Admissibility and Event-Rationality*

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Abstract

We develop an approach to providing epistemic conditions for admissible behavior in games. Instead of using lexicographic beliefs to capture infinitely less likely conjectures, we postulate that players use tie-breaking sets to help decide among strategies that are outcome-equivalent given their conjectures. A player is event-rational if she best responds to a conjecture and uses a list of subsets of the other players' strategies to break ties among outcome-equivalent strategies. Using type spaces to capture interactive beliefs, we show that event-rationality and common belief of event-rationality (RCBER) imply $S^{\infty}W$, the set of admissible strategies that survive iterated elimination of dominated strategies. By strengthening standard belief to *validated belief*, we show that event-rationality and common validated belief of event-rationality (RCvBER) imply IA, the iterated admissible strategies. We show that in complete, continuous and compact type structures, RCBER and RCvBER are nonempty, hence providing epistemic criteria for $S^{\infty}W$ and IA.

Keywords: Epistemic game theory; Admissibility; Iterated weak dominance; Common Knowledge; Rationality; Completeness.

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1 Introduction

As noted by Samuelson (1992) and many others, there is a potential problem in dealing with common knowledge of admissibility in games, which is known as the inclusion-exclusion problem. The reason is that, under the assumptions of probabilistic beliefs and expected utility, a strategy is admissible if and only if it is a best response to a belief with full support. So a natural way of obtaining the prediction of admissible choices is to require that players consider all strategies of their opponents to be possible. But then the prediction of an admissible choice for a player is accompanied by a belief that does not exclude any strategy of the player's opponents from consideration, in particular it does not exclude strategies that are not admissible. So a player cannot be certain that the opponents do not play inadmissible strategies.

Recently, Brandenburger et al. (2008), henceforth BFK, provided a way of dealing with the inclusion-exclusion issue, by using lexicographic expected utility (LEU) and the notion of *assumption* in the place of *certainty*. Roughly speaking, a player with a list of probabilistic beliefs can have a fully supported overall belief while "assuming" certain events that are not equal to the whole state space. BFK show that strategies that survive $m+1$ rounds of iterated elimination of inadmissible strategies are the strategies compatible with Rationality and mth-order Assumption of Rationality (RmAR), for every natural number m. However, the limiting construction as $m \to \infty$, RCAR, is empty in complete and continuous type structures. Therefore, BFK do not provide an epistemic characterization of IA. Keisler and Lee (2011) and Yang (2009) have recently extended BFK's analysis and obtained nonemptiness of RCAR. The former allows for discontinuous type mappings, and the latter uses a weaker notion of assumption. Perea (2012) shows that common assumption of rationality is always possible in finite structures.

We propose an alternative route. Instead of an LEU-based analysis, we use event-rationality to allow for players to break ties with lists of subsets of opponents' strategies. That is, we use a different notion of rationality: the LEU-based approaches assume that players are lexicographic expected utility maximizers. We assume that players are event-rational. The two notions of rationality are equally capable of reconciling "belief of rationality" with "admissible choice". The difference comes into play in the analysis of interactive beliefs. Interactive beliefs are described by type spaces. In our framework, a type of a player determines her beliefs over the strategies and types of the other players (as in the standard framework) and in addition it determines the tie-breaking list of events that the (event-rational) type uses. As a result, common belief of event-rationality bypasses the inclusion-exclusion issue. In contrast, in an LEU-based analysis a type of a player determines her lexicographic beliefs over the strategies and types of the other players, and the inclusion-exclusion tension is bypassed by the use of "assumption" in the place of certainty. Under our approach, we provide epistemic foundations for both the solution concept proposed by Dekel and Fudenberg (1990) ($S^\infty W$) and iterated admissibility (IA).

We consider finite two-player games in strategic form. The two players are Ann and Bob, denoted by superscripts "a" and "b". In order to provide some intuition about event-rationality, note that if a strategy s^a of Ann's is (expected utility) rational then it is a best response to some probabilistic belief, $v \in \Delta(S^b)$, where S^b is the set of Bob's strategies. If s^a is inadmissible and therefore weakly dominated by some (mixed) strategy σ^a, then s^a and σ^a give the same payoff for all strategies of Bob in the support of v while σ^a is strictly better than s^a for all probability measures with support equal to the complement of the support of v. Hence, when Ann chooses an admissible strategy, it is as if Ann optimizes given the belief v, as usual, but when she is completely indifferent between two strategies, she compares their expected utilities with respect to a probability measure with support equal to the complement of the support of v. We say that Ann "breaks ties" using the event that is the complement of the support of v.

Event-rationality does not require that Ann breaks ties only with respect to the complement of the support of her belief. Ann can conceivably break ties using any other set, as long as it is outside her current frame of mind, that is, disjoint from the support of v.[1] Furthermore, Ann need not use a single such tie-breaking set. She may well have many such sets, each providing extra validation for the chosen strategy. We refer to a collection of tie-breaking sets as a tie-breaking list.

The principle behind event-rationality is, therefore, the following: if two strategies are outcome-equivalent given Ann's belief, then Ann has no way of deciding among them within her frame of mind: the two strategies yield the same outcome for whichever strategy of Bob she considers possible. Ann must, therefore, resort to information beyond her frame of mind to make a decision. For instance, she could resort to fully external means, like coin flips. However, Ann would be neglecting information about the two strategies under consideration,

[1]But note that, for the purpose of breaking ties, it suffices to consider only subsets of Bob's strategies. In particular, when we introduce the formal model of interactive beliefs, it is without loss of generality to assume that Ann uses only lists of Bob's strategies to break ties, because lists that include the types of Bob only matter for breaking ties through the strategies of Bob that they are related to.

namely how they fare against strategies of Bob that are considered impossible by her belief. Event-rationality postulates that Ann does not neglect this information and, at the same time, she does not change what she thinks about Bob's choices.

Turn now to interactive beliefs, captured by type structures. Let T^a and T^b be the sets of types of Ann and Bob. A type $t^a \in T^a$ determines Ann's conjectures over Bob's choices, Ann's beliefs over Bob's types and so on, together with the tie-breaking list. A state for Ann is a strategy-type pair (s^a, t^a) and the beliefs over Bob are given by probability measures over $S^b \times T^b$. A strategy-type pair (s^a, t^a) of Ann's is called event-rational if s^a is optimal given t^a's belief over S^b and breaks ties for all sets in t^a's tie-breaking list. States where event-rationality and common belief of event-rationality obtain are captured as the intersection of infinitely many events: Ann is event-rational, and so is Bob; Ann is certain that Bob is event-rational and Bob is certain that Ann is event-rational. And so on. This yields our RCBER ((Event) Rationality and Common Belief of Event-Rationality) set of states.

Event-rationality captures the idea of choosing a strategy with extra validation, in the sense that a strategy has to be optimal under one's belief and in addition it has to pass a series of validating tie-breaking tests. We also introduce the idea of extra validation of a belief. Consider a type t^a that believes that an event $E \in S^b \times T^b$ is true, and is associated with a list ℓ of subsets of S^b. The belief on the event E will be validated by the list ℓ if there is an element of the list, say $E^b \in \ell$, that is equal to the projection of E on S^b.

