Dependent risk modelling in (re)insurance and ruin

by

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of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t,\ c_r = 0.5$.

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Table 2. *Optimal values of M and m, maximizing* $E[R_x^c \mid (T^c > x,\ T^r > x)]$ *or* $E[R_x^r \mid (T^c > x,\ T^r > x)]$ *respectively subject to* $P(T^c > x,\ T^r > x) = 1 - p$, *in the case of dependent claim severities,* $C^{RCl}(F(w_1), \ldots, F(w_k);\ \theta)$ *distributed with* $F \equiv Weibull(2.12,\ 1.14)$ *marginals and* $\theta = 1$, *with* $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t,\ c_r = 0.775$.

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Declaration

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Abstract

The work presented in this dissertation is motivated by the observation that the classical (re)insurance risk modelling assumptions of independent and identically distributed claim amounts, Poisson claim arrivals and premium income accumulating linearly at a certain rate, starting from possibly non-zero initial capital, are often not realistic and violated in practice. There is an abundance of examples in which dependence is observed at various levels of the underlying risk model.

Developing risk models which are more general than the classical one and can successfully incorporate dependence between claim amounts, consecutively arriving at the insurance company, and/or dependence between the claim inter-arrival times, is at the heart of this dissertation. The main objective is to consider such general models and to address the problem of (non-) ruin within a finite-time horizon of an insurance company.

Furthermore, the aim is to consider general risk and performance measures in the context of a risk sharing arrangement such as an excess of loss (XL) reinsurance contract. There are two parties involved in an XL reinsurance contract and their interests are contradictory, as has been first noted by Karl Borch in the 1960s. Therefore, we define joint, between the cedent and the reinsurer, risk and performance measures, both based on the probability of ruin, and show how the latter can be used to optimally set the parameters of an XL reinsurance treaty. Explicit expressions for the proposed risk and performance measures are derived and are used efficiently in numerical illustrations.

Chapter 1

Introduction

The core part of this dissertation is given in chapters 2 - 5 and is based on four pieces of research in the field of ruin theory and reinsurance. The purpose of the current introduction is to give an overview of the structure of the dissertation, to describe briefly the motivation behind the problems considered in each of the four chapters and to provide some background information about the research presented therein.

This work is motivated by the observation that the classical (re)insurance risk modelling assumptions of independent and identically distributed claim amounts, Poisson claim arrivals and premium income accumulating linearly at a certain rate, starting from possibly non-zero initial capital, are often not realistic and do not hold in practice. There is an abundance of examples in which dependence is observed at various levels of the underlying risk model. Developing risk models which are more general than the classical one and can successfully incorporate dependence between claim amounts, consecutively arriving at the insurance company, and/or dependence between the claim inter-arrival times, is at the heart of this dissertation.

The main objective of the research presented in this dissertation is to consider such general models and to address the problem of (non-) ruin within a finite-time horizon of an insurance company. Furthermore, the aim is to consider general risk and performance measures in the context of a risk sharing arrangement such as an excess of loss (XL) reinsurance contract. There are two parties involved in an XL reinsurance contract and their interests are contradictory, as has been first noted by Karl Borch in the 1960s. Therefore, we define joint, between the cedent and the reinsurer, risk and performance measures, both based on the probability of ruin, and illustrate how these measures can be used to optimally set the parameters of an XL reinsurance treaty.

The dissertation is structured as follows.

Chapter 2, entitled *"Finite-time ruin probability in the case of continuous claim severities"*, provides an introduction to the subject of (classical) ruin theory with references to relevant research. Under the classical assumption of i.i.d. claim sizes, we have investigated the use of the method of local moment matching, to discretize the individual claim amount distribution, in combination with known explicit results for the finite probability of (non-) ruin for discrete claim amounts. Further, a more general risk model is introduced, according to which the premium income of an insurance company is represented by any non-decreasing, positive, real-valued function, the claim severities are modelled by any continuous joint distribution, claim arrivals follow a Poisson process and claim severities are independent of the claims arrival process. Under this model, a formula for the finite-time probability of ruin of an insurance company is obtained and its numerical performance is investigated.

In Chapter 3, entitled *"Excess of loss reinsurance under joint survival optimality"*, explicit expressions for the probability of joint survival up to time x of the cedent and the reinsurer, under an excess of loss reinsurance contract with a limiting and a retention level are obtained, under the reasonably general assumptions of the risk model of Chapter 2. By stating appropriate optimality problems, we show that these results can be used to set the limiting and the retention levels in an optimal way with respect to the probability of joint survival. Alternatively, for fixed retention and limiting levels, the results yield an optimal split of the total premium income between the two parties in the excess of loss contract. This methodology is illustrated numerically on several examples of independent and dependent claim severities. The latter are modelled by a copula function. The effect of varying its dependence parameter and the marginals, on the solutions of the optimality problems and the joint survival probability, has also been explored.

In Chapter 4, entitled *"Optimal joint survival reinsurance: an efficient frontier approach"*, the problem of optimal excess of loss reinsurance with a limiting and a retention level is considered. It is demonstrated that this problem can be solved, combining specific risk and performance measures, under the general risk model of Chapter 2. As a performance measure, we define the expected profits at time x of the direct insurer and the reinsurer, given their joint survival up to x, and derive explicit expressions for their numerical evaluation. The probability of joint survival of the direct insurer and the reinsurer up to the finite time horizon x is employed as a risk measure. An efficient frontier type approach to setting the limiting and the retention levels, based on the probability of joint survival considered as a risk measure and on the expected profit given joint survival, considered as a performance measure is introduced. Several optimality problems are defined and their solutions are illustrated numerically on several examples of appropriate claim amount distributions, both for the case of dependent and independent claim severities.

In Chapter 5, entitled *"Reinsurance and ruin under dependence of the claim inter-arrival times"*, a framework which generalizes the risk model considered in Chapters 2, 3 and 4 is introduced. We first consider independent, non-identically Erlang distributed claim inter-arrival times. Then, we allow for modelling dependence between the claim inter-arrival times by assuming that the latter are Erlang distributed with a random shape parameter. Explicit expressions for the probability of joint survival of the cedent and the reinsurer up to time x and the expected profit at x, given joint survival up to x, are obtained in both cases.

Chapter 6 summarizes the conclusions and indicates directions for future research.

The research presented in Chapter 3 has been published recently in the *Insurance: Mathematics and Economics* journal (see Kaishev and Dimitrova 2006). This work has also been presented at the 9th International Congress on Insurance: Mathematics and Economics, Quebec city, Canada in 2005.

Results presented in Chapter 4 is based on a paper co-authored with Dr Vladimir Kaishev which is currently under review in the *Journal of Risk and Insurance* (see Dimitrova and Kaishev 2007). This work has been presented at the 4th Conference in Actuarial Science and Finance, Samos, Greece in 2006.

Chapter 2

Finite-time ruin probability in the case of continuous claim severities

Summary

An introduction to the subject of (classical) ruin theory with references to relevant research is provided. Under the classical assumption of i.i.d. claim sizes, we have investigated the use of the method of local moment matching, introduced by Gerber and Jones (1976) and Gerber (1982), in discretizing the individual claim amount distribution, in combination with known explicit results for the finite probability of (non-) ruin for discrete claim amounts, e.g. the formulae of Picard and Lefèvre (1997) and Ignatov and Kaishev (2000). Further, a more general risk model is introduced, according to which the premium income of an insurance company is represented by any non-decreasing, positive, real-valued function, the claim severities are modelled by any continuous joint distribution and claim arrivals follow a Poisson process. Under this model, a formula for the finite-time probability of ruin of an insurance company is obtained and its numerical performance is investigated.

2.1 Introduction

The business activity of an insurance company is characterized by two major cash flows. One incoming flow of premiums, charged to policyholders, and a second one, outgoing and comprised by the claim amounts, paid by the company in the case of occurrence of insurance events. Since, in most cases premiums are charged on preliminary known days and since, usually the number of policies in the insurance portfolios is considerable, it is natural to assume that the premium income of the company can be modelled by a positive, real-valued, deterministic function. As for the claims paid by the company, it is realistic to assume that such payments occur at random moments in time and their sizes are not known in advance and hence, they can also be modelled as a certain random quantities.

Thus, an important problem which arises in practice is the problem of appropriately matching the aggregate premium income to the aggregate flow of claim payments. If these two cash flows are not appropriately matched, there may be a high chance that the company becomes insolvent. Insolvency is of course a broader concept. It has recently been at the focus of the attention of Regulators of Insurance and Financial businesses, in connection with their efforts to introduce a common platform of methods for estimating risk capital requirements based on Basel II and Solvency II, (see e.g. Basel Committee on Banking Supervision 2006, and Linder and Ronkainen 2004). For the purpose of this dissertation, we will restrict our attention to considering the so-called technical ruin of an insurance company. Technical ruin, occurs when the company's outgoing flow of aggregate claim payments exceeds its incoming aggregate premium income. The actuarial literature devoted to investigating and modelling technical ruin is vast and its importance in developing systems of early warning for possible insolvency has been widely recognized. Recently, the probability of ruin has also been used as a risk measure in determining capital requirements for mitigating operational risk (see Embrechts, Kaufmann and Samorodnitsky 2004, Kaishev, Dimitrova and Ignatov 2007), and in estimation of the risk solvency margin in the spirit of Solvency II (see Loisel, Mazza and Rullière 2007).

If we denote by $T > 0$ the moment of ruin, and by $h(t)$ and S_t the total amount of premiums and claims up to time $t \geq 0$ respectively, we can illustrate the technical ruin by the following Fig. 1.

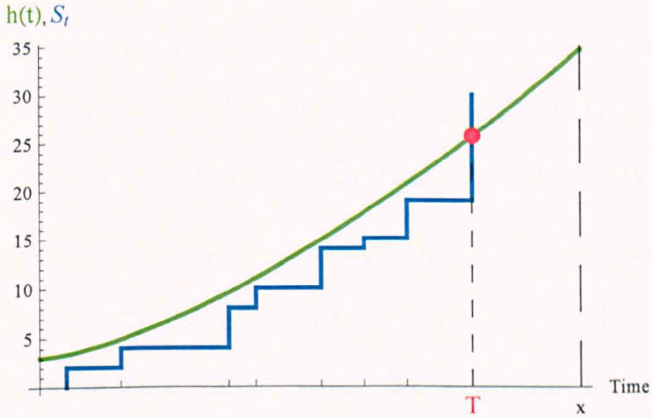

Fig. 1. The total premium income function $h(t)$, the aggregate claim amount process S_t, and the moment of ruin, T.

From a practical point of view, the probability that (technical) ruin of an insurance company will (not) occur up to a finite moment of time x is more interesting than the case of infinite time horizon. The time interval $[0, x]$ can be viewed as the management planning horizon and the finite-time probability of ruin within $[0, x]$ can be used as a risk measure and its values can be regularly observed. Thus, changes in its level may trigger different managerial decisions, for example increase of the premiums charged by the company. Since, the planning horizon may be thought of as the sum of the time until the risk business is found to behave 'badly', the time until the management reacts and the time until a decision of a premium increase takes effect, it may be natural to regard x equal to four or five years as reasonable (see Burnecki, Mišta and Weron 2005, and Grandell 1991).

So, clearly, it is important for an insurance company to be able to assess the probability that ruin will occur (or, respectively, will not occur) up to any a priori defined time horizon, x. This problem has been at the focus of the attention of a large number of academic and applied actuaries, and mathematicians since the beginning of the last century. Contributions to the subject have been made by F. Lundberg (1903), O. Lundberg (1948), Cramer (1955), Seal (1969, 1978), Wikstad (1971), Gerber (1979), Bühlmann (1982), De Vylder and Goovaerts (1988), Dickson and Waters (1991), Dickson, Egídio dos Reis and Waters (1995), Willmot (1993), Grandell (1991), Picard and Lefèvre (1997), Asmussen (1984, 1987, 2000), Ignatov and Kaishev (2000, 2006), Nyrhinen (2001), Paulsen (2002), Albrecher and Boxma (2004), Pitts and Politis (2007), to mention only a few.

However, it has to be noted that a vast proportion of the papers and monographs devoted to the evaluation of the probability of ruin have been restricted to the classical risk model when the premium income is modelled by a positive linear function, the claims are assumed independent of each other and identically distributed, and the time horizon has been considered infinity. In spite of the large number of research performed in this area, there are very few explicit ruin probability formulae (e.g. see Seal 1969, De Vylder and Goovaerts 1999, Picard and Lefèvre 1997, Asmussen 2000, Ignatov, Kaishev and Krachunov 2001, 2004) and not very many are the efficient numerical procedures to calculate ruin probabilities, developed in the actuarial literature. In this connection, we will mention Wikstad (1971), Seal (1978), De Vylder and Goovaerts (1988), Dickson and Waters (1991), Kling and Goovaerts (1991), De Vylder (1999), Barndorff-Nielsen and Schmidli (1995) and Rullière and Loisel (2004).

The first objective of the present work (see section 2.2) is to review the literature and to assess the numerical efficiency of some of the methods for the evaluation of the finite-time ruin probability for the case of continuous claim severities developed in the literature. Further, our aim is to propose alternative methods for numerical evaluation of finite-time ruin probabilities in the case of the more general risk model of an arbitrary, non-decreasing, positive, real function, modeling the premium income, claim severities following any continuous joint distribution (i.e. both dependent or independent) and claim arrivals according to a Poisson point process.

Section 2.3 is devoted to numerical methods in the case when the claims are assumed independent, identically distributed, having an arbitrary continuous distribution. The method proposed therein is to discretize the density function of the claim amounts by matching its first $p \geq 1$ moments to the corresponding p moments of the resulting discrete distribution and then to apply directly the finite-horizon ruin probability formula of Picard and Lefèvre (1997) or of Ignatov and Kaishev (2000). *Mathematica* modules implementing the proposed algorithm have been developed and used to produce numerical and graphical illustrations. The proposed procedure is compared numerically with the methods of De Vylder and Goovaerts (1988), Dickson and Waters (1991), Kling and Goovaerts (1991), Barndorff-Nielsen and Schmidli (1995) and De Vylder (1999), is performed.

In Section 2.4, we look at new representations and numerical procedures for the evaluation of finite-time ruin probability in the case of dependent, continuous claim severities. A new explicit expression is obtained, which can be viewed as a continuous version of the formula of Ignatov and Kaishev (2000). Based on it, an alternative method for calculating ruin probabilities is given and compared with the existing competitors.

2.2 An overview of methods for evaluation of finite-time ruin probabilities

2.2.1 The basic model

We will consider the following reasonable general finite-time ruin probability model. Denote by R_t, $t \geq 0$, the risk reserve process

$$R_t = h(t) - S_t,$$

where $h(t)$ is a positive, non-decreasing, real function defined on $\mathbb{R}_+$, representing the total premium income of an insurance company up to time t and

$$S_t = \sum_{i=1}^{N_t} Y_i,$$

is the aggregate claims amount at time t. The consecutive individual claims $Y_1, Y_2, \ldots$ arrive at the insurance company at random moments in time $T_1, T_2, \ldots$ with inter-occurrence times, $\tau_1 = T_1$, $\tau_2 = T_2 - T_1, \ldots$, assumed exponentially distributed r.v.s. with parameter $\lambda > 0$, i.e. it is assumed that the number of claims up to time t is represented by a homogeneous Poisson process, N_t, with parameter λ. The claim severities $Y_1, Y_2, \ldots$are assumed to be independent of N_t. The function $h(t)$ is such that $\lim_{t\to\infty} h(t) = \infty$. The latter is required so that the insurance company will not get ruined with probability 1 within an infinite time horizon. The function $h(t)$ may be continuous or discontinuous, in which case $h^{-1}(y) = \inf\{z : h(z) \geq y\}$. The time of ruin, T, is defined as

$$T := \inf\{t : t \geq 0,\ R_t < 0\}$$

and we will be interested in the probability of non-ruin, $P(T > x)$, in a finite time interval $[0,\ x], x > 0$.

Let us note that in the classical setting we have $h(t) = u + c\,t$, where $u \geq 0$ is the initial reserve and $c > 0$ is the premium income rate, and the consecutive individual claim amounts $Y_1, Y_2, \ldots$ are assumed to be independent and identically distributed (i.i.d.) random variables.

The probability of ruin in the classical context is traditionally denoted as $\psi(u, x)$, and defined as

$$\psi(u, x) = P(R_t < 0,\ 0 \leq t \leq x) = P(T < x),$$

and the non-ruin probability is respectively

$\phi(u, x) = 1 - \psi(u, x) = \Pr(T > x)$.

A significant amount of research has been devoted to the study of infinite horizon probability of ruin

$\psi(u) = \Pr(R_t < 0,\ t \geq 0)$.

In this thesis, we will be interested in methods and explicit expressions for calculating the probability of ruin within a finite-time interval as a more practically appealing risk measure.

2.2.2 Overview of existing methods for evaluation of $P(T > x)$

Since, explicit closed-form expressions for the finite horizon probability of ruin are difficult to obtain in the general case, approximate solutions have been looked for. Some of the important results in this direction of research are those of Thorin and Wikstad (1973, 1977), Seal (1974), De Vylder and Goovaerts (1988), Dickson and Waters (1991), Kling and Goovaerts (1991), Barndorff-Nielsen and Schmidli (1995), De Vylder (1999).

Wikstad (1971) was one of the first to give values for the finite-time ruin probabilities for continuous i.i.d. claim amounts. He based his numerical algorithm on the explicit formula of Thorin (1971) for (mixture of) exponential claim severities and his ideas have been used later by other authors using inversion of Fourier/Laplace transform when solving the ruin problem.

A very popular method for calculating the ruin probability is the model in which time is discretized and approximate values of the unknown probability are obtained fairly easily. De Vylder and Goovaerts (1988) derived a recursive approximation method which involves discretizing and re-scaling the risk process. Dickson and Waters (1991) improved the algorithm of De Vylder and Goovaerts (1988) by introducing an arbitrary discretization span, $\beta > 0$, and an alternative way of re-scaling the time unit.

Let us briefly describe the method of De Vylder and Goovaerts (1988) which is developed within the classical ruin theory framework. Namely, the counting process, N_t, is Poisson with parameter $\lambda > 0$. The claim severities $Y_1, Y_2, \ldots$ are i.i.d. and independent of N_t. The risk process is

$$R_t = u + c\,t - S_t,$$

where $c > 0$ is the premium income rate per unit of time such that

$$c = \lambda\,\mu\,(1 + \eta),$$

where $\eta \geq 0$ is the so called security loading factor and $\mu = E(Y_k)$, $\lambda = E(\tau_i)$. The probability of non-ruin in $[0, x]$ which corresponds to initial risk reserve u is

$$\phi_x(u) = P(\forall\, s \leq x : R_s(u) \geq 0).$$

where we use the alternative notation, $R_s(u)$, for the risk process corresponding to the initial reserve u. The authors propose to discretize the time as follows. For $n = 1, 2, \ldots$, let

$$\phi_{1,n}(u) = P(R_1(u) \geq 0,\ R_2(u) \geq 0,\ \ldots,\ R_n(u) \geq 0)$$

be the probability of non-ruin at the end of each of the first n years. Obviously, we have

$$\phi_{1,n}(u) = P(Y_1 \leq u + c,\ Y_1 + Y_2 \leq u + 2\,c,\ \ldots,\ Y_1 + Y_2 + \ldots + Y_n \leq u + n\,c) \tag{2.1}$$

and

$$\phi_{1,n}(u - c) \leq \phi_n(u) \leq \phi_{1,n}(u), \tag{2.2}$$

where $\phi_{1,n}(u - c) = 0$ if $u - c < 0$.

Taking into account the above inequalities (2.2), De Vylder and Goovaerts (1988) use the approximation

$$\phi_n(u) \cong \frac{1}{2}\left(\phi_{1,n}(u - c) + \phi_{1,n}(u)\right). \tag{2.3}$$

The quantity $\phi_{1,n}(.)$ involved in (2.3) can be evaluated as follows. Let $G(y)$ be the distribution function of Y_i, $i = 1, 2, \ldots$. Then, from (2.1) it is clear that

$$\phi_{1,1}(u) = G(u + c)$$

$$\phi_{1,n}(u) = \int_0^{u+c} \phi_{1,n-1}(u + c - y)\, dG(y), \; (n \geq 2). \tag{2.4}$$

If the claim severities are assumed to have a discrete distribution, we have

$$\phi_{1,n}(u) = \sum_{j=0}^{u+c} \phi_{1,n-1}(u + c - j)\, P_j, \tag{2.5}$$

where $P_j = P(Y_i = j)$.

The method presented above, for the case of continuous claim amounts, has been implemented in *Mathematica* following the recursive formula (2.4) and estimate (2.3). In their article, De Vylder and Goovaerts (1988) propose first to discretize the underlying continuous distribution following a certain algorithm (see equation (2.9)) and then to apply formula (2.5) and the estimate (2.3) in order to avoid the integration involved in evaluating (2.4). In Tables 1 and 2 these two approaches are compared with each other and also with the results obtained by Wikstad (1971), which have four correct digits after the decimal point.

Table 1. $P(T < x)$ for different values of the premium income rate ($c = 1 + \eta$). $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 0$, $x = 1$.

η	Wikstad (1971)	De Vylder and Goovaerts (1988), (2.5)	De Vylder and Goovaerts (1988), (2.4)
0.05	0.4698	0.66497	0.674969
0.10	0.4634	0.65989	0.666436
0.15	0.4572	0.65495	0.658318
0.20	0.4510	0.65015	0.650597
0.25	0.4450	0.64549	0.643252
0.30	0.4391	0.64096	0.636266
1.00	0.3662	0.58982	0.567668

Table 2. $P(T < x)$ for different values of the premium income rate ($c = 1 + \eta$). $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 1$, $x = 1$.

η	Wikstad (1971)	De Vylder and Goovaerts (1988), (2.5)	De Vylder and Goovaerts (1988), (2.4)
0.05	0.2420	0.58693	0.564367
0.10	0.2381	0.58412	0.561228
0.15	0.2342	0.58141	0.558242
0.20	0.2305	0.57878	0.555402
0.25	0.2268	0.57622	0.552700
0.30	0.2232	0.57374	0.550129
1.00	0.1800	0.54615	0.524894

As can be seen from Tables 1 and 2, the approximations, base both on (2.4) and (2.5), have very low accuracy for certain (small) values of the initial reserve u and the time horizon x, and hence, are not useful in such cases. A more extensive comparison for different choices of u, x and η is given in Table 1 in De Vylder and Goovaerts (1988). The authors provide no estimates of the error of approximation.

In Barndorff-Nielsen and Schmidli (1995) a saddlepoint technique is applied to obtain approximations of the probability of ruin in a finite-time interval in the classical risk model. This method is reasonably accurate (as it can be seen from Table 6 in section 2.3.1) but it is more difficult to use because it requires many preliminarily calculations and verifications, and besides that, it is not valid for arbitrary continuous distribution of the claim amounts.

De Vylder and Goovaerts (1999) obtain the following explicit analytic expression for the finite-time ruin probability in the classical risk model, where Y_1, Y_2, ... are assumed Exp(1) distributed

$$\phi(u,\ x) =$$

$$1 - e^{-u-\lambda x} \sum_{i\geq 1} \sum_{0\leq k\leq i-1} q^i (c\,x)^{i/} u^{k/} + e^{-u-\lambda x} \sum_{i\geq 1} \sum_{0\leq k\leq i-1} q^i (u + c\,x)^{i/} u^{k/} -$$

$$e^{-u-\lambda x - c\,x} \sum_{i\geq 1} \sum_{0\leq k\leq i-1} q^i (u + c\,x)^{i/} (u + c\,x)^{k/} -$$

$$e^{-u-\lambda x} \sum_{i\geq 1} \sum_{0\leq k\leq i} q^i (u + c\,x)^{i-1/} u^{k/} +$$

$$e^{-u-\lambda x - c\,x} \sum_{i\geq 1} \sum_{0\leq k\leq i} q^i (u + c\,x)^{i-1/} (u + c\,x)^{k/} +$$

$$e^{-u-\lambda x} \sum_{i\geq 1} \sum_{j\geq 1} \sum_{0\leq k\leq j-1} \sum_{0\leq n\leq j} q^{i+j} (-1)^i \binom{i+n+k}{i} \binom{n+k}{n} (c\,x)^{j-n/} u^{2\,i+n+k/} - \tag{2.6}$$

$$e^{-u-\lambda x - c\,x}$$

$$\sum_{i\geq 1} \sum_{j\geq 1} \sum_{0\leq k\leq j-1} \sum_{0\leq n\leq j+k} q^{i+j} (-1)^i \binom{j+k}{k} \binom{i+n}{i} (c\,x)^{j+k-n/} u^{2\,i+n/} - e^{-u-\lambda x}$$

$$\sum_{i\geq 1} \sum_{j\geq 1} \sum_{0\leq k\leq j} \sum_{0\leq n\leq j-1} q^{i+j} (-1)^i \binom{i+n+k}{i} \binom{n+k}{n} (c\,x)^{j-1-n/} u^{2\,i+n+k/} +$$

$$e^{-u-\lambda x - c\,x}$$

$$\sum_{i\geq 1} \sum_{j\geq 1} \sum_{0\leq k\leq j} \sum_{0\leq n\leq j-1+k} q^{i+j} (-1)^i \binom{j-1+k}{i} \binom{i+n}{i} (c\,x)^{j-1+k-n/} u^{2\,i+n/}$$

In (2.6) the notation $a^{k/} = a^k \big/ k!$ and $q = \frac{1}{1+\eta}$ is used.

In their paper, De Vylder and Goovaerts (1999) give a couple of calculated values with precision up to twelve digits after the decimal point. Clearly, the implementation of (2.6) is hindered by some serious difficulties since (2.6) contains many infinite sums and requires the calculation of binomial coefficients for large values i. Furthermore, explicit formula (2.6) is valid in the case of exponentially distributed claim amounts with parameter $\alpha = 1$ only. Formula (2.6) is derived from an integral equation, but in the general case of arbitrary continuous distribution of $Y_1, Y_2, \ldots$ this equation is difficult to solve analytically as pointed out by the authors.

Two alternative explicit formulae for exponentially distributed claims have been obtained by Seal (1972) and more recently by Asmussen (1984). Both expressions involve numerical integration. However, as noted by Asmussen (2000), Seal's formula may be unstable for large x. Here, we present the explicit result of Asmussen (1984), in the simplified case of $c = 1$ and $Y_i \sim \text{Exp}(1)$,

$$\psi(u, x) = \lambda\, e^{-(1-\lambda)\,u} - \frac{1}{\pi}\int_0^{\pi} \frac{f_1(\theta)\, f_2(\theta)}{f_3(\theta)}\, d\theta$$

where

$$f_1(\theta) = \lambda \exp\left(2\sqrt{\lambda}\; x\cos\theta - (1+\lambda)\,x + u\left(\sqrt{\lambda}\,\cos\theta - 1\right)\right)$$

$$f_2(\theta) = \cos\left(u\sqrt{\lambda}\,\sin\theta\right) - \cos\left(u\sqrt{\lambda}\,\sin\theta + 2\,\theta\right)$$

$$f_3(\theta) = 1 + \lambda - 2\sqrt{\lambda}\,\cos\theta.$$

If $c \neq 1$ and $Y_i \sim \text{Exp}(\alpha)$, one can use the relations $\psi_{\lambda,c}(u, x) = \psi_{\frac{\lambda}{c},1}(u, c\,x)$ and $\psi_{\lambda,\alpha}(u, x) = \psi_{\frac{\lambda}{\alpha},1}(\alpha\,u, \alpha\,x)$.

An alternative approach to calculating ruin probabilities for continuous claim severities is to discretize the assumed continuous distribution and then, apply one of the known formulae which are valid for the discrete case. Following this approach, Kling and Goovaerts (1991) propose the following method for calculating ruin probabilities for continuously distributed claim amounts in a finite time interval.

Let us consider the system of equations (see Seal 1969, and Gerber 1979)

$$\begin{aligned} \phi(0, x) &= \frac{1}{c\,x}\int_0^{c\,x} G(s, x)\, ds \\ \phi(u, x) &= G(u + c\,x, x) - c\int_0^{x} \phi(0, \tau)\, g(u + c\,(x-\tau), x-\tau)\, d\tau, \end{aligned} \tag{2.7}$$

where $G(s, x)$ represents the cumulative distribution function of the aggregate claim amount up to time x and $g(s, x)$ is the corresponding aggregate claim density function if $G(s, x)$ is absolutely continuous, or the frequency function if it is discrete.

Obviously, if $G_h(s, x)$ is a lattice distribution function with span $h \geq 0$, then the integrals appearing in the right-hand side of the equations (2.7) constitute finite summations, i.e.

$$\phi\left(0, \frac{h\,t_0}{c}\right) = \frac{1}{t_0} \sum_{j=0}^{t_0-1} G_h\left(h\,j, \frac{h\,t_0}{c}\right)$$

$$\phi\left(h\,u_0, \frac{h\,t_0}{c}\right) = \tag{2.8}$$

$$G_h\left(h\,(u_0 + t_0 - 1), \frac{h\,t_0}{c}\right) - \sum_{\tau_0=1}^{t_0-1} \phi\left(0, \frac{h\,\tau_0}{c}\right) g_h\left(h\,(u_0 + t_0 - \tau_0), \frac{h\,(t_0 - \tau_0)}{c}\right)$$

where $t_0 = 1, 2, \ldots$.

As $h \to 0$ expressions (2.8) tend to the exact value for the probability of non-ruin given by (2.7), i.e. one can improve the precision of the numerical evaluation of formula (2.8) only by decreasing the span of the discretization. Kling and Goovaerts (1991) used the same discretization method as De Vylder and Goovaerts (1988) to find a lattice distribution. Namely, if $F(y)$ is the generic cumulative distribution function of Y_1, Y_2, ..., the value $F_h(k\,h)$ of the discrete cdf F_h on the interval $[k\,h, (k+1)\,h)$ is fixed in such a way that

$$h\,F_h(k\,h) = \int_{k\,h}^{(k+1)\,h} F(y)\,dy,$$

i.e.,

$$p_0 + p_{1\,h} + \ldots + p_{k\,h} = \int_{k\,h}^{(k+1)\,h} F(y)\,dy, \tag{2.9}$$

where $p_{kh} = P(Y_i = kh)$, $k = 0, 1, 2, \ldots$.

Discretization (2.9) is a straightforward guess if one is to decide on how to discretize a continuous distribution and has been used also by De Vylder (1999) and others, as we will see in the next section. However, it has to be noted that by using (2.9) only the first moment of the corresponding continuous and discrete distributions are equated. In the next section, we consider another method of discretization which overcomes this restriction.

A comparison of the methods discussed above is presented in Table 3. We see that among those methods the one of Kling and Goovaerts (1991) is the most accurate and its accuracy can be improved by decreasing further the span h since its behaviour is stable for relatively small values of h. The method of Dickson and Waters (1991), which is in one aspect a refinement and in another aspect a simplification of the method of De Vylder and Goovaerts (1988) (see section 2 of Dickson and Waters 1991), may become unstable as noted by Dickson (2005). For details of how to decrease the span h and how to change the monetary unit and the time unit respectively, can be found in Dickson and Waters (1991).

