Quasi-exactly solvable quantum systems with explicitly time-dependent Hamiltonians

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Abstract: For a large class of time-dependent non-Hermitain Hamiltonians expressed in terms linear and bilinear combinations of the generators for an Euclidean Lie-algebra respecting different types of PT-symmetries, we find explicit solutions to the time-dependent Dyson equation. A specific Hermitian model with explicit time-dependence is analyzed further and shown to be quasi-exactly solvable. Technically we constructed the Lewis-Riesenfeld invariants making use of the metric picture, which is an equivalent alternative to the Schrödinger, Heisenberg and interaction picture containing the time-dependence in the metric operator that relates the time-dependent Hermitian Hamiltonian to a static non-Hermitian Hamiltonian.

1. Introduction

Quasi-exactly solvable (QES) quantum systems are characterized by the feature that only part of their infinite energy spectrum and corresponding eigenfunctions can be calculated analytically. Systematic studies of such type of systems have been carried out by casting them into the form of Lie algebraic quantities [1, 2] and making use of the property that the eigenfunctions of the corresponding Hamiltonian systems form a flag which coincides with the finite dimensional representation space of the associated Lie algebras. QES systems that can be cast into such a form are usually referred to as QES models of Lie algebraic type [3, 4]. The relevant underlying algebras are either of $sl_2(\mathbb{C})$-type, with their compact and non-compact real forms $su(2)$ and $su(1,1)$, respectively [5], or of Euclidean Lie algebras type [6, 7, 8]. The latter class was found to be particularly useful when dealing with certain types of non-Hermitian systems.

While many QES models have been studied in stationary settings, little is known for time-dependent systems. So far a time-dependence has only been introduced into the eigenfunctions in form of a dynamical phase [9, 10]. However, no QES systems with explicitly time-dependent Hamiltonians have been considered up to now. The main purpose of this article is to demonstrate how they can be dealt with and to initiate further studies of such type of systems. We provide the analytical solutions to a QES Hamiltonian quantum system with explicit time-dependence. As a concrete example we consider QES systems of E_2-Lie algebraic type. Technically we make use of the metric picture [11, 12], which is an alternative to the Schrödinger, Heisenberg and interaction picture. It will allow us to solve a Hermitian time-dependent Hamiltonian system by solving first a static non-Hermitian system as an auxiliary problem with a time-dependence in the metric operator.

Systems build up from Euclidean Lie algebras, in particular of E_2, have a wide range of physical applications. They have been employed for instance in the formal quantisation of strings on tori [13]. Depending on the chosen representation of the algebra one can describe a large number of concrete physical systems. Common representations for E_2 may lead to two dimensional systems or most commonly in optical settings, the trigonometric representation, see below, correspond to Mathieu potentials and variations thereof. The latter have proven be useful and accurate in the decription of energy band structures in crystals [14] and especially in the experimental and theoretical study of optical solitons [15, 16, 17, 18, 19, 20]. Here we consider explicitly time-dependent versions of these type of systems and keep our discussion generic, that is independent of the choice a concrete representation for the underlying algebra.

The Hermitian Hamiltonian systems we study here are of the general form

$$h(t) = \mu_{JJ}(t)J^2 + \mu_J(t)J + \mu_u(t)u + \mu_v(t)v + \mu_{uu}(t)u^2 + \mu_{vv}(t)v^2 + \mu_{uv}(t)uv, \qquad (1.1)$$

where the time-dependent coefficient functions μ_i, $i \in \{J, JJ, u, v, uu, vv, uv\}$, are real and u, v and J denote the three generators that span the Euclidean-algebra E_2. They obey the commutation relations

$$[u, J] = iv, \qquad [v, J] = -iu, \qquad \text{and} \qquad [u, v] = 0. \qquad (1.2)$$

Considering here only Hermitian representations with $J^\dagger = J$, $v^\dagger = v$ and $u^\dagger = u$, the Hamiltonian in equation (1.1) is clearly Hermitian. Standard representation are for instance the trigonometric representation $J := -i\partial_\theta$, $u := \sin\theta$ and $v := \cos\theta$ or a two-dimensional representation $J := yp_x - xp_y$, $u := x$ or $v := y$ with x, y, p_x, p_y denoting Heisenberg canonical variables with non-vanishing commutators $[x, p_x] = [y, p_y] = i$. We have set here and mostly in what follows to $\hbar = 1$.

We briefly recall from [11, 12] what is meant by the *metric picture*. It is well known that the Schrödinger and the Heisenberg picture are equivalent with the former containing the time-dependence entirely in the states and the latter entirely in the operators. $\mathcal{PT}$-symmetric/quasi-Hermitian systems [21, 22, 23] allow for yet another equivalent variant in which the time-dependence is contained entirely in the metric operator. In order to see that we first need to solve the time-dependent Dyson relation [24, 25, 26, 27, 28, 11, 12, 29, 30, 31] which in general reads

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t), \qquad (1.3)$$

involving a time-dependent non-Hermitian Hamiltonian $H(t) \neq H^\dagger(t)$ and the Dyson operator η related to the metric operator ρ as $\rho = \eta^\dagger\eta$. For our purposes we will eventually take the Hamiltonian to be time-independent $H(t) \to H$, with $h(t)$ satisfying the time-dependent Schrödinger equation $h(t)\phi(t) = i\hbar\partial_t\phi(t)$ and H the time-independent Schrödinger equation $H\psi = E\psi$ with energy eigenvalue E. The corresponding wavefunctions are related as $\phi(t) = \eta(t)\psi$.

Before we solve a concrete system in a quasi-exactly solvable fashion we consider first the fully time-dependent Dyson relation with time-dependent non-Hermitian Hamiltonian $H(t)$ and investigate which type of Hamiltonians can be related to the Hermitian Hamiltonian $h(t)$ in (1.1). We will see that in some cases we are even forced to take $H(t)$ or part of it to be time-independent. As not many explicit solutions to the time-dependent Dyson relation are known, this will be a valuable result in itself.