States where event-rationality and common validated belief of event-rationality obtain are again captured as the intersection of infinitely many events: Ann and Bob are event-rational. Ann has a validated belief that Bob is event-rational and Bob has a validated belief that Ann is event-rational. And so on. This yields our RCvBER ((Event) Rationality and Common validated Belief of Event-Rationality) set of states.

Our results are as follows. We show that in a complete structure, RCBER produces the set of strategies that survive one round of elimination of inadmissible strategies followed by iterated elimination of strongly dominated strategies ($S^\infty W$), whereas RCvBER produces the set of iterated admissible strategies (IA). We then show that strategies played under RCvBER constitute a self-admissible set (SAS), but the converse is not necessarily true. Because BFK have shown that every SAS is the implication of RCAR in some type structure, the RCvBER construction is more restrictive than the RCAR construction of BFK. Nevertheless, we show that the RCBER and the RCvBER are nonempty whenever the type structure is complete, continuous and compact, therefore providing epistemic criteria for $S^\infty W$ and IA.

Our approach provides an alternative and effective perspective to deal with common "knowledge" of admissibility in games. A solution to the inclusion-exclusion problem is obtained by using event-rationality together with having S^b (from Ann's perspective) as one of the tie-breaking sets. LEU-based approaches also obtain a solution to the inclusion-exclusion problem. But some conclusions coming from the LEU-based approach are functions of the notions of rationality and beliefs adopted by the approach. For instance, from BFK and Keisler and Lee (2011) we get that either continuity or completeness have to be dropped for an epistemic characterization of IA to be obtained. Our results show that, using a different notion of rationality, neither continuity nor completeness have to be dropped for such a characterization to be obtained. We should also note that completeness captures the idea that players have no prior knowledge about each other, so it is a desirable property in an epistemic analysis. Robustness with respect to continuity of the type structure is another desirable property, which is satisfied by our construction.

1.1 Related Literature

Bernheim (1984) and Pearce (1984) argue that common knowledge of rationality implies (in terms of behavior) the iteratively undominated (IU) set, that is, the set of strategy profiles surviving iterated deletion of strongly dominated strategies. Tan and Werlang (1988) provide epistemic conditions for IU by characterizing RCBR (rationality and common belief of rationality). Admissibility, or the avoidance of weakly dominated strategies, has a long history in decision and game theory (see Wald (1939), Luce and Raiffa (1957) and Kohlberg and Mertens (1986)). However, Samuelson (1992) shows that common knowledge of admissibility is not equivalent to iterated admissibility and does not always exist. Foundations for the $S^\infty W$ strategies (Dekel and Fudenberg (1990)) are provided by Börgers (1994) (using approximate common knowledge), Brandenburger (1992) (using LEU (Blume et al. (1991)) and 0-level belief) and Ben-Porath (1997) (in extensive form games). Stahl (1995) defines the notion of lexicographic rationalizability and shows that it is equivalent to iterated admissibility.

BFK use LEU and characterize rationality and common assumption of rationality (RCAR) by the solution concept of self-admissible sets. They show that rationality and mth order assumption of rationality is characterized by the strategies that survive $m+1$ rounds of elimination of inadmissible strategies, in complete type structures.[2] Finally, RCAR is empty in a complete and continuous lexicographic type structure when the agent is not completely

[2]See Section 5.1 for the formal definition of "assumption".

indifferent. Hence, although the IA set can be captured by RmAR (rationality and mth order assumption of rationality) for big enough m (note that games are finite), BFK do not provide an epistemic criterion for IA. Keisler and Lee (2011) show that RCAR is nonempty if one drops continuity. Yang (2009) provides an epistemic criterion for IA, with an analogous version of BFK's RCAR, that makes use of a weaker notion of "assumption". The message from Keisler and Lee (2011) and Yang (2009) is that continuity strengthens the notion of caution implied by fully supported LPS.[3] The notion of caution implied by event-rationality is independent of continuity.

The paper is organized as follows. In the following section we illustrate the differences between the various notions of rationality and belief through examples. In Sections 3 and 4 we set up the framework and provide the relevant definitions, including event-rationality, RCBER and RCvBER. In Section 5 we show that RmBER (m rounds of mutual belief) generates $S^\infty W$ and that RmvBER (m rounds of mutual validated belief) generates the IA set, for big enough m. Moreover, we show that RmvBER is more restrictive than RCAR of BFK. In Section 6 we show that RCBER and RCvBER are always nonempty in compact, complete and continuous type structures, therefore providing epistemic criteria for $S^\infty W$ and IA. Finally, the Appendices A-C provide decision theoretic foundations for event-rationality and validated beliefs, and characterize RCBER and RCvBER in type structures that are not necessarily complete but satisfy a richness condition.

2 Examples

In order to illustrate the differences between the BFK approach and that of the present paper, consider the following game from Samuelson (1992) and BFK. There are two players, Ann and Bob.

		1	[1]
		L	R
1	U	1, 1	0, 1
[1]	D	0, 2	1, 0

From the literature we know that, under expected utility, rationality and common belief of rationality (RCBR) is characterized by the best response sets (BRS) and, in a complete

[3] Perea (2012) shows that common assumption of rationality is always possible in finite structures.

structure, the strategies that survive iterated deletion of strongly dominated strategies.[4] Can we get a similar result for the admissible strategies and the iteratively admissible strategies if we modify the notions of belief and of rationality? Recall that a strategy is admissible if and only if it is a best response to a probability measure with full support (no strategy of the other player is excluded). Then, the obvious solution is to specify that rationality incorporates full support beliefs.

But such a specification does not always work. In the game above, if Ann is rational, she assigns positive probability to Bob playing L and R. If Bob is rational, he assigns positive probability to Ann playing U and D. Hence, Bob plays L. If Ann knows that Bob is rational, she assigns positive probability only on Bob playing L. But then, Ann is not rational! In other words, the RCBR set is empty for this game.

One solution is obtained using LEU. Suppose Ann's primary belief assigns probability 1 to Bob playing L, and her secondary belief assigns probability 1 to Bob playing R. Bob's primary belief assigns 1 to Ann playing U and his secondary belief assigns 1 to Ann playing D. Then, Bob playing L is (lexicographic expected utility) rational because he is indifferent between L and R given his primary belief, but strictly prefers L given his secondary belief.[5] Ann playing U is rational because U is the best response given her primary belief. She assumes that Bob is rational, because she considers Bob playing L infinitely more likely than Bob playing R.[6] Similarly, Bob assumes that Ann is rational. As a result, rationality and common assumption of rationality (RCAR) is nonempty.

A similar result can be obtained if we use the definition of event-rationality in the context of type structures augmented with tie-breaking lists. Suppose Ann's belief assigns probability 1 to Bob playing L and Bob's belief μ assigns probability 1 to Ann playing U. Moreover, Bob has the set $S^a \setminus \operatorname{supp} \mu$ in his tie-breaking list. Bob playing L is event-rational because he plays best response given his beliefs and, although L and R are outcome-equivalent at supp μ, L is better under at least one probability measure with support equal to $S^a \setminus \operatorname{supp} \mu$. Similarly, Ann is event-rational since, under her belief, she does not need to break ties. Finally, Ann believes that Bob is event-rational and Bob believes that Ann is event-rational. Hence, rationality and common belief of event-rationality (RCBER) is nonempty.

In the game above RCAR and RCBER produce the same strategies because the IA and the $S^\infty W$ sets are equal. However, this is not always true. Consider the following game

[4] $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

[5] That is, the associated sequence of payoffs under L is lexicographically greater than the sequence under R.

