PART I

A Gentle Introduction to Default Risk and Counterparty Credit Modelling

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Abstract

In this paper we introduce the reader to the basic tools for the computation of Counterparty Credit Risk such as Credit Value Adjustment and Debt Value Adjustment. We also present the effect of mitigating clauses, like netting and collateral, in reducing the credit exposure. Detailed numerical examples are presented with reference to commodity derivatives.

Contents

1 Introduction

The expansion of the liberalized physical commodity markets has led to the development of financial derivative instruments linked to energy commodities; this was followed by an increase in the number of energy companies acting as market operators both with hedging and trading purposes. It is acknowledged by energy companies that their activities expose them to relevant market and credit risks. For this reason the companies should measure, manage and limit these risks to maintain both the stability of cash flows, generated by the assets and contracts in the portfolio, and the company economic-financial statements.

The most used derivative instruments for hedging purposes by oil, gas and power producers are commodity and interest rate swaps; in these contracts, the floating leg is usually indexed either to the price of energy products, such as oil, natural gas and power, or to a LIBOR rate.

Commodity swaps are instruments traded Over the Counter (OTC), ie the position is not managed by a Central Clearing House.[1] Therefore the two parties entering into the contract, in general a bank and a corporate firm, are exposed to the risk of default of the other party, the so-called Counterparty Credit Risk (CCR), which needs to be priced in a suitable manner. This can be done either by entering specific agreements such as the Margining Agreement, the Exchange of Futures for Physical, the Additional Collateralisation or by simply adjusting the OTC derivative instrument risk free value with a metric called Credit Value Adjustment (CVA).

With the introduction of the new Basel III regime and the new International Financial Reporting Standards (IFRS) 13, adjustments in the market consistent value of derivatives transactions have become common for all financial institutions. Further, capital requirements have been linked to more sophisticated measures of counterparty credit risk such as the above mentioned CVA, Debit Value Adjustment (DVA) and, more recently, their potential volatility captured via VAR models in conjunction with stress testing under extreme market scenarios (VAR of CVA). These changes in regulation have a significant impact on banks behaviour and on the pricing of specific types of trades where the underlying exposure profile is significantly large from the bank's perspective. Particularly, uncollateralized long dated trades (cross currency swaps, long dated foreign exchange forwards, interest rate swaps with significant carry) became more expensive and banks have either shifted focus on trades that are less credit intensive or tried to mitigate the exposure via mandatory breaks (that reduce the effective duration of the trades) or asked for collateral protection. Among bank counterparties the most affected group has been corporates. Corporates typically engage in derivatives transactions for hedging purposes and traditionally do not have the ability to face a high frequency of margin calls against derivatives trades. Typically these

[1] However, in the United States, most OTC derivatives must now be traded on organized trading venues and must be centrally cleared. In addition, the impact of trade reporting requirements, anti-evasion authority, business conduct standards and Basel III capital requirements suggests a likely increase in the costs of trading these products and a trend towards futurization of swaps. See for example Litan [18].

trades are aimed at hedging relatively long dated liabilities against interest rate and foreign exchange risk, hence falling within the category of transactions that are the most credit intensive for the banks. The immediate implications on pricing of these instruments have affected the behavior of corporates themselves, leading them to opt for shorter dated hedges, to match assets and liabilities currencies to the greatest possible extent, and to enact enhanced pricing search across relationship banks (through credit auction methodology) in order to limit overall costs.

To this purpose, we note that accurate quantification of CCR through metrics such as CVA is a complex task requiring a sound appreciation of default risk on the one hand, its modelling and quantification in accordance to existing market quotes; and identification of the exposure on the other hand, which turns out to be connected with option pricing theory.

In light of the above, the aim of this paper is to illustrate CVA computation, and how mitigating clauses, like netting and collateral, can help in reducing the credit exposure. This will be done with concrete reference to interest rate and commodity swap contracts.

The paper is organized as follows. Section 2 introduces the main tools in Credit Risk Analysis, such as Recovery Ratio, Default and Survival Probability and their use in pricing risky instruments like defaultable bonds and Credit Default Swaps. These financial instruments are very important in a reverse engineering procedure that allows to extract market default probability from their quotations. This is illustrated in Section 3. Then, Section 4 takes a closer look at Counterparty Credit Risk, providing the definition and its quantitative estimation. Section 5 investigates the so called bilateral adjustment and the computation of the Debt Value Adjustment (DVA). Section 6 introduces the concept of Wrong Way Risk (WWR) discussing how relaxing the independence hypothesis between market value of the derivative and default event can affect the CVA. Two possible modelling approaches, ie structural models and intensity based models, are presented as possible ways of capturing WWR. Section 7 presents the so-called mitigating clauses, ie collateral and netting agreements, which contribute to CVA reduction. Finally, in Section 8, Value at Risk of CVA is discussed.

2 Credit Risk

Credit risk refers to financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement. It is related to the possibility that an unexpected change in a counterparty's creditworthiness may generate a corresponding unexpected change in the market value of the associated credit exposure. Examples range from agency downgrades to failure to service debt to liquidation. In computing credit risk, the following elements play important roles

- Recovery ratio;
- Probability of default;
- Expected loss under default.

2.1 Recovery Ratio

The first ingredient in credit risk modelling is the recovery ratio. In general, the default event can be defined in very different ways, from real default to missed interest payments. Accordingly, the loss on default can vary significantly. In practice, simplifying assumptions are made concerning the recovered amount given a default. The most common ones are: a) Receive cash in proportion of market value, and b) Receive cash in proportion of principal. In both cases, it is customary to define the recovery ratio R to be such a proportion. The recovery ratio is therefore a real number in $(0, 1)$. A common assumption, based on historical records, is to assume $R = 0.4$, that means that we recover 40% of the market value (or face value) and we lose 60%. The quantity $1 - R$ is then termed Loss Given Default, or uncovered loss on default. In reality, historical defaults show that

- The recovery rate is a function of the seniority of the obligation.
- The recovery rate distribution is asymmetrical and skewed with a trailing right side tail.
- The data exhibits a considerable amount of dispersion around the mean for each class of debt.
- Bank debt has the highest recovery rate and is generally less volatile than the other debt classes.

Altman [1] and Keisman and Marshella [15] provide an excellent review on this issue. In the following, we will assume that the recovery ratio is a fixed number, eventually stressing where this assumption is important.

2.2 Default and Survival Probability

Given that the default event is unpredictable, it is common practice to introduce a random variable τ denoting the default time. Accordingly, the cumulative default probability $PD(t, T)$ is defined as

$$PD(t, T) = \Pr(\tau \leq T \,|\tau > t),$$

ie $PD(t, T)$ is the probability of having default before T given that up to date it has not yet happened.

The cumulative survival probability is defined as

$$Q(t, T) = 1 - PD(t, T) = \Pr(\tau > T \,|\tau > t).$$

Related quantities are also

- The probability of default between T_1 and T_2

$$\begin{aligned} PD(t, T_1, T_2) &= \Pr(T_1 < \tau \leq T_2 \,|\tau > t)) \\ &= PD(t, T_2) - PD(t, T_1) \\ &= Q(t, T_1) - Q(t, T_2). \end{aligned}$$

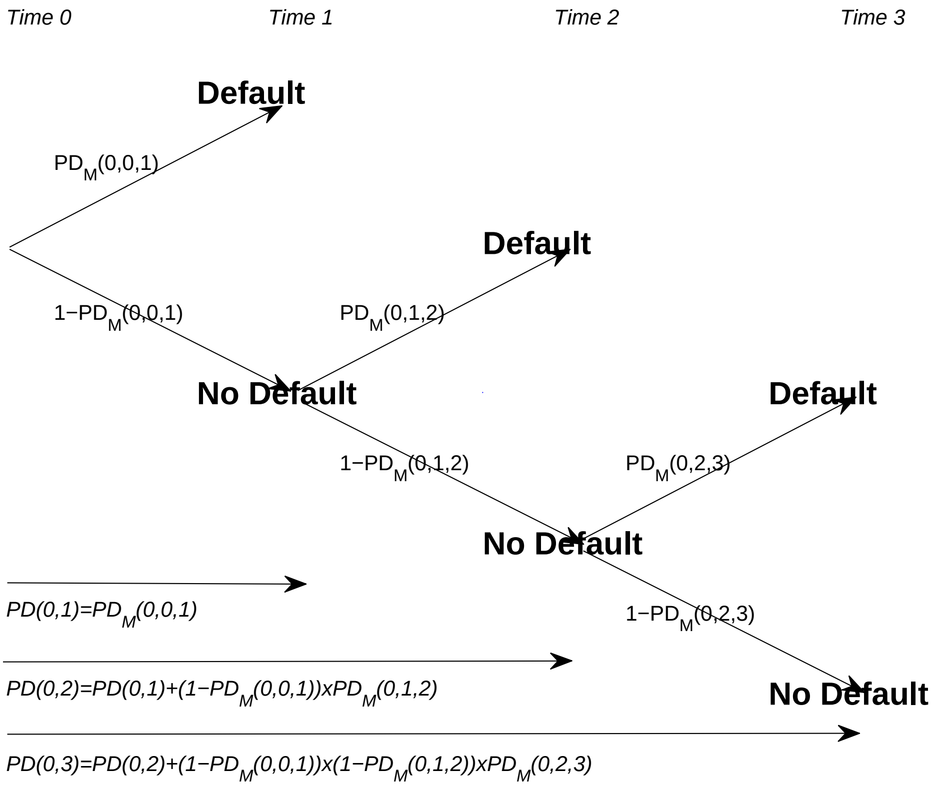

Figure 1: The sequential default process. The relationship 3 between cumulative and marginal default probabilities, is also illustrated.

- The marginal default probability, denoted by $PD_M(t, T_1, T_2)$, is the probability of default in the period $(T_1, T_2]$ conditioned on having survived until the beginning of year T_1

$$PD_M(t, T_1, T_2) = \Pr(T_1 < \tau \leq T_2 \,|\tau > T_1) \tag{1}$$
$$= \frac{PD(t, T_2) - PD(t, T_1)}{1 - PD(t, T_1)}. \tag{2}$$

The relationship between marginal and cumulative default probabilities is as illustrated here below. Clearly, we have

$$PD(0,1) = PD_M(0,0,1).$$

We observe that over a 2-year horizon the following events can occur: either default occurs within the first year, or default occurs in the second year, given the entity survived in the first year. Therefore, we have

$$\underbrace{PD(0,2)}_{default\ within\ 2\ years} = \underbrace{PD(0,1)}_{default\ within\ 1st\ year} \underbrace{+}_{OR} \underbrace{(1 - PD_M(0,0,1))}_{survive\ 1st\ year} \times \underbrace{PD_M(0,1,2)}_{default\ in\ the\ 2nd\ year}.$$

Solving for $PD_M(0,1,2)$, we get

$$PD_M(0,1,2) = \frac{PD(0,2) - PD(0,1)}{1 - PD(0,1)}.$$

that agrees with the formula in (2). Similarly to the above, we observe that over a 3-year horizon the following events can occur: either default occurs within the first year, or default occurs in the second year, given the entity survived in the first year, or default occurs in the third year, given the entity survived in the first two years. Therefore, we have

$$\underbrace{(1 - PD_M(0,0,1))}_{survive\ 1st\ year} \times \underbrace{(1 - PD_M(0,1,2))}_{survive\ 2nd\ year} \times \underbrace{PD_M(0,2,3)}_{default\ in\ the\ 3rd\ year}.$$

Again, if we solve for $PD_M(0,2,3)$ we can verify that the resulting expression agrees with formula (2). Notice that in writing the above expressions we have assumed that survival over a given year is independent of surviving over the successive years. A graphical illustration of the sequential default process is given in Figure 1. In the figure, the following general relationship between cumulative and marginal default probabilities is also illustrated

$$PD(t,T_2) = PD(t,T_1) + (1 - PD_M(t,t,T_1)) \times PD_M(t,T_1,T_2), t < T_1 < T_2. \quad (3)$$

By using the quantities just introduced, and by defining the third element, ie the expected loss under default, we can consider the pricing of very simple securities, such as risky zero-coupon bonds, risky coupon bonds and credit default swaps. These products are very important in credit markets because they can also be used to extract information concerning the default probability of the issuer.

2.3 Pricing a risky zero-coupon bond

A risk-free zero coupon bond (zcb) expiring at T and having notional value N is priced according to

$$V_{RF}(t) = P(t,T) \times N,$$

where $P(t,T)$ is the risk-free discount factor, ie the t value of 1 unit of account due at expiry T. We can now price a risky (defaultable) zero-coupon bond as

$$\begin{aligned} V_D(t) &= P(t,T) \times \left(\underbrace{(1 - PD(t,T)) \times N}_{\text{expected payoff if alive}} + \underbrace{PD(t,T) \times R \times N}_{\text{expected payoff on default}} \right) \\ &= \underbrace{P(t,T) \times N}_{V_{RF}(t)} - \underbrace{P(t,T)}_{\text{discount factor}} \times \underbrace{PD(t,T) \times (1-R) \times N}_{\text{expected loss given default}} \end{aligned}$$

In the above expression, in the first row, we have replaced the notional at maturity by the expected payment at maturity, depending on the fact that the issuer will survive (event that will happen with probability $1 - PD(t,T)$) or will default (event that will happen with probability $PD(t,T)$ and in this case a fraction of

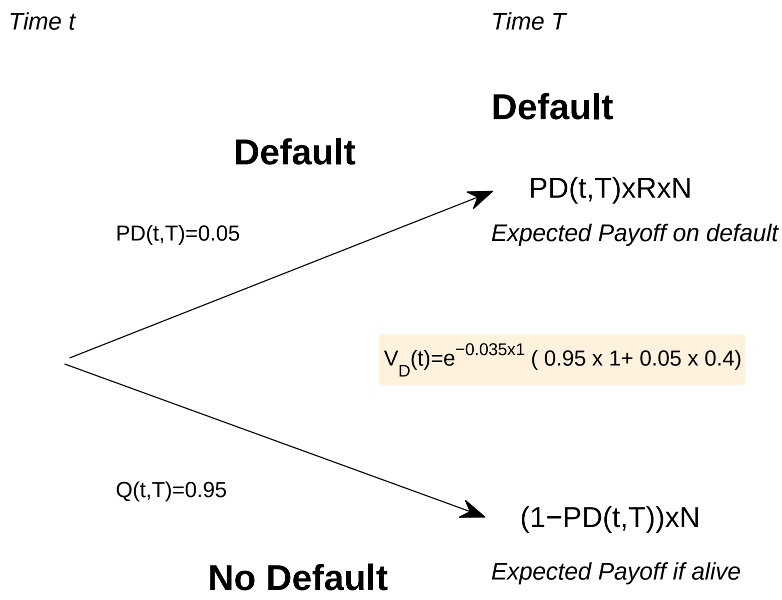

Figure 2: Valuation of a risky zero-coupon bond with time to maturity $T - t = 1$, recovery rate $R = 0.4$, one-year risk-free discount factor equal to $e^{-0.035}$ and one year default probability of 5%.

the notional will be recovered). In the second row, we see that the value of the risky zero-coupon bond is equal to the value of the corresponding risk-free zero-coupon bond minus an amount equal to the present value of the expected loss on default. As discussed later on, this amount is called credit value adjustment. The equation above also tells us that the expected loss from credit risk for a nominal exposure equal to N is given by $PD \times (1 - R) \times N$. An illustrative explanation is given in Figure 2.

Notice that in the above computation we have assumed that the recovery amount and the default event are independent random variables. This is a common simplifying assumption in most credit risk models, even if empirical evidence suggests a plausible negative dependence between these two variables.

