Structure Assignment Problems in Linear Systems: Algebraic and Geometric Methods

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Abstract

The Determinantal Assignment Problem (DAP) is a family of synthesis methods that has emerged as the abstract formulation of pole, zero assignment of linear systems. This unifies the study of frequency assignment problems of multivariable systems under constant, dynamic centralized, or decentralized control structure. The DAP approach is relying on exterior algebra and introduces new system invariants of rational vector spaces, the Grassmann vectors and Plücker matrices. The approach can handle both generic and non-generic cases, provides solvability conditions, enables the structuring of decentralisation schemes using structural indicators and leads to a novel computational framework based on the technique of Global Linearisation. DAP introduces a new approach for the computation of exact solutions, as well as approximate solutions, when exact solutions do not exist using new results for the solution of exterior equations. The paper provides a review of the tools, concepts and results of the DAP framework and a research agenda based on open problems.

Keywords: Linear multivariable control, Structural Control Methods, Output feedback, Pole placement, Frequency assignment, Algebraic-Geometry methods

1. Introduction

Systems and Control provide a paradigm that introduces many open problems of mathematical nature (Rosenbrock, 1970), (Kailath, 1980), (Wonham, 1979). We distinguish two main approaches in Control Theory, the design methodologies (based on performance criteria and structural characteristics) are mostly of iterative nature and the synthesis methodologies (based on the use of structural characteristics, invariants) linked to well defined mathematical problems. Of course, there exist variants of the two aiming to combine the best features of the two approaches. The Determinantal Assignment Problem (DAP) is a synthesis method and has emerged as the unified abstract problem formulation of pole, zero assignment of linear systems (Karcanias and Giannakopoulos, 1984, 1989), (Karcanias et al., 1988). DAP unifies the study of (pole, zero) frequency assignment problems of multivariable systems under constant, dynamic centralized, or decentralized control structures. There are two approaches developed for the study of frequency assignment problems which are: (i) the affine space approach; (ii) the projective geometry approach. The first approach was introduced in (Hermann and Martin, 1977), (Martin and Hermann, 1978), (Brockett and Byrnes, 1981), (Byrnes, 1989), and deals with the formulation of the problem in an affine space as an intersection problem of the Grassmannian with a linear space. The DAP approach, as it has been developed in Karcanias and Giannakopoulos (1984), Karcanias et al. (1988), Leventides and Karcanias (1995) is based on the Plücker embedding (Hodge and Pedoe, 1952) of the Grassmannian of the affine space into an appropriate projective space and then deals with finding solutions as the real intersections of a linear space with the Grassmann variety (Hodge and Pedoe, 1952) of the corresponding projective space. The DAP approach relies on exterior algebra (Marcus, 1973) and on the explicit description of the Grassmann variety, in terms of the

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Quadratic Plücker Relations (QPR) (Hodge and Pedoe, 1952). There are many approaches dealing with specific frequency assignment problems (pole-zero), but they rely on specific system representations and they cannot be easily extended to deal with the whole family of constant, dynamic, decentralised problems. The affine geometry approach deals with generic cases only and it does not provide computations for exact, as well as, approximate problems. The DAP approach has the advantage of introducing new system invariants of rational vector spaces in terms Grassmann vectors and Plücker matrices (Karcanias and Giannakopoulos, 1984) providing a matrix characterisation of decomposability in terms of the Grassmann matrices (Karcanias and Giannakopoulos, 1988), (Karcanias and Leventides, 2015) and developing a novel computational framework based on the technique of Global Linearisation (GL) (Leventides and Karcanias, 1995). GL is based on the notion of degenerate feedback (Brockett and Byrnes, 1981) and apart from establishing solvability conditions (Leventides and Karcanias, 1995), also provides a linearisation of the inherently nonlinear equations and leads to the computation of solutions (when such solutions exist). Within the DAP framework a number of solvability conditions have been established (Leventides and Karcanias, 1995, 1992, 1993; Karcanias and Leventides, 1996) for the exact and generic frequency assignment problem. The GL methodology has in general high sensitivity leading to high gains in the compensation. Techniques such as, homotopy continuation and Newton-type schemes (Leventides et al., 2014a,b) have been used in order to be able to achieve solutions with much better sensitivity properties.

The DAP framework has been used for the study of constant and dynamic pole assignment, where low complexity solutions have been established (Leventides and Karcanias, 1998b), as well as for problems of zero assignment by squaring down (Karcanias and Giannakopoulos, 1989), (Leventides and Karcanias, 2009). Degenerate feedback gains (Karcanias et al., 2016b) are defined for both constant and dynamic assignment problems. Parametrising the family of degenerate feedbacks gives extra degrees of freedom in computing appropriate controllers that linearise asymptotically DAP and enabling the selection of solutions with reduced sensitivity This parametrisation methodology plays a key role in selecting feasible structures for decentralized control problems. The selection of a decentralisation scheme has been handled mostly using conditions derived from the nature and spatial arrangement of subprocess units (Siljak, 1991). DAP can provide an algebraic framework for selection of the desirable decentralisation (Karcanias et al., 2016c) aiming at developing schemes that allow the satisfaction of generic solvability conditions and shaping the parametric invariants linked to solvability of decentralised control problems. DAP framework provides simple tests for avoiding fixed modes by exploiting the relationship of algebraic invariants (Plücker matrices) to decentralised Markov parameters (Leventides and Karcanias, 1998a). The overall philosophy is to devise methods for design that facilitate the solvability of decentralised control problems. Amongst the problems considered are: (i) Define the desirable cardinality of input, output structures to permit satisfaction of generic solvability conditions, (ii) Design the structure of input, output maps (matrices B, C) to eliminate the fixed modes and guarantee full rank properties to the decentralised Plücker matrices (Leventides and Karcanias, 2006).

A significant advantage of the DAP framework is that it introduces a new approach for the computation of exact and approximate solutions of DAP. This is based on an alternative, linear algebra type, criterion for decomposability of multivectors to that defined by the QPRs, in terms of the rank properties of structured matrices, referred to as Grassmann matrices (Karcanias and Giannakopoulos, 1988), (Karcanias and Leventides, 2015). The development of the new computational framework requires the study of the properties of Grassmann matrices, which are further developed by using the Hodge duality (Hodge and Pedoe, 1952) leading to the definition of the Hodge-Grassmann matrix (Karcanias and Leventides, 2015). Computing solutions (exact, or approximate) to DAP requires the investigation of distance problems, such as: (i) distance of a point from the Grassmann variety; (ii) distance of a linear variety from the Grassmann variety; (iii) parametrisation of families of linear varieties with a given distance from the Grassmann variety; (iv) relating the latter distance problems with properties of the stability domain. The distance problems extend the exact intersection problem between the Grassmann and the linear space varieties and are related to classical problems, such as spectral analysis of tensors (Lathauwer et al., 2000), homotopy and constrained optimization methods (Absil et al., 2008), theory of algebraic invariants etc.

This paper provides a review of the concepts, methodology and results of the DAP framework, as well as relevant results that complement those of the current approach. The review is then completed by providing a number of challenges for the DAP approach which form a research agenda for future activities.
The paper is structured as follows: Section 2, deals with the frequency assignment problems in Control Theory,

whereas Section 3 presents the abstract DAP framework which is reduced to the study of polynomial combinants and the problem of decomposability of multivectors. The solvability is equivalent to finding real intersections between a linear variety and the Grassmann variety of a projective space. DAP introduces new invariants of rational vector spaces defined by the Grassmann vectors and the Plücker matrices. Section 4, deals with the decomposability of multivectors where first the Quadratic Plücker Relations and then a new test for decomposability provided by the rank properties of the Grassmann matrix is presented. Section 5, examines the solvability of DAP under the genericity assumption. We focus on determining real solutions and a number of results are reviewed derived from the DAP framework and other related approaches. Section 6, deals with the GL methodology by examining the properties of the pole placement map, the notion of degenerate compensators with their parametrisation and the sensitivity of such solutions. Section 7, extends the GL methodology to decentralized control problems. The parametrisation of the family of decentralized degenerate schemes is linked to the selection of appropriate asymptotic linearising decentralized schemes capable of assigning the closed loop poles. Section 8, deals with the computation of exact and approximate solutions of DAP, in terms of the properties of structured matrices, the Grassmann and Hodge-Grassmann matrices. Finally, Section 8 provides a list of open questions related to the DAP framework which form a future research agenda.

Notation: Throughout the paper the following notation is adopted: If $\mathbb{F}$ is a field, or ring then $\mathbb{F}^{m\times n}$ denotes the set of $m \times n$ matrices over $\mathbb{F}$. $\mathbb{R}[s]$ is the ring of polynomials and $\mathbb{R}(s)$ is the field of rational functions over $\mathbb{R}$ respectively. If H is a map, then $\mathcal{R}(H)$, $\mathcal{N}_r(H)$, $\mathcal{N}_l(H)$ denote the range, right, left null-spaces respectively. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from the set $\bar{n} = \{1, 2, \ldots, n\}$. If $\mathcal{V}$ is a vector space and $\{\underline{v}_{i_1}, \ldots, \underline{v}_{i_k}\}$ are vectors of $\mathcal{V}$ then $\underline{v}_{i_1} \wedge \ldots \wedge \underline{v}_{i_k} = \underline{v}_{\omega} \wedge$, $\omega = (i_1, \ldots, i_k)$ denotes their exterior product and $\wedge^r\mathcal{V}$ the $r-$th exterior power of $\mathcal{V}$ (Marcus, 1973). If $H \in \mathbb{F}^{m\times n}$ and $r \leqslant min\{m, n\}$ then $C_r(H)$ denotes the $r-$th compound matrix of H (Marcus and Minc, 1964). In most of the following, we assume that $\mathbb{F} = \mathbb{R}$, or $\mathbb{C}$.

2. The Determinantal Assignment Problems in Control Theory

2.1. Introduction

The DAP methodology (Karcanias and Giannakopoulos, 1984) has been formulated as a unifying approach for all pole, zero frequency assignment problems with constant and dynamic compensators. This framework may be also applied to the case of decentralised control problems.

2.2. Control Problems leading to the DAP formulation

Consider the linear system, $S(A, B, C)$, described by:

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + B\underline{u},\ A \in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times p} \\ \underline{y} &= C\underline{x},\ C \in \mathbb{R}^{m\times n} \end{aligned} \tag{1}$$

where (A, B) is controllable, (A, C) is observable, or equivalently the transfer function matrix $G(s) = C(sI - A)^{-1}B$ has $rank_{\mathbb{R}(s)}\{G(s)\} = \min(m, p)$. In terms of Left, Right Coprime Matrix Fraction Descriptions (LCMFD, RCMFD) (Kailath, 1980), $G(s)$ may be represented as

$$G(s) = D_l(s)^{-1} \cdot N_l(s) = N_r(s) \cdot D_r(s)^{-1} \tag{2}$$

where, $N_l(s), N_r(s) \in \mathbb{R}[s]^{m\times p}$ and $D_l(s) \in \mathbb{R}[s]^{m\times m}$, $D_r(s) \in \mathbb{R}[s]^{p\times p}$. The following frequency assignment problems are defined:

(i) Pole assignment by state feedback: Consider $L \in \mathbb{R}^{n\times p}$, where L is a state feedback applied on system (1). The state feedback design involves finding $L \in \mathbb{R}^{n\times p}$ assigning the closed loop characteristic polynomial:

$$p_L(s) = \det\{sI - A - BL\} = \det\{B(s) \cdot \tilde{L}\} \tag{3}$$

where, $B(s) = [sI - A, -B]$ and $\tilde{L} = [I_n, L^t]^t$.

(ii) Design of an $n-$state observer: The design problem of an $n-$state observer for system (1) involves finding an output injection, $T \in \mathbb{R}^{n\times m}$, such that the characteristic polynomial of the observer is:

$$p_T(s) = \det\{sI - A - TC\} = \det\{\tilde{T} \cdot C(s)\} \tag{4}$$

where, $\tilde{T} = [I_n, T]$ and $C(s) = [sI - A^t, -C^t]^t$.

