = x^{a+i} y^{b+j} = f_{a+i,b+j}(x)$. Clearly if $a+1+i$ or $b+j$ is greater than p we have zero. Similarly for $g_{c,d}$. □

We can now prove the main theorem of this section

Theorem 3.8.5. *With the notation and assumptions above, we have that all derivations on A are* 1*-integrable.*

Since X is a k-basis in $\mathrm{HH}^1(A)$ where X_1, X_2 are from Lemma 3.8.3, our aim is to prove that all elements of $X = X_1 \cup X_2$ are 1-integrable derivations.

Let us first consider $f_{1,0} \in X_1$. Set $\alpha : A \to A[[t]]$ defined as $\alpha(x) = x(1+t)$ and $\alpha(y) = y$ for x, y in A. In order to show that α extends to an automorphism in $\mathrm{Aut}_1(A[[t]])$ we need to check that the following relations hold:

$$\begin{aligned} \alpha(x)^p &= 0 \\ \alpha(y)^p &= 0 \\ \alpha(x)\alpha(y) &= q\alpha(y)\alpha(x). \end{aligned} \tag{3.8.6}$$

Since $\alpha(x)^p = (x(1+t))^p = 0 = x^p = \alpha(x^p)$. The second equation is straightforward. For the last one we have: $\alpha(x)\alpha(y) = x(1+t)y = xy(1+t) = qyx(1+t) = q\alpha(y)\alpha(x)$. Hence $f_{1,0}$ is integrable. Now we consider $f_{a,b}$ such that $a \neq 0$ and e divides $a-1$ and b. Every element of this form can be obtained from $f_{1,0}$ by the multiplication by $x^{a-1}y^b \in Z(A)$. Since the integrable derivations are a $Z(A)$-module we have that they are integrable. All the elements of the form $f_{a,p-1}$ where $a \neq 0$ can be obtained as $f_{a,p-1} = x^{a-1}y^{p-1} \cdot f_{1,0}$. Since $x^{a-1}y^{p-1}$ is an element of the centre they are also integrable. The only remaining element, using Lemma 3.8.4, is $f_{0,p-1}$. If we consider the automorphism $\alpha(x) = (x + x^{p-1}t)$, then it is easy to check that α verifies the conditions 3.8.6. Similarly we can do the same for $g_{c,d}$. Consequently, all the elements of the basis of X are 1-integrable. Any other derivation is represented by a class of linear combinations of them, hence the result.

Corollary 3.8.6. *Let A be a quantum complete intersection. Then*

$$\mathrm{HH}^1_1(A) = \mathrm{HH}^1(A).$$

In particular, $\mathrm{HH}^1_1(A)$ is a restricted Lie algebra

Using Theorem 3.5.2 we have also the following:

Corollary 3.8.7. *Let A be a quantum complete intersection and let B be symmetric a k-algebra. Let M, N be an A-B-bimodule, B-A-bimodule, respectively, inducing a stable equivalence of Morita type between A and B. Then*

$$\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$$

as restricted Lie algebras. In addition $\mathrm{HH}^1(B) \cong \mathrm{HH}^1_1(B)$.

Proof. We have $\mathrm{HH}^1_1(A) \cong \mathrm{HH}^1_1(B)$ from Theorem 3.5.2 . From Corollary 3.8.6 it follows that $\mathrm{HH}^1_1(A) = \mathrm{HH}^1(A)$. We also have that $\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$ as k-vector spaces (from the stable equivalence of Morita type). Hence the statement. □

Chapter 4

Block algebras with HH^1 a simple Lie algebra

4.1 Introduction

Let p be a prime and k an algebraically closed field of characteristic p. The purpose of this chapter is to illustrate close connections between the Lie algebra structure of $\mathrm{HH}^1(B)$ and the structure of B, where B is a block of a finite group algebra kG. We consider two extreme cases for blocks with a single isomorphism class of simple modules. The main result of this chapter is from a joint paper with Markus Linckelmann [27].

Theorem 4.1.1. *Let G be a finite group and let B be a block algebra of kG having a unique isomorphism class of simple modules. Then $\mathrm{HH}^1(B)$ is a simple Lie algebra if and only if B is nilpotent with an elementary abelian defect group P of order at least 3. In that case, we have a Lie algebra isomorphism $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(kP)$.*

In particular, Theorem 4.1.1 implies that no other simple modular Lie algebra occurs as $\mathrm{HH}^1(B)$ for B a block with a single isomorphism class of simple modules. See [35], [36] for details and further references on the classification of simple Lie algebras in positive characteristic. We do not know whether the hypothesis on B to have a single isomorphism class of simple modules is necessary in Theorem 4.1.1.

Theorem 4.1.2. *Let G be a finite group and let B be a block algebra of kG having a nontrivial defect group and a unique isomorphism class of simple modules. Then*

$$\dim_k(\mathrm{HH}^1(B)) \geq 2.$$

We introduce some background material in the next four sections. In section 4.6 we prove some intermediate results which are fundamental for the proofs of the main theorems which are proved in the last section.

4.2 Block theory background

The background for this section can be found in [37].

4.2.1 The G-Algebra Structure and the Trace Map

If k has characteristic zero or positive characteristic not dividing the order of a finite group G, then kG is semisimple, or equivalently, its block algebras are matrix algebras over division algebras. If $char(k) = p > 0$ and p divides the order of G, then the structure of the block algebras of kG is more complicated. A first measure for how far off a block is from being a matrix algebra is encapsuled in the defect groups of a block, a concept due to Brauer. We will see that the defect groups of a block form a conjugacy class of p-subgroups of G and in some sense control the complexity of the representation theory of the block algebra. There are many different ways to characterise defect groups of blocks; we will follow the approach due to Green in [14] using the notion of a G-algebra.

Let G be a finite group. The group algebra kG is endowed with an action of G, given by the conjugation action ${}^x a = xax^{-1}$, where $x \in G$, $a \in kG$. In addition, the map sending $a \in kG$ to ${}^x a$ is a k-algebra automorphism of kG for all $x \in G$. In this sense, kG is called a *G-algebra* over k.

Definition 4.2.1. Let G be a finite group. For every subgroup P of G we denote by $(kG)^P$ the subalgebra of all P-fixed points in kG; that is,

$$(kG)^P = \{a \in kG |\ {}^y a = a \ for\ all\ y \in P\}.$$

Example 4.2.2. Let G be a finite group. We have $(kG)^G = Z(kG)$.