Example 1 *Let us assume that the one-year risk-free discount factor is 0.99; the one-year survival probability is 0.98; the recovery rate is set at 0.4. Then, the expected payment at expiry is*

$$0.98 \times N + (1 - 0.98) \times 0.4 \times N = 0.988 \times N$$

and therefore the price of the risky zero-coupon bond is obtained by discounting this amount by using the risk free discount factor:

$$V_D(0) = 0.99 \times 0.988 \times N = 0.97812 \times N.$$

In this example, the expected loss on default is simply

$$(1 - 0.4) \times 0.98 \times N.$$

2.3.1 Credit Spread

The continuously compounded (c.c.) spot rate on a risk-free zcb is

$$y_{RF}(t, T) = -\frac{1}{T - t} \ln P(t, T).$$

The continuously compounded spot rate on a risky zcb is

$$\begin{aligned} y_D(t, T) &= -\frac{1}{T - t} \ln \left(P(t, T) \left(Q(t, T) \times 1 + (1 - Q(t, T)) \times R\right)\right) \\ &= y_{RF}(t, T) - \frac{1}{T - t} \ln \left(Q(t, T) \times 1 + (1 - Q(t, T)) \times R\right). \end{aligned}$$

If we look at the difference between y_D and y_{RF} we have the so called credit spread, *cs* say, that is nothing else than a different representation of the riskiness of the issuer

$$\begin{aligned} cs(t, T) &= y_D(t, T) - y_{RF}(t, T) \\ &= -\frac{1}{T - t} \ln \left(Q(t, T) \times (1 - R) + R\right). \end{aligned}$$

Figure 3 illustrates the inverse relationship between credit spread and survival probability, given an horizon of two years and a recovery ratio of 40%. Notice that, even if default will happen for sure, ie the survival probability is 0, the credit spread takes a finite value equal to $-\frac{1}{T-t} \ln(R)$ and becomes infinite if and only if the recovery ratio is 0.

Example 2 *Let us suppose that* $Q(0, 2) = 0.98$*, therefore*

$$cs(0, 2) = -\frac{1}{2} \ln (0.98 \times (1 - 0.4) + 0.4) = 0.006,$$

ie a 60 basis points credit spread. If the two years risk free rate is at 1.50%, the rate on a corporate bond of similar maturity is

$$1.50\% + 0.6\%,$$

and the price of a two year risky zero-coupon bond is

$$e^{-(0.0150+0.006)\times 2} = 0.9588.$$

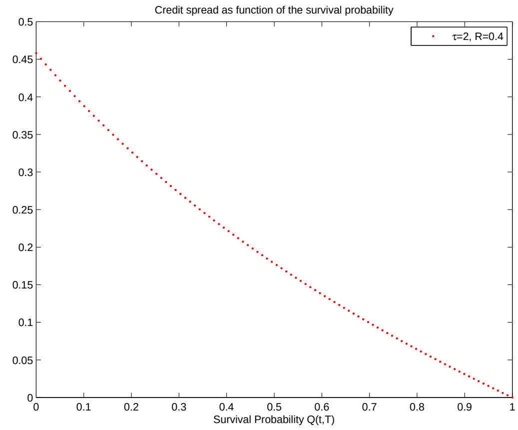

Figure 3: Credit spread as function of the survival probability (Parameters: $T - t = 2$ years, $R = 0.4$).

Given the relationship between cs and Q, we can solve it for Q to obtain the following expression

$$Q\left(t,T\right) = \frac{e^{-cs(t,T)\times(T-t)} - R}{1-R},$$

that relates credit spread and the survival probability. In addition, if $R = 0$, we have

$$Q\left(t,T\right) = e^{-cs(t,T)\times(T-t)},$$

and the price of the risky zcb simplifies into

$$V_D\left(t\right) = P\left(t,T\right) \times Q\left(t,T\right) = e^{-(y_{RF}(t,T)+cs(t,T))(T-t)} \times N,$$

ie we can price a risky zero-coupon bond discounting the notional using the rate y_D which is a sum of the credit spread cs and the risk-free rate y_{RF}. Therefore, assuming zero recovery, the credit spread quantifies the extra discounting required by the market as a compensation for default risk.

2.4 Pricing Risky Coupon Bonds

Let us consider the pricing of a risk-free coupon bond. Let c be the annual coupon, m the number of coupons per year, n the total number of payments, N the principal value. The risk-free coupon bond is priced according to

$$V_{RF}\left(t\right) = \left(\frac{c}{m} \times \sum_{i=1}^{n} P\left(t,T_i\right) + P\left(t,T_n\right)\right) \times N.$$

In order to price a risky bond, we observe that the payment at time T_i will be made only if the issuer is still alive, which occurs with probability $Q(t,T_i)$. However, if the issuer defaults in the period (T_{i-1},T_i), conditioned on the fact that she has survived up to time T_{i-1}, the bond holder will receive only a fraction of the notional, ie $R \times N$. The probability of this event is given by

$$Q(t,T_{i-1}) \times PD_M(t,T_{i-1},T_i) = Q(t,T_{i-1}) - Q(t,T_i). \tag{4}$$

Event	Payoff	Probability	PV x Prob x Payoff
Default in T_1	R	$\underbrace{PD_M(0,0,T_1)}_{1-Q(0,T_1)}$	$P(0,T_1) \times PD_M(0,0,T_1) \times R$
Survival in T_1	$C(T_1)$	$Q(0,T_1)$	$P(0,T_1) \times Q(0,T_1) \times C(T_1)$
Survival in T_1 and default in T_2	R	$\underbrace{Q(0,T_1) \times PD_M(0,T_1,T_2)}_{Q(0,T_1)-Q(0,T_2)}$	$P(0,T_2) \times Q(0,T_1) \times PD_M(0,T_1,T_2) \times R$
Survival in T_2	$C(T_2)$	$Q(0,T_1) \times (1-PD_M(0,T_1,T_2))$	$P(0,T_2) \times Q(0,T_1) \times (1-PD_M(0,T_1,T_2)) \times C(T_2)$

Table 1: Cash flows on a risky coupon bond depending on the default time. In particular, $C(T_1) = c/m$, and $C(T_2) = (1 + c/m)$. First column: event (default or surivival). Second column: payoff given the event (if default occurs: R, otherwise coupon or principal payment). Third column: probability of the event in the first column. Fourth column: discounted payoff weighted by the corresponding probability.

Table 1, with reference to a one year bond with semiannual coupons, illustrates the sequence of payments with their associated probability of occurrence and the present value of each probability weighted cash flow.

Table 1 and the above reasoning help us to get the general formula for pricing a risky coupon bond:

$$
\begin{aligned}
V_D(t) = {} & \underbrace{\frac{c}{m} \times \sum_{i=1}^{n} Q(t,T_i) \times P(t,T_i) \times N}_{\text{coupon payment if survives up to } T_i} \\
& + \underbrace{Q(t,T_n) \times P(t,T_n) \times N}_{\text{notional payment if survives up to } T_n} \\
& + N \times R \times \sum_{i=1}^{n} \underbrace{(Q(t,T_{i-1}) - Q(t,T_i))}_{prob.\ of\ default\ in\ (T_{i-1},T_i]} \times P(t,T_i), \qquad (5)
\end{aligned}
$$

In the above formula, we are assuming that, if default occurs, we can recover a proportion of the face value and that default can occur only at the coupon dates. Little amendements, related to the treatment of the accrued coupon, are otherwise needed if we assume that default can occur between two successive coupon payment dates.

Example 3 (Pricing a Risk Free Coupon Bond) *Let us suppose that we have the zero spot rates as in Table 2. We have to price a coupon bond expiring in one year and paying a semiannual coupon. The annual coupon is 6%. The risk-free bond is priced according to*

$$
\begin{aligned}
V_{RF}(t) &= \frac{0.06}{2} \times \left(e^{-0.03\times 0.5} + e^{-0.035\times 1}\right) + e^{-0.035\times 1} \\
&= 0.03 \times 1.950717 + 0.9656054 \\
&= 1.024127.
\end{aligned}
$$

Time	0.5	1
Spot (c.c.)	0.03	0.035
$Q(t,T)$	0.98	0.94

Table 2: Term Structure of risk-free spot rates (continuously compounded) and survival probabilities.

Event	Payoff	Probability	PV x Prob x Payoff
Default in $T_1 = 0.5$	0.4	$\underbrace{0.02}_{1-0.98}$	$e^{-0.03\times0.5} \times 0.02 \times 0.4 = 0.00788$
Survival in $T_1 = 0.5$	0.03	0.98	$e^{-0.03\times0.5} \times 0.98 \times 0.03 = 0.02896$
Survival in $T_1 = 0.5$ and default in $T_2 = 1$	0.4	$0.98 \times \frac{0.04}{0.98} = \underbrace{0.04}_{0.98-0.94}$	$e^{-0.035\times1} \times 0.04 \times 0.4 = 0.01545$
Survival in $T_2 = 1$	1 + 0.03	0.94	$e^{-0.035\times1} \times 0.94 \times 1.03 = 0.934899$

Table 3: Cash flows on a risky coupon bond. First column gives the relevant payment dates. The second column the corresponding payoff. The third column gives the probability of each payment. These probabilities are computed according to formula (4). Last column gives the discounted payoff, weighted by the corresponding probabilities. Discount factors and default probabilities are obtained from Table 2.

Let us consider the price of a risky bond having the same characteristics. Let us assume a recovery rate equal to 0.4. The payoffs of the bond are given in Table 3. The sequential default process and the corresponding payments are illustrated in Figure 4. The price of the risky bond is

$$0.00788 + 0.02896 + 0.01545 + 0.934899 = 0.987192.$$

The yield to maturity (ytm) of the government bond referred above is solution of the following equation

$$1.024127 = \frac{0.06}{2} e^{-ytm_{RF}\times0.5} + \left(1 + \frac{0.06}{2}\right) e^{-ytm_{RF}\times1} \Rightarrow ytm_{RF} = 3.4927\%.$$

Similarly, the yield to maturity of the corporate bond solves

$$0.987192 = \frac{0.06}{2} e^{-ytm_{D}\times0.5} + \left(1 + \frac{0.06}{2}\right) e^{-ytm_{D}\times1} \Rightarrow y_D = 7.2199\%.$$

The one year credit spread (expressed as difference of ytms) is

$$cs = ytm_D - ytm_{RF} = 7.2199\% - 3.4927\% = 3.7272\%.$$

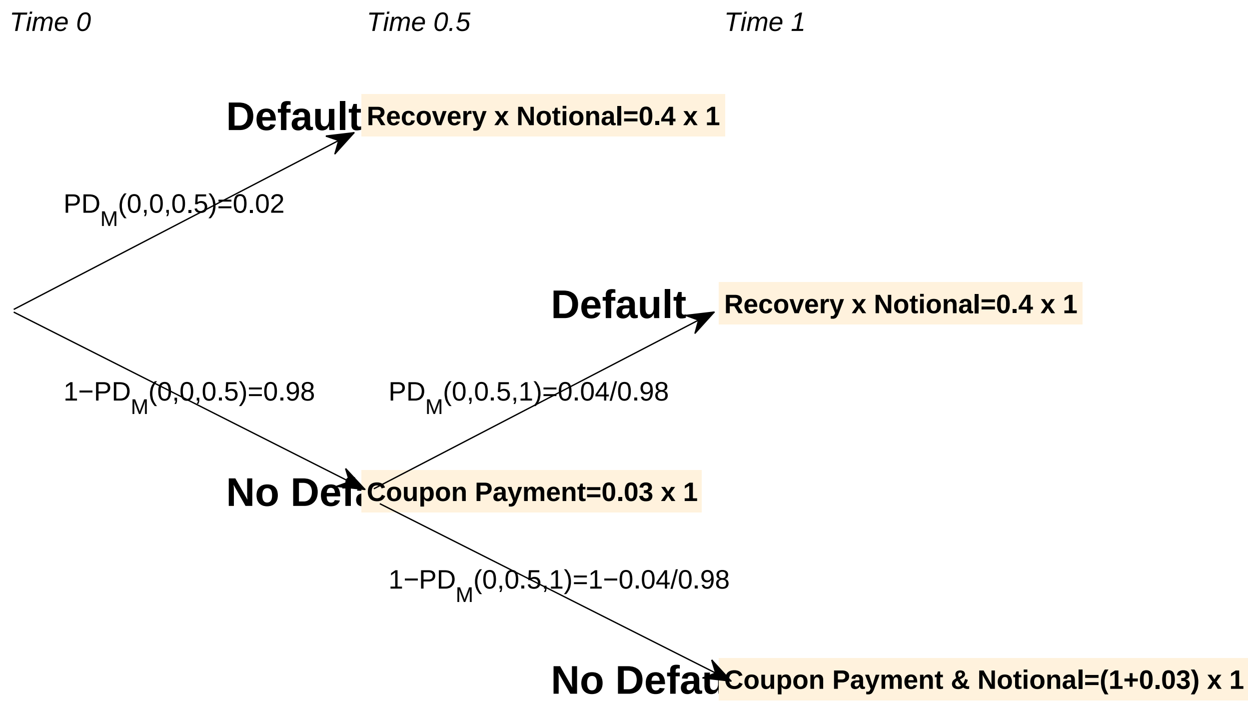

Figure 4: Sequential default process and corresponding cash flows of a risky coupon bond.

2.5 Credit Default Swap

A Credit Default Swap (CDS) **gives the right to the buyer to be compensated** for a loss, given the default of a reference security. The protection buyer pays a periodic fee (the CDS spread) and will receive the compensation on default of the reference entity. The protection seller will receive the periodic fee and will pay the uncovered loss to the buyer. Let T be the maturity of the CDS and $s(T)$ the CDS spread. Let T_i, $i = 1, \cdots, n$ be the periodic payment dates, with $T = T_n$. These dates are typically fixed as quarterly dates during the year and are called IMM-Dates[2], where adjustments are made if a date falls on a holiday.

Fixed Leg The present value of the payment of the protection buyer is

$$
\begin{aligned}
&PV_{fix}(T) \\
&= s(T) \times \sum_{i=1}^{n} \alpha_{i-1,i} \times P(t, T_i) \times Q(t, T_i) + \frac{s(T)}{2} \times \sum_{i=1}^{n} \alpha_{i-1,i} \times P(t, T_i) \times \left(Q(t, T_{i-1}) - Q(t, T_i)\right)\right),
\end{aligned}
$$

where

- the first term refers to the present value of the payments to be made if the reference entity survives;
- the second term refers to the fact that default is assumed to occur midway in (T_{i-1}, T_i), but the payment is made at the end of the period[3];
- and $\alpha_{i-1,i}$ refers to year fraction between calendar dates T_{i-1} and T_i, and follows the day count convention actual/360.

Floating Leg The present value of the payment of the protection seller is

$$
PV_{float}(T) = (1 - R) \times \sum_{i=1}^{n} \alpha_{i-1,i} \times P(t, T_i) \times \left(Q(t, T_{i-1}) - Q(t, T_i)\right).
$$

In the above two formulas, we have assumed independence between the default process and the interest rate process.

Determining the CDS spread At inception, the counterparties exchange expected cash-flows with equal values, so that the cost of entering a CDS must be zero. This implies that the spread $s(T)$ must be chosen so that

$$
PV_{float}(T) = PV_{fix}(T), \quad \text{for all } T.
$$

This gives us an equation to be solved with respect to the CDS spread for maturity T_n

$$
s(T_n) = \frac{(1 - R) \times \left(\hat{B}_n - B_n\right)}{0.5 \times (\hat{A}_n + A_n)}, \tag{6}
$$

[2]The International Monetary Market dates are the four quarterly dates of each year at which most futures contracts and option contracts expiry. CDS all mature on one of the four days of 20 March, 20 June, 20 September and 20 December.

[3]This term is due to the fact that the protection buyer pays *accrued* spread if a default happens during a quarter.

where

$$A_n := A(T_n) = \sum_{i=1}^{n} \alpha_{i-1,i} \times P(t, T_i) \times Q(t, T_i),$$
$$B_n := B(T_n) = \sum_{i=1}^{n} P(t, T_i) \times Q(t, T_i),$$

and

$$\hat{A}_n := \hat{A}(T_n) = \sum_{i=1}^{n} \alpha_{i-1,i} \times P(t, T_i) \times Q(t, T_{i-1}),$$
$$\widehat{B}_n := \widehat{B}(T_n) = \sum_{i=1}^{n} P(t, T_i) \times Q(t, T_{i-1}),$$

Example 4 *Table 4 provides relevant market information to determine the spread of a 1 year CDS with quarterly payments. Assuming a recovery ratio of 0.4, we obtain*

$$s = \frac{(1-0.6)\times(3.88072 - 3.85844)}{0.5\times(0.97018 + 0.96461)} = 0.0092,$$

ie 92 basis points.

