The indecomposability of a certain bimodule given by the Brauer construction

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Abstract.

Broué's abelian defect conjecture [3, 6.2] predicts for a p-block of a finite group G with an abelian defect group P a derived equivalence between the block algebra and its Brauer correspondent. By a result of Rickard [11], such a derived equivalence would in particular imply a stable equivalence induced by tensoring with a suitable bimodule - and it appears that these stable equivalences in turn tend to be obtained by "gluing" together Morita equivalences at the local levels of the considered blocks; see e.g. [4, 6.3], [8, 3.1], [12, 4.1], and [13, 5.6, A.4.1]. This note provides a technical indecomposability result which is intended to verify in suitable circumstances the hypotheses that are necessary to apply gluing results as mentioned above. This is used in [7] to show that Broué's abelian defect group conjecture holds for nonprincipal blocks of the simple Held group and the sporadic Suzuki group.

Keywords: Broué's conjecture; Brauer construction; block; Brauer pair

Throughout this note, p is a prime and $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field k of characteristic p. We allow the case $\mathcal{O} = k$. We state our result and explain the terminology below.

Theorem. *Let G be a finite group, let b be a block of $\mathcal{O}G$ and let (P, e) be a maximal b-Brauer pair. Set $H = N_G(P, e)$. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$ satisfying $(Q, e_Q) \subseteq (P, e)$ and $(Q, f_Q) \subseteq (P, e)$, respectively. Let f be a primitive idempotent in $(\mathcal{O}Gb)^{\Delta H}$ such that $\mathrm{Br}_{\Delta P}(f)e = e$ and set $X = \mathcal{O}Gf$. Then, as $\mathcal{O}(G \times H)$-module X is indecomposable with vertex ΔP, and for any subgroup Q of $Z(P)$ the $k(C_G(Q) \times C_H(Q))$-module $e_Q X(\Delta Q) f_Q$ is up to isomorphism the unique indecomposable direct summand of $e_Q kC_G(Q) f_Q$ with vertex ΔP.*

This Theorem is used in [7] to verify Broué's abelian defect group conjecture for nonprincipal blocks of the simple Held group and the sporadic Suzuki group.

Given a finite group G, a *block of* $\mathcal{O}G$ is a primitive idempotent in $Z(\mathcal{O}G)$. We denote by ΔG the diagonal subgroup $\Delta G = \{(g,g) \mid g \in G\}$ of $G \times G$. Unless stated otherwise, modules are left modules. If G and H are two finite groups, by an $(\mathcal{O}G, \mathcal{O}H)$-bimodule we mean a bimodule whose left and right $\mathcal{O}$-module structure coincide, so that we can view any such bimodule X as $\mathcal{O}(G \times H)$-module via $(g,h)x = gxh^{-1}$ for any $(g,h) \in G \times H$ and any $x \in X$. If furthermore Q is a common subgroup of G and H, we set $X^{\Delta Q} = \{x \in X \mid (u,u)x = x, \forall u \in Q\} = \{x \in X \mid uxu^{-1} = x, \forall u \in Q\}$. If Q is actually a p-group, the *Brauer construction* is defined to be the quotient $X(\Delta Q) = X^{\Delta Q}/(\sum_{Q'} \mathrm{Tr}_{Q'}^{Q}(X^{\Delta Q'}) + J(\mathcal{O})X^{\Delta Q})$, where in the sum Q' runs over the set of proper subgroups of Q, and where $\mathrm{Tr}_{Q'}^{Q}$ is the usual relative trace map. This construction is functorial in X. Moreover, since $C_{G\times H}(\Delta Q) = C_G(Q) \times C_H(Q) \subseteq N_{G\times H}(\Delta Q)$, we can regard $X(\Delta Q)$ as a $(kC_G(Q), kC_H(Q))$-bimodule. When applied to $X = \mathcal{O}G$, there is a canonical isomorphism $(\mathcal{O}G)(\Delta Q) \cong kC_G(Q)$, and the map $\mathrm{Br}_{\Delta Q} : (\mathcal{O}G)^{\Delta Q} \longrightarrow kC_G(Q)$ obtained from composing the canonical epimorphism $(\mathcal{O}G)^{\Delta Q} \to (\mathcal{O}G)(\Delta Q)$ with this isomorphism is in fact an algebra homomorphism, called the *Brauer homomorphism*. More explicitly, every element in $(\mathcal{O}G)^{\Delta Q}$ is an $\mathcal{O}$-linear combination of Q-conjugacy class sums of elements in G, and $\mathrm{Br}_{\Delta Q}$ maps the Q-conjugacy class sum of an element $x \in G$ to zero unless $x \in C_G(Q)$, in which case x is mapped to its canonical image in $kC_G(Q)$.