It is easy to check that $(kG)^P$ is an $N_G(P)$-algebra. If $Q \subseteq P$ are subgroups of G, then there is an inclusion map $(kG)^P \to (kG)^Q$. Conversely, we define a k-linear map $\mathrm{Tr}_Q^P : (kG)^Q \to (kG)^P$ as follows. If $a \in (kG)^Q$ and $x \in P$, then ${}^x a$ depends only on the coset xQ, not on the choice of x because a is fixed by Q. Thus, if we denote by $[P/Q]$ any set of representatives in P of the Q-cosets $P/Q = \{xQ|\ x \in P\}$, then the expression

$$\mathrm{Tr}_Q^P(a) = \sum_{x \in [P/Q]} {}^x a$$

does not depend on the choice of $[P/Q]$. Moreover, for any $y \in P$, we have

$${}^y\mathrm{Tr}_Q^P(a) = \sum_{x \in [P/Q]} {}^{yx} a$$

As x runs over a set of representatives of the cosets P/Q, so does yx, and hence this expression is again equal to $\mathrm{Tr}_Q^P(a)$. Thus defined k-linear map $\mathrm{Tr}_Q^P : (kG)^Q \to (kG)^P$ is called the *trace map* from Q to P on kG. We set $(kG)_Q^P = \mathrm{Im}(\mathrm{Tr}_Q^P)$.

We outline some properties of the trace map:

Proposition 4.2.3. *Let G be a finite group and let S, Q, R, P be subgroups of G such that $Q \leq P$ and $R \leq S \leq P$. Then we have:*

- *For any $a \in (kG)^P$ and $b \in (kG)^R$ we have*

$$a\mathrm{Tr}_R^P(b) = \mathrm{Tr}_R^P(ab),\ \mathrm{Tr}_R^P(b)a = \mathrm{Tr}_R^P(ba)$$

  In particular, $(kG)_R^P$ is a two-sided ideal in $(kG)^P$.

- *We have $\mathrm{Tr}_S^P\mathrm{Tr}_R^S = \mathrm{Tr}_R^P$.*

- *If $a \in (kG)^R$ then*

$$\mathrm{Tr}_R^P(a) = \sum_{x\in[Q\backslash P/R]} \mathrm{Tr}_{Q\cup^x R}^Q({}^x a)$$

  which is called Mackey formula.

Proof. The first two parts are straightforward, hence we just prove the last. In the disjoint union $P = \cup_{x\in[Q\backslash P/R]}QxR$ any double coset QxR is again a disjoint union $QxR = \cup_{y\in[Q/Q\cap^x R]}yxR$. In other words, we can take for $[P/R]$ the set of all yx, with x running over $[Q\backslash P/R]$ and, for any such x, with y running over $[Q/Q\cap^x R]$, which implies the formula. □

Proposition 4.2.4. *Let Q, P be subgroups of G such that $Q \leq P$. Then $(kG)_Q^P$ is spanned by elements of the form $\mathrm{Tr}_{C_Q(x)}^P(x)$, $x \in G$.*

4.2.2 The Brauer Homomorphism

Let G be a finite group and let H be a subgroup of G. In general the canonical k-linear projection $kG \to kH$ sending $\sum_{x\in G}\lambda_x x$ to $\sum_{x\in H}\lambda_x x$ is not an algebra homomorphism. However, the next theorem illustrates that this map does restrict to be an algebra homomorphism for suitable subalgebras of fixed points:

Theorem 4.2.5. *Let G be a finite group and let P be a p-subgroup of G. The canonical k-linear projection $kG \to kC_G(P)$ induces a split surjective homomorphism of $N_G(P)$-*

algebras over k,

$$\begin{aligned}\mathrm{Br}_P^{kG} : (kG)^P &\to kC_G(P) \\ \sum_{x\in G}\lambda_x x &\mapsto \sum_{x\in C_G(P)}\lambda_x x\end{aligned} \tag{4.2.1}$$

where $\lambda_x \in k$. The kernel of (Br_P^{kG}) is given by

$$\sum_{Q<P}(kG)_Q^P$$

where in the sum Q runs over all proper subgroups.

The $N(P)$-algebra homomorphism Br_P^{kG} is called the *Brauer homomorphism* for P on kG. We write Br_P instead of Br_P^{kG} if it causes no confusion. The following two propositions analyse the interaction between the Brauer homomorphism and the trace map.

Lemma 4.2.6. *Let P, Q be p-subgroups of G. Suppose that $a \in (kG)_Q^G$ and $\mathrm{Br}_Q(a) \neq 0$. Then there exists $x \in G$ such that $Q \subseteq^x P$.*

Proof. We have $a = \mathrm{Tr}_P^G(c)$ for some $c \in (kG)^P$. By Mackey's formula, Proposition 4.2.3, we have

$$\mathrm{Br}_Q(a) = \mathrm{Br}_Q\mathrm{Tr}_P^G(c) = \sum_{x\in[P\backslash G/P]}\mathrm{Br}_Q\mathrm{Tr}_{Q\cap^x P}^Q({}^x c)$$

Since $\mathrm{Br}_Q(a) \neq 0$, there exists $x \in G$ such that $Q \cup^x P = Q$, that is, $Q \subseteq^x P$. □

Proposition 4.2.7. *Let P be a p-subgroup of G. Then for $a \in (kG)^P$ we have*

$$\mathrm{Br}_P\mathrm{Tr}_P^G(a) = \mathrm{Tr}_P^{N_G(P)}\mathrm{Br}_P(a).$$

In particular, $\mathrm{Br}_P((kG)_P^G) = (kC_G(P))_P^{N_G(P)}$.

Proof. By Mackey's formula, Proposition 4.2.3, we have

$$\mathrm{Br}_P\mathrm{Tr}_P^G(a) = \sum_{x\in[P\backslash G/P]}\mathrm{Br}_P\mathrm{Tr}_{P\cap^x P}^P({}^x a).$$

But $P \cap^x P = P$ iff $x \in N_G(P)$. Thus

$$\mathrm{Br}_P \mathrm{Tr}_P^G(a) = \sum_{x \in [N_G(P)/P]} \mathrm{Br}_P(^x a) = \mathrm{Tr}_P^{N_G(P)} \mathrm{Br}_P(a).$$

□

We are ready to define the defect groups of a block:

4.2.3 Defect Groups of a Block

Definition 4.2.8. Let b be a block of kG. A defect group P of the block b is a minimal subgroup of G such that $b \in (kG)_P^G$.

Using Brauer homomorphisms, we can give alternative characterisations of defect groups of a block.

Theorem 4.2.9. *Let b be a block of kG. For a p-subgroup P of G, the following conditions are equivalent:*

- *P is a defect group of b.*
- *P is a maximal subgroup of G such that $\mathrm{Br}_P(b) \neq 0$.*
- *We have $b \in (kG)_P^G$ and $\mathrm{Br}_P(b) \neq 0$.*

Since b is G-invariant, any G-conjugate of P is again a defect group of b. In fact the converse is also true:

Proposition 4.2.10. *Let G be a finite group, b a block of kG and let P be a defect group of b. If char$(k) = 0$, then the defect groups of b are trivial. If char$(k) = p > 0$, then the defect groups of the block b of kG form a single G-conjugacy class of p-subgroups of G.*

Definition 4.2.11. Let G be a finite group. The kG-module k endowed with the identity action of all group elements is called the *trivial* kG-module. The surjective algebra homomorphism $\epsilon : kG \to k$ defined by

$$\epsilon\Big(\sum_{x\in G}\lambda_x x\Big) = \sum_{x\in G}\lambda_x$$

is called the *augmentation homomorphism* and the ideal $I(kG) = \mathrm{Ker}(\epsilon)$ is called the augmentation ideal of kG.