T_i	$\alpha_{i-1,i}$	$P(0,T_i)$	$Q(0,T_i)$	$P(0,T_i)Q(0,T_i)\alpha_{i-1,i}$	$P(0,T_i)Q(0,T_{i-1})\alpha_{i-1,i}$
0		1	1		
0.25	0.25	0.99	0.999	0.247253	0.247500
0.5	0.25	0.98	0.994	0.243530	0.244755
0.75	0.25	0.97	0.987	0.239348	0.241045
1	0.25	0.96	0.977	0.234480	0.236880
				$A_4 = 0.96461$	$\hat{A}_4 = 0.97018$

Table 4: Computation of the quantities A_n and $\hat{A}_n$ for determining the fair spread for a one year CDS with quarterly payments, given the payment dates, the discount curve and the survival probability term structure, with $B_4 = 3.85844$ and $\hat{B}_4 = 3.88072$.

3 Estimating default probability

In the previous section we have considered the pricing of very common risky products. In addition to the knowledge of the term structure of risk free discount factors, which can be recovered either from market quotations of government bonds or the swap market[4], we need the term-structure of risk-neutral survival probabilities. In general, industry practice is to infer such probabilities either from liquid quotes of risky bonds, aggregated by sector and rating if necessary,

[4]For details on the procedure, see, e.g., Kienitz [17] and references therein.

or, if available, from CDS quotes, which typically are more reactive to incorporate market views on the obligor's creditworthiness. Another possibility is to estimate the term structure by using the historical frequency of default. With the caveat that in this situation we extract the so called real-word term structure of survival probability which can be significantly different from the risk-neutral one computed from market prices of credit instruments. In this section, we illustrate how to boostrap risk-neutral survival probabilities using market quotes of risky coupon bonds and credit default swaps spreads. In Appendix A, we discuss the use of historical default frequencies and transition matrices to construct the real-world term structure of survival probability.

3.1 Default probability from prices of risky bonds

Let us consider the pricing of a risky bond, Formula 5. Let us suppose that the term structure of survival probability is known up to time T_{n-1} and we have the price of risky bond expiring in T_n. We can use the pricing formula 5 to solve it with respect to $Q(t, T_n)$ and we obtain

- for $n = 1$
$$Q\left(t, T_1\right) = \frac{\frac{V_D}{N}\left(t\right) - R\,P\left(t, T_1\right)}{P\left(t, T_1\right)\left(1 + \frac{c}{m} - R\right)}, \tag{7}$$

- for $n > 1$
$$Q\left(t, T_n\right) = \tag{8}$$
$$\frac{\frac{V_D}{N}\left(t\right) - \left(\frac{c}{m} - R\right)\sum_{i=1}^{n-1} Q\left(t, T_i\right) \times P\left(t, T_i\right) - R \times \sum_{i=1}^{n} Q\left(t, T_{i-1}\right) \times P\left(t, T_i\right)}{P\left(t, T_n\right) \times \left(1 + \frac{c}{m} - R\right)} \tag{9}$$

Example 5 (Bootstrapping survival probability from bonds) *Let us consider two risky coupon bond, expiring in one and two-years, respectively. They pay yearly coupons equal to 4% and 5%, respectively and their market price is 0.97 and 0.96. Let us suppose that the risk-free discount factors for one and two years maturities are 0.99 and 0.98. Set the recovery ratio R at 0.4. The one year bond is priced according to the formula*

$$0.97 = 0.99 \times \left(Q\left(0,1\right) \times 1.04 + \left(1 - Q\left(0,1\right)\right) \times 0.4\right).$$

Solving it with respect to $Q\left(0,1\right)$ we get (see formula (7))

$$Q\left(0,1\right) = \frac{\frac{0.97}{0.99} - 0.4}{1.04 - 0.4} = 0.905934.$$

The second risky bond is priced according to

$$\begin{aligned}
0.96 = {} & 0.99 \times Q\left(0,1\right) \times 0.05 + 0.98 \times Q\left(0,2\right) \times 1.05 \\
& + 0.99 \times 0.4 \times \left(1 - Q\left(0,1\right)\right) + 0.98 \times 0.4 \times \left(Q\left(0,1\right) - Q\left(0,2\right)\right) \\
& = 0.99 \times 0.905934 \times 0.05 + 0.4 \times \left(1 - 0.905934\right) \times 0.99 \\
& + 0.98 \times Q\left(0,2\right) \times 1.05 + 0.4 \times \left(0.905934 - Q\left(0,2\right)\right) \times 0.98,
\end{aligned}$$

and therefore

$$0.96 = 0.43722 + Q\left(0,2\right) \times 0.98 \times \left(1.05 - 0.4\right).$$

Solving with respect to $Q(0,2)$, *we get*

$$Q\left(0,2\right) = \frac{0.96 - 0.43722}{0.98 \times \left(1.05 - 0.4\right)} = 0.81501.$$

The main limit of this approach is that market quotations at the desired maturities are not always available. Therefore, we need to aggregate bonds according to the issuer's rating or other characteristics. Sometimes, some kind of interpolation of quotations is also needed.

3.2 Default Probability from Credit Default Swap Spreads

Suppose we have a term structure of CDS spreads $s(T_1), ..., s(T_n)$ relative to maturities $T_1, ..., T_n$, and given that $t = T_0$ and $Q(t,t) = 1$, we can use the equilibrium condition (6) to extract sequentially $Q(t,T_1), \cdots, Q(t,T_n)$ from the quoted spreads. In the following, we adopt the shortand notation $Q_i = Q(t,T_i)$, $P_i = P(t,T_i)$, $s_i = s(T_i)$, and we proceed as follows:

1. Given $s(T_1)$ and $P(t,T_1)$, the fair value of the contract is zero if

$$s_1 \alpha_{0,1} P_1 Q_1 + \frac{s_1}{2} \alpha_{0,1} P_1 (1 - Q_1) = (1 - R) P_1 \left(Q_0 - Q_1\right),$$

   so that

$$Q_1 = \frac{1 - R - \frac{s_1 \alpha_{0,1}}{2}}{\frac{s_1}{2}\alpha_{0,1} + (1 - R)}.$$

2. In general, for $n > 1$, given s_n, and $\{P_i\}_{i=1}^{n}$, and $\{Q_i\}_{i=1}^{n}$, the equation can be solved recursively for Q_n returning

$$Q_n = \frac{0.5\left(\hat{A}_n + A_{n-1}\right) s_n - (1 - R)\left(\widehat{B}_n - B_{n-1}\right)}{\left(0.5\alpha_{n-1,n} s_n - (1 - R)\right) P_n}.$$

Example 6 *Let us consider the following market information relative to the term structure of risk-free discount factors and CDS spreads. Assume a recovery ratio of 0.4.*

Term	0	1	2	3	4	5
P	1	0.987	0.98	0.975	0.97	0.963
CDS		0.020	0.025	0.031	0.037	0.045

Table 5: Market quotes of risk-free discount factors and CDS spreads.

For $n = 1$, *we have,* $\hat{A}_1 = (1 - 0) \times 0.987 \times 1 = 0.9870$, *and*

$$Q_1 = \frac{1 - 0.4 - \frac{0.02}{2}}{1 - 0.4 + \frac{0.02}{2}} = 0.96721,$$

and

$$A_1 = (1-0) \times 0.987 \times 0.96721 = 0.9546.$$

For $n = 2$, $\hat{A}_2 = \hat{A}_1 + (2-1) \times 0.98 \times 0.9870 = 1.9349$,

$$Q_2 = \frac{0.5(1.9349 + 1.8559) \times 0.025 - (1-0.4) \times (1.9349 - 1.8559)}{(0.5 \times (2-1) \times 0.025 - (1-0.4)) \times 0.98} = 0.9196,$$

and

$$A_2 = A_1 + (2-1) \times 0.98 \times 0.9196 = 1.8559.$$

If we continue the procedure, we obtain Table 6, where last column refers to the bootstrapped survival probabilities

Term T_i	CDS	$\alpha_{i-1,i}$	$P(t,T_i)$	A_i	$\hat{A}_i$	$Q(t,T_i)$
0			1			1
1	0.02	1	0.987	0.9546	0.9870	0.9672
2	0.025	1	0.98	1.8559	1.9349	0.9196
3	0.031	1	0.975	2.6889	2.8315	0.8544
4	0.037	1	0.97	3.4413	3.6603	0.7757
5	0.045	1	0.963	4.0886	4.4072	0.6722

Table 6: Bootstrapped Term Structure of Default Probabilities.

More details on bootstrapping default probabilities from CDS spreads can be found in Beumee et al. [5] and in O'Kane and Turnbull [22]. Standard CDS specification adopted by ISDA can be found at http://www.cdsmodel.com.

4 Credit Value Adjustment for derivatives

This section will draw on the basic ingredients previously presented to illustrate the basic methodologies for measuring the Counterparty Credit Risk and the Credit Value Adjustment (CVA)

Counterparty Credit Risk (CCR) is the risk that the counterparty of an over-the-counter (OTC) deal will default before the maturity of the contract. Credit Value Adjustment (CVA) tries to measure the expected loss due to missing the remaining payments. Counterparty Value Adjustment has become an integral part of IAS 39 accounting rules and Basel III regulatory requirements.

Credit Value Adjustment is defined as the difference between the risk-free value and the risky value of one or more trades or, alternatively, the expected loss arising from a future counterparty default:

$$CVA = \text{Risk Free Mark to Market} - \text{Risky Mark to Market}.$$

As illustration, if we consider the price difference between a risk-free bond and a risky zero-coupon one, we have

$$CVA\,(t) = V_{RF}(t) - V_D(t) = P\,(t, T_n) \times (1 - Q\,(t, T_n)) \times (1 - R) \times N,$$

which is nothing else than the present value of the expected loss on default.

In particular, in the following, with reference to bilateral contracts such as forwards and swaps, we discuss the relationship between CVA and the value of a (portfolio of) options. The main ingredients for its determination turn out to be the default probabilities of the two parties entering into the contract, in first instance, and the volatility of the asset underlying their trade. Additional elements are related to the joint dependence between default events and value of the trade. This feature is related to the so called wrong or right way risk. Finally, mitigating clauses such as netting and collateral are also part of the discussion. In particular, CVA computation with a netting agreement requires to price an option on a portfolio rather than a portfolio of options, whilst assessing the risk-mitigation due to the presence of collateral is related to the pricing of calendar spread options.

4.1 Unilateral Adjustment

To determine the loss arising from the counterparty's default, it is convenient to assume that the bank is default-free and enters, when a default happens, into a similar contract with another counterparty in order to maintain its market position. Since the bank's market position is unchanged after replacing the contract, the loss is determined by the contract's replacement cost at the time of default. For example, as soon as market spot prices change after a swap is entered, the fixed and floating sides of the swap do not have equal value. Thus, the swap has value to one of the counterparties. Therefore

- If the **contract value is negative for the bank** at the time of default, the bank
  - closes out the position by paying the defaulting counterparty the market value of the contract;
  - enters into a similar contract with another counterparty and receives the market value of the contract; and
  - has a **net loss of zero**.
- If the **contract value is positive** for the bank at the time of default, the bank
  - closes out the position, but receives nothing from the defaulting counterparty;
  - enters into a similar contract with another counterparty and pays the market value of the contract; and
  - has a **net loss equal to the contract's market value**.

Thus, the credit exposure of a bank that has a single derivative contract with a counterparty is the maximum of the contract's market value and zero.

It is common practice to compute the exposure using simulation, discretizing the time to the expiry of the contract using a finite sets of times $t_0, t_1, \cdots, t_N = T$, and generating possible scenarios ω_s occurring with probability p_s. Let $V(t, \omega_s)$

be the value of the contract at time t in the scenario ω_s. Then few quantities are defined

1. **Future exposure (E)**: the contract-level exposure to occur on default at a future date for a given scenario
$$E\left(t_j, \omega_s\right) = \max\left(V\left(t_j, \omega_s\right), 0\right).$$
2. **Worst potential future exposure (PFE)**: the maximum exposure, with respect to all possible scenarios, at the default time
$$PFE(t_j) = \max_{\omega_s}(E(t_j, \omega_s)).$$
3. **Expected positive exposure (EPE)**: the probability-weighted average exposure (with respect to all possible scenarios) estimated to exist on a future date as seen from starting time t
$$EPE(t_j) = \mathbb{E}_t\left(E\left(t_j, \omega\right)\right) = \sum_{s=1}^{M} E\left(t_j, \omega_s\right) \times p_s,$$
where M is the number of simulated scenarios.

The above quantities are computed using either Monte Carlo simulation or option valuation models, and other statistical techniques.

Let us now consider unilateral CVA, ie the case in which only the bank's counterparty can default (in the following we omit the dependence of the contract value on the scenario ω, ie we write $V(t)$ instead of $V(t, \omega)$). The loss to the bank at the default time τ on a derivative contract expiring in T is
$$Loss(\tau) = \mathbf{1}_{\tau \le T} \times (1 - R) \times E(\tau),$$
where

- $\mathbf{1}_{\tau \le T}$ is the default indicator function:
  - If default occurs before maturity, then $\mathbf{1}_{\tau \le T} = 1$;
  - If default does not occur, then $\mathbf{1}_{\tau \le T} = 0$, and there is no loss to the bank.
- R is the fraction of the exposure that the bank recovers if default occurs;
- $E(\tau) = \max(V(\tau), 0) \equiv V^+(\tau)$ is the contract exposure on default, defined before.

The **unilateral CVA** is obtained by taking the risk-neutral expectation of the discounted loss given above
$$CVA(t) = \mathbb{E}_t\left(e^{-\int_t^\tau r(s)ds} \times \mathbf{1}_{\tau \le T} \times (1 - R) \times E(\tau)\right),$$

where $r(t)$ is the instantaneous (continuously compounded) risk-free rate. The computation of the expectation is not trivial at all, because it requires to know the joint distribution of the default time, the contract value, the recovery rate and the stochastic discount factor up to the default time.

Let us assume, for the moment, that **default can be observed only at the contract expiry** (even if it can occur before) and that the random variables involved in the expectation are independent, so that the expectation above factorizes into a product of expectations. More specifically, we assume that recovery rate, interest rates, exposure and default are independent. We have

$$CVA(t) = \underbrace{(1 - \mathbb{E}_t(R))}_{\text{expected loss given default}} \times \underbrace{\mathbb{E}_t(\mathbf{1}_{\tau \leq T})}_{\text{default probability}} \times \underbrace{\mathbb{E}_t\left(e^{-\int_t^T r(s)ds} V^+(T)\right)}_{\text{price of an option expiring in } T}.$$

Therefore, the CVA depends on

- The **loss given default** (LGD), ie the percentage of the exposure expected to be lost on default,
$$1 - \mathbb{E}_t(R),$$
but in general the recovery ratio is assumed to be constant, so this term simplifies into $1 - R$.

- The **probability of defaulting** at or before T, given that the entity is still alive
$$PD(t, T) = \mathbb{E}_t(\mathbf{1}_{\tau \leq T}).$$

- The **discounted expected positive exposure on maturity**, corresponding to the price of an option with maturity T
$$\text{option price}\,(t, T) = \mathbb{E}_t\left(e^{-\int_t^T r(s)ds+} V^+(T)\right),$$
i.e. option price(t, T) measures the expected present value of the exposure if default occurs at time T and is independent on the contract value.