(iii) Pole assignment by constant output feedback: For the system described by (2), the constant output feedback design problem requires finding a matrix $K \in \mathbb{R}^{m\times p}$ that assigns the closed loop pole polynomial:

$$p_K(s) = \det\{D_l(s) + N_l(s) \cdot K\} = \det\{D_r(s) + K \cdot N_r(s)\}$$

or, equivalently

$$p_K(s) = \det\{T_l(s) \cdot \tilde{K}_l\} = \det\{\tilde{K}_r \cdot T_r(s)\} \tag{5}$$

by defining the composite matrices $T_l(s) \in \mathbb{R}[s]^{m\times(m+p)}$, $T_r(s) \in \mathbb{R}[s]^{(m+p)\times p}$, $\tilde{K}_l \in \mathbb{R}^{(m+p)\times m}$, $\tilde{K}_r \in \mathbb{R}^{p\times(m+p)}$

$$T_l(s) = [D_l(s), N_l(s)] \qquad \tilde{K}_l = \left[I_m, K^t\right]$$

$$T_r(s) = \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix} \qquad \tilde{K}_r = \left[I_p, K^t\right]$$

(iv) Zero assignment by squaring down: For a system with $m > p$ we can expect to have independent control over at most $p-$linear combinations of m outputs. If $\underline{c} \in \mathbb{R}^p$ is the vector of the variables which are to be controlled, then, $\underline{c} = Hy$, where $H \in \mathbb{R}^{p\times m}$ is a squaring down post-compensator, and $G^{'}(s) = H \cdot G(s)$ is the squared down transfer function matrix (Karcanias and Giannakopoulos, 1989). A right MFD for $G^{'}(s)$ is defined by $G^{'}(s) = H \cdot N_r(s)D_r(s)^{-1}$ where $G(s) = N_r(s)D_r(s)^{-1}$. Finding H such that $G^{'}(s)$ has assigned zeros is defined as the zero assignment by squaring down problem. The zero polynomial of $S(A, B, HC, HD)$ is given by:

$$z_k(s) = \det\{H \cdot N_r(s)\} \tag{6}$$

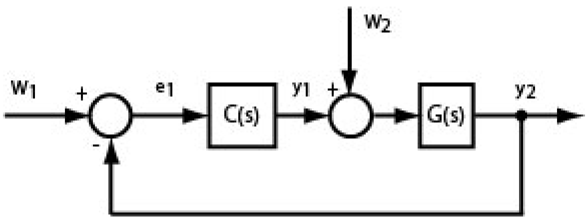

Figure 1: Feedback configuration.

(v) Dynamic Compensation Problems: Consider the standard feedback configuration (Kucera, 1979) shown in Fig.1. If $G(s) \in \mathbb{R}(s)^{m\times p}_{pr}$, $C(s) \in \mathbb{R}(s)^{p\times m}_{pr}$, and assume coprime MFDs as in (2) and

$$C(s) = A_l(s)^{-1} \cdot B_l(s) = B_r(s) \cdot A_r(s)^{-1} \tag{7}$$

the closed loop characteristic polynomial is

$$f(s) = \det\left\{[D_\ell(s),\ N_\ell(s)] \begin{bmatrix} A_r(s) \\ B_r(s) \end{bmatrix}\right\} \tag{8}$$

$$f(s) = \det\left\{[A_l(s),\ B_l(s)] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \tag{9}$$

If $p \geq m$, the $C(s)$ may be interpreted as pre-compensator (8); whereas, if $p \leq m$, then $C(s)$ may be interpreted as feedback compensator (9). The above general dynamic formulation covers a number of important families of $C(s)-$compensators as:

Constant Controllers: If $p \leq m$, $A_l = I_p$, $B_l = K \in \mathbb{R}^{p\times m}$, then (9) expresses the constant output feedback case; whereas if $p \geq m$, $A_r = I_m$, $B_r = K \in \mathbb{R}^{p\times m}$ expresses the constant pre-compensation.

Proportional plus Integral Controllers:

$$C(s) = K_0 + \frac{1}{s}K_1 = \left[sI_p\right]^{-1}[sK_0 + K_1] \tag{10}$$

where, $K_0, K_1 \in \mathbb{R}^{p\times m}$ and the left MFD for $C(s)$ is coprime, if and only if, $rank(K_1) = p$. Then, $f(s)$ is:

$$\begin{aligned} f(s) &= \det\left\{[sI_p, sK_0 + K_1] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \qquad (11) \\ &= \det\left\{[I_p, K_0, K_1] \begin{bmatrix} sD_r(s) \\ sN_r(s) \\ N_r(s) \end{bmatrix}\right\} \end{aligned}$$

Proportional plus Derivative Controllers:

$$C(s) = sK_0 + K_1 = \left[I_p\right]^{-1}[sK_0 + K_1] \tag{12}$$

where, $K_0, K_1 \in \mathbb{R}^{p\times m}$ and the left MFD for $C(s)$ is coprime for finite s and also for $s = \infty$ if $rank(K_0) = p$. Then, $f(s)$ is:

$$\begin{aligned} f(s) &= \det\left\{[I_p, sK_0 + K_1] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \qquad (13) \\ &= \det\left\{[I_p, K_1, K_0] \begin{bmatrix} D_r(s) \\ N_r(s) \\ sN_r(s) \end{bmatrix}\right\} \end{aligned}$$

PID Controllers:

$$\begin{aligned} C(s) &= K_0 + \frac{1}{s}K_1 + sK_2 \qquad (14) \\ &= \left[sI_p\right]^{-1}\left[s^2K_2 + sK_0 + K_1\right] \end{aligned}$$

where, $K_0, K_2 \in \mathbb{R}^{p\times m}$ and the left MFD is coprime with the only exception possibly at $s = 0$, $s = \infty$ (coprimeness at $s = 0$ is guaranteed by $rank(K_1) = p$ and at $s = \infty$ by $rank(K_2) = p$). Then, $f(s)$ is expressed as:

$$\begin{aligned} f(s) &= \det\left\{[sI_p, s^2K_2 + sK_0 + K_1] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \\ &= \det\left\{[I_p, K_0, K_1, K_2] \begin{bmatrix} sD_r(s) \\ sN_r(s) \\ N_r(s) \\ s^2N_r(s) \end{bmatrix}\right\} \qquad (15) \end{aligned}$$

The problems introduced here belong to the same family, DAP, involving solving the following equation with respect to polynomial matrix $H(s)$:

$$\det\{H(s) \cdot M(s)\} = f(s) \tag{16}$$

where, $f(s)$ is a polynomial of an appropriate degree (n) and $M(s)$ a polynomial matrix defined by the system. Existence of solutions of the problems stated above are reduced to finding real intersections between the Grassmann variety of a projective space and a linear variety as discussed in Section3. Such conditions deal with generic and exact problems and the results depend on the specific design problem formulation. Such conditions are given in Section 5.

3. The Abstract DAP, the Projective Geometry Approach and Grassmann Invariants

3.1. Introduction

The determinantal nature of DAP demonstrates that it is of a multilinear nature. Such problems may be naturally split into a linear and multilinear problem (decomposability of multivectors). The final solution is thus reduced to the solvability of a set of linear equations together with quadratics (characterising the multilinear problem of decomposability).

3.2. The Decomposition of DAP

The family of DAP problems requires solving

$$\det\{H(s) \cdot M(s)\} = f(s) \tag{17}$$

with respect to polynomial matrix $H(s)$, where $f(s) \in \mathbb{R}[s]$ with $deg(f(s)) = n$ and $M(s)$ is a polynomial matrix related to the system and the problem under study. The difficulty in solving DAP is due to the multilinear nature of the problem. Note that all dynamic problems may be reduced to equivalent constant DAP by shifting all dynamics from $H(s)$ to $M(s)$ and defining an equivalent matrix $M(s)^*$.

Remark 1. The reduction of dynamic DAP problems to equivalent constant is evident from the formulation of general dynamic DAP problems, as indicated for instance, by conditions (11), (13), (15) for the PI, PD and PID design problems respectively. The shifting of dynamics implies development of appropriate augmented design matrices $M(s)$ ($M(s)^*$).

Let $M(s) \in \mathbb{R}[s]^{p\times r}$, $r \leq p$, such that $rank\{M(s)\} = r$ and let $\mathcal{H}$ be a family of full rank $r\times p$ constant matrices H having a certain structure defined by the nature of the system and the type of compensation. The degree of $f(s)$ depends upon the degree of $M(s)$ and the structure of $H \in \mathcal{H}$. Hence, (17) is equivalent to

$$f_M(s, H) = \det\{H \cdot M(s)\} = f(s) \tag{18}$$

If $\underline{h}_i^t$, $\underline{m}_i(s)$, $i \in \tilde{r}$, denote the rows of $H \in \mathbb{R}^{r\times p}$, columns of $M(s) \in \mathbb{R}[s]^{p\times r}$ respectively, then

$$C_r(H) = \underline{h}_1{}^t \wedge \ldots \wedge \underline{h}_r{}^t = \underline{h}^t \wedge \in \mathbb{R}^{1\times\sigma}$$
$$C_r(M(s)) = \underline{m}_1(s) \wedge \ldots \wedge \underline{m}_r(s) = \underline{m}\wedge \in \mathbb{R}^{\sigma}[s]$$

where $\sigma = \binom{p}{r}$. By Binet-Cauchy theorem (Marcus and Minc, 1964) we have that (Karcanias and Giannakopoulos, 1984):

$$f_M(s, H) = C_r(H) \cdot C_r(M(s)) = \left\langle \underline{h}\wedge, \underline{m}(s)\wedge \right\rangle = \sum_{\omega\in Q_{r,p}} h_\omega \cdot m_\omega(s) \tag{19}$$

where, $\langle\cdot,\cdot\rangle$ denotes the inner product, $\omega = (i_1, \ldots, i_r) \in Q_{r,p}$, and $h_\omega, m_\omega(s)$ are the coordinates of $\underline{h}\wedge, \underline{m}(s)\wedge$ respectively. Note that h_ω is the $r \times r$ minor of H which corresponds to the ω-set of columns of H and thus h_ω is a multilinear alternating function of the h_{ij} entries. The multilinear nature of DAP suggests that the natural framework for its study is of exterior algebra. The essence of exterior algebra is that it reduces the study of multilinear skew-symmetric functions to the simpler study of linear functions. An example on how to compute the exterior product for a set of vectors is given below.

Example 1. For a set of 2 vectors in $\mathbb{R}^4$ with coordinates

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

the exterior product can be computed by taking all the 2×2 minors lexicographically ordered:

$$C_2(A) = (|A(1,2)|, |A(1,3)|, |A(1,4)|, |A(2,3)|, |A(2,4)|, |A(3,4)|)$$

where, $|A(i, j)|$ denote the 2×2 minors of matrix A consists of the i, j-th columns lexicographically ordered. □

The study of the zero structure of $f_M(s, H)$ may thus be reduced to a linear subproblem and a standard multilinear algebra subproblem:

Linear sub-problem of DAP: Set $\underline{m}(s)\wedge = \underline{p}(s) \in \mathbb{R}^{\sigma}[s]$. Determine whether there exists a $\underline{k} \in \mathbb{R}^{\sigma}$, $\underline{k} \neq 0$, such that for $f(s) \in \mathbb{R}[s]$

$$f_M\left(s, \underline{k}\right) = \underline{k}^t \cdot \underline{p}(s) = \sum_{i\in\sigma} k_i \cdot p_i(s) = f(s) \tag{20}$$

Multilinear sub-problem of DAP: Assume that $\mathcal{K}$ is the family of solution vectors $\underline{k}$ of (20). Determine whether there exists $H^t = \left[\underline{h}_1, \ldots, \underline{h}_r\right] \in \mathbb{R}^{p\times r}$ such that

$$\underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{h}\wedge = \underline{k},\ \underline{k} \in \mathcal{K} \tag{21}$$

Polynomials defined by (20) are called *polynomial combinants* (Karcanias and Giannakopoulos, 1984) and the zero assignability of them provides necessary conditions for the solution of DAP. The solution of the exterior equation (21) is a standard problem of exterior algebra known as decomposability of multivectors. The essence of the DAP approach is projective. We use a natural embedding for determinantal problems to embed the space of the unknown, H, of DAP, the rows of which define an r−space of the Grassmanian $\mathcal{H}$ (Griffiths and Harris, 1978) of the r−dimensional subspaces

$\mathcal{V}$ into an appropriate projective space $\mathcal{P}^{\sigma-1}$, $\sigma = \binom{p}{r}$. This map is referred to as the Plücker Embedding (Marcus, 1973) of the Grassmanian $\mathcal{H}$ into a projective space $\mathcal{P}^{\sigma-1}$. Thus, we are searching for common solutions of some set of linear equations and another set of second order polynomial equations, i.e. the set of Quadratic Plücker Relations (QPRs) characterising the Grassman variety of $\mathcal{P}^{\sigma-1}$ (Hodge and Pedoe, 1952), (Marcus, 1973). This also allows us to compactify H into $\bar{H}$ and then use algebraic geometric, or topological intersection theory methods to determine existence of solutions for the above sets of equations. The current framework allows the use of algebraic geometry and topological methods (Brockett and Byrnes, 1981), (Byrnes, 1989), (Leventides and Karcanias, 1992) for the study of solvability conditions but also computations.