Proposition 4.2.12. *Let G be a finite group. The augmentation ideal $I(kG)$ is free as a k-module and the set $\{x - 1|\ x \in G\backslash\{1\}\}$ is a k-basis of $I(kG)$.*

Proof. Let $\sum_{x\in G}\lambda_x x \in I(kG)$; that is, $\sum_{x\in G}\lambda_x = 0$. Then

$$\sum_{x\in G}\lambda_x x = \sum_{x\in G}\lambda_x(x-1) = \sum_{x\in G, x\neq 1}\lambda_x(x-1)$$

which shows that the set $\{x - 1|\ x \in G\backslash\{1\}\}$ generates $I(kG)$ as a k-module. Since G is a k-basis of kG one easily sees that this set is k-linearly independent, hence a k-basis of $I(kG)$. □

Proposition 4.2.13. *Let G, H be finite groups. Let $\phi : G \to H$ be a group homomorphism and let $\alpha : kG \to kH$ be the induced algebra homomorphism. Set $N = \ker(\phi)$. We have $\ker(\alpha) = kG \cdot I(kN) = I(kN) \cdot kG$.*

Proof. Let $\sum_{y\in N}\mu_y y \in I(kN)$; that is, $\sum_{y\in N}\mu_y = 0$. Since N is normal in G we have $\sum_{y\in N}\mu_y xyx^{-1} \in I(kN)$, hence $xI(kN) = I(kN)x$ for all $x \in G$, from which we get the equality $kG \cdot I(kN) = I(kN) \cdot kG$. Since ϕ maps all elements in N to 1 we get $I(kN) \subseteq \ker(\alpha)$. As α is an algebra homomorphism, its kernel is an ideal and thus contains the ideal $I(kN) \cdot kG$ generated by $I(kN)$ in kG. Let $\sum_{x\in G}\lambda_x x \in \ker(\alpha)$. Denote by $[G/N]$ a system of coset representatives G/N in G. Thus $\alpha(\sum_{x\in G}\lambda_x x) = \sum_{x\in G}\lambda_x\phi(x) =$

$\sum_{x\in[G/N]}\sum_{y\in N}\lambda_{xy}\phi(xy)=\sum_{x\in[G/N]}\sum_{y\in N}\lambda_{xy}\phi(x)=0$ if and only if $\sum_{y\in N}\lambda_{xy}=0$ for any $x\in G$. This means that $\sum_{y\in N}\lambda_{xy}y\in I(kN)$ for any $x\in G$, and hence $\sum_{x\in G}\lambda_x x=\sum_{x\in[G/N]}\sum_{y\in N}\lambda_{xy}xy=\sum_{x\in[G/N]}x(\sum_{y\in N}\lambda_{xy}y)$ belongs to $kG\cdot I(kN)$ □

Clearly $\epsilon(Z(kG))=k$. Since the algebra k has a unique block 1_k, the Idempotent Lifting Theorem, Proposition 2.1.6, tells us that there exists a unique block b_0 of kG such that $\epsilon(b_0)=1_k$, that is, b_0 is not contained in $I(kG)$. The block b_0 is called the *principal block* of kG.

Proposition 4.2.14. *Let b_0 be the principal block of kG. The defect groups of b_0 are the Sylow p-subgroups of G.*

Theorem 4.2.15 (Brauer's First Main theorem)**.** *Let G be a finite group, let P be a p-subgroup of G,and let b be a block of kG with P as a defect group. Then There is a unique block c of $kN_G(P)$ with P as defect group such that $\mathrm{Br}_P(b)=\mathrm{Br}_P(c)$, and this correspondence defines a bijection between the sets of blocks of kG and of $kN_G(P)$ with P as defect group.*

Definition 4.2.16. Let G be a finite group, let b be a block of kG and let P be a defect group of b. The unique block c of $kN_G(P)$ with P as defect group satisfying $\mathrm{Br}_P(b)=\mathrm{Br}_P(c)$ is called the *Brauer correspondent* of b.

4.2.4 Brauer Pairs and Nilpotent blocks

Definition 4.2.17. A Brauer pair for kG is a pair (P,e) consisting of a p-subgroup P of G and a block e of $kC_G(P)$. The set of Brauer pairs for kG admits the natural conjugation action by G: for a Brauer pair (P,e) and $x\in G$, then

$$^{x}(P,e)=(^{x}P,^{x}e).$$

Definition 4.2.18. Let G be a finite group, let b be a block of kG and let P be a defect group of b. The unique block c be the Brauer correspondent of b. If e is a block of $kC_G(P)$ satisfying $ec = e$ then the group $E = N_G(P,e)/PC_G(P)$ is called the *inertial quotient* of b.

Definition 4.2.19 ([1], [9])**.** Let (P,e), (Q,f) be Brauer pairs for kG. We say that (P,e) contains (Q,f) and write $(Q,f) \leq (P,e)$ if $Q \leq P$ and for every primitive idempotent i for $(kG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$ we have $\mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i)$.

It is possible to prove that the set of Brauer pairs is partially ordered using the Brauer map with respect this inclusion. The Brauer pair (P,e) belongs to a block b if b is the unique block such that $(1,b) \leq (P,e)$.

Definition 4.2.20 ([10])**.** Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group. A block b of kG with defect group D is *nilpotent* if whenever there is a Brauer pair (Q,e_Q) that belongs to b and satisfies $(Q,e_Q) \leq (D,e)$, then $(Q,e_Q)^g \leq (D,e)$ implies that there is a $c \in C_G(Q)$ and $u \in D$ such that $g = cu$.

Nilpotent blocks with abelian defect can be characterised as follow:

Theorem 4.2.21 (Okuyama and Tsushima [29])**.** *Let G be a finite group and B a block algebra of kG. Then the following are equivalent:*

(i) B is a nilpotent block with an abelian defect group.

(ii) $J(B) = J(Z(B))B$.

(iii) B has abelian defect group and trivial inertial quotient.

4.3 Basic algebras

Background for this section can be found in [2].

Let A be a k-algebra. The aim of this section is to introduce the concept of a basic algebra of A which allows us to reduce the study of the representation theory of A to a smaller algebra.