Hence, we have a very simple formula for the CVA

$$CVA = (1 - R) \times PD(t, T) \times \text{option price}\,(t, T).$$

Example 7 (CVA of a forward contract with default observed at expiry) *Let us consider a forward contract with a risky counterparty, whose default can be observed only at the contract expiry (but it can occur before) and which is independent of the underlying exposure.*

The current underlying oil price today is 100 USD and in one period the stock price can take the values of 112, 104 or 90 with equal (risk-neutral) probabilities. Recalling that the forward price is equal, under no-arbitrage, to the risk-neutral expectation of the terminal oil price, we have that at inception the risk-free forward contract has a forward price equal to

$$\frac{1}{3}112 + \frac{1}{3}104 + \frac{1}{3}90 = \frac{306}{3} = 102.$$

Exposure	Probability
$\max(112-102,0)=10$	$\frac{1}{3}$
$\max(104-102,0)=2$	$\frac{1}{3}$
$\max(90-102,0)=0$	$\frac{1}{3}$

Table 7: Computation of the exposure on a forward contract if default can occurr only at expiry.

The 1-period discount factor is[5]

$$\frac{100}{102} = 0.9804.$$

Let us also assume that the 1-period survival probability of the counterparty is 0.95 and the LGD is 0.6. The computation of the CVA proceeds as follows

- ***STEP 1: Pricing the risk-free forward****. The forward price is set equal to 102, so that the current value of the contract is fair*

$$\left(\frac{1}{3}(112-102)+\frac{1}{3}(104-102)+\frac{1}{3}(90-102)\right)\times 0.9804 = 0.$$

- ***STEP 2: Computing the exposure at time 1****. The exposure in one period can be computed using the information in Table 7. We compute the payoff of the forward contract and if negative we set it equal to 0.*

- ***STEP 3: Computing the expected exposure****. The expected exposure is*

$$\frac{1}{3}\times 10+\frac{1}{3}\times 2+\frac{1}{3}\times 0 = 4.$$

- ***STEP 4: Computing the CVA (under the independence assumption)****. As the LGD is 0.6, ie the recovery rate is 0.4, the CVA of the contract will be the present value of the expected loss given the counterparty default*

$$\begin{aligned} CVA &= \underbrace{(1-0.4)}_{LGD}\times\underbrace{(1-0.95)}_{def.\ prob.}\times\underbrace{0.9804}_{disc.\ fact.}\times\underbrace{4}_{expected\ exposure} \\ &= 0.6\times 0.05\times 0.9804\times 4 \\ &= 0.117648. \end{aligned}$$

- ***STEP 5: Pricing the risky forward****. The value of the risky forward contract is the risk-free value of the contract minus the CVA:*

$$0-0.117648 = -0.117648.$$

  So this contract has a negative fair value to the bank, which will be charged to the client.

[5] The discount factor must guarantee that the return obtained investing for 1 period in the money market account is equal to the expected return on the commodity. In fact, the forward price can be also determined according to the cash and carry formula, so that it is equal to 100/0.98.

In practice, it is common to assume that default can be observed at discrete dates (e.g. end of each quarter). The CVA computation now requires to evaluate the following expression

$$CVA(t) = \sum_{i=1}^{n}\left[\mathbb{E}_t\left(e^{-\int_t^{T_i} r(s)ds} \times \mathbf{1}_{T_{i-1}<\tau\leq T_i} \times (1-R) \times \max(V(T_i),0)\right)\right]$$

and under independence

$$\begin{aligned}
&= \sum_{i=1}^{n} \mathbb{E}_t\left(\mathbf{1}_{T_{i-1}<\tau\leq T_i}\right) \times (1-\mathbb{E}_t(R)) \times \text{option price}(t,T_i) \\
&= \sum_{i=1}^{n} \left[PD(t,T_{i-1},T_i) \times (1-\mathbb{E}_t(R)) \times \text{option price}(t,T_i)\right] \\
&= \sum_{i=1}^{n} CVA(T_{i-1},T_i),
\end{aligned}$$

where $CVA(T_{i-1},T_i)$ denotes the credit adjustment due to the counterpart's default in the subperiod (T_{i-1},T_i). We also remark that in the above formula the probability of defaulting between T_{i-1} and T_i, given survival up to time T_{i-1} has been computed as (see formula (4))

$$PD(t,T_{i-1},T_i) = PD(t,T_i) - PD(t,T_{i-1}).$$

Example 8 (Credit exposure on a forward on a commodity) *Let us consider a 1 year forward contract on a commodity, which does not pay implied dividends (ie the convenience yield is zero). Let us set the risk-free rate at 3%. The forward price is determined according to the usual no-arbitrage cash-and-carry formula, ie*

$$F(t,T) = S(t)e^{r(T-t)} = 100 \times e^{0.03\times 1} = 103.0454.$$

The value of the forward contract at any time $\tau, t < \tau < T$, is

$$V(\tau;T) = S(\tau) - F(t,T)e^{-r(T-\tau)}.$$

The positive part of this gives the current exposure at time τ in case of default

$$E(\tau) = \max(V(\tau;T),0).$$

Commodity price dynamics are assumed to be described according to a binomial tree with quarterly time steps with up factor u, down factor d, and risk-neutral probability π given by

$$u = 1.1; d = \frac{1}{u} = 0.9091; \pi = \frac{e^{0.03\times 0.25} - d}{u-d} = 0.5156.$$

The commodity price dynamics is shown in Panel 1 of Table 8. Panel 2 shows the corresponding dynamics of the value of the forward contract. In Panel 3, we have the exposure, ie the loss if default occurs when the contract has a positive value. Finally, Panel 4 provides the risk-neutral probability of each state of the world. Given this information, we can compute, at each possible default date the following quantities

Panel 1: Commodity Tree

Time	0	0.25	0.5	0.75	1
					146.41
				133.1	
			121		121
		110		110	
	100		100		100
		90.91		90.91	
			82.64		82.64
				75.13	
					68.30

Panel 2: Forward Contract Value Tree $V(\tau)$

Time	0	0.25	0.5	0.75	1
					43.36
				30.82	
			19.49		17.95
		9.25		7.72	
	0		-1.51		-3.05
		-9.84		-11.37	
			-18.87		-20.40
				-27.14	
					-34.74

Panel 3: Positive Exposure $\max(V(\tau), 0)$

Time	0	0.25	0.5	0.75	1
					43.36
				30.82	
			19.49		17.95
		9.25		7.72	
	0		0		0
		0		0	
			0		0
				0	
					0

Panel 4: Risk-Neutral Probabilities of Each Node

Time	0	0.25	0.5	0.75	1
					0.0707
				0.1371	
			0.2659		0.2656
		0.5156		0.3863	
	1		0.4995		0.3743
		0.4844		0.3629	
			0.2346		0.2344
				0.1136	
					0.0550

Time	0	0.25	0.5	0.75	1	
PFE	0	9.25	19.49	30.82	43.36	**Using Panel 3**
EPE	0	4.77	5.18	7.21	7.83	**Using Panel 3 and Panel 4**

Table 8: Trees to compute PFE (worst case) and EE on a forward contract on a commodity.

- *the worst potential future exposure, PFE, for example in nine months,*

$$PFE(0.75) = \max(30.82; 7.72; 0; 0) = 30.82.$$

- *the expected exposure, EE, for example in nine months,*

$$EE(0.75) = 0.1371 \times 30.82 + 0.3863 \times 7.72 + 0.3629 \times 0 + 0.1136 \times 0 = 7.21.$$

This quantity has been computed considering the exposure in nine months (Panel 3) and the corresponding risk-neutral probabilities (Panel 4).

The PFE and the EPE are given in the last two rows of Table 8. Then, Table 9 provides a detailed computation of the CVA period by period. For example the CVA relative to the first quarter is computed according to

$$CVA(0, 0.25) = 4.7681 \times 0.9925 \times 0.00025 \times 0.6 = 0.0007,$$

where in particular $4.7681 \times 0.9925 = 4.7324$ *is the price of the option expiring in 3 months.*

Similar computations are possible for the other subperiods. Summing up, we obtain the full contract CVA

$$CVA = 0.0007 + 0.0008 + 0.0010 + 0.0011 = 0.003673.$$

In conclusion, the contract has a negative value to the bank of 0.003673.

Positive Exposure of a Long Forward					
Time	0	0.25	0.5	0.75	1
					43.36
				30.82	
			19.49		17.95
		9.25		7.72	
	0		0		0
		0		0	
			0		0
				0	
					0
$EPE(T_i)$	0	4.7681	5.1814	7.2099	7.8341
$P(0, T_i)$		0.9925	0.9851	0.9778	0.9704
Option(t, T_i)		**4.7324**	**5.1043**	**7.0495**	**7.6026**
$PD(0, T_i)$	0	0.025%	0.050%	0.075%	0.10%
PD($0, T_{i-1}, T_i$)	**0**	**0.025%**	**0.025%**	**0.025%**	**0.025%**
LGD		**0.6000**	**0.6000**	**0.6000**	**0.6000**
$CVA(T_{i-1}, T_i)$		0.0007	0.0008	0.0011	0.0011

Table 9: Computation of the CVA for a long forward with intermediate default.

In practice, the above computations are done very often via Monte Carlo simulation. This is illustrated in Figure 5. In the top left panel, we have simulated the underlying commodity value according to the preferred model, that can be either a simple Geometric Brownian motion or a more sophisticated process such as a mean-reverting model with jumps and stochastic volatility. In the top right panel, for each simulated scenario and each time step we reprice the forward contract. This contract at inception has zero value, but along its life can take positive and negative values. Then in the bottom left panel, we have the simulated contract exposure, ie we set at zero the negative values of the contract. Finally, in the bottom right panel we compute the expected exposure, by taking, at each time step, the average, across all simulated scenarios, of the exposure. We also plot the quantiles at different confidence levels of the simulated exposures. These profiles represent one input for the CVA formula. Additional ingredients (assuming independence) are the default probability, the recovery ratio and the term structure of discount factors. Notice that the expected exposure for a forward contract typically increases over time: the longer the maturity, the larger the expected exposure. This is due to the presence of a unique cash-flow at the forward expiry.

4.2 Bilateral Adjustment

In the bilateral adjustment, we consider the possibility that the bank, ie the firm's counterpart, can default as well. In such a case, the bilateral CVA from the point of view of the corporate is defined as the present value of the expected loss in which firm incurs if the bank defaults, corporate survives at the moment in which the default of the bank occurs, and the contract written on the reference name has

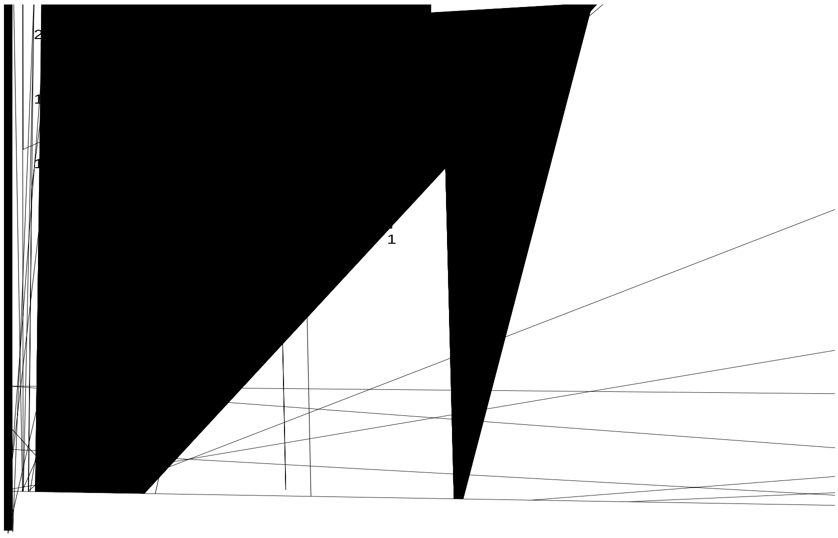

Figure 5: Credit Exposure on a long forward contract expiring in 1 year computed via Monte Carlo. **Top Left**: Commodity Price MC simulation (10 simulations). **Top Right**: Forward Value MC simulation (10 simulations). **Bottom Left**: Forward Exposure (10 simulations). **Bottom Right**: Expected Exposure and Potential Future Exposure at different confidence levels (10,000 simulations).

a positive value to the corporate. A symmetric valuation is done by the bank as well. For this reason, we now need two adjustments

- CVA computation, conditional to the bank's survival and counterparty's defaulting;
- DVA (debt value) computation, conditional to the bank's default and counterparty's survival.

The value of the risky contract will be now:

$$\text{Risk-free value} - \mathbf{CVA} + \mathbf{DVA}$$

A naive approach estimates the bilateral adjustment as difference of unilateral adjustments where only the other side can default. The correct approach must instead model the joint default process of the two parties, as we now describe.

Let τ_j be the default time of party j, and R_j its recovery rate, where $j = 1$ is the corporate and $j = 2$ is the bank. The CVA formula now becomes:

$$CVA(t) = \sum_{i=1}^{n} \mathbb{E}_t \left(e^{-\int_t^{T_i} r(s)ds} \times \mathbf{1}_{T_{i-1} < \tau_1 \leq T_i, \tau_2 > T_i} \times (1 - R_1) \times \max(V(T_i), 0) \right),$$

whilst the DVA formula is given by

$$DVA(t) = \sum_{i=1}^{n} \mathbb{E}_t \left(e^{-\int_t^{T_i} r(s)ds} \times \mathbf{1}_{T_{i-1} < \tau_2 \leq T_i, \tau_1 > T_i} \times (1 - R_2) \times \max(-V(T_i), 0) \right).$$

Notice that in quantifying the DVA, we are considering the expected loss to the counterparty due to the bank's default and therefore the exposure is now given by $\max(-V(T_i), 0)$. Assuming that the two parties default independently and that their default is independent on the contract exposure, we need to compute

$$\Pr(\tau_1 = T_i < \tau_2 \,|\, \tau_1 > T_{i-1}) = (PD_1(t, T_i) - PD_1(t, T_{i-1})) \times (1 - PD_2(t, T_i)).$$

or using survival probabilities

$$\Pr(\tau_1 = T_i < \tau_2 \,|\, \tau_1 > T_{i-1}) = (Q_1(t, T_{i-1}) - Q_1(t, T_i)) \times Q_2(t, T_i).$$

Consequently,

$$CVA(t) = (1 - R_1) \sum_{i=1}^{n} \mathbb{E}_t \left(e^{-\int_t^{T_i} r(s)ds} \times \max(V(T_i), 0) \right) \times (Q_1(t, T_{i-1}) - Q_1(t, T_i)) \times Q_2(t, T_i),$$

and similarly for the DVA

$$DVA(t) = (1 - R_2) \sum_{i=1}^{n} \mathbb{E}_t \left(e^{-\int_t^{T_i} r(s)ds} \times \max(-V(T_i), 0) \right) \times (Q_2(t, T_{i-1}) - Q_2(t, T_i)) \times Q_1(t, T_i).$$

Example 9 *Let us consider again the Example 8 on the commodity forward contract. We are interested in computing the bilateral CVA and DVA. Let us start computing the bilateral CVA. In Table 10 we have*

- *the expected positive exposure (first row). These numbers are the same as in the corresponding row of Table 9.*
- *the discount curve (second row).*
- *the present value of the expected positive exposure (third row), ie the price of a call option written on the forward contract.*
- *the term structure of default probability of counterparty 1 (fourth row).*
- *the term structure of marginale default probability of counterparty 1 (fifth row).*
- *the term structure of default probability of counterparty 2 (sixth row).*
- *the term structure of survival probability of counterparty 2 (seventh row).*
- *the loss given default of counterparty 1 (eighth row).*
- *the term structure of bilateral CVA (last row). For example the bilateral CVA of the last period is obtained according to*

$$7.6026 \times 0.025\% \times 0.9975 \times 0.6 = 0.001138.$$

(Notice that in the unilateral case, the CVA of the last period is simply $7.6026 \times 0.025\% \times 0.6 = 0.001140$*). The bilateral CVA is then obtained as*

$$0.0009 + 0.0021 + 0.0042 + 0.0055 = 0.003666,$$

Time	0.25	0.5	0.75	1
$EPE(T_i)$	4.7681	5.1814	7.2099	7.8341
$P(0, T_i)$	0.9925	0.9851	0.9778	0.9704
option(t, T_i)	4.7324	5.1043	7.0495	7.6026
$PD_1(0, T_i)$	0.025%	0.050%	0.075%	0.10%
$PD_1(0, T_{i-1}, T_i)$	0.025%	0.025%	0.025%	0.025%
$PD_2(0, T_i)$	0.1%	0.14%	0.20%	0.25%
$Q_2(0, T_i)$	0.999	0.9986	0.9980	0.9975
LGD_1	0.6	0.6	0.6	0.6
Bilateral CVA	0.0009	0.0021	0.0042	0.0055

Table 10: Computing bilateral CVA (default of counterparty 1, i.e. the firm, and survival of counterparty 2, i.e. the bank).