[6] For more information on the notions of "assumption" and "infinitely more likely", see BFK.

which illustrates the difference between RCBER (which yields the $S^{\infty}W$ set) and RCvBER (which yields the IA set):

	L	R
U	$1,0$	$1,3$
M	$0,2$	$2,2$
D	$0,4$	$1,1$

Because D is strongly dominated and Ann is event-rational, she will not play D. In a complete structure though, event-rational Ann will play U or M, while event-rational Bob will play L or R. For example, Ann's type playing U is event-rational if she assigns probability 1 to Bob playing L. Ann's type playing M is also event-rational if she assigns probability 1 to Bob playing R. Note that Ann never needs to break ties. Moreover, for both U and M there are event-rational types of Ann's who assign positive probability to event-rational types of Bob playing L or R. And similarly for Bob. In other words, these types of Ann believe the event "Bob is event-rational", Bob's types believe the event "Ann is event-rational", and so on for every finite order of beliefs about beliefs. Hence, RCBER yields the $S^{\infty}W$ set, $\{\mathrm{U},\mathrm{M}\} \times \{\mathrm{L},\mathrm{R}\}$.

Now repeat the same procedure but impose a stronger form of belief. Take an event $E \subseteq S^b \times T^b$, where S^b and T^b are the set of Bob's strategies and types, respectively. A type t^a of Ann is associated with a belief over $S^b \times T^b$ and a list ℓ of subsets of S^b. We say that t^a has a validated belief in an event E if it assigns probability 1 to E and there exists an element E^b of the list ℓ that is equal to the projection of E on S^b. Imposing event-rationality and common validated belief of event-rationality gives us RCvBER.

Which strategies are generated by RCvBER? The first round of RCvBER yields the set of event-rational types for Ann and event-rational types for Bob, just like RCBER. But the second round of RCvBER requires that each of Ann's types has a validated belief in the event "Bob is rational", and similarly for Bob. Then, all types playing L are excluded. To see this, note that if Bob is event-rational and has a validated belief in the event "Ann is event-rational", then the strategies played by event-rational types of Ann's, namely $\{\mathrm{U},\mathrm{M}\}$, must belong to his list. The only event-rational types of Bob playing L (and having a validated belief that Ann is event-rational) are the ones that assign probability 1 on Ann playing M. In order to have a validated belief in $\{\mathrm{U},\mathrm{M}\} \times T_0^a$, where T_0^a is Ann's event-rational types, Bob must have U as a tie-breaking set in his list. Moreover, he assigns probability 1 to M and therefore has to break ties, because L and R are outcome equivalent given M. But L is never a best response for any conjecture with support on U. Hence, Bob, assigning probability one to M, cannot have a validated belief that Ann is event-rational.

In the third round of RCvBER, Ann has a validated belief that Bob has a validated belief that Ann is event-rational. This means that Ann's types playing U are excluded, because those types assign positive probability to Bob's types playing L, and none of them has a validated belief that Ann is event-rational. The only event-rational types of Ann playing M and of Bob playing R survive all rounds of RCvBER and generate the IA set, $\{M\} \times \{R\}$.

3 Setup

Let (S^a, S^b, π^a, π^b) be a two-player finite strategic form game, with $\pi^a : S^a \times S^b \to \mathbb{R}$, and similarly for b (as usual, a stands for Ann, and b stands for Bob). In what follows we sometimes present definitions and results focusing only on player a. In these cases, the definitions and results for player b are analogous. For any given topological space X, let $\Delta(X)$ denote the space of probability measures defined on the Borel subsets of X, endowed with the weak* topology. We extend π^a to $\Delta(S^a) \times \Delta(S^b)$ in the usual way: $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b)\in S^a\times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$. A (possibly mixed) strategy $\sigma^a \in \Delta(S^a)$ is a best response to a conjecture $v \in \Delta(S^b)$ if $\pi^a(\sigma^a, v) \geq \pi^a(\hat{s}^a, v)$ for every $\hat{s}^a \in S^a$.[7] It is denoted by $\sigma^a \in BR^a(v)$.

3.1 Admissibility and Event-Rationality

The following definition and Lemma are taken from BFK.

Definition 1. *Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is* **weakly dominated** *with respect to $X \times Y$ if there exists $\sigma^a \in \Delta(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$ and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is* **admissible** *with respect to $X \times Y$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is admissible.*

Lemma 1. *A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there exists $\sigma^b \in \Delta(S^b)$, with supp $\sigma^b = Y$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

[7] In what follows, we will use the term "conjecture" to refer to a probabilistic belief over the opponent's strategy choices.

Lexicographic beliefs have been used in dealing with the inclusion-exclusion issue identified by Samuelson (1992) (see BFK, Brandenburger (1992), Stahl (1995), Keisler and Lee (2011) and Yang (2009)). We follow an alternative approach, based on "tie-breaking lists." We stress that our approach is a way of capturing admissible behavior (Lemma 3 below) and at the same time dealing with belief of rationality. Admissible behavior can be viewed as the requirement that ties are broken by events outside the conjecture of a player. This leads us to consider tie-breaking events, as follows.

Let $\ell = \{F_1, ..., F_k\}$ be a **list of subsets** of S^b, with $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = S^b$, for some $k \geq 1$. The collection of all such lists, L^b, is a set of finite cardinality, because S^b is a finite set.

For a given conjecture $v \in \Delta(S^b)$, let $\sigma^a \sim_{\text{supp } v} s^a$ denote that the mixed strategy $\sigma^a \in \Delta(S^a)$ satisfies $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for every $s^b \in \text{supp } v$. Therefore, $\sigma^a \sim_{\text{supp } v} s^a$ denotes that σ^a is outcome equivalent to s^a in supp v.

Definition 2. *Given a pair $(v, \ell) \in \Delta(S^b) \times L^b$, we say that a strategy $s^a \in S^a$ is event-preferred to a strategy $r^a \in S^a$ with respect to (v, ℓ) if either*

- *$\pi^a(s^a, v) \geq \pi^a(r^a, v)$, and it is not the case that $r^a \sim_{supp\ v} s^a$, or*
- *$r^a \sim_{supp\ v} s^a$ and for each $F \in \ell$ with $F \setminus supp\ v \neq \emptyset$, there exists $v' \in \Delta(S^b)$ with $supp\ v' = F \setminus supp\ v$ and $\pi^a(s^a, v') \geq \pi^a(r^a, v')$.*

A pure strategy $s^a \in S^a$ being preferred to a mixed strategy $\sigma^a \in \Delta(S^a)$ is similarly defined. A strategy is event-rational if it is maximal with respect to the event-preferred preference relation.

Definition 3. *Given a pair $(v, \ell) \in \Delta(S^b) \times L^b$, we say that a strategy $s^a \in S^a$ is event-rational with respect to (v, ℓ) if it is event-preferred with respect to (v, ℓ) to every mixed strategy $\sigma^a \in \Delta(S^a)$.*

Since we introduce a new notion of being preferred to, it is important to verify the following.