Our manuscript is organized as follows: In section 2 we explore various types of $\mathcal{PT}$-symmetries that leave the Euclidean E_2-algebra invariant and investigate time-dependent non-Hermitian Hamiltonians in terms E_2-algebraic generators that respect these symmetries. We find new solutions to the time-dependent Dyson relation for those type of Hamiltonians by computing the corresponding Hermitian Hamiltonians and the Dyson map. In section 3 we provide analytical solutions for a concrete model respecting a particular $\mathcal{PT}$-symmetry. We compute the eigenstates of the Lewis-Riesenfeld invariants and the time-dependent Hermitian Hamiltonian in a quasi-exactly solvable fashion. A three-level system is presented in more detail. Our conclusions are stated in section 4.

2. Solutions to the time-dependent Dyson equation for E_2-Hamiltonians

A key property in the study and classification of Hamiltonian systems related to the E_2-algebra are the antilinear symmetries [32] that leave the algebra (1.2) invariant. Given the general context of $\mathcal{PT}$-symmetric/quasi-Hermitian systems we call these symmetries $\mathcal{PT}_i, i = 1, 2, \ldots$ As discussed in more detail in [33, 34], there are many options which all give rise to models with qualitatively quite distinct features. It is easy to see that each of the following antilinear maps leave all the commutation relations (1.2) invariant

$$\begin{array}{lllll} \mathcal{PT}_1: & J \to -J, & u \to -u, & v \to -v, & i \to -i, \\ \mathcal{PT}_2: & J \to -J, & u \to u, & v \to v, & i \to -i, \\ \mathcal{PT}_3: & J \to J, & u \to v, & v \to u, & i \to -i, \\ \mathcal{PT}_4: & J \to J, & u \to -u, & v \to v, & i \to -i, \\ \mathcal{PT}_5: & J \to J, & u \to u, & v \to -v, & i \to -i. \end{array} \tag{2.1}$$

Next we seek non-Hermitian Hamiltonians that respect either of these symmetries. Focussing here on time-dependent Hamiltonians consisting entirely of linear and bilinear combinations of E_2-generators they can all be cast into the general form

$$\begin{aligned} H_{\mathcal{PT}_i}(t) = {} & \mu_{JJ}(t)J^2 + \mu_J(t)J + \mu_u(t)u + \mu_v(t)v + \mu_{uJ}(t)uJ + \mu_{vJ}(t)vJ \\ & + \mu_{uu}(t)u^2 + \mu_{vv}(t)v^2 + \mu_{uv}(t)uv. \end{aligned} \tag{2.2}$$

Demanding that $[H_{\mathcal{PT}_i}(t), \mathcal{PT}_i] = 0$, the symmetries are implemented by taking the coefficient functions to be either real, purely imaginary or relate different functions to each other by conjugation. For the different symmetries in (2.1) we are forced to take

$$\begin{array}{lll} \mathcal{PT}_1: & (\mu_J, \mu_u, \mu_v) \in i\mathbb{R}, & (\mu_{JJ}, \mu_{uJ}, \mu_{vJ}, \mu_{uu}, \mu_{vv}, \mu_{uv}) \in \mathbb{R}, \\ \mathcal{PT}_2: & (\mu_J, \mu_{uJ}, \mu_{vJ}) \in i\mathbb{R}, & (\mu_u, \mu_v, \mu_{JJ}, \mu_{uu}, \mu_{vv}, \mu_{uv}) \in \mathbb{R}, \\ \mathcal{PT}_3: & (\mu_{JJ}, \mu_J, \mu_{uv}) \in \mathbb{R}, & \mu_u = \mu_v^*, \mu_{uJ} = \mu_{vJ}^*, \mu_{uu} = \mu_{vv}^* \\ \mathcal{PT}_4: & (\mu_u, \mu_{uJ}, \mu_{uv}) \in i\mathbb{R}, & (\mu_J, \mu_v, \mu_{JJ}, \mu_{vJ}, \mu_{uu}, \mu_{vv}) \in \mathbb{R}, \\ \mathcal{PT}_5: & (\mu_v, \mu_{vJ}, \mu_{uv}) \in i\mathbb{R}, & (\mu_J, \mu_u, \mu_{JJ}, \mu_{uJ}, \mu_{uu}, \mu_{vv}) \in \mathbb{R}. \end{array} \tag{2.3}$$

Except for very specific combinations of the coefficient functions, the Hamiltonians $H_{\mathcal{PT}_i}(t)$ are non-Hermitian in general.

We now solve the time-dependent Dyson relation (1.3) for $\eta(t)$ by mapping different $\mathcal{PT}_i$-symmetric versions of $H(t)$ to a Hermitian Hamiltonian $h(t)$ of the form (1.1). For the time-dependent Dyson map we make an Ansatz in terms of all the E_2-generators

$$\eta(t) = e^{\tau(t)v} e^{\lambda(t)J} e^{\rho(t)u}. \tag{2.4}$$

At this point we allow $\lambda, \tau, \rho \in \mathbb{C}$, keeping in mind that $\eta(t)$ does not have to be Hermitian. We exclude here unitary operators, i.e. $\lambda, \tau, \rho \in i\mathbb{R}$, as in that case $\eta(t)$ just becomes a gauge transformation. The adjoint action of this operator on the E_2-generators is computed by using the standard Baker-Campbell-Haussdorff formula

$$\eta J \eta^{-1} = J + i\rho\cosh(\lambda)v - [i\tau + \rho\sinh(\lambda)]u, \tag{2.5}$$

$$\eta u \eta^{-1} = \cosh(\lambda)u - i\sinh(\lambda)v, \tag{2.6}$$

$$\eta v \eta^{-1} = \cosh(\lambda)v + i\sinh(\lambda)u. \tag{2.7}$$

The gauge-like term in (1.3) acquires the form

$$i\dot{\eta}\eta^{-1} = i\dot{\lambda}J + \left[i\dot{\rho}\cosh(\lambda) + \tau\dot{\lambda}\right]u + [\dot{\rho}\sinh(\lambda) + i\dot{\tau}]\,v. \tag{2.8}$$

As common, we abbreviate here time-derivatives by overdots. For the computation of the time-dependent energy operator $\tilde{H}(t)$, see below, we also require the term

$$i\eta^{-1}\dot{\eta} = i\dot{\lambda}J + [i\dot{\rho} + \dot{\tau}\sinh(\lambda)]\,u + \left[\rho\dot{\lambda} + i\dot{\tau}\cosh(\lambda)\right]v. \tag{2.9}$$

Using (2.5)-(2.7) we calculate next the adjoint action of η on $H(t)$ and add the expression in (2.8). Demanding that the result is Hermitian will constrain the time-dependent functions $\mu_i(t)$, $\lambda(t)$, $\tau(t)$ and $\rho(t)$. We need to treat each $\mathcal{PT}$-symmetry separately.