(slightly) smaller than the unitaleral CVA computed in Example 8 (and equal to 0.003673).

In order to compute the bilateral DVA, we now need the losses to counterparty 1 due to the default of counterparty 2. To do this, we can consider Panel 2 of Table 8 and then compute the exposure to counterparty 1, ie $\max(-V(\tau), 0)$. *These losses are given in Table 11. At each period we compute their expected value, the so called expected negative exposure (ENE). For example, the ENE relative to the last period is obtained by using the risk-neutral probabilities in last column in Panel 4 of Table 8 and the losses in last column of Table 11:*

$$ENE(1) = 0 \times 0.0707 + 0 \times 0.2656 + 3.05 \times 0.3743 + 20.40 \times 0.2344 + 34.74 \times 0.0550 = 7.8341.$$

By repeating the computation over all dates we obtain the row named ENE in Table 12. Then using the discount factors given in the second row of Table 12, we have the present value of the expected losses to counterparty 1 if counterparty 2 (the bank) defaults. These amounts are nothing else than prices of put options expiring in 3, 6, 9 and 12 months respectively. Then using the term structure of survival probabilities of counterparty 1 and the term structure of default probabilities of the bank, we can compute the DVA period by period and then the bilateral DVA

$$0.0028 + 0.0012 + 0.0025 + 0.0023 = 0.0089.$$

Finally, given the bilateral CVA and DVA we can compute the risky adjusted value of the forward contract (from the bank's point of view)

$$0 - 0.0037 + 0.0089 = -0.0052$$

Notice, if the bank shows a worsened credit quality, then its default probabilities will be higher and therefore the DVA will be lower. This will generate a profit to the bank on the risky forward, which needs to be recognized in the

0	0.25	0.5	0.75	1
				0
			0	
		0		0
	0		0	
0.00		1.51		3.05
	9.84		11.37	
		18.87		20.40
			27.14	
				34.74

Table 11: Losses to counterparty 1 due to the default of counterparty 2.

Time	0.25	0.5	0.75	1
$ENE(T_i)$	4.7681	5.1814	7.2099	7.8341
$P(0, T_i)$	0.9925	0.9851	0.9778	0.9704
$Option(t, T)$	4.7324	5.1043	7.0495	7.6026
$PD_1(0, T_i)$	0.025%	0.05%	0.075%	0.10%
$Q_1(0, T_i)$	0.9998	0.9995	0.9993	0.9990
$PD_2(0, T_i)$	0.10%	0.14%	0.20%	0.25%
$PD_2(0, T_{i-1}, T_i)$	0.10%	0.04%	0.06%	0.05%
LGD_2	0.6	0.6	0.6	0.6
Bilateral DVA	0.0028	0.0012	0.0025	0.0023

Table 12: Computing bilateral DVA (default of counterparty 2, i.e. the bank, and survival of counterparty 1, i.e. the firm).

balance sheet according to the International Accounting Standards. The possibility of making profits on own debt out of a lower credit quality has happened in reality in the recent years, as reported in Keoun and Henry [16][6].

4.3 A Case Study: Estimating the Bilateral CVA of an Interest Rate Swap via Monte Carlo simulation

We are interested in computing the CVA of an Interest Rate Swap (IRS), with maturity T and equally spaced payment dates T_j, $j = 1, \cdots, n$. Let f be the fixed rate whilst the LIBOR rate with tenor $\Delta = T_j - T_{j-1}$, for all $j = 1, \cdots, n$, is the reference rate. For illustrative purposes, the Vasicek one-factor model is used to model the term structure evolution. More sophisticated interest rate models can also be used, at the cost of an increase of the computational time. According to the mean-reverting Vasicek model, the short rate dynamics is

$$dr(t) = \lambda(\mu - r(t))dt + \sigma dW(t).$$

Here, $W(t)$ is a Brownian motion under the risk-neutral measure, σ is the (annualized) volatility of interest rate changes, μ is the long-run interest rate level and λ is usually referred to as speed of mean reversion. The above stochastic differential equation has the solution

$$r(s) = e^{-\lambda(s-t)} \times (r(t) - \mu) + \mu + \sigma\eta(t,s).$$

where

$$\eta(t,s) = \int_t^s e^{-\lambda(s-u)} dW(u) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda(s-t)}\right)\right).$$

Given the short rate at time $s > t$, we can generate the future discount curve according to

$$P(s, s+\tau) = e^{A(\tau) - B(\tau) \times r(s)}, \tag{10}$$

where $B(\tau) = \frac{1-e^{-\lambda\tau}}{\lambda}$ and $A(\tau) = (B(\tau) - \tau)\left(\mu - \frac{\sigma^2}{2\lambda^2}\right) - \frac{\sigma^2 B^2(\tau)}{4\lambda}$.

Computation of the bilateral CVA and DVA via Monte Carlo simulation is performed as follows

1. Draw a standard Gaussian random variable ϵ_i, $i = 1, \cdots, n$ and recursively simulate $r(T_i)$ on the swap payment dates according to

   $$r^{(k)}(T_i) = \mu + e^{-\lambda\Delta}\left(r^{(k)}(T_{i-1}) - \mu\right) + \sqrt{\frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda(\Delta)}\right)}\epsilon_i^{(k)},$$

   where the index k refers to the simulation $k = 1, \cdots, M$.

[6]Results may include gains taken under a U.S. accounting rule known as Statement 159, adopted by the Financial Accounting Standards Board in 2007, which allows banks to book profits when the value of their bonds falls from par. The rule expanded the daily marking of banks? trading assets to their liabilities, under the theory that a profit would be realized if the debt were bought back at a discount.

2. Define the money market account (MMA)

$$MMA(T) = MMA(t) \times e^{\int_t^T r(s)ds},$$

with $MMA(0) = 1$ and simulate it recursively approximating the integral using the trapezoidal rule

$$\begin{aligned} MMA^{(k)}(T_i) &= MMA^{(k)}(T_{i-1})e^{\int_{T_{i-1}}^{T_i} r^{(k)}(s)ds} \\ &\cong MMA^{(k)}(T_{i-1})e^{\frac{\Delta}{2}\left(r^{(k)}(T_i)+r^{(k)}(T_{i-1})\right)}. \end{aligned}$$

3. At each future date s (which for simplicity we take to be a payment date, ie $s = T_1, ..., T_n$) simulate the discount curve using the pricing formula (10)

$$P^{(k)}(s, T_i) = e^{A(T_i - s) - B(T_i - s)r^{(k)}(s)},$$

and compute the fair value FV of the swap according to

$$FV^{(k)}(s) = (1 - P^{(k)}(s, T_n)) - \frac{f(t)}{12} \sum_{T_i > s} P^{(k)}(s, T_i).$$

4. The simulated exposure for the long side is

$$E_A^{(k)}(s) = \max\left(FV^{(k)}(s), 0\right),$$

which is equivalent to the payoff of a swaption. In some models, this swaption can be priced by closed form, but in general we have to resort to Monte Carlo simulation.

5. Similarly, the exposure for the short side B

$$E_B^{(k)}(s) = \max\left(-FV^{(k)}(s), 0\right).$$

6. Given the recovery rates of A and B, we compute the present value of the losses occurring at time T_i for the two sides of the swap according to the formula

$$CVA_A^{(k)}(T_i) = (1 - R_B) \times Q_A(t, T_i) \times (Q_B(t, T_{i-1}) - Q_B(t, T_i))) \times \frac{E_A^{(k)}(T_i)}{MMA(T_i)},$$

and similarly for B:

$$CVA_B^{(k)}(T_i) = (1 - R_A) \times Q_B(t, T_i) \times (Q_A(t, T_{i-1}) - Q_A(t, T_i))) \times \frac{E_B^{(k)}(T_i)}{MMA(T_i)},$$

where we assume independence between default events and interest rate process.

7. At the last payment date, we can compute the simulated CVA

$$CVA_A^{(k)} = \sum_{i=1}^{n} CVA_A^{(k)}(T_i),$$

and DVA

$$DVA_A^{(k)} = CVA_B^{(k)} = \sum_{i=1}^{n} CVA_B^{(k)}(T_i).$$

8. Repeat steps 1-7 M times and obtain M simulated values of the CVA of the i-th period, $CVA^{(k)}(T_i)$, and then the M values $CVA_A^{(k)}$.

9. Compute the average across M simulations to obtain the estimated full contract CVA

$$CVA_A = \frac{1}{M}\sum_{k=1}^{M} CVA^{(k)},$$

10. We proceed in a similar way for side B and compute

$$DVA_A = \frac{1}{M}\sum_{k=1}^{M} DVA_A^{(k)}.$$

11. The Fair Value to A of the risky swap is
Risk-Free Value - $\mathbf{CVA_A}$ + $\mathbf{CVA_B}$.

Let us consider a concrete numerical example.

Example 10 *Let us consider a swap with the following characteristics. Swap Tenor: one year; Payment Frequency: monthly; Notional: 1ml Euros. The initial market discount curve is given in Table (13). According to the given term structure, the present value of*

τ_i	0.08	0.17	0.25	0.33	0.42	0.50
$P^{mkt}(t, t+\tau_i)$	99.9881%	99.9672%	99.9379%	99.9003%	99.8550%	99.8022%
τ_i	0.58	0.67	0.75	0.83	0.92	1.00
$P^{mkt}(t, t+\tau_i)$	99.7422%	99.6753%	99.6019%	99.5222%	99.4366%	99.3452%

Table 13: Euro Discount Curve on November 17, 2015.

the floating leg is

$$(1 - P(t, t+1)) \times N = (1 - 99.3452\%) \times N = 0.6548 \times N.$$

whilst the annuity (present value of the unit monthly payment) is

$$\frac{1}{12}\sum_{i=1}^{12} P\left(t, t+\frac{i}{12}\right) \times N = \frac{1}{12} \times 11.9677 \times N.$$

The fixed swap rate $f(t)$ at inception is chosen so that the present value of the two legs is the same, so that:

$$f(t) = \frac{0.6548 \times N}{\frac{1}{12} \times 11.9677 \times N} = 0.65661\%.$$

The Vasicek model can be calibrated to the observed market discount curve in Table 13 by solving the following minimization problem

$$\min_{r(t),\mu,\sigma,\lambda} MSE = \sum_{i=1}^{N} \left(P^{mkt}(t, t+\tau_i) - P^{vas}(t, t+\tau_i) \right)^2,$$

which provides the following calibrated parameters

$$r(t) = 0.088\%, \mu = 2.768\%, \sigma = 1.028\%, \lambda = 0.49886.$$

The market and the calibrated values as well as the functions A and B are given in Table 14. We can now start the simulation of the term structure of interest rates. Set $t = 0$,

τ_i	$P^{mkt}(t, t+\tau)$	$P^{vas}(t, t+\tau)$	$B(\tau)$	$A(\tau)$
0.00274	1	1	0.0027	0.0000
0.0247	1.0000	1.0000	0.0245	0.0000
0.0438	1.0000	0.9999	0.0434	0.0000
0.0630	0.9999	0.9999	0.0620	0.0000
0.0932	0.9999	0.9999	0.0910	-0.0001
0.1726	0.9997	0.9997	0.1654	-0.0002
0.2575	0.9995	0.9993	0.2417	-0.0004
0.3315	0.9991	0.9990	0.3055	-0.0007
0.4192	0.9985	0.9985	0.3783	-0.0011
0.5014	0.9979	0.9980	0.4436	-0.0016
0.5836	0.9973	0.9974	0.5063	-0.0021
0.6685	0.9967	0.9967	0.5685	-0.0028
0.7534	0.9960	0.9960	0.6280	-0.0035
0.8411	0.9952	0.9951	0.6869	-0.0043
0.9205	0.9943	0.9943	0.7381	-0.0050
1.0055	0.9934	0.9934	0.7907	-0.0059
MSE		4.8×10^{-8}		

Table 14: Calibration of the Vasicek model to market price of discount factors.

$r(0) = 0.088\%$ and the time step $\Delta = 1/12$ (swap payment dates tenor is assumed indeed to be monthly). Therefore, $\sqrt{\frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda\times\Delta}\right)} = 0.00291$ and assuming that we have simulated a standard Gaussian pseudo-random number $\epsilon_1 = 1.426$, we obtain

$$r^{(1)}\left(\frac{1}{12}\right) = 0.02768 + e^{-0.49886\times\frac{1}{12}} \times (0.088\% - 0.02768) + 0.00291 \times (1.426) = 0.612\%.$$

In addition

$$MMA^{(1)}\left(\frac{1}{12}\right) = MMA(0) \times e^{\frac{1}{24}\times(0.088\%+0.612\%)} = 1.0003.$$

Then if we draw a new Gaussian random variable and, for the sake of illustration, we get -1.891, the simulated value of the short rate on the second month is

$$r^{(1)}\left(\frac{2}{12}\right) = 0.02768 + e^{-0.49886\times\frac{1}{12}} \times (0.612\% - 0.02768) + 0.00291 \times (-1.891) = 0.15\%$$

and then

$$MMA^{(1)}\left(\frac{2}{12}\right) = 1.0003 \times e^{\frac{1}{24}(0.612\%+0.15\%)} = 1.0006.$$

The simulated short rate provides a simulated value in one month of the floating leg equal to 9459 Euros and a value of the fixed leg equal to 5991 Euros. Therefore the swap has a positive value of 3468 Euros. The simulated exposure for A due to the default of B is 3468 Euros. Viceversa, the exposure for B due to the default of A is 0. In the second month, the simulated value of the floating leg is 5196 Euros, the value of the fixed leg is 5459 Euros and the swap value is -263 Euros. The exposure for A due to the default of B will be now equal to 0 Euros. Viceversa, the exposure for B due to the default of A is 263 Euros.

Let us suppose that the constant monthly survival probability for A is 0.9989 and the one for B is 0.9983. Assume equal recovery rate and set it at the default value of 0.4. The simulated present value of the loss for A due to the default of B is

$$CVA_A^{(1)}\left(\frac{1}{12}\right) = (1-0.4) \times 0.9989 \times (1-0.9983) \times \frac{3468}{1.003} = 5.8248\ EUR.$$

For B we have $CVA_B\left(\frac{1}{12}\right) = 0$ *and*

$$CVA_B^{(1)}\left(\frac{2}{12}\right) = (1-0.4) \times (0.9983)^2 \times 0.9989 \times (1-0.9989) \times \frac{263}{1.006} = 0.1712\ EUR.$$

Therefore given a simulated path of the term structure, we compute at each step the present value of the loss to one counterparty given default of the other and the present value of the total loss. Table 15 illustrates a simulated path of the swap value and, in the last row, the corresponding simulated losses for the two sides. Notice that at inception and at expiry the swap value is zero, so we do not consider these dates. Table 16 illustrates simulated values of the CVA for different months. Then in Table 17, we average across simulations, and we obtain the present value of the expected loss to the two parties for each month and then the total CVA and DVA (last row). The difference between simulated CVA and simulated DVA gives us the value of the risky swap: it has a negative value to A of 22.14 Euros. The profile of the present value of the expected losses to the two parties and the net effect is shown in Figure 6.