Lemma 2. *For each pair $(v, \ell) \in \Delta(S^b) \times L^b$, there exists $s^a \in S^a$ which is event-rational with respect to (v, ℓ).*

Proof. As S^a is finite, $BR^a(v) \neq \emptyset$. We show that there exists $s^a \in BR^a(v)$ that is not weakly dominated given $F \setminus \text{supp } v \neq \emptyset$ by any $\sigma^a \in BR^a(v)$, for all $F \in \ell$. Using Lemma 3 below, s^a is then event-rational with respect to (v, ℓ).

Let $F_l \in \ell$ be the smallest set such that $F_l \setminus \text{supp } v \neq \emptyset$. If there does not exist such a set then supp $v = S^b$ and event-rationality is trivially satisfied. Suppose without loss of generality that $l = 1$. Let $H^a_0(v) = BR^a(v)$ and recall that $BR^a(v)$ includes all pure and mixed strategies that are best responses to v. Let $H^a_1(v) \subseteq H^a_0(v)$ be the set of pure and mixed strategies that are weakly undominated by any $\sigma^a \in H^a_0(v)$, given $F_1 \setminus \text{supp } v$. Because S^a is a finite set, $H^a_1(v) \neq \emptyset$. Moreover, it cannot be that $H^a_1(v)$ contains only mixed strategies, because if all strategies in the support of a mixed strategy are weakly dominated given some set E, then the same is true for the mixed strategy. Let $H^a_2(v) \subseteq H^a_0(v)$ be the set of pure and mixed strategies that are weakly undominated by any $\sigma^a \in H^a_0(v)$, given $F_2 \setminus \text{supp } v$. We claim that $\in H^a_1(v) \cap H^a_2(v) \neq \emptyset$, and by the same argument as above, $H^a_1(v) \cap H^a_2(v)$ contains at least one pure strategy. In fact, pick $\sigma^a_1 \in H^a_1(v) \setminus H^a_2(v)$, so σ^a_1 is weakly dominated by some $\sigma^a_2 \in H^a_2(v)$ given $F_2 \setminus \text{supp } v$. Because σ^a_1 is weakly undominated by σ^a_2 given $F_1 \setminus \text{supp } v$, and $F_1 \setminus \text{supp } v \subseteq F_2 \setminus \text{supp } v$, it must be that $\sigma^a_1 \sim_{F_1 \setminus \text{supp } v} \sigma^a_2$ and therefore $\sigma^a_2 \in H^a_1(v)$. Therefore, $\sigma^a_2 \in H^a_1(v) \cap H^a_2(v)$, as claimed. Continuing, let $H^a_l(v) \subseteq H^a_0(v)$ be the set of pure and mixed strategies that are weakly undominated by any $\sigma^a \in H^a_0(v)$, given $F_l \setminus \text{supp } v$, for $l = 3, ..., k$. By induction, say that $\bigcap_{l=1}^m H^a_l(v) \neq \emptyset$ for $m < k$. Pick $\sigma^a_m \in \bigcap_{l=1}^m H^a_l(v) \setminus H^a_{m+1}(v)$, so σ^a_m is weakly dominated by some $\sigma^a_{m+1} \in H^a_{m+1}(v)$ given $F_{m+1} \setminus \text{supp } v$. Because σ^a_m is weakly undominated by σ^a_{m+1} given $F_l \setminus \text{supp } v$, $l = 1, ..., m$, and $F_1 \setminus \text{supp } v \subseteq \cdots F_m \setminus \text{supp } v \subseteq F_{m+1} \setminus \text{supp } v$, it must be that $\sigma^a_m \sim_{F_l \setminus \text{supp } v} \sigma^a_{m+1}$ for $l = 1, ..., m$ and therefore $\sigma^a_{m+1} \in \bigcap_{l=1}^m H^a_l(v)$. Hence $\bigcap_{l=1}^k H^a_l(v) \neq \emptyset$, and the same argument above shows existence of a pure strategy in that set. That is, there exists s^a that is weakly undominated by any $\sigma^a \in H^a_0(v) = BR^a(v)$, given $F_l \setminus \text{supp } v$, for $l = 1, \ldots, k$. □

The following Lemma shows the connection between admissibility and event-rationality.

Lemma 3. *For each pair $(v, \ell) \in \Delta(S^b) \times L^b$ and each $F \in \ell$, if s^a is event-rational with respect to (v, ℓ) and supp $v \subseteq F$, then s^a is admissible with respect to $S^a \times F$. Conversely, if s^a is admissible with respect to $S^a \times F$, for each $F \in \ell$, then, for each $F \in \ell$ there exists v with supp $v = F$, such that s^a is event-rational with respect to (v, ℓ).*

Proof. Suppose that s^a is event-rational for v such that supp $v \subseteq F$. If supp $v = F$ then the result is immediate so suppose supp $v \subset F$ and $F \setminus \text{supp } v \neq \emptyset$. Suppose there exists $\sigma^a \in \Delta(S^a)$ with $\pi(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in F$, with strict inequality for some $s^b \in F$. Because $s^a \in BR^a(v)$, we have $s^a \sim_{\text{supp } v} \sigma^a$, which implies that there exists v' with supp $v' = F \setminus \text{supp } v$ and $\pi(s^a, v') \geq \pi(\sigma^a, v')$, a contradiction. Conversely, suppose s^a is admissible with respect to $S^a \times F$, for each $F \in \ell$. Pick a set $F \in \ell$. Since, s^a is admissible with respect to $S^a \times F$, there exists v with supp $v = F$ such that $s^a \in BR(v)$. For $F' \in \ell$ such that $F' \subsetneq F$ we have $F' \setminus F = \emptyset$ and the definition for event-rationality of s^a is trivially satisfied. For $F' \in \ell$ such that $F \subsetneq F'$, take σ^a such that $s^a \sim_F \sigma^a$ and suppose that there does not exist v' with supp $v' = F' \setminus F$ such that $\pi(s^a, v') \geq \pi(\sigma^a, v')$. Then, σ^a weakly dominates s^a on $F' \setminus F$, and therefore also on F', a contradiction.

□

3.2 Interpretation of Event-Rationality

The idea of event-rationality is that Ann uses each of the sets in the list ℓ to break ties. Whenever Ann has a conjecture $v \in \Delta(S^b)$ over Bob's choices under which s^a is optimal and s^a is outcome-equivalent to a (mixed) strategy σ^a given any s^b in supp v, Ann uses each $F \in \ell$ as a "tie-breaking experiment", by checking whether there exists at least one probability measure v' with support on $F \setminus$ supp v that validates the choice of s^a. Ann is fully confident in the conjecture v and in the best response s^a to v, as long as there is no σ^a that is outcome equivalent to s^a in supp v. In that case, the probabilistic assessments captured by v are irrelevant, because whichever other conjecture $\hat{v}$ with supp $\hat{v} =$ supp v would not help Ann breaking ties between s^a and σ^a. Ann then uses the tie-breaking list ℓ as we just described.

It is important to note that, although the "tie-breaking experiments" are additional thought experiments that Ann uses to guide her choices, they do not play the role of additional hypotheses, as one would have if we were in a LEU framework. If s^a is indifferent to σ^a according to v, but not outcome equivalent in supp v, then event-rationality does not require that the tie-breaking list be invoked to decide between s^a and σ^a.

3.2.1 Thought Experiments

As suggested above, the "tie-breaking experiments" are thought experiments used by the decision maker to help making decisions. As with standard expected utility preferences, when Ann is event-rational with respect to (v, ℓ), she considers possible only the events that are given positive probability by v. Intuitively speaking, the support of v is the largest possible event that does not contain an impossible event. The events in the list ℓ are not considered possible, but may nevertheless be relevant for Ann's decisions.