3.3. The Grassmann and Plücker Invariants of a Rational Vector Space

The DAP framework uses the natural embedding of a Grassmannian into a projective space and this in turn defines new sets of invariants characterising the solvability of the different DAP problems. We may summarise the main results next (Karcanias and Giannakopoulos, 1984):

Let $T(s) = \left[\underline{t}_1(s), \ldots, \underline{t}_r(s)\right] \in \mathbb{R}^{p\times r}(s)$, $r \leq p$, $rank\{T(s)\} = r$ and $\mathcal{X}_t = rowspan_{\mathbb{R}(s)}\{T(s)\}$. If $T(s) = M(s)D(s)^{-1}$ is a RCMFD of $T(s)$, then $M(s)$ is a polynomial basis for $\mathcal{X}_t$. If $Q(s)$ is a greatest right divisor of $M(s)$ then $T(s) = \tilde{M}(s)Q(s)D(s)^{-1}$, where $\tilde{M}(s)$ is a least degree polynomial basis of $\mathcal{X}_t$ (Rosenbrock, 1970). A Grassmann Representative (GR) for $\mathcal{X}_t$ is defined by

$$\begin{aligned}\underline{t}(s)\wedge &= \underline{t}_1(s) \wedge \ldots \wedge \underline{t}_r(s)\\ &= \underline{\tilde{m}}_1(s) \wedge \ldots \wedge \underline{\tilde{m}}_r(s) \cdot z_t(s)/p_t(s)\end{aligned}$$

where, $z_t(s) = \det\{Q(s)\}$, $p_t(s) = \det\{D(s)\}$ are the zero, pole polynomials of $T(s)$ and $\underline{\tilde{m}}(s)\wedge = \underline{\tilde{m}}_1(s) \wedge \ldots \wedge \underline{\tilde{m}}_r(s) \in \mathbb{R}^{\sigma}[s]$, is also a GR of $\mathcal{X}_t$. Since, $\tilde{M}(s)$ is a least degree polynomial basis for $\mathcal{X}_t$, the polynomial entries of $\underline{\tilde{m}}(s)\wedge$ are coprime and it will be referred to as a *reduced polynomial* GR of $\mathcal{X}_t$. If $\delta = deg\{\underline{\tilde{m}}(s)\wedge\}$, then δ is the Forney dynamical order (Forney, 1975) of $\mathcal{X}_t$. $\underline{\tilde{m}}(s)\wedge$ may always be expressed as

$$\underline{\tilde{m}}(s)\wedge = \underline{p}(s) = \underline{p}_0 + s\underline{p}_1 + \cdots + s^{\delta}\underline{p}_{\delta} = P_{\delta} \cdot \underline{e}_{\delta}(s) \quad (22)$$

where, $P_{\delta} \in R^{\sigma\times(\delta+1)}$ is a basis matrix for $\underline{\tilde{m}}(s)\wedge$ and $\underline{e}_{\delta}(s) = \left[1, s, \ldots, s^{\delta}\right]^t$. By choosing an $\underline{\tilde{m}}(s)\wedge$ with $\|P_{\delta}\| = 1$, a canonical $\mathbb{R}[s]$−GR of $\mathcal{X}_t$ is defined denoted by $g(\mathcal{X}_t)$. The basis matrix P_{δ} of $g(\mathcal{X}_t)$ is defined as the Plücker matrix of $\mathcal{X}_t$. The following properties hold true (Karcanias and Giannakopoulos, 1984):

Theorem 1. *The $\mathbb{R}[s]$−GR, $g(\mathcal{X}_t)$, or the associated Plücker matrix, P_{δ}, is a complete invariant of $\mathcal{X}_t$.*

Remark 2. Let $T(s) = \left[\underline{t}_1(s), \ldots, \underline{t}_r(s)\right] \in \mathbb{R}^{p\times r}(s)$, $r \leq p$, $rank\{T(s)\} = r$, and $z_t(s)$, $p_t(s)$ be the monic zero, pole polynomials of $T(s)$ and let $g(\mathcal{X}_t) = \underline{p}(s)$ be the $C-\mathbb{R}[s]$−GR of $\mathcal{X}_t$. $\underline{t}(s)\wedge$ may be uniquely decomposed as

$$\underline{t}(s)\wedge = c \cdot p(s) \cdot z_t(s)/p_t(s) \quad (23)$$

and the linear part of DAP is thus reduced to

$$f_M\left(s, \underline{k}\right) = \underline{k}^t \underline{p}(s) z_m(s) = \underline{k}^t \cdot P_{\delta} \cdot \underline{e}_{\delta}(s) \cdot z_m(s) \quad (24)$$

The zeros of $T(s)$ are fixed zeros of all combinants of $\underline{t}(s)\wedge$.

The freely assigned zeros of $f_M\left(s, \underline{k}\right)$ are those of the combinant $f_M\left(s, \underline{k}\right) = \underline{k}^t \cdot \underline{m}(s)\wedge$, where $\underline{m}(s)\wedge$ is reduced. If $a(s) = \underline{a}^t_{\delta}\underline{e}_{\delta}(s) = a_0 + a_1 s + \cdots + a_{\delta}s^{\delta} \in R[s]$ is the polynomial to be assigned, then $max(deg(a(s)) = \delta$, where δ is the Forney dynamical order of $\mathcal{X}_t$, and finding $\underline{k} \in \mathbb{R}^{\sigma}$, such that $f_M\left(s, \underline{k}\right) = a(s)$, is reduced to:

$$P^t_{\delta} \cdot \underline{k} = \underline{a} \quad (25)$$

Remark 3. Let $M(s) \in \mathbb{R}^{p\times r}[s]$ be a least degree matrix, P_{δ} be the Plücker matrix of $\mathcal{X}_m$ and let $\pi = rank\,(P_{\delta})$. Then, necessary and sufficient condition for $M(s)$ to generate a DAP that is Linearly Assignable (LA) (no decomposability constraints) is that $\pi = \delta + 1$.

3.4. The Grassmann and Plücker Invariants of a system

Plücker type matrices associated with state space descriptions are defined (Karcanias and Leventides, 1996):

Controllability Plücker Matrix: For the pair (A, B), $\underline{b}(s)^t\wedge$ denotes the exterior product of the rows of $B(s) = [sI - A, -B]$ and $P(A, B)$ is the $(n+1)\times\binom{n+p}{n}$ basis matrix of $\underline{b}(s)^t\wedge$. $P(A, B)$ is the *Controllability Plücker* matrix.

Corollary 2. *The system $S(A, B)$ is controllable if and only if $\underline{b}(s)^t\wedge$ is coprime, or equivalently $P(A, B)$ has full rank.*

Example 2. Consider the system $S(A, B)$ described by the pencil $[sI - A, -B] = R(s)$

$$\left[\begin{array}{ccc|c} s & -1 & 0 & 0\\ 0 & s & -1 & 0\\ 0 & 0 & s & -1\end{array}\right] = \left[\begin{array}{c}\underline{r}_1(s)^t\\ \underline{r}_2(s)^t\\ \underline{r}_3(s)^t\end{array}\right]$$

The exterior product of the rows of $R(s)$ is defined by the minors of maximal order of $R(s)$ lexicographically ordered, i.e.

$$\underline{r}(s)^t\wedge = \underline{r}_1{}^t(s)\wedge\underline{r}_2{}^t(s)\wedge\underline{r}_3{}^t(s) = \left[\begin{array}{cccc} s^3 & -s^2 & s & -1\end{array}\right]$$

and the Controllability Plücker matrix, $P(A,B)$, is then

$$P(A,B)\cdot \underline{e}_3(s) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{P(A,B)} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

Clearly, $P(A,B)$ has full rank and hence the system is controllable. □

A similar result for observability may be stated using duality principle.

Transfer Function Matrix Plücker Matrices: For the transfer function matrix $G(s)$ represented by the RCMFD, LCMFD we define by $\underline{t}_r(s)\wedge$, $\underline{t}_l(s)^t\wedge$ the exterior product of the columns of $T_r(s)$, rows of $T_l(s)$ respectively , where $T_r(s)$, $T_l(s)$ are defined by (2). By $P(T_r)$ we denote the $\binom{m+p}{p}\times(n+1)$ basis matrix of $\underline{t}_r(s)\wedge$, and by $P(T_l)$ the $(n+1)\times\binom{m+p}{p}$ basis matrix for $\underline{t}_l(s)^t\wedge$. $P(T_r)$, $P(T_l)$ will be referred to as *right, left fractional Plücker* matrices respectively. Such matrices provide the necessary conditions for the solvability of pole assignment problems by output feedback.

Proposition 3 (Leventides and Karcanias (1995))**.** *For a generic system with $mp > n$, then the Plücker matrices $P(T_r)$, $P(T_l)$ have full rank.*

Column, Row Plücker Matrices: For the transfer function $G(s)$, with $m \geq p$, we denote by $\underline{n}(s)\wedge$ the exterior product of the columns of the numerator $N_r(s)$, of a RCMFD and by $P(N)$ the $\binom{m}{l}\times(d+1)$ basis matrix of $\underline{n}(s)\wedge$. Note that $d=\delta$, the Forney order of $\mathcal{X}_t$, if $G(s)$ has no finite zeros and $d=\delta+k$, where k is the number of finite zeros of $G(s)$ otherwise. If $N_r(s)$ is least degree, then $P_c(N)$ will be called the *column space Plücker* matrix of the system. The *row space Plücker* matrix $P_r(N)$ may be similarly defined when $m \leq p$. Such matrices play a key role in the study of squaring down problems (Karcanias and Giannakopoulos, 1989).

Proposition 4. *For a generic system with $m > p$, for which $p(m-p) > \delta + 1$, where δ is the Forney order, $P_c(N)$ has full rank.*

Similar definitions and invariant may be defined for dynamic compensation transfer function matrices.

4. Decomposability of Multivectors and the Grassmann Variety

4.1. Introduction: Decomposability of Multivectors

The solution of DAP is reduced to finding amongst the family of solutions, $\mathcal{K}$, of the linear problem in (20), at least a solution $\underline{k}\in\mathcal{K}$ that also satisfies the exterior equation (21). The set of $r-$dimensional subspaces of $\mathbb{R}^p$ is referred to as the $r-$ Grassmaniann and the row space of H, $\mathcal{H}$, defines a basis for such subspaces. The mapping of each $r-$dimensional subspace $\mathcal{H}$ expressed by $\underline{h}_1\wedge\ldots\wedge\underline{h}_r = \underline{h}\wedge^r = \underline{k}$, where $\underline{h}_i$ are the row vectors of H, is a vector $\underline{k}\in\mathbb{R}^\sigma$, $\underline{k}\neq 0$ that defines a point in the projective space $\mathcal{P}^{\sigma-1}(\mathbb{R})$, $\sigma=\binom{p}{r}$; for some $H\in\mathbb{R}^{r\times p}$, the points of $\mathcal{P}^{\sigma-1}$ which satisfy (21) are those which belong to the Grassmann variety $\Omega(r,p)$ of $\mathcal{P}^{\sigma-1}(\mathbb{R})$. The coordinates k_ω, $\omega=(i_1,....,i_r)\in Q_{r,p}$ are referred to as the *Plücker coordinates* of $\underline{k}\in\mathbb{R}^\sigma$, and the mapping of $\mathcal{H}$ through $\wedge^r$ is known as the *Plücker Embedding* of the $r-$Grassmaniann into $\mathcal{P}^{\sigma-1}(\mathbb{R})$. The characterisation of the $\Omega(r,p)$ variety and the construction of the subspaces $\mathcal{H}$ corresponding to $\underline{k}\in\Omega(r,p)$ are considered next.

4.2. The Grassmann Variety and the Quadratic Plücker Relations

The variety $\Omega(r,p)$ is characterised by the result (Marcus, 1973):

Theorem 5. *Necessary and sufficient condition for an $H=\left[\underline{h}_1,\ldots,\underline{h}_r\right]^t\in\mathbb{R}^{r\times p}$ to exist, such that*

$$\underline{h}\wedge=\underline{h}_1\wedge\ldots\wedge\underline{h}_r=\underline{k}=[\ldots,k_\omega,\ldots] \quad (26)$$

is that the coordinates k_ω satisfy the following quadratic relations

$$\sum_{k=1}^{r+1}(-1)^{\nu-1}k_{i_1,\ldots,i_{r-1}},j_\nu^k j_1,\ldots,j_{\nu-1},j_{\nu+1},j_{r+1}=0 \quad (27)$$

where, $1\leq i_1<i_2<\ldots<i_{r-1}\leq n$ and $1\leq j_1<j_2<\ldots<j_{r+1}\leq n$.