2.1 Time-dependent $\mathcal{PT}_1$-invariant Hamiltonians

For convenience we take the coefficient function μ_{JJ} to be time-independent. Of course the general scenario with $\mu_{JJ}(t)$ is also possible to consider, but leads to more cumbersome expressions. For the $\mathcal{PT}_1$-invariant Hamiltonian with coefficient functions as specified in (2.3) we have to be aware that for $\mu_J = \mu_{uJ} = \mu_{vJ} = 0$ the Hamiltonian $H_{\mathcal{PT}_1}(t)$ becomes Hermitian. Substituting the general form for $H_{\mathcal{PT}_1}(t)$ into (1.3), using (2.5)-(2.7), (2.8), reading off the coefficients in front of the generators and demanding that the right hand side becomes Hermitian enforces to take the functions $\lambda, \tau, \rho \in \mathbb{R}$ in (2.4). The resulting Hermitian Hamiltonian is

$$\begin{aligned} h_{\mathcal{PT}_1} &= J^2\mu_{JJ} + \frac{[\mu_{vJ}\tanh\lambda - \mu_J\mu_{vJ}]\sinh\lambda}{2\mu_{JJ}}u - \frac{\mu_J\mu_{uJ}\tanh\lambda\,\mathrm{sech}\,\lambda}{2\mu_{JJ}}v \\ &+ \left(\mu_{uu} - \frac{\mu_{uJ}^2\tanh^2\lambda}{4\mu_{JJ}}\right)u^2 + \left(\mu_{uu} + \frac{\cosh^2(\lambda)\mu_{vJ}^2 - \mu_{uJ}^2}{4\mu_{JJ}}\right)v^2 + \mu_{uv}uv, \\ &+ \frac{\mu_{uJ}}{2}\,\mathrm{sech}\,\lambda\{u, J\} + \frac{\mu_{vJ}}{2}\cosh\lambda\{v, J\} \end{aligned} \tag{2.10}$$

with 7 constraining relations

$$\lambda = -\int^t \mu_J(s)ds, \quad \tau = \frac{\mu_{vJ}\sinh\lambda}{2\mu_{JJ}}, \quad \rho = \frac{\mu_{uJ}\tanh\lambda}{2\mu_{JJ}}, \quad \mu_{vv} = \mu_{uu} + \frac{\mu_{vJ}^2 - \mu_{uJ}^2}{4\mu_{JJ}}, \tag{2.11}$$
$$\mu_{uv} = \frac{\mu_{uJ}\mu_{vJ}}{2\mu_{JJ}}, \quad \mu_u = \frac{\mu_J\mu_{uJ} - \dot{\mu}_{uJ}\tanh\lambda}{2\mu_{JJ}} + \frac{\mu_{vJ}}{2}, \quad \mu_v = \frac{\mu_J\mu_{vJ} - \dot{\mu}_{vJ}\tanh\lambda}{2\mu_{JJ}} - \frac{\mu_{uJ}}{2}.$$

Thus from the original 12 free parameters, i.e. the 9 coefficient functions μ_i and the 3 functions λ, τ, ρ in the Dyson map, we can still freely choose 5. In comparison with the other $\mathcal{PT}_i$-symmetries, this is the most constrained case. We also note that this system is the only one in which all three functions in the Dyson map are constrained when we take the coefficient functions μ_i as primary quantities.

2.2 Time-dependent $\mathcal{PT}_2$-invariant Hamiltonians

The Hamiltonian $H_{\mathcal{PT}_2}(t)$ becomes Hermitian for $\mu_J = 0$, $\mu_{uJ} = 2\mu_u$, $\mu_{vJ} = -2\mu_u$, but is non-Hermitian otherwise. Preceding as in the previous section the implementation of (1.3) enforces to take $\tau, \rho \in \mathbb{R}$ and $\lambda \in i\mathbb{R}$ in (2.4), which makes the Dyson map $\mathcal{PT}_2$-symmetric. The Hermitian Hamiltonian is computed to

$$\begin{aligned} h_{\mathcal{PT}_2} &= \mu_{JJ}J^2 + \dot{\lambda}J + \left[\left(\mu_u + \frac{\mu_{vJ}}{2}\right)\cos\lambda + \left(\frac{\mu_{uJ}}{2} - \mu_v\right)\sin\lambda\right]u \\ &+ \left[\left(\mu_v - \frac{\mu_{uJ}}{2}\right)\cos\lambda + \left(\mu_u + \frac{\mu_{vJ}}{2}\right)\sin\lambda\right]v + \left[\left(\frac{\mu_{uJ}^2 - \mu_{vJ}^2}{8\mu_{JJ}} + \frac{\mu_{uu} - \mu_{vv}}{2}\right)\cos(2\lambda)\right. \\ &- \left.\left(\frac{\mu_{uJ}\mu_{vJ}}{4\mu_{JJ}} + \frac{\mu_{uv}}{2}\right)\sin(2\lambda) + \frac{\mu_{uJ}^2 + \mu_{vJ}^2}{8\mu_{JJ}} + \frac{\mu_{uu} + \mu_{vv}}{2}\right]u^2 \\ &+ \left[\left(\frac{\mu_{uJ}^2}{4\mu_{JJ}} + \mu_{uu}\right)\sin^2\lambda + \left(\frac{\mu_{uJ}\mu_{vJ}}{4\mu_{JJ}} + \frac{\mu_{uv}}{2}\right)\sin 2\lambda + \left(\frac{\mu_{vJ}^2}{4\mu_{JJ}} + \mu_{vv}\right)\cos^2\lambda\right]v^2 \\ &+ \left[\left(\frac{\mu_{uJ}^2 - \mu_{vJ}^2}{4\mu_{JJ}} + \mu_{uu} - \mu_{vv}\right)\sin(2\lambda) + \left(\frac{\mu_{uJ}\mu_{vJ}}{2\mu_{JJ}} + \mu_{uv}\right)\cos(2\lambda)\right]uv, \end{aligned} \tag{2.12}$$

with 5 constraining relations

$$\tau = \frac{\mu_{uJ}}{2\mu_{JJ}}\sec\lambda, \qquad \rho = -\frac{\mu_{vJ} + \mu_{uJ}\tan\lambda}{2\mu_{JJ}}, \qquad \mu_J = \dot{\mu}_{uJ} = \dot{\mu}_{vJ} = 0. \tag{2.13}$$

We note that we have less constraints as in the previous section, but some of the coefficient functions can no longer be taken to be time-dependent and one even has to vanish. One of the three functions in the Dyson map, e.g. λ, can be freely chosen. Compared to the other cases this is the only one for which η has the same $\mathcal{PT}_i$-symmetry as the corresponding non-Hermitian Hamiltonian $H_{\mathcal{PT}_i}(t)$ when taking the constraints on τ, ρ, λ into account.