One way to understand the ideas involved here is as follows.[8] Let a pair (v, ℓ) be given, and consider the events

$$L(v, \ell) = \{F \setminus \operatorname{supp} v : F \in \ell\} \setminus \{\emptyset\}.$$

By construction, (v, ℓ) and $(v, L(v, \ell))$ represent the same event-rational preferences.[9] One can interpret event-rationality by viewing the elements of $L(v, \ell)$ as the *objects of the thought experiments*, and the probability measures μ_F on $F \in L(v, \ell)$ that break ties in favor of some candidate strategy s^a as the *outputs of the thought experiments*. Thus, an event-rational strategy with respect to (v, ℓ) is one that is optimal under v and has successful outputs against all thought experiments in the list $L(v, \ell)$.

In particular, a probability measure μ_F on $F \in L(v, \ell)$ is not actually a conjecture held by Ann (it is just the output of some experiment), and the thought experiment is the act of finding such probability measures on each $F \in L(v, \ell)$ that break ties.

Using a thought experiment to break ties and yet considering the events in the experiments impossible is simple to grasp when dealing with past events/actions: for instance, one may wonder what would have happened if Germany had won World War II, and use it to help deciding whether to move to Germany or not. But one knows that Germany did not win. So the thought experiment "what if Germany had won" is simply a mental construct, and the decision maker is sure that it is impossible. Still, this experiment may tip the scale in favor of moving or not moving to Germany. When we deal with future rather than past actions the same line of reasoning goes through, as illustrated in the following example.

Consider an upcoming football (soccer) match between the teams of coach Ann and coach Bob. Their relevant strategies are the possible line-ups for their teams. Coach Bob has a star defender who is disqualified from playing because of a red card in a previous match. Coach Ann knows this and believes with probability one that the star defender will not play. Moreover, coach Ann has two star strikers who, absent the star defender of coach Bob's team, are outcome equivalent given all the possible strategies (line-ups) that coach Ann believes coach Bob can choose from. However, coach Ann chooses to put in the striker who would be better if the star defender actually played. An impossible event in coach Ann's mind helps her break ties in favor of one of her strategies. Moreover, coach Ann can potentially think of many other sets of "impossible" line-ups of coach Bob's and evaluate how her outcome

[8] We thank the editor in charge for providing this interpretation.

[9] Observe that we necessarily have redundant preference representations. These redundancies lead to the existence of redundant hierarchies of preferences if the standard universal construction, as in Appendix B, is followed.

equivalent strikers will perform. We require that these sets are nested, so that they always contain a "core" object of a thought experiment, which in this example is the existence of the star defender. Moreover, there is no presumption (as in the lexicographic approach) that Ann is ranking these impossible scenarios in terms of how unlikely they are.

The following example, suggested by an anonymous referee, illustrates this point further:

	L	C	R
U	$4, 6$	$0, 0$	$4, 3$
M	$0, 0$	$4, 6$	$0, 3$
D	$2, 3$	$2, 3$	$0, 0$

Suppose that Ann is event-rational with respect to (v, ℓ), with $v(\mathrm{L}) = v(\mathrm{C}) = 1/2$ and $L(v, \ell) = \{\mathrm{R}\}$.[10] The unique experiment considered is $\{\mathrm{R}\}$, so the only possible outcome of the experiment is the probability measure assigning probability 1 to R. Strategy D is outcome equivalent to a coin-flip between U and M under supp v, so Ann cannot decide between D and this coin-flip, and resorts to the experiment $\{\mathrm{R}\}$ for help. Under the unique outcome of the experiment, D is strongly dominated by the coin-flip, so the coin-flip is event-preferred to D with respect to (v, ℓ). Note that R is weakly dominated by a coin-flip between L and C. So Ann resorts to a thought experiment composed of an inadmissible strategy for Bob. But, as we indicated above, her theory only considers possible that Bob plays either L or C, which are admissible. So Ann believes that Bob plays admissibly.

3.2.2 Nested Thought Experiments

The lists used by an event-rational Ann are composed of strictly nested subsets of S^b, $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = S^b$, for $k \geq 1$. The requirement that $F_k = S^b$ is needed to capture admissible behavior (that is, admissibility with respect to $S^a \times S^b$), as is clear from Lemma 3. The nestedness requirement ensures existence of an event-rational choice for any pair (v, ℓ), as verified in Lemma 2. But beyond this agnostic justification for the requirement, it reveals a particularity of event-rationality that is quite different from lexicographic models. In these models, the hypotheses are disjoint events and lexicographic expected utility proceeds sequentially, checking one hypothesis at a time, in lexicographic order. Here, the thought experiments are not alternative hypotheses to be checked sequentially. There is no ranking

[10] To connect it with the previous example, one can think of R as Bob's strategy of including his star defender in the line-up.

in terms of how unlikely each experiment is, and the experiments are to be performed all at once.

The thought experiments can be viewed as having F_1 as the anchor or target, that is, the part of S^b that Ann targets for her experiment, and successive enlargements $F_2, ..., F_k$, with $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$, as robustness checks all the way to the most imprecise superset of G_1, $F_k = S^b$. Dually, one can view the thought experiments as starting from the most imprecise experiment $F_k = S^b$ and moving down with successively more precise experiments in a definite direction $F_k \supsetneq F_{k-1} \supsetneq \cdots \supsetneq F_1$ towards the most precise experiment, the target F_1. Going back to the football example in Section 3.2.1, the target experiment for coach Ann could be the presence of the star defender in coach Bob's line up. We stress that what we just described are two ways of interpreting the tie-breaking list, or how the decision maker would design the experiments $F_1, ..., F_k$. As all checks must be passed, they can be performed in any order.

3.2.3 Decision Theoretic Considerations

Turn now to decision theoretic considerations. We postulate that a decision maker (Ann) has a theory captured by her preference relation $\succsim$ and the resulting probability measure μ. Let $F_0 = \text{supp}\ \mu$ and write $\succsim$ as $\succsim_0$. Moreover, when faced with a comparison between two acts that are completely indifferent according to her theory, Ann resorts to thought experiments to break ties. This is captured by a list of conditional preferences, where the conditioning events are outside F_0. Formally, Ann's choices are determined by a list of preferences $(\succsim_0, \succsim_1, \ldots, \succsim_k)$ and the resulting supports $(F_0, ..., F_k)$. F_0 represents Ann's theory, while $(F_1, ..., F_k)$ with $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = S^b$ are thought experiments, used only for the purposes of breaking complete indifference. Thus F_0 describes Ann's frame of mind, as it contains the states that Ann considers possible, and $(F_1, ..., F_k)$ describe zero probability "counter-factuals" as $F_0 \cap F_i = \emptyset$ for each $i = 1, \ldots, k$. Ann prefers an act x to an act y if $x \succsim_0 y$ and if x is outcome-equivalent to y in F_0, then $x \succsim_i y$ for all $i = 1, ..., k$. Appendix A provides a more detailed exposition and shows that the notion just defined is equivalent to event-rationality.