Proposition 4.3.1. *Let A be a k-algebra. Let e an idempotent in A and let U be an A-module. The correspondence sending U to $eA \otimes_A U$ yields a functor, called Schur functor,*

$$\mathrm{Mod}(A) \to \mathrm{Mod}(eAe)$$

which sends an A-module homomorphism $\phi : U \to V$ to the eAe-module homomorphism $\mathrm{Id}_{eA} \otimes \phi : eA \otimes_A U \to eA \otimes_A V$ defined by $(\mathrm{Id}_{eA} \otimes \phi)(ea \otimes u) = ea \otimes \phi(u)$ for all $a \in A$ and $v \in U$.

If the algebra A and the module U are finite-dimensional, then eAe and eU are finite-dimensional, and hence this functor restricts to a functor

$$\mathrm{mod}(A) \to \mathrm{mod}(eAe)$$

where by $\mathrm{mod}(A)$ we denote the category of finitely generated left A-modules. The fundamental question regarding the Schur functor is to understand for which idempotent the functor is an equivalence.

Theorem 4.3.2. *Let A be a k-algebra and e an idempotent in A. The following are equivalent.*

(i) The functor $eA \otimes_A - : \mathrm{Mod}(A) \to \mathrm{Mod}(eAe)$ is an equivalence.

(ii) We have $AeA = A$.

(iii) For every simple A-module S we have $eS \neq \{0\}$.

If one of three equivalent statements holds, then the functors $eA \otimes_A -$ and $Ae \otimes_{eAe} -$ are equivalences between $\mathrm{Mod}(A)$ and $\mathrm{Mod}(eAe)$ which are inverse to each other.

Proof. Suppose that $A = AeA$. We are going to show that the functor

$$eA \otimes_A - : \mathrm{Mod}(A) \to \mathrm{Mod}(eAe)$$

has as inverse the functor

$$Ae \otimes_{eAe} - : \mathrm{Mod}(eAe) \to \mathrm{Mod}(A)$$

sending an eAe-module N to the A-module $Ae \otimes_{eAe} N$. Here Ae is considered as an A-eAe-bimodule. The functor $Ae \otimes_{eAe} -$ followed by $eA \otimes_A -$ is given by tensoring with $eA \otimes_A Ae \cong eAe$, and clearly $eAe \otimes_{eAe} -$ is the identity functor on $\mathrm{Mod}(eAe)$. Conversely, the functor $eA \otimes_A -$ followed by $Ae \otimes_{eAe} -$ is given by tensoring with $Ae \otimes_{eAe} eA$. In order to show that this is the identity functor on $\mathrm{Mod}(A)$ we have to show that $Ae \otimes_{eAe} eA \cong A$ as A-A-bimodule. What we are going to show that the map $\mu : Ae \otimes_{eAe} eA \to A$ sending $ce \otimes ed$ to ced is an isomorphism, where $c, d \in A$. Clearly this map is a homomorphism of A-A-bimodules. Its image is $\mathrm{Im}(\mu) = AeA = A$, so this map is surjective. It remains to see that μ is injective. Since $AeA = A$ there is a finite set J and elements $x_j \in Ae$, $y_j \in eA$, for any $j \in J$, such that $\sum_{j \in J} x_j y_j = 1$. Let $\sum_{s \in S} c_s \otimes d_s$ be in the kernel of this map, where S is a finite indexing set and $c_s \in Ae$, $d_s \in eA$, for $s \in S$. That means that we have $\sum_{s \in S} c_s d_s = 0$. But then also $\sum_{s \in S} y_j c_s d_s = 0$ for any $j \in J$. Note that $y_j c_s \in eAe$. Therefore, tensoring with x_j and taking the sum over all j yields

$$0 = \sum_{j \in J, s \in S} x_j \otimes y_j c_s d_s = \sum_{j \in J, s \in S} x_j y_j c_s \otimes d_s = \sum_{s \in S} c_s \otimes d_s.$$

Thus $\ker(\mu) = 0$. This shows the implication (ii) $\implies$ (i). The implication (i) $\implies$ (iii) is trivial. For the implication (iii) $\implies$ (ii), suppose that AeA is a proper ideal. The nonzero A-module A/AeA is finitely generated as a left A-module (by the image of 1). This module has a maximal submodule, hence a simple quotient S. In particular, S is annihilated by e, which completes the proof. $\square$

The following Lemma give us some criteria for when the Schur functor is an equivalence.

Lemma 4.3.3. *Let A be a finite-dimensional k-algebra, e an idempotent in A, I a primitive decomposition of 1 and J a primitive decomposition of e. The following are equivalent.*

(i) We have $AeA = A$.

(ii) Every $i \in I$ is conjugate to an element in J.

(iii) Every primitive idempotent in A is conjugate to a primitive idempotent in eAe.

(iv) For every simple A-module S we have $eS \neq \{0\}$.

We recall from Chapter 2 that f k is algebraically closed, then

$$A/J(A) \cong \prod M_{n_i}(k) \cong \prod \mathrm{End}_k(S_i).$$

which shows that the integer n_i is both equal to $\dim_k(S_i)$ and equal to the multiplicity of Ai as a direct summand of the regular A-module A.

The point of the following definition is to bring the dimensions of the simple modules to be equal to 1, without changing the equivalence class of the module category of A.

Definition 4.3.4. Let A be a finite-dimensional k-algebra. We say that A is a *basic* k-algebra if in any primitive decomposition I of 1_A, the elements of I are pairwise non-conjugate, or equivalently, if the summands in $A = \oplus_{i \in I} Ai$ are pairwise non-isomorphic.

Proposition 4.3.5. *Let A be a finite-dimensional k-algebra. The following are equivalent.*

(i) The algebra A is basic.

(ii) As a left A-module, $A/J(A)$ is a direct sum of pairwise non-isomorphic simple A-modules.

(iii) As an algebra, $A/J(A)$ is a finite direct product of division algebras.

Proof. Consider a module decomposition $A = \oplus_{i \in I} Ai$ for some primitive decomposition of 1. Then the elements in I are pairwise non-conjugate if and only if the A-modules Ai, $i \in I$, are pairwise non-isomorphic. This holds if and only if the simple modules $S_i = Ai/J(A)i$, $i \in I$, are pairwise non-isomorphic. This shows the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from Wedderburn's theorem. □

Corollary 4.3.6. *Suppose that k is algebraically closed. The following are equivalent for a finite dimensional k-algebra A.*

- *The algebra A is basic.*
- *The algebra $A/J(A)$ is a finite direct product of copies of k.*
- *For any simple A-module S we have $dim_k(S) = 1$.*

Proof. This is an easy consequence of Wedderburn'ss theorem and Proposition 4.3.5. □

Theorem 4.3.7. *Let A be a finite-dimensional k-algebra. There is an idempotent e in A such that the algebra eAe is basic and Morita equivalent to A via the Schur functor given by e. Moreover, e is then unique up to conjugation, and hence eAe is unique up to isomorphism.*

Proof. Let I be a primitive decomposition of 1 in A. Choose in I a set J of representatives of the conjugacy classes of elements in I. Set $e = \sum_{j \in J} j$. The result follows from combining Proposition 4.3.2 and Lemma 4.3.3. □

Definition 4.3.8. Let A be a k-algebra and let e be an idempotent in A such that eAe is basic. Then we define eAe to be a *basic algebra of A*.