Table 3. $P(T<x)$ for different values of the premium income rate c and the initial capital u. $Y_i \sim Exp(1)$, $\lambda = 1$, $x = 1$. ($h = 0.05$)*

c / u	Wikstad (1971)	De Vylder & Goovaerts (1988)*, (2.5)	Dickson & Waters (1991)*	Kling & Goovaerts (1991)*	PL_MLMM $h = 0.1$
1.1 / 0	0.4634	0.6599	0.4485	0.4634	0.463383
1.1 / 1	0.2381	0.5841	0.2301	0.2381	0.238160
1.2 / 0	0.4510	0.6502	0.4364	0.4510	0.451004
1.2 / 1	0.2305	0.5788	0.2228	0.2305	0.230589

In the next section 2.3, a method for discretizing the distribution in the case of independent continuous claim severities is presented. In contrast with (2.9), this method allow for matching the moments of the discrete and the continuous distributions of the claim amounts up to an order higher than one (see section

2.3.1.1). Furthermore, based on a numerical study, it will be shown that this method combined with the formula of Picard and Lefèvre (1997), gives faster convergence to the true value of the probability of non-ruin, compared to the methods discussed in this section. For comparison, some preliminary results are presented in the last column of Table 3, abbreviated PL_MLMM.

2.3 Discretizing continuous independent claims

In this section, we introduce two formulae for $P(T > x)$, which are valid when the claim severities are i.i.d. r.v.s., independent of the counting process N_t. The purpose here is to develop appropriate numerical methods which are based on the discretization of the distribution of the claim amounts and on the subsequent use of exact survival probability formulae for discretely distributed claims.

The two formulae are the one of Picard and Lefèvre (1997) and the formula of Ignatov and Kaishev (2000) which give the survival probability for an arbitrary, increasing function of the premium income and an arbitrary, discretely distributed, independent (Picard-Lefèvre, Ignatov-Kaishev) or dependent (Ignatov-Kaishev) claims.

Let us note that De Vylder (1999) propose to use the discretization method (2.9) in combination with the Picard-Lefèvre formula for the calculation of $P(T > x)$. In his paper, De Vylder (1999) discusses the classical case of $h(t) = u + c\,t$ and gives the corresponding special case of the Picard-Lefèvre formula.

2.3.1 The formula of Picard-Lefèvre

Picard and Lefèvre (1997) consider the case when claim severities are modeled by integer valued r.v.s. Y_1, Y_2, ... assumed i.i.d. with distribution function $P(Y_i = j) = P_j$, $j = 1, 2, \ldots$. In this case they obtained the following expression for the finite-time survival probability

$$P(T > x) = e^{-\lambda x} \sum_{i=0}^{\infty} A_i(x)\, I_{\{x \geq v_i\}}, \tag{2.10}$$

where $I_{\{.\}}$ is the indicator of the event $\{.\}$, $v_i = h^{-1}(i)$, $i = 0, 1, 2, \ldots$ and $A_i(x)$, $i = 1, 2, \ldots.$ are the generalized Appell polynomials defined as

$$A_i'(x) = \sum_{j=0}^{i-1} \lambda\, P_j\, A_{i-j}(x), \quad A_0(x) = 1$$

with

$$A_i(v_i) = 0, i > 0.$$

The generalized Appell polynomials, $A_i(x)$, $i = 1, 2, \ldots$ are expressed as

$$A_i(x) = \sum_{r=0}^{i} b_r\, e_{i-r}(x), \text{ where}$$

$$e_i(x) = \sum_{k=0}^{i} \frac{(\lambda x)^k}{k!} q_i^{*k}, i \geq 0\,, e_0 = 1, q_j^{*k} = P(Y_1 + \ldots + Y_k = j), \quad k, j = 0, 1, 2, \ldots,$$

$q_j^{*0} = \delta_{j0}$, $q_j^{*i} = 0$ for $i > j$, and b_r, $r = 0, 1, \ldots, i$ are unknown coefficients.

To find the values of b_r, $r = 0, 1, \ldots, i$, one has to solve the system

$$\sum_{r=0}^{i} b_r\, e_{i-r}(v_i) = \delta_{i0}\,.$$

In his paper, De Vylder (1999) gives a simplified and numerically efficient version of formula (2.10) for the ruin probability in the special case of a linear premium income function, $h(t) = u + c\,t$,

$$\psi(u, x) = 1 -$$

$$e^{-\lambda x} \sum_{0 \leq j \leq u} \left(e_j(c\,x) + e_j(j-u) \times \sum_{u+1 \leq i \leq [u+c\,x]} e_{i-j}(u + c\,x - j)\, \frac{(u + c\,x - i)}{(u + c\,x - j)} \right), \tag{2.11}$$

where $[u + c\,x]$ is the integer part of $u + c\,x$.

The functions e_j occurring in (2.11) are the polynomials with values

$$e_j(\tau) = \sum_{0 \le i \le j} \left[\frac{(\lambda / \tau c)^i}{i!}\right] q_j^{*i} \quad , \; j = 0, 1, 2, \ldots; \; -\infty < \tau < +\infty.$$

There has been a debate in the literature on the numerical properties of formula (2.11). For example, De Vylder (1999) found a critical value for u around 22 above which formula (2.11) behaved unstable. Ignatov, Kaishev and Krachunov (2001) found no critical values for u up to 120 using *Mathematica*. More recently, Rullière and Loisel (2004) explained the inconsistency in opinion by the different software used in implementing (2.11). *Mathematica* is capable of adjusting the number of internal digits used in a calculation and returns an answer with a very-high precision.

Thus, the exact finite-time ruin probability formula (2.11) for i.i.d. integer valued claim amounts and linear premium income function is efficient and stable for numerical evaluations using *Mathematica*. There are other alternatives and as noted by Rullière and Loisel (2004), depending on the parameters involved, e.g. u, x, c, λ etc., different formulas are the most appropriate. For further comments and comparisons, we refer the reader to Rullière and Loisel (2004).

In order to calculate $P(T < x)$ in the case of continuous claim severities, one can discretize the continuous distribution of the claim sizes and then use (2.11).

De Vylder (1999) proved that for any claim size distribution $F(y)$,

$\lim_{h \downarrow 0} \psi_h(u, x) = \psi(u, x) \quad (x > 0, \; \lambda > 0, \; u \ge 0, \; c > 0)$,

where $\psi_h(u, x)$ is the finite-time ruin probability corresponding to the discretized claim size distribution F_h and $\psi(u, x)$ is the ruin probability corresponding to the continuous claim size distribution F.

In the following section, we will present an alternative method for discretizing F, which allows for matching higher moments of the continuous and the discrete distributions for any chosen discretization span h.

2.3.1.1 Discretization by the Method of Local Moment Matching (MLMM)

In this section, we will show how to apply a method of discretization which matches higher order local moments of the continuous and discrete distribution and then use the Picard-Lefèvre formula to obtain an (approximate) value for the ruin probability in the continuous case. We suggest to discretize the density function of the individual claim amounts by the method of local moment matching (MLMM) proposed in Gerber and Jones (1976) and Gerber (1982), (see also Klugman, Panjer and Willmot 1998).

The idea is to construct a discrete distribution whose first $p \geq 1$ moments are matched with, correspondingly, the first p moments of the true continuous distribution of the claims. The method can be described as follows.

Consider an arbitrary interval $[x_k, x_k + p\,h)$, $k = 0, 1, \ldots$, which consists of p sub-intervals $[x_k, x_k + h)$, $[x_k, x_k + 2\,h)$, ..., $[x_k, x_k + p\,h)$. Clearly, the first p moments will be preserved, if masses, $m_0^k, m_1^k, \ldots, m_p^k$, are located at the beginning of each sub-interval, i.e. at the points x_k, $x_k + h$, ..., $x_k + p\,h$, which satisfy the following system of $p + 1$ equations

$$\sum_{j=0}^{p} (x_k + j\,h)^r\, m_j^k = \int_{x_k}^{x_k + p\,h} y^r\, dF(y),\ r = 0, 1, 2, \ldots, p. \tag{2.12}$$

Arranging the successive intervals so that $x_{k+1} = x_k + p\,h$, $k = 0, 1, \ldots$ with $x_0 = 0$, and summing (2.12) over all $k = 0, 1, \ldots$ will guarantee that p moments are preserved for the entire distribution. Furthermore, the probabilities

$$m_0^0,\ m_1^0,\ \ldots,\ m_{p-1}^0,\ m_p^0 + m_0^1,\ m_1^1,\ \ldots,\ m_{p-1}^1,\ m_p^1 + m_0^2,\ \ldots \tag{2.13}$$

add to one.

It is not difficult to prove (see e.g. Klugman, Panjer and Willmot 1998) that the solution of the system (2.12) is given by

$$m_j^k = \int_{x_k}^{x_k+p\,h} \prod_{i\neq j} \frac{y - x_k - i\,h}{(j-i)\,h}\, dF(y),\ j = 0,\ 1,\ \ldots,\ p. \tag{2.14}$$

The densities of Exp(0.1) and Gamma(2, 0.1) distributions and the respective discrete distributions, obtained using MLMM with span $h = 1$ and by matching only the first or the first and second moments, i.e. $p = 1, 2$, are given in Fig. 2 and Fig. 3.

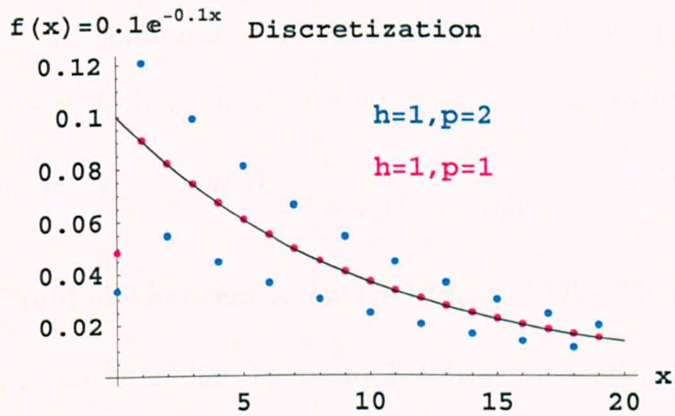

Fig.2 Exp(0.1) distribution and the discrete distributions, obtained through MLMM by matching respectively the first or the first and the second moments.

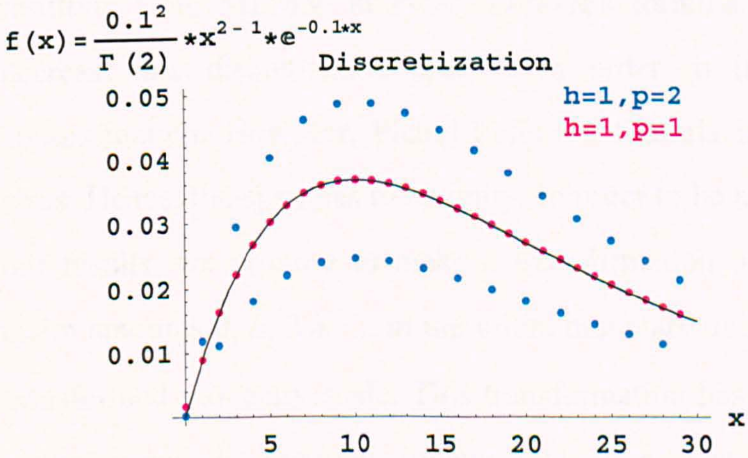

Fig.3 Gamma(2, 0.1) distribution and the discrete distributions, obtained through MLMM by matching respectively the first or the first and the second moments.

Obviously, applying the MLMM method, a discrete integer-valued distribution is obtained with $P(Y_i = 0) > 0$. In order to be able to use it in the formula of Picard and Lefèvre (2.11) one has to make sure that the assumption of having zero claim amounts with a probability zero is satisfied. The following theorem due to De Vylder (1999) indicates an elegant way of overcoming this drawback by modifying the resulting discrete claim size distribution.

Theorem 1. *The Picard-Lefèvre formula (11) can be used in the case of partial claim amounts $Y_1, Y_2, \ldots$ with values $0, 1, 2, \ldots$ and $q_i = P(Y_1 = i) \geq 0$ $(i = 0, 1, 2, \ldots)$. Assuming $q_0 \neq 1$, it is sufficient to replace the probabilities $q_0, q_1, q_2, \ldots$ with $0, \dfrac{q_1}{(1-q_0)}, \dfrac{q_2}{(1-q_0)}, \ldots$ and λ with $\lambda(1-q_0)$.*

Proof of Theorem 1. See De Vylder (1999), Theorem 3.□

Let us note that Theorem 1 follows a well established approach of modifying a distribution known in statistics as *zero-truncation* (see e.g. Johnson, Kotz and Balakrishnan 1997).

There is a second problem related to the direct use of the discrete distribution resulting from MLMM in Picard-Lefèvre's formula. Following MLMM, one can decrease the discretization span h in order to increase the accuracy of the approximation. However, Picard-Lefèvre's formula is valid only for integer claim sizes. Hence, the span has to be unity. In order to be able to increase the precision of our results, we propose to make a transformation of the monetary unit so as the claim amounts $0, h, 2h, \ldots$ in the initial monetary unit will become $0, 1, 2, \ldots$ in the transformed monetary scale. This transformation has to be performed on the initial capital u and the premium income rate c, i.e. u and c from the initial scale will correspondingly become u/h and c/h in the transformed monetary scale. Since most often the premium income rate c satisfy the assumption

$$c = \lambda \mu (1 + \eta),$$

where $\mu = E(Y_i)$ and $\eta > 0$ is the security loading factor, to preserve the required proportion μ has to become μ / h. Obviously λ and η do not depend on the monetary unit.

Table 4. P(T<x) for different values of the premium income rate ($c = 1 + \eta$). $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 10$, $x = 10$, $h = 0.25$.

η	De Vylder and Goovaerts (1999)	PL – MLMM ($p = 2$)	Time, seconds	De Vylder (1999)
0.05	0.0366941	0.0367234	4.66	0.037067
0.10	0.0319030	0.0319261	3.83	0.032238
0.15	0.0277248	0.0277431	4.00	0.028025
0.20	0.0240873	0.0241016	4.13	0.024356
0.25	0.0209252	0.0209364	4.34	0.021165
0.30	0.0181799	0.0181887	4.55	0.018394

Table 5. P(T<x) for different values of the premium income rate ($c = 1 + \eta$). $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 10$, $x = 10$, $h = 0.1$.

η	De Vylder and Goovaerts (1999)	PL – MLMM ($p = 2$)	Time, seconds	De Vylder (1999)
0.05	0.0366941	0.0366989	31.42	0.036754
0.10	0.0319030	0.0319068	31.08	0.031957
0.15	0.0277248	0.0277279	33.06	0.027773
0.20	0.0240873	0.0240898	34.94	0.024130
0.25	0.0209252	0.0209272	37.06	0.020964
0.30	0.0181799	0.0181815	39.28	0.018214

The approach proposed above is implemented in *Mathematica* and the results are shown in Tables 4 and 5 in the case of exponential claims amounts, discretized using MLMM with $p = 2$ and span values of $h = 0.25$ and $h = 0.1$, and combined with the formula of Picard-Lefèvre (2.11). For convenience, we shall abbreviate this approach as PL-MLMM. The ruin probability values calculated by De Vylder (1999) using the discretization method (2.9) with the same span values $h = 0.25$ and

$h = 0.1$, and combined with the formula of Picard-Lefèvre (2.11), are also presented. In Tables 4 and 5, the exact values for $P(T < x)$ calculated by De Vylder and Goovaerts (1999) using (2.6), are also given for comparison.

As can be seen, for one and the same value of the discretization span, PL-MLMM with $p = 2$ produces more accurate values than those obtained by De Vylder (1999) using (2.9). It can also be seen that decreasing the discretization span from $h = 0.25$ to $h = 0.1$ increases the accuracy of the results calculated using the PL-MLMM method but increases significantly the computational time. A serious weakness of both the method proposed by De Vylder (1999) and the PL-MLMM method is that in neither of the cases one can calculate the ruin probability with a predetermined accuracy.

In Table 6, we give ruin probability values for different choices of the time interval x, obtained applying PL-MLMM with $p = 1$ and $h = 0.5$ in the case of gamma distributed claim severities and compare them with the corresponding values, obtained by Barndorff-Nielsen and Schmidli (1995) using the saddlepoint approximation, and values obtained via Monte Carlo (MC) simulations.

Obviously, one can increase the accuracy of the ruin probabilities presented in Table 6 for the PL-MLMM method by decreasing the span h. However, using a relatively rough span of $h = 0.5$ we already get values for the ruin probability with the same accuracy or even better than those reported by Barndorff-Nielsen and Schmidli (1995).

Table 6. P(T<x) for different values of the time interval x. $Y_i \sim Gamma(0.5, 0.5)$, $\lambda = 0.2$, $u = 3.74$, $c = 1$, $h = 0.5$.

x	MC	Barndorff – Nielsen and Schmidli (1995)	PL – MLMM ($p = 1$)
0.5	0.0049	0.0050	0.0048420
1.0	0.0087	0.0090	0.0087822
1.5	0.0119	0.0123	0.0120020
2.0	0.0144	0.0150	0.0146468
2.5	0.0165	0.0172	0.0168312
3.0	0.0184	0.0190	0.0186443
3.5	0.0199	0.0205	0.0201564
4.0	0.0211	0.0217	0.0214231
4.5	0.0222	0.0227	0.0224885
5.0	0.0231	0.0236	0.0233879
5.5	0.0239	0.0243	0.0241497
6.0	0.0245	0.0249	0.0247971
6.5	0.0251	0.0255	0.0253488
7.0	0.0256	0.0259	0.0258203
7.5	0.0260	0.0263	0.0262243
8.0	0.0263	0.0266	0.0265711
8.5	0.0266	0.0269	0.0268696
9.0	0.0268	0.0271	0.0271270
9.5	0.0271	0.0273	0.0273494
10.0	0.0273	0.0275	0.0275418

To summarize, the following comments with respect to the efficiency of the proposed PL-MLMM algorithm can be made.

The PL-MLMM method is valid for any continuous claim severity distribution. It is relatively simple to implement and fast to compute. Hence, it is an attractive alternative. Its major disadvantage is related to the exponential growth of the computational time as the discretization span decreases, in the cases when higher accuracy of the results are required. The computational time may also be prohibitive for high values of the initial capital u and the time horizon x. Our experience also shows that the method of local moment matching (2.12) may become unstable for $p \geq 4$ and $h \leq 0.04$.

2.3.2 The formula of Ignatov-Kaishev

As noted already, most of the methods for evaluation of ruin probabilities consider the classical linear premium income function $h(t) = u + c\,t$. The formula of Picard and Lefèvre (1997) is valid for any increasing function $h(t)$ such that $\lim_{t\to\infty} h(t) = \infty$ and any i.i.d. positive integer-valued claim sizes, but its simplified version (2.11) has been derived under the classical assumption of $h(t) = u + c\,t$. In this section, we present an alternative explicit expression for $P(T > x)$, the formula of Ignatov and Kaishev (2000), which also holds under the general assumptions of non-decreasing $h(t)$ but allows dependence in that it assume integer-valued claim sizes having any joint distribution. As we will see, when the claim amounts are assumed to be independent, not necessarily identical, random variables, the latter formula can be used in combination with the MLMM method to calculate $P(T > x)$ for continuous claim severities, as described in the previous section.

The formula of Ignatov-Kaishev (see Ignatov and Kaishev 2000, and Ignatov, Kaishev and Krachunov 2001) is valid for discrete claim amounts, assumed either dependent or independent, and any non-decreasing real function $h(t)$ modeling the incoming flow of premiums up to time t. It has the following form

$P(T > x) =$

$$e^{-x\lambda}\sum_{k=1}^{n}\left(\sum_{\substack{y_1\ge 1,\dots,y_{k-1}\ge 1\\ y_1+\dots+y_{k-1}\le n-1}} P(Y_1 = y_1,\ \dots,\ Y_{k-1} = y_{k-1};\ Y_k \ge n - y_1 - \dots - y_{k-1}) \right. \tag{2.15}$$

$$\left. \sum_{j=0}^{k-1}(-1)^j b_j(z_1,\ \dots,\ z_j)\,\lambda^j \sum_{m=1}^{k-j-1}\frac{(x\lambda)^m}{m!}\right),$$

where $n = [h(x)] + 1$, $[h(x)]$ is the integer part of $h(x)$, $v_{n-1} \le x < v_n$, $v_i = h^{-1}(i)$, for $i = 0, 1, 2, \dots,$ noting that $0 = v_0 \le v_1 \le v_2 \dots,$ and k is such that $y_1 + \dots + y_{k-1} \le n - 1$, $y_1 + \dots + y_k \ge n$, $(1 \le k \le n)$, $z_l = v_{y_1+\dots+y_l}$, $l = 1, 2, \dots$ and $b_j(z_1,\ \dots,\ z_j)$ is defined recurrently as

$b_j(z_1,\ \dots,\ z_j) =$

$$(-1)^{j+1}\frac{z_j^{\,j}}{j!} + (-1)^{j+2}\frac{z_j^{\,j-1}}{(j-1)!}b_1(z_1) + \dots + (-1)^{j+j}\frac{z_j^{\,1}}{1!}b_{j-1}(z_1,\ \dots,\ z_{j-1}), \tag{2.16}$$

with $b_0 \equiv 1$, $b_1(z_1) = z_1$.

Since, in the case of independent claim severities the probability $P(Y_1 = y_1,\ \dots,\ Y_{k-1} = y_{k-1};\ Y_k \ge n - y_1 - \dots - y_{k-1})$ is in fact a product of the individual probabilities, we can again apply the discretization method MLMM with formula (2.15). We shall abbreviate the latter approach as IK_MLMM

Tables 7 and 8 compare ruin probability values, calculated following the PL-MLMM and the IK_MLMM methods. Our numerical study suggests that the computation time of PL-MLMM and IK-MLMM significantly depends on the size of discretization step, the time interval x and especially on the size of the initial capital u. In particular, the running time for IK-MLMM may increase dramatically for large x and/or u but one has to bear in mind that the Ignatov-Kaishev's formula is more general than the one due to Picard and Lefèvre (1997) and hence, is not 'optimized' for the special case of i.i.d. claim amounts. For small values of x and u the efficiency of the two methods, in terms of time and accuracy is comparable and we can successfully use both modules.

Table 7. P(T<x) for different values of the premium income rate c. $Y_i \sim Exp(0.1)$, $\lambda = 1$, $u = 1$, $x = 0.5$, $h = 0.2$.

c	PL – MLMM $p = 1$	Time, seconds	IK – MLMM	Time, seconds
1.00	0.356980	0.28	0.359827	0.28
1.05	0.356664	0.28	0.359510	0.28
1.10	0.356378	0.27	0.359221	0.30
1.15	0.356116	0.27	0.358959	0.28
1.20	0.355877	0.31	0.358718	0.33

Table 8. P(T<x) for different values of the time interval x. $Y_i \sim Exp(0.1)$, $\lambda = 1$, $u = 0$, $c = 1.1$, $h = 0.1$. *The values obtained with IK_C have at least four correct digits after the decimal point.*

x	PL – MLMM $p = 2$	Time, seconds	IK – MLMM	Time, seconds	IK_C	Time, seconds
0.5	0.385221	0.14	0.389395	0.11	0.385243	1.04
1.0	0.612306	0.30	0.617426	0.33	0.612255	1.15
1.5	0.749636	0.42	0.754590	0.61	0.749644	1.43
2.0	0.834932	0.61	0.839226	1.04	0.834929	1.10
2.5	0.889127	0.75	0.892700	2.53	0.889131	1.15
3.0	0.924329	0.95	0.927220	8.35	0.924324	1.10
3.5	0.947614	1.14	0.949887	17.02	0.947617	1.21
4.0	0.963298	1.36	0.965076	77.56	0.963299	1.10

In the next section, we derive an explicit expression for the probability of ruin in the case of any continuous claim amounts distributions. This expression can be viewed as a 'continuous' generalization of the formula of Ignatov and Kaishev (2000). In the last column of Table 8, we give the corresponding results obtained with this 'continuous version' of Ignatov-Kaishev's formula (abbreviated as IK_C). As can be seen, in some cases it is even more time efficient than the methods based on MLMM for independent claim severities and as it will be shown in the next section, it produces ruin probability values with a preliminary chosen precision. The IK_C values given in Table 8 have four accurate digits after the decimal point.

2.4 Evaluation of ruin probabilities for continuous, dependent claims

Our main objective in this section is to obtain a finite-time ruin probability formula and develop an appropriate numerical method based on this formula in a risk model, where the severities of individual claims may possibly be dependent, i.e. can have any joint continuous distribution, their arrival times follow a Poisson process and $h(t)$ is modelled by a non-decreasing, positive real function. Within this framework, an explicit expression for the probability of ruin has been derived by Ignatov and Kaishev (2004). We use the latter to test and compare the numerical efficiency of the alternative expression which we present here.

2.4.1 An extension of the Ignatov-Kaishev's formula to the continuous case

In what follows, we show how the ruin probability formula (2.15) can be extended to cover the case of any continuous individual claim severities distribution. Further an algorithm which allows to calculate $P(T < x)$ with any preassigned accuracy is developed. We illustrate the algorithm numerically on the example of exponentially and Inverted Dirichlet distributed claims severities.

The Ignatov-Kaishev's formula given by (2.15) has been shown to be exact and numerically efficient in the case when the claims are assumed to have any discrete distribution (see Ignatov, Kaishev, Krachunov 2001, and Rullière and Loisel 2004).

Having this in mind, we state the following theorem where an extension to the case of continuous claims severities $Y_1, Y_2, \ldots, Y_k$ with joint density $f_k(y_1, \ldots, y_k)$ is presented.

Theorem 2. The probability of survival within a finite-time horizon x for continuous claim amounts is given by

$$P(T > x) =$$

$$e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-2}} \int_{h(x)-y_1-\ldots-y_{k-1}}^{\infty} \left(\sum_{j=0}^{k-1} (-1)^j b_j(z_1, \ldots, \right. \tag{2.17}$$

$$\left. z_j) \lambda^j \left(\sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!} \right) \right) f_k(y_1, \ldots, y_k)\, d y_k \ldots d y_2\, d y_1,$$

where $z_j = h^{-1}(y_1 + \ldots + y_j)$, $j = 1, 2, \ldots,$ $f_k(y_1, \ldots, y_k)$ is the probability density function of $Y_1, Y_2, \ldots, Y_k$, and $b_j(z_1, \ldots, z_j)$ is defined recurrently as in (2.16).

Proof of Theorem 2. A straightforward representation of $P(T > x)$ is given by

$$P(T > x) = \sum_{k=0}^{\infty} P(N_x = k)\, P(T > x \mid N_x = k)$$

$$= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} P(T > x \mid \{T_k \le x\} \cap \{T_{k+1} > x\})$$

since $\{N_x = k\} \equiv \{T_k \le x\} \cap \{T_{k+1} > x\}$. Utilizing the fact that

$$\{T > x\} = \bigcap_{j=1}^{\infty} \left[\left\{ h^{-1}(Y_1 + \ldots + Y_j) < T_j \right\} \cup \{x < T_j\} \right]$$

and that (see e.g. Ignatov and Kaishev 2004)

$$\{T > x\} \cap \{T_k \le x\} \cap \{T_{k+1} > x\}$$

$$= \left[\bigcap_{j=1}^{\infty} \left\{ h^{-1}(Y_1 + \ldots + Y_j) < T_j \right\} \cup \{x < T_j\} \right] \cap \{T_k \le x\} \cap \{T_{k+1} > x\}$$

$$= \left[\bigcap_{j=1}^{k}\{h^{-1}(Y_1 + \ldots + Y_j) < T_j\}\right] \cap \{T_k \le x\} \cap \{T_{k+1} > x\}$$

and using the property of conditional probabilities $P(A \mid B) = P(A \cap B \mid B)$, we obtain

$$P(T > x \mid \{T_k \le x\} \cap \{T_{k+1} > x\})$$

$$= P\left(\left[\bigcap_{j=1}^{\infty}\{h^{-1}(Y_1 + \ldots + Y_j) < T_j\} \cup \{x < T_j\}\right] \cap \{T_k \le x\} \cap \{T_{k+1} > x\} \middle| \{T_k \le x\} \cap \{T_{k+1} > x\}\right)$$

$$= P\left(\left[\bigcap_{j=1}^{k}\{h^{-1}(Y_1 + \ldots + Y_j) < T_j\}\right] \cap \{T_k \le x\} \cap \{T_{k+1} > x\} \middle| \{T_k \le x\} \cap \{T_{k+1} > x\}\right)$$

$$= P\left(\left[\bigcap_{j=1}^{k}\{h^{-1}(Y_1 + \ldots + Y_j) < T_j\}\right] \middle| \{T_k \le x\} \cap \{T_{k+1} > x\}\right)$$

Therefore,

$$P(T > x) = \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} P(T > x \mid \{T_k \le x\} \cap \{T_{k+1} > x\})$$

$$= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} P\left(\left[\bigcap_{j=1}^{k}\{h^{-1}(Y_1 + \ldots + Y_j) < T_j\}\right] \middle| \{T_k \le x\} \cap \{T_{k+1} > x\}\right)$$

$$= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} \int \cdots \int_{\mathcal{D}_k}$$

$$P\left(\left[\bigcap_{j=1}^{k} \{z_j < T_j\} \right] \middle| \{T_k \le x\} \cap \{T_{k+1} > x\} \right) f_k(y_1, \ ..., \ y_k)\, d y_k \cdots d y_1$$

where $z_j = h^{-1}(y_1 + ... + y_j)$ and $\mathcal{D}_k \equiv \begin{pmatrix} 0 \le y_1, \ ..., \ 0 \le y_k \\ y_1 + \ ... \ + y_k \le h(x) \end{pmatrix}$.

Now, it can be shown that (see Ignatov and Kaishev 2004)

$$P\left(\left[\bigcap_{j=1}^{k} \{z_j < T_j\} \right] \middle| \{T_k \le x\} \cap \{T_{k+1} > x\} \right) = \frac{k!}{x^k} A_k(x; z_1, \ ..., \ z_k)$$

where $A_k(x; z_1, \ ..., \ z_k)$, $k = 1, 2, \ ...$ are the Appell polynomials defined as $A_0(x) = 1$, $A_k'(x) = A_{k-1}(x)$ and $A_k(z_k) = 0$, $k = 1, 2, \ ...$, hence

$$P(T > x) = e^{-\lambda x} \sum_{k=0}^{\infty} \lambda^k \int \cdots \int_{\mathcal{D}_k} A_k(x; z_1, \ ..., \ z_k)\, f_k(y_1, \ ..., \ y_k)\, d y_k \cdots d y_1$$

$$= e^{-\lambda x} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots$$

$$\int_0^{h(x)-y_1-\ldots-y_{k-2}} \lambda^{k-1} A_{k-1}(x; z_1, \ ..., \ z_k)\, f_{k-1}(y_1, \ ..., \ y_{k-1})\, d y_{k-1} \cdots d y_1$$

Denote

$$C_{k-1} := \sum_{j=0}^{k-1} (-1)^j\, b_j(z_1, \ ..., \ z_j)\, \lambda^j \left(\sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!} \right)$$

From Ignatov and Kaishev (2000), we see that

$$\lambda^{k-1} A_{k-1}(x; z_1, \ ..., \ z_k) = \left(\sum_{j=0}^{k-2} (-1)^j \frac{b_j(z_1, \ ..., \ z_j)\, \lambda^j}{(k-j-1)!} \left((x\lambda)^{k-j-1} - (\lambda z_k)^{k-j-1} \right) \right)$$

$= C_{k-1} - C_{k-2}$

so that

$$P(T > x) = e^{-\lambda x} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots$$

$$\int_0^{h(x)-y_1-\ldots-y_{k-2}} (C_{k-1} - C_{k-2}) f_{k-1}(y_1, \ldots, y_{k-1})\, d y_{k-1} \cdots d y_1 \tag{2.18}$$

Now, it remains to show that expression (2.17) coincides with (2.18).