2.3 Time-dependent $\mathcal{PT}_3$-invariant Hamiltonians

The Hamiltonian $H_{\mathcal{PT}_3}(t)$ becomes Hermitian for $\mu_{vJ} = \mu_{uu} = 0$ and $\mu_{uJ} = 2\mu_v$. Using the same arguments as above, we are forced to take $\tau, \rho \in \mathbb{R}$ and $\lambda \in i\mathbb{R}$ in (2.4). The Hermitian Hamiltonian is computed to

$$\begin{aligned} h_{\mathcal{PT}_3} = {} & J^2\mu_{JJ} + \left(\mu_J - \dot{\lambda}\right) J + \cos\lambda \left(\mu_u - \frac{\mu_{vJ}}{2}\right)(u+v) + \sin\lambda \left(\mu_u - \frac{\mu_{vJ}}{2}\right)(v-u) \\ & + \left(\mu_{vv} + \frac{\mu_{vJ}^2}{4\mu_{JJ}}\right)\left(u^2+v^2\right) + \left(\frac{\mu_{vJ}^2}{4\mu_{JJ}} - \frac{\mu_{uv}}{2}\right)\sin(2\lambda)\left(u^2-v^2\right) \\ & + \frac{\mu_{uJ}}{2}\cos\lambda\left[\{v,J\} + \{u,J\}\right] + \frac{\mu_{uJ}}{2}\sin\lambda\left[\{v,J\} - \{u,J\}\right] \\ & + \cos(2\lambda)\left(\mu_{uv} - \frac{\mu_{vJ}^2}{2\mu_{JJ}}\right) uv, \end{aligned} \tag{2.14}$$

with 5 constraining relations

$$\tau = \frac{\mu_{vJ}}{2\mu_{JJ}}\sec\lambda, \quad \rho = \frac{\mu_{vJ} - \mu_{vJ}\tan\lambda}{2\mu_{JJ}}, \quad \mu_v = \frac{\mu_{vJ}}{2} + \frac{\mu_J\mu_{vJ}}{2\mu_{JJ}}, \quad \mu_{uv} = -\frac{\mu_{vJ}\mu_{uJ}}{2\mu_{JJ}}, \quad \dot{\mu}_{vJ} = 0. \tag{2.15}$$

Once again one of the coefficient functions has to be time-independent and one of the three functions in the Dyson map can be chosen freely.

2.4 Time-dependent $\mathcal{PT}_4$-invariant Hamiltonians

The Hamiltonian $H_{\mathcal{PT}_4}(t)$ becomes Hermitian for $\mu_{uJ} = \mu_{uv} = 0$ and $\mu_{vJ} = 2\mu_u$. By the same reasoning as above we have to take $\tau, \rho \in \mathbb{R}$ and $\lambda \in i\mathbb{R}$ in (2.4). The Hermitian Hamiltonian results to to

$$\begin{aligned} h_{\mathcal{PT}_4} = {} & J^2\mu_{JJ} + \left(\mu_J - \dot{\lambda}\right) J + \sin\lambda\left(\frac{\mu_{uJ}}{2} - \mu_v\right) u + \cos\lambda\left(\mu_v - \frac{\mu_{uJ}}{2}\right) v \\ & + \left(\mu_{uu} - \mu_{vv} + \frac{\mu_{uJ}^2}{4\mu_{JJ}}\right)\sin(2\lambda) uv - \frac{\mu_{vJ}}{2}\sin\lambda\{u,J\} + \frac{\mu_{vJ}}{2}\cos\lambda\{v,J\} \\ & + \left[\left(\frac{\mu_{uu} - \mu_{vv}}{2} + \frac{\mu_{uJ}^2}{8\mu_{JJ}}\right)\cos(2\lambda) + \left(\frac{\mu_{uu} + \mu_{vv}}{2}\right) + \frac{\mu_{uJ}^2}{8\mu_{JJ}}\right] u^2 \\ & + \left[\left(\mu_{uu} + \frac{\mu_{uJ}^2}{4\mu_{JJ}}\right)\sin^2\lambda + \cos^2\lambda\,\mu_{vv}\right] v^2, \end{aligned} \tag{2.16}$$

with 5 constraining relations

$$\tau = \frac{\mu_{uJ}}{2\mu_{JJ}}\sec\lambda, \quad \rho = -\frac{\mu_{uJ}\tan\lambda}{2\mu_{JJ}}, \quad \mu_u = \frac{\mu_{vJ}}{2} + \frac{\mu_J\mu_{uJ}}{2\mu_{JJ}}, \quad \mu_{uv} = \frac{\mu_{vJ}\mu_{uJ}}{2\mu_{JJ}}, \quad \dot{\mu}_{uJ} = 0. \tag{2.17}$$

This case is similar to the previous one with one of the coefficient functions forced to be time-independent and one of the three functions in the Dyson map being freely choosable.