The vectors $\underline{k}$ which satisfy (27) are known as decomposable and the set of quadratics defined by (27) as Quadratic Plücker Relations (QPR) (Hodge and Pedoe, 1952), (Marcus, 1973) and they define the Grassmann variety of $\mathcal{P}^{\sigma-1}(\mathbb{R})$. Interesting questions are: (i) Defining alternative conditions for decomposability; (ii) Reconstructing the matrix H for a decomposable $\underline{k}$; (iii) Characterising the distance of a general $\underline{k}\in\mathcal{P}^{\sigma-1}(\mathbb{R})$ from the Grassmann variety $\Omega(r,p)$. The reconstruction of H from the decomposable $\underline{k}$ is given in Giannakopoulos et al. (1985) and in Section 4.3.

Corollary 6. *Let $\underline{k}=[\ldots,k_\omega,\ldots]^t\in\mathbb{R}^\sigma$, be a decomposable vector and let $k_{a_1,\ldots,a_r}$ be a non-zero coordinate of $\underline{k}$. If we define by*

$$h_{ij}=k_{a_1,\ldots,a_{i-1},j,a_{i+1},\ldots,a_r},\ i\in\bar{r},j\in\bar{p} \quad (28)$$

then for the matrix $H=\left[h_{ij}\right]$, $C_r(H)=k$.

The procedure for constructing H for a decomposable $\underline{k}$ also requires writing down an independent set of QPRs which completely describes $\Omega(r, p)$.

Example 3. Assume $p = 4$, $r = 2$ and let $\{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\}$ be the coordinates of a vector in $\wedge^2\mathbb{R}^4$. The Grassmann variety $\Omega(2, 4)$ is defined by the single QPR:

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

4.3. The Grassmann Matrix and Decomposability of Multivectors

The Grassmann matrix of $\underline{z} \in \wedge^r(\mathcal{H}_r)$ (Karcanias and Giannakopoulos, 1988) is introduced and a number of its properties are examined. This matrix provides an alternative test for decomposability of $\underline{z}$, which also allows the computation of the $\mathcal{H}_r$ solution space in an easy manner.

Proposition 7 (Marcus (1973))**.** *Let $\mathcal{U}$ be a $p-$dimensional vector space over $\mathcal{F}$ and let $\underline{0} \neq \underline{z} \in \wedge^r\mathcal{U}$. Then, $\underline{z}$ is decomposable, if and only if, there exists a set of linearly independent vectors $\{h_i, i \in \bar{r}\}$ in $\mathcal{U}$ such that*

$$\underline{v}_i \wedge \underline{z} = \underline{0}, \forall i \in \bar{p} \tag{29}$$

Lemma 8. *Let $B_{\mathcal{U}} = \{\underline{u}_i, i \in \bar{p}\}$ be a basis of $\mathcal{U}$, $B^r_{\mathcal{U}} = \{\underline{u}_\omega\wedge, \omega \in Q_{r,p}\}$ be a basis of $\mathcal{U}$ the corresponding basis of $\wedge^r\mathcal{U}$ and let $\underline{v} = \sum_{t=1}^{r} c_t\,\underline{u}_t$, $\underline{z} = \sum_{\omega\in Q_{r,p}} a_\omega\,\underline{u}_\omega\wedge$. Then,*

$$\underline{v} \wedge \underline{z} = \sum_{\gamma\in Q_{r+1,p}} b_\gamma\,\underline{u}_\gamma\wedge \quad b_\gamma = \sum_{k=1}^{r+1} (-1)^{k-1} c_{\gamma(k)} a_{\gamma(\hat{k})}$$

where, $\gamma(k)$ denotes the $k-$th element of $\gamma(k) \in Q_{r+1,p}$ and $\gamma(\hat{k})$ is the sequence $\{\gamma(1), \ldots, \gamma(k-1), \gamma(k+1), \ldots, \gamma(r+1)\} \in Q_{r,p}$.

Notation: Let $\gamma = (j_1, j_2, \ldots, j_k, j_{r+1}) \in Q_{r+1,p}$ with $r + 1 \leqslant p$. We denote by $Q^\gamma_{r,r+1}$ the subset of $Q_{r,p}$ sequences with elements taken from the γ set of integers. $Q^\gamma_{r,r+1}$ has $r + 1$ elements and the sequences in it are defined from γ by deleting an index in γ. Thus, for all $k \in \overline{r+1}$

$$Q^\gamma_{r,r+1} = \left\{\rho_\gamma[\hat{j}_k] = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{r+1})\right\}$$

Definition 1. Let $\{a_\omega, \omega \in Q_{r,p}\}$ be the coordinates of $\underline{z} \in \wedge^r\mathcal{U}$ with respect to a basis $B^r_{\mathcal{U}}$ of $\wedge^r\mathcal{U}$, $r + 1 \leqslant p$, $\gamma = (j_1, \ldots, j_k, j_{r+1}) \in Q_{r+1,p}$. We define

$$\phi : \{i : i = 1, \ldots, p\} \times \left\{\gamma, \gamma \in Q_{r+1,p}\right\} \to \mathcal{F} \tag{30}$$

with, $\rho_\gamma[\hat{j}_k] = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{r+1}) \in Q^\gamma_{r,r+1}$

$$\begin{aligned} \phi^i_\gamma &= \phi_\gamma(i) = 0 & , if\ i \neq \gamma \\ \phi^i_\gamma &= \phi_\gamma(i) = sign\left(j_k : \rho_\gamma[\hat{j}_k]a_{\rho_\gamma[\hat{j}_k]}\right) & , if\ i = j_k \in \gamma \end{aligned}$$

where,

$$sign\left(j_k : \rho_\gamma[\hat{j}_k]\right) = sign\,(j_k, j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{r+1})\,.$$

Theorem 9 (Karcanias and Giannakopoulos (1988))**.** *If $B_{\mathcal{U}} = \{\underline{u}_i, i \in \bar{p}\}$, $B^r_{\mathcal{U}} = \{\underline{u}_\omega\wedge, \omega \in Q_{r,p}\}$ are bases of $\mathcal{U}$, $\wedge^r\mathcal{U}$, $\underline{v} = \sum_{i=1}^{n} c_i\,\underline{u}_i \in \mathcal{U} : \underline{v} \neq \underline{0}$, and $\underline{z} = \sum_{\omega\in Q_{r,p}} a_\omega\,\underline{u}_\omega \wedge \in \wedge^r\mathcal{U} : \underline{z} \neq \underline{0}$. Then, $\underline{v} \wedge \underline{z} = \underline{0}$, if and only if,*

$$\sum_{i=1}^{n} \phi^i_\gamma c_i = 0, \textit{ for all } \gamma \in Q_{r+1,p} \tag{31}$$

If we denote by γ_t the elements of $Q_{r+1,p}$ (lexicographically ordered), with $t = 1, 2, \ldots, \binom{p}{r+1} = \tau$, then (31) may be expressed as

$$\underbrace{\begin{bmatrix} \phi^1_{\gamma_1}, & \phi^2_{\gamma_1}, & \ldots, & \phi^i_{\gamma_1}, & \ldots, & \phi^p_{\gamma_1} \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi^1_{\gamma_t}, & \phi^2_{\gamma_t}, & \ldots, & \phi^i_{\gamma_t}, & \ldots, & \phi^p_{\gamma_t} \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi^1_{\gamma_\tau}, & \phi^2_{\gamma_\tau}, & \ldots, & \phi^i_{\gamma_\tau}, & \ldots, & \phi^p_{\gamma_\tau} \end{bmatrix}}_{\Phi^r_p(\underline{z})} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \\ c_p \end{bmatrix}}_{\underline{c}} = \underline{0} \tag{32}$$

The matrix $\Phi^r_p(\underline{z}) \in \mathbb{F}^{\tau\times p}$ is a structured matrix (has zeros in fixed positions), it is called the *Grassmann Matrix* (GM) of $\underline{z}$ and it is defined by the pair (r, p) and the coordinates $\{a_\omega, \omega \in Q_{r,p}\}$ of $\underline{z} \in \wedge^r\mathcal{U}$.

Theorem 10 (Karcanias and Giannakopoulos (1988))**.** *Let $\mathcal{U}$ be an $n-$dimensional vector space over $\mathcal{F}$, $B_{\mathcal{U}}$ a basis of $\mathcal{U}$, $\underline{0} \neq \underline{z} \in \wedge^r\mathcal{U}$, $\Phi^r_p(\underline{z})$ the GM of $\underline{z}$) with respect to $B_{\mathcal{U}}$ and let $\mathcal{N}^r_p(\underline{z}) = \mathcal{N}_r\{\Phi^r_p(\underline{z})\}$. Then,*

(i) $dim\mathcal{N}^r_p(\underline{z}) \leqslant r$ and equality holds, if and only if $\underline{z}$ is decomposable.

(ii) If $dim\mathcal{N}^r_p(\underline{z}) = r$, then a representation of the solution space, $\mathcal{H}_z$ of $\underline{h}_1 \wedge \ldots \wedge \underline{h}_r = \underline{z}$ with respect to $\mathcal{U}$ is given by $\mathcal{N}^r_p(\underline{z})$.

This result provides an alternative characterisation for decomposability of multivectors, and a simple procedure for reconstruction of the solution space of the exterior equation.

Corollary 11. *Let $\Phi_p^r(\underline{z})$ be the GR of $\underline{z} \in \wedge^r \mathcal{U}$, $\underline{z} \neq 0$. Then,*

i) If $r = 1$ then for all p, $\Phi_p^1(\underline{z})$ is always canonical; furthermore, if $p \geqslant 3$ $rank_{\mathcal{F}} \left\{\Phi_p^1(\underline{z})\right\} = p - 1$.

ii) If $r = p - 1$, then $\Phi_p^{p-1}(\underline{z}) \in \mathcal{F}^{1\times p}$ and it is always canonical with $rank_{\mathcal{F}} \left\{\Phi_p^{p-1}(\underline{z})\right\} = 1$.

iii) If $r = p - \rho$, $p > 1$, and $\rho \geq 2$, then for all $\underline{z}$, $rank_{\mathcal{F}} \left\{\Phi_p^1(\underline{z})\right\} \geq p - \rho$, where equality holds, if and only if, $\Phi_p^r(\underline{z})$ is canonical.

5. Real Intersections of the Grassmann Variety and Linear Space: Generic Solvability Conditions

DAP can be formulated as an intersection problem between a linear variety, $L\mathbb{R}$, and the Grassmann variety, $\mathcal{G}_p(\mathbb{F}^{m+p})$ of a projective space, where the field $\mathbb{F}$ is considered to be either $\mathbb{R}$ (real) or $\mathbb{C}$ (complex).

Proposition 12. *The set of (finite and infinite) real solutions of the constant pole assignment problem is given by*

$$L\mathbb{R} \cap \mathcal{G}_p\left(\mathbb{R}^{p+m}\right) \tag{33}$$

where, $L\mathbb{R}$ is a linear variety of co-dimension (n) in $P(\mathbb{R})^{\sigma-1}$ defined by the linear DAP sub-problem.

The real constant pole assignment problem is generically solvable if and only if the intersection (33) is nonvoid. (Similarly is defined the generic solvability for the complex case). The real solvability of the intersection problem (33) is challenging due to the lack of strong intersection theorems when the definition field of the problem is not algebraically closed. For instance, in the simple case of one polynomial equation with one unknown real solvability is not guaranteed. The only thing we can say is that when the degree of the polynomial is odd there exists at least one real solution. In contrast, regarding the complex solvability case, we know that we have always as many roots as the degree of the polynomial to be assigned. The main results on the solvability of the output feedback pole assignment problem via the intersection theory are summarised below:

Theorem 13 (Leventides and Karcanias (1992))**.** *A sufficient condition for the existence of real solutions of the output feedback pole assignment problem for a generic proper system ($p-$inputs, $m-$outputs, $n-$states) is*

$$h(p, m) \geqslant n \tag{34}$$

where, $h(p, m)$ is the height of the first Whitney class (w) (Hiller, 1980) of a real Grassmannian $\mathcal{G}_p\left(\mathbb{R}^{p+m}\right)$.