Proposition 4.3.9. *Let A be a finite dimensional k-algebra and let eAe be a basic algebra of A. Then A and eAe have the same Cartan matrix.*

Proof. Since Morita equivalence preserves projective indecomposables and simple modules the result follows. □

Definition 4.3.10. Let A be a finite dimensional k-algebra. Then A is *local* if the k-algebra $A/J(A)$ is isomorphic to k.

Theorem 4.3.11. *Let A be a finite-dimensional split k-algebra, and let I be a set of representatives of the conjugacy classes of primitive idempotents in A. Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of A. For any $i, j \in I$ we have $c_{ij} = \dim_k(iAj)$.*

Proposition 4.3.12. *Let A be a symmetric local k-algebra. If A is Morita equivalent to a block algebra B of kG for some finite group G then $\dim_k(A) = |P|$, where P is a defect group of B.*

Proof. The proof uses the fact, due to Brauer, that the elementary divisors of the Cartan matrix of B, hence of A, divide $|P|$ and exactly one of them is equal to $|P|$. Thus if A is local, its Cartan matrix consists of the single entry $|P|$, which then by Theorem 4.3.11 is the dimension of A. □

4.4 Uniserial algebras

Background for this section can be found in [2] and [41].

Definition 4.4.1. Let A be a k-algebra. An A-module M is uniserial if it has only one composition series. An algebra A is uniserial if each indecomposable projective A-module is uniserial.

The following theorem is well known:

Theorem 4.4.2 ([3, Theorem VI.2.1]). *Let A be a uniserial algebra. Then every indecomposable A-module is uniserial, and A has finite representation type.*

Theorem 4.4.3 ([22, Theorem 2.1]). *Let k be a field and let A be a finite dimensional symmetric k-algebra. Then the following statements are equivalent.*

(i) There is a $t \in A$ such that $J(A) = A \cdot t$ or $J(A) = t \cdot A$.

(ii) The A-modules $A/J(A)$ and $J(A)/J^2(A)$ are isomorphic.

4.5 Bounds and simplicity of HH^1 for elementary abelian p-groups

We collect in this section results needed for the proof of Theorem 4.1.1.

Theorem 4.5.1 ([6, Theorem 3.1]). *Let A be a symmetric k-algebra and let E be a maximal semisimple subalgebra. Let $f\colon A \to A$ be an E-E-bimodule homomorphism satisfying $E + J(A)^2 \subseteq \ker(f)$ and $\mathrm{Im}(f) \subseteq \mathrm{soc}(A)$. Then f is a derivation on A in $\mathrm{soc}_{Z(A)}(\mathrm{Der}(A))$, and if $f \neq 0$, then f is an outer derivation of A. In particular, we have*

$$\sum_{S} \dim_k(\mathrm{Ext}^1_A(S,S)) \leq \dim_k(\mathrm{soc}_{Z(A)}(\mathrm{HH}^1(A)))$$

where in the sum S runs over a set of representatives of the isomorphism classes of simple A-modules.

Corollary 4.5.2 ([6, Corollary 3.2]). *Let A be a local symmetric k-algebra. Let $f\colon A \to A$ be a k-linear map satisfying $1 + J(A)^2 \subseteq \ker(f)$ and $\mathrm{Im}(f) \subseteq \mathrm{soc}(A)$. Then f is a derivation on A in $\mathrm{soc}_{Z(A)}(\mathrm{Der}(A))$, and if $f \neq 0$, then f is an outer derivation of A. In particular, we have*

$$\dim_k(J(A)/J(A)^2) \leq \dim_k(\mathrm{soc}_{Z(A)}(\mathrm{HH}^1(A))) \ .$$

Theorem 4.5.3 (Jacobson [18, Theorem 1]). *Let P be a finite elementary abelian p-group of order at least 3. Then $\mathrm{HH}^1(kP)$ is a simple Lie algebra.*

The following holds as well.

Proposition 4.5.4. *Let P be a finite abelian p-group. If $\mathrm{HH}^1(kP)$ is a simple Lie algebra, then P is elementary abelian of order at least 3.*

Proof. Suppose that P is not elementary abelian; that is, its Frattini subgroup $Q = \Phi(P)$ is nontrivial. Let where $I(kQ)$ be the augmentation ideal of kQ. We will show that the set of derivations with image contained in $I(kQ)kP$, which is equal to $\ker(kP \to kP/Q)$ by Proposition 4.2.13, is a nonzero Lie ideal in $\mathrm{Der}(kP)$. Indeed, every element in Q is equal to x^p for some $x \in P$. Consequently, using Proposition 4.2.12 and the fact that k has characteristic p, every element in $I(kQ)$ is a linear combination of elements of the form $(x-1)^p$, where $x \in P$. Every derivation on kP preserves $I(kQ)kP$. In fact let $(x-1)^p \in I(kG)$, $y \in P$ and let $f : kP \to kP$ be a derivation on kP. Using the fact that k has characteristic p we have: $f((x-1)^p y) = f((x-1)^p)y + (x-1)^p f(y) = f(x^p - 1)y + (x-1)^p f(y) = f(x^p)y + (x-1)^p f(y) = pf(x)x^{p-1}y + (x-1)^p f(y) = (x-1)^p f(y) \in I(kQ)P$. Thus there is a canonical Lie algebra homomorphism $\varphi : \mathrm{Der}(kP) \to \mathrm{Der}(kP/Q)$. In order to prove that φ is nonzero with nonzero kernel we first let $P = C_{p^n} =< y >$ for some $y \in C_{p^n}$. Then $Q = C_{p^{n-1}} =< y^p >$ and $P/Q \cong C_p$. Hence $\varphi : \mathrm{Der}(k[x]/(x^{p^n})) \to \mathrm{Der}(k[x]/x^p)$. If we consider the derivation $f(x) = x$ we have $\varphi \circ f$ is nonzero whilst the image $g(x) = x^p$ under φ is zero. We apply the same idea when P be a finite abelian p-group. We have $P = C_{p^{n_1}} \times \cdots \times C_{p^{n_r}}$ for some positive integers n_i. Then the group algebra kP is isomorphic to $kC_{p^{n_1}} \otimes \cdots \otimes kC_{p^{n_r}}$. Using the notation from Corollary 2.3.25

we have $kP = \prod_{i=1}^{r} kC_{p^{n_i}}$ and $kP/Q = \prod_{i=1}^{r} kC_p$. By Corollary 2.3.25 we have