Expression (2.17) can be re-written as follows

$$P(T > x) = e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots$$

$$\int_0^{h(x)-y_1-\ldots-y_{k-2}} \int_{h(x)-y_1-\ldots-y_{k-1}}^{\infty} C_{k-1} f_k(y_1, \ldots, y_k)\, d y_k \ldots d y_2\, d y_1$$

$$= e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-2}} C_{k-1} \left(\int_0^{\infty} f_k(y_1, \ldots, y_k)\, d y_k - \right.$$

$$\left. \int_0^{h(x)-y_1-\ldots-y_{k-1}} f_k(y_1, \ldots, y_k)\, d y_k \right) d y_{k-1} \ldots d y_2\, d y_1$$

$$= e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-2}} (C_{k-1} - C_{k-2} + C_{k-2})$$

$$f_{k-1}(y_1, \ldots, y_{k-1})\, d y_{k-1} \ldots d y_2\, d y_1$$

$$-e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-1}} C_{k-1} f_k(y_1, \ldots, y_k)\, d y_k \ldots d y_2\, d y_1$$

$$= e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots$$

$$\int_0^{h(x)-y_1-\ldots-y_{k-2}} (C_{k-1} - C_{k-2})\, f_{k-1}(y_1, \ldots, y_{k-1})\, dy_{k-1} \ldots dy_1$$

$$+\, e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-2}} C_{k-2}\, f_{k-1}(y_1, \ldots, y_{k-1})\, dy_{k-1} \ldots dy_1$$

$$-e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-1}} C_{k-1}\, f_k(y_1, \ldots, y_k)\, dy_k \ldots dy_1$$

Noting that

$$e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-2}} C_{k-2}\, f_{k-1}(y_1, \ldots, y_{k-1})\, dy_{k-1} \ldots dy_1$$

$$-e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-1}} C_{k-1}\, f_k(y_1, \ldots, y_k)\, dy_k \ldots dy_1$$

$$= 0$$

$$+e^{-x\lambda} \sum_{k=2}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-2}} C_{k-2}\, f_{k-1}(y_1, \ldots, y_{k-1})\, dy_{k-1} \ldots dy_1$$

$$-e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots \int_0^{h(x)-y_1-\ldots-y_{k-1}} C_{k-1}\, f_k(y_1, \ldots, y_k)\, dy_k \ldots dy_1 = 0$$

for (2.17) we finally obtain

$$P(T > x) = e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-y_1} \cdots$$

$$\int_0^{h(x)-y_1-\ldots-y_{k-2}} (C_{k-1} - C_{k-2})\, f_{k-1}(y_1, \ldots, y_{k-1})\, dy_{k-1} \ldots dy_2\, dy_1$$

which coincides with (2.18) and hence, the proof is completed. □

Expression (2.17) involves infinite summation. Obviously, for numerical calculations it is necessary to truncate the summation with respect to k up to a finite integer n and give some estimate of the truncation error. The following theorem helps in determining the integer n for a given required accuracy $\epsilon > 0$.

Theorem 3. Assume that the individual claim amounts Y_1, Y_2, ... are modelled by i.i.d. random variables. Then,for every $\epsilon > 0$ there exists an integer $n > 0$ such that

$$P(T > x) - P_n(T > x) = P(Y_1 + \ldots + Y_n \le h(x)) \le \epsilon,$$

where

$$P_n(T > x) =$$

$$e^{-x\lambda} \sum_{k=1}^{n} \int \cdots \int_{\mathbb{D}_k} \left(\sum_{j=0}^{k-1} (-1)^j b_j(z_1, \ldots, z_j) \lambda^j \left(\sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!} \right) \right) f_k(y_1, \ldots, y_k) \tag{2.19}$$

$$dy_k \ldots dy_2 \, dy_1$$

and

$$\mathbb{D}_k = \begin{pmatrix} y_1 > 0, \ \ldots, \ y_{k-1} > 0, \ y_k > 0 \\ y_1 + \ldots + y_{k-1} \le h(x) \\ y_1 + \ldots + y_k > h(x) \end{pmatrix} = \begin{pmatrix} 0 \le y_1 < h(x) \\ 0 \le y_2 < h(x) - y_1 \\ \ldots \\ 0 \le y_{k-1} \le h(x) - y_1 - \ldots - y_{k-2} \\ h(x) - y_1 - \ldots - y_{k-1} \le y_k < \infty \end{pmatrix}.$$

Proof of Theorem 3. It is not difficult to see that the difference between (2.17) and (2.19) can be rewritten as

$$P(T > x) - P_n(T > x) =$$

$$\sum_{k=n+1}^{\infty} \int \cdots \int_{\mathbb{D}_k} \left(\sum_{j=0}^{k-1} (-1)^j b_j(z_1, \ldots, z_j) \lambda^j \left(\sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!} e^{-x\lambda} \right) \right) \tag{2.20}$$

$$f_k(y_1, \ldots, y_k) \, dy_k \ldots dy_2 \, dy_1 .$$

We recall that (see Ignatov and Kaishev 2000) the expression

$$\left(\sum_{j=0}^{k-1}(-1)^j\, b_j(z_1, \ldots, z_j)\,\lambda^j\left(\sum_{m=0}^{k-j-1}\frac{(x\,\lambda)^m}{m!}\, e^{-x\lambda}\right)\right)$$

can be viewed as certain conditional probability and hence, we can replace it with unity in (2.20) and obtain the bound

$$P(T > x) - P_n(T > x) \le \sum_{k=n+1}^{\infty} \int_{\mathbb{D}_k} \cdots \int f_k(y_1, \ldots, y_k)\, d y_k \ldots d y_2\, d y_1 . \tag{2.21}$$

Let us now introduce the notation

$$\hat{\mathbb{C}}_n = \begin{pmatrix} y_1 > 0, \; y_2 > 0, \; \ldots \\ y_1 + \ldots + y_n \le h(x) \end{pmatrix}$$

and observe that $\mathbb{D}_{n+1} \subset \hat{\mathbb{C}}_n, \mathbb{D}_{n+2} \subset \hat{\mathbb{C}}_n, \ldots$,

$\bigcup_{k=n+1}^{\infty} \mathbb{D}_k = \hat{\mathbb{C}}_n$ and $\mathbb{D}_k \bigcap \mathbb{D}_{k+1} = \emptyset$.

We can now rewrite (2.21) as

$$\begin{aligned}
&P(T > x) - P_n(T > x) \\
&\le \sum_{k=n+1}^{\infty} \int_{\mathbb{D}_k} \cdots \int f_k(y_1, \ldots, y_k)\, d y_k \ldots d y_2\, d y_1 \\
&= \int_{\bigcup_{k=n+1}^{\infty} \mathbb{D}_k} \cdots \int f_n(y_1, \ldots, y_n)\, d y_n \ldots d y_2\, d y_1 \\
&= \int_{\hat{\mathbb{C}}_n} \cdots \int f_n(y_1, \ldots, y_n)\, d y_n \ldots d y_2\, d y_1 \\
&= P(Y_1 + \ldots + Y_n \le h(x)) .
\end{aligned} \tag{2.22}$$

Further, we have that

$$P(Y_1 \le h(x)) \ge P(Y_1 + Y_2 \le h(x)) \ge \ldots \ge P(Y_1 + \ldots + Y_n \le h(x)) \xrightarrow[n \to \infty]{} 0 , \tag{2.23}$$

since, the more claims occur up to time x, the less probable it is that their sum will remain below $h(x)$. From (2.22) and (2.23) it is not difficult to deduce that there exist n such that

$$P(T > x) - P_n(T > x) \leq P(Y_1 + \ \dots\ + Y_n \leq h(x)) \leq \epsilon$$

which completes the proof of the theorem.□

Based on Theorems 2 and 3, we propose the following numerical method for computing ruin probabilities with any required accuracy.

Step 1. Choose $\epsilon > 0$ and let $k = 1$.

Step 2. Calculate

$$P(Y_1 + \ \dots\ + Y_k \leq h(x)) \tag{2.24}$$

Step 3. If $P(Y_1 + \ \dots\ + Y_k \leq h(x)) \leq \epsilon$ then set $n = k$ and go to step 4. Otherwise, set $k := k + 1$ and go back to step 2.

Step 4. Calculate $P_n(T > x)$ using (2.19).

As an illustration of the above proposed algorithm, let us consider the case of independent, identically Exp(α) distributed claim amounts, i.e. $Y_i \sim Exp(\alpha)$, $i = 1, 2, \dots$. Substituting the exponential density in (2.23) for $n = 1, 2, 3, \dots$ we get

$$P(Y_1 + \dots + Y_n \leq h(x)) = 1 - e^{-\alpha h(x)} \sum_{j=0}^{n-1} \frac{(\alpha h(x))^j}{j!} \tag{2.25}$$

The value n, found following the algorithm with (2.24) replaced by (2.25), should be substituted in (2.19) in order to obtain the non-ruin probability with the required accuracy ϵ.

Our empirical observations show that the ruin probability values obtained with formula (2.19) usually have more accurate digits than those guaranteed by the above algorithm.

In the classical risk model, when $h(t) = u + c\,t$, the 'continuous version' of Ignatov-Kaishev's formula given in Theorem 2 can be simplified for $u = 0$ and the numerical evaluation of finite-time ruin probabilities can be further speeded up by making a change of variables $(y_1,\ ...,\ y_k) \longrightarrow (u_1,\ ...,\ u_k)$ as follows,

$$\begin{array}{lcl} u_1 = y_1 & & y_1 = u_1 \\ u_2 = y_2 + y_1 & & y_2 = u_2 - u_1 \\ \cdots & \longleftrightarrow & \cdots \\ u_k = y_1 + \ldots + y_k & & y_k = u_k - u_{k-1} \end{array} \tag{2.26}$$

The Jacobian of the transformation $|\ J\ | = \det\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 1 \end{pmatrix} = |\ 1\ |$ is non-singular and formula (2.17) becomes

$P(T > x) =$

$$e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^{h(x)} \int_{u_1}^{h(x)} \ldots \int_{u_{k-2}}^{h(x)} \int_{h(x)}^{\infty} \left(\sum_{j=0}^{k-1} (-1)^j\, b_j(z_1,\ ...,\ z_j)\, \lambda^j \left(\sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!} \right) \right) \tag{2.27}$$

$$f_k(u_1,\ u_2 - u_1,\ ...,\ u_k - u_{k-1})\, du_k\, du_{k-1} \ldots du_2\, du_1,$$

where $z_j = h^{-1}(u_j)$ and u_j can be interpreted as the partial sums of the j-th consecutive individual claim amounts.

We perform a second change of variables $(u_1,\ ...,\ u_k) \longrightarrow (v_1,\ ...,\ v_k)$ in (2.27) as follows,

$$\begin{array}{lcl} v_1 = h^{-1}(u_1) & & u_1 = h(v_1) \\ v_2 = h^{-1}(u_2) & & u_2 = h(v_2) \\ \cdots & \longleftrightarrow & \cdots \\ v_k = h^{-1}(u_k) & & u_k = h(v_k) \end{array} \tag{2.28}$$

Since, we assume that $h(t) = c\,t$, we have $h^{-1}(t) = \frac{t}{c}$ and the Jacobian of the transformation

$$| J |=\begin{pmatrix} c & 0 & \dots & 0 & 0 \\ 0 & c & \dots & 0 & 0 \\ : & : & \dots & : & : \\ 0 & 0 & \dots & 0 & c \end{pmatrix} = | c^k |$$ is again non-singular. After this second change of

variables, expression (2.27) becomes

$P(T > x) =$

$$e^{-x\lambda} \sum_{k=1}^{\infty} \int_0^x \int_{v_1}^x \dots \int_{v_{k-2}}^x \int_x^{\infty} c^k \left(\sum_{j=0}^{k-1} (-1)^j b_j(v_1, \dots, v_j) \lambda^j \left(\sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!} \right) \right) \tag{2.29}$$

$$f_k(h(v_1), h(v_2) - h(v_1), \dots, h(v_k) - h(v_{k-1}))\, dv_k\, dv_{k-1} \dots dv_2\, dv_1$$

So, in the special case when $h(t) = c\,t$, formula (2.17) simplifies to formula(2.29) which is easier to implement and use for numerical calculations.

Table 9. P(T>x) for different values of the safety loading factor η. $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 0$, $x = 0.5$. The precision of IK_C is at least four digits after the decimal point.

η	IK_C	Time, seconds	PL_MLMM $h = 0.05$, $p = 1$	Time, seconds
0.05	0.676611	2.20	0.676744	0.33
0.10	0.679518	2.25	0.679529	0.44
0.15	0.682389	3.62	0.682507	0.39
0.20	0.685225	2.26	0.685237	0.49
0.25	0.688026	2.25	0.688134	0.44
0.30	0.690794	2.30	0.690808	0.49

In Tables 9 and 10, ruin probability values calculated using (2.29) and the PL_MLMM method are listed along with the corresponding computational times. It has to be noted that the results obtained using (2.29), i.e. column headed IK_C, have guaranteed precision of four correct digits after the decimal point. The latter is achieved by evaluating (2.29) up to $k = 6$, since following the algorithm described in this section with $\epsilon = 0.00001$, we stopped at $n = 6$ in (2.24).

Table 10. $P(T>x)$ for different values of the time interval x. $Y_i \sim Exp(0.1)$, $\lambda = 1$, $u = 0$, $c = 1.15$. The precision of IK_C is at least four digits after the decimal point.

x	IK_C	Time, seconds	PL_MLMM $h = 0.1$, $p = 1$	Time, seconds	PL_MLMM $h = 0.05$, $p = 1$	Time, seconds
1	0.388631	2.31	0.388670	1.27	0.388631	3.46
2	0.166419	2.42	0.166421	3.46	0.166419	11.48
3	0.076906	3.29	0.076916	6.48	0.076908	34.22

In addition to the guaranteed accuracy the IK_C approach of calculating $P(T > x)$ has yet another advantage. As we can see from Table 10, for particular set of values of the parameters of the risk model, IK_C is faster in achieving six digits accuracy than PL_MLMM for values of $x > 1$. This is remarkable because IK_C turns out to be more general and more efficient than PL_MLMM for large values of x.

2.4.2 The formula of Ignatov and Kaishev (2004)

In this section, we will present the formula of Ignatov and Kaishev (2004). It is valid under the general assumptions of any joint continuous distribution of the claims severities (either dependent or independent), arbitrary non-decreasing income function and Poisson claim arrivals. Our purpose here will be to investigate the numerical efficiency of the latter formula and compare it with IK_C. Thus, the formula of Ignatov and Kaishev (2004) has the following form

$$P(T > x) =$$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_{u_1}^{h(x)} \cdots \int_{u_{k-1}}^{h(x)} \psi_k(x, u_1, \ \ldots, u_k)\, du_k \ldots du_2\, du_1\right), \tag{2.30}$$

where

$\psi_0 = 1$

$\psi_k = A_k\left(x, h^{-1}(u_1), \ ..., h^{-1}(u_k)\right) \times \varphi_k(u_1, \ ..., u_k)$,

U_1, U_2, ... are the partial sums of the individual claim amounts Y_1, Y_2, ..., $\varphi_k(u_1, ..., u_k)$ is the probability density function of U_1, ..., U_k and $A_k\left(x, h^{-1}(u_1), \ ..., h^{-1}(u_k)\right)$, $k = 1, 2, ...$ are the Appell polynomials defined as

$A_0(x) = 1$

$A_k{}'(x) = A_{k-1}(x)$

$A_k\left(h^{-1}(u_k)\right) = 0, \quad k = 1, 2, \ ...$

Obviously, if $f_k(y_1, \ ..., y_k)$ is the density function of the individual claims, then

$\varphi_k(u_1, \ ..., u_k) = \ f_k(u_1, u_2 - u_1, \ ..., u_k - u_{k-1})$.

As in the previous section, in the special case of $h(t) = c\,t$, in formula (2.30) we can make the same two changes of variables as given by (2.26) and (2.28), and rewrite (2.30) as

$$P(T > x) =$$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^x \int_{v_1}^x \cdots \int_{v_{k-1}}^x c^k \, P_k(x, v_1, \ ..., v_k)\, \varphi_k(h(v_1), \ ..., h(v_k))\, d v_k \cdots \right. \tag{2.31}$$

$$\left. d v_2 \, d v_1 \right)$$

Clearly, (2.31) is relatively simple and easy to evaluate. For example, in the case of independent, exponentially distributed claim amounts we have

$\varphi_k(h(v_1), \ ..., h(v_k)) = f_k(h(v_1), h(v_2) - h(v_1), \ ..., h(v_k) - h(v_{k-1})) =$

$\alpha^k \, e^{-\alpha\,(h(v_1) + h(v_2) - h(v_1) + ... + h(v_k) - h(v_{k-1}))} = \alpha^k \, e^{-\alpha\, h(v_k)}$.

In Table 11, numerical results obtained using (2.31) and PL_MLMM are given, along with the corresponding computational times. It has to be noted that the precision of the results obtained via PL_MLMM can not be assessed as in the case of IK_C.

Table 11. $P(T>x)$ for different values of the security loading factor η. $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 0$, $x = 1$.

η	Ignatov and Kaishev (2004)	Time, Seconds	PL_MLMM $h = 0.05$, $p = 1$	Time, seconds
0.05	0.530242	3.52	0.530263	0.93
0.10	0.536596	3.13	0.536617	0.99
0.15	0.542840	3.52	0.542861	1.10
0.20	0.548974	3.62	0.548996	1.15
0.25	0.555002	3.35	0.555024	1.27
0.30	0.560925	3.35	0.560947	1.26

In the next section, we perform a more detailed comparison of the ruin probability values obtained by using the formula of Ignatov and Kaishev (2004) and the one proposed in the previous section (see (2.29)), both in the case of independent and dependent claim amounts.

2.4.3 A numerical study

In this section, we compare the numerical efficiency of different methods for computing of probabilities of ruin under the assumption of independent or dependent continuous claim severities. Namely, we compare the PL_MLMM method, we proposed in section 2.3, the extension of the formula of Ignatov and Kaishev (2000) which we proposed in section 2.4.1 and the explicit formula developed in Ignatov and Kaishev (2004).

2.4.3.1 Comparison - independent case

In Table 12, we present ruin probability values calculated using the three different methods and compliment them with the corresponding computational times. The precision of IK_C is at least four digits after the decimal point.

Table 12. P(T>x) for different values of the time interval x. $Y_i \sim Exp(0.1)$, $\lambda = 1$, $u = 0$, $c = 1.1$.

x	IK_C	Time, seconds	Ignatov and Kaishev (2004)	Time, seconds	PL_MLMM $h = 0.1$, $p = 1$	Time, seconds
0.5	0.6147570	0.39	0.6147570	0.60	0.6148240	0.17
1.0	0.3877450	0.77	0.3877450	3.68	0.3877470	0.44
2.0	0.1650710	0.77	0.1650710	3.52	0.1650730	0.99
3.0	0.0756765	0.82	0.0756768	3.57	0.0756791	1.97
5.0	0.0185692	1.32	0.0185694	10.9	0.0185708	5.50

The numerical results indicate that for values of the parameter $\alpha \le 0.5$ of the exponential distribution and sizes of the time interval $x \ge 2$ the IK_C method is faster than the one of Ignatov and Kaishev (2004). Same is confirmed when we evaluate ruin probabilities with Pareto and Weibull distributed claim amounts (results not presented here).

Table 13. P(T>x) for different values of the time interval x. $Y_i \sim Exp(1)$, $\lambda = 1$, $u = 0$, $c = 1.1$.

x	IK_C	Time, seconds	Ignatov and Kaishev (2004)	Time, seconds	PL_MLMM $h = 0.05$, $p = 1$	Time, seconds
0.5	0.679519	11.31	0.679519	4.29	0.679529	1.27
1.0	0.536599	19.99	0.536599	14.39	0.536617	3.35
1.5	0.457648	19.72	0.457652	14.34	0.457677	6.10
2.0	0.407053	19.77	0.407077	14.44	0.407158	10.65

Table 13 contains numerical results for $P(T > x)$, obtained with IK_C, PL_MLMM and the formula of Ignatov and Kaishev (2004) as well as their running times. The precision of IK_C is guaranteed up to the third digit after the decimal point. Our numerical experience shows that for $\alpha \geq 1$ the formula of Ignatov and Kaishev (2004) is more efficient in terms of time and accuracy than the other two alternatives, IK_C and PL_MLMM.

2.4.3.2 Comparison - dependent case

Finally, we illustrate the performance of the two explicit expressions for calculating finite-time survival probabilities assuming dependent claim severities, namely the IK_C formula (2.29) and the one of Ignatov and Kaishev (2004). Following Ignatov and Kaishev (2004), we use the Inverted Dirichlet distribution which has the following density

$$f_k(y_1, \ldots, y_k) = \frac{\Gamma\left(\sum_{j=0}^{k} g_j\right)}{\prod_{j=0}^{k} \Gamma(g_j)} \frac{\prod_{j=1}^{k} (y_j)^{g_j - 1}}{\left(1 + \sum_{i=1}^{k} y_i\right)^{g_0 + g_1 + \ldots + g_k}}, \quad y_j > 0, \; j = 1, \ldots, k,$$

where $g_i > 0$, $i = 0, 1, \ldots, k$, are the parameters of the Inverted Dirichlet distribution (see Johnson and Kotz 1994) and $\Gamma(.)$ is the gamma function.

For the purpose of our numerical calculations, we set $g_i = 2$, $i = 0, \ldots, k$. The probability density function of the two dimensional Inverted Dirichlet distribution, InvDir(2, 2, 2), is illustrated in Fig. 4.

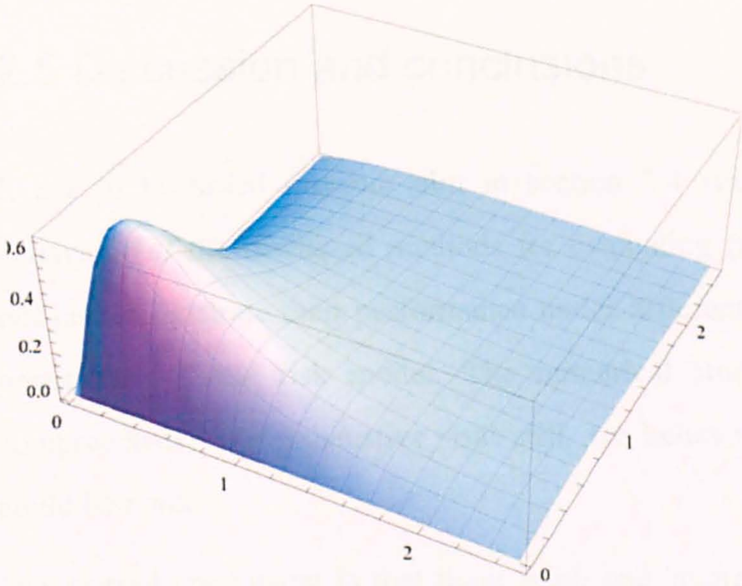

Fig. 4. The probability density function of the two dimensional Inverted Dirichlet distribution with parameters $g_j = 2$, $j = 0, 1, 2$, i.e. InvDir(2, 2, 2).

In Table 14, values of the survival probabilities, calculated via 20 000 Monte Carlo simulations (see the column headed Simul.) and with the exact explicit formulae IK_C and the one of Ignatov and Kaishev, are presented. For the chosen set of parameters, IK_C is less computationally involved than the alternative.

Table 14. $P(T > x)$ for different values of the safety loading factor η. $(Y_1, \ldots, Y_k) \sim InvDir(2, \ldots, 2)$, $\lambda = 1$, $u = 0$, $x = 0.5$.

η	Simul.	IK_C	Time, seconds	Ignatov and Kaishev (2004)	Time, seconds
0.0	0.641302	0.641178	9.72	0.641180	29.44
0.1	0.645116	0.645938	9.39	0.645942	51.52
0.2	0.651066	0.650711	15.92	0.650712	54.37
0.3	0.652333	0.655465	18.62	0.655467	53.77
0.4	0.661201	0.660185	15.05	0.660190	56.19

It has to be noted that there are different ways of modelling the dependence between the claim amounts. For example by using copula functions. The latter is illustrated in Chapter 3 and 4.

2.5 Discussion and conclusions

It has to be noted that our aim in section 2.4 was not to study the numerical behaviour of the discussed methods for evaluating (non-) ruin probabilities in full but just to illustrate their performance under different assumption and values of the parameters of the risk model. The numerical study performed here is neither comprehensive nor exhaustive. But still, we believe that the following comments could be made.

Our overall conclusion is that there is no one 'numerically most efficient' formula which is the 'best' choice for any set of parameters of the risk model. Depending on the specific assumptions one may need to use a different explicit expression or even a discretization method. A similar conclusion has been reached by Rullière and Loisel (2004) for the case of discrete claim sizes.

Chapter 3

Excess of loss reinsurance under joint survival optimality

Summary

Explicit expressions for the probability of joint survival up to time x of the cedent and the reinsurer, under an excess of loss reinsurance contract with a limiting and a retention level are obtained, under the reasonably general assumptions of any non-decreasing premium income function, Poisson claim arrivals and continuous claim amounts, modelled by any joint distribution. By stating appropriate optimality problems, we show that these results can be used to set the limiting and the retention levels in an optimal way with respect to the probability of joint survival. Alternatively, for fixed retention and limiting levels, the results yield an optimal split of the total premium income between the two parties in the excess of loss contract. This methodology is illustrated numerically on several examples of independent and dependent claim severities. The latter are modelled by a copula function. The effect of varying its dependence parameter and the marginals, on the solutions of the optimality problems and the joint survival probability, has also been explored.

3.1 Introduction

Several approaches to optimal reinsurance have been attempted in the actuarial literature, based on risk theory, economic game theory and stochastic dynamic control. Examples of research in each of these directions are the papers by Dickson and Waters (1996, 1997), Centeno (1991, 1997), Andersen (2000), Krvavych

(2001), by Aase (2002), Suijs, Borm and De Waegenaere (1998), and by Schmidli (2001, 2002), Hipp and Vogt (2001), Taksar and Markussen (2003). A common feature of most of the quoted works is that optimality is considered with respect to the interest of solely the direct insurer, minimizing his (approximated) ruin probability, under the classical assumptions of linearity of the premium income function and independent, identically distributed claim severities.

Recently, a different reinsurance optimality model, which takes into account the interests of both the cedent and the reinsurer, has been considered by Ignatov, Kaishev and Krachunov (2004). As a joint optimality criterion they introduce the direct insurer's and the reinsurer's probability of joint survival up to a finite time horizon. Under this model, a volume of risks is insured by a direct insurer, who is entitled to receiving certain premium income in return for the obligation to cover individual claims. The latter are assumed to have any discrete joint distribution and Poisson arrivals. It is further assumed that the cedent is seeking to share claims and premium income with a reinsurer under a simple excess of loss contract with a retention level M, taking integer values. In their paper, Ignatov, Kaishev and Krachunov (2004) have derived expressions for the probability of joint survival of the cedent and the reinsurer and have demonstrated its applicability in the context of optimal reinsurance.

Catastrophic events in recent years have caused insurance and reinsurance losses of increasing frequency and severity. As a result, some reinsurance companies have been downgraded with respect to their credit rating while others, such as the 6-th largest reinsurer worldwide Gerling Global Re, even became insolvent and went out of business. The latter developments have motivated even stronger the proposed idea of considering reinsurance not solely from the point of view of the direct insurer, but taking into account the contradicting interests of the two parties, by jointly measuring the risk they share.

Our aim in this paper is to generalize the joint survival optimality reinsurance model, introduced by Ignatov, Kaishev and Krachunov (2004). We extend it here by considering an excess of loss (XL) contract in which the reinsurer covers each individual claim in excess of a retention level M, but up to a limiting level L and individual claim severities are not discrete but are modelled by continuous (dependent) random variables, with any joint distribution. Under these reasonably general assumptions we give closed form expressions for the probability of joint survival of the cedent and the reinsurer up to a fixed future moment in time. Based on these expressions, we state two optimality problems, according to which optimal values of M and L or alternatively, an optimal split of the total premium income, maximizing the probability of joint survival, can be obtained. These problems have been solved numerically, due to the infeasibility of their analytical solution. The derived joint survival probability formulae, conveniently allow the use of copula functions in modelling the dependency between claim severities. We have shown how varying the degree of dependence through the copula parameter(s) affects the optimal choice of the retention and the limiting levels, the optimal sharing of the premium income and also the probability of joint survival.

The results presented in this paper comprise an extension of the model considered by Ignatov, Kaishev and Krachunov (2004), to the practically more important case of continuous, dependent claim severities. In addition, the more general XL contract considered here gives a refined control over the optimal structure of this risk sharing arrangement. For further details on XL contracts with one or more layers, see e.g. Bugmann (1997).

The paper is organized as follows. In Section 3.2 we introduce the XL contract and the related joint survival probability model, considered further. Our main results are stated in Section 3.3 and illustrated numerically in Section 3.4, where we have introduced the copula approach to modelling dependence of consecutive claim severities under reinsurance. The final Section 3.5 provides some concluding remarks and indicates questions for further research.

3.2 The XL contract

We will consider an insurance portfolio, generating claims with inter-occurrence times τ_1, τ_2, ..., assumed identically, exponentially distributed r.v.s with parameter λ. Denote by $T_1 = \tau_1$, $T_2 = \tau_1 + \tau_2$, ... the sequence of random variables representing the consecutive moments of occurrence of the claims. Let $N_t = \# \{i : T_i \leq t\}$, where $\#$ is the number of elements of the set $\{.\}$. The claim severities are modeled by the non-negative continuous r.v.s. W_1, W_2, ..., W_k, ..., with joint density function $\psi(w_1, ..., w_k)$. It will be convenient to introduce the random variables $Y_1 = W_1$, $Y_2 = W_1 + W_2$, ... representing the partial sums of consecutive claim severities.

The r.v.s W_1, W_2, ..., are assumed to be independent of N_t. Then, the risk (surplus) process R_t, at time t, is given by $R_t = h(t) - Y_{N_t}$, where $h(t)$ is a nonnegative, non-decreasing, real function, defined on $\mathbb{R}_+$, representing the aggregated premium income up to time t, to be received for carrying the risk associated with the entire portfolio. The function $h(t)$ may be continuous or not. If $h(t)$ is discontinuous we will define $h^{-1}(y) = \inf \{z : h(z) \geq y\}$. Clearly, $h(t)$ represents a rather general class of functions and the classical case, $h(t) = u + c\,t$, with initial reserve u and premium rate c, is of course included. We will assume that the premium has been determined in such a way that the premium income defined by the function $h(t)$ adequately corresponds to the aggregate claim amount, generated by the portfolio up to time t. For the purpose, the various premium rating principles (see e.g., Gerber, 1979 and Wang, 1995) or other practical rating techniques can be used.

Without reinsurance, explicit formulae for the probability of non-ruin (survival) $P(T > x)$ of the direct insurer, in a finite time interval $[0, x]$, $x > 0$, with the time T

of ruin, defined as

$$T := \inf\{t : t > 0,\ R_t < 0\}, \tag{3.1}$$

were derived by Ignatov and Kaishev (2004) and by Kaishev and Dimitrova (2006).