3.3 Type Structures and Beliefs

Type structures are used to describe interactive beliefs. Because event-rationality has players using tie-breaking sets, a type of a player must determine a conjecture and a list of tie-breaking sets. Observe that we assign a list of tie-breaking sets for each type, thereby fixing that type's thought experiments. An (event)-irrational type may not use the assigned tie-breaking list, in the same way that an irrational type in the standard type space construction may not choose based on expected utility maximization given his/her beliefs. Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

Definition 4. *An (S^a, S^b)-based type structure with tie-breaking lists is a structure*

$$\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle,$$

where $\lambda^a : T^a \to \Delta(S^b \times T^b) \times L^b$, and similarly for b. Members of T^a, T^b are called types, members of L^a, L^b are called lists and members of $S^a \times T^a \times S^b \times T^b$ are called states.

We refer to an (S^a, S^b)-based type structure with tie-breaking lists as simply a type structure. The types spaces T^a and T^b are assumed topological. The sets S^a, S^b, L^a, L^b are finite, and we endow each with the discrete topology so that they are compact spaces. The belief mappings λ^a and λ^b are assumed Borel measurable. A type structure is: **complete** when λ^a and λ^b are surjective (c.f. Brandenburger (2003)); **continuous** when λ^a and λ^b are continuous; and **compact** when T^a and T^b are compact spaces.

The standard construction of all coherent hierarchies of "beliefs about beliefs" yields a complete, continuous and compact type structure. So existence of such structures (which we assume in some of our results below) is guaranteed. Some details are provided in Appendix B.

We use the notation $\lambda^a(t^a) = (\mu^a(t^a), \ell^a(t^a))$, with $\mu^a(t^a) \in \Delta(S^b \times T^b)$ and $\ell^a(t^a) \in L^b$. Similarly for b. Fix an event $E \subseteq S^b \times T^b$ and write

$$B^a(E) = \{t^a \in T^a : \mu^a(t^a)(E) = 1\}$$

as the set of types that are certain of the event E. This is the standard definition of certainty (as 1-belief): the states of Bob are the strategy type pairs in $S^b \times T^b$, and Ann's beliefs are over Bob's states. Note that B^a satisfies monotonicity: if Ann is certain of E and $E \subset F$ then Ann is also certain of F. Note also that, coupled with event-rationality, the behavioral implications of 1-belief are different than under expected utility: the complement of a probability 1 event may not be irrelevant for choices.

Fix $E \subseteq S^b \times T^b$ and define the following operator

$$B^a_*(E) = \{t^a \in T^a : \mathrm{proj}_{S^b} E \in \ell^a(t^a)\},$$

mapping an event E to the set of Ann's types specifying a list that contains the projection of E to the set of Bob's strategies. We say that a type of Ann's has a **validated belief** in an event $E \subseteq S^b \times T^b$ if the type belongs to the set

$$B_v^a(E) = B^a(E) \cap B_*^a(E).$$

In other words, Ann has a validated belief in E if she believes it and $\mathrm{proj}_{S^b} E$ is one of the objects of her thought experiments. Appendix A provides a preference based characterization of validated beliefs.

3.3.1 Lists Made of Subsets of Strategies Suffice for Breaking Ties

Before proceeding further, let us stress the following important property. The principle behind event-rationality is that a player goes beyond her "frame of mind" to break ties. With a formal type structure, the frame of mind is given by a type t^a and the associated assessment $\mu^a(t^a)$ over $S^b \times T^b$ (note that the list $\ell^a(t^a)$ captures what is beyond the frame of mind). Hence, one could argue that we should consider lists over subsets of $S^b \times T^b$, thereby treating strategies and types symmetrically. In fact, the inclusion/exclusion tension identified by Samuelson (1992) could be interpreted as requiring that the player includes "everything else" in her thought experiments.[11]

However, it is redundant to include lists of subsets of $S^b \times T^b$ for tie-breaking purposes: a list ℓ made of subsets E^b of S^b breaks ties between s^a and σ^a if and only if a list $\hat{\ell}$ made of subsets E of $S^b \times T^b$ whose projections on S^b are given by the subsets E^b of the list ℓ also breaks ties between s^a and σ^a. This is obvious, as types are payoff irrelevant.

Moreover, if one insists in using lists $\hat{\ell}$ of subsets of $S^b \times T^b$, the analysis below would follow on exactly the same lines, defining validated beliefs using the operator

$$\hat{B}_*^a(E) = \{t^a \in T^a : E \in \hat{\ell}^a(t^a)\}$$

in the place of the operator B_*^a, where $\hat{\ell}^a(t^a)$ would denote the list of subsets of $S^b \times T^b$ associated with type t^a. In fact, as we just argued, tie-breaking purposes would not restrict the "type" component of the lists $\hat{\ell}$. In Appendix B.1, we show that nothing relevant would be changed in the analysis below. Thus, the seemingly asymmetric treatment of strategies and types is irrelevant, as a symmetric analysis can be provided with the appropriate changes in notation.

[11] This logic is employed in BFK.

3.4 RCBER - Rationality and Common Belief of Event-Rationality

With type structures, a state for Ann is a pair (s^a, t^a) determining what she plays (s^a) and her state of mind (t^a). A strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is event-rational if s^a is event-rational with respect to $\lambda^a(t^a) = (\text{marg}_{S^b}\mu^a(t^a), \ell^a(t^a))$. We therefore have the following definition.

Definition 5. *Strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is event-rational if*

- $s^a \in BR^a(v)$, *for* $v = marg_{S^b}\mu^a(t^a)$,
- *for each* $F \in \ell^a(t^a)$ *with* $F \setminus supp\ v \neq \emptyset$ *and mixed strategy* $\sigma^a \in \Delta(S^a)$ *with* $\sigma^a \sim_{supp\ v} s^a$, *there exists a* $v' \in \Delta(S^b)$ *with* $supp\ v' = F \setminus supp\ v$ *such that* $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$.

Let R_1^a be the set of event-rational strategy-type pairs (s^a, t^a). For finite m, define R_m^a inductively by

$$R_{m+1}^a = R_m^a \cap [S^a \times B^a(R_m^b)].$$

Similarly for b.

Definition 6. *If $(s^a, t^a, s^b, t^b) \in R_{m+1}^a \times R_{m+1}^b$, say there is event-rationality and mth-order belief of event-rationality (RmBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} R_m^a \times \bigcap_{m=1}^{\infty} R_m^b$ say there is event-rationality and common belief of event-rationality (RCBER) at this state.*

In words, there is RCBER at a state if Ann is event-rational, Ann believes that Bob is event-rational, Ann believes that Bob believes that Ann is event-rational, and so on. Similarly for Bob. Believing that Bob is event-rational means that Ann is certain that Bob only chooses strategies that are best responses to Bob's conjectures that Ann considers possible, and that Bob breaks ties using the sets of strategies in his list.

Note that for a strategy-type pair (s^a, t^a) to belong to R_m^a the following conditions are satisfied. Strategy s^a is a best response to $v = \text{marg}_{S^b}\mu^a(t^a)$, $\mu^a(t^a)(R_{m-1}^b) = 1$ and whenever $\sigma^a \sim_{\text{supp}\ v} s^a$, for each $E^b \in \ell^a(t^a)$, there exists a probability measure v' in $E^b \setminus \text{supp}\ v$ for which $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. Notice that Ann is certain that the conjectures of Bob are of the form $v = \text{marg}_{S^a}\mu^b(t^b)$, for $t^b \in \text{proj}_{T^b} R_{m-1}^b$, and knows that, for each such conjecture, Bob breaks each tie using some v' with support in $E^b \setminus \text{supp}\ v$. We show below that this flexibility implies that the set of strategies compatible with RCBER are the ones that survive one round of elimination of inadmissible strategies, followed by iterated elimination of strongly dominated strategies.