Given a finite group G and a block b of $\mathcal{O}G$, a *b-Brauer pair* is a pair (Q,f) consisting of a p-subgroup Q of G and a block f of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(b)f = f$. By results of Alperin and Broué [1], the set of b-Brauer pairs is a G-poset with a single G-conjugacy class of maximal b-Brauer pairs. If (P,e) is such a maximal b-Brauer pair then P is a defect group of b. A primitive idempotent $i \in (\mathcal{O}Gb)^{\Delta P}$ satisfying $\mathrm{Br}_{\Delta P}(i) \neq 0$ is then called a *source idempotent* of the block b. Since $\mathrm{Br}_{\Delta P}$ is a surjective algebra homomorphism, $\mathrm{Br}_{\Delta P}(i)$ is a primitive idempotent in $kC_G(P)$, and we may thus always choose i such that $\mathrm{Br}_{\Delta P}(i)e \neq 0$. By [5, 1.8], for any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$, and then e_Q is the unique block of $kC_G(Q)$ such that $(Q, e_Q) \subseteq (P,e)$; in particular, $e_P = e$. See [9] and [14] for more details and background information. For the proof of the above Theorem we need the following Lemma, the first part of which is well-known.

Lemma. *Let G be a finite group, let b be a block of $\mathcal{O}G$, let (P, e) be a maximal b-Brauer pair, and let $i \in (\mathcal{O}Gb)^{\Delta P}$ be a source idempotent of b such that* $\mathrm{Br}_{\Delta P}(i)e \neq 0$*. Let Q be a subgroup of $Z(P)$ and let e_Q be the unique block of $kC_G(Q)$ satisfying $(Q, e_Q) \subseteq (P, e)$. Then P is a defect group of e_Q and* $\mathrm{Br}_{\Delta Q}(i)$ *is a source idempotent of the block e_Q in $(kC_G(Q)e_Q)^{\Delta P}$.*

Proof. Since $Q \subseteq Z(P)$ we have $P \subseteq C_G(Q)$, and hence P is a defect group of e_Q by [8, 7.6]. Now $\mathrm{Br}_{\Delta Q}$ maps $(\mathcal{O}G)^{\Delta Q}$ onto $kC_G(Q)$; since P normalises $C_G(Q)$, any P-conjugacy class of elements in G is either contained in $C_G(Q)$ or in $G - C_G(Q)$. Hence $\mathrm{Br}_{\Delta Q}$ maps $(\mathcal{O}G)^{\Delta P}$ onto $(kC_G(Q))^{\Delta P}$. This implies that $\mathrm{Br}_{\Delta Q}(i)$ is a primitive idempotent in $(kC_G(Q))^{\Delta P}$. Moreover, by [5, 1.8] we have $\mathrm{Br}_{\Delta Q}(i) \in kC_G(Q)e_Q$ and clearly $\mathrm{Br}_{\Delta P}(\mathrm{Br}_{\Delta Q}(i)) = \mathrm{Br}_{\Delta P}(i) \neq 0$, which proves the second statement of the Lemma. □