Here, we will be concerned with the case when the direct insurer wishes to reinsure his portfolio of risks by concluding an XL contract with a retention level M and a limiting level L, $M \geq 0$, $L \geq M$. In other words, the cedent reinsures the part of each claim which hits the layer $m = L - M$, i.e., each individual claim W_i is shared between the two parties so that $W_i = W_i^c + W_i^r$ $i = 1, 2, \ldots$ where W_i^c and W_i^r denote the parts covered respectively by the cedent and the reinsurer. Clearly, we can write

$$W_i^c = \min(W_i, M) + \max(0, W_i - L)$$

and

$$W_i^r = \min(L - M, \max(0, W_i - M)).$$

Denote by $Y_1^c = W_1^c$, $Y_2^c = W_1^c + W_2^c$, ... and by $Y_1^r = W_1^r$, $Y_2^r = W_1^r + W_2^r$, ... the consecutive partial sums of claims to the cedent and to the reinsurer, respectively. Under our XL reinsurance model, the total premium income $h(t)$ is also divided between the two parties so that $h(t) = h_c(t) + h_r(t)$, where $h_c(t)$, $h_r(t)$ are the premium incomes of the cedent and the reinsurer, assumed also non-negative, non-decreasing functions on $\mathbb{R}_+$. As a result, the risk process, R_t, can be represented as a superposition of two risk processes, that of the cedent

$$R_t^c = h_c(t) - Y_{N_t}^c \tag{3.2}$$

and of the reinsurer

$$R_t^r = h_r(t) - Y_{N_t}^r \tag{3.3}$$

i.e., $R_t = R_t^c + R_t^r$.

There are two alternative optimization problems which may be stated in connection with an XL contract as the one described here. The first is, given M and m are fixed, how should then the premium income $h(t)$ be divided between the two parties, so as to optimize a certain criterion measuring their joint risk or performance. And alternatively, if the total premium income $h(t)$ is divided in an agreed way between the cedent and the reinsurer, i.e., $h_c(t)$ and $h_r(t) = h(t) - h_c(t)$ are fixed, how should the parameters M and L of the XL contract be optimally set so as to minimize (maximize) the chosen joint risk or performance criterion.

3.3 The probability of joint survival optimality

In this section we will introduce some risk measures, assuming both the cedent and the reinsurer jointly survive up to time x.

Define the moments, T^c and T^r, of ruin of correspondingly the cedent and the reinsurer as in (3.1), replacing R_t with R_t^c and R_t^r respectively. Clearly, the two events $(T^c > x)$ and $(T^r > x)$, of survival of the cedent and the reinsurer are dependent since the two risk processes R_t^c and R_t^r are dependent through the common claim arrivals and the claim severities W_i, $i = 1, 2, \ldots$ as seen from (3.2) and (3.3). Hence, as has been proposed in Ignatov, Kaishev and Krachunov (2004), it is meaningful to consider the probability of joint survival, $P(T^c > x, T^r > x)$, as a measure of the risk the two parties share and jointly carry. The two optimization problems we have stated can now be formulated more precisely as follows.

Problem 1. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$, find

$$\max_{L,M} P(T^c > x, T^r > x).$$

Problem 2. For fixed M, L and $h(t)$, find

$$\max_{\substack{h_c(t),\\ h(t)=h_c(t)+h_r(t)}} \quad P(T^c > x,\, T^r > x)\,.$$

Problems 1 and 2 may be given the following interpretation. In Problem 1, the ceding company may wish to retain a certain fixed part, $h_c(t)$, of the premium income, $h(t)$, and then to find values for M and L, defining the corresponding optimal portion of the risk it would need to accept, so as to have maximum chances of joint with the reinsurer survival, up to a finite time x. Alternatively, the values M and L may be fixed, according to the ceding company's risk aversion and/or according to decisions, driven by negotiations with the reinsurer or other market conditions, after which the optimal split of $h(t)$, between the two parties would need to be defined, solving Problem 2. To explore Problems 1 and 2, next we will derive closed form expressions for the probability $P(T^c > x, T^r > x)$.

Theorem 1. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$P(T^c > x, T^r > x) =$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k) \right. \tag{3.4}$$

$$\left. \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1 \right)$$

where

$\tilde{v}_j = \min(\tilde{z}_j, x)$, $\tilde{z}_j = \max\left(h_c^{-1}\left(y_j^c\right), h_r^{-1}\left(y_j^r\right)\right)$, $y_j^c = \sum_{i=1}^{j} w_i^c$, $y_j^r = \sum_{i=1}^{j} w_i^r$, $j = 1, \ldots, k$,

$w_i^c = \min(w_i, M) + \max(0, w_i - L)$, $w_i^r = \min(L - M, \max(0, w_i - M))$, *and*

$A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)$, $k = 1, 2, \ldots$ *are the classical Appell polynomials* $A_k(x)$ *of degree* $k \geq 1$, *defined by*

$A_0(x) = 1, \quad A_k'(x) = A_{k-1}(x), \quad A_k(\tilde{v}_k) = 0.$

Remark 1. Appell polynomials were introduced by P.E. Appell (1880) and up to a normalization, contain many classical sequences of polynomials, among which the Bernoulli, Hermite and Laguerre polynomials. The sequence of Appell polynomials $\{A_k(x): k = 0, 1, ...\}$ are alternatively defined by a generating function

$$f(y)\, e^{xy} = \sum_{k=0}^{\infty} A_k(x) \left(y^k / k!\right),$$

where $f(y) = \sum_{k=0}^{\infty} A_k(0) \left(y^k / k!\right)$, $(f(0) \neq 0)$. and the values $A_k(0)$, $k = 0, 1, ...$ uniquely determine $\{A_k(x): k = 0, 1, ...\}$.

Clearly, Theorem 1 establishes a promising link of the survival probability $P(T^c > x, T^r > x)$ to the wide and important class of Appell polynomials. This link, worth further exploration, may give new insights into the properties of formula (3.4), and in particular may lead to a substantial improvement of its numerical efficiency. For a more detailed account on Appell polynomials we refer to Kaz'min (2002).

Proof of Theorem 1. The event of joint survival $\{T^c > x, T^r > x\}$ can be expressed as

$$\{T^c > x, T^r > x\} = \bigcap_{j=1}^{\infty} \left[\left\{ \left(h_c^{-1}(Y_j^c) < T_j\right) \cap \left(h_r^{-1}(Y_j^r) < T_j\right) \right\} \cup \{x < T_j\} \right]$$

$$= \bigcap_{j=1}^{\infty} \left[\left\{ \max\left(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)\right) < T_j \right\} \cup \{x < T_j\} \right] \tag{3.5}$$

Noting that $\Omega = \bigcup_{k=0}^{\infty} \{N_x = k\}$, applying the partition theorem we have

$$P(T^c > x, T^r > x) = \sum_{k=0}^{\infty} P(N_x = k)\, P(T^c > x, T^r > x \mid N_x = k)$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P(T^c > x, T^r > x \mid \{T_k \leq x\} \cap \{T_{k+1} > x\}) \tag{3.6}$$

In (3.6), we have used the fact that the event $\{N_x = k\} \equiv \{T_k \leq x\} \cap \{T_{k+1} > x\}$.

If we now express $\{T^c > x, T^r > x\}$ in (3.6) using its representation given by (3.5) we obtain

$$P(T^c > x, T^r > x) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x}$$
$$P\left(\bigcap_{j=1}^{\infty} \left[\left\{\max\left(h_c^{-1}\left(Y_j^c\right), h_r^{-1}\left(Y_j^r\right)\right) < T_j\right\} \cup \{x < T_j\}\right] \,\middle|\, \{T_k \le x\} \cap \{T_{k+1} > x\}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\left(\bigcap_{j=1}^{\infty} \left[\left\{\max\left(h_c^{-1}\left(Y_j^c\right), h_r^{-1}\left(Y_j^r\right)\right) < T_j\right\} \cup \{x < T_j\}\right]\right) \cap \right.$$
$$\left. \{T_k \le x\} \cap \{T_{k+1} > x\} \,\middle|\, \{T_k \le x\} \cap \{T_{k+1} > x\}\right) \tag{3.7}$$

where in the last equality we have used that $P(A \mid B) = P(A \cap B \mid B)$. Applying some algebraic manipulations on the event in (3.7) it can be shown that

$$\left(\bigcap_{j=1}^{\infty} \left[\left\{\max\left(h_c^{-1}\left(Y_j^c\right), h_r^{-1}\left(Y_j^r\right)\right) < T_j\right\} \cup \{x < T_j\}\right]\right) \cap \{T_k \le x\} \cap \{T_{k+1} > x\}$$
$$= \left(\bigcap_{j=1}^{k} \left\{\max\left(h_c^{-1}\left(Y_j^c\right), h_r^{-1}\left(Y_j^r\right)\right) < T_j\right\}\right) \cap \{T_k \le x\} \cap \{T_{k+1} > x\} \tag{3.8}$$

Substituting (3.8) back in (3.7) leads to

$$P(T^c > x, T^r > x)$$
$$= \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\bigcap_{j=1}^{k} \left[\left\{\max\left(h_c^{-1}\left(Y_j^c\right), h_r^{-1}\left(Y_j^r\right)\right) < T_j\right\} \cap \{T_k \le x\} \cap \{T_{k+1} > x\}\right] \,\middle|\, \{T_k \le x\} \cap \{T_{k+1} > x\}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\bigcap_{j=1}^{k} \left\{\max\left(h_c^{-1}\left(Y_j^c\right), h_r^{-1}\left(Y_j^r\right)\right) < T_j\right\} \,\middle|\, \{T_k \le x\} \cap \{T_{k+1} > x\}\right) \tag{3.9}$$

It is known that (see Karlin and Taylor, 1981)

$$P(T_1 \le t_1, \ldots, T_k \le t_k \mid \{T_k \le x\} \cap \{T_{k+1} > x\}) = P\left(\tilde{T}_1 \le t_1, \ldots, \tilde{T}_k \le t_k\right) \tag{3.10}$$

where $\tilde{T}_1 \le \ldots \le \tilde{T}_k$ are the order statistics of k independent, uniformly distributed random variables in the interval $(0, x)$. From the independence of the two sequences of random variables Y_j^c, Y_j^r, $j = 1, 2, \ldots$ and T_k, $k = 1, 2, \ldots$ and applying (3.10) we can rewrite (3.9) as

$$P(T^c > x,\, T^r > x) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\bigcap_{j=1}^{k} \max\left(h_c^{-1}\left(Y_j^c\right),\, h_r^{-1}\left(Y_j^r\right)\right) < \tilde{T}_j \right) \tag{3.11}$$

The random variables $\tilde{T}_1 \le \ldots \le \tilde{T}_k$ have a joint density (see Karlin and Taylor, 1981)

$$f_{\tilde{T}_1,\ldots,\tilde{T}_k}(t_1,\, \ldots,\, t_k) = \begin{cases} \frac{k!}{x^k} & \text{if} \quad 0 \le t_1 \le \ldots \le t_k \le x \\ 0 & \text{otherwise} \end{cases}$$

hence, introducing the notation

$$\mathcal{D}_k \equiv \begin{pmatrix} 0 \le w_1,\, \ldots,\, 0 \le w_k \\ w_1 + \ldots + w_k \le h(x) \end{pmatrix},$$

we can express the probability on the right-hand side of (3.11) as

$$P\left(\bigcap_{j=1}^{k} \max\left(h_c^{-1}\left(Y_j^c\right),\, h_r^{-1}\left(Y_j^r\right)\right) < \tilde{T}_j \right)$$

$$= \int \cdots \int_{\mathcal{D}_k} \psi(w_1,\, \ldots,\, w_k) \tag{3.12}$$

$$\int \cdots \int_{\substack{\min\left[\max\left(h_c^{-1}(y_1^c),\, h_r^{-1}(y_1^r)\right),\, x\right] < t_1 < x \\ \cdots \\ \min\left[\max\left(h_c^{-1}(y_k^c),\, h_r^{-1}(y_k^r)\right),\, x\right] < t_k < x \\ t_1 \le \ldots \le t_k}} \frac{k!}{x^k}\, dt_k \cdots dt_1\, dw_k \cdots dw_1$$

where $\min\left[\max\left(h_c^{-1}\left(y_j^c\right),\, h_r^{-1}\left(y_j^r\right)\right),\, x\right]$, $j = 1, 2, \ldots, k$ appear as lower limits of integration since $\max\left(h_c^{-1}\left(y_j^c\right),\, h_r^{-1}\left(y_j^r\right)\right)$ can in general exceed x for some value $y_j = y_j^c + y_j^r = w_1^c + \ldots + w_j^c + w_1^r + \ldots + w_j^r = w_1 + \ldots + w_j$, $j = 1, 2, \ldots, k$. In this case $\min\left[\max\left(h_c^{-1}\left(y_j^c\right),\, h_r^{-1}\left(y_j^r\right)\right),\, x\right] = x$, i.e., the integral in (3.11) vanishes as is necessary, since such trajectories $t \mapsto y_j$ cause ruin of at least one of the parties and therefore should not contribute to the probability of their joint survival. To simplify notation, we let $\tilde{v}_j = \min[\tilde{z}_j, x]$, $\tilde{z}_j = \max\left(h_c^{-1}\left(y_j^c\right), h_r^{-1}\left(y_j^r\right)\right)$, $j = 1, 2, \ldots, k$ and use (3.12) to rewrite (3.11) as

$$P(T^c > x, T^r > x)$$

$$= e^{-\lambda x} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} \int \cdots \int_{\mathcal{D}_k} \psi(w_1, \ldots, w_k) \int \cdots \int_{\substack{\tilde{v}_1 < t_1 < x \\ \cdots \\ \tilde{v}_k < t_k < x \\ t_1 \leq \ldots \leq t_k}} \frac{k!}{x^k} \, dt_k \cdots dt_1 \, dw_k \cdots dw_1$$

$$= e^{-\lambda x} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} \int \cdots \int_{\mathcal{D}_k} \psi(w_1, \ldots, w_k)$$

$$\frac{k!}{x^k} \int_{\tilde{v}_1}^{x} \int_{\max[\tilde{v}_2, t_1]}^{x} \cdots \int_{\max[\tilde{v}_k, t_{k-1}]}^{x} dt_k \cdots dt_2 \, dt_1 \, dw_k \cdots dw_1$$

$$= e^{-\lambda x} \sum_{k=0}^{\infty} \lambda^k \int \cdots \int_{\mathcal{D}_k} \psi(w_1, \ldots, w_k) \, A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k) \, dw_k \cdots dw_1 \tag{3.13}$$

where we have set

$$A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k) = \int_{\tilde{v}_1}^{x} \int_{\max[\tilde{v}_2, t_1]}^{x} \cdots \int_{\max[\tilde{v}_k, t_{k-1}]}^{x} dt_k \cdots dt_2 \, dt_1.$$

It can be seen directly that $A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k)$ is a polynomial of degree k with a coefficient at the highest degree $1/k!$. Moreover, applying similar reasoning as in Theorem 1 of Ignatov and Kaishev (2004) it can be shown that $A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k)$, $k = 1, 2, \ldots$ are the classical Appell polynomials.

The asserted joint survival probability formula now follows, appropriately rewriting the multiple integral in (3.13).□

An alternative formula for $P(T^c > x, T^r > x)$ is provided by the following

Theorem 2. *The probability of joint survival is*

$P(T^c > x, T^r > x) =$

$$e^{-\lambda x}\left(\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)\right. \tag{3.14}$$

$$\left.\psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\right)$$

where

$B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x) = \sum_{j=0}^{l-1}(-\lambda)^j\, b_j(\tilde{z}_1, \ldots, \tilde{z}_j)\left(\sum_{m=0}^{l-j-1}\frac{(x\lambda)^m}{m!}\right)$, with $B_0(\cdot) \equiv 0,\ B_1(\cdot) = 1$,

l is such that $\tilde{z}_1 \le \ldots \le \tilde{z}_{l-1} \le x < \tilde{z}_l$,

$b_j(\tilde{z}_1, \ldots, \tilde{z}_j) = \sum_{i=1}^{j}(-1)^{j+i}\frac{\tilde{z}_j^{\,j-i+1}}{(j-i+1)!}\, b_{i-1}(\tilde{z}_1, \ldots, \tilde{z}_{i-1})$, *with* $b_0 \equiv 1$,

$\tilde{z}_j$ *and* $\psi(w_1, \ldots, w_k)$ *are defined as in Theorem 1.*

Proof of Theorem 2. The probability of survival of the cedent without reinsurance (see Theorem 2 of Chapter 2, section 2.4.1) is given by

$P(T > x) =$

$$\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} P(T > x \mid W_1 = w_1, \ldots, \tag{3.15}$$

$$W_{k-1} = w_{k-1};\, W_k \ge h(x) - w_1 - \ldots - w_{k-1}) \times$$

$$\psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1$$

where

$$P(T > x \mid W_1 = w_1, \ldots, W_{k-1} = w_{k-1};\, W_k \ge h(x) - w_1 - \ldots - w_{k-1}) \\ = e^{-\lambda x} B_k(z_1, \ldots, z_{k-1}, x) \tag{3.16}$$

and $z_j = h^{-1}(w_1 + \ldots + w_j)$, provided that $h^{-1}(w_1 + \ldots + w_{k-1}) \le x < h^{-1}(w_1 + \ldots + w_k)$.

By analogy with the reasoning in deriving (3.15) we can write

$P(T^c > x,\ T^r > x) =$

$$\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \dots \int_0^{h(x)-w_1-\dots-w_{k-2}} \int_{h(x)-w_1-\dots-w_{k-1}}^{\infty} P(T^c > x,\ T^r > x \mid W_1 =$$

$$w_1,\ \dots,\ W_{k-1} = w_{k-1};\ W_k \geq h(x) - w_1 - \dots - w_{k-1})$$

$$\psi(w_1,\ \dots,\ w_k)\, dw_k\, dw_{k-1} \dots dw_2\, dw_1$$

Following equality (10) of Ignatov, Kaishev and Krachunov (2004), it is possible to show that

$$P(T^c > x,\ T^r > x \mid W_1 = w_1,\ \dots,\ W_{k-1} = w_{k-1};\ W_k \geq h(x) - w_1 - \dots - w_{k-1})$$

$$= P\left(\bigcap_{j=1}^{k-1} \left\{ \max\left(h_c^{-1}(y_j^c),\ h_r^{-1}(y_j^r)\right) \leq T_j \right\} \cap \{T_k > x\} \right) \tag{3.18}$$

From (3.16) and (3.18) it can be concluded that

$$P\left(\bigcap_{j=1}^{k-1} \left\{ \max\left(h_c^{-1}(y_j^c),\ h_r^{-1}(y_j^r)\right) \leq T_j \right\} \cap \{T_k > x\} \right) = e^{-\lambda x} B_k(\tilde{z}_1,\ \dots,\ \tilde{z}_{k-1},\ x) \tag{3.19}$$

where $\tilde{z}_j = \max\left(h_c^{-1}(y_j^c),\ h_r^{-1}(y_j^r)\right)$, $j = 1,\ \dots,\ k$. It is not difficult to see that there should exist an index $1 \leq l \leq k$, such that $\tilde{z}_1 \leq \dots \leq \tilde{z}_{l-1} \leq x < \tilde{z}_l$ and since we consider the events of ruin of the cedent and the reinsurer up to time x only, hence we can rewrite (3.19) as

$$P\left(\bigcap_{j=1}^{k-1} \{\tilde{z}_j \leq T_j\} \cap \{T_k > x\} \right) = e^{-\lambda x} B_l(\tilde{z}_1,\ \dots,\ \tilde{z}_{l-1},\ x) \tag{3.20}$$

Formula (3.14) now follows from (3.18), (3.20) and (3.17) which completes the proof of Theorem 2.□

The use of formulae (3.4) and (3.14) to compute $P(T^c > x,\ T^r > x)$ is discussed in Section 3.4 where the case of independent and dependent claim severities are thoroughly explored.

3.4 Computational considerations and results

In this section we demonstrate that using the results of Theorem 1 and 2, one can successfully find solutions to Problems 1 and 2, stated in Section 3.3, and optimally determine the parameters of an XL contract. A quick analysis of formulae (3.4) and (3.14) reveals that an attempt to use them in solving the optimization Problems 1 and 2 analytically is confronted with considerable difficulties. For example formula (3.4) requires the maximization of a complex functional with respect to the function $h_c(t)$, with the constraint $h(t) = h_c(t) + h_r(t)$, and under the additional assumption of invertibility of $h_c(t)$ and $h_r(t)$. This is a task which is hardly feasible, at least under the rather general definitions of $h(t)$, $h_c(t)$ and $h_r(t)$ assumed here. For this reason, in what follows we will use (3.4) and (3.14) to solve Problems 1 and 2 numerically.

Formulae (3.4) and (3.14) have been implemented in *Mathematica* in the case of any joint distribution of the original claims and linear premium income function $h(t) = u + c\,t$, where u is the total initial reserve and c is the total premium rate. Thus, Problems 1 and 2 have been solved with different joint distributions for the claim amounts and different choices for the rest of the model parameters. In the independent case, results for Exponential, Pareto and Weibull claim amount distributions are presented and the effect of their varying tail behavior on the probability of joint survival is assessed. In order to model dependence between claim severities, copula functions have been successfully used. The copula approach has allowed us to study how the assumption of dependence affects the solutions to Problems 1 and 2 and the probability of joint survival. For the purpose, a combination of Rotated Clayton copula with Weibull marginals has been implemented.

In general, our experience has shown that expression (3.4) is computationally more efficient than (3.14) since it converges faster with respect to k, i.e., a small number of terms is required in the summation in order to reach a desired accuracy of the result. The multiple integration is less computationally involved and hence faster, since all limits of integration in (3.4) are finite whereas in (3.14) the inner most integral is infinite. However, it should be noted that the derived expressions for $P(T^c > x, T^r > x)$ are rather general and that in each particular case, when the input parameters are fixed, both formulae could be simplified and of course, depending on the software used for the implementation, the computational efficiency may turn to be in favour of (3.14).

3.4.1 Independent claim severities

Here, we have assumed that claim amounts are independent and have three alternative distributions: lighter tailed Exponential and heavier tailed Pareto and Weibull distributions. The optimization Problems 1 and 2 have been solved in each of these cases and the effect of the different tail behaviour of the claim distributions on the optimal solutions have been studied. Sensitivity results with respect to the choice of other model parameters are also presented.

The solution of the optimization Problem 2 in the case of exponentially distributed claim severities with parameter $\alpha = 1$, Poisson intensity $\lambda = 1$, finite time interval $x = 2$ and $h(t) = u + c\,t$, with total initial reserve $u = 0$ and premium rate $c = 1.55$, is illustrated in Fig. 1. For fixed combinations of values of the levels M and L, an optimal reinsurance premium rate, c_r, is found, which maximizes $P(T^c > x, T^r > x)$, given that $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$. This is achieved by varying the proportion, $h_r(t) = c_r\,t$, of the premium income, given to the reinsurer from 1% to 99%, i.e., c_r is varied from 0.1 to 1.5 with a step 0.1. In the left panel of Fig. 1 we present results for the case of an XL contract without a limiting level, i.e. $L = \infty$, while the right panel refers to a retention level M and a limiting level $L = M + 0.5$. In both cases, the optimal premium rate c_r decreases when the retention level M increases. This complies well with the market principle that a smaller reinsurance premium should be charged for a smaller proportion of the risk, taken by the reinsurer. Comparing the two cases $L = \infty$ and $L = M + 0.5$, it can be seen that, in the latter case, the optimal solutions for c_r are shifted to the left, since there is a fixed non-zero layer $m = L - M = 0.5$, covered by the reinsurer.

From both panels of Fig. 1 it can also be seen that each curve has a single global maximum of the joint survival probability. This suggests that the optimization Problem 2 has a unique solution, at least for the classical linear $h(t)$. The proof of this interesting conjecture is hindered by the complexity of formulae (3.4) and (3.14) and in particular of the definitions of $\tilde{v}_j$, $\tilde{z}_j$, w_i^c, w_i^r, and is a subject of current investigation.

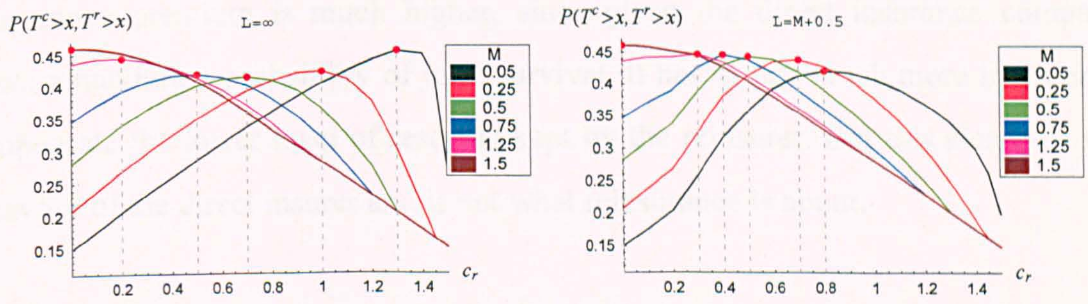

Fig. 1. Solutions to the optimality Problem 2: independent claim severities, Exp(1) distributed, $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$.

Problem 2 has also been solved for different choices of the total initial reserve u and the initial reserves of the cedent, u_c and the reinsurer, u_r. The impact of different initial reserves on $P(T^c > x, T^r > x)$ and hence on the optimal value of c_r is illustrated in the left panel of Fig 2, for fixed levels $M = 0.5$, $L = \infty$ and parameters as in Fig 1, i.e., Exp(1) distributed claim severities, $\lambda = 1$ and $x = 2$. For this set of parameters, an optimal value, c_r, is found, which maximizes $P(T^c > x, T^r > x)$, given that $h(t) = u + c\,t$, $h_c(t) = u_c + (1.55 - c_r)\,t$, $h_r(t) = u_r + c_r\,t$, with $u = u_c + u_r$ and $c = c_c + c_r = (1.55 - c_r) + c_r$. Five curves are given in the left panel of Fig 2 which correspond to five different choices of the pair of values u_c, u_r, for which the total reserve $u = u_c + u_r$ is correspondingly equal to 0.0, 1.0, 0.5, 1.0, 1.0. There are two effects which can be observed. First, with the increase of the total reserve u, given $u_c = u_r$, (see curves corresponding to $(u_c, u_r) = \{(0, 0), (0.25, 0.25), (0.5, 0.5)\}$), the probability of joint survival increases as can be expected. The second effect is that, for fixed value of the total reserve $u = 1$, the optimal reinsurance premium c_r is lower if $u_c < u_r$, increases when $u_c = u_r$, and goes further up if $u_c > u_r$. Hence, the conclusion is that, if a direct insurance company wants to pay less in reinsurance premium and at the same time wants to maximize its and the reinsurer's chances of survival, the company should seek for a reinsurer with initial reserves higher than its own reserves, which is a practically meaningful business strategy. In the alternative case, $u_c > u_r$, the optimal reinsurance premium is much higher, since given the direct insurance company wants a maximum probability of joint survival, it has to pay much more in order to compensate the lower level of reserves kept by the reinsurer. But this clearly is not in favour of the direct insurer and is not what reinsurance is about.

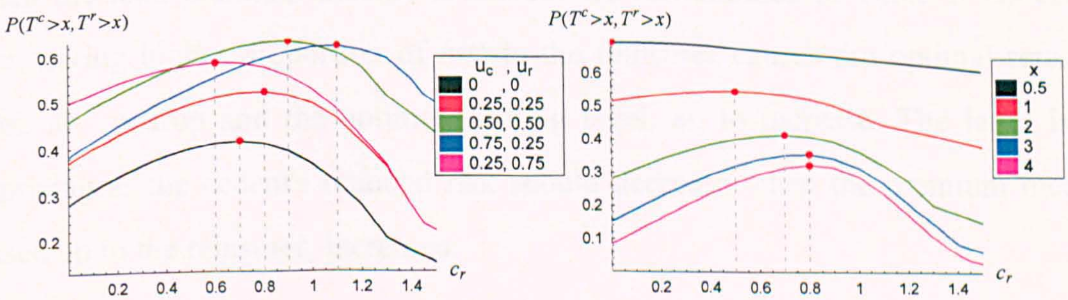

Fig. 2. Solutions to the optimality Problem 2: independent claim severities, Exp(1) distributed, $\lambda = 1$, $x = 2$, $c = 1.55$, $L = \infty$, $M = 0.5$; Left panel: $u \geq 0$, Right panel: $u = u_c = u_r = 0$, $x = 0.5, 1, 2, 3, 4$.

In the right panel of Fig. 2, we illustrate the impact of the time horizon x on the probability of joint survival and c_r. As can be seen, $P(T^c > x, T^r > x)$ decreases for longer time horizons, which is natural to expect. On the other hand, increasing x

from 0.5 to 3 results in higher reinsurance premium, whereas further increase of x does not affect c_r. This can be explained with the higher possibility of arrival of large claims to the reinsurer as x initially goes up.

The solution of the optimization Problem 1 has been performed in the case of exponentially and Pareto distributed claim severities, both with unit mean, $\lambda = 1$, $x = 2$ and $h(t) = 1.55\,t$. Thus, in Fig. 3 two 3D plots are given, which illustrate the behaviour of the probability of joint survival as a function of M and $m = L - M$ when the premium income is equally shared, i.e. $h_c(t) = h_r(t)$ for any $t \geq 0$. The left panel of Fig. 3 refers to the case of exponentially distributed claim amounts, W_i, $i = 1, 2, \ldots$ with mean and variance $E(W) = V(W) = 1$, whereas the plot in the right panel is for Pareto claims with $E(W) = 1$ and $V(W) = 3$. As seen from both panels of Fig. 3, $P(T^c > x, T^r > x)$ has a single global maximum with respect to M and m. As with Problem 2, the existence of a unique solution of Problem 1 can be conjectured, but the proof is related with similar difficulties.

Solutions of Problem 1 for different choices of c_r, i.e., for different proportions in which the total premium income is shared, are summarized in Table 1. As can be seen, giving higher proportion of $h(t)$ to the reinsurer causes the optimal retention level, M, to drop and the optimal limiting level, m, to increase. The latter is not surprising as the cedent's retained risk should decrease when the premium income, passed on to the reinsurer, increases.

Table 1. Optimal values of M and m, maximizing $P(T^c > x, T^r > x)$ in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$.

$\max_{M,m} P(T^c > x, T^r > x)$	$c_r = 0.25$	$c_r = 0.50$	$c_r = 0.775$	$c_r = 1.00$	$c_r = 1.25$
M	0.4	0.3	0.3	0.2	0.001
m	0.1	0.3	0.7	1.2	> 1.5

As can also be seen from Fig. 3, although the implemented Exponential and Pareto distributions have different variance and imply lighter and heavier tails of the claim severities, the two surfaces are very similar and the optimal values of M and m, which maximize $P(T^c > x, T^r > x)$ in each case, are very close. This is explained by the similarity in the shape of the Exponential and Pareto densities, as can be seen from the left panel of Fig. 4, since all other model parameters are the same. We have also implemented Weibull distributed claims, which does not affect the form of the surface as well. It is interesting to note that the probability of joint survival is higher for Pareto distributed claim amounts, compared with the exponential case, given that other model parameters coincide. The probability $P(T^c > x, T^r > x)$ is even higher if the claim size follows Weibull distribution with the same mean, $E(W) = 1$, and $V(W) = 2.2$. An illustration of the latter phenomenon is given in the right panel of Fig. 4. It can be explained by the fact that the time interval, [0, 2], is relatively short and $P(T^c > x, T^r > x)$ is affected most significantly by the distribution of the smaller but more probable claims rather than by the less probable extreme claims in the tail. This is in compliance with the order of the probabilities 0.955, 0.940, 0.917, computed as $P(W \leq h(2)) = P(W \leq 3.1)$ correspondingly for exponentially, Pareto and Weibull distributed claims. The shape of the three densities, given in the left panel of Fig. 4, are also in support of this explanation. Our experience shows that for higher x the tail behaviour is of more importance for $P(T^c > x, T^r > x)$ and the order may reverse.