2.5 Time-dependent $\mathcal{PT}_5$-invariant Hamiltonians

The Hamiltonian H becomes Hermitian for $\mu_{vJ} = \mu_{uv} = 0$ and $\mu_{uJ} = -2\mu_v$. Here we have to take $\rho \in \mathbb{R}$ and $\lambda, \tau \in i\mathbb{R}$ in (2.4). The Hermitian Hamiltonian is computed to

$$\begin{aligned} h_{\mathcal{PT}_5} &= J^2\mu_{JJ} + \left(\mu_J - \dot{\lambda}\right) J + \left(\tau\mu_J + \frac{\mu_{uJ}}{2}\cos\lambda\right)\{u, J\} + \frac{\mu_{uJ}}{2}\sin\lambda\{v, J\} \\ &+ \left[\tau\left(\mu_J - \dot{\lambda}\right) + \cos\lambda\left(\mu_u + \frac{\mu_{vJ}}{2}\right)\right] u + \left[\sin\lambda\left(\mu_u + \frac{\mu_{vJ}}{2}\right) - \dot{\tau}\right] v \\ &+ \left[\tau^2\mu_{JJ} + \sin^2\lambda\left(\frac{\mu_{vJ}^2}{4\mu_{JJ}} + \mu_{vv}\right) + \tau\cos\lambda\mu_{uJ} + \cos^2\lambda\mu_{uu}\right] u^2 \\ &+ \sin\lambda\left[2\cos\lambda\left(\mu_{uu} - \mu_{vv} - \frac{\mu_{vJ}^2}{4\mu_{JJ}}\right) + \tau\tau\mu_{uJ}\right] uv \\ &+ \left[\left(\frac{\mu_{vJ}^2}{4\mu_{JJ}} + \mu_{vv}\right)\cos^2\lambda + \mu_{uu}\sin^2\lambda\right] v^2, \end{aligned} \tag{2.18}$$

with only 4 constraining relations

$$\rho = -\frac{\mu_{vJ}}{2\mu_{JJ}}, \quad \mu_v = -\frac{\mu_{uJ}}{2} + \frac{\mu_J\mu_{vJ}}{2\mu_{JJ}}, \quad \dot{\mu}_{vJ} = 0, \quad \mu_{uv} = \frac{\mu_{vJ}\mu_{uJ}}{2\mu_{JJ}}. \tag{2.19}$$

In comparison with the other symmetries, this is the least constraint case. From the three functions in the Dyson map only one is constraint and the others can be chosen freely. However, one of the coefficient functions needs to be time-independent.

3. Time-dependent quasi-exactly solvable systems

We will now specify one particular model and show how it can be quasi-exactly solved in the metric picture. Since the $\mathcal{PT}_2$ symmetry appears to be somewhat special, in the sense that it is the only case for which the Dyson map respects the same symmetry as the Hamiltonian, we consider a particular non-Hermitian $\mathcal{PT}_2$-symmetric time-independent Hamiltonian of the form

$$\hat{H} = m_{JJ}J^2 + m_v v + m_{vv}v^2 + im_{uJ}uJ. \tag{3.1}$$

Given the constraining equations (2.13), we could in principle take m_v, m_{vv} to be time dependent, but to enforce the metric picture we take here all four coefficients m_{JJ}, m_v, m_{vv} and m_{uJ} to be time-independent real constants. According to the analysis in section 2.2, the time-dependent Dyson map

$$\eta(t) = e^{\tau(t)v}e^{i\lambda(t)J}e^{\varrho(t)u}, \quad \tau(t) = \frac{\mu_{uJ}}{2\mu_{JJ}}\sec\lambda(t), \quad \varrho(t) = -\frac{\mu_{uJ}}{2\mu_{JJ}}\tan\lambda(t), \tag{3.2}$$

with $\lambda, \tau, \rho \in \mathbb{R}$, maps the time-independent non-Hermitian Hamiltonian $\hat{H}$ to the time-dependent Hermitian Hamiltonian

$$\begin{aligned} \hat{h}(t) &= m_{JJ}J^2 - \dot{\lambda}J + \sin\lambda\left(\frac{m_{uJ}}{2} - m_v\right) u + \cos\lambda\left(m_v - \frac{m_{uJ}}{2}\right) v \\ &+ \left[\cos(2\lambda)\left(\frac{m_{uJ}^2}{8\mu_{JJ}} - \frac{m_{vv}}{2}\right) + \frac{m_{uJ}^2}{8\mu_{JJ}} + \frac{m_{vv}}{2}\right] u^2 \\ &+ \left[\frac{m_{uJ}^2}{4\mu_{JJ}}\sin^2\lambda + m_{vv}\cos^2\lambda\right] v^2 + \sin(2\lambda)\left(\frac{m_{uJ}^2}{4\mu_{JJ}} - m_{vv}\right) uv. \end{aligned} \tag{3.3}$$

Here we are free to chose the time-dependent function $\lambda(t)$. As previously pointed out for non-Hermitian systems with time-dependent metric, one needs to distinguish between the Hamiltonian, that is a non-observable operator, and the observable energy operator. This feature remains also true when the non-Hermitian Hamiltonian is time-independent, but the metric is dependent on time. In reverse, it simply means that when one identifies the non-Hermitian Hamiltonian with the energy operator one has made the choice for the metric to be time-independent. With $\eta(t)$ as specified in (3.2), the energy operator is computed with the help of (2.9) to

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = \hat{H} + i\hbar\eta^{-1}(t)\partial_t\eta(t) \tag{3.4}$$

$$= m_{JJ}J^2 + m_v v + m_{vv}v^2 + im_{uJ}uJ - \dot{\lambda}J \; - i\frac{m_{uJ}}{m_{JJ}}\dot{\lambda}u. \tag{3.5}$$

We note that $\tilde{H}(t)$ is also $\mathcal{PT}_2$-symmetric when we include $\partial_t \to -\partial_t$ into the symmetry transformation. In order to demonstrate that this system is quasi-exactly solvable we specify the constants in the Hamiltonian (3.1) further to $m_{JJ} = 4$, $m_{uJ} = 2(1-\beta)\zeta$, $m_{vv} = -\beta\zeta^2$, $m_v = 2\zeta N$ so that it becomes

$$H(N,\zeta,\beta) = 4J^2 + i2(1-\beta)\zeta uJ - \beta\zeta^2 v^2 + 2\zeta N v, \qquad \beta,\zeta,N \in \mathbb{R}. \tag{3.6}$$

This Hamiltonian can be obtained from one discussed in [8] by transforming $\theta \to \theta/2$, $J \to 2J$ in the trigonometric representation. The constants in $H(N,\zeta,\beta)$ are chosen so that it exhibits an interesting double scaling limit $\lim_{\zeta\to 0, N\to\infty} H(N,\zeta,\beta) = 4J^2 + 2gv$ when assuming that $g := \zeta N$. In the trigonometric representation this limiting Hamiltonian is the Mathieu Hamiltonian.