Proof of the Theorem. Let $\hat{e}$ be the block of $\mathcal{O}C_G(P)$ which corresponds to the block e of $kC_G(P)$. Note first that $\hat{e}$ is still a block of $\mathcal{O}H$ with (P, e) as unique maximal Brauer pair. Let $j \in (\mathcal{O}H\hat{e})^{\Delta P}$ be a source idempotent of $\hat{e}$ as block of $\mathcal{O}H$. Then, by [6, 4.10] (or also [2, Theorem 5(ii) and p.265, line 3]) the idempotent $i = jf$ is a source idempotent of the block b in $(\mathcal{O}Gb)^{\Delta P}$, and since f was chosen such that $\mathrm{Br}_{\Delta P}(f)e = e$ we have $\mathrm{Br}_{\Delta P}(i)e \neq 0$. Let Q be a subgroup of $Z(P)$. By the above Lemma, $i_Q = \mathrm{Br}_{\Delta Q}(i)$ is a source idempotent of the block e_Q, and $j_Q = \mathrm{Br}_{\Delta Q}(j)$ is a source idempotent of the block f_Q. Since $i = jf = fj$ we have $i_Q = \mathrm{Br}_{\Delta Q}(f)j_Q$, and this is therefore in particular a primitive idempotent in $(kC_G(Q)e_Q)^{\Delta P}$.

Since $X = \mathcal{O}Gf$ we have $X(\Delta Q) = kC_G(Q)\,\mathrm{Br}_{\Delta Q}(f)$, and therefore

$$e_Q X(\Delta Q) j_Q = e_Q kC_G(Q)\,\mathrm{Br}_{\Delta Q}(f) j_Q = e_Q kC_G(Q) i_Q \ .$$

As $i_Q \in kC_G(Q)e_Q$ this implies in particular that $e_Q X(\Delta Q) j_Q$ is non zero. The point now is that since i_Q is primitive in $(kC_G(Q)e_Q)^{\Delta P}$, the $(kC_G(Q)e_Q, kP)$-bimodule $e_Q kC_G(Q) i_Q$ is indecomposable. Since kP is isomorphic to a subalgebra of the source algebra $j_Q kC_H(Q) j_Q$ via multiplication by j_Q, it follows that $e_Q X(\Delta Q) j_Q$ is indecomposable as $(kC_G(Q)e_Q, j_Q kC_H(Q) j_Q)$-bimodule. By [10, 3.4], the block algebra $kC_H(Q)f_Q$ and its source algebra $j_Q kC_H(Q) j_Q$ are Morita equivalent, which implies that indeed $e_Q X(\Delta Q) f_Q$ is indecomposable as $k(C_G(Q) \times C_H(Q))$-module.

Since X is a direct summand of $\mathcal{O}Gb$ as $\mathcal{O}(G \times H)$-module, $X(\Delta Q)$ is a direct summand of $kC_G(Q)\,\mathrm{Br}_{\Delta Q}(b)$ as $k(C_G(Q) \times C_H(Q)$-module, and hence $e_Q X(\Delta Q) f_Q$ is a direct summand of $e_Q kC_G(Q) f_Q$.

Since f is primitive in $(\mathcal{O}G)^{\Delta H}$, the $\mathcal{O}(G \times H)$-module X is indecomposable. As $\mathcal{O}(G \times G)$-module, $\mathcal{O}Gb$ has ΔP as vertex. Thus X has a vertex contained in a $(G \times G)$-conjugate of ΔP. Since $\mathrm{Br}_{\Delta P}(f)e = e \neq 0$, we have $X(\Delta P) \neq 0$ and thus X has ΔP as a vertex by [14, 27.7]. Similarly, we have $e = e\,\mathrm{Br}_{\Delta P}(e_Q) = e\,\mathrm{Br}_{\Delta P}(f_Q) = e\,\mathrm{Br}_{\Delta P}(f)$ by [5, 1.8(3)] and the assumption. Thus, if we denote by $\bar{f}$ the canonical image of f in $(kG)^{\Delta H}$, we get $e\,\mathrm{Br}_{\Delta P}(e_Q \bar{f} f_Q) = e \neq 0$, so that $\mathrm{Br}_{\Delta P}(e_Q \bar{f} f_Q) \neq 0$, hence $(e_Q X(\Delta Q) f_Q)(\Delta P) \neq 0$, and so ΔP is a vertex of $e_Q X(\Delta Q) f_Q$.