The general conclusion based on these examples is that $P(T^c > x, T^r > x)$ is a relevant reinsurance risk optimization criterion, which complies with some basic principles driving reinsurance risk assessment and pricing decisions.

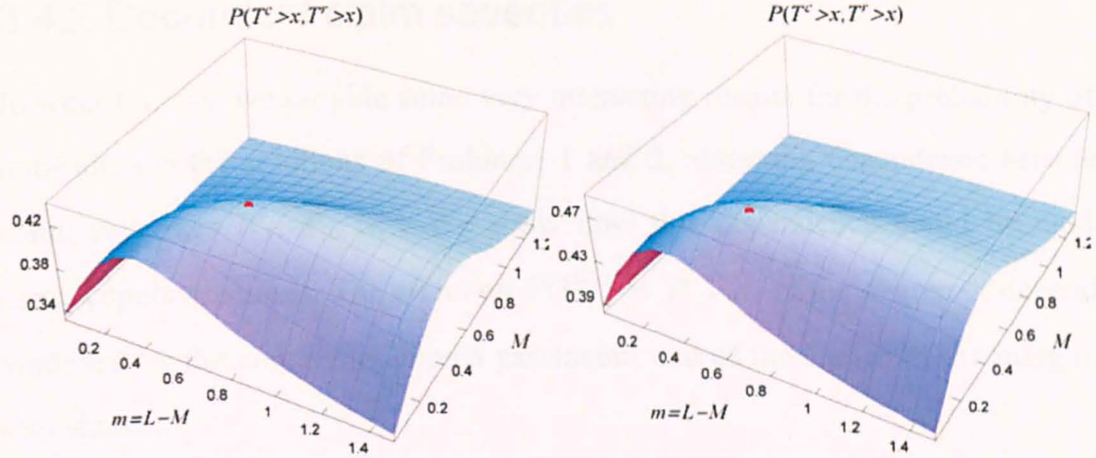

Fig. 3. Solutions to the optimality Problem 1: independent claim severities, $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$. Left panel - exponentially distributed, $E(W) = V(W) = 1$; Right panel - Pareto distributed, $E(W) = 1$, $V(W) = 3$.

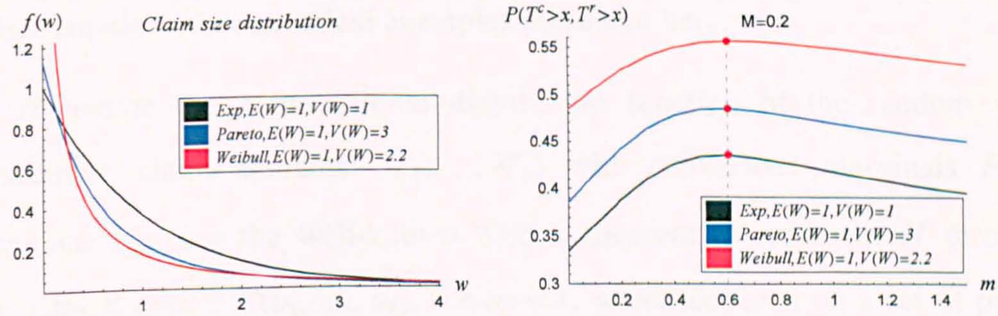

Fig. 4. Left panel - assumed probability density functions for the claim amounts W_i, $i = 1, 2, \ldots$; Right panel - $P(T^c > x, T^r > x)$ as a function of the layer m, $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$.

3.4.2 Dependent claim severities

In what follows, we provide some very interesting results for the probability of joint non-ruin and the solutions of Problems 1 and 2, assuming dependence between the claim severities W_1, W_2, ... We show how this dependence could be modelled, using copula functions. The effect on $P(T^c > x, T^r > x)$ of the degree of dependence, modelled by the underlying copula parameter, and of the choice of the marginals, is also studied.

A difficulty, related to the copula approach is that, in general, a large number of consecutive claims may arrive at the insurance company and modelling their joint distribution will require highly multivariate copulas. The curse of dimensionality is overcome here due to the fast convergence of formula (3.4), for which only the first few terms in the summation with respect to k are needed, in order to compute $P(T^c > x, T^r > x)$ with a reasonable accuracy. This allows us to use up to a five-variate copula in the numerical examples presented here.

Let H denote the k-dimensional distribution function of the random vector of consecutive claim amounts $(W_1, ..., W_k)$ with continuous marginals $F_1, ..., F_k$. Then, one can use the well-known Sklar's theorem to represent H through a k-dimensional copula $C(u_1, ..., u_k)$, $0 \le u_j \le 1$, which depends on a set of parameters $\boldsymbol{\theta}$, as $H(w_1, ..., w_k) = C(F_1(w_1), ..., F_k(w_k))$. By changing the values of $\boldsymbol{\theta}$ within a specified range, one can control the degree of dependence, in general, from extreme negative, through independence, to extreme positive dependence. To measure the dependence in the tails of the distributions of two consecutive claims W_1 and W_2, one can use the upper and lower tail dependence coefficients, defined as

$$\lambda_L = \lim_{u \to 0^+} C(u, u) / u$$

$$\lambda_U = \lim_{u \to 1^-} (1 - 2u + C(u, u)) / (1 - u)$$

where $\lambda_L \in (0, 1]$, $\lambda_U \in (0, 1]$. The copula C has no upper (lower) tail dependence iff $\lambda_U = 0$ ($\lambda_L = 0$). For example, in our context, $\lambda_U > 0$ would mean that extremely large insurance losses are likely to occur jointly. For further properties of copulas and related dependence measures we refer to Joe (1997). An extensive account on some actuarial applications of copulas can be found in Frees and Valdez (1998).

It should be noted that in most practical cases dependence between the components of the random vector $(W_1, ..., W_k)$ would imply dependence between the components of the random vector $\left(W_1^c, ..., W_k^c\right)$ and also between the components of $\left(W_1^r, ..., W_k^r\right)$, since $W_i = W_i^c + W_i^r$. So, the two risk processes, R_t^c and R_t^r, which implicitly define $P(T^c > x, T^r > x)$, would also incorporate dependent claims, namely $\left(W_1^c, ..., W_k^c\right)$ and $\left(W_1^r, ..., W_k^r\right)$. However, since formulae (3.4) and (3.14) involve the joint density function $\psi(w_1, ..., w_k)$ of the random vector $(W_1, ..., W_k)$, in order to compute $P(T^c > x, T^r > x)$ under dependence, we express this density through the copula function as

$$
\begin{aligned}
\psi(w_1, ..., w_k) &= \frac{\partial^k C(F_1(w_1), ..., F_k(w_k))}{\partial w_1 ... \partial w_k} \\
&= \frac{\partial^k C(u_1, ..., u_k)}{\partial u_1 ... \partial u_k} \prod_{i=1}^{k} \frac{\partial F_i(w_i)}{\partial w_i} = c(F_1(w_1), ..., F_k(w_k)) \prod_{i=1}^{k} f_{W_i}(w_i)
\end{aligned}
\tag{3.21}
$$

where $c(u_1, ..., u_k)$ is the density of the copula C and $f_{W_i}(w_i)$, $i = 1, ..., k$ are the marginal density functions. As can be seen from (3.21), the copula approach to modelling dependence between claim amounts is very convenient since it separates the dependence structure, incorporated into the copula, from the marginals. Thus, one can independently choose the copula and its parameter(s), and the marginals, and study separately the effect of these two choices on $P(T^c > x, T^r > x)$ and on the solutions of the optimality Problems 1 and 2. For the purpose, we have chosen C to be the k-dimensional Rotated Clayton copula, C^{RCl}, and F_1, ..., F_k to be identical Weibull(α, β) marginals.

Clayton and Rotated Clayton copulas are suitable for modelling dependence between claim severities. To see this, let us first introduce the Clayton copula, which is an Archimedean copula, with generator $\phi(t) = t^{-\theta} - 1, \theta > 0$, defined as

$$C^{\mathrm{Cl}}(u_1, \ldots, u_k; \theta) = \left(\sum_{i=1}^{k} u_i^{-\theta} - k + 1\right)^{-1/\theta} ,$$

where $0 \leq u_i \leq 1, i = 1, \ldots, k$ and $\theta \in (0, \infty)$ is a parameter. Its density is given by

$$c^{\mathrm{Cl}}(u_1, \ldots, u_k; \theta) = \theta^k \frac{\Gamma(1/\theta+k)}{\Gamma(1/\theta)} \left(\prod_{i=1}^{k} u_i^{-\theta-1}\right)\left(\sum_{i=1}^{k} u_i^{-\theta} - k + 1\right)^{-1/\theta-k} .$$

As $\theta \to 0$, the Clayton copula converges to the product copula with density $c(u_1, \ldots, u_k) = 1$, which, as seen from (3.21), corresponds to independent claim amounts. The degree of dependence increases as θ increases. Further properties of the Clayton copula and its application in finance can be found in Cherubini et al. (2004).

In the general insurance context, it is of interest to consider the case in which the occurrence of large claims is highly correlated with the emergence of further large claims. Hence, it is meaningful to use a copula with upper tail dependence. However, the Clayton copula has lower tail dependence with coefficient $\lambda_L = 2^{-1/\theta}$, which makes it convenient for modeling dependence in the left tails of the marginal distributions, i.e. between very small claims. A typical example would be the joint occurrence of a large number of small motor insurance claims caused by a common (catastrophic) event, e.g. hail or bad driving conditions.

Based on the Clayton copula, one can model upper tail dependence using the multivariate Rotated Clayton copula, defined as

$$C^{\mathrm{RCl}}(u_1, \ldots, u_k; \theta) = \sum_{i=1}^{k} u_i - k + 1 + \left(\sum_{i=1}^{k} (1 - u_i)^{-\theta} - k + 1\right)^{-1/\theta} \tag{3.22}$$

with density $c^{\mathrm{RCl}}(u_1, \ldots, u_k; \theta) = c^{\mathrm{Cl}}(1 - u_1, \ldots, 1 - u_k; \theta)$ and $\theta \in (0, \infty)$. The value $\theta = 0$ corresponds to independence as for C^{Cl}. A two dimensional version of (3.22) has been considered by Patton (2004). The Rotated Clayton copula has upper tail dependence with coefficient $\lambda_U = 2^{-1/\theta}$ and is suitable for modeling dependence between extreme insurance losses. The dependence structure, defined by a Rotated Clayton copula with parameter $\theta = 5$, is illustrated in the left panel of Fig. 5 through a random sample of 500 simulated pairs (u_1, u_2). In the right panel, we give the corresponding simulated claim amounts with joint distribution function $H(w_1, w_2) = C^{\mathrm{RCl}}(F_1(w_1), F_2(w_2); \theta)$ and identical Weibull(1, 1) marginals. The presence of positive dependence, determined by $\theta = 5$, and of upper tail dependence, $\lambda_U = 2^{-1/5}$, are clearly visible.

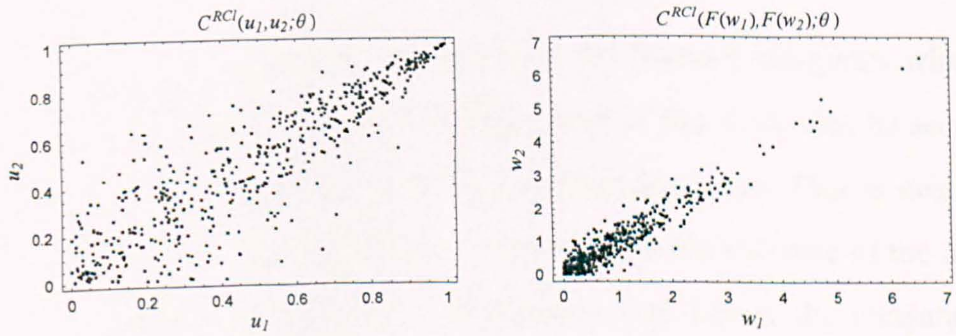

Fig. 5. A random sample of 500 simulations from a bivariate Rotated Clayton copula, with dependence parameter $\theta = 5$, marginals $F \equiv$ Weibull(1, 1) $\equiv$ Exp(1).

With the increase of θ, the solution of the optimality Problem 2 does not change, as illustrated in the left panel of Fig. 6 for fixed Weibull marginals with unit mean and variance. It can also be seen that, for any c_r, $P(T^c > x, T^r > x)$ goes up as θ deviates from zero. This may seem unexpected but it should be mentioned that, as θ increases, not only the tail dependence increases but so does the dependence throughout the whole range of claim amounts. As a result of this, jointly small claims occur with higher probability and through the risk processes, R_t^c and R_t^r, affect more significantly $P(T^c > x, T^r > x)$ than the occurrence of jointly large claims.

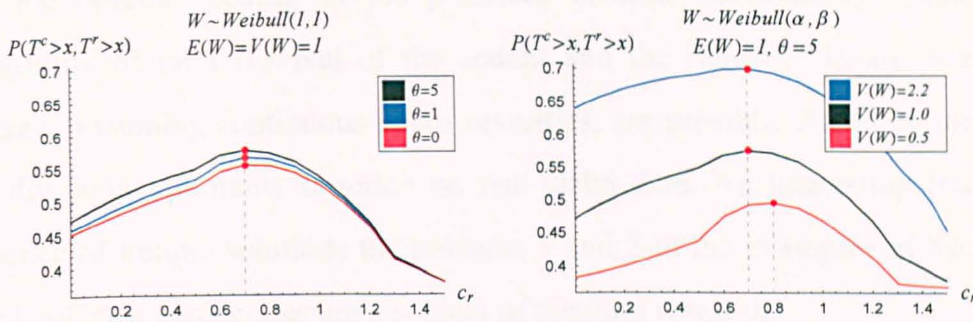

Fig. 6. Solutions to the optimality Problem 2: dependent claim severities, $C^{RCl}(F(w_1), \ldots, F(w_k); \theta)$ distributed, marginals $F \equiv Weibull(\alpha, \beta)$, $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $M = 0.25$, $L = M + 0.5$.

The solution of the optimality Problem 2 for Weibull marginals with mean 1 and increasing variance is given in the right panel of Fig. 6. As can be seen, the optimal value for c_r slightly decreases as the variance increases. This is meaningful, since the variance of the cedent's claims increases with the variance of the original claims more significantly than that of the reinsurer and hence, the reinsurance premium should decrease. The latter effect is due to the fact that the reinsurer's liability is limited within the layer m. It can also be seen from the right panel of Fig. 6 that $P(T^c > x, T^r > x)$ increases as the variance increases which is a phenomenon, similar to the one illustrated in Fig. 4 and can be explained applying similar reasoning.

3.5 Conclusions and comments

In this paper, we have demonstrated that the optimal retention and limiting levels and the optimal sharing of the premium income, obtained by maximizing the probability of joint survival of the cedent and the reinsurer in an excess of loss contract, assuming continuous claim severities, are sensible. It will be instructive to test this joint optimality criterion on real claim data. An interesting finding is the presence of unique solutions to Problems 1 and 2 in the examples of Section 3.4.1. Proofs of such conjectures are a subject of ongoing research.

Let us note that in the model presented here the initial capital $u_c = h_c(0)$ and $u_r = h_r(0)$ should not necessarily be shared between the two parties. It has to be noted also that a reinsurance company has typically many clients. However, often some of these clients choose to work (exclusively) with one particular big reinsurance company, such as for example Swiss Re, Munich Re etc., and they form a substantial part of the total business underwritten by the reinsurer. In such cases, when the joint survival of the two parties is critical, the model considered here can be applied on a bilateral basis. It is also appropriate and applicable in cases where the two parties involved in the contract are for example represented by e.g. a company (not necessarily an insurance company) and its captive or two parties exchanging risk in a syndicate like Lloyds.

We have demonstrated that formulae (3.4) and (3.14), through their reasonable generality, conveniently allow to implement copulas in modelling dependence between consecutive claim severities. These are only first steps in this important new direction of research and a variety of open problems arrises. For example, it is interesting to explore how the solutions of Problems 1 and 2, and also $P(T^c > x, T^r > x)$, will be affected by different dependence structures. In particular, will the upper and lower Fréchet bounds lead to upper and lower bounds for $P(T^c > x, T^r > x)$?

Finally, viewing $P(T^c > x, T^r > x)$ as a risk measure, one could define a performance measure based on the expected profits, at the end of the time horizon x, of the insurer and the reinsurer and consider an optimality criterion which combines these measures and could be used to optimally set the parameters of a reinsurance contract. The latter is a subject of future investigation.

Chapter 4

Optimal joint survival reinsurance: an efficient frontier approach

Summary

The problem of optimal excess of loss reinsurance with a limiting and a retention level is considered. It is demonstrate that this problem can be solved, combining specific risk and performance measures, under some relatively general assumptions for the risk model, incorporating any non-decreasing premium income function, Poisson claim arrivals and continuous claim amounts, modelled by any joint distribution. As a performance measure, we define the expected profits at time x of the direct insurer and the reinsurer, given their joint survival up to x, and derive explicit expressions for their numerical evaluation. The probability of joint survival of the direct insurer and the reinsurer up to the finite time horizon x is employed as a risk measure. An efficient frontier type approach to setting the limiting and the retention levels, based on the probability of joint survival considered as a risk measure and on the expected profit given joint survival, considered as a performance measure is introduced. Several optimality problems are defined and their solutions are illustrated numerically on several examples of appropriate claim amount distributions, both for the case of dependent and independent claim severities.

4.1 Introduction

An upward trend in insurance and reinsurance claims frequency and severity has recently been observed, mostly due to catastrophic events, such as hurricane Katrina in the USA in 2005 and the winterstorm Kirill over northern Europe in 2007, causing enormous damage to households and infrastructure, measured in billions of dollars. As a result of this, both the insurance and reinsurance industry suffered severe losses, (see e.g. Zanetti, Schwarz and Lindemuth 2007 for an up-to-date account on world largest losses), and some companies became even insolvent. In order to cope with increasing future catastrophic risk, the industry faces the necessity of improving their internal risk models and especially, their implementation and use in the context of reinsurance. In particular, it becomes more clear that such models have to incorporate the interests of both insurance and reinsurance companies in order for them to maximize their chances of (joint) survival.

Coherent with these developments are the recent attempts in the actuarial literature to introduce joint risk and performance measures (see papers by Ignatov, Kaishev and Krachunov 2004, and Kaishev and Dimitrova 2006) which can be used in determining the parameters of a reinsurance contract. These studies are preceded by extensive research on optimal reinsurance performed in previous years, solely from the point of view of the direct insurer. Recent examples in this direction are the papers by Kaluszka (2004) and Verlaak and Beirlant (2003), who study mean-variance optimality criteria, Gajek and Zagrodny (2004a) and Cao and Zhang (2007) who look at general risk measures, and Liang and Guo (2007), Gajek and Zagrodny (2004b), and Schmidli (2004) where the risk is measured by the probability of ruin. A summary on the variety of research techniques used in setting optimal reinsurance arrangements and further references can be found in Centeno (2004), Aase (2002) and Ignatov, Kaishev and Krachunov (2004).

Recently, Ignatov, Kaishev and Krachunov (2004) and Kaishev and Dimitrova (2006) considered a reinsurance optimality model, which combines the (contradicting) interests of both the cedent and the reinsurer under an excess of loss contract. Under this model, claims generated by a volume of risks arrive according to a Poisson process and the two parties share each individual claim and the total premium income in such a proportion that a certain joint optimality criterion is maximized (minimized). In their paper, Ignatov, Kaishev and Krachunov (2004), assumed that claim severities have any discrete joint distribution and considered a simple excess of loss without a policy limit. As a joint risk measure they proposed to use the probability of joint survival of the cedent and the reinsurer up to a finite time horizon and derived explicit expressions for this probability. As a joint performance measure, the expected profit of each of the parties at a finite-time horizon, given their joint survival up to this instant has also been considered.

The model has been extended further in the paper by Kaishev and Dimitrova (2006), where it was assumed that claim amounts have any continuous (dependent) joint distribution and the excess of loss has a retention and a policy limit. Under these assumptions, closed form expressions for the probability of joint survival have been derived. Based on these expressions, it was demonstrated that retention and limiting levels could be optimally set by maximizing the probability of joint survival, given the premium income is split in a preassigned proportion or alternatively, an optimal split of the premium income between the two parties could be determined, given fixed retention and limiting levels.

In the present paper, we consider the model of Kaishev and Dimitrova (2006) and propose a Markowitz type efficient frontier solution to the problem of optimally setting the retention and limiting levels M and L, so that for a given level of the probability of joint survival the expected profits of the two parties are maximized. As an alternative, it is proposed to use an optimality criterion which provides for 'fair' distribution of the expected profits based on the agreed allocation of the premium income. In order to implement these ideas, we derive explicit expressions for the expected profit of the cedent and the reinsurer at some future moment in time, given their joint survival up to this instant.

The paper is organized as follows. In section 4.2, we briefly introduce the model and recall the formulae for the probability of joint survival of Kaishev and Dimitrova (2006). In section 4.3, explicit expressions for the expected profits of the direct insurer and the reinsurer are derived. The optimality problems, which incorporate these joint risk and performance measures, are formulated in section 4.4 and their efficient frontier solutions are illustrated. Section 4.5 concludes the paper with some comments on the results and possibilities of future research.

4.2 The excess of loss (XL) risk model of joint survival

4.2.1 The model

We consider an insurance portfolio, generating claims at some random moments of time. The claims inter-arrival times $\tau_1, \tau_2, \ldots$ are assumed identically, exponentially distributed r.v.s with parameter λ. Denote by $T_1 = \tau_1$, $T_2 = \tau_1 + \tau_2$, ... the sequence of random variables representing the consecutive moments of occurrence of the claims. Let $N_t = \#\{i : T_i \leq t\}$, where # is the number of elements of the set {.}. The claim severities are modeled by the continuous r.v.s. W_1, W_2, ..., W_k, ... with joint density function $\psi(w_1, \ldots, w_k)$. For convenience, we will introduce also the random variables $Y_1 = W_1$, $Y_2 = W_1 + W_2$, ... representing the partial sums of consecutive claim amounts.

It is assumed that the r.v.s W_1, W_2, ... are independent of N_t. Then, the risk (surplus) process R_t, at time t, is given by $R_t = h(t) - Y_{N_t}$, where $h(t)$ is a nonnegative, non-decreasing, real function, defined on $\mathbb{R}_+$, representing the aggregate premium income up to time t. The function $h(t)$ may be continuous or not. If $h(t)$ is discontinuous, we define $h^{-1}(y) = \inf\{z : h(z) \geq y\}$. Note that the classical case $h(t) = u + ct$, with initial reserve u and premium rate c, is included in this rather general class of functions $h(t)$.

In this paper, we will be concerned with the case when the insurance company wants to reinsure its portfolio of risks by concluding an XL contract with a retention level $M \geq 0$ and a limiting level $L \geq M$. In other words, the cedent wants to reinsure the part of each claim which hits the layer $m = L - M$, i.e. each individual claim W_i is shared between the two parties so that $W_i = W_i^c + W_i^r$, $i = 1, 2, \ldots$, where W_i^c and W_i^r denote the parts covered respectively by the cedent and the reinsurer. Clearly, we can write

$$W_i^c = \min(W_i, M) + \max(0, W_i - L)$$

and

$$W_i^r = \min(L - M, \max(0, W_i - M)).$$

Denote by $Y_1^c = W_1^c$, $Y_2^c = W_1^c + W_2^c$, ... and by $Y_1^r = W_1^r$, $Y_2^r = W_1^r + W_2^r$, ... the consecutive partial sums of claims to the cedent and to the reinsurer, respectively. Under our XL reinsurance model, the total premium income $h(t)$ is also divided between the two parties so that $h(t) = h_c(t) + h_r(t)$, where $h_c(t)$, $h_r(t)$ are the premium incomes of the cedent and the reinsurer, assumed also non-negative, non-decreasing functions on $\mathbb{R}_+$. As a result, the risk process, R_t, can be represented as a superposition of two risk processes, that of the cedent

$$R_t^c = h_c(t) - Y_{N_t}^c \tag{4.1}$$

and of the reinsurer

$$R_t^r = h_r(t) - Y_{N_t}^r \tag{4.2}$$

i.e., $R_t = R_t^c + R_t^r$. Note that the two risk processes R_t^c and R_t^r are dependent through the common claim arrivals and the claim severities W_i, $i = 1, 2, \ldots$, as seen from (4.1) and (4.2).

Under this model, explicit formulae for the probability of joint survival, $P(T^c > x, T^r > x)$, of the cedent and the reinsurer within a finite time interval

$[0, x]$, $x > 0$, were derived by Kaishev and Dimitrova (2006). The moments, T^c and T^r, of ruin of correspondingly the cedent and the reinsurer are defined as

$T^c := \inf\{t : t > 0,\ R_t^c < 0\}$,

$T^r := \inf\{t : t > 0,\ R_t^r < 0\}$.

Clearly, the two events $(T^c > x)$ and $(T^r > x)$, of survival of the cedent and the reinsurer are dependent and hence, $P(T^c > x, T^r > x)$, is a meaningful measure of the risk the two parties share and jointly carry.

In section 4.2.3, we will define the expected profit for each of the two parties, given joint survival up to time x, and show how this performance measure can be used in combination with the risk measure $P(T^c > x, T^r > x)$ in finding the optimal set of parameters related to an XL reinsurance contract.

4.2.2 The probability of joint survival

There are two alternative optimization problems which have been stated in connection with the XL contract, considered here. The first is, given M and m are fixed, divide the premium income $h(t)$ between the two parties, so as to maximize the probability of joint survival, $P(T^c > x, T^r > x)$. And alternatively, if the total premium income, $h(t)$, is divided in an agreed way between the cedent and the reinsurer, i.e. $h_c(t)$ and $h_r(t) = h(t) - h_c(t)$ are fixed, set the parameters M and L of the XL contract so as to maximize $P(T^c > x, T^r > x)$. Obviously, both optimization problems are based solely on the joint risk measure $P(T^c > x, T^r > x)$. To address these problems, Kaishev and Dimitrova (2006) derived explicit expressions for $P(T^c > x, T^r > x)$ given by the following theorems.

Theorem 1. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$P(T^c > x, T^r > x) =$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k) \psi(w_1, \ldots, w_k)\, dw_k \ldots dw_2\, dw_1\right) \tag{4.3}$$

where

$\tilde{v}_j = \min(\tilde{z}_j, x)$, $\tilde{z}_j = \max\left(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r)\right)$, $y_j^c = \sum_{i=1}^{j} w_i^c$, $y_j^r = \sum_{i=1}^{j} w_i^r$, $j = 1, \ldots, k$,

$w_i^c = \min(w_i, M) + \max(0, w_i - L)$, $w_i^r = \min(L - M, \max(0, w_i - M))$, *and*

$A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k)$, $k = 1, 2, \ldots$ *are the classical Appell polynomials* $A_k(x)$ *of degree* k, *defined by*

$$A_0(x) = 1, \qquad A_k'(x) = A_{k-1}(x), \qquad A_k(\tilde{v}_k) = 0.$$

For further properties of Appell polynomials we refer to Kaz'min (2002). An alternative formula for $P(T^c > x, T^r > x)$ is provided by the following

Theorem 2. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$P(T^c > x, T^r > x) =$

$$e^{-\lambda x}\left(\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x) \psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\right) \tag{4.4}$$

where

$B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x) = \sum_{j=0}^{l-1} (-\lambda)^j b_j(\tilde{z}_1, \ldots, \tilde{z}_j)\left(\sum_{m=0}^{l-j-1} \frac{(x\lambda)^m}{m!}\right)$, with $B_0(\cdot) \equiv 0$, $B_1(\cdot) = 1$,

l is such that $\tilde{z}_1 \le \ldots \le \tilde{z}_{l-1} \le x < \tilde{z}_l$,

$b_j(\tilde{z}_1, \ldots, \tilde{z}_j) = \sum_{i=1}^{j} (-1)^{j+i} \frac{\tilde{z}_j^{j-i+1}}{(j-i+1)!} b_{i-1}(\tilde{z}_1, \ldots, \tilde{z}_{i-1})$, *with* $b_0 \equiv 1$,

$\tilde{z}_j$ *are defined as in Theorem 1.*

As noted in Kaishev and Dimitrova (2006), the above two expressions can be used interchangeably and depending on the specified parameters and the software used for implementation either (4.3) or (4.4) can be faster and less computationally involved.

In the next section, we will supplement the risk measure $P(T^c > x, T^r > x)$ by a performance measure and in section 4.3 we will demonstrate how the two measures can be combined into a single optimization problem, which incorporates the contradictory goals of maximizing the profit and minimizing the risk of the cedent and the reinsurer.

4.3 The expected profit given joint survival

Under the general model of an XL contract with a retention level M and a limiting level L, and assuming claims have any continuous joint distribution, we will be concerned here with the profit at time x, each of the parties are expected to make, given they both survive up to x. Considering a joint optimality criterion, based on expected profit given joint survival, is reasonable since with the eventual ruin of either of the parties the XL reinsurance contract will cease and this will affect the risk and profitability of the surviving party. So, obviously the two parties have mutually dependent performance with respect not only to the risk they carry but also with respect to their expected profits. Expected profit assuming joint survival was first considered by Ignatov, Kaishev and Krachunov (2004) in the case of a simple XL contract with one retention level and discrete integer-valued claims.

In what follows, we will present some explicit expressions for these quantities and a result establishing the existence of values of M and L such that the expected profits of the two parties are in the same proportion as their premium incomes. First, we will introduce some useful definitions and notation. Following Ignatov, Kaishev and Krachunov (2004), we will define the profits at time x of the cedent and the reinsurer, correspondingly as the values, R_x^c and R_x^r, of their risk processes, given by (4.1) and (4.2), at time x. Denote by I_A and I_B the indicator random variables of the events $A = \{T^c > x\}$ and $B = \{T^r > x\}$. There exists a suitable function $\phi(u, v)$ such that the conditional expectation $E(R_x^c \mid I_A, I_B) \overset{a.s.}{=} \phi(I_A, I_B)$. When $I_A \equiv 1$ and $I_B \equiv 1$, we obtain $\phi(1, 1) = E[R_x^c \mid (T^c > x, T^r > x)]$ which we will call the expected profit of the cedent at time x, given the two parties' joint survival up to time x. Similarly, $E[R_x^r \mid (T^c > x, T^r > x)]$ denotes the reinsurer's expected profit at time x, given its and the insurer's joint survival up to time x.

The following two theorems give explicit expressions for $E[R_x^c \mid (T^c > x, T^r > x)]$ and $E[R_x^r \mid (T^c > x, T^r > x)]$ correspondingly.