The first important result on this problem was given in Kimura (1975) and Davison and Wang (1975), where they showed that for a strictly proper system a sufficient condition for generic pole placement is that

$$m + p - 1 \geqslant n$$

Using tools from algebraic geometry (Hermann and Martin, 1977), (Giannakopoulos and Karcanias, 1985) showed that a necessary and sufficient condition for arbitrary pole assignment by complex (constant) output feedback is $m \cdot p \geqslant n$, whereas, a special case ($m \cdot p = n$ and $d(m, p) =$ odd) was proved to be a sufficient condition for generic pole assignment via real output feedback (Brockett and Byrnes, 1981).
A similar nature condition for arbitrary pole assignment of real poles using other topological invariants of the Grassmannian were given in Byrnes (1983) in terms of

$$LS\,cat(p, m) \geqslant n \tag{35}$$

where, *LScat* is the Ljusterning Snirelman category of the Grassmannian $\mathcal{G}_p\left(\mathbb{R}^{p+m}\right)$. In both complex and real cases the intersections (33) may be represented as certain elements of the corresponding cohomology ring, i.e. $H^*\left(G_p(R^{p+m}); Z_2\right)$ where the existence of intersection is then reduced to whether these elements are nonzero. The intersection points are considered mod-2 (i.e. odd number of points correspond to one and even number of points to zero). This element is of the form w^n, where w is the first Whitney class (Hiller, 1980) of the Grassmannian and n is the number of poles to be placed. Hence, for a generic real solution we require $w^n \neq 0$. If we let $h(p, m)$ the highest exponent, h, of $w \in H^*\left(G_p(R^{p+m}); Z_2\right)$ so that $w^h \neq 0$, then a sufficient condition for real solvability of DAP is $n \leqslant h(p, m)$. It is worth noting that not all intersections (33) correspond to w^n. It is only the intersections for which some regularity condition holds true. Note that, since

$$m + p - 1 \leqslant h(p, m) \leqslant m \cdot p \tag{36}$$

where the upper bound is the best possible bound (complex case) for the degree of the polynomial and in most cases $h(p, m)$ is closer to the lowest bound (Leventides and Karcanias, 1992). These approaches are of very general and qualitative and tackle only the existence problem. They do not consider the special nature of the problem and they do not provide computation of solutions. A considerable improvement which overcomes the above deficiencies has been given by the GL method (Leventides and Karcanias, 1995).

6. The Global Linearisation Methodology

The solvability of DAP may be seen as a problem of finding real intersections between the linear variety and the Grassmann variety of an appropriate projective space. An approach that applies to generic and given systems which leads to establishing of existence results and also provides a computation scheme, has been based on the notion of degenerate feedback solutions; this is referred to as Global Linearisation (GL) (Leventides and Karcanias, 1995). Degenerate solutions have been introduced in (Brockett and Byrnes, 1981) as the compensation solutions where the feedback configuration vanishes. Such solutions have the significant property that linearise asymptotically the multilinear nature of DAP and thus lead to the computation of solutions. The GL approach leads to numerical methods to design feedback laws for DAP applications capable to handle dynamic schemes, as well as structurally constrained compensation schemes (Leventides et al., 2014a,b). Furthermore, it has provided new solvability of control problems for both generic and non-generic cases.

6.1. The Pole Placement Map (Primitive Results)

Consider the pole placement problem via constant output feedback. The closed-loop pole polynomial is

$$\det\left([I,K]\cdot\begin{bmatrix} D(s) \\ N(s)\end{bmatrix}\right) = \det\left([I,K]\cdot M(s)\right) = p(s) \tag{37}$$

where, $M(s) \in \mathbb{R}[s]^{(p+m)\times p}$ is the composite MFD of the open-loop system transfer function $G(s) = N(s)\cdot D^{-1}(s)$, $[I, K] \in \mathbb{R}^{p\times(m+p)}$ are the generalised finite feedback compensators with $K \in \mathbb{R}^{p\times m}$ and $p(s)$ the target polynomial to be assigned.

Solvability conditions of the pole placement problem under complex and real output feedback have been established in Leventides and Karcanias (1992) based on the dimension of the *Pole Placement Map* (PPM) and in particularly the image of the maps:

Pole Placement Map (PPM): The PPM under real (or complex) output feedback which maps every K to $\underline{p} = (p_n, \dots, p_1)$ under the relation (37) is defined by

$$PPM : \mathbb{F}^{p\times m} \to \mathbb{F}^n : PPM(K) = \underline{p} \tag{38}$$

where, $\mathbb{F}$ is either $\mathbb{R}$ (for real) or $\mathbb{C}$ (for complex). The solvability of the arbitrary pole placement problem is translated into onto properties of the related PPM. For the complex case there are results based on Shard Theorem (Byrnes, 1983). In Leventides and Karcanias (1993) the global onto properties of a polynomial map can be proved by the local onto properties of the map (linear) i.e. the differential of the mapping at a point is onto, which requires only to test the rank of a jacobian matrix (differential).

Procedure: The computational procedure involves: (i) express the PPM; (ii) calculate the differential at any point; (iii) select a specific point that the differential is easily calculated (linear map); (iv) calculate the rank of the differential of the related PPM ; (v) if the rank of the differential is full then the PPM is almost onto. □

It has been proved that for a generic proper system arbitrary pole placement is solvable with complex controllers when $m \cdot p \geqslant n$ (Byrnes, 1983), (Leventides and Karcanias, 1992). However, this procedure answers only the existence of solutions and does not lead to a construction of such solutions. Next, we summarize the main early results as far as sufficient conditions for generic pole placement.

Theorem 14. *Sufficient conditions of generic systems for pole placement via complex controllers are given:*

(i) $m + p - 1 \geqslant n$*; in Kimura (1975)*

(ii) $mp \geqslant n$*; in Byrnes (1983); Leventides and Karcanias (1992)*

whereas, for pole placement via real controllers the main results have been given by:

(i) $LS\,cat(p, m) \geqslant n$*; in Byrnes (1983)*

(ii) $h(p, m) \geqslant n$*; in Leventides and Karcanias (1992)*

(iii) $mp = n$ *(holds when the degree of* $\mathcal{G}_p(\mathbb{R}^{p+m})$ *is odd); in Brockett and Byrnes (1981).*

A first breakthrough regarding the sufficiency of the static generic pole assignability was established by Wang (1992) as, $m \cdot p > n$. Using geometric techniques Rosenthal and Wang (1996, 1997) derive that, $q \cdot max(m, p) + mp > n$, implies generic assignability over the reals with dynamic compensators of $q-$degree. Moreover, by using the linearisation procedure around a degenerate point (Leventides and Karcanias, 1995, 1998b) derive not only sufficient conditions but also closed formulas and a procedure for construction of feedback compensators. Similar results have been given by (Rosenthal et al., 1995), (Wang, 1996), (Ravi et al., 1996), (Rosenthal and Sottile, 1998), (Ariki, 1998), (Sottile, 2000), (Eremenko and Gabrielov, 2002), (Huber and Verschelde, 2000), whereas for a comprehensive review on the open pole placement problems see Rosenthal and Willems (1999).

6.2. Degenerate Solutions

Degenerate gains were first introduced by Brockett and Byrnes (1981) in their generalized form as follows:

Definition 2. A generalized gain, $D = rowspan\,(A, K)$, is degenerate if and only if satisfies equation:

$$\det([A, K] \cdot M(s)) \equiv 0 \tag{39}$$

Degenerate gains can be constructed easily from the null-spaces of certain matrices (Wang, 1992), (Leventides and Karcanias, 1995). In the following we denote by $\mathcal{M} = colsp\,(M(s))$, the $\mathbb{R}[s]-$module generated by the columns of the system composite MFD $M(s)$.

Theorem 15. *For the system represented by composite MFD, $M(s) \in \mathbb{R}[s]^{(m+p)\times p}$, a $p-$dimensional space, $D = rowspan\,(A, K)$, corresponds to degenerate gain if and only if either of the following two conditions hold true:*

(i) There exists an $(m+p)\times 1$ polynomial vector, $\underline{m}(s) \in \mathcal{M}$, such that $(A, K) \cdot \underline{m}(s) = 0, \forall s \in \mathbb{C}$

(ii) There exists an $(m + p) \times 1$ polynomial vector, $\underline{m}(s) \in \mathcal{M}$, with coefficient matrix $P_{\underline{m}}$ such that $rank(P_{\underline{m}}) \leqslant m$.

The following example illustrates the standard procedure for constructing degenerate points.

Example 4. Consider a $p = 2-$input, $m = 3-$output system with $n = 5$ states represented by $M(s) = \begin{bmatrix} D(s)^t & N(s)^t \end{bmatrix}^t$

$$M(s) = \left[\begin{array}{cc} s^3 & 0 \\ 1 & s^2 \\ \hline s^2+1 & s+1 \\ s+3 & s \\ s+1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \underline{m}_1(s) & \underline{m}_2(s) \end{array}\right]$$

To construct a degenerate point, we select a polynomial vector $\underline{m}(s) \in \mathcal{M}$ with the lowest (column) degree, i.e. $\underline{m}(s) = \underline{m}_2(s) = \begin{bmatrix} 0 & s^2 & s+1 & s & 1 \end{bmatrix}^t$ and extract its coefficient matrix, hence we express it as:

$$\underline{m}(s) = P_{\underline{m}} \cdot \underline{e}_2(s) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s^2 \\ s \\ 1 \end{bmatrix}$$

A p=2-dimensional subspace that corresponds to a degenerate point can be found by constructing a basis for the left null-space of P_m, such that $D \cdot P_{\underline{m}} = \underline{0}_{2\times 3}$. Thus, a degenerate gain which satisfies conditions (i), (ii) of Th.16 is given by

$$D = rowspan\,(A, K) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

□

A degenerate solution for the feedback configuration is a gain where the closed-loop system has a singularity, in the sense that the feedback system is not well posed i.e. $p(s) \equiv 0$. In many cases, especially when the open loop system is strictly proper, degeneracy can occur only when $K \to \infty$ (since if K is finite, $p(s)$ is not identically 0). To cover also this case ($K \to \infty$), the gain space has been extended to the Grassmannian (the set of all $p-$dimensional subspaces of $\mathbb{F}^{p+m}$).
The extended PPM (where $\mathbb{F} = \mathbb{R}$: real or $\mathbb{C}$: complex) to the projective space is

$$\tilde{x} : \ \mathrm{G}_p\left(\mathbb{F}^{m+p}\right) \to P^n\left(\mathbb{F}\right) \tag{40}$$

This extension introduces new generalized controllers for output feedback, apart from the standard finite (bounded) controllers (37) appear in in the original formulation, which have the following form:

- *Infinite controllers*: $\tilde{K} = [A, K] \in \mathbb{F}^{m\times(m+p)}$, where $\det(A) = 0$
- *Degenerate controllers*: $\tilde{K}_d = [A, K] \in \mathbb{F}^{m\times(m+p)}$, for which the closed-loop polynomial is not defined, i.e.
$$\det\left(\tilde{K}_d \cdot M(s)\right) \equiv 0$$

The family of degenerate controllers are crucial for the development of the GL method. The GL method introduces new solvability conditions and provides the means for the parametrisation of the families of degenerate controllers using the theory of minimal bases (Karcanias et al., 2013), (Karcanias et al., 2016a).

6.3. The Global Linearisation Methodology

The GL methodology is an algebro-geometric method that tackles the problem of pole placement for generic and given systems under output feedback controllers. This method is based on an asymptotic linearisation of the pole placement map. The output feedback problem is reduced to solving a set of linear equations whereas the asymptotic solution of the problem (as $\varepsilon \to 0$) is given in closed form in terms of a one-parameter family of feedback compensators, i.e. $A(s)+\varepsilon\cdot B(s)$ where, $A(s)$ is the so-called degenerate compensator and $B(s)$ is the solution from the set of linear equations. We can construct a degenerate gain for the GL methodology (Leventides and Karcanias, 1995, 1998b) and consider sequences of generalized gains such as

$$S_\varepsilon = [A, K] + \varepsilon \cdot [A_1, K_1]$$

that converge to the degenerate gain $[A, K]$ as $\varepsilon \to 0$. It has been shown that for the standard feedback configuration and using the gain matrix $[A + \varepsilon \cdot A_1]^{-1} [K + \varepsilon \cdot K_1]$, the closed-loop pole polynomial has the same roots as:

$$p_\varepsilon(s) = \det\left\{ S_\varepsilon \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \right\} = \det\{S_\varepsilon \cdot M(s)\} \qquad (41)$$

where, $p_\varepsilon(s)$ tends to $p(s)$ as $\varepsilon \to 0$. The polynomial in (41) is called the prime polynomial with respect to the degenerate point $D = rowspan\,(A, K)$. The relationship between the perturbed direction and the pole polynomial (Leventides and Karcanias, 1995) is described as:

Theorem 16. *For a given degenerate point of a system, $rowspan\,(A, K)$, and a sequence of gains, S_ε, converging to it, the linear function that maps the direction $(A_1, K_1) = [b_{ij}]$ to the coefficient vector $\underline{p}$ of the prime polynomial $p(s)$ has a matrix representation denoted by L_D which is the $p(m + p) \times (n + 1)$ coefficient matrix of the polynomial vector $\left[p_{11}(s), \ldots, p_{ij}(s), \ldots, p_{p(m+p)}(s)\right]$ and $\underline{p}$ can be written as:*

$$\underline{p} = vec(b_{ij}) \cdot L_D \qquad (42)$$

where, $vec(b_{ij})$ is the vector formed by stacking all the rows of $(A_1, K_1) = [b_{ij}]$.