The Hermitian Hamiltonian (3.3) simplifies in this case to

$$h(t,N,\zeta,\beta) = 4J^2 - \dot{\lambda}J + \zeta\left(2N+\beta-1\right)\left(\cos\lambda v - \sin\lambda u\right) + \frac{\gamma^2}{4}\left(\cos\lambda u + \sin\lambda v\right)^2 + \beta\zeta^2 C \tag{3.7}$$

where we denoted the Casimir operator by $C := v^2 + u^2$ and abbreviated $\gamma := (1+\beta)\zeta$. In the aforementioned double scaling limit we obtain a time-dependent Hamiltonian of the form $\lim_{\zeta\to 0, N\to\infty} h(t,N,\zeta,\beta) = 4J^2 - \dot{\lambda}J + 2g\left(\cos\lambda v - \sin\lambda u\right)$.

3.1 Quasi-exactly solvable Lewis-Riesenfeld invariants

The most efficient way to solve the time-dependent Dyson equation (1.3) is to use the Lewis-Riesenfeld approach [35] and compute at first the respective time-dependent invariants $I_h(t)$ and $I_H(t)$ for the Hamiltonian $h(t)$ and $H(t)$, see [36, 37, 30], by solving the equations

$$\partial_t I_H(t) = i\hbar\left[I_H(t), H(t)\right], \quad \text{and} \quad \partial_t I_h(t) = i\hbar\left[I_h(t), h(t)\right]. \tag{3.8}$$

Unlike the corresponding Hamiltonians that have to obey (1.3), the invariants are related by a similarity transformation

$$I_h(t) = \eta(t) I_H(t) \eta^{-1}(t). \tag{3.9}$$

Computing the eigenstates of the invariants

$$I_h(t)\left|\tilde{\phi}(t)\right\rangle = \Lambda\left|\tilde{\phi}(t)\right\rangle, \qquad I_H(t)\left|\tilde{\psi}(t)\right\rangle = \Lambda\left|\tilde{\psi}(t)\right\rangle, \qquad \text{with } \dot{\Lambda}=0 \tag{3.10}$$

the solutions to the time-dependent Schrödinger equations for $|\phi(t)\rangle$, $|\psi(t)\rangle$ are simply related by a phase factor to the eigenstates of the invariants $|\phi(t)\rangle = e^{i\alpha_h(t)/\hbar}\left|\tilde{\phi}(t)\right\rangle$, $|\psi(t)\rangle = e^{i\alpha_H(t)/\hbar}\left|\tilde{\psi}(t)\right\rangle$. It is easy to to derive that the two phase factors have to be identical $\alpha_h=\alpha_H=\alpha$. They can be determined from

$$\dot{\alpha} = \left\langle\tilde{\phi}(t)\right| i\hbar\partial_t - h(t)\left|\tilde{\phi}(t)\right\rangle = \left\langle\tilde{\psi}(t)\right| \eta^\dagger(t)\eta(t)\left[i\hbar\partial_t - H(t)\right]\left|\tilde{\psi}(t)\right\rangle. \tag{3.11}$$

Taking now H to be time-independent, we may assume $I_H = H + c\mathbb{I}$ with c being some constant. The Lewis-Riesenfeld then just becomes a dynamical phase factor

$$\dot{\alpha} = \left\langle\tilde{\psi}\right| \rho(t)\left[i\hbar\partial_t - H\right]\left|\tilde{\psi}\right\rangle = \left\langle\tilde{\psi}\right| \rho(t)\left[c\mathbb{I} - I_H\right]\left|\tilde{\psi}\right\rangle = c - \Lambda = -E, \tag{3.12}$$

such that $\alpha(t) = -Et$.

Next we quasi-exactly construct the Lewis-Riesenfeld invariants together with its eigenstates for the time-dependent Hermitian and time-independent non-Hermitian systems (3.3) and (3.1), respectively.

3.1.1 The quasi-exactly solvable symmetry operator $I_{\hat{H}}$

We make a general Ansatz for the invariant of $\hat{H}$ of the form

$$I_{\hat{H}} = \nu_{JJ}J^2 + \nu_J J + \nu_u u + \nu_v v + \nu_{uJ}uJ + \nu_{vJ}vJ + \nu_{uu}u^2 + \nu_{vv}v^2 + \nu_{uv}uv, \tag{3.13}$$

with unknown constants ν_i. The invariant for the time-independent system is of course just a symmetry and we only need to compute the commutator of $I_{\hat{H}}$ with $\hat{H}$ to determine the coefficients in (3.13). We find the most general symmetry or invariant to be

$$I_{\hat{H}} = \nu_{JJ}J^2 + m_v\frac{\nu_{JJ}}{m_{JJ}}v + im_{uJ}\frac{\nu_{JJ}}{m_{JJ}}uJ + \left(\nu_{vv} - m_{vv}\frac{\nu_{JJ}}{m_{JJ}}\right)u^2 + \nu_{vv}v^2 \tag{3.14}$$

$$= \hat{H} + (\beta\zeta^2 + \nu_{vv})C, \tag{3.15}$$

where in the last equation we have taken $\nu_{JJ} = m_{JJ}$. Since the last term only produces an overall shift in the spectrum we set $\nu_{vv}=0$ for convenience.

Next we compute the eigensystem for $I_{\hat{H}}$ by solving (3.10). Assuming the two linear independent eigenfunctions to be of the general forms

$$\tilde{\psi}^c_{\hat{H}}(\theta) = \psi_0\sum_{n=0}^{\infty} c_n P_n(\Lambda)\cos(n\theta), \quad \text{and} \quad \tilde{\psi}^s_{\hat{H}}(\theta) = \psi_0\sum_{n=1}^{\infty} c_n Q_n(\Lambda)\sin(n\theta), \tag{3.16}$$

with constants $c_n = 1/\zeta^n(N+\beta)(1+\beta)^{n-1}\left[(1+N+2\beta)/(1+\beta)\right]_{n-1}$ where $[a]_n := \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol. The ground state $\psi_0 = e^{-\frac{1}{2}\zeta\cos(\theta)}$ is taken to be $\mathcal{PT}_2$-symmetric. The constants c_n are chosen conveniently to ensure the simplicity of the polynomials $P_n(\Lambda)$, $Q_n(\Lambda)$ in the eigenvalues Λ. We then find that the functions $\tilde{\psi}_{\hat{H}}^{c}$ and $\tilde{\psi}_{\hat{H}}^{s}$ satisfy the eigenvalue equation provided the coefficient functions $P_n(\Lambda)$ and $Q_n(\Lambda)$ obey the three-term recurrence relations