Theorem 3. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E[R_x^c \mid (T^c > x, T^r > x)] =$$

$$h_c(x) - \Big\{\sum_{k=1}^{\infty}\lambda^k \int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^c A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k)\, \psi(w_1, \ldots, w_k)\, dw_k \ldots dw_2\, dw_1\Big\} \Big/ \tag{4.5}$$

$$\Big\{1 + \sum_{k=1}^{\infty}\lambda^k \int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k)\, \psi(w_1, \ldots, w_k)\, dw_k \ldots dw_2\, dw_1\Big\}$$

where y_k^c, $\tilde{v}_j$, $j = 1, \ldots, k$ and $A_k(x; \tilde{v}_1, \ldots, \tilde{v}_k)$ *are defined as in Theorem 1.*

Proof. In view of the definitions (4.1) and (4.2) of the risk processes R_t^c and R_t^r, and expression (4.3) for the probability of joint survival, we can express the unconditional expectation $E(R_x^c \cdot I_A \cdot I_B)$ as

$$E(R_x^c \cdot I_A \cdot I_B) =$$

$$e^{-x\lambda}\left\{h_c(x) + \sum_{k=1}^{\infty}\lambda^k \int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-1}}\left(h_c(x) - \sum_{i=1}^{k} w_i^c\right)\right.$$
$$\left. A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)\ \psi(w_1,\ \ldots,\ w_k)\,dw_k \ldots dw_2\,dw_1\right\}$$

Note that in equality (4.6), if k claims have occurred up to time x, where $k = 1, 2, \ldots$, the profit of the cedent at the end of the time horizon $[0, x]$ is equal to $h_c(x) - \sum_{i=1}^{k} w_i^c$, and if no claims have occurred, i.e. $k = 0$, the profit is equal to the premium income at time x, i.e. $h_c(x)$, which is accounted for by the first term of the sum in (4.6). The unconditional expectation (4.6) can be rewritten as

$$E(R_x^c \cdot I_A \cdot I_B) =$$

$$e^{-x\lambda}\, h_c(x)\left\{1 + \sum_{k=1}^{\infty}\lambda^k \int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)\right.$$
$$\left.\psi(w_1,\ \ldots,\ w_k)\,dw_k \ldots dw_2\,dw_1\right\} - \tag{4.7}$$
$$e^{-x\lambda}\left\{\sum_{k=1}^{\infty}\lambda^k \int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-1}}\left(\sum_{i=1}^{k} w_i^c\right) A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)\right.$$
$$\left.\psi(w_1,\ \ldots,\ w_k)\,dw_k \ldots dw_2\,dw_1\right\}$$

For the conditional expectation $E[R_x^c \mid (T^c > x,\ T^r > x)]$ we have

$$E[R_x^c \mid (T^c > x,\ T^r > x)] = \frac{E\,(R_x^c \cdot I_A \cdot I_B)}{P(T^c > x,\ T^r > x)} \tag{4.8}$$

Substituting (4.7) and (4.3) in (4.8), and after cancelling appropriate terms, recalling the notation $\sum_{i=1}^{k} w_i^c = y_k^c$, we obtain the assertion of the theorem.□

Similarly, for the expected profit of the reinsurer we have

Theorem 4. *The expected profit of the reinsurer at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E[R_x^r \mid (T^c > x,\ T^r > x)] =$$

$$h_r(x) - \left\{\sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^r\, A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)\right.$$

$$\left.\psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1\right\} \Big/ \tag{4.9}$$

$$\left\{1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)\right.$$

$$\left.\psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1\right\}$$

where w_i^r, $\tilde{v}_j$, $j = 1,\ \ldots,\ k$ and $A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)$ *are defined as in Theorem 1.*

Proof. The proof follows the same lines of reasoning as in Theorem 3, replacing the premium income and the claims to the cedent with the ones to the reinsurer.□

Alternative formulae for $E[R_x^c \mid (T^c > x,\ T^r > x)]$ and $E[R_x^r \mid (T^c > x,\ T^r > x)]$ can be derived using expression (4.4) for $P(T^c > x,\ T^r > x)$ and its derivation. They are given in the next two theorems.

Theorem 5. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E[R_x^c \mid (T^c > x, T^r > x)] =$$

$$h_c(x) - \Big\{\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} y_k^c$$

$$B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)\,\psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\Big\} \Big/ \quad (4.10)$$

$$\Big\{\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)$$

$$\psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\Big\}$$

where $\tilde{z}_j$ and $B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)$ *are defined as in Theorem 2.*

Theorem 6. *The expected profit of the reinsurer at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E[R_x^r \mid (T^c > x, T^r > x)] =$$

$$h_r(x) - \Big\{\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} y_k^r$$

$$B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)\,\psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\Big\} \Big/ \quad (4.11)$$

$$\Big\{\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)$$

$$\psi(w_1, \ldots, w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\Big\}$$

where $\tilde{z}_j$ and $B_l(\tilde{z}_1, \ldots, \tilde{z}_{l-1}, x)$ *are defined as in Theorem 2.*

As with (4.3) and (4.4) for $P(T^c > x, T^r > x)$, the expressions (4.5), (4.9) and (4.10), (4.11) can be used interchangeably and depending on the specified parameters and the software used for implementation either of them can converge faster and be less computationally involved.

4.4 Combining the risk and performance measures in setting an optimal XL contract

In this section, we will illustrate how the probability of joint survival up to time x and the expected profits at time x, given joint survival of the cedent and the reinsurer up to x, can be used in combination, correspondingly as risk and performance measures, in order to set (optimally) the parameters of an XL reinsurance contract. Our approach is motivated by the mean-variance, portfolio optimization model of Markowitz (1952), in which an efficient frontier is found where the expected return from an investment portfolio over the investment horizon x is maximized for a given level of risk, measured by the variance of the portfolio return.

We outline and discuss several alternative approaches of solving the optimal XL reinsurance problem. The solution under any of them is obtained as a reasonable compromise between the contradictory risk and performance optimality criteria. On one hand, it is in the interest of the direct insurance company to possibly maximize the risk and minimize the premium income it transfers to the reinsurer. On the other hand, the reinsurance company aims at minimizing the risk and maximizing the portion of the premium it charges. In this way, both companies are aiming at optimizing their individual risk and performance measures. At the same time, it is reasonable to assume that the two parties are rational investors and hence, are interested in decreasing their joint probability of ruin and increasing their expected profits, given joint survival. Here, we state three problems which illustrate different approaches for determining the values of the retention and the limiting levels, M and L, given a split of the premium income $h(t) = h_c(t) + h_r(t)$, which balances the conflicting goals of the cedent and the reinsurer. The complexity of the expressions derived in Theorems 1 to 6 precludes the possibility of solving the stated problems analytically but as we will see, finding the numerical solutions is straightforward.

For convenience, throughout this section we will use the notation $m = L - M$ for the layer covered by the reinsurer.

In order to exemplify these approaches, formulae (4.3), (4.4), (4.5), (4.9), (4.10) and (4.11), given by Theorems 1 to 6, were implemented in *Mathematica* under two sets of model assumptions: one with independent exponentially distributed claim amounts and one with dependent claim severities, modelled by a Rotated Clayton Copula, $C^{\mathrm{RCl}}(F(w_1), ..., F(w_k); \theta)$, with $F \equiv \mathrm{Weibull}(\alpha, \beta)$ marginals and dependence parameter θ. In this way, we are able to study also the effect of dependence on the choice of the parameters of an XL contract. In both cases, we have assumed linear premium income function $h(t) = u + c\,t$, where u is the total initial reserve and c is the total premium rate per unit of time.

A random sample of 500 simulated data points from a bivariate Rotated Clayton copula, with dependence parameter $\theta = 1$ and Weibull(2.12, 1.14) marginals is presented in Fig.1. One of the properties of this particular type of copula is that it has an upper tail dependence and therefore, in our context it models positive dependence between large claim amounts. We refer the reader to Kaishev and Dimitrova (2006), where the expressions for a multidimensional Rotated Clayton copula and its density, together with some further applications in modelling dependence among claims severities, can be found.

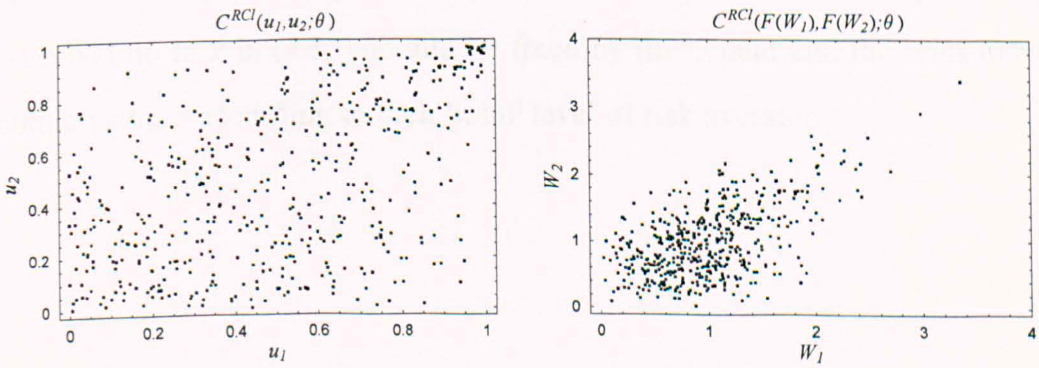

Fig. 1. A random sample of 500 simulations from a bivariate Rotated Clayton copula, with dependence parameter $\theta = 1$, marginals $F \equiv$ Weibull(2.12, 1.14).

Being able to calculate $P(T^c > x, T^r > x)$, $E[R_x^c \mid (T^c > x, T^r > x)]$ and $E[R_x^r \mid (T^c > x, T^r > x)]$, the 'individual' approach of the cedent and the reinsurer for finding optimal values of M and m, given $h(t) = h_c(t) + h_r(t)$, can be formulated as follows.

Problem 1. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$, find

$$\begin{aligned} &\max_{M,m} \; E[.\mid (T^c > x, T^r > x)] \\ &\text{subject to} \;\; P(T^c > x, T^r > x) = p\,. \end{aligned} \tag{4.12}$$

The expectation $E[.\mid (T^c > x, T^r > x)]$ in (4.12) is taken with respect to either R_x^c or R_x^r.

Solving Problem 1 simply means that the cedent and the reinsurer would choose points (M^c, m^c) and (M^r, m^r) respectively from their 'individual' efficient frontiers. The efficient frontier in our context is the set of dominant pairs of retention and limiting levels, (M, L), in the sense that the latter provide the highest return, measure by $E[.\mid (T^c > x, T^r > x)]$, for a chosen level of risk, measure by $1 - P(T^c > x, T^r > x)$.

The solution of Problem 1 is illustrated in Fig. 2, where it is assumed that the risk for each of the two parties of the XL reinsurance contract is measured by the complement of the probability of their joint survival up to time x. The probability of joint survival up to x in (4.12) should be fixed by the cedent and the reinsurer to an acceptable value p according to their 'joint' level of risk aversion.

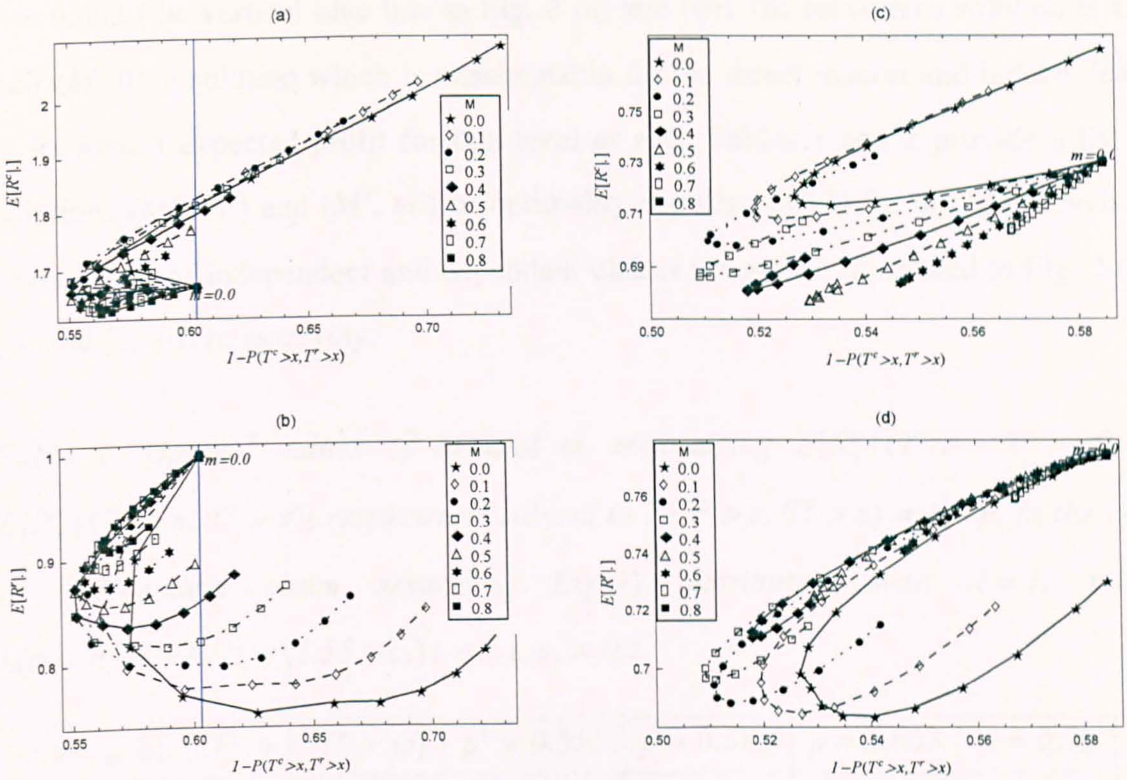

Fig. 2. $E[R^c_x \mid (T^c > x,\ T^r > x)]$ *and* $E[R^r_x \mid (T^c > x,\ T^r > x)]$ *respectively plotted against* $1 - P(T^c > x,\ T^r > x)$ *in the case of: (a) and (b) - independent claim severities,* $Exp(1)$ *distributed,* $m = 0.0,\ 0.1,\ 0.2,\ \ldots,\ 1.5,$ *with* $\lambda = 1,$ $x = 2,$ $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t,$ $c_r = 0.5$*; (c) and (d) - dependent claim severities,* $C^{RCl}(F(w_1),\ \ldots,\ F(w_k);\ \theta)$ *distributed with* $F \equiv Weibull(2.12,\ 1.14)$ *marginals and* $\theta = 1,$ $m = 0.0,\ 0.05,\ 0.1,\ \ldots,\ 0.8,$ *with* $\lambda = 1,$ $x = 1,$ $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t,$ $c_r = 0.775.$

It is obvious that, given $h(t) = h_c(t) + h_r(t)$ and fixed level p, such an 'individual' approach may not lead to one and the same optimal solution (M, m), since the interests of the two parties are contradictory. As can be seen from Fig. 2, if $p = p^* = \max_{M,m} P(T^c > x,\ T^r > x) = \min_{M,m}(1 - P(T^c > x,\ T^r > x))$ the solution to Problem 1 will be one and the same for the two parties and will coincide with the solution of Problem 1 of Kaishev and Dimitrova (2006). However, as seen from Fig. 2 (a) and (b), in the case of i.i.d. Exp (1) distributed claim amounts for instance, if

$p = 0.603$ (the vertical blue line in Fig. 2 (a) and (b)), the reinsurer's solution is any pair $(M, 0)$, a solution which is unacceptable for the direct insurer and indeed, leads to its lowest expected profit for this level of risk. Tables 1 and 2 provide a list of solutions (M^c, m^c) and (M^r, m^r) of optimality problem (4.12) for different levels p, in the cases of independent and dependent claims severities, illustrated in Fig. 2 (a), (b) and (c), (d) respectively.

Table 1. Optimal values of M and m, maximizing $E[R_x^c \mid (T^c > x, T^r > x)]$ *or* $E[R_x^r \mid (T^c > x, T^r > x)]$ *respectively subject to* $P(T^c > x, T^r > x) = 1 - p$*, in the case of independent claim severities, Exp(1) distributed, with* $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)t + c_r t$, $c_r = 0.5$.

$max_{M,m}$ $E[. \mid (T^c > x, T^r > x)]$	$p^* = 0.551$	$p = 0.585$	$p = 0.603$	$p = 0.70$
(M^c, m^c)	(0.3, 0.3)	(0.2, 0.4)	(0.1, 0.4)	(0.1, 1.5)
(M^r, m^r)	(0.3, 0.3)	(0.8, 0.2)	$(M, 0)$	(0.1, 1.5)

Table 2. Optimal values of M and m, maximizing $E[R_x^c \mid (T^c > x, T^r > x)]$ *or* $E[R_x^r \mid (T^c > x, T^r > x)]$ *respectively subject to* $P(T^c > x, T^r > x) = 1 - p$*, in the case of dependent claim severities,* $C^{RCl}(F(w_1), \ldots, F(w_k); \theta)$ *distributed with* $F \equiv Weibull(2.12, 1.14)$ *marginals and* $\theta = 1$, *with* $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)t + c_r t$, $c_r = 0.775$.

$max_{M,m}$ $E[. \mid (T^c > x, T^r > x)]$	$p^* = 0.509$	$p = 0.515$	$p = 0.54$	$p = 0.56$
(M^c, m^c)	(0.3, 0.5)	(0.2, 0.4)	(0.1, 0.6)	(0.1, 0.8)
(M^r, m^r)	(0.3, 0.5)	(0.3, 0.4)	(0.4, 0.3)	(0.5, 0.2)

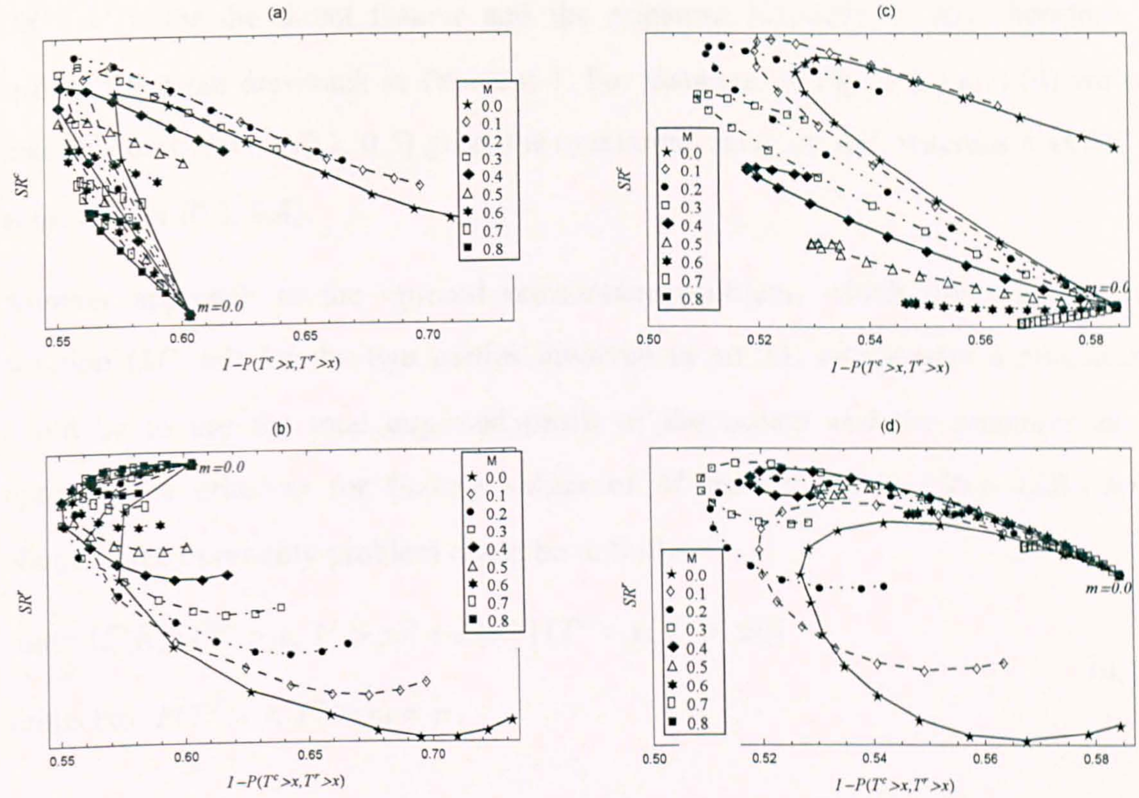

Fig. 3. SR^c and SR^r respectively plotted against $1 - P(T^c > x,\ T^r > x)$ in the case of: (a) and (b) - independent claim severities, Exp(1) distributed, $m = 0.0, 0.1, 0.2, \ldots, 1.5$, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.5$; (c) and (d) - dependent claim severities, $C^{RCl}(F(w_1), \ldots, F(w_k); \theta)$ distributed with $F \equiv Weibull(2.12, 1.14)$ marginals and $\theta = 1$, $m = 0.0, 0.05, 0.1, \ldots, 0.8$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$.

It has to be noted that, instead of solving (4.12), an alternative 'individual' approach for each of the two parties could be to try and find their set of values (M', m') which gives the highest 'return per unit of risk taken'. The latter means that (M', m') would provide the highest Sharpe ratio, defined as $\mathrm{SR}^c = E[R_x^c \mid (T^c > x,\ T^r > x)]/(1 - P(T^c > x,\ T^r > x))$ and $\mathrm{SR}^r = E[R_x^r \mid (T^c > x,\ T^r > x)]/(1 - P(T^c > x,\ T^r > x))$ respectively. However, this would again lead to possibly two different optimal solutions, $(M^{c\prime}, m^{c\prime})$ and

$(M^{r\prime}, m^{r\prime})$, for the direct insurer and the reinsurer respectively and therefore, it suffers the same drawback as Problem 1. For instance, in Fig. 3 (c) and (d) we see that the combination (0.1, 0.5) gives the maximum value of SR^c, whereas max SR^r is achieved for (0.3, 0.4).

Another approach to the optimal reinsurance problem, which gives a common solution (M', m') for the two parties involved in an XL reinsurance arrangement, could be to use the total expected profit of the cedent and the reinsurer as an optimization criterion for finding values of M and m, given $h(t) = h_c(t) + h_r(t)$. Namely, the optimality problem could be to find

$$\begin{aligned}&\max_{M,m} \{E[R_x^c \mid (T^c > x, T^r > x)] + E[R_x^r \mid (T^c > x, T^r > x)]\}\\&\text{subject to } P(T^c > x, T^r > x) = p.\end{aligned} \tag{4.13}$$

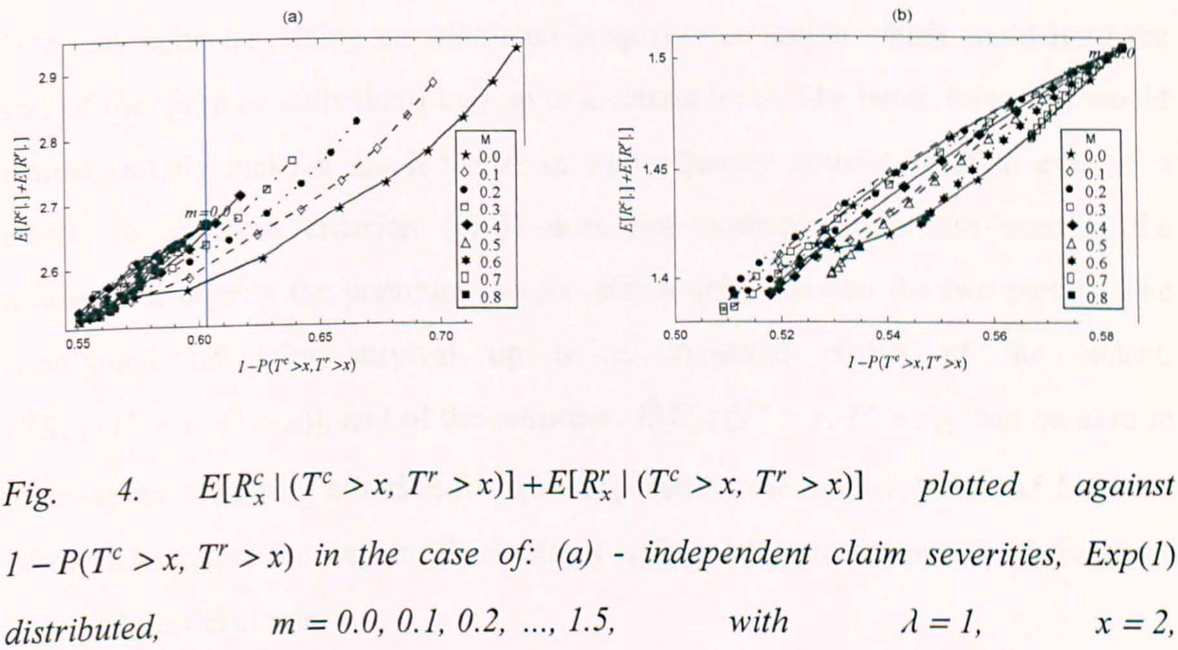

Fig. 4. $E[R_x^c \mid (T^c > x, T^r > x)] + E[R_x^r \mid (T^c > x, T^r > x)]$ *plotted against* $1 - P(T^c > x, T^r > x)$ *in the case of: (a) - independent claim severities,* $Exp(1)$ *distributed,* $m = 0.0, 0.1, 0.2, \ldots, 1.5,$ *with* $\lambda = 1,$ $x = 2,$ $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)t + c_r t,$ $c_r = 0.5;$ *(b) - dependent claim severities,* $C^{RCl}(F(w_1), \ldots, F(w_k); \theta)$ *distributed with* $F \equiv Weibull(2.12, 1.14)$ *marginals and* $\theta = 1,$ $m = 0.0, 0.05, 0.1, \ldots, 0.8,$ *with* $\lambda = 1,$ $x = 1,$ $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)t + c_r t,$ $c_r = 0.775.$

However, such a criterion seems not to be 'fair' with respect to both the cedent and the reinsurer, since as can be seen form Fig. 4, depending on the level p, (4.13) could be maximized due to maximizing the expected profit of only one of the two parties at the expense of the other. For example, when $p = 0.603$ (the vertical blue line in Fig. 4 (a)) a solution of (4.13) is any point $(M, 0)$, which is not adequate for the cedent, as has been already mentioned with respect to Problem 1, since it pays a non-zero reinsurance premium against zero reinsurance coverage. In Fig. 4 (a) and (b), the contradictory goals of maximizing $P(T^c > x, T^r > x)$ and maximizing $E[R_x^c \mid (T^c > x, T^r > x)] + E[R_x^r \mid (T^c > x, T^r > x)]$, as functions of M and m, are also illustrated.

In fact, optimality problem (4.13) does not explicitly manage the size of a possible loss and as such, does not prevent the two parties from taking very risky positions. One may consider adding an additional inequality condition which could limit the size of the (joint or individual) loss up to a certain level. The latter, however, would almost certaily make it much harder to find a jointly optimal solution even if it exists. In addition, criterion (4.13) does not explicitly take into account the information of how the premium income $h(t)$ is split between the two parties. The conditional on joint survival up to x, expected profits of the cedent, $E[R_x^c \mid (T^c > x, T^r > x)]$, and of the reinsurer, $E[R_x^r \mid (T^c > x, T^r > x)]$, can be used in defining the following criterion for optimally setting the XL levels M and L, which takes into account the way in which $h(t)$ is split and transfers it into the ratio of the expected profits at time x.

Problem 2. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$ with $h_c(t) = \alpha\, h(t)$, $h_r(t) = (1 - \alpha)\, h(t)$, $0 \le \alpha \le 1$, i.e. given that at any $t \ge 0$ the cedent retains $100\,\alpha\,\%$ of $h(t)$ and the rest $100\,(1 - \alpha)\,\%$ is taken by the reinsurer, find values of M and m such that

$$\frac{E[R_x^c \mid (T^c > x,\, T^r > x)]}{E[R_x^r \mid (T^c > x,\, T^r > x)]} = q \tag{4.14}$$

where

$$q = \frac{h_c(t)}{h_r(t)} = \frac{\alpha\, h(t)}{(1-\alpha)\, h(t)} = \frac{\alpha}{1-\alpha}\,. \tag{4.15}$$

In order to be able to address this optimality problem, we will use the explicit formulae for the corresponding expected profits given in Theorems 3 to 6. First, we will prove the following theorem, which states the existence of a solution to Problem 3.

Theorem 7. *If the total premium income, $h(t) = h_c(t) + h_r(t)$, is shared between the cedent and the reinsurer in such a way that $h_c(t)/h_r(t) = q$, for any $t \geq 0$, where $q \geq 0$, then there always exist $M \geq 0$ and $L \geq M$, such that*

$$E[R_x^c \mid (T^c > x,\, T^r > x)] / E[R_x^r \mid (T^c > x,\, T^r > x)] = q. \tag{4.16}$$

Proof. Varying $0 \leq \alpha \leq 1$ in (4.15) one can see that $0 \leq q \leq \infty$. Applying equations (4.5) and (4.9), established by Theorems 3 and 4 respectively, to express the numerator and the denominator of the ratio in (4.16), it is easy to verify that, given $h_c(t)/h_r(t) = q$ for any $t \geq 0$, the expected profits of the two parties will be in the same proportion, q, if and only if

$$\left(\sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^c\, A_k(\,;\tilde{v}_1,\, \ldots,\, \tilde{v}_k) \right.$$
$$\left. \psi(w_1,\, \ldots,\, w_k)\, dw_k \ldots dw_2\, dw_1 \right) \Bigg/$$
$$\left(\sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^r\, A_k(x;\tilde{v}_1,\, \ldots,\, \tilde{v}_k) \right.$$
$$\left. \psi(w_1,\, \ldots,\, w_k)\, dw_k \ldots dw_2\, dw_1 \right) = q \tag{4.17}$$

Note that the numerator and the denominator in (4.17) depend on M and L through y_k^c, y_k^r and $A_k(x\,;\tilde{v}_1,\ ...,\ \tilde{v}_k)$. From their definitions, given in Theorem 1, it can be seen that y_k^c, y_k^r and $A_k(x;\tilde{v}_1,\ ...,\ \tilde{v}_k)$ are continuous functions of M and L, and hence both the numerator and the denominator in (4.17) are also continuous functions of M and L.

Varying $M \geq 0$ and $L \geq M$, the left-hand side of (4.17) takes the whole range of values from 0 to ∞, e.g. when $M = 0$, $L = \infty$ we have $y_k^c = 0$, $0 < y_k^r < \infty$ for every $k = 1, 2, \ldots$ and hence the left-hand side of (4.17) is zero. On the other extreme when $M = L$, we have $y_k^r = 0$, $0 < y_k^c < \infty$ for every $k = 1, 2, \ldots$ and hence the left-hand side of (4.17) is infinity. Therefore, there should exist a pair M and L, for which the left-hand side of (4.17) will be equal to q and so, the ratio of the cedent's and the reinsurer's expected profits will be equal to q. This completes the proof of the theorem.□

In summary, Theorem 7 states that there always exist a solution to Problem 2, however the following remarks should be made.

Remark 1. The solution to Problem 2 may not be unique. There may exist a whole curve of combinations of M and m, for which the ratio of the expected profits of the cedent and the reinsurer is equal to q. We will refer to it as the 'fair' curve. For an illustration of this phenomenon see the right panels in Fig. 5, 6 and 7, where the 'fair' curve is the intersection between the plane $q = h_c(t)\,/\,h_r(t) = \text{const}$ and the surface $E[R_x^c \mid (T^c > x,\ T^r > x)]\,/\,E[R_x^r \mid (T^c > x,\ T^r > x)]$ as a function of M and m.

Remark 2. The numerator and the denominator in (4.17) coincide with the unconditional expectations $E\left[Y_{N_x}^c \cdot I_A \cdot I_B\right]$ and $E\left[Y_{N_x}^r \cdot I_A \cdot I_B\right]$ which in fact are the unconditional expected aggregate claim amounts at time x of the cedent and the reinsurer respectively, assuming they both survive up to x. So, as is natural to expect, in order for the expected profits to be in proportion q, it is necessary for the expected aggregate claim amounts to be in proportion q, since the premium income, $h(t)$, has been shared in the same proportion.