$$\mathrm{Der}(kP) = \mathrm{HH}^1(\prod_{i=1}^{r} kC_{p^{n_i}}) \cong \sum_{\substack{i_1,\dots,i_r\geq 0,\\ i_1+\dots+i_r=1}} \prod_{j=1}^{r} \mathrm{HH}^{i_j}(kC_{p^{n_j}}) = \sum_{\substack{i_1+\dots i_r\geq 0,\\ i_1+\dots+i_r=1}} \prod_{j=1}^{r} \mathrm{HH}^{i_j}(k[x_j]/(x_j^{p^{n_j}})) \tag{4.5.1}$$

and

$$\mathrm{Der}(kP/Q) = \mathrm{HH}^1(kP/Q) \cong \sum_{\substack{i_1,\dots,i_r\geq 0,\\ i_1+\dots+i_r=1}} \prod_{j=1}^{n} \mathrm{HH}^{i_j}(kC_p) = \sum_{\substack{i_1,\dots,i_r\geq 0,\\ i_1+\dots+i_r=1}} \prod_{j=1}^{n} \mathrm{HH}^{i_j}(k[x_j]/x_j^p) \tag{4.5.2}$$

Let $\hat{f} = f \otimes x_2 \otimes \dots \otimes x_r \in \mathrm{Der}(k[x_1]/x_1^{p^{n_1}}) \otimes k[x]/x_2^{p^{n_2}} \otimes \dots \otimes k[x]/x_r^{p^{n_r}}$ where $f(x_1) = x_1$. Then the image of $\hat{f}$ under φ is non-zero. Similarly, if we let $\hat{g} = g \otimes x_2 \otimes \dots \otimes x_r \in \mathrm{Der}(k[x_1]/x_1^p) \otimes k[x_2]/x_2^p \otimes \dots \otimes k[x_r]/x_r^p$, where $g(x_1) = x_1^p$, then the image of $\hat{g}$ under φ is zero. Hence $\mathrm{HH}^1(kP)$ is not simple. Note that the order of P cannot be 2. In fact in this case we have $kP = kC_2 \cong k[x]/x^2$ and $\mathrm{Der}(k[x]/x^2)$ is not simple. The result follows. $\square$

Remark 4.5.5. Theorem 4.1.1 implies that the hypothesis on P being abelian is not necessary in the statement of Proposition 4.5.4.

4.6 Further results on derivations

Let B be a block having a unique isomorphism class of simple modules and let A be the basic algebra of B. In order to exploit the hypothesis of HH^1 being simple in the statement of Theorem 4.1.1, we consider Lie algebra homomorphisms into the HH^1 of subalgebras and quotients of A.

Lemma 4.6.1. *Let A be a finite-dimensional k-algebra and f be a derivation on A. Then f sends $Z(A)$ to $Z(A)$, and the map sending f to the induced derivation on $Z(A)$ induces a Lie algebra homomorphism* $\mathrm{HH}^1(A) \to \mathrm{HH}^1(Z(A))$.

Proof. Let $z \in Z(A)$. For any $a \in A$ we have $az = za$, hence $f(az) = f(a)z + af(z) = f(z)a + zf(a) = f(za)$. Comparing the two expressions, using $zf(a) = f(a)z$, yields $af(z) = f(z)a$, and hence $f(z) \in Z(A)$. The result follows. □

Lemma 4.6.2. *Let A be a local symmetric k-algebra such that $J(Z(A))A \neq J(A)$. Then the canonical Lie algebra homomorphism $\mathrm{HH}^1(A) \to \mathrm{HH}^1(Z(A))$ is not injective.*

Proof. Since $J(Z(A))A < J(A)$, it follows from Nakayama's lemma that $J(Z(A))A + J(A)^2 < J(A)$. Thus there is a nonzero linear endomorphism f of A which vanishes on $J(Z(A))A + J(A)^2$ and on $k \cdot 1_A$, with image contained in $\mathrm{soc}(A)$. In particular, f vanishes on $Z(A) = k \cdot 1_A + J(Z(A))$. By Corollary 4.5.2, the map f is an outer derivation on A. Thus the class of f in $\mathrm{HH}^1(A)$ is nonzero, and its image in $\mathrm{HH}^1(Z(A))$ is zero, whence the result. □

Lemma 4.6.3. *Let A be a local symmetric k-algebra and let f be a derivation on A such that $Z(A) \subseteq \ker(f)$. Then $f(J(A)) \subseteq J(A)$.*

Proof. Since A is local and symmetric, we have $\mathrm{soc}(A) \subseteq Z(A)$, and $J(A)$ is the annihilator of $\mathrm{soc}(A)$. Let $x \in J(A)$ and $y \in \mathrm{soc}(A)$. Then $xy = 0$ and hence $0 = f(xy) = f(x)y + xf(y)$. Since $y \in \mathrm{soc}(A) \subseteq Z(A)$, it follows that $f(y) = 0$, hence $f(x)y = 0$. This shows that $f(x)$ annihilates $\mathrm{soc}(A)$, and hence that $f(x) \in J(A)$. □

Lemma 4.6.4. *Let A be a finite-dimensional k-algebra and J be an ideal of A.*

(i) Let f be a derivation on A such that $f(J) \subseteq J$. Then $f(J^n) \subseteq J^n$ for any positive integer n.

(ii) Let f, g be derivations on A and let m, n be positive integers such that $f(J) \subseteq J^m$ and $g(J) \subseteq J^n$. Then $[f, g](J) \subseteq J^{m+n-1}$.

Proof. In order to prove (i), we argue by induction over n. For $n = 1$ there is nothing to prove. If $n > 1$, then $f(J^n) \subseteq f(J)J^{n-1} + Jf(J^{n-1})$. Both terms are in J^n, the first by the assumptions, and the second by the induction hypothesis $f(J^{n-1}) \subseteq J^{n-1}$. Let $y \in J$. Then $[f, g](y) = f(g(y)) - g(f(y))$. We have $g(y) \in J^n$; that is, $g(y)$ is a sum of products of n elements in J. Applying f to any such product shows that the image is in J^{m+n-1}. A similar argument applied to $g(f(y))$ implies (ii). $\square$

Proposition 4.6.5. *Let A be a finite-dimensional k-algebra. For any positive integer m denote by $\mathrm{Der}_{(m)}(A)$ the k-subspace of derivations f on A satisfying $f(J(A)) \subseteq J(A)^m$.*

(i) For any two positive integers m and n we have $[\mathrm{Der}_{(m)}(A), \mathrm{Der}_{(n)}(A)] \subseteq \mathrm{Der}_{(m+n-1)}(A)$.

(ii) The space $\mathrm{Der}_{(1)}(A)$ is a Lie subalgebra of $\mathrm{Der}(A)$.