As discussed in Pykhtin and Zhu [23] there are two main effects that determine the credit exposure over time for a single transaction: diffusion and amortization. As time passes, the diffusion effect tends to increase the exposure: at longer horizons there is *a larger uncertainty and therefore a larger expected exposure; the amortization effect, in contrast, tends to decrease the exposure over time, because it reduces the remaining cash flows that are exposed to default. In the case of a forward contract with a unique cash flow at maturity, there is no amortization effect and therefore the expected exposure increases over time as observed in the previous example. Viceversa, for the swap case considered here, as we approach the swap maturity there are less and less cash flows, so that the amortization prevails on the diffusion effect, which generates the profile of the CVA shown in Figure 6. In general, depending on the characteristics of the bilateral contract, the exposure profile can largely vary from contract to contract.*

Month	**Floating Leg (EUR)**	**Fixed Leg (EUR)**	**Swap (EUR)**	**MMA**	**PV($Loss_A$)**	**PV($Loss_B$)**
0	6548	6548	0	1.0000	0.0000	
1	9459	5991	3468	1.0003	5.8248	0.0000
2	5196	5459	-263	1.0006	0.0000	0.1712
3	6756	4908	1848	1.0009	3.0855	0.0000
4	8638	4357	4281	1.0016	7.1222	0.0000
5	9347	3810	5537	1.0026	9.1771	0.0000
6	8008	3268	4740	1.0038	7.8246	0.0000
7	6589	2725	3864	1.0050	6.3531	0.0000
8	4325	2183	2142	1.0061	3.5093	0.0000
9	3403	1638	1765	1.0071	2.8798	0.0000
10	1511	1093	418	1.0080	0.6789	0.0000
11	855	547	308	1.0088	0.4987	0.0000
					CVA=46.95	**DVA=0.17**

Table 15: Simulated CVA and DVA exposures and present values of the losses to the two sides.

Sim.	cva	cva(0)	cva(1)	cva(2)	...	cva(10)	cva(11)	cva(12)
1	15.750	0.000	1.203	0.520	4.489	0.000	0.000	0.000
2	11.724	0.000	3.654	2.161	0.603	0.000	0.000	0.000
3	14.724	0.000	0.000	2.398	3.065	0.000	0.012	0.000
4	1.209	0.000	1.209	0.000	0.000	0.000	0.000	0.000
5	34.817	0.000	0.000	0.909	0.000	3.504	1.596	0.000
6	33.520	0.000	0.000	1.348	0.000	3.961	1.831	0.000
...	...	...	...	...	...	...	...	...
995	67.414	0.000	5.027	3.522	4.345	3.083	2.429	0.000
996	37.850	0.000	0.000	0.058	0.000	3.459	1.004	0.000
997	0.824	0.000	0.000	0.000	0.000	0.000	0.024	0.000
998	16.603	0.000	0.000	0.000	1.673	0.000	0.040	0.000
999	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1000	2.647	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 16: Simulated CVA for the full contract (second column) and for each month ($M = 1000$).

5 Wrong Way Risk (WWR)

In the previous examples, we have assumed independence between exposure and default event. We say that the risk is

Period	CVA A	CVA B	Net (A-B)
0	0.0000	0.0000	0.0000
1	1.7137	0.4289	1.2848
2	2.3815	0.4964	1.8852
3	2.8944	0.5080	2.3865
4	3.0943	0.4606	2.6337
5	3.1815	0.4560	2.7255
6	3.1206	0.4253	2.6953
7	2.8327	0.3487	2.4840
8	2.4941	0.2723	2.2217
9	2.0159	0.2078	1.8082
10	1.4427	0.1365	1.3062
11	0.7787	0.0714	0.7073
12	0.0000	0.0000	0.0000
CVA	25.9501	3.8118	22.1383

Table 17: Period CVA (and DVA) and total CVA (DVA) of the swap.

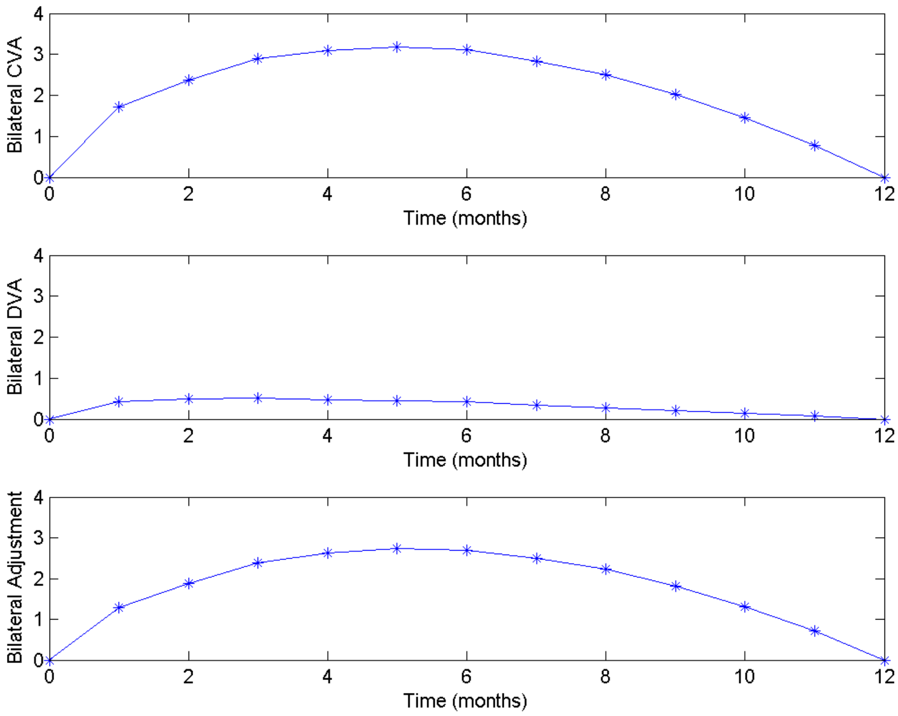

Figure 6: Top: Bilateral CVA profile of the swap. Middle: Bilateral DVA profile of the swap. Bottom: Bilateral adjustment of the 1 year swap considered in the Case Study.

- **wrong way** if exposure tends to increase when counterparty credit quality worsens;
- **right way** if exposure tends to decrease when counterparty credit quality worsens.

Example 11 *A company writing put options on a highly correlated firm creates wrong-way exposure for the buyer. An oil producer selling oil in a swap creates right-way exposures for the buyer.*

To better appreciate the relevance of the dependence between underlying (the price of a commodity) and default event in the energy sector we can refer to a Financial Times article by Rodrigues and Crooks [24]. They report that the rating agency Standard & Poor's, since October 2014 up to February 2015, and following the plunge in oil prices, has downgraded 19 high-yield oil and gas companies, with eight of these sent deeper into junk territory in January 2015. There were more downgrades in oil and gas than in any other sector during the same period. The 50% drop in oil prices since June 2014 has strongly affected smaller companies, which led the US shale revolution relied on steady inflows of capital to finance their drilling programmes. Indeed, their cash flows did not cover their capital expenditure. Also exploration and production companies that borrowed about 163bn $ in high-yield debt during the period 2000-14 have been affected by the oil price plunge.

In order to model wrong-way risk different approaches are possible. In the following we consider

- the structural approach;
- the reduced form model.

and we illustrate concrete examples of using the above models to capture wrong way risk in a commodity oil swap. Whatever the default model, we assume that the oil price evolves according to a geometric Brownian motion[7],

$$S\left(T_{i+1}\right) = S\left(T_i\right) \times \frac{F\left(0, T_{i+1}\right)}{F\left(0, T_i\right)} \times e^{-\frac{\sigma^2}{2}\left(T_{i+1}-T_i\right)+\sigma\left(W_S\left(T_{i+1}\right)-W_S\left(T_i\right)\right)},$$

where $F(t, T)$ refers to the quoted futures price for expiry at T. The current date is time t and $S(t) = F(t, t)$. In addition, W_S is a Brownian motion under the risk-neutral measure, ie

$$W_S\left(T_{i+1}\right) - W_S\left(T_i\right) \sim \mathcal{N}\left(0, T_{i+1} - T_i\right).$$

Notice that if we need to simulate at dates at which no futures quote is available, some interpolation of quoted futures prices is needed. In the numerical example, we have adopted linear interpolation of log-futures prices, ie let $T_{i-1} <$

[7] For a more general framework, see Marena et al. [19].

$T < T_i$ and let us assume that we have the quotations $F(t, T_{i-1})$ and $F(t, T_i)$. If we need to interpolate at date T, we use the following formula

$$\log\left(F(t,T)\right) = w\log\left(F(t,T_{i-1})\right) + (1-w)\log\left(F(t,T_i)\right),$$

where

$$w = \frac{T_i - T}{T_i - T_{i-1}}.$$

5.1 CVA of a commodity swap

To make the presentation concrete, we consider a commodity swap with trade date on June 17, 2014, tenor one year, monthly payments (at the end of each month, starting in July 2014), swap fixed rate F, notional of 1000 barrels, party A as floating price payer and party B as fixed price payer. Table 18 shows payment dates (in days), corresponding futures prices and discount factors at the inception date.

T_i (in days)	$F(0, T_i)$	$P(0, T_i)$
0	61.02	1
37	61.52	0.99980
68	61.81	0.99955
98	62.09	0.99923
128	62.43	0.99880
159	62.76	0.99826
190	63.00	0.99763
219	63.05	0.99696
250	63.07	0.99615
281	63.15	0.99525
310	63.36	0.99432
342	63.57	0.99320
372	63.42	0.99211

Table 18: Term structure of interpolated futures prices and discount factors on the swap payment dates.

Given the time grid $T_i = \frac{i}{12}$, $i = 0, \ldots, 12$, the fair value of the swap at time 0 for party A which is receiving floating and paying fixed, assuming deterministic interest rates, is given by

$$\begin{aligned} V_A(t) &= \sum_{i=1}^{12} \left(\mathbb{E}_t\left(S\left(T_i\right)\right) - F\right) \times P\left(t, T_i\right) \\ &= \sum_{i=1}^{12} \left(F\left(t, T_i\right) - F\right) \times P\left(t, T_i\right). \end{aligned}$$

Here, we exploit the fact that futures prices are risk neutral expectations of future spot prices, ie $\mathbb{E}_t\left(S\left(T_i\right)\right) = F(t, T_i)$. The fair fixed price F which makes the present value of the payments on the fixed leg equal to the present value of the expected payments on the floating leg at inception is

$$F = \frac{\sum_{i=1}^{12} F\left(t, T_i\right) \times P\left(t, T_i\right)}{\sum_{i=1}^{12} P\left(t, T_i\right)}.$$

The value of the indexed floating payoff $S\left(T_i\right)$ at time $s, 0 \leq s \leq T_i \leq T_{12}$, can be computed as follows

$$\mathbb{E}_s\left(S\left(T_i\right)\right) = \mathbb{E}_s\left(S\left(s\right) \times \frac{F\left(t, T_i\right)}{F\left(t, s\right)} \times e^{-\frac{\sigma^2}{2}(T_i - s) + \sigma(W(T_i) - W(s))}\right) = S\left(s\right) \times \frac{F\left(t, T_i\right)}{F\left(t, s\right)}.$$

Therefore, the fair value at time s for party A is

$$V_A(s) = \sum_{i:T_i > s}^{12} \left[S\left(s\right) \times \frac{F\left(t, T_i\right)}{F\left(t, s\right)} - F\right] \times \frac{P\left(t, T_i\right)}{P\left(t, s\right)}. \tag{11}$$

The CVA of party A is

$$CVA_A(t) = \sum_{i=1}^{12} \mathbb{E}\left[(1 - R_B) \times \max(V_A(T_i), 0) \times \mathbf{1}_{T_{i-1} < \tau_B \leq T_i} \times \mathbf{1}_{\tau_A > T_i} \times e^{-\int_t^{T_i} r(u) du}\right].$$

If we assume that the recovery rate is deterministic, interest rates and default events are independent, and default of party A is independent of the exposure, then we obtain

$$CVA_A(t) = \sum_{i=1}^{12} (1 - R_B) \times \mathbb{E}\left[\max(V_A(T_i), 0) \times \mathbf{1}_{T_{i-1} < \tau_B \leq T_i}\right] \times Q_A\left(t, T_i\right) \times P\left(t, T_i\right). \tag{12}$$

We notice that representation (12) allows to take into account wrong and right-way risk, by jointly simulating the time to default of party B and the spot commodity price, which determines $V_A(T_i)$, according to formula (11). To this purpose, we introduce two different approaches in the following sections. Both models will be calibrated to survival probabilities bootstrapped from CDS quotes of the risky party, reported in Table 19, jointly with the corresponding survival probabilities[8].

5.2 Structural Model

According to the structural approach, default occurs as soon as a state variable, usually refereed in the financial literature as the firm value, falls below a given barrier. In the basic Merton [20] model, this can or cannot happen only at a fixed future date. In the first-passage time approach due to Black and Cox [6], default

[8] Bootstrapped probabilities in Table 19 have been computed using the procedure described in O'Kane and Turnbull [22] and presented in a simplified way in Section 3.2

Expiry	CDS (bps)	Def. Prob. $(1-Q)$
6M	13.327	0.11%
1Y	16.133	0.27%
2Y	22.635	0.76%
3Y	30.673	1.55%
4Y	41.054	2.78%
5Y	52.100	4.44%

Table 19: CDS and Default probabilities of an oil company at swap's inception.

can happen at any instant before a given time, usually referred to as the debt maturity. Therefore, if we let A be the firm value and we assume that, under the risk-neutral measure, it evolves according to a Geometric Brownian Motion process

$$dA\left(t\right) = rA\left(t\right)dt + \sigma A\left(t\right)dW_A\left(t\right),$$

default will occur at the first instant the firm value falls below a given barrier B, ie

$$\tau = \inf\{s \le T : A(s) \le B\}.$$

The corresponding model survival probability is given by

$$Q(t,T) = \Pr\left(\tau > T | \tau > t\right) = N\left(\frac{x(t) + \nu(T-t)}{\sigma\sqrt{T-t}}\right) - e^{-2\alpha x(t)}N\left(\frac{-x(t) + \nu(T-t)}{\sigma\sqrt{T-t}}\right)$$

where $N(.)$ is the cumulative Gaussian distribution and

$$x(t) = \ln\left(\frac{A(t)}{B}\right), \nu = r - \sigma^2/2, \text{ and } \alpha = \frac{\nu}{\sigma^2}.$$

In order to capture wrong-way risk in this framework, we correlate the Brownian motions affecting the dynamics of the counterparty firm value and the underlying of the contract:

$$corr(dW_A, dW_S) = \rho_{A,S}dt.$$

Wrong-way risk is captured by large negative values of $\rho_{A,S}$, which make more likely the simultaneous event that the firm value moves below the barrier and oil price increases. In this case, the long position in the swap takes positive value as soon as default occurs. Calibration of the correlation parameter is in general quite difficult. A possibility is to use as proxy of the firm value the quoted equity price and then estimating $\rho_{A,S}$ using the sample correlation between the log-returns of the equity value and the underlying. This value can be used as reference value to assess the importance of wrong/right-way risk.

By way of illustration, let us assume that we are interested in computing the CVA of an oil swap with quarterly payments and one-year tenor. The Black-Cox model parameters have been calibrated to the term structure of default probabilities given in Table 19, by using a non-linear least squares procedure. This returns the calibrated parameters

$$\hat{x} = 0.9355, \hat{\sigma} = 0.0188.$$

Then we proceed as follows:

1. Given the initial value of the log-ratio

$$x(0) = 0.9355,$$

   we jointly simulate the process for x and for the oil log-price with a time step Δ, so that default occurs as soon as $x(t)$ becomes negative;

2. at each time step we verify if the quantity $x(t)$ falls below 0. If this is the case, it means that the counterparty is defaulting and we compute the exposure to the bank, ie the present value of the remaining cash flows of the swap and then we stop the simulation; if there is no default, we move to the next time step;

3. we average across simulations the losses occurring to the bank at each time step;

4. the CVA to the bank is given by the sum over the different time steps of the simulated average exposures.