3.5 RCvBER - Rationality and Common validated Belief of Event-Rationality

Let $\overline{R}_1^a$ be the set of event-rational strategy-type pairs (s^a, t^a). For finite m, define $\overline{R}_m^a$ inductively by

$$\overline{R}_{m+1}^a = \overline{R}_m^a \cap [S^a \times B_v^a(\overline{R}_m^b)].$$

Similarly for b.

The only difference with RCBER is that we use the validated belief operator instead of the standard one.

Definition 7. *If $(s^a, t^a, s^b, t^b) \in \overline{R}_{m+1}^a \times \overline{R}_{m+1}^b$, say there is event-rationality and mth-order validated belief of event-rationality (RmvBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \bigcap_{m=1}^{\infty} \overline{R}_m^b$ say there is event-rationality and common validated belief of event-rationality (RCvBER) at this state.*

Because validated beliefs are stronger than standard beliefs, RCvBER $\subseteq$ RCBER. Note again that RCBER and RCvBER avoid the inclusion-exclusion tension. What a type t^a of Ann believes about Bob's choices is given by the marginal of $\mu^a(t^a)$ over S^b. Moreover, a type that knows that Bob's strategy-type pairs are in $\overline{R}_m^b$ is a type that assigns positive probability only to the strategies that are consistent with $\overline{R}_m^b$. Therefore, many of Bob's strategies can be excluded from t^a's consideration, without causing any contradiction in the construction. The event-rational strategy-type pair (s^a, t^a) resorts to the tie-breaking list $\ell^a(t^a)$ to handle counter-factuals, without having to believe that the counter-factuals are a real possibility.

4 Solution Concepts

Consider the following generalization of the definition in BFK of the support of a strategy s^a, which they denote $\mathrm{su}(s^a)$.

Definition 8. *Say that r^a supports s^a given Q^b if there exists some $\sigma^a \in \Delta(S^a)$ with $r^a \in$ supp σ^a and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in Q^b$. Write $su_{Q^b}(s^a)$ for the set of $r^a \in S^a$ that supports s^a given Q^b. Likewise for b.*

Therefore, $\mathrm{su}_{S^b}(s^a) = \mathrm{su}(s^a)$. BFK characterize rationality and common assumption of rationality (RCAR) by the solution concept of a self-admissible set (SAS).

Definition 9. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an SAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{S^b}(s^a)$, then $r^a \in Q^a$.*

Likewise for b.

In particular, BFK show that the projection of the RCAR into $S^a \times S^b$ is an SAS. Conversely, given an SAS $Q^a \times Q^b$, there is a type structure such that the projection of RCAR into $S^a \times S^b$ is equal to $Q^a \times Q^b$. BFK discuss the need for the third requirement in the definition of an SAS. In particular, consider the weak best response sets (WBRS), which does not include a restriction on convex combinations.

Definition 10. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is a WBRS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is not strongly dominated with respect to $S^a \times Q^b$.*

Likewise for b.

As Brandenburger (1992) and Börgers (1994) show, if common assumption of rationality is relaxed to common belief at level 0 of rationality (RCB0R) (that is, believing E means $\mu_0(E) = 1$, where μ_0 is the first measure of the agent's LPS), then the projection of RCB0R into $S^a \times S^b$ is a WBRS. Conversely, given a WBRS $Q^a \times Q^b$, there is a type structure such that $Q^a \times Q^b$ is contained in (but not necessarily equal to) the projection of RCB0R into $S^a \times S^b$.[12]

Our main result is the characterization of RCBER and RCvBER in complete, compact and continuous type structures, with two solution concepts, $S^\infty W$ and IA, respectively.[13] The first, $S^\infty W$, is the set of strategies that survive one round of deletion of inadmissible strategies followed by iterated deletion of strongly dominated strategies (Dekel and Fudenberg (1990)).

[12] See Section 11 in BFK.

[13] In Appendix C we also characterize RCBER and RCvBER in the case where the type spaces are not complete but satisfy a richness condition.

Definition 11. *Let* $SW_1^i = S_1^i$, *for* $i = a, b$ *be the set admissible strategies and define inductively, for* $m \geq 1$,

$$SW_{m+1}^i = \{s^i \in SW_m^i : s^i \text{ is not strongly dominated with respect to } SW_m^a \times SW_m^b\}.$$

Let $S^\infty W = \bigcap_{m=1}^\infty SW_m^a \times \bigcap_{m=1}^\infty SW_m^a$.

The second, IA, is the set of strategies that survive iterated deletion of weakly dominated strategies.

Definition 12. *Set* $S_0^i = S^i$ *for* $i = a, b$ *and define inductively, for* $m \geq 0$,

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is admissible with respect to } S_m^a \times S_m^b\}.$$

A strategy $s^i \in S_m^i$ *is called m-admissible. A strategy* $s^i \in \bigcap_{m=0}^\infty S_m^i$ *is called iteratively admissible (IA).*

With a view to compare RCvBER with RCAR of BFK, we introduce the following generalization of the SAS.

Definition 13. *The set* $Q^a \times Q^b \subseteq S^a \times S^b$ *is an* $SAS_{P^a \times P^b}$ *if:*

- *each* $s^a \in Q^a$ *is admissible with respect to* $S^a \times S^b$,
- *each* $s^a \in Q^a$ *is admissible with respect to* $S^a \times Q^b$,
- *for any* $s^a \in Q^a$, *if* $r^a \in su_{P^b}(s^a)$ *and* r^a *is admissible with respect to* $S^a \times S^b$, *then* $r^a \in Q^a$.

Likewise for b.

Note that the only difference with an SAS is that the support $\mathrm{su}_{P^b}(s^a)$ is with respect to an abstract set P^b, not S^b. This means that the SAS is equivalent to the $\mathrm{SAS}_{S^a \times S^b}$.[14] Moreover, if $Q^a \times Q^b \subseteq P^a \times P^b$ then an $\mathrm{SAS}_{Q^a \times Q^b}$ is also an $\mathrm{SAS}_{P^a \times P^b}$, but the reverse may not hold. This means that for any $P^a \times P^b$, an $\mathrm{SAS}_{P^a \times P^b}$ is also an SAS.

[14]Note that if $r^a \in \mathrm{su}_{S^b}(s^a)$ and s^a is admissible, then r^a is also admissible. Hence, the third condition for an $\mathrm{SAS}_{S^a \times S^b}$ is identical to the third condition for a SAS.

5 Characterization of RCBER and RCvBER

Propositions 1 and 2 below show that, in a complete type structure and for big enough m, RCmBER generates the $S^{\infty}W$ set and RmvBER generates the IA set.