(iii) For any positive integer m, the space $\mathrm{Der}_{(m)}(A)$ is an ideal in $\mathrm{Der}_{(1)}(A)$.

(iv) Let A be a local finite dimensional k-algebra. The space $\mathrm{Der}_{(2)}(A)$ is a nilpotent Lie subalgebra of $\mathrm{Der}(A)$.

Proof. Statement (i) follows from Lemma 4.6.4 (ii). The statements (ii) and (iii) are immediate consequences of (i). In order to prove statement (iv) we observe that derivations on a local algebra are non-zero only on the Jacobson radical. Then the proof follows from (i) and the fact that $J(A)$ is nilpotent. $\square$

4.7 Proofs of main Theorems

Proof of Theorem 4.1.1. Let G be a finite group and B a block of kG. Suppose that B has a single isomorphism class of simple modules. If B is nilpotent and P a defect group of B, then by [30], B is Morita equivalent to kP, and hence there is a Lie algebra isomorphism

$\mathrm{HH}^1(B) \cong \mathrm{HH}^1(kP)$. Thus if B is nilpotent with an elementary abelian defect group P of order at least 3, then $\mathrm{HH}^1(B)$ is a simple Lie algebra by Theorem 4.5.3.

Suppose conversely that $\mathrm{HH}^1(B)$ is a simple Lie algebra. If $J(B) = J(Z(B))B$, then B is nilpotent with an abelian defect group P by Theorem 4.2.21. As before, we have $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(kP)$, and hence Proposition 4.5.4 implies that P is elementary abelian of order at least 3.

Let e be an idempotent in B such that the algebra $A = eBe$ is a basic algebra of B. Suppose that $J(Z(B))B \neq J(B)$. Then $J(Z(A))A \neq J(A)$. We give a proof by contraposition: assume $J(Z(A))A = J(A)$. By Lemma 4.3.3 it follows that $BeB = B$. By Proposition 4.3.7 we have that A and B are Morita equivalent hence they have isomorphic centers, that is, $Z(A) \cong Z(B)e$. It is easy to prove that $J(A) = eJ(B)e$ and $J(B) = BJ(A)B$. Consequently, $J(B) = BJ(A)B = BJ(Z(A))AB \cong BJ(Z(B)e)AB = BJ(Z(B))eAB = J(Z(B))BeBeB = J(Z(B))B$. Hence $J(Z(B))B = J(B)$.

Since B has a single isomorphism class of simple modules and since the set of representatives of the isomorphism classes of simple B-modules is in bijection with the conjugacy classes of primitive idempotents we have that there is just one conjugacy class of primitive idempotents. By Corollary 4.3.6 and Wedderburn's Theorem we have that A is local. By Proposition 2.1.11 we have that A is symmetric. Thus $\mathrm{soc}(A)$ is the unique minimal ideal of A. We have $J(A)^2 \neq \{0\}$. Indeed, if $J(A)^2 = \{0\}$, then $\mathrm{soc}(A)$ contains $J(A)$, and hence $J(A)$ has dimension 1, implying that A has dimension 2 by Wedderburn-Malcev's theorem and by the fact that A is local. By Proposition 4.3.9 A and B have the same Cartan matrix C. Since A is basic, local and has dimension 2, then $A = k[x]/x^2$. Consequently the Cartan matrix C has one single entry which is equal to 2. By Theorem 4.3.11 and Proposition 4.3.12 it follows that B is a block with defect group of order 2.

Then the number of simple modules equals the order of the inertial quotient E (see Theorem 6.5.5 in [4] for example), hence by Theorem 4.2.21 and by the fact that B has a unique simple module then B is nilpotent. Consequently B Morita equivalent to kC_2. Hence $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(kC_2)$ which is not a simple Lie algebra, a contradiction. Thus $J(A)^2 \neq \{0\}$, and hence $\mathrm{soc}(A) \subseteq J(A)^2$. By Lemma 4.6.2, the canonical Lie algebra homomorphism $\mathrm{HH}^1(A) \to \mathrm{HH}^1(Z(A))$ is not injective. Since $\mathrm{HH}^1(A)$ is a simple Lie algebra, it follows that this homomorphism is zero. In other words, every derivation on A has $Z(A)$ in its kernel. It follows from Lemma 4.6.3 that every derivation on A sends $J(A)$ to $J(A)$. Thus, by Lemma 4.6.4, every derivation on A sends $J(A)^2$ to $J(A)^2$. This implies that the canonical surjection $A \to A/J(A)^2$ induces a Lie algebra homomorphism $\mathrm{HH}^1(A) \to \mathrm{HH}^1(A/J(A)^2)$. Let a, b be two elements in A. Using the fact that A is split local, there are $r, s \in J(A)$ and $\lambda, \mu \in k$ such that $a = \lambda \cdot 1_A + r$ and $b = \mu \cdot 1_A + s$. Then $ab = \lambda\mu \cdot 1_A + \mu \cdot r + \lambda \cdot s + rs$. Since $rs \in J(A)^2$, the algebra $A/J(A)^2$ is commutative. Since $J(A)^2$ contains $\mathrm{soc}(A)$, it follows that the kernel of the canonical map $\mathrm{HH}^1(A) \to \mathrm{HH}^1(A/J(A)^2)$ contains the classes of all derivations with image in $\mathrm{soc}(A)$. By Corollary 4.5.2 there are outer derivations with this property. It follows from the simplicity of $\mathrm{HH}^1(A)$ that the canonical map $\mathrm{HH}^1(A) \to \mathrm{HH}^1(A/J(A)^2)$ is zero. Using the fact that $A/J(A)^2$ is commutative, that is $\mathrm{Inn}(A) = \{\mathrm{Id}\}$, this implies that every derivation on A has image in $J(A)^2$. But then Proposition 4.6.5 implies that $\mathrm{Der}(A) = \mathrm{Der}_{(2)}(A)$ is a nilpotent Lie algebra. Thus $\mathrm{HH}^1(A)$ is nilpotent. By Proposition 4.3.7 A and B are Morita equivalent hence $\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$ which contradicts the simplicity of $\mathrm{HH}^1(B)$. □

Proof of Theorem 4.1.2. Denote by A a basic algebra of B. Since B has a unique isomorphism class of simple modules and a nontrivial defect group, it follows by Corollary 4.3.6, Proposition 2.1.11 and Weddernburn's Theorem that A is a local symmetric algebra. Since