For the last part we observe that the $k(C_G(Q) \times C_H(Q))$-module $e_Q k C_G(Q) f_Q$ is a direct summand of $kC_G(Q)f_Q = \mathrm{Ind}_{C_H(Q)\times C_H(Q)}^{C_G(Q)\times C_H(Q)}(kC_H(Q)f_Q)$. Moreover, the $k(C_H(Q) \times C_H(Q))$-module $kC_H(Q)f_Q$ is indecomposable with ΔP as vertex, and the normaliser of ΔP in $C_G(Q) \times C_H(Q)$ is contained in $C_H(Q) \times C_H(Q)$. Thus the Green correspondence implies that the $k(C_G(Q) \times C_H(Q))$-module $kC_G(Q)f_Q$ has exactly one indecomposable direct summand with ΔP as vertex, up to isomorphism. The result follows. $\square$

Remark. With the notation of the Theorem, if Q is a subgroup of $Z(P)$ then $f_Q = e$. Indeed, P is normal in H, hence in $C_H(Q)$, and thus every block of $kC_H(Q)$ is contained in $kC_H(P) = kC_G(P)$. The last argument in the proof of the Theorem shows the seemingly stronger statement that $e_Q X(\Delta Q) f_Q$ is the unique direct summand with vertex ΔP of the $k(C_G(Q) \times C_H(Q))$-module $kC_G(Q)f_Q$, but since $\mathrm{Br}_{\Delta P}(e_Q)e = e = f_Q$, every direct summand of $kC_G(Q)f_Q$ with vertex ΔP is already a direct summand of $e_Q k C_G(Q) f_Q$.

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References

[1] J. L. Alperin, M. Broué, Local methods in block theory, Ann. of Math. (2) **110** (1979), 143–157.

[2] J. L. Alperin, M. Linckelmann, R. Rouquier, Source algebras and source modules, J. Algebra **239** (2001), 262–271.
[3] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque **181/182** (1990), 61–92.
[4] M. Broué, Equivalences of blocks of group algebras, in: "Finite Dimensional Algebras and Related Topics", V. Dlab and L. L. Scott (eds), Kluwer Acad. Pub. (1994), 1–26.
[5] M. Broué, L. Puig, Characters and local structure in G-algebras, J. Algebra **73** (1980), 306–317.
[6] Y. Fan, L. Puig, On blocks with nilpotent coefficient extensions, Algebras and Representation Theory **2** (1999).
[7] S. Koshitani, N. Kunugi, K. Waki, Broué's abelian defect group conjecture for the Held group and the sporadic Suzuki group, J. Algebra, in press (2004).
[8] M. Linckelmann, On splendid derived and stable equivalences between blocks of finite groups, J. Algebra **242** (2001), 819–843.
[9] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, New York, 1988.
[10] L. Puig, Pointed groups and construction of characters, Math. Z. **176** (1981), 265–292.
[11] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. **43** (1991), 37–48.
[12] J. Rickard, Splendid equivalences: derived categories and permutation modules, Proc. London Math. Soc. **72** (1996), 331–358.
[13] R. Rouquier, Block theory via stable and Rickard equivalences, in: "Modular Representation Theory of Finite Groups", M. J. Collins, B. J. Parshall and L. L. Scott (eds.), de Gruyter, Berlin, (2001), 101–146.
[14] J. Thévenaz, G-Algebras and Modular Representation Theory, Clarendon Press, Oxford, 1995.