$$\begin{aligned} P_2 &= (\Lambda-4)P_1 + \mathbf{2}\zeta^2 (N-1)(N+\beta) P_0, &(3.17)\\ P_{n+1} &= (\Lambda-4n^2)P_n - \zeta^2 [N+n\beta+(n-1)][N-(n-1)\beta-n] P_{n-1}, &(3.18)\\ Q_2 &= (\Lambda-4)Q_1, &(3.19)\\ Q_{m+1} &= (\Lambda-4m^2)Q_m - \zeta^2 [N+m\beta+(m-1)][N-(m-1)\beta-m] Q_{m-1}, &(3.20)\end{aligned}$$

for $n = 0, 2, \ldots$ and for $m = 2, 3, 4, \ldots$ Setting $P_0 = 1$ and $Q_1 = 1$, the first solutions for (3.17) - (3.20) are found to be

$$\begin{aligned} P_1 &= \Lambda, &(3.21)\\ P_2 &= \Lambda^2 - 4\Lambda - 2\zeta^2(N-1)(\beta+N),\\ P_3 &= \Lambda^3 - 20\Lambda^2 + \left[\zeta^2\left(2\beta^2+7\beta-3N^2-3(\beta-1)N+2\right)+64\right]\Lambda + 32\zeta^2(N-1)(\beta+N),\end{aligned}$$

and

$$\begin{aligned} Q_2 &= (\Lambda-4), &(3.22)\\ Q_3 &= (\Lambda-20)\Lambda + \zeta^2(\beta-N+2)(2\beta+N+1)+64,\\ Q_4 &= \Lambda^3 - 56\Lambda^2 + \left[2\zeta^2\left(4\beta^2+9\beta-N^2-\beta N+N+4\right)+784\right]\Lambda\\ &\quad +8\zeta^2\left[5N^2+5(\beta-1)N-12-\beta(12\beta+29)\right]-2304.\end{aligned}$$

The well-known and crucial feature responsible for a system to be quasi-exactly solvable is the occurrence of the three-term recurrence relations and that they can be forced to terminate at certain values of n. This is indeed the case and for our relations (3.18), (3.20) and can be achieved for some specific values $n = \hat{n}$ or $m = \hat{n}$, respectively. To see this we take $N = \hat{n} + (\hat{n}-1)\beta$ and note that the polynomials P_n and Q_m factorize for $n \geq \hat{n}$, $m \geq \hat{n}$ as

$$P_{\hat{n}+\ell} = P_{\hat{n}} R_\ell \qquad \text{and} \qquad Q_{\hat{n}+\ell} = Q_{\hat{n}} R_\ell, \tag{3.23}$$

where the first R_ℓ-polynomials are

$$\begin{aligned} R_1 &= \Lambda - 4\hat{n}^2, &(3.24)\\ R_2 &= 16\hat{n}^2(\hat{n}+1)^2 + \Lambda[\Lambda-4-8\hat{n}(\hat{n}+1)] + 2\hat{n}\gamma^2. &(3.25)\end{aligned}$$

Since according to (3.23) the polynomials $P_{\hat{n}}$ and $Q_{\hat{n}}$ are factor in all P_n and Q_m for $n \geq \hat{n}$ and $m \geq \hat{n}$, respectively, all higher order polynomial vanish when setting $P_{\hat{n}}(\Lambda) = Q_{\hat{n}}(\Lambda) = 0$. These latter constraints are the quantization conditions for Λ. Thus setting $P_{\hat{n}}(\Lambda) = 0$ at the different levels $\hat{n}$, we find the real eigenvalues

$$\begin{aligned} \hat{n} = 1: &\quad \Lambda_1^c = 0, &(3.26)\\ \hat{n} = 2: &\quad \Lambda_2^{c,\pm} = 2 \pm 2\sqrt{1+\gamma^2}, &(3.27)\\ \hat{n} = 3: &\quad \Lambda_3^{c,\ell=0,\pm1} = \frac{4}{3}\left\{5 + 2\kappa\cos\left[\frac{\ell\pi}{3} - \frac{1}{3}\arccos\left(\frac{35-18\gamma^2}{\kappa^3}\right)\right]\right\}, &(3.28)\end{aligned}$$

with $\kappa = \sqrt{13+3\gamma^2}$, and from $Q_{\hat{n}}(\Lambda) = 0$ we find the real eigenvalues

$$\hat{n} = 2: \quad \Lambda_2^s = 4, \tag{3.29}$$

$$\hat{n} = 3: \quad \Lambda_3^{s,\pm} = 10 \pm 2\sqrt{9+\gamma^2}, \tag{3.30}$$

$$\hat{n} = 4: \quad \Lambda_4^{s,\ell=0,\pm 1} = \frac{8}{3}\left\{7 + \tilde{\kappa}\cos\left[\frac{\ell\pi}{3} - \frac{1}{3}\arccos\left(\frac{143-18\gamma^2}{\tilde{\kappa}^3}\right)\right]\right\}, \tag{3.31}$$

with $\tilde{\kappa} = \sqrt{49+3\gamma^2}$.

Thus $\hat{H}$ is a QES system with eigenfunctions identical to those in (3.16) and energies $E = \Lambda - \beta\zeta^2$.