In order to assess the relevance of wrong-way risk on the CVA of the contract, we jointly simulate the firm value and the oil price for different values of the correlation coefficient $\rho_{A,S}$. Given our parametrization and 10,000 Monte Carlo simulations, the CVA in Figure 7 illustrates the percentage change in CVA varying the correlation coefficient with respect to the case of independence between market and credit variables. In this example by considering extreme cases of wrong-way risk, we can have up to a 500% change in CVA with respect to the case of independence.

The fundamental shortcoming of diffusion structural models is the underprediction of both credit spreads and default probabilities for low risk debt and for short horizons. This problem is easily dealt with considering structural models with jumps. An integrated approach in the context of CVA computation is presented in Ballotta and Fusai [2] and in Ballotta et al. [3] and references therein.

5.3 Reduced form Model

In the reduced form approach, the default time is set equal to the first jump time of a non-homogeneous Poisson process with intensity $\lambda(t)$, known as the hazard rate or default intensity. Assuming the hazard rate process is deterministic, the default probability $PD(t,T)$ is given by

$$PD(t,T) = 1 - e^{-\int_t^T \lambda(u)du}.$$

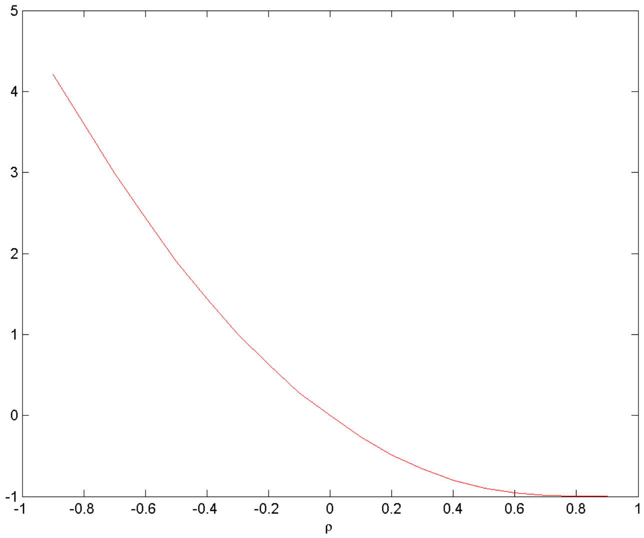

Figure 7: CVA percentage change with respect to the independence case as function of the correlation coefficient capturing the wrong way risk in structural models

In particular, the probability of a default occurring within the time interval $[t, t + dt)$, conditional on surviving to time t, is approximately

$$\Pr(\tau < t + dt | \tau \geq t) \approx \lambda(t) dt.$$

The function $\lambda(t)$ can be interpreted as the conditional instantaneous default probability. The survival probability is

$$Q(t,T) = 1 - PD(t,T) = e^{-\int_t^T \lambda(u) du}.$$

In this section, in order to account for the dependence between exposure and default event, we assume a stochastic default intensity following a square-root Cox-Ingersoll-Ross process

$$d\lambda(t) = \kappa_\lambda (\mu_\lambda - \lambda(t))\, dt + \sigma_\lambda \sqrt{\lambda(t)} dW_\lambda(t), \quad \lambda(0) = \lambda_0,$$

with

$$\kappa_\lambda, \mu_\lambda, \sigma_\lambda, \lambda_0 > 0.$$

The intensity turns out to be positive if the so-called Feller condition is satisfied

$$\kappa_\lambda \mu_\lambda \geq \sigma_\lambda.$$

The survival probability is now computed by the following expectation

$$\Pr(\tau > T | \tau > t) = \mathbb{E}_t \left(e^{-\int_t^T \lambda(u) du} \right) \tag{13}$$

and it can be shown that

$$\Pr(\tau > T | \tau > t) = e^{\alpha(t,T) + \beta(t,T)\lambda(t)}, \tag{14}$$

where by setting $\gamma = \frac{1}{2}\sqrt{\kappa_\lambda^2 + 2\sigma_\lambda}$, we have

$$\alpha(t,T) = \frac{\gamma \exp^{0.5\kappa_\lambda(T-t)}}{\gamma\cosh(\gamma(T-t)) + 0.5\kappa_\lambda \sinh(\gamma(T-t))},$$

and

$$\beta(t,T) = \frac{-\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + 0.5\kappa_\lambda \sinh(\gamma(T-t))}.$$

Table 20 shows the calibrated CIR parameters, obtained by matching model default probabilities from (14) to market default probabilities in Table 19.

Parameters	Estimates
κ_λ	0.0497
μ_λ	0.0656
σ_λ	0.0218
λ_0	0.0010

Table 20: CIR calibrated parameters, obtained by mimimizing the sum of the absolute differences between model and market default probabilities.

We correlate the default intensity to the commodity price by correlating the increments of the corresponding Brownian motions. We have

$$corr(dW_\lambda, dW_S) = \rho_{\lambda,S}dt.$$

To compute the expectations in the CVA formula, we jointly simulate the commodity price process and the default intensity, using a Euler discretization scheme with monthly time step [9] for the default intensity

$$\lambda(T_i) = \lambda(T_{i-1}) + \kappa_\lambda \times \left(\mu_\lambda - \lambda^+(T_{i-1})\right)) \times (T_i - T_{i-1}) + \sigma_\lambda \times \sqrt{\lambda^+(T_{i-1})(T_i - T_{i-1})} \times Z_\lambda(T_i),$$

$$S(T_i) = S(T_{i-1}) \times \frac{F(0,T_i)}{F(0,T_{i-1})} \times e^{-\frac{\sigma^2}{2}(T_i - T_{i-1}) + \sigma\sqrt{T_i - T_{i-1}}Z_S(T_i)},$$

where $Z_\lambda(T_i)$ and $Z_S(T_i)$ are two correlated Gaussian random variables with correlation coefficient $\rho_{\lambda,S}$, and $i = 1, \ldots, 12$. We set

$$\lambda^+(T_i) = \max(\lambda(T_i), 0),$$

to guarantee the positivity of the intensity. Let $\Gamma(t)$ denotes the so-called hazard function

$$\Gamma(T) = \int_t^T \lambda(s)ds.$$

We can approximate the integral using the trapezoidal rule

$$\Gamma(T_i) = \Gamma(T_{i-1}) + 0.5 \times (\lambda(T_i) + \lambda(T_{i-1})) \times (T_i - T_{i-1}).$$

[9] A finer time step, or a more efficient simulation scheme, should be used to achieve better accuracy (see Glasserman [13]).

The simulation of the default time can be performed by drawing a uniform random variable $U \in [0,1]$, so that default occurs according to

$$\tau = \inf\{t > 0 : \exp\left(-\Gamma(t)\right) < U\}.$$

We investigate the impact of the correlation parameter between the default intensity and the commodity price on the CVA. For different correlation coefficients, we re-price the swap and compute CVA of party A, using 1,000,000 Monte Carlo simulations. We implement here the unilateral CVA, assuming that the bank survives with probability 1, ie $Q_A\left(0, T_i\right) = 1$, for all $i = 1, \ldots, 12$. In the independence case, corresponding to $\rho_{\lambda,S} = 0$, the CVA is 24.287[10]. As $\rho_{\lambda,S}$ goes from negative to positive values, right way risk becomes wrong way risk and, as consequence, the CVA increases. The change in the CVA can be significant: if $\rho_{\lambda,S} = -0.9$, the CVA is equal to 19.404, whilst if $\rho_{\lambda,S} = 0.9$ the CVA can reach the value of 31.413. If $\rho_{\lambda,S}$ has a high negative value, it means that there is a strong negative correlation between the exposure on default and the energy company's default: the higher the exposure for the bank, the less likely the default of the energy company, and vice versa. Therefore the CVA will be low. If $\rho_{\lambda,S}$ has a high positive value, it means that there is a strong positive correlation between the exposure on default and the energy company's default: the higher the exposure for the bank, the more likely the default of the energy company, and vice versa. Therefore the CVA will be high.

5.4 Other approaches to wrong way risk

Other approaches to incorporate wrong way risk into the CVA calculation have been proposed in the literature. We give here a short review of the most popular ones.

- Cespedes et al. [9] propose an ordered scenario copula model. A simplified copula[11] approach introduces direct correlation between exposure and default events. Default events and exposures are driven by factor models, while a Gaussian copula is used to correlate exposure and credit events. The approach builds on existing exposure scenarios by a non-parametric sampling of exposure via the factor model. This framework is applied to CVA computations under wrong way risk by Rosen and Saunders [25] to compute the joint probabilities of exposure's scenarios and the counterparty's default for different spacing.

- Cherubini [10] uses general copulas to account for the dependence between the underlying asset and the couterparty's default time, while Bocker and

[10]In fact, our MC simulation of the joint distribution of default event and oil prices provides a CVA equal to 25.249, due to Monte Carlo and approximation errors.

[11]Copulas are popular in multivariate statistics as they allow to easily model and estimate the distribution of random vectors by estimating marginals and copulae separately. A copula function is used to model the dependence among random variables once the marginals have been fixed. In other words, copulas allow the modelling of the marginals and dependence structure of a multivariate probability model separately.

Brunnbauer (2014) define a general copula model linking the couterparty's default time with the discounted portfolio value.

- Hull and White [14] model the hazard rate as a deterministic monotonic function of the value of the contract. Wrong-way (right-way) risk is obtained by making the hazard rate to be an increasing (decreasing) function of the contract's value.

6 Mitigating Counterparty Exposure

The exposure can be greatly reduced by means of netting agreements and collateral clauses. We briefly illustrate these contractual features in the following subsection.

6.1 Netting

In presence of multiple trades with a counterparty, mitigation of the investor's exposure can be achieved through netting agreements, ie the two parties , in the event of default of one of the counterparty, agree on the aggregation of transaction before settling claims. Therefore, under netting agreements, the value of all trades are added together so that the resulting portfolio value is settled as a single trade.

In general, if there is more than one trade with a defaulted counterparty and counterparty risk is not mitigated in any way, the maximum loss for the bank is equal to the sum of the contract-level credit exposures:

$$E(\tau) = \sum_{k=1}^{n} q_k E_k(\tau) = \sum_{k=1}^{n} q_k \max(V_k(\tau), 0).$$

where n is the number of contracts, q_k are the asset quantities, and E_k the corresponding exposures.

A *netting agreement* is a legally binding contract between two counterparties based on which, in the event of default, the exposure results

$$E^{netting}(\tau) = \max\left(\sum_{k=1}^{n} q_k V_k(\tau), 0\right).$$

Example 12 *If a counterparty holds a currency option written by its bank with a market value of 50, while the bank has an interest rate swap with the same counterparty having a marked to market value in favour of the bank of 80, then the exposure of the bank to the counterparty is 80. The exposure of the counterparty to the bank is 50. The exposure of the bank to the counterparty, with netting, is 30.*

Table 21 shows an example of the time paths of the values of five trades as well as the future exposures to the counterparty, with and without netting. Figure 8 shows the simulated CVA of a portfolio of long forward contracts, assuming that the underlyings are homogeneous and characterized by the same cross-correlation levels. Larger the cross correlation among assets, the smaller the benefit of the netting clause. The percentage reduction in the CVA due to the netting

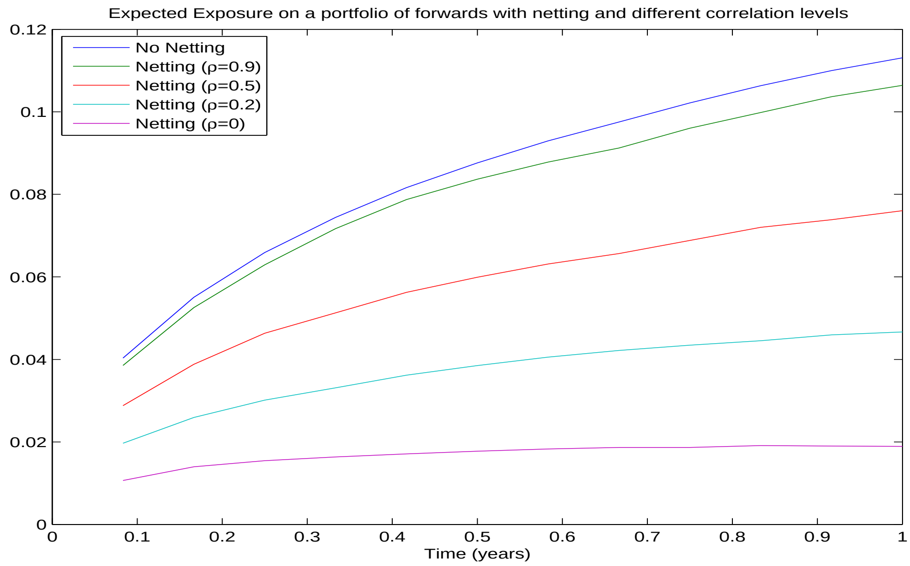

Figure 8: Credit Exposure on a long forward contract expiring in 1 year via Monte Carlo simulation with netting. An homogeneous portfolio is considered: 10 assets having same implied volatility (0.38) and varying their cross-correlation ρ as indicated in the legend.

clause is shown in Table 22. For very low values of the cross-correlation amongst trades, the benefit of netting agreement can achieve a 80% reduction of the expected exposure.

	Time (Months)				
Trade nr.	1	2	3	4	5
1	10	-7	8	-6	-2
2	9	0	4	-2	2
3	7	7	5	10	-8
4	-7	-6	3	-6	-6
5	-5	-5	3	6	-6
Exposure (US$)					
No Netting	26	7	23	16	2
With Netting	14	0	23	2	0

Table 21: Example of one path of counterparty's exposures in presence of netting.

6.2 Collateral

Collateralization is one of the most important techniques of mitigation of counterparty risk, and indeed, more and more OTC transactions use collateral and margin agreements to reduce counterparty credit risk. A collateral account is a contractual clause aimed at reducing potential losses that may be incurred by investors in case of the default of the counterparty, while the contract is still alive.

	CVA	CVA Reduction
No netting	0.01518	
$\rho = 0.9$	0.01449	4.59%
$\rho = 0.5$	0.01055	30.48%
$\rho = 0.2$	0.00649	57.23%
$\rho = 0$	0.00304	80.00%

Table 22: CVA reduction due to netting by varying the cross-correlation (10 homogeneous assets with implied volatility equal to 38%, marginal default probability = 0.025%, risk free rate $r = 0.03$ and dividend yield $q = 0.01$).

In the following, we illustrate a common procedure used by many banks to model the inclusion of collateral in the calculation of the exposure. Let us recall that the future exposure is defined as

$$E(t) = V^{+}(t),$$

where V denotes the value of the contract under discussion. Let us consider the point of view of the bank, and let $C(t)$ denote the (cash) collateral amount posted at time t. As this is aimed at reducing the value of the exposure, the bank has no exposure to the contract up to the collateral amount itself, whilst its losses are reduced exactly by the collateral amount in the case in which the exposure exceeds it. Consequently, the resulting collateralized exposure $E_C(t)$ can be defined as

$$E_C(t) = (E(t) - C(t))^{+}.$$

From the above formula, it follows that the collateralized exposure can be decomposed as

$$E_C(t) = E(t) - \left(C(t) - (C(t) - E(t))^{+}\right),$$

ie the posting of collateral allows a mitigation of the exposure in favour of the part receiving it. This mitigation is positive and equal to the amount

$$C(t) - (C(t) - E(t))^{+}.$$

The actual amount of the collateral available at time t depends on the contractual agreement between the parties, specifically the posting threshold, the minimum transfer amount, the call frequency and the inclusion of downgrade triggers.

In more details, the threshold H triggers the posting of the collateral: if the exposure is positive but less than the threshold no collateral is posted, otherwise the full collateral amount is called. Under this assumption, the collateral is defined as

$$C(t) = (E(t - \delta t) - H)^{+}. \tag{15}$$

In the above expression, δt refers to the so called remargin period, ie the interval at which margin is monitored and called for. Further, the standard assumption is that if the counteparty defaults at time t, the last time the counterparty is in the position of answering the collateral posting call is at $t - \delta t$. Most collateral agreements require daily calculations and collecting/returning of collateral. However, some collateral agreements can specify weekly or monthly calculations which can result in increased credit risk with reduced operational requirements.