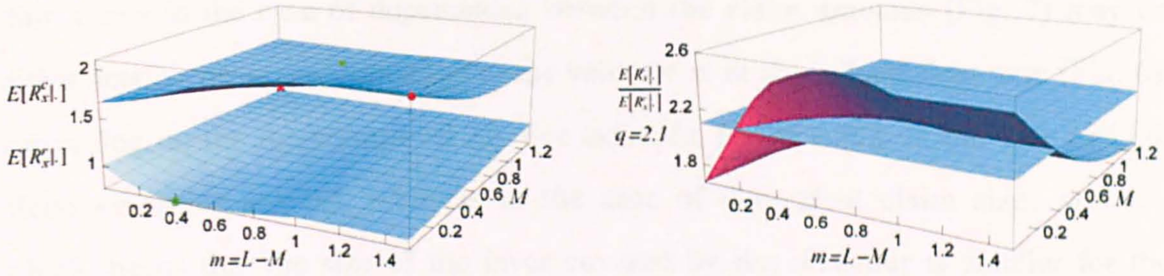

Fig. 5. Solutions to the optimality Problem 2, in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.5$, $q = 2.1$.

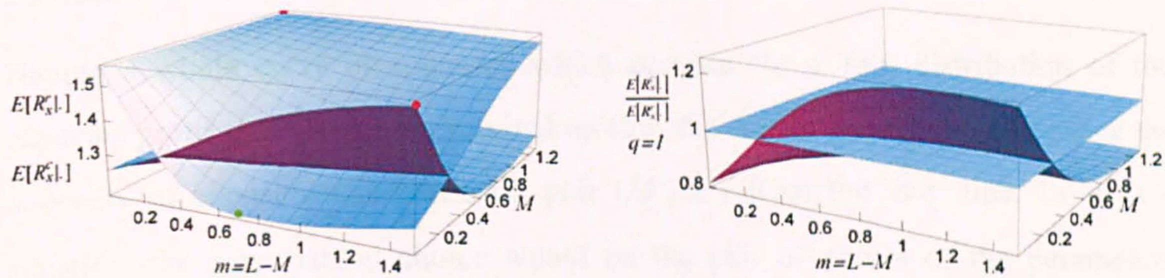

Fig. 6. Solutions to the optimality Problem 2, in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$, $q = 1$.

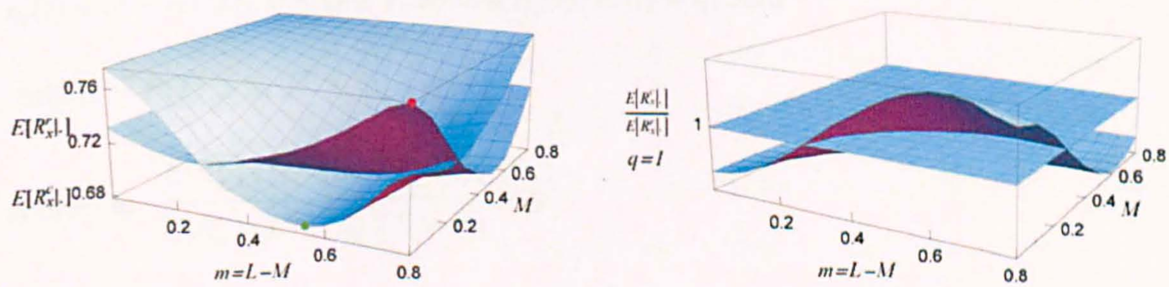

Fig. 7. Solutions to the optimality Problem 2, in the case of dependent claim severities, $C^{RCl}(F(w_1), \ldots, F(w_k); \theta)$ *distributed with* $F \equiv Weibull(2.12, 1.14)$ *marginals and* $\theta = 1$, *with* $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$, $q = 1$.

As can be seen from the right panels of Fig. 6 and 7, given a fixed split of the premium income $q = h_c(t)/h_r(t) = 1$ for all $t \geq 0$, the value of m which lies on the 'fair' curve in the case of dependence between the claim amounts (Fig. 7) may be either smaller or larger, compared to the value of m in the independent case (Fig. 6), depending on the retention level M. For example, for $M = 0.2$, in the case of i.i.d. claim severities $m = 0.5$, whereas in the case of dependent claim sizes $m = 0.4$, which means that the size of the layer covered by the reinsurer is smaller for the same fixed split of $h(t)$. Our experience shows that the effect of dependence modelled through a copula function is complex and may be different for different choices of copulas, marginals and values of the dependence parameter (for further comments see Kaishev and Dimitrova 2006).

Having a whole curve of solutions which provide for a 'fair' distribution of the expected profit at x, given joint survival up to x, the cedent and the reinsurer face the necessity of choosing one particular pair (M', m') from the 'fair' line. In such a situation, the most natural choice would be the pair of values of the parameters (M, m) with the highest probability of joint survival, i.e. the solution of the following problem.

Problem 3. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$ with $h_c(t) = \alpha\, h(t)$, $h_r(t) = (1-\alpha)\, h(t)$, $0 \leq \alpha \leq 1$, so that $h_c(t)/h_r(t) = q$, find

$$\begin{aligned} &\min_{M,m}\, [1 - P(T^c > x, T^r > x)] \\ &\text{subject to } \frac{E[R_x^c \mid (T^c > x, T^r > x)]}{E[R_x^r \mid (T^c > x, T^r > x)]} = q\,. \end{aligned} \tag{4.18}$$

It is clear that there always exists a unique solution to Problem 3. As illustrated in Fig. 8 (a) and (b), it is $(0.2, 0.3)$ in the case of i.i.d. claim sizes and $q = h_c(t)/h_r(t) = 1.05\,t/0.5\,t = 2.1$, and $(0.25, 0.5)$ in the dependent case with $q = h_c(t)/h_r(t) = 0.775\,t/0.775\,t = 1$.

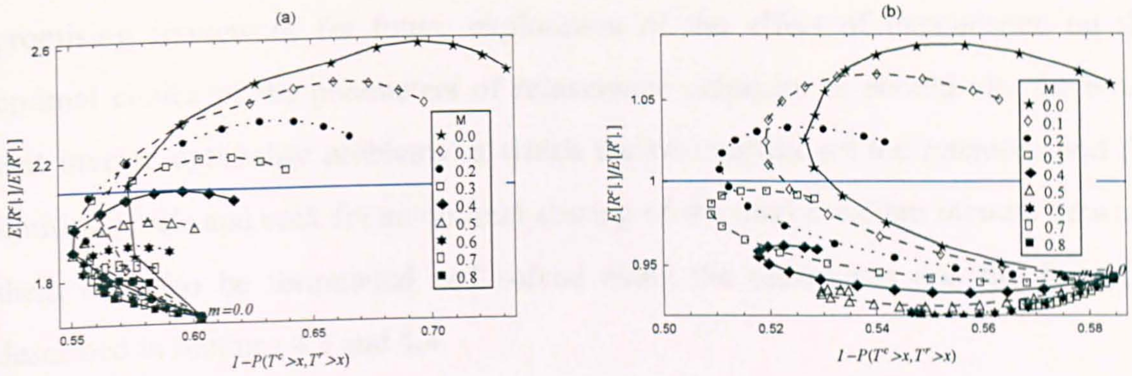

Fig. 8. Solutions to the optimality Problem 3, in the case of: (a) - independent claim severities, Exp(1) distributed, $m = 0.0, 0.1, 0.2, ..., 1.5,$ *with* $\lambda = 1,$ $x = 2,$ $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t,$ $c_r = 0.5,$ $q = 2.1;$ *(b) - dependent claim severities,* $C^{RCl}(F(w_1), ..., F(w_k); \theta)$ *distributed with* $F \equiv Weibull(2.12, 1.14)$ *marginals and* $\theta = 1,$ $m = 0.0, 0.05, 0.1, ..., 0.8,$ *with* $\lambda = 1,$ $x = 1,$ $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t,$ $c_r = 0.775,$ $q = 1.$

4.5 Comments and conclusions

In the present paper, we have shown how the problem of optimal XL reinsurance can be solved, combining specific risk and performance measures, under a relatively general assumptions for the risk model. As a performance measure, we have defined the expected profits at time x of the direct insurer and the reinsurer given their joint survival up to x, and derived explicit expressions for their numerical evaluation. The results of Kaishev and Dimitrova (2006) for the probability of joint survival of the direct insurer and the reinsurer up to time x have been recalled and employed as a risk measure. Three optimality problems have been defined and their solutions have been numerically illustrated and discussed under the assumption of both dependent and independent claim severities. It is interesting to mention that the effect of dependence of the claim severities is rather complex and difficult to predict based on purely intuitive reasoning. Henceforth, the model presented here provides a very promising framework for future exploration of the effect of dependence on the optimal choice of the parameters of reinsurance contracts. It should also be noted that inverse optimality problems in which the two parties set the retention and the limiting levels and seek for an optimal sharing of the total premium income between them can also be formulated and solved using the techniques and the formulae described in sections 4.3 and 4.4.

Chapter 5

Reinsurance and ruin under dependence of the claim inter-arrival times

Summary

A framework which generalizes the model considered in Chapters 2, 3 and 4 is introduced. We first consider independent, non-identically Erlang distributed claim inter-arrival times. Then, we allow for modelling dependence between the claim inter-arrival times by assuming that the latter are Erlang distributed with a random shape parameter. Explicit expressions for the probability of joint survival of the cedent and the reinsurer up to time x and the expected profit at x, given joint survival up to x, are derived in both cases.

5.1 Introduction

The excess of loss (XL) reinsurance model, considered in Chapter 3 and Chapter 4, incorporates any non-decreasing premium income function and continuous claim amounts, modelled by any joint distribution. The latter are relatively general assumptions, compared to the classical risk model of linear premium income function and independent, identically distributed claim severities. However, under both models claim arrivals follows a homogeneous Poisson process with parameter λ, i.e. the claim inter-arrival times are Exp(λ) distributed. In this paper, we deviate from this classical assumption and study the case of independent, non-identically Erlang distributed claim inter-arrival times. Then, the latter assumption is generalized by introducing dependence between the claim inter-occurrence times. Such models have been considered recently by Ignatov and Kaishev (2007). Under both risk models, we derive explicit expressions for the probability of joint survival of the cedent and the reinsurer up to time x and the expected profit at time x, given joint survival up to time x. It is shown, that these expressions can be used in finding the optimal parameters of an XL reinsurance treaty, considering optimality Problems similar to the ones defined in Chapters 3 and 4.

It has to be noted that the two risk models specified here are not Sparre Andersen models since the premium income is assumed to follow a positive, non-decreasing function and claim severities are assumed to have any continuous joint distribution. Furthermore, the second model deals with dependence between the claim arrivals. As is well-known, the Sparre Andersen model assumes independent, identically distributed claim inter-occurrence times, with a general distribution (not necessarily exponential), and independent, identically distributed claim sizes with premium income modelled by a straight line. A great deal of research in the area of ruin theory has been performed under the Sparre Andersen framework and different results have been obtained in the case of independent Erlang(2) or Erlang(n) distributed claim inter-arrival times. Some recent examples include Dickson (1998), Gerber and Shiu (1998), Dickson and Hipp (1998, 2001), Cheng and Tang (2003), Sun and Yang (2004), Li and Garrido (2004), Wei and Yang (2004), Gerber and Shiu (2005) and Li and Dickson (2006). Recently, a general Sparre Andersen model with any inter-arrival claim density has been considered by Pitts and Politis (2007) and generalizations of the Gerber and Shiu (1997) results have been obtained.

The paper is organized as follows. In the next section, the risk model with non-identical, independent Erlang distributed claim inter-arrival times and the related notations are introduced. Then, formulae for the probability of joint survival and the expected profits are derived in sections 5.2.2 and 5.2.3 respectively. In section 5.3, the risk model with dependent claim inter-arrival times is defined and in sections 5.3.2 and 5.3.3, expressions for the joint risk and performance measures are obtained. Section 5.4 discusses the problems which can be formulated in order to find the optimal parameters of an XL reinsurance treaty within the presented general risk models.

5.2 The risk model with independent Erlang distributed claim inter-arrival times

5.2.1 The model

We consider an insurance portfolio, generating claims at some random moments of time. The claim severities are modeled by the continuous random variables $\tilde{W}_1$, $\tilde{W}_2$, ..., $\tilde{W}_k$, ... with joint density function $\psi(\tilde{w}_1, ..., \tilde{w}_k)$ and cumulative distribution function $F_{\tilde{W}_1,...,\tilde{W}_k}(\tilde{w}_1, ..., \tilde{w}_k)$ or briefly $F(\tilde{w}_1, ..., \tilde{w}_k)$. For convenience, we will introduce also the random variables $\tilde{Y}_1 = \tilde{W}_1$, $\tilde{Y}_2 = \tilde{W}_1 + \tilde{W}_2$, ... representing the partial sums of consecutive claim amounts.

The claims inter-arrival times $\tilde{\tau}_1$, $\tilde{\tau}_2$, ... are assumed independent, gamma distributed r.v.s with parameters $g_i \in \mathbb{N}$ and $\lambda > 0$, i.e. $\tilde{\tau}_i \sim \text{Gamma}(g_i, \lambda)$ with density

$$f_{\tilde{\tau}_i}(t) = \frac{\lambda^{g_i} t^{g_i - 1} e^{-\lambda t}}{\Gamma(g_i)},\ i = 1, 2, \ldots .$$

This means that the claim inter-occurrences are assumed to have an Erlang distribution, each with a shape parameter g_i and rate λ, i.e. $\tilde{\tau}_i \sim \text{Erlang}(g_i)$. Denote by $\tilde{T}_1 = \tilde{\tau}_1$, $\tilde{T}_2 = \tilde{\tau}_1 + \tilde{\tau}_2$, ... the sequence of random variables representing the consecutive moments of occurrence of the claims. Let $\tilde{N}_t = \#\left\{i : \tilde{T}_i \leq t\right\}$, where # is the number of elements of the set {.}. It is assumed that the random variables $\tilde{W}_1$, $\tilde{W}_2$, ... are independent of $\tilde{N}_t$.

Then, the risk (surplus) process $\tilde{R}_t$, at time t, is given by $\tilde{R}_t = h(t) - \tilde{Y}_{\tilde{N}_t}$, where $h(t)$ is a nonnegative, non-decreasing, real function, defined on $\mathbb{R}_+$, representing the aggregate premium income up to time t. The function $h(t)$ may be continuous or not. If $h(t)$ is discontinuous, we will define $h^{-1}(y) = \inf\{z : h(z) \geq y\}$.

In this paper, we will be concerned with the case when the insurance company wants to reinsure its portfolio of risks by concluding an XL contract with a retention level $M \geq 0$ and a limiting level $L \geq M$. In other words, the cedent wants to reinsure the part of each claim which hits the layer $m = L - M$, i.e. each individual claim $\tilde{W}_i$ is shared between the two parties so that $\tilde{W}_i = \tilde{W}_i^c + \tilde{W}_i^r$, $i = 1, 2, \ldots$, where $\tilde{W}_i^c$ and $\tilde{W}_i^r$ denote the parts covered respectively by the cedent and the reinsurer. Clearly, we can write

$$\tilde{W}_i^c = \min\left(\tilde{W}_i, M\right) + \max\left(0, \tilde{W}_i - L\right)$$

and

$$\tilde{W}_i^r = \min\left(L - M, \max\left(0, \tilde{W}_i - M\right)\right).$$

Denote by $\tilde{Y}_1^c = \tilde{W}_1^c$, $\tilde{Y}_2^c = \tilde{W}_1^c + \tilde{W}_2^c$, ... and by $\tilde{Y}_1^r = \tilde{W}_1^r$, $\tilde{Y}_2^r = \tilde{W}_1^r + \tilde{W}_2^r$, ... the consecutive partial sums of claims to the cedent and to the reinsurer, respectively. Obviously, $\tilde{Y}_i = \tilde{Y}_i^c + \tilde{Y}_i^r$, $i = 1, 2, \ldots$. Under our XL reinsurance model, the total premium income $h(t)$ is also divided between the two parties so that $h(t) = h_c(t) + h_r(t)$, where $h_c(t)$, $h_r(t)$ are the premium incomes of the cedent and the reinsurer, assumed also non-negative, non-decreasing functions on $\mathbb{R}_+$. As a result, the risk process, $\tilde{R}_t$, can be represented as a superposition of two risk processes, that of the cedent

$$\tilde{R}_t^c = h_c(t) - \tilde{Y}_{\tilde{N}_t}^c \tag{5.1}$$

and of the reinsurer

$$\tilde{R}_t^r = h_r(t) - \tilde{Y}_{\tilde{N}_t}^r \tag{5.2}$$

i.e., $\tilde{R}_t = \tilde{R}_t^c + \tilde{R}_t^r$. Note that the two risk processes $\tilde{R}_t^c$ and $\tilde{R}_t^r$ are dependent through the common claim arrivals and the claim severities $\tilde{W}_i$, $i = 1, 2, \ldots$, as seen from (5.1) and (5.2).

The moments, $\tilde{T}^c$ and $\tilde{T}^r$, of ruin of correspondingly the cedent and the reinsurer are defined as

$$\tilde{T}^c := \inf\{t : t > 0,\ \tilde{R}_t^c < 0\},$$

$$\tilde{T}^r := \inf\{t : t > 0,\ \tilde{R}_t^r < 0\}.$$

Under this model, explicit formulae for $P(\tilde{T}^c > x, \tilde{T}^r > x)$, $x > 0$, and for $E[\tilde{R}_x^c \mid (\tilde{T}^c > x, \tilde{T}^r > x)]$ and $E[\tilde{R}_x^r \mid (\tilde{T}^c > x, \tilde{T}^r > x)]$, are derived in the next two sections.

In order to do this, we need to introduce the sequence $\tau_1, \tau_2, \ldots$ of independent, $\mathrm{Exp}(\lambda)$ distributed random variables with mean $1/\lambda$, such that

$$\left(\tau_1 + \ldots + \tau_{g_1}, \tau_{g_1+1} + \ldots + \tau_{g_1+g_2}, \ldots\right) \stackrel{d}{=} (\tilde{\tau}_1, \tilde{\tau}_2, \ldots). \tag{5.3}$$

Denote $T_1 = \tau_1$, $T_2 = \tau_1 + \tau_2$, $\ldots$. Clearly, we have that $T_{g_1+\ldots+g_i} = \tilde{T}_i$, $i = 1, 2, \ldots$. Recall that (5.3) follows from the fact that a $\mathrm{Gamma}(g_i, \lambda)$ distributed random variable, where g_i is a positive integer, can be expressed as a sum of g_i independent $\mathrm{Exp}(\lambda)$ distributed random variables.

Let us also introduce the random variables $W_1, W_2, \ldots$ independent of $\tau_1, \tau_2, \ldots$, such that

$$W_l = \begin{cases} \tilde{W}_i, & \text{if } \ l = g_1 + \ldots + g_i,\ i = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}. \tag{5.4}$$

If we then define

$$W_l^c = \min(W_l, M) + \max(0, W_l - L),$$

$$W_l^r = \min(L - M, \max(0, W_l - M)),\ l = 1, 2, \ldots,$$

and

$Y_l^c = \sum_{j=1}^{l} W_j^c,$

$Y_l^r = \sum_{j=1}^{l} W_j^r, l = 1, 2, ...,$

it is not difficult to see that we will have $Y_1^c \leq Y_2^c \leq ...$ and $Y_1^r \leq Y_2^r \leq ...$, both independent of $\tau_1, \tau_2, ...,$ such that $Y_{g_1+...+g_i}^c = \tilde{Y}_i^c$ and $Y_{g_1+...+g_i}^r = \tilde{Y}_i^r$, $i = 1, 2, ...,$ and $\tilde{Y}_i = \tilde{Y}_i^c + \tilde{Y}_i^r = Y_{g_1+...+g_i}^c + Y_{g_1+...+g_i}^r, i = 1, 2, ...$

5.2.2 The probability of joint survival under independent Erlang inter-arrival times

The following theorem gives the probability of joint survival of the cedent and the reinsurer up to time x, under the model of any non-decreasing premium income function, independent Erlang(g_i) distributed claim inter-arrivals and continuous claim severities, modelled by any joint distribution. Within this framework, an explicit formula for the probability of non-ruin of the direct insurer only has been recently obtained by Ignatov and Kaishev (2007).

Theorem 1. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$$P\left(\tilde{T}^c > x, \tilde{T}^r > x\right) =$$

$$e^{-\lambda x}\left(1 + \sum_{l=1}^{g_1-1} \lambda^l \frac{x^l}{l!} + \sum_{k=1}^{\infty} \sum_{l=g_1+...+g_k}^{g_1+...+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-...-\tilde{w}_{k-1}} A_l(x; \tilde{v}_1, ..., \tilde{v}_l)\, \psi(\tilde{w}_1, ..., \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1\right) \tag{5.5}$$

where $\tilde{v}_j = \min(\tilde{z}_j, x)$,

$\tilde{z}_j =$

$$\begin{cases} 0 & \text{if} \quad 1 \le j < g_1, \ (g_1 \ne 1) \\ \max\left(h_c^{-1}(\tilde{y}_i^c),\ h_r^{-1}(\tilde{y}_i^r)\right) & \text{if} \quad g_1 + \ldots + g_i \le j < g_1 + \ldots + g_{i+1}, \ i = 1, 2, \ldots \end{cases}$$

$\tilde{y}_j^c = \sum_{i=1}^{j} \tilde{w}_i^c$, $\qquad \tilde{y}_j^r = \sum_{i=1}^{j} \tilde{w}_i^r$, $\qquad \tilde{w}_i^c = \min(\tilde{w}_i, M) + \max(0, \tilde{w}_i - L)$,

$\tilde{w}_i^r = \min(L - M, \max(0, \tilde{w}_i - M))$, $j = 1, \ldots, l$, and

$A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)$, $l = 1, 2, \ldots$ *are the classical Appell polynomials* $A_l(x)$ *of degree* l, *defined by*

$A_0(x) = 1, \quad A_l'(x) = A_{l-1}(x), \quad A_l(\tilde{v}_l) = 0.$

Remark 1. It is straightforward to verify that in the case of $g_i = 1$, $i = 1, 2, \ldots$ formula (5.5) coincides with (3.4).

Proof of Theorem 1. For the event of joint survival $\{\tilde{T}^c > x, \tilde{T}^r > x\}$ we have

$$\{\tilde{T}^c > x, \tilde{T}^r > x\} = \bigcap_{j=1}^{\infty} [\{(h_c^{-1}(\tilde{Y}_j^c) < \tilde{T}_j) \cap (h_r^{-1}(\tilde{Y}_j^r) < \tilde{T}_j)\} \cup \{x < \tilde{T}_j\}]$$

$$= \bigcap_{j=1}^{\infty} [$$

$$\{(h_c^{-1}(Y_{g_1+\ldots+g_j}^c) < T_{g_1+\ldots+g_j}) \cap (h_r^{-1}(Y_{g_1+\ldots+g_j}^r) < T_{g_1+\ldots+g_j})\} \cup \{x < T_{g_1+\ldots+g_j}\}]$$

$$= \bigcap_{j=1}^{\infty} [\{\max(h_c^{-1}(Y_{g_1+\ldots+g_j}^c), h_r^{-1}(Y_{g_1+\ldots+g_j}^r)) < T_{g_1+\ldots+g_j}\} \cup \{x < T_{g_1+\ldots+g_j}\}] \tag{5.6}$$

Denote

$$Q_l^c = \begin{cases} 0 & \text{if} \quad 1 \le l < g_1, \ (g_1 \ne 1) \\ Y_{g_1+\ldots+g_i}^c & \text{if} \quad g_1 + \ldots + g_i \le l < g_1 + \ldots + g_{i+1}, \ i = 1, 2, \ldots \end{cases} \tag{5.7}$$

and

$$Q_l^r = \begin{cases} 0 & \text{if} \quad 1 \le l < g_1, \ (g_1 \ne 1) \\ Y_{g_1+\ldots+g_i}^r & \text{if} \quad g_1 + \ldots + g_i \le l < g_1 + \ldots + g_{i+1}, \ i = 1, 2, \ldots \end{cases} . \tag{5.8}$$

Note that from (5.7) and (5.8) it follows that $Q_l^c \neq 0$ for $l \geq g_1$, whereas Q_l^r could be zero for $1 \leq l < g_1 + ... + g_{k+1}$, $k > 0$ if for example the first k claims $\tilde{W}_1, ..., \tilde{W}_k$ are smaller than the retention level M and hence, $Y_{g_1}^r = \tilde{Y}_1^r = 0, ..., Y_{g_1+...+g_k}^r = \tilde{Y}_k^r = 0$. For the j-th event in (5.6) we have that

$$\left\{\max\left(h_c^{-1}\left(Y_{g_1+...+g_j}^c\right), h_r^{-1}\left(Y_{g_1+...+g_j}^r\right)\right) < T_{g_1+...+g_j}\right\} \cup \left\{x < T_{g_1+...+g_j}\right\} \subseteq$$
$$\left\{\max\left(h_c^{-1}\left(Y_{g_1+...+g_j}^c\right), h_r^{-1}\left(Y_{g_1+...+g_j}^r\right)\right) < T_{g_1+...+g_j+s}\right\} \cup \left\{x < T_{g_1+...+g_j+s}\right\}$$

for any $s = 0, 1, ..., g_{j+1} - 1$, which is equivalent to

$$\left\{\max\left(h_c^{-1}\left(Y_{g_1+...+g_j}^c\right), h_r^{-1}\left(Y_{g_1+...+g_j}^r\right)\right) < T_{g_1+...+g_j}\right\} \cup \left\{x < T_{g_1+...+g_j}\right\} \subseteq$$
$$\left\{\max\left(h_c^{-1}\left(Q_l^c\right), h_r^{-1}\left(Q_l^r\right)\right) < T_l\right\} \cup \{x < T_l\}$$

for any $g_1 + ... + g_j \leq l < g_1 + ... + g_{j+1}$. Therefore, for any $j = 1, 2, ...$

$$\left\{\max\left(h_c^{-1}\left(Y_{g_1+...+g_j}^c\right), h_r^{-1}\left(Y_{g_1+...+g_j}^r\right)\right) < T_{g_1+...+g_j}\right\} \cup \left\{x < T_{g_1+...+g_j}\right\} \subseteq$$
$$\bigcap_{l=g_1+...+g_j}^{g_1+...+g_{j+1}-1} \left[\left\{\max\left(h_c^{-1}\left(Q_l^c\right), h_r^{-1}\left(Q_l^r\right)\right) < T_l\right\} \cup \{x < T_l\}\right]. \tag{5.9}$$

In addition, we also have that for $1 \leq l < g_1$, ($g_1 \neq 1$),

$$\left\{\max\left(h_c^{-1}\left(Q_l^c\right), h_r^{-1}\left(Q_l^r\right)\right) < T_l\right\} \cup \{x < T_l\} =$$
$$\left\{\max\left(h_c^{-1}(0), h_r^{-1}(0)\right) < T_l\right\} \cup \{x < T_l\} = \{0 < T_l\} \cup \{x < T_l\} = \Omega$$

and hence

$$\bigcap_{l=1}^{g_1-1} \left[\left\{\max\left(h_c^{-1}\left(Q_l^c\right), h_r^{-1}\left(Q_l^r\right)\right) < T_l\right\} \cup \{x < T_l\}\right] = \Omega\ . \tag{5.10}$$

Thus, from (5.6), (5.9) and (5.10) we obtain

$$\left\{\tilde{T}^c > x, \tilde{T}^r > x\right\} = \bigcap_{l=1}^{\infty} \left[\left\{\max\left(h_c^{-1}\left(Q_l^c\right), h_r^{-1}\left(Q_l^r\right)\right) < T_l\right\} \cup \{x < T_l\}\right] \tag{5.11}$$

Note that (5.11) has the same form as equality (3.5). From (5.7), we see that the sequence

$$Q_1^c \le Q_2^c \le \dots \le Q_{g_1-1}^c \le Q_{g_1}^c \le Q_{g_1+1}^c \le \dots \le Q_{g_1+g_2-1}^c \le Q_{g_1+g_2}^c \le Q_{g_1+g_2+1}^c \le \dots$$

can be alternatively expressed as

$$\underbrace{0 \le 0 \le \dots \le 0}_{g_1-1} \le \underbrace{Y_{g_1}^c \le Y_{g_1}^c \le \dots \le Y_{g_1}^c}_{g_2} \le \underbrace{Y_{g_1+g_2}^c \le Y_{g_1+g_2}^c \le \dots \le Y_{g_1+g_2}^c}_{g_3} \le \dots$$

or as

$$\underbrace{0 \le 0 \le \dots \le 0}_{g_1-1} \le \underbrace{\tilde{Y}_1^c \le \tilde{Y}_1^c \le \dots \le \tilde{Y}_1^c}_{g_2} \le \underbrace{\tilde{Y}_2^c \le \tilde{Y}_2^c \le \dots \le \tilde{Y}_2^c}_{g_3} \le \dots \ . \tag{5.12}$$

Similarly, from (5.8) the sequence

$$Q_1^r \le Q_2^r \le \dots \le Q_{g_1-1}^r \le Q_{g_1}^r \le Q_{g_1+1}^r \le \dots \le Q_{g_1+g_2-1}^r \le Q_{g_1+g_2}^r \le Q_{g_1+g_2+1}^r \le \dots$$

can be expressed as

$$\underbrace{0 \le 0 \le \dots \le 0}_{g_1-1} \le \underbrace{Y_{g_1}^r \le Y_{g_1}^r \le \dots \le Y_{g_1}^r}_{g_2} \le \underbrace{Y_{g_1+g_2}^r \le Y_{g_1+g_2}^r \le \dots \le Y_{g_1+g_2}^r}_{g_3} \le \dots$$

or as

$$\underbrace{0 \le 0 \le \dots \le 0}_{g_1-1} \le \underbrace{\tilde{Y}_1^r \le \tilde{Y}_1^r \le \dots \le \tilde{Y}_1^c}_{g_2} \le \underbrace{\tilde{Y}_2^r \le \tilde{Y}_2^r \le \dots \le \tilde{Y}_2^r}_{g_3} \le \dots \ . \tag{5.13}$$

Note that from (5.12) and (5.13) we see that both sequences of random variables Q_l^c and Q_l^r, $l = 1, 2, \dots$, are independent of T_l, $l = 1, 2, \dots$ and are also non-decreasing. Hence, (5.11) has the same form as equality (3.5) and the random variables Q_l^c and Q_l^r, $l = 1, 2, \dots$ fulfill the same requirements as Y_j^c and Y_j^r, $j = 1, 2, \dots$ from (3.5).

Therefore, from (5.11), (3.5) and (3.13) it follows that

$$P\left(\tilde{T}^c > x, \tilde{T}^r > x\right) = e^{-\lambda x}\left(\sum_{l=0}^{g_1-1} \lambda^l A_l(x\,;\tilde{v}_1, \;..., \tilde{v}_l) + \right. \tag{5.14}$$

$$\left. \sum_{l=g_1}^{\infty} \lambda^l \int \cdots \int_{\mathcal{D}_l} A_l(x\,;\tilde{v}_1, \;..., \tilde{v}_l)\, d F_{W_1,...,W_l}(w_1, \;..., w_l)\right)$$

where

$$\mathcal{D}_l \equiv \begin{pmatrix} 0 \le w_1, \;..., 0 \le w_l \\ w_1 + \;...\; + w_l \le h(x) \end{pmatrix},$$

$A_l(x\,; v_1, \;..., v_l)$ are classical Appell polynomials $A_l(x)$ of degree l, and $\tilde{v}_j = \min[\tilde{z}_j, x]$, $\tilde{z}_j = \max\left(h_c^{-1}\left(y_j^c\right), h_r^{-1}\left(y_j^r\right)\right)$, $y_j^c = \sum_{i=1}^j w_i^c$, $y_j^r = \sum_{i=1}^j w_i^r$, $j = 1, 2, \;..., l$.