Theorem 17. *Let $D = rowspan\,(A, K)$ be a degenerate gain defined by the composite MFD representation $M(s)$. The target polynomial of the given system with respect to D and the direction $[A_1, K_1] = \left[b_{ij}\right]$ can be written as:*

$$p(s) = \sum \left(b_{ij} \cdot p_{ij}(s)\right) \qquad (43)$$

where, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, p + m$ and p_{ij} is the determinant of the $p \times p$ polynomial matrix $D_{ij}(s)$ having the same rows as the matrix $[A \cdot D(s) + K \cdot N(s)]$ apart from the $i-$th which is replaced by the $j-$th row of $M(s)$.

Note that in the characterization of degenerate gains we consider all possible gains (bounded and unbounded) which are further classified as:

(i) Regular (bounded) degenerate compensators: $rank(L) = n + 1$

(ii) Non-regular (unbounded) degenerate compensators: $rank(L) < n + 1$

Global Linearisation Method (Leventides and Karcanias, 1995)

1) Construct a degenerate point: $D = rowspan\,(A, K)$
2) Calculate the matrix L_D (Theorem 16)
3) If $rank\,(L_D) = n + 1$, then solve the linear equation (42) with the direction $(A_1, K_1) = [b_{ij}]$ else return to Step (1)
4) The one parameter family of $m \times p$ matrices, $K_\varepsilon = [A + \varepsilon \cdot A_1]^{-1} [K + \varepsilon \cdot K_1]$, are the real constant output feedback compensators placing the poles at the given set as $\varepsilon \to 0$.
5) Select a small enough ε (in K_ε), to approach the given closed-loop pole polynomial as close as it is desirable. □

The computational approach of GL is based on the assumption that we can select degenerate points for which the map L_D (related to the Plücker invariant of the system) has full rank. There exists a non-trivial family of systems for which this property can be satisfied:

Corollary 18 ((Leventides and Karcanias, 1995))**.** *For a generic proper system of $p-$inputs, $m-$outputs, $n-$states for which the condition $m \cdot p \geq n$ is satisfied, the following hold true:*

i) There always exist a degenerate compensator $D = rowspan\,(A, K)$ such as the matrix, L_D, has full rank.

ii) Every closed-loop polynomial of an appropriate degree (n) can be approximately assigned by sequences of feedback controllers converging to a degenerate gain.

iii) A generic closed-loop polynomial of an appropriate degree (n) can be exactly assigned by a real constant output feedback compensator.

A distinct advantage of the GL framework is that it permits the calculations of feedback compensators as solutions of a simple linear set of equations and allows the parametrisation of all such solutions based on the (restricted) system Plücker matrix, L_D, and another matrix associated with the degenerate point. The numerical aspects of this scheme such as sensitivity, robustness and limitation of high gains have been examined with various numerical schemes based on the GL framework aiming to improve significantly the sensitivity properties of the solutions. The numerical methods are based: (a) on a predictor-corrector scheme (Leventides et al., 2014b), and (b) on a modified quasi-Newton method (Leventides et al., 2014a). The iterative numerical schemes have been applied to the output feedback pole placement problem. The framework applied here for the output feedback pole placement problem may also be extended to the other DAP variants, such as dynamic (Leventides and Karcanias, 1998b) and decentralised (Karcanias and Leventides, 2005).

7. Decentralised DAP and Selection of the Decentralisation

We specialize now the previous results on the centralized DAP, to the case of the structured frequency assignment problems (decentralised control problems) and we review the main results on the structural characteristics and diagnostics for the selection of the possible decentralisation schemes. Central to this approach is the notion of decentralisation characteristic, which expresses the effect of decentralisation on the design problem and the resulting structural invariants that predict properties of the decentralised control schemes (Karcanias et al., 1988), (Karcanias and Leventides, 2005), such as fixed and almost-fixed modes.

7.1. The decentralised pole assignment problem

We consider linear systems described by a proper transfer function matrix $G(s) \in \mathbb{R}(s)^{m\times p}$ of McMillan degree n. We assume that we have a $k-$channel decentralisation scheme, where $k \leqslant \min(m, p)$, defined by the $k-$partition of the input, output vectors $\underline{u} \in \mathbb{R}^p$ and $\underline{y} \in \mathbb{R}^m$. For a given pair (m, p) we also define the sets of integers, introduced by partitioning of m, p as:

$$\{m\} = \{m_i,\ m_i \geqslant 1,\ \sum_{i=1}^{k} m_i = m\}$$

$$\{p\} = \{p_i,\ p_i \geqslant 1,\ \sum_{i=1}^{k} p_i = p\}$$

where it is also assumed that $m_i \geqslant p_i,\ \forall i \in \tilde{k}$. The set $\mathcal{I}_D = (\{m\}, \{p\}; k)$ will be called a *decentralisation index*. If local feedback laws of the following type

$$\underline{u}_i(s) = C_i(s) \cdot \underline{y}_i(s)$$

are applied to each channel, $i = 1, 2, \ldots, k$, then the closed loop transfer function is $G(s)[I + C(s)G(s)]^{-1}$, with $C(s) = diag\{C_1(s), \ldots, C_k(s)\} \in \mathbb{R}^{p\times m}(s)$, $C_i(s) \in \mathbb{R}^{p_i\times m_i}(s)$ representing the controller. The closed loop pole polynomial is

$$p(s) = det\left\{[A(s), B(s)] \cdot \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}\right\} = det\{H(s) \cdot M(s)\}$$

where, $A(s)^{-1}B(s)$ is a left coprime MFD for $C(s)$, $N(s)D(s)^{-1}$ is a right coprime MFD of $G(s)$ and $M(s) = [D^t(s), N^t(s)]^t$, $H(s) = [A(s), B(s)]$ are the composite descriptions of $G(s)$ and $C(s)$ respectively. The structured controller matrix $H(s)$ can be written as $\left[\bar{A}(s); \bar{B(s)}\right]$, where $\bar{A}(s) = bl.diag\{A_1(s), \ldots, A_k(s)\}$ and $\bar{B}(s) = bl.diag\{B_1(s), \ldots, B_k(s)\}$. By a simple reordering of the blocks the following problems are defined:

Problem 1 (Dynamic Dec. Pole Assignment). Given an arbitrary set of poles by $p(s)$, solve the equation

$$\begin{aligned} p(s) &= det\left\{bl.diag\,[H_1(s), \ldots, H_k(s)] \cdot \begin{bmatrix} M_1(s) \\ \vdots \\ M_k(s) \end{bmatrix}\right\} \\ &= det\,\{H_{dec}(s) \cdot M_{dec}(s)\} \end{aligned} \tag{44}$$

with respect to the decentralised controller $H_{dec}(s)$, where $H_i(s) = [A_i(s), B_i(s)]$ and $M_i(s) = \left[D_i^t(s), N_i^t(s)\right]^t$.

Problem 2 (Constant Dec. Pole Assignment). Given an arbitrary polynomial $p(s)$, solve the equation

$$p(s) = \det\left\{\left[I_p; H_{dec}\right] \cdot M_{dec}(s)\right\} \tag{45}$$

with respect to the constant structured matrix $\left[I_p; H_{dec}\right]$, where $H_{dec} = \text{bl.diag}\,(H_1, \ldots, H_k)$.

7.2. Parameterisation of decentralised degenerate compensators

The selection of a decentralisation scheme is a problem that has not been properly addressed as a structural design and control theory issue with the exception of the graph methodologies (Siljak, 1991). DAP can provide an algebraic framework for selection of the desirable decentralisation (Karcanias et al., 2016a) aiming at developing schemes that allow the satisfaction of generic solvability conditions and shaping the parametric invariants linked to solvability of decentralised control problems The structural indicators suggest the desirable values of inputs, outputs and their partitioning. The results of the exterior algebra framework provide the means for simple tests for avoiding fixed modes (Karcanias et al., 1988), whereas the link of Plücker matrices to decentralised Markov parameters (Leventides and Karcanias, 1998a) allow the linking of the algebraic invariants to state space design. The overall philosophy aims to devise methods for design, or redesign the system in order to facilitate the solvability of decentralised control problems. Amongst the specific problems considered are: (i) Define the desirable cardinality of input, output structures to permit satisfaction of generic solvability conditions, (ii) Design the structure of input, output maps (matrices B, C) to eliminate the existence of fixed modes and guarantee full rank properties to the decentralised Plücker matrices (Leventides and Karcanias, 2006). By extending the results from the centralised DAP case for degenerate feedback gains, we have:

Definition 3. A decentralised controller $H_{dec}(s)$ is *degenerate* if the closed loop system is not well posed, i.e

$$det\{H_{dec}(s) \cdot M_{dec}(s)\} \equiv 0 \tag{46}$$

The existence of dynamic decentralised degenerate gains (DDG) are given below (Leventides and Karcanias, 2006). Let us denote by $\mathcal{M} = col.span\{M(s)\}$.

Proposition 19 ((Leventides and Karcanias, 2006)). *A polynomial matrix $H_{dec}(s) = bl.diag\,[H_1(s), \cdots, H_k(s)]$ corresponds to a degenerate compensator of the feedback configuration, if and only if, either of the following equivalent conditions holds true:*

(i) There exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s) \in \mathcal{M}$, such that, $H_{dec}(s) \cdot \underline{m}(s) = 0,\ \forall s \in \mathbb{C}$.

(ii) There exists an $(m+p)\times 1$ polynomial vector $\underline{m}(s) \in \mathcal{M}$, which if partitioned (conformally with the decentralised controller) into the set of $(m_i + p_i) \times 1$ polynomial vectors, we have that, $H_i(s) \cdot m_i(s) = 0$ where, $\underline{m}_i(s) \in \mathcal{M}_i,\ i = 1, \ldots, k$.

For constant structured matrices we may define for any given $\mathcal{I}_D$, the corresponding composite output feedback constant decentralised gain as

$$\left[I_p; H_{dec}\right] = \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & H_1 & 0 & 0 & 0 \\ 0 & I_{p_2} & 0 & 0 & 0 & H_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & I_{p_k} & 0 & 0 & 0 & H_k \end{bmatrix}$$

where, $H_i \in \mathbb{R}^{p_i \times m_i},\ \forall i \in \tilde{k}$. For a generator $\underline{m}(s) \in \mathcal{M}$ and a decentralisation index $\mathcal{I}_D$, $\underline{m}^*(s)$ denotes the corresponding permuted vector. The family of all generators that lead to degenerate gains is denoted by $\mathcal{D}$.

Theorem 20. *For a system with dimensions (n, m, p) and decentralisation index $\mathcal{I}_D = (\{m\}, \{p\}; k)$, let $\underline{m}(s) \in \mathcal{D}$ and denote by $\underline{m}^*(s) = P^* \underline{e}_\delta(s) \in \mathcal{M}^*$ the corresponding permuted generator vector and let us consider $P^* \in \mathbb{R}^{(p+m)\times(\delta+1)}$ partitioned into k−blocks, according to $\mathcal{I}_D$, as indicated below:*

$$P^* = \begin{bmatrix} P_1 \\ \vdots \\ P_i \\ \vdots \\ P_k \end{bmatrix} \begin{matrix} \updownarrow p_1 + m_1 \\ \\ \updownarrow p_i + m_i \\ \\ \updownarrow p_k + m_k \end{matrix} \tag{47}$$

If $\mathcal{L}_{\underline{m}^*}$ is the $\underline{m}^*(s)$−DDG family, then $\mathcal{L}_{\underline{m}^*}$ contains decentralised gains with $\mathcal{I}_D$−characteristic if and only if, $m_i \geqslant rank(P_i),\ \forall i \in \tilde{k}$. This family is defined by

$$\mathcal{L}_{\underline{m}^*} = \{H_{dec} : H_{dec} = bl.diag\{\cdots; H_i; \cdots\} :\ H_i P_i = 0\}$$

where, $rank(H_i) = P_i,\ \forall i \in \tilde{k}$.