The underlying commercial reason for a threshold is that often parties will be willing to take a certain amount of credit risk (equal to the threshold) before requiring collateral to cover any additional risk. Sometimes, the level of the threshold varies according to the credit rating of the counterparty.

The Minimum Transfer Amount (MTA), denoted in the following as M, represents the amount below which no margin transfer is made: the collateral is set to zero if it is less than MTA. The presence of the MTA avoids the operational costs of small transactions and it contributes to reduce the frequency of collateral exchanges. Hence, in presence of MTA the collateral function is re-defined as

$$C(t) = (E(t - \delta t) - H)^+ \, 1_{(E(t-\delta t)-H>M)}. \tag{16}$$

Sometimes, the threshold and the MTA vary during the lifetime of the contract if the parties agree on the inclusion of downgrade triggers, sometimes known as rating-based collateral calls. These clauses typically force a firm to post more collateral to their counterparty, if they are downgraded below a certain level. For example, AIG failed because downgrade triggers in its credit default swap (CDS) contracts forced it to post $15 billion in collateral when Moody's and S&P downgraded its credit rating immediately after Lehman failed. AIG could not collect the required funds on such a short notice. Similarly, in June 2012 Moody's downgraded three major derivatives dealers (Citigroup, Morgan Stanley and Royal Bank of Scotland) below the crucial single A threshold, which has led to collateral calls from counterparties.

Example 13 *Let us consider Table 23. The second row of the Table provides the exposure of a bilateral contract at different dates. The remaining rows illustrate the collateral amount under different assumption on the threshold and the MTA. These are computed according to formula (15) and (16).*

Time (months)	0	2	4	6	8	10	12
$E(t)$	0	3	12	19	25	26	0
$C(t)$ $(H = 0, M = 0)$	0	0	3	12	19	25	26
$E_c(t)$	0	3	9	7	6	1	0
$C(t)$ $(H = 1, M = 0)$	0	0	2	11	18	24	25
$E_c(t)$	0	3	10	8	7	2	0
$C(t)$ $(H = 1, M = 2)$	0	0	0	11	18	24	25
$E_c(t)$	0	3	12	8	7	2	0

Table 23: Computing the collateralized exposure.

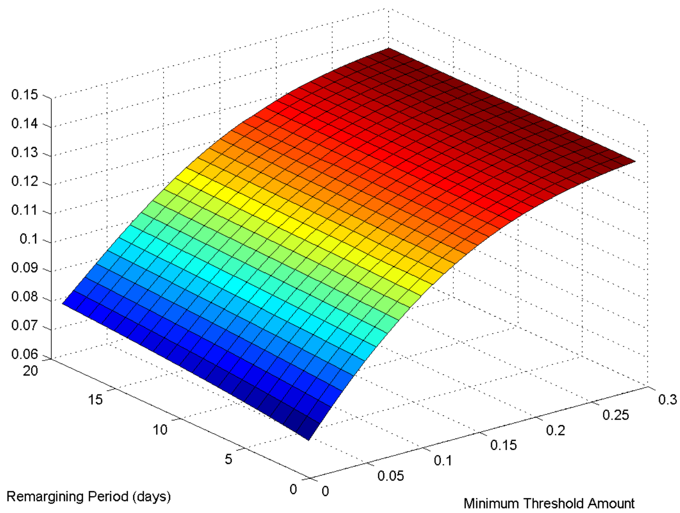

Figure 9: Collateralized expected positive exposure as function of both threshold and remargining period.

As shown in the last example, the risk mitigation due to presence of the collateral is largely affected by the threshold H, and the length δt of the margining period. The higher the threshold, the less effective the protection. The longer the margining period, the higher the risk of an upward movement in the value of the contract, and ultimately in the investor's CVA. Figure 9 illustrates how the presence of collateral guarantees a significant reduction in the expected exposure, only if the margining period or the threshold amount are not too large. In the figure, the expected uncollateralized exposure is 0.128. With perfect collateralization, ie threshold level set to zero and daily remargining, the collateralized exposure is reduced to 0.062, allowing a 50% reduction in the exposure. The collateralized exposure quickly increases as both variables increase. The most important feature affecting the expected positive exposure is the threshold amount.

The effect of threshold and margin period has been examined by Gibson [12] in a simplified setup, and by Ballotta et al. [3] in a more general structural model with jumps. The main insight is that collateralization is not able to fully eliminate counterparty risk, especially in presence of market shocks. As discussed in Ballotta et al. [3], in fact, the presence of jumps introduces the so-called gap risk. Sudden movements can increase both the exposure since the time of the last collateral exchange, and the probability of the relevant default event. This originates gap risk, which is the risk that the corporate defaults, the bank survives and the contract moves in the money, given that at the last margining period the counterparty was solvent and the exposure out-of-the money. The model presented in Ballotta et al. [3] is able to quantify this specific risk.

We note that the treatment of the collateral discussed above is for the case in which a unilateral agreement is in place, ie only one party is subject to collateral posting. Alternatively, both parties could be required to post collateral, originating a bilateral agreement. Further details on the bilateral agreement can be found in Ballotta et al. [3] and Pykthin and Zhu [23].

An important issue related to the presence of collateral is its use by banks not only as a way of reducing credit risk, but as a funding mitigant, typically through re-hypothecation, ie in the case in which the assets are reused, sold or lent out to a third party. According to a survey on margin published by the International Swaps and Derivatives Association in March 2010, 82% of large dealers reported rehypothecating collateral received in connection with OTC derivatives transactions, see Whittall [26].

If collateral is segregated and not available for rehypothecation, banks have to assume that they need to raise funding to meet the cashflows over the life of the trade using their own internal funding curves, see Cameron [8]. This suggests that in the future the pricing of derivatives transactions will also take into account whether a dealer has full rehypothecation rights or the collateral is segregated within the dealer, with a third party or managed under a full tri-party arrangement.

A final issue about the posting of collateral is that even if the CVA charge shrinks dramatically when exposure is collateralised, very few corporates post collateral because they do not have enough liquid assets for the purpose. In addition, for a corporate the operational complexity associated with collateralization (negotiating a legal document, monitoring exposures, making cash transfers, etc.) may significantly increase the cost and resource requirements. As a result, hedging with derivative can become so expensive that corporates will choose to accept higher levels of exposure instead (Risk Magazine, October 2011).

7 CVA Value at Risk

We conclude this paper mentioning briefly the possible approaches for measuring the Value at Risk (VaR) of the CVA exposure. We recall that VaR refers to the maximum loss at a given confidence level over a given horizon.The term VaR of CVA refers to the potential losses that a bank is facing for a deterioration of the credit quality of its counterparty, at a given confidence level and within a given horizon. Under Basel III, the requlatory framework for banks, the advanced approach for determining capital for CVA risk requires to examine how changes in spreads, and not in market variables, such as changes in the value of the reference asset, commodity, currency or interest rate of a derivative, affect CVA exposures. This VaR model is restricted to changes in the counterparties? credit spreads and does not model the sensitivity of CVA to changes in other market factors.

In addition, Basel III refers to unilateral CVA and does not allow to adjust the risk-free value of the contract by considering the DVA. We refer to the Basel III documentation [4] for fuller details.

In general, aside from a regulatory requirements but with reference to internal risk management puroposes, VaR estimation requires to compute the distribution of CVA at the prefixed horizon. This can be achieved by adopting one of the proposed models (structural, reduced form, copula models), simulating the relevant market and credit variables up to the VaR horizon and then recomputing the risky value of the contract and the CVA value. This has to be done under the real world measure. By repeating the simulation M times we can simulate M possible CVA changes out of which to compute the VaR.

8 Conclusions

This paper has reviewed the most important tools in credit risk analysis and discussed how to apply them to the quantification of counterparty credit risk.

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A Default Probability from Transition matrix

Major ratings agencies report the historical average incidence of transitions among credit ratings and into default in the form of a matrix of average transition frequencies.

An example from Fitch is given in Table 24:

- The row heading corresponds to the rating at the beginning of a year;
- The column heading gives the end-of-year rating.
- For example, on average over this sample, 86.56% of firms rated BBB at the beginning of a year retained this rating, while approximately 3.13% made a transition to BB.
- The fractions of transitions to WD correspond to withdrawn ratings. This effect is often ignored by normalizing each transition frequency by the total fraction of bonds that do not have a withdrawn rating.

%	AAA	AA	A	BBB	BB	B	CCC to C	D	WD
AAA	**87.18**	5.3	0.22	0	0	0	0	0.11	7.18
AA	0.12	**85.44**	8.61	0.35	0.02	0.02	0	0.03	5.42
A	0.01	1.84	**87.55**	5.02	0.41	0.06	0.03	0.07	5
BBB	0	0.13	3.09	**86.56**	3.13	0.47	0.13	0.17	6.31
BB	0.02	0.03	0.08	7.32	**75.52**	5.63	1.22	0.94	9.24
B	0	0	0.18	0.35	7.53	**75.5**	4.56	1.93	9.95
CCC to C	0	0	0	0.29	1.74	17.01	**46.08**	23.69	11.19

Table 24: Corporate Finance Average Annual Transition Matrix: 1990-2014. Source: Needham and Carter [21].

In industry practice the annual average transition frequency matrix of the sort shown in Table 24 is called matrix Π of transition probabilities, with π_{ij} denoting the probability that a firm rated i at the beginning of the year is rated j at the end of the year. It is implicitly assumed that transition probabilities are constant over time and that the issuers current rating is the unique determinant of its default. This strong assumption would allow one to treat the rating of a firm as a Markov chain.

Example 14 *Let us consider the transition matrix in Table 26. In addition to the default state, we have only two possible rating classes: A and B. There is a 3% probability of defaulting in 1 period given that at the beginning of the period we had rating B. Therefore, there is a 3% probability of defaulting in 1 year, given that we started in class B.*

We can compute the probability of being in the different classes in 1 period, starting from B

$$\begin{bmatrix} 0.9 & 0.02 & 0 \\ 0.09 & 0.95 & 0 \\ 0.01 & 0.03 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.02 \\ 0.95 \\ 0.03 \end{bmatrix}$$

	Rating at the end of the period		
	A	B	Default
Rating at start of the period — A	Prob of staying in A	Prob of migrating from A to B	Prob of default from A
Rating at start of the period — B	Prob of migrating from B to A	Prob of staying in B	Prob of default from B

Table 25: Structure of a transition matrix.

	A	B	D
A	0.9	0.09	0.01
B	0.02	0.95	0.03
D	0	0	1

Table 26: An example of transition matrix.

Similarly, the probability of being in the different classes in 2 periods, starting from B at time 0:

$$\begin{bmatrix} 0.9 & 0.02 & 0 \\ 0.09 & 0.95 & 0 \\ 0.01 & 0.03 & 1 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.95 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.037 \\ 0.9043 \\ 0.0587 \end{bmatrix}$$

Therefore, there is a 5.87% probability of defaulting within two years, given that we started in class B.

The probability of being in the different classes in 3 periods, starting from B at time 0 is:

$$\begin{bmatrix} 0.9 & 0.02 & 0 \\ 0.09 & 0.95 & 0 \\ 0.01 & 0.03 & 1 \end{bmatrix} \begin{bmatrix} 0.037 \\ 0.9043 \\ 0.0587 \end{bmatrix} = \begin{bmatrix} 0.0514 \\ 0.8624 \\ 0.0862 \end{bmatrix}$$

We have therefore the term structure of default probabilities in 1-2-3-etc. years starting from B. This is also illustrated in Figure 10.

Years	0	1	2	3	4	5	6	7
Def. Prob.	0.00%	3.00%	5.87%	8.62%	11.26%	13.79%	16.23%	18.58%

Table 27: Cumulative default probabilities starting from rating B.

A.1 Matrix Factorization and Default Probability

This section illustrates how to construct the term structure of default probabilities at any horizon (multiple of 1 period), given the 1 period transition matrix.

Let T be the transpose of a transition matrix. Then it can be factorized as

$$\mathrm{T} = \mathbf{ADA}^{-1}$$

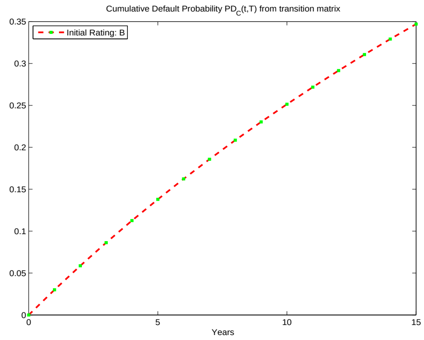

Figure 10: Term structure of cumulative default probabilities starting from rating B.

where $\mathbf{D}$ is the diagonal matrix of the eigenvalues of T and $\mathbf{A}$ is the corresponding matrix of eigenvectors. In particular, T has always (at least) an unit eigenvalue and all the remaining eigenvalues are, in absolute value, less than 1. If we compute T^n, we have

$$\mathrm{T}^n = (\mathbf{A}\mathbf{D}\mathbf{A}^{-1})^n = \mathbf{A}\mathbf{D}^n\mathbf{A}^{-1},$$

and the vector giving the probabilities of being in the different ratings after n periods is

$$\mathbf{x}(n) = \mathbf{A}\mathbf{D}^n\mathbf{A}^{-1}\mathbf{x}(0).$$

If we let the number of periods diverge, ie $n \to \infty$, then we get the so called stationary distribution

$$\mathbf{x}(\infty) = \mathrm{T}_{\infty}\mathbf{x}(0) = \mathbf{A}\mathbf{D}_{\infty}\mathbf{A}^{-1}\mathbf{x}(0)$$

where $\mathbf{D}_{\infty}$ has the (1,1) entry equal to 1 and all the remaining components are zero.

It is easy to verify that the so-called stationary distribution, ie the long-run probability of being in one rating class whatever the starting condition, is the eigenvector associated to the unit eigenvalue (the eigenvectors is chosen such that the sum of his components is 1):

$$\mathbf{x}(\infty) = \mathbf{A}_{(1)}.$$

Example 15 *Let us consider the matrix*

$$\mathrm{T} = \begin{bmatrix} 0.9 & 0.02 & 0 \\ 0.09 & 0.95 & 0 \\ 0.01 & 0.03 & 1 \end{bmatrix}$$

Its eigenvalues are 1, 0.97424 and 0.87576. The matrix of the eigenvectors, and its inverse, is

$$\mathbf{A} = \begin{bmatrix} 0 & 0.164431 & 0.630629 \\ 0 & 0.610404 & -0.76446 \\ 1 & -0.77484 & 0.133829 \\ & & \end{bmatrix}, \textit{and } \mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1.49706 & 1.23498 & 0 \\ 1.195372 & -0.32201 & 0 \\ & & \end{bmatrix}.$$

We can verify that

$$\mathrm{T} = \mathbf{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.97424 & 0 \\ 0 & 0 & 0.87576 \\ & & \end{bmatrix} \mathbf{A}^{-1}.$$

Therefore

$$\mathrm{T}^n = \mathbf{A} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 0.97424^n & 0 \\ 0 & 0 & 0.87576^n \\ & & \end{bmatrix} \mathbf{A}^{-1}.$$

Therefore, if at the initial time we are in class B,

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and we are interested in the default probability after 10 periods, we have

$$\mathbf{x}(10) = \mathbf{A} \begin{bmatrix} 1^{10} & 0 & 0 \\ 0 & 0.97424^{10} & 0 \\ 0 & 0 & 0.87576^{10} \\ & & \end{bmatrix} \mathbf{A}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10.73\% \\ 47.89\% \\ 41.37\% \end{bmatrix}$$

and we obtain that the 10-years default probability is 41.37%. In this example, given that there is no resurrection from default, the default will occurr with certainty. For example, the default probability after 50 periods will be 84.19%, after 100 periods 95.71% and 99.01% after 156 periods.