From the definition (5.4) it follows that the sequence

$$W_1, W_2, \;..., W_{g_1-1}, W_{g_1}, W_{g_1+1}, \;..., W_{g_1+g_2-1}, W_{g_1+g_2}, W_{g_1+g_2+1}, \;...$$

can be expressed as

$$0, 0, \;..., 0, W_{g_1}, 0, \;..., 0, W_{g_1+g_2}, 0, \;...$$

or

$$\underbrace{0, 0, \;..., 0,}_{g_1-1} \underbrace{\tilde{W}_1, 0, \;..., 0,}_{g_2} \underbrace{\tilde{W}_2, 0, \;..., 0,}_{g_3} \;... \;.$$

Hence,

$$d F_{W_1,...,W_l}(w_1, \;..., w_l) =$$

$$d F_{\underbrace{0,0,...,0,}_{g_1-1}\underbrace{\tilde{W}_1,0,...,0,}_{g_2}\underbrace{\tilde{W}_2,0,...,0,}_{g_3}...,\underbrace{\tilde{W}_k,0,...,0}_{l-(g_1+...+g_k)+1}}\left(\underbrace{0, 0, \;..., 0,}_{g_1-1} \underbrace{\tilde{w}_1, 0, \;..., 0,}_{g_2} \;..., \right. \tag{5.15}$$

$$\left. \underbrace{\tilde{w}_k, 0, \;..., 0}_{l-(g_1+...+g_k)+1}\right) = d F_{\tilde{W}_1,...,\tilde{W}_k}(\tilde{w}_1, \;..., \tilde{w}_k)$$

and $\tilde{z}_j$ can be expressed as

$$\tilde{z}_j = \begin{cases} \max\left(h_c^{-1}(0),\, h_r^{-1}(0)\right) & \text{if} \quad 1 \le j < g_1,\ (g_1 \ne 1) \\ \max\left(h_c^{-1}(\tilde{y}_i^c),\, h_r^{-1}(\tilde{y}_i^r)\right) & \text{if} \quad g_1 + \ldots + g_i \le j < g_1 + \ldots + g_{i+1},\ i = 1, 2, \ldots \end{cases} \tag{5.16}$$

$j = 1, 2, \ldots, l.$

So, in view of (5.15) and (5.16), we can re-write formula (5.14) in the terms of the original claim severities as follows

$$P\left(\tilde{T}^c > x, \tilde{T}^r > x\right) = e^{-\lambda x}\left(\sum_{l=0}^{g_1-1} \lambda^l A_l(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_l) + \sum_{k=1}^{\infty}\sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int\cdots\int_{\tilde{\mathcal{D}}_k} A_l(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_l)\,\psi(\tilde{w}_1,\ \ldots,\ \tilde{w}_k)\,d\tilde{w}_k \cdots d\tilde{w}_1\right) \tag{5.17}$$

where

$$\tilde{\mathcal{D}}_k \equiv \begin{pmatrix} 0 \le \tilde{w}_1,\ \ldots,\ 0 \le \tilde{w}_k \\ \tilde{w}_1 + \ \ldots\ + \tilde{w}_k \le h(x) \end{pmatrix}.$$

The asserted formula (5.5) now follows, appropriately rewriting the multiple integral in (5.17) and noting that $\max\left(h_c^{-1}(0),\, h_r^{-1}(0)\right) = 0$, and that $A_l(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_l) = A_l(x\,;0,\ \ldots,\ 0) = x^l\big/l!$, for $1 \le l < g_1$.□

In the next section, we give expressions for the expected profit of the cedent and the reinsurer respectively under the risk model considered here.

5.2.3 The expected profit given joint survival under independent Erlang inter-arrival times

In this section, we will present some explicit results for the performance measures of the direct insurer and the reinsurer, as defined in Chapter 4, section 4.3, under the risk model described in section 5.2.1. Following the notation introduced in section 4.3 and section 5.2.1, we will define the profits at time x of the cedent and the reinsurer, correspondingly as the values, $\tilde{R}_x^c$ and $\tilde{R}_x^r$, of their risk processes, given by (5.1) and (5.2), at time x. Denote by $E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]$ the expected profit of the cedent at time x, given the two parties' joint survival up to time x. Similarly, $E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]$ denotes the reinsurer's expected profit at time x, given its and the insurer's joint survival up to time x.

The following two theorems give explicit expressions for $E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]$ and $E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]$ correspondingly.

Theorem 2. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right] =$$

$$h_c(x) - \left(\sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} \tilde{y}_k^c \right.$$

$$\left. A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \right) \Big/ \tag{5.18}$$

$$\left(1 + \sum_{l=1}^{g_1-1} \lambda^l \frac{x^l}{l!} + \sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} A_l(\right.$$

$$\left. x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \right)$$

where $\tilde{y}_k^c$, $\tilde{v}_j$, $j = 1, \ldots, l$ and $A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)$ *are defined as in Theorem 1.*

Proof of Theorem 2. Denote by I_A and I_B the indicator random variables of the events $A = \{\tilde{T}^c > x\}$ and $B = \{\tilde{T}^r > x\}$. In view of the definitions (5.1) and (5.2) of the risk processes $\tilde{R}_t^c$ and $\tilde{R}_t^r$, expression (5.5) for the probability of joint survival and its derivation, we can express the unconditional expectation $E(\tilde{R}_x^c \cdot I_A \cdot I_B)$ as

$$E(\tilde{R}_x^c \cdot I_A \cdot I_B) = e^{-\lambda x} \Bigg(h_c(x) + \sum_{l=1}^{g_1 - 1} h_c(x) \lambda^l \frac{x^l}{l!} + \sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} \left(h_c(x) - \sum_{i=1}^{k} \tilde{w}_i^c \right) A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \Bigg) \tag{5.19}$$

The unconditional expectation (5.19) can be rewritten as

$$E(\tilde{R}_x^c \cdot I_A \cdot I_B) = e^{-\lambda x} h_c(x) \Bigg(1 + \sum_{l=1}^{g_1-1} \lambda^l \frac{x^l}{l!} + \sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \Bigg) - e^{-\lambda x} \Bigg(\sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} \left(\sum_{i=1}^{k} \tilde{w}_i^c \right) A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \Bigg) \tag{5.20}$$

For the conditional expectation $E[\tilde{R}_x^c \mid (\tilde{T}^c > x, \tilde{T}^r > x)]$ we have

$$E[\tilde{R}_x^c \mid (\tilde{T}^c > x, \tilde{T}^r > x)] = \frac{E(\tilde{R}_x^c \cdot I_A \cdot I_B)}{P(\tilde{T}^c > x, \tilde{T}^r > x)} \tag{5.21}$$

Substituting (5.20) and (5.5) in (5.21), and after cancelling appropriate terms, recalling the notation $\sum_{i=1}^{k} \tilde{w}_i^c = \tilde{y}_k^c$, we obtain the assertion of the theorem.□

Similarly, for the expected profit of the reinsurer we have

Theorem 3. *The expected profit of the reinsurer at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right] =$$

$$h_r(x) - \left(\sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} \tilde{y}_k^r \right.$$

$$\left. A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \right) \Big/ \tag{5.22}$$

$$\left(1 + \sum_{l=1}^{g_1-1} \lambda^l \frac{x^l}{l!} + \sum_{k=1}^{\infty} \sum_{l=g_1+\ldots+g_k}^{g_1+\ldots+g_{k+1}-1} \lambda^l \int_0^{h(x)} \int_0^{h(x)-\tilde{w}_1} \cdots \int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} A_l(\right.$$

$$\left. x; \tilde{v}_1, \ldots, \tilde{v}_l)\, \psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1 \right)$$

where $\tilde{y}_k^r$, $\tilde{v}_j$, $j = 1, \ldots, l$ and $A_l(x; \tilde{v}_1, \ldots, \tilde{v}_l)$ *are defined as in Theorem 1.*

Proof of Theorem 3. The proof follows the same lines of reasoning as in Theorem 2, replacing the premium income and the claims to the cedent with the ones to the reinsurer.□

In the next section, we will look at a generalization of the risk model described in section 5.2.1 to the case of dependent inter-arrival times.

5.3 The risk model with dependent claim inter-arrival times

5.3.1 The model

The risk model, as specified in section 5.2.1, can be further generalized by introducing a dependence between the Erlang claim inter-arrival times, $\tilde{\tau}_i \sim \text{Erlang}(g_i)$, $i = 1, 2, \ldots$ through a randomization of the shape parameters g_i. This can be done as follows.

Recall that g_i, $i = 1, 2, \ldots$ are positive integers. Then, consider the integer-valued random variables $G_1, G_2, \ldots$, independent of $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$, with joint probability mass function

$$p_{g_1,\ldots,g_s} = P(G_1 = g_1, \ \ldots, G_s = g_s), \ \text{for}\, g_1 \geq 1, \ g_2 \geq 1, \ \ldots, g_s \geq 1\,. \tag{5.23}$$

Now, it is not difficult to see that the claim inter-occurrence times $\tilde{\tau}_1 = \tau_1 + \ldots + \tau_{G_1}$, $\tilde{\tau}_2 = \tau_{G_1+1} + \ldots + \tau_{G_1+G_2}$, ... are dependent random variables. For instance, we have that

$$E(\tilde{\tau}_1) = E(E(\tilde{\tau}_1 \,|\, G_1)) = E(\tau_1)\, E(G_1) = \frac{1}{\lambda}\, E(G_1),$$

$$E(\tilde{\tau}_2) = E(E(\tilde{\tau}_2 \,|\, G_2)) = E(\tau_1)\, E(G_2) = \frac{1}{\lambda}\, E(G_2),$$

$$\begin{aligned}
E(\tilde{\tau}_1\, \tilde{\tau}_2) &= E\Big(\big(\tau_1 + \ldots + \tau_{G_1}\big)\big(\tau_{G_1+1} + \ldots + \tau_{G_1+G_2}\big)\Big) = \\
&\quad E\Big(E\Big(\big(\tau_1 + \ldots + \tau_{G_1}\big)\big(\tau_{G_1+1} + \ldots + \tau_{G_1+G_2}\big)\,\Big|\, G_1, G_2\Big)\Big) = \\
&\quad \sum_{g_1}\sum_{g_2} p_{g_1,g_2}\, E\big((\tau_1 + \ldots + \tau_{g_1})\big(\tau_{g_1+1} + \ldots + \tau_{g_1+g_2}\big)\big) = \\
&\quad \sum_{g_1}\sum_{g_2} p_{g_1,g_2}\, \frac{1}{\lambda}\, g_1\, \frac{1}{\lambda}\, g_2 = \frac{1}{\lambda^2}\, E(G_1\, G_2),
\end{aligned}$$

so that

$$\begin{aligned}
&\mathrm{Cov}(\tilde{\tau}_1, \tilde{\tau}_2) = \\
&\quad E(\tilde{\tau}_1\, \tilde{\tau}_2) - E(\tilde{\tau}_1)\, E(\tilde{\tau}_2) = \frac{1}{\lambda^2}\,(E(G_1\, G_2) - E(G_1)\, E(G_2)) = \frac{1}{\lambda^2}\,\mathrm{Cov}(G_1\, G_2)\,.
\end{aligned}$$

Hence,

$\mathrm{Corr}(\tilde{\tau}_1, \tilde{\tau}_2) =$

$$\frac{\frac{1}{\lambda^2}\,\mathrm{Cov}(G_1\, G_2)}{\frac{1}{\lambda}\sqrt{\mathrm{Var}(G_1) + E(G_1)}\;\frac{1}{\lambda}\sqrt{\mathrm{Var}(G_2) + E(G_2)}} < \frac{\mathrm{Cov}(G_1\, G_2)}{\sqrt{\mathrm{Var}(G_1)}\;\sqrt{\mathrm{Var}(G_2)}} =$$

$$\mathrm{Corr}(G_1\, G_2).$$

since

$$\mathrm{Var}(\tilde{\tau}_1) = \mathrm{Var}\big(E\big(\tau_1 + \ldots + \tau_{G_1} \,\big|\, G_1\big)\big) + E\big(\mathrm{Var}\big(\tau_1 + \ldots + \tau_{G_1} \,\big|\, G_1\big)\big) =$$

$$\mathrm{Var}(E(\tau_1)\, G_1) + E(\mathrm{Var}(\tau_1)\, G_1) =$$

$$(E(\tau_1))^2\,\mathrm{Var}(G_1) + \mathrm{Var}(\tau_1)\, E(G_1) = \frac{1}{\lambda^2}\,\mathrm{Var}(G_1) + \frac{1}{\lambda^2}\, E(G_1)$$

$$\mathrm{Var}(\tilde{\tau}_2) = \frac{1}{\lambda^2}\,\mathrm{Var}(G_2) + \frac{1}{\lambda^2}\, E(G_2).$$

In principle, a large class of multivariate discrete distributions can be used to introduce dependence in the model through (5.23), e.g. the *Dirichlet-compound multinomial distribution* (see Johnson, Kotz and Balakrishnan 1997, p.80), the *multivariate logarithmic series distribution* (see Johnson, Kotz and Balakrishnan 1997, p.158), and the *multivariate Pólya-Eggenberger distributions* (see Johnson, Kotz and Balakrishnan 1997, p.200), subject to appropriate 'zeros-truncation' as described in Johnson, Kotz and Balakrishnan (1997, p.21). As an example we will consider the *'zeros-truncated' multinomial distribution* ($\mathrm{MD}_{\mathrm{ZT}}$) of Ignatov, Kaishev and Krachunov (2001).

The joint probability mass function of the $\mathrm{MD}_{\mathrm{ZT}}$ distribution with parameters m and $d_1, \ldots, d_s$ is defined as

$$P(G_1 = g_1, \ldots, G_s = g_s) = \frac{m!}{(g_1 - 1)! \ldots (g_s - 1)!\,(m + s - g_1 - \ldots - g_s)!}\, d_1^{g_1 - 1} \ldots$$

$$d_s^{g_s - 1}\,(1 - d_1 - \ldots - d_s)^{m + s - g_1 - \ldots - g_s},$$

for $g_i \geq 1$, $i = 1, 2, \ldots, s$, positive integers, $g_1 + \ldots + g_s \leq m + s$ and $P(G_1 = g_1, \ldots, G_s = g_s) = 0$ otherwise, where $m \geq 1$ is a positive integer and $d_i \in \mathbb{R}_+$, $i = 1, \ldots, s$, are such that $d_1 + \ldots + d_s < 1$.

We have

$$E(G_1) = \sum_{g_1=1}^{m+1} g_1 \frac{m!}{(g_1 - 1)!\,(m + 1 - g_1)!} d_1^{g_1 - 1} (1 - d_1)^{m+1-g_1} = 1 + m\, d_1,$$

$$E(G_2) = 1 + m\, d_2,$$

$$E(G_1\, G_2) = \sum_{g_1=1}^{m+1} \sum_{g_2=1}^{m+2-g_1} g_1\, g_2 \frac{m!}{(g_1 - 1)!\,(g_2 - 1)!\,(m + 2 - g_1 - g_2)!} d_1^{g_1 - 1} d_2^{g_2 - 1}$$

$$(1 - d_1 - d_2)^{m+2-g_1-g_2} = m\,((m - 1)\, d_2 + 1)\, d_1 + m\, d_2 + 1\,.$$

Hence,

$$\mathrm{Cov}(\tilde{\tau}_1, \tilde{\tau}_2) = \frac{1}{\lambda^2} \mathrm{Cov}(G_1\, G_2) = \frac{1}{\lambda^2} (E(G_1\, G_2) - E(G_1)\, E(G_2)) =$$

$$\frac{1}{\lambda^2} (m\,((m - 1)\, d_2 + 1)\, d_1 + m\, d_2 + 1 - (1 + m\, d_1)\,(1 + m\, d_2)) =$$

$$-\frac{1}{\lambda^2} m\, d_1\, d_2,$$

and

$$\mathrm{Corr}(\tilde{\tau}_1, \tilde{\tau}_2) < \mathrm{Corr}(G_1\, G_2) = -\sqrt{\frac{d_1\, d_2}{(1 - d_1)\,(1 - d_2)}}\,.$$

Obviously, after the 'zeros-truncation' the covariance matrix $\{\mathrm{Cov}(G_i, G_j)\}_{i,j=1}^{s}$ coincides with the covariance matrix of the standard (non-truncated) multinomial distribution.

The joint probability mass function of the MD_{ZT} distribution with parameters $m = 15$ and $d_1 = d_2 = 1/3$ is plotted in Fig. 1.

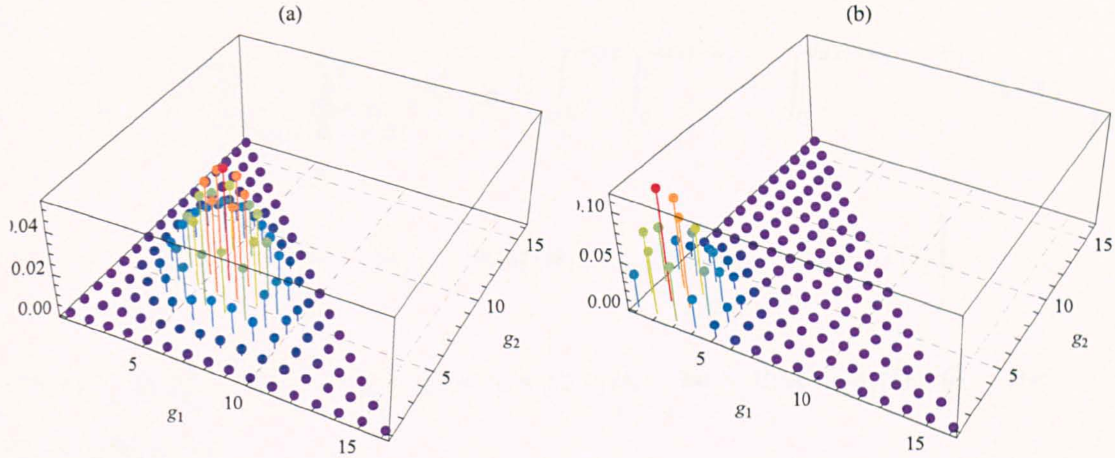

Fig. 1. The joint probability mass function of the MD_{ZT} distribution for: (a) - $m = 15$ and $d_1 = d_2 = 1/3$; (b) - $m = 15$ and $d_1 = d_2 = 1/10$;

5.3.2 The probability of joint survival under dependent inter-arrival times

Under the general risk model, specified in the previous section, which involves dependence between the claim sizes as well as between the claim inter-occurrence times, an expression for the probability of joint survival of the cedent and the reinsurer can be obtained. Within this more general framework, an explicit formula for the probability of non-ruin of the direct insurer only is derived in Ignatov and Kaishev (2007).

Following the notations, introduced in sections 5.2.1 and 5.3.1, we can state the following theorem.

Theorem 4. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$P\left(\tilde{T}^c > x, \tilde{T}^r > x\right) =$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty}\sum_{l=1}^{k}\sum_{(g_1,\ldots,g_s)\in\mathcal{G}_s(l)} p_{g_1,\ldots,g_s}\,\lambda^l \int_0^{h(x)}\int_0^{h(x)-\tilde{w}_1}\cdots\int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} A_l(x\,; \right. \tag{5.24}$$

$$\left. \tilde{v}_1,\ \ldots,\ \tilde{v}_l)\,\psi(\tilde{w}_1,\ \ldots,\ \tilde{w}_k)\,d\tilde{w}_k\cdots d\tilde{w}_2\,d\tilde{w}_1\right),$$

where $\mathcal{G}_s(l) = \{(g_1,\ \ldots,\ g_s) : g_1 + \ldots + g_s = l\}$, so that $\mathcal{G}_s(l) \equiv \emptyset$ for $l < g_1$, $\tilde{v}_j = \min(\tilde{z}_j,\ x)$,

$\tilde{z}_j =$

$$\begin{cases} 0 & \text{if } \ 1 \le j < g_1,\ (g_1 \neq 1) \\ \max\left(h_c^{-1}(\tilde{y}_i^c),\ h_r^{-1}(\tilde{y}_i^r)\right) & \text{if } \ g_1 + \ldots + g_i \le j < g_1 + \ldots + g_{i+1},\ i = 1, 2,\ \ldots \end{cases}$$

$\tilde{y}_j^c = \sum_{i=1}^{j} \tilde{w}_i^c$, $\quad \tilde{y}_j^r = \sum_{i=1}^{j} \tilde{w}_i^r$, $\quad \tilde{w}_i^c = \min(\tilde{w}_i,\ M) + \max(0,\ \tilde{w}_i - L)$,

$\tilde{w}_i^r = \min(L - M,\ \max(0,\ \tilde{w}_i - M))$, $j = 1,\ \ldots,\ l$, and

$A_l(x\,; \tilde{v}_1,\ \ldots,\ \tilde{v}_l)$, $l = 1, 2,\ \ldots$ *are the classical Appell polynomials* $A_l(x)$ *of degree* l, *defined by*

$A_0(x) = 1,\ \ A_l'(x) = A_{l-1}(x),\ \ A_l(\tilde{v}_l) = 0.$

Proof of Theorem 4. The proof follows using the same reasoning as in the proof of Theorem 1, conditioning on the random variables G_1, G_2, ... and applying the total probability formula.□

5.3.3 The expected profit given joint survival under dependent inter-arrival times

The following two theorems give explicit expressions for $E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x,\ \tilde{T}^r > x\right)\right]$ and $E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x,\ \tilde{T}^r > x\right)\right]$ correspondingly, under the general risk model with dependent claim arrivals, introduced in section 5.3.1.

Theorem 5. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right] =$$

$$h_c(x) - \left(\sum_{k=1}^{\infty} \sum_{l=1}^{k} \sum_{(g_1, \ldots, g_s) \in \mathcal{G}_s(l)} p_{g_1, \ldots, g_s} \lambda^l \int_0^{h(x)} \int_0^{h(x) - \tilde{w}_1} \cdots \int_0^{h(x) - \tilde{w}_1 - \ldots - \tilde{w}_{k-1}} \tilde{y}_k^c \right.$$

$$\left. A_l(x ; \tilde{v}_1, \ldots, \tilde{v}_l) \psi(\tilde{w}_1, \ldots, \tilde{w}_k) \, d\tilde{w}_k \cdots d\tilde{w}_2 \, d\tilde{w}_1 \right) \Big/ \quad (5.25)$$

$$\left(1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \sum_{(g_1, \ldots, g_s) \in \mathcal{G}_s(l)} p_{g_1, \ldots, g_s} \lambda^l \int_0^{h(x)} \int_0^{h(x) - \tilde{w}_1} \cdots \int_0^{h(x) - \tilde{w}_1 - \ldots - \tilde{w}_{k-1}} A_l(\right.$$

$$\left. x ; \tilde{v}_1, \ldots, \tilde{v}_l) \psi(\tilde{w}_1, \ldots, \tilde{w}_k) \, d\tilde{w}_k \cdots d\tilde{w}_2 \, d\tilde{w}_1 \right)$$

where $\tilde{y}_k^c$, $\mathcal{G}_s(l)$, $\tilde{v}_j$, $j = 1, \ldots, l$ and $A_l(x ; \tilde{v}_1, \ldots, \tilde{v}_l)$ *are defined as in Theorem 4.*

Proof of Theorem 5. Follows the same line of conclusions as in the proof of Theorem 2.□

Theorem 6. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right] =$$

$$h_r(x) - \left(\sum_{k=1}^{\infty}\sum_{l=1}^{k}\sum_{(g_1,\ldots,g_s)\in\mathcal{G}_s(l)} p_{g_1,\ldots,g_s}\lambda^l \int_0^{h(x)}\int_0^{h(x)-\tilde{w}_1}\cdots\int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} \tilde{y}_k^r\right.$$

$$\left. A_l(x;\tilde{v}_1, \ldots, \tilde{v}_l)\,\psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1\right)\Big/ \qquad (5.26)$$

$$\left(1 + \sum_{k=1}^{\infty}\sum_{l=1}^{k}\sum_{(g_1,\ldots,g_s)\in\mathcal{G}_s(l)} p_{g_1,\ldots,g_s}\lambda^l \int_0^{h(x)}\int_0^{h(x)-\tilde{w}_1}\cdots\int_0^{h(x)-\tilde{w}_1-\ldots-\tilde{w}_{k-1}} A_l(\right.$$

$$\left. x;\tilde{v}_1, \ldots, \tilde{v}_l)\,\psi(\tilde{w}_1, \ldots, \tilde{w}_k)\, d\tilde{w}_k \cdots d\tilde{w}_2\, d\tilde{w}_1\right)$$

where $\tilde{y}_k^r$, $\mathcal{G}_s(l)$, $\tilde{v}_j$, $j = 1, \ldots, l$ and $A_l(x;\tilde{v}_1, \ldots, \tilde{v}_l)$ *are defined as in Theorem 4.*

Proof of Theorem 6. Follows the same line of conclusions as in the proof of Theorem 3.□

5.4 The optimal XL reinsurance contract

The results obtained in Theorems 1 to 6 can be used to find the optimal values of the parameters of an XL reinsurance contract, considered under the risk models described in section 5.2.1, (M1), and section 5.3.1, (M2). Furthermore, any of the optimality Problems 1 and 2 defined in section 3.3, and Problems 1, 2 and 3 defined in section 4.4, can be re-formulated here within the framework of both models (M1) and (M2), as follows.

Problem 1. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$, find

$$\max_{L,M} P\left(\tilde{T}^c > x, \tilde{T}^r > x\right).$$

Problem 2. For fixed M, L and $h(t)$, find

$$\max_{\substack{h_c(t),\\ h(t)=h_c(t)+h_r(t)}} P\left(\tilde{T}^c > x, \tilde{T}^r > x\right).$$

Problem 3. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$, find

$$\begin{aligned}&\max_{M,m} E\left[.\mid\left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]\\ &\text{subject to } P\left(\tilde{T}^c > x, \tilde{T}^r > x\right) = p.\end{aligned} \tag{5.27}$$

Problem 4. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$ with $h_c(t) = \alpha\, h(t)$, $h_r(t) = (1-\alpha)\, h(t)$, $0 \le \alpha \le 1$, i.e. given that at any $t \ge 0$ the cedent retains $100\,\alpha\,\%$ of $h(t)$ and the rest $100\,(1-\alpha)\,\%$ is taken by the reinsurer, find values of M and m such that

$$\frac{E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]}{E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]} = q \tag{5.28}$$

where

$$q = \frac{h_c(t)}{h_r(t)} = \frac{\alpha\, h(t)}{(1-\alpha)\, h(t)} = \frac{\alpha}{1-\alpha}. \tag{5.29}$$

Problem 5. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$ with $h_c(t) = \alpha\, h(t)$, $h_r(t) = (1-\alpha)\, h(t)$, $0 \le \alpha \le 1$, so that $h_c(t)\,/\,h_r(t) = q$, find

$$\begin{aligned}&\min_{M,m}\left[1 - P\left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]\\ &\text{subject to } \frac{E\left[\tilde{R}_x^c \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]}{E\left[\tilde{R}_x^r \mid \left(\tilde{T}^c > x, \tilde{T}^r > x\right)\right]} = q.\end{aligned} \tag{5.30}$$

Due to the high complexity of explicit formulae (5.5), (5.18), (5.19) and (5.24), (5.25), (5.26), solving Problems 1-5 numerically seems to be the only feasible approach.

5.5 Comments and conclusions

In this paper, we introduce two models, (M1) and (M2), which generalize the classical assumption of Poisson claim arrivals. The first model, (M1), assumes that claims arrive at random moments $\tilde{T}_i$, such that $\tilde{\tau}_i = \tilde{T}_i - \tilde{T}_{i-1} \sim \text{Erlang}(g_i)$, $i = 1, 2, \ldots$ with possibly different shape parameters g_i, $i = 1, 2, \ldots$. In the second model, (M2), we introduce dependence between the claim inter-arrival times by randomizing the Erlang parameters g_i through a multivariate integer-valued distribution.

An excess of loss reinsurance with a retention and a limiting level is considered, and explicit expressions for the probability of joint survival and the expected profits of the direct insurer and the reinsurer are obtained under both models (M1) and (M2). It is shown how these risk and performance measures can be used in optimally setting the parameters of an XL reinsurance treaty.

Chapter 6

Conclusions and Future Research

In this thesis we have considered general risk models which incorporate dependence between claim amounts and/or dependence between the claim inter-arrival times. Under such models, we have addressed the problem of (non-) ruin within a finite-time horizon of an insurance company.

In Chapter 2, we have provided an overview of some existing approaches to evaluating the probability of finite-time ruin in the classical framework. We have investigated the use of the method of local moment matching for discretizing the individual claim amount distribution and then combined it with the formulae of Picard and Lefèvre (1997) and Ignatov and Kaishev (2000) in order to evaluate ruin probabilities for continuous claim amount. Further, under a more general risk model, an extension of the formula of Ignatov and Kaishev (2000) to the case of continuous case has been obtained and its numerical performance has been investigated.

In Chapter 3, we have derived explicit expressions for the probability of joint survival up to time x of the cedent and the reinsurer, under an XL reinsurance contract with a limiting and a retention level, under the reasonably general assumptions of the risk model of Chapter 2. We have stated some optimality problems, and have shown how the latter results can be used to set the limiting and the retention levels in an optimal way with respect to the probability of joint survival or how, for fixed retention and limiting levels, the results can yield to an optimal split of the total premium income between the two parties. This methodology was illustrated numerically on several examples of independent and dependent claim severities.

Under a general risk model, in Chapter 4, we have demonstrated how the problem of optimal reinsurance can be solved, combining the expected profits at time x of the direct insurer and the reinsurer, given their joint survival up to x, and the probability of joint survival of the direct insurer and the reinsurer up to the finite time horizon x. Explicit expressions have been derived and used for their numerical evaluation. We have introduced an efficient frontier type approach to setting the limiting and the retention levels, based on the probability of joint survival considered as a risk measure and on the expected profit given joint survival, considered as a performance measure. Several optimality problems are defined and their solutions are illustrated numerically, both for the case of dependent and independent claim severities.

In Chapter 5, we further generalized the risk model considered in Chapters 2, 3 and 4. We first looked at the case of independent, non-identically Erlang distributed claim inter-arrival times and then, we allowed for modelling dependence between the claim inter-arrival times by assuming that the latter are Erlang distributed with a random shape parameter. Explicit expressions for the probability of joint survival of the cedent and the reinsurer up to time x and the expected profit at x, given joint survival up to x, were obtained in both cases.

The research presented in the current thesis forms part of a continuous research programme which has led to a number of publications in the area of Actuarial Science and Insurance. These include the papers by Kaishev and Dimitrova (2007), Kaishev, Dimitrova and Haberman (2007), Dimitrova, Kaishev and Penev (2008), Kaishev, Dimitrova, Haberman and Verrall (2007), Kaishev, Dimitrova and Ignatov (2007).

Future research may look at an even more general risk model where cross-over dependence between the claim inter-occurrence times and claim sizes is allowed for. A model which incorporates such dependence but assumes that the claim amounts are i.i.d. random variables has recently been considered by Albrecher and Boxma (2004).

Another possible direction of expansion of the risk model is to introduce a deterministic or stochastic interest in the risk model and look for appropriate generalization of the presented results.

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