Proposition 21. *Given that $rank\{P_i\} \leqslant rank\{P\} \leqslant \delta + 1$ where, $\delta = \partial\{m(s)\}$, a sufficient condition for the existence of a decentralised degenerate gain in $\mathcal{L}_{\underline{m}^*}$, or equivalently $\mathcal{L}_{\underline{m}}$, is that:*

$$m_i \geqslant \delta + 1,\ \forall i \in \tilde{k}. \tag{48}$$

Obviously, the smaller the degree of $\underline{m}(s)$, easier it is to find decentralised degenerate solutions. The presence of decentralized elements are implied by the Gain Degeneracy Set $< \mathcal{L} >$. For the case where $m \geqslant p$ and for a given generator vector, $\underline{m}(s) \in \mathcal{D}$, the conditions for the set $\mathcal{L}_{\underline{m}}$ to contain at least one decentralised element are given in (Karcanias et al., 2016a).

7.3. The set of Structurally Compatible Partitions

For any generator vector $\underline{m}(s) \in \mathcal{D}$ corresponding to a system with dimensions (n, p, m) and with $r = rank\{P\}$, the existence of a set of non-trivial compatible partitions $(k \neq 1)$ $\mathcal{I}_D$−CP, which are independent from the numerical values of the corresponding partitioned matrices P_i, is given by the following result.

Proposition 22. *Let $\underline{m}(s) = P \cdot \underline{e}_\delta(s) \in \mathcal{D}$ be a system with dimensions (n, p, m), $r = rank\{P\} \leqslant \delta + 1$ and let $\bar{k}$ be the integer defined by $\bar{k} = \max\{k \in \mathbb{Z}_{>0} : k \leqslant m/r\}$. If $\bar{k} \geqslant 2$, then for any $k : 2 \leqslant k \leqslant \bar{k}$ there exist $\mathcal{I}_D$−CP, $\mathcal{I}_D = (\{m\}, \{p\}; k)$, defined by certain k−partitions of m, p and satisfying the following conditions:*

$$m_i \geqslant r, \quad m_i \geqslant p_i,\ \forall i = 1, 2, \ldots, k. \tag{49}$$

Such a set will be denoted by $\{\mathcal{I}_D; \underline{m}\}$ and referred to as the set of *Structurally Compatible Partitions* (SCP) of $\underline{m}(s)$, and this description does not depend on the values of elements of P, but only on (m, p, r) numbers.

Theorem 23. *For every system with dimensions (n, p, m) and any generator $\underline{m}(s) = P \cdot \underline{e}_{\delta}(s) \in \mathcal{D}$ with $r = rank\{P\} \leqslant \delta + 1$, the set of all Structurally Compatible Partitions of the (m, p) pair is given by*

$$\{m, p; r\} = \mathfrak{X}(m, p; r) = \bigcup_{k=1}^{k=\bar{k}} [\mathfrak{X}(m), \mathfrak{X}(p)] \tag{50}$$

The study of properties on a given $\underline{m}(s) \in \mathcal{D}$ depends only on its degree, rank and (p, m) number and not on n, from which the only numerically dependent parameter is r. Given that $r \leqslant \delta + 1$, a numerically independent subset of $\{m, p; r\}$, is the set $\{m, p; \delta + 1\}$.

8. Exact and Approximate Solutions of DAP

A direct solution to the computation of exact, as well as approximate solutions of DAP, has been proposed recently in Leventides et al. (2014c), Karcanias and Leventides (2015). The exact DAP is to find a decomposable l-vector $\underline{k}^t$ that satisfies (20) and is an intersection problem between a linear variety and the Grassmann variety. In the approximate DAP (which is addressed when the exact problem is not solvable) we aim to minimise the distance between the linear variety defined by (20) and the Grassmann variety of all decomposable vectors. This new approach is based on a linear algebra type, criterion for decomposability of multivectors stems from the properties of the Grassmann matrices.

8.1. The Grassmann and Hodge-Grassmann matrices and the canonical representation of multivectors

The Hodge-Grassmann matrix
The Hodge-Grassmann matrix is the Grassmann matrix of the Hodge dual of the multivector $\underline{z}$ and its properties are dual to those of the Grassmann matrix. In fact decomposability turns out to be an image problem for the transpose of the Hodge-Grassmann matrix (Karcanias and Leventides, 2015).

Definition 4. The Hodge $*-$operator, for a oriented n-dimensional vector space $\mathcal{U}$ equipped with an inner product $< .,. >$, is an operator defined as: $* : \wedge^m\mathcal{U} \to \wedge^{n-m}\mathcal{U}$ such that $\underline{a} \wedge \left(\underline{b*}\right) = \langle \underline{a}, \underline{b} \rangle \underline{w}$ where $\underline{a}, \underline{b} \in \wedge^m\mathcal{U}$, $\underline{a} \in \wedge^n\mathcal{U}$ defines the orientation on $\mathcal{U}$ and $m < n$.

Definition 5. The Hodge-Grassmann matrix of a multivector $\underline{z}, \underline{z} \in \wedge^m\mathcal{U}, \underline{z} \neq 0$ is the Grassmann matrix of the Hodge dual of $\underline{z}, \underline{z}^*$, i.e. it is the matrix $\Phi_n^{n-m}(\underline{z}^*)$ representing the linear map $\wedge^R_{\underline{z}^*} : \mathcal{U} \to \wedge^{n-m+1}\mathcal{U}$ defined as the representation of: $\wedge^R_{\underline{z}^*}(\underline{u}) = \underline{u} \wedge \underline{z}^*, \forall \underline{u} \in \mathcal{U}$.

A procedure to calculate the Hodge star of a multivector in $\wedge^m\mathcal{U}$ and the main properties of the Hodge-Grassmann matrix of a multivector $\underline{z}$ are given in Karcanias and Leventides (2015).

Proposition 24. *For any $\underline{z} \in \wedge^m\mathcal{U}$ the following are equivalent: (i) $\underline{z}$ is decomposable; (ii) $\underline{z}^*$ is decomposable.*

Furthermore, the following statements hold true:

(i) $dim\mathcal{N}_r\{\Phi_n^{n-m}(\underline{z}^*)\} \leqslant n - m$ equality holding iff $\underline{z}$ is decomposable.

(ii) $dimrowspan\{\Phi_n^{n-m}(\underline{z}^*)\} \geqslant m$ equality holding iff $\underline{z}$ is decomposable.

Theorem 25. *(a) For $\underline{z} \in \wedge^m\mathcal{U}$, $\underline{z} \neq 0$ the matrix $\Phi_n^m(\underline{z})^T$ is the representation of the map ${}^T\wedge^R_{\underline{z}} : \wedge^{m+1}\mathcal{U} \to \mathcal{U}$ given by:*

$${}^T\wedge^R_{\underline{z}}(\underline{y}) = (-1)^{n-1}(\underline{z} \wedge \underline{y}^*)^* \text{ ,where } \underline{y} \in \wedge^{m+1}\mathcal{U}$$

b) The matrix $\Phi_n^{n-m}(\underline{z}^*)^T$ is the representation of the map ${}^T\wedge^R_{\underline{z}^*} : \wedge^{n-m+1}\mathcal{U} \to \mathcal{U}$ given by:

$${}^T\wedge^R_{\underline{z}^*}(\underline{y})) = (-1)^{n-1}(\underline{z}^* \wedge \underline{y}^*)^* \text{ ,where } \underline{y} \in \wedge^{m+1}\mathcal{U}$$

The above lead to a new test for decomposability in terms of the Grassmann and Hodge-Grassmann matrices (Karcanias and Leventides, 2015):

Theorem 26. *For any $\underline{z} \in \wedge^m\mathcal{U}$ the following conditions are equivalent:*

i) $\underline{z}$ is decomposable

ii) $\Phi_n^m(\underline{z}) \cdot (\Phi_n^{n-m}(\underline{z}^*))^T = \underline{0} \in \mathbb{R}^{\binom{n}{m+1} \times \binom{n}{n-m+1}}$

Theorem 27. *Let $\underline{z} \in \wedge^m\mathcal{U}$, then the following holds true*

$$\mathcal{N}_r\left\{\Phi_n^m(\underline{z})\right\} \subseteq rowspan\left\{\Phi_n^{n-m}(\underline{z}^*)\right\} = R\left\{\Phi_n^{n-m}(\underline{z}^*)^T\right\}$$

Two fundamental spaces associated with $\underline{z}$ are

$$\mathcal{D}_1(\underline{z}) = \mathcal{N}_r(\Phi_n^m(\underline{z})) \text{ with } d_1(\underline{z}) = \dim \mathcal{N}_r(\Phi_n^m(\underline{z}))$$
$$\mathcal{D}_2(\underline{z}) = \mathcal{R}(\Phi_n^{n-m}(\underline{z}^*)^T) \text{ with } d_2(\underline{z}) = \dim \mathcal{R}(\Phi_n^{n-m}(\underline{z}^*)^T)$$
$$\{0\} \subseteq \mathcal{D}_1(\underline{z}) \subseteq \mathcal{D}_2(\underline{z}) \subseteq \mathcal{U}$$

where, $0 \leqslant d_1(\underline{z}) \leqslant d_2(\underline{z}) \leqslant m$.

Theorem 28. *For a $\underline{z} \in \wedge^m\mathcal{U}$ we have: (i) Let $\{\underline{u}_1, \ldots, \underline{u}_{d_1}\}$ be a basis for $\mathcal{D}_1(\underline{z})$ then $\underline{z}$ can be written as $\underline{z} = \underline{u}_1 \wedge \ldots \wedge \underline{u}_{d_1} \wedge \underline{z}_1$; (ii) $\underline{z} \in \wedge^m\mathcal{D}_2(\underline{z})$.*

Corollary 29. *If* $\{\underline{u}_1, \ldots, \underline{u}_{d_1}\}$ *is a basis for* $\mathcal{D}_1(\underline{z})$*, then the multivector* $\underline{z}$ *can be represented as* $\underline{z} = \underline{u}_1 \wedge \ldots \wedge \underline{u}_{d_1} \wedge \underline{z}_1$ *where,* $\underline{z}_1 \in \wedge^{m-d_1}\mathcal{D}_3(\underline{z})$ *, where* $\mathcal{D}_3(\underline{z})$ *is the orthogonal complement of* $\mathcal{D}_1(\underline{z})$ *in* $\mathcal{D}_2(\underline{z})$*.*

A fundamental relationship between the singular vectors and the singular values of the Grassmann and Hodge-Grassmann matrices is given by (Karcanias and Leventides, 2015):

Theorem 30. *For any* $\underline{z} \in \wedge^m\mathbb{R}^n$ *the following holds true*

$$\Phi_n^m(\underline{z})^T\Phi_n^m(\underline{z}) + \Phi_n^{n-m}(\underline{z}^*)^T\Phi_n^{n-m}(\underline{z}^*) = \left\|\underline{z}\right\|^2 I_n$$

Corollary 31. *(i) The vector* $\underline{z} \in \wedge^m\mathbb{R}^n$ *is decomposable iff the matrix* $\Phi_n^m(\underline{z})$ *has m singular values equal to* 0 *and* $n-m$ *singular values equal to* $\left\|\underline{z}\right\|$*.*
(ii) The vector $\underline{z} \in \wedge^m\mathbb{R}^n$ *is decomposable iff the matrix* $\Phi_n^{n-m}(\underline{z}^*)$ *has* $n-m$ *singular values equal to* 0 *and m singular values equal to* $\left\|\underline{z}\right\|$*.*
(iii) The vector $\underline{z} \in \wedge^m\mathbb{R}^n$ *is decomposable iff*

$$\mathcal{N}_r\{\Phi_n^m(\underline{z})\} = colspan\{\Phi_n^{n-m}(\underline{z}^*)^T\} = span\left(\underline{x}_1, \ldots, \underline{x}_m\right)$$

where, $\{\underline{x}_1, \ldots, \underline{x}_m\}$ *are the right singular vectors of the Grassmann matrix corresponding to its* 0 *singular value, or the right singular vectors of the Hodge-Grassmann matrix corresponding to its singular value that equals to* $\left\|\underline{z}\right\|$*.*