3.1.2 The quasi-exactly solvable invariant $I_{\hat{h}}$

Next we construct the invariant $I_{\hat{h}}$ together with their eigenfunctions. In principle we have to solve the second equation in (3.8) for this purpose, however, since we already know the Dyson map we can simply use (3.9) and act adjointly with $\eta(t)$, as given in (3.2), on $I_{\hat{H}}$ as specified in (3.14). This yields the time-dependent invariant for $\hat{h}(t)$ as

$$I_{\hat{h}} = \eta(t) I_{\hat{H}}(t)\eta^{-1}(t) = \hat{h} + \dot{\lambda}J + \beta\zeta^2 C \tag{3.32}$$

We convince ourselves that the relation (3.8) is indeed satisfied by $I_{\hat{h}}$ as given in (3.32) and $\hat{h}(t)$ as in (3.7). The eigenfunctions for $I_{\hat{h}}$ are then simply obtained as $\tilde{\phi} = \eta\tilde{\psi}$. From (3.16) we compute

$$\tilde{\phi}^c_{\hat{h}}(\theta) = \phi_0 \sum_{n=0}^{\infty} c_n P_n(\Lambda)\cos\left[n(\theta+\lambda)\right], \quad \tilde{\phi}^s_{\hat{h}}(\theta) = \phi_0 \sum_{n=1}^{\infty} c_n Q_n(\Lambda)\sin\left[n(\theta+\lambda)\right]. \tag{3.33}$$

with ground state wavefunction $\phi_0 = e^{-\frac{1}{4}\zeta(1+\beta)\cos(\theta+\lambda)}$ and coefficients c_n, $P_n(\Lambda)$, $Q_n(\Lambda)$ as defined above. According to the above arguments, the solutions to the time-dependent Schrödinger equation are $\phi^{c,s}_{\hat{h}}(\theta) = e^{-iEt/\hbar}\tilde{\phi}^{c,s}_{\hat{h}}(\theta)$.

3.2 A time-dependent three level system

For each integer value of $\hat{n}$ we have now obtained a time-dependent QES system with a finite dimensional Hilbert space. Since it is the easiest non-trivial example and time-dependent three-level systems are of some interest in the literature [38, 39, 40] we present here the case for $\hat{n} = 2$ in more detail. From (3.33) we obtain three orthonormal wavefunctions

$$\phi_\pm(\theta,t) = \frac{\sqrt{\gamma}}{2\sqrt{\pi N_\pm}} e^{-\frac{1}{4}\gamma\cos[\theta+\lambda(t)]-iE_\pm t}\left[\gamma + (1\pm\sqrt{1+\gamma^2})\right]\cos\left[\theta+\lambda(t)\right], \tag{3.34}$$

$$\phi_0(\theta,t) = \frac{\sqrt{\gamma}}{2\sqrt{\pi N_0}} e^{-\frac{1}{4}\gamma\cos[\theta+\lambda(t)]-iE_0 t}\sin\left[\theta+\lambda(t)\right], \tag{3.35}$$

with normalization constants

$$N_\pm = \gamma\left(1+\gamma^2 \pm \sqrt{1+\gamma^2}\right) I_0(\gamma/2) - \left[2+2\gamma^2 \pm (2+\gamma^2)\sqrt{1+\gamma^2}\right] I_1(\gamma/2), \tag{3.36}$$

$$N_0 = I_1(\gamma/2), \tag{3.37}$$

and eigenenergies $E_0 = 4 - \beta\zeta^2$, $E_\pm = 2 - \beta\zeta^2 \pm 2\sqrt{1+\gamma^2}$. The $I_n(z)$ denote here the modified Bessel function of the first kind. The functions in (3.34) and (3.35) solve the time-dependent Schrödinger equation for $\hat{h}(t)$ and are orthonormal on any interval $[\theta_0, \theta_0 + 2\pi]$

$$\langle \phi_n(\theta,t) | \phi_m(\theta,t) \rangle =: \int_{\theta_0}^{\theta_0+2\pi} \phi_n^*(\theta,t)\phi_m(\theta,t)d\theta = \delta_{n,m} \qquad n,m \in \{0,\pm\}. \tag{3.38}$$

We may now compute analytically all time-dependent quantities of physical interest. For instance, the expectation values for the generators in the trigonometric representation result to

$$\left\langle \phi_\pm(\theta,t) \right| u \left| \phi_\pm(\theta,t) \right\rangle = -\frac{M_\pm}{N_\pm} \sin[\lambda(t)], \quad \langle \phi_0(\theta,t) | u | \phi_0(\theta,t) \rangle = \frac{I_2(\gamma/2)}{I_1(\gamma/2)} \sin[\lambda(t)], \tag{3.39}$$

$$\left\langle \phi_\pm(\theta,t) \right| v \left| \phi_\pm(\theta,t) \right\rangle = \frac{M_\pm}{N_\pm} \cos[\lambda(t)], \quad \langle \phi_0(\theta,t) | v | \phi_0(\theta,t) \rangle = -\frac{I_2(\gamma/2)}{I_1(\gamma/2)} \cos[\lambda(t)], \tag{3.40}$$

$$\langle \phi_\ell(\theta,t) | J | \phi_\ell(\theta,t) \rangle = 0, \qquad \ell \in \{0,\pm\}, \tag{3.41}$$

where we abbreviated

$$M_\pm = \gamma\left(1 - \gamma^2 \pm \sqrt{1+\gamma^2}\right) I_1(\gamma/2) + \left[2 + 2\gamma^2 \pm (2+\gamma^2)\sqrt{1+\gamma^2}\right] I_2(\gamma/2). \tag{3.42}$$

Similarly we may obtain any kind of n-level system from (3.33).

4. Conclusions

We have provided new analytical solutions for the time-dependent Dyson equation. The time-dependent non-Hermitian Hamiltonians (2.2) considered are expressed in terms linear and bilinear combinations of the generators for an Euclidean E_2-algebra respecting the $\mathcal{PT}_i$-symmetries defined in (2.3). Restricting the coefficient functions appropriately, the corresponding time-dependent Hermitian Hamiltonians were constructed. We expect a different qualitative behaviour for Hamiltonians belonging to different symmetry classes.

A specific $\mathcal{PT}_2$-symmetric system was analyzed in more detail. For that model we assumed the non-Hermitian Hamiltonian to be time-independent so that we could employ the metric picture. This enabled us to compute the corresponding eigensystems in a quasi-exactly solvable fashion using Lewis-Riesenfeld invariants. Thus we found for the first time quasi-exactly solvable systems for Hamiltonians with explicit time-dependence. A time-dependent Hermitian three-level system is presented in more detail.

Evidently there are many open issues and problems for further investigations left. Having solved the time-dependent Dyson equation for a large class of models in section 2, it would be interesting to solve their corresponding time-dependent Schrödinger equation as carried out for the model in section 3. Furthermore, it is desirable in this type of analysis to allow an explicit time-dependence also in the non-Hermitian Hamiltonians. Clearly one may also generalize these studies to Euclidean algebras of higher rank and other types of Lie algebras.

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