B has nontrivial defect, the dimension of A is at least 2, otherwise B is Morita equivalent to a simple algebra which is a contradiction. By Corollary 4.5.2 we have $\dim_k(\mathrm{HH}^1(A))) \geq \dim_k(J(A)/J(A)^2)$. Thus $\dim_k(\mathrm{HH}^1(A)) \geq 1$. Moreover, if $\dim_k(\mathrm{HH}^1(A)) = 1$, then $\dim_k(J(A)/J(A)^2) = 1$. Consequently $J(A)/J(A)^2$ is a simple module which is isomorphic to $A/J(A)$, hence by Theorem 4.4.3 A is a uniserial algebra. By Theorem 4.4.2 we have that A has finite representation type which implies that B is a block with a cyclic defect group P. By assumption B has a unique isomorphism class of simple modules, hence by Theorem 4.2.21 B is a nilpotent block. Note that A and kP are both basic algebras, hence by Theorem 4.3.7 we have $A \cong kP$. We have $\dim_k \mathrm{HH}^1(A) = \dim_k(\mathrm{HH}^1(kP)) = |P|$, a contradiction. The result follows. □

Remark 4.7.1. The hypothesis that B has a single isomorphism class of simple modules is necessary in Theorem 4.1.2; for instance, if P is cyclic of order $p \geq 3$ and if E is the cyclic automorphism group of order $p-1$ of P, then $\mathrm{HH}^1(k(P \rtimes E))$ has dimension one.

Remark 4.7.2. All finite-dimensional algebras in this paper are split thanks to the assumption that k is algebraically closed. It is not hard to see that one could replace this by an assumption requiring k to be a splitting field for the relevant algebras. Lemma 4.6.1 and Lemma 4.6.4 do not require any hypothesis on k.

Bibliography

[1] J. Alperin, M. Broué, *Local Methods in Block Theory,* Ann. Math. **110**, (1979), 143–157.

[2] I.Assem, D.Simson, A.Skowroński, *Elements of the representation theory of associative algebras. Techniques of representation theory*, Cambridge University Press **1** (2006).

[3] M. Auslander, I. Reiten, and S. Smalø, *Representation Theory of Artin Algebras.* Cambridge University Press, (1995).

[4] D. J. Benson, *Representations and Cohomology: Volume 1, Basic Representation Theory of Finite Groups and Associative Algebras.* Cambridge Studies Adv. Math. **30** (1991), Cambridge University Press.

[5] D. J. Benson, *Representations and Cohomology: Volume 2, Cohomology of groups and modules.* Cambridge Studies Adv. Math. **30** (1991), Cambridge University Press.

[6] D.J. Benson, R. Kessar, M. Linckelmann, *On blocks of defect two and one simple module, and Lie algebra structure of* HH^1, Preprint (2016), Arxiv.org/abs/1604.04437.

[7] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées,* Astrisque (1990), 61–92.

[8] M. Broué, *Equivalences of blocks of group algebras. In Finite dimensional algebras and related topics,* Kluwer (1994), 1–26.

[9] M. Broué, L. Puig, *Characters and Local Structure in G-Algebras,* J. Algebra **63** (1980), 306–317.

[10] M. Broué, L. Puig, *A Frobenius theorem for blocks,* Invent. Math. **56** (1980), 117–128.

[11] D. R. Farkas, C. Geiss, E. N. Marcos, *Smooth automorphism group schemes.* Representations of algebras (São Paulo, 1999), 71–89, Lecture Notes in Pure and Appl. Math., **224** Dekker, New York (2002).

[12] M. Gerstenhaber, *The cohomology structure of an associative ring.* Ann. of Math. (2) **78** (1963), 267–288.

[13] M. Gerstenhaber, *On the deformations of rings and algebras.* Ann. Math. **79** (1964), 59–103.

[14] J. Green, *Some remarks on defect groups*, Math. Z. **107** (1968), 133–150.

[15] D. Happel, *On the derived category of a finite-dimensional algebra.* Comment. Math. Helv. **62** (1987), 339–389.

[16] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras,* LMS Lecture Note Series 119, Cambridge University Press 1988.

[17] N. Jacobson, *Restricted Lie algebras of characteristic p,* Trans. Amer. Math. Soc. **50** (1941), no.1, 15–25.

[18] N. Jacobson, *Classes of restricted Lie algebras of characteristic p, II*, Duke Math. J. **10** (1943), 107–121.

[19] B.Keller, *Deriving DG categories,* Ann. Sci. cole Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[20] S. Koenig, Y. Liu, and G. Zhou, *Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander-Reiten Conjecture.* Trans. Amer. Math. Soc. **364** (2012), no. 1, 195–232.

[21] J.Le and G. Zhou, *On the Hochschild cohomology ring of tensor products of algebras*, J. of Pure and Appl. Algebra **218** (2014) 1463–1477.

[22] M. Linckelmann, *Variations sur les blocs á groupes de défaut cyclique, in Séminaire sur les groupes finis* Publications mathmatiques de l'université Paris VII.

[23] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups,* Math. Z. 223 (1996) 87–100.

[24] M. Linckelmann, *Transfer in Hochschild Cohomology of Blocks of Finite Groups*, Algebras and Representation Theory 2 (1999), 107–135.

[25] M. Linckelmann, *Hochschild and block cohomology varieties are isomorphic*, J. London Math. Soc. **81** (2010) 389–411.

[26] M. Linckelmann, *Integrable derivations and stable equivalences of Morita type*, Preprint (2015), Arxiv.org/abs/1506.04676.

[27] M. Linckelmann, L. Rubio y Degrassi, *Block algebras with* HH^1 *a simple Lie algebra*, Preprint Arxiv.org/abs/1611.08556.

[28] H. Matsumura, *Integrable derivations* , Math. J. **87** (1982), 227–245.

[29] T. Okuyama and Y. Tsushima, *Local properties of p-block algebras of finite groups*, Osaka J. Math. **20** (1983), 33–41.

[30] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.

[31] J.Rickard, *Morita theory for derived categories*, J. London Math. Soc. **39** (1989), 436–456.

[32] J.Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **39** (1989), 303–317.

[33] J.Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.

[34] L. Rubio y Degrassi, *Invariance of the restricted p-power map on integrable derivations under stable equivalences*, Journal of Algebra **469** (2017), 288–301.

[35] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic, I.* de Gruyter Expositions in Mathematics **38** (2004), viii 540pp.

[36] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic, II.* de Gruyter Expositions in Mathematics **42** (2009), vi 385pp.

[37] J. Thévenaz, *G-algebras and modular representation theory*, Oxford Mathematical Mono graphs, The Clarendon Press Oxford University Press, New York, (1995), Oxford Science Publications.

[38] T. Tradler, *The BV algebra on hochschild cohomology induced by infinity inner products*, Ann. Inst. Fourier (Grenoble)**58** (2008), no. 7, 2351–2379.

[39] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies Adv. Math. **38** (1994), Cambridge University Press.

[40] A. Zimmermann, *Fine Hochschild invariants of derived categories for symmetric algebras*, Journal of Algebra **308** (2007), 350–367.

[41] A. Zimmermann, *Representation theory: A Homological Algebra Point of View*, Springer, **12** (2014).