8.2. The solution of the exact and approximate DAP

As described in Section 3.2, DAP can be decomposed into a linear and a multilinear problem. Assume that $a(s) = \underline{a}^t\underline{e}(s)$, $\underline{e}(s)^t = \left[1, s, \ldots, s^d\right]$, is the polynomial to be assigned, where d is the degree of $a(s)$. Let A be a right annihilator matrix of $\underline{a}^t$ (i.e. $\underline{a}^tA = 0$), then (20) may be expressed as

$$\underline{k}^tPA = \underline{0}$$

If V is an orthonormal basis matrix for the left kernel of PA, then $\underline{k}^t$ equals to $\underline{k}^t = \underline{x}^tV$, $V \in \mathbb{R}^{p\times q}$, where p is the dimension of the left kernel of PA. Thus, for $\underline{k}^t$ to be decomposable, or to be the closest to decomposability, we require that either

a) the QPRs are exactly zero, that is,

$$\Phi_m^l(\underline{k}) \cdot \Phi_{m-l}^l(\underline{k}^*)^T = \underline{0}$$

b) the square norm of the QPRs is minimum, that is, minimise

$$\left\|\Phi_m^l(\underline{k}) \cdot \Phi_{m-l}^l(\underline{k}^*)^T\right\|$$

Therefore, for both exact and approximate DAP, the following optimisation problem has to be solved

Problem 3. $\min \left\|\Phi_m^l(\underline{k}) \cdot \Phi_{m-l}^l(\underline{k}^*)^T\right\|$ subject to, $\underline{k}^t = \underline{x}^tV$ and $\left\|\underline{x}\right\| = 1$. □

which can be rewritten as a maximisation problem

Problem 4. $\max tr\left(\Phi_m^l(\underline{x}^tV)^T \cdot \Phi_m^l(\underline{x}^tV)\right)^2$ subject to, $\left\|\underline{x}\right\| = 1$. □

The objective function of the new optimisation problem is a homogeneous polynomial in p variables $\underline{x} = \left(x_1, x_2, \ldots, x_p\right)$ under the constraint $\left\|\underline{x}\right\| = 1$. This is a nonlinear maximisation problem which can be solved using standard optimisation methods and algorithms. An iterative method resembling the power method (Kolda and Mayo, 2011) for finding the largest modulus eigenvalue and its corresponding eigenvector of a matrix that solves the above problem, is suggested in (Karcanias and Leventides, 2015).

Iterative method for computing solutions: Let Φ be the matrix

$$\Phi = \cdot\Phi_m^l(\underline{x}^tV) \cdot \Phi_m^l(\underline{x}^tV)^T = \begin{pmatrix} & \vdots & \\ \cdots & \phi_{ij}(\underline{x}) & \cdots \\ & \vdots & \end{pmatrix}$$

where, $\phi_{ij}(\underline{x}) = \underline{x}^tA_{ij}\underline{x}$ a quadratic function in x. Then, the objective function is $tr(\Phi)^2 = \sum_{i,j=1}^m \phi_{ij}^2(\underline{x})$ and the Lagrangian of the problem is given by

$$\mathcal{L}(\underline{x}, \lambda) = \sum\nolimits_{i,j=1}^m \phi_{ij}^2(\underline{x}) - \lambda\left(\left\|\underline{x}\right\|^2 - 1\right) \tag{51}$$

The first-order conditions can be expressed as a nonlinear eigenvalue problem defined by

$$A(\underline{x}) \cdot \underline{x} = \frac{\lambda}{2}\underline{x} \tag{52}$$

where, $A(\underline{x})$ is the $p \times p$ matrix, $A(\underline{x}) = \sum_{i,j=1}^m \phi_{ij}(\underline{x})A_{ij}$. The solution of the problem can be found by applying the following iteration for $N = 0, 1, 2, \ldots, N_{max}$:

$$\underline{x}_{N+1} = A(\underline{x}_N)\cdot\underline{x}_N \Big/ \left\|A(\underline{x}_N) \cdot \underline{x}_N\right\| \tag{53}$$

The stopping criteria are $\left\|\underline{x}_{N+1} - \underline{x}_N\right\| < \varepsilon$. □
The iterative method described above can be applied to both exact and approximate DAP.

Example 5. Consider a system with $p = 3$ inputs, $m = 3$ outputs and $n = 7$ states given by:

$$M(s)^T = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}^T =$$
$$= \begin{bmatrix} s^3 & s^2 & s & s+1 & 1 & 1 \\ 0 & s^2+1 & s^2-s-2 & 2s+1 & s & 1 \\ 0 & 0 & s^2 & s-1 & s-3 & 1 \end{bmatrix}$$

The open-loop system is not BIBO stable since it has 5 poles at $s = 0$ and 2 poles at $s = \pm j$. We would like to place its poles at $(-1, -2, ..., -7)$ and we are seeking an output feedback $K \in \mathbb{R}^{3\times3}$ such that

$$\det([I_3, K] \cdot M(s)) = (s+1)(s+2)\cdots(s+7) = a(s)$$

By applying the Binet-Cauchy theorem we get $k^t P = \underline{a}^t$, with $k^t \in \mathbb{R}^{20}$, $P \in \mathbb{R}^{20\times8}$ where $\underline{a}^t$ is the coefficient vector of $a(s)$. The solution of the linear problem is of the form

$$\underline{k}^t = \underline{x}^t V, \underline{x}^t \in \mathbb{R}^{13}, V \in \mathbb{R}^{13\times20}$$

Starting from an appropriate selected vector $\underline{x}_0 \in \mathbb{R}^{13}$, we apply the iteration (53) and after a sufficiently large number of iterations we stopped when the value of the objective function becomes $(m + p) - p = 6 - 3 = 3$, that indicates exact pole placement. The final solution is given by the decomposable vector $\underline{k}^t$ which gives rise to the feedback controller

$$K = \begin{bmatrix} -958.381 & 1309.17 & -117.214 \\ 239.588 & -326.091 & 29.119 \\ 576.064 & -786.652 & 70.971 \end{bmatrix}$$

9. Open Problems and Suggestions for Further Research

The development of structural methodologies for linear systems has many open challenges, however this paper has focused to those which may be seen through the algebro-geometric DAP framework. The DAP approach for the solution of frequency assignment problems has provided a new set of system invariants in terms of the Grassmann vectors and Plücker matrices, new solvability conditions and a computational framework based on the GL methodology. Furthermore, it has also provided a methodology for computing approximate solutions when exact solutions cannot be found. The framework is by no means complete and a number of open issues remain which define a research agenda for the future. Areas for future research within the DAP framework deal with advanced control design and areas of system structural synthesis as considered below.

Advanced Control Design based on DAP

The general framework of DAP has already being developed but some key problems need further consideration by using results from other control and mathematics areas. The main areas of research include:

Selection of best degenerate compensators for Global Linearisation of DAP: The overall methodology of GL is based on defining a degenerate compensator. In general, there is no unique degenerate compensator. The problem of classifying such compensators for the constant DAP case has been considered in (Karcanias et al., 2013) but the extension to dynamic compensation remains open.

Least sensitivity solutions of the Global Linearisation Methodology: An inherent feature of the GL is its sensitivity. Alternative methods for overcoming the sensitivity for a given degenerate compensator have been considered in (Leventides and Karcanias, 1996), (Leventides et al., 2014a). Improving the sensitivity of the method for a given degenerate compensator is an area worth investigating. A major challenge is the study of sensitivity for the different degenerate compensators and finding the least sensitivity solution. Such an investigation may be also extended to dynamic degenerate compensators, which may offer reduction to sensitivity.

Robustness of solutions under model uncertainty: The study of DAP so far has assumed a fixed model for the system. Transforming the DAP from synthesis to a design methodology requires handling the problem of model uncertainty. This problem is open and involves as important sub-problems the effect of model uncertainty on (i) the Grassmann and Plücker invariants; (ii) the family of degenerate compensators (crucial for GL); (iii) the linear variety of DAP. The latter implies linking model uncertainty to the family of resulting linear varieties and then studying DAP for such families. This area requires enriching the algebraic framework by using results from robustness and system properties with their appropriate modification in the context of the DAP formulation.

Approximate solutions of families of Dynamic DAP: A framework for finding approximate solutions of DAP has been developed in (Karcanias and Leventides, 2015). The solution is based on an iterative method resembling the power method (Kolda and Mayo, 2011) for finding the largest modulus eigenvalue and its corresponding eigenvector of a matrix that solves the above problem iteratively. The power method may be applied using a shifted variant, as in the symmetric case, which guarantees convexity and hence convergence of the method. These developments imply use of results from optimization, numerical computations and approximation and express another enrichment of the original algebraic framework. Development of techniques that avoid the iterative nature is an additional challenge.

DAP and the stabilisation problem: Frequency assignment may also guarantee stabilisation but it is rather restrictive. There exist some stabilisation results for pole assignment (Byrnes, 1983), but the stabilisation version of DAP takes a different form. In this case we deal with semi-algebraic sets (stability domain) and the linear variety of DAP becomes a semi-algebraic set and this is a study in the field of semi-algebraic geometry. This is an open field for further development of DAP.

Partial Decomposability of multivectors and restricted DAP: The study of DAP assumes that the design parameter is entirely free. However, in many practical cases parts of the design matrix (a row, or a number of them) are fixed. This is equivalent to decomposing a multivector as a product of lower dimensional multivectors. In case that this is not possible we examine the problem of approximate partial decomposability. This introduces a new dimension to the study of DAP which is relevant to design problems.

System Structural Synthesis

Network Re-engineering and DAP: The problem of redesigning autonomous (no inputs or outputs) passive electric networks (Karcanias et al., 2014) aims to change the network (natural frequencies) by modification of the types of elements, possibly their values, interconnection topology and possibly addition, or elimination of parts of the network. This problem differs considerably from a standard control problem, but may be reduced to an equivalent DAP problem where the topology of the network, types and values of physical elements become design parameters. A new family of DAP problems may be introduced when the system cardinality changes. In this case the DAP system operator described by the impedance or admittance matrix may be expanded, or reduced.

Parametrisation of Decentralized DAP schemes: The selection of a decentralization scheme has not been properly addressed as a design issue and has been handled mostly using process heuristics, and conditions derived from the spatial arrangement of sub-process units. The only exception is the use of Graph theory, which however has not developed to a systematic methodology for selection and parameterization of decentralized structures. This problem may be considered within the framework of structural methodologies for linear systems (Siljak, 1991), (Leventides and Karcanias, 1998a), (Karcanias and Leventides, 2005). Addressing the design of decentralisation schemes in order to guarantee solvability of families of control problems and exclude undesirable characteristics, such as fixed modes, requires a systematic methodology for synthesis of decentralisation schemes (Karcanias and Leventides, 2005) based on structural methodologies. DAP based criteria may be deployed on existing generic solvability conditions and on the DAP diagnostics (Plücker matrices, decentralized Markov parameters) linked to solvability conditions and avoidance of fixed modes. The parametrisation of degenerate compensations for decentralized schemes (Karcanias et al., 2016b) together with the DAP invariants and their properties also provides a possible route for the classification of decentralization schemes with good control potential.

Minimal design problem: Dynamic compensation problems may be reduced to constant DAP problems. Amongst the open issues in the area of dynamic frequency assignment problems, is defining the least complexity compensator (this is frequently defined by the McMillan degree), for which we may have solvability of the arbitrary spectrum assignment of the corresponding DAP. This is referred to as the minimal design DAP problem (Karcanias and Galanis, 2010) and it has been based on the linear sub-problem of DAP by using the properties of dynamic polynomial combinants. These results define lower bounds to the minimal design problem since the approach has ignored the decomposability of multivectors constraints. Defining upper bounds for dynamic compensators solving DAP and parameterising such families based on their fixed McMillan degree are challenging open issues.

Synthesis of Grassmann invariants: The solvability of DAP depends on the properties of the Grassmann vectors and their respective Plücker matrices. These invariants are functions of the state space parameters. Although the state matrix may be fixed the input and output matrices may be considered as design parameters. Defining such schemes to well condition the solvability of a variety of constant, or dynamic centralized or decentralized DAP problems introduces new challenges for the DAP methodology. Within this family of problems we can consider the problem of reducing the bounds on the minimal design. Of course, some of the design parameters may be fixed and such issues may be considered within the framework partial decomposability considered before. For the case of decentralised control the selection of decentralised Markov parameters is linked to the design of the matrices C, and B of the state space model.

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