Proposition 1. *Fix a complete structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. Then, for each m,*

$$proj_{S^a} R^a_m \times proj_{S^b} R^b_m = SW^a_m \times SW^b_m.$$

Proof. Let T^a_0 be the set of types t^a such that $\ell^a(t^a) = \{S^b\}$. From Lemma 3 we have that $(s^a, t^a) \in R^a_1$ implies s^a is admissible. Conversely, since we have a complete structure, if s^a is admissible then there exists $t^a \in T^a_0$ such that $(s^a, t^a) \in R^a_1$. Hence, $\text{proj}_{S^a} R^a_1 = S^a_1 = SW^a_1$ and $\text{proj}_{S^b} R^b_1 = S^b_1 = SW^b_1$. Suppose that for up to m we have that $\text{proj}_{S^a} R^a_m = SW^a_m$ and $\text{proj}_{S^b} R^b_m = SW^b_m$. Suppose $s^a \in SW^a_{m+1}$. Then, $s^a \in SW^a_m = \text{proj}_{S^a} R^a_m$. Because s^a is not strongly dominated with respect to $SW^a_m \times SW^b_m$, it is also not strongly dominated with respect to $S^a \times SW^b_m$. Hence, there is a v with supp $v \subseteq SW^b_m$ under which s^a is optimal. We take (s^a, t^a), $t^a \in T^a_0$, with supp $\mu^a(t^a) \subseteq R^b_m$ and $\text{marg}_{S^b} \mu^a(t^a) = v$. Because s^a is admissible with respect to S^b, (s^a, t^a) is event-rational. Because $t^a \in B^a(R^b_m)$ and $R^b_m \subseteq R^b_k$, $1 \le k \le m$, we have that $(s^a, t^a) \in R^a_{m+1}$ and $s^a \in \text{proj}_{S^a} R^a_{m+1}$.

Suppose $s^a \in \text{proj}_{S^a} R^a_{m+1}$. Then, $s^a \in SW^a_m = \text{proj}_{S^a} R^a_m$ and supp $\text{marg}_{S^b} \mu^a(t^a) \subseteq SW^b_m = \text{proj}_{S^b} R^b_m$. Because s^a is optimal under v, where supp $v \subseteq SW^b_m$, s^a is not strongly dominated with respect to SW^b_m and therefore $s^a \in SW^a_{m+1}$. $\square$

Proposition 2. *Fix a complete type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. Then, for each m,*

$$proj_{S^a} \overline{R}^a_m \times proj_{S^b} \overline{R}^b_m = S^a_m \times S^b_m.$$

Proof. For $m = 1$, Lemma 3 and a complete structure imply $\text{proj}_{S^a} \overline{R}^a_1 = S^a_1$. Suppose that for up to m we have that $\text{proj}_{S^a} \overline{R}^a_m = S^a_m$ and $\text{proj}_{S^b} \overline{R}^b_m = S^b_m$. Suppose $s^a \in S^a_{m+1}$. Then, $s^a \in S^a_m = \text{proj}_{S^a} \overline{R}^a_m$. Because s^a is admissible with respect to $S^a_m \times S^b_m$, it is also admissible with respect to $S^a \times S^b_m$. Note that $S^b_m \subseteq \dots S^b_1 \subseteq S^b$ and take t^a such that $\text{marg}_{S^b} \mu^a(t^a) = v$, $\ell^a(t^a) = \{S^b, S^b_1, \dots, S^b_m\}$. Because s^a is admissible with respect to $S^a \times S^b_m$, we can choose v such that supp$v = S^b_m$ and s^a is best response to v. Therefore, supp$\mu^a(t^a) = \overline{R}^b_m$. Take any set $S^b_i \in \ell(t^a)$ with $S^b_i \setminus S^b_m \neq \emptyset$ and mixed strategy σ^a such that $\sigma^a \sim_{S^b_m} s^a$. Suppose there exists no measure v', with supp$v' = S^b_i \setminus$ supp$v$, such that $\pi^a(s^a, v') \ge \pi^a(\sigma^a, v')$. Then, σ^a weakly dominates s^a on S^b_i, which implies that s^a is not admissible with respect to $S^a \times S^b_i$, a contradiction. Therefore, (s^a, t^a) is event-rational and $t^a \in B^a_v(\overline{R}^b_k)$ for all $k \le m$, which implies that $(s^a, t^a) \in \overline{R}^a_{m+1}$ and $s^a \in \text{proj}_{S^a} \overline{R}^a_{m+1}$.

Suppose $s^a \in \text{proj}_{S^a}\overline{R}^a_{m+1}$. Then, $s^a \in S^a_m = \text{proj}_{S^a}\overline{R}^a_m$ and there exists t^a such that $(s^a, t^a) \in \overline{R}^a_{m+1}$ and supp $\text{marg}_{S^b}\mu^a(t^a) \subseteq S^b_m = \text{proj}_{S^b}\overline{R}^b_m$. Because $t^a \in B^a_v(\overline{R}^b_m)$, $S^b_m \in \ell^a(t^a)$. Hence, we have that s^a is admissible with respect to $S^a_m \times S^b_m$ and $s^a \in S^a_{m+1}$. □

5.1 Comparison with BFK

BFK's LEU-based approach uses the following construction. Let $\mathcal{L}^+(X)$ be the space of fully supported lexicographic probability systems over X, that is, the space of finite sequences $\sigma = (\mu_0, \ldots, \mu_{n-1})$, for some integer n, where $\mu_i \in \Delta(X)$ and $\bigcup_{i=0}^{n-1} \text{supp } \mu_i = X$. In addition, the probability measures μ_i in σ are required to be non-overlapping, that is, mutually singular. A lexicographic type structure is a type structure where $\lambda^a : T^a \to \mathcal{L}^+(S^b \times T^b)$, and similarly for b. An event E is **assumed** by type t^a of Ann if and only if there is a level j such that $\lambda^a(t^a)$ assigns probability one to the event E for all levels $k \leq j$, and assigns probability zero to the event for all levels $k > j$. Yang (2009) uses a weaker notion that allows the levels higher than j to assign positive (and strictly smaller than 1) weights to the event. The use of lexicographic beliefs is to be contrasted with our use of standard beliefs.

RCAR in BFK is characterized by the SAS and RmAR (m levels of mutual assumption) produces the IA set in a complete structure, for big enough m. Since RmvBER generates the IA set as well, it is important to study the relationship between RCAR and RCvBER in terms of the solution concepts they generate. The following Proposition and examples show that RCvBER generates a strict subclass of SAS, hence it is a more restrictive notion than RCAR. However, as we show in the following section, RCvBER and RCBER are always nonempty in a complete, continuous and compact structure, unlike RCAR. Let A^a and A^b be the set of Ann's and Bob's admissible strategies, respectively.

Proposition 3.

(i) Fix a type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$ is an $SAS_{A^a \times A^b}$.

(ii) Fix an $SAS_{Q^a \times Q^b}$ $Q^a \times Q^b$. Then there is a type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$.

Proof. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$ is empty, then the conditions for $\text{SAS}_{A^a \times A^b}$ are satisfied, so suppose that it is nonempty. By definition of event-rationality and Lemma 3, each $s^a \in Q^a = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a$ is admissible with respect to $S^a \times S^b$ and $S^a \times Q^b$.

Suppose $s^a \in Q^a$, $r^a \in \text{su}_{A^b}(s^a)$ and r^a is admissible. This implies that for any t^a, $(s^a, t^a) \in \bigcap_{m=1}^{\infty} \overline{R}_m^a$ implies that supp $\text{marg}_{S^b}\mu^a(t^a) \subseteq A^b$ and r^a is optimal under $v = \text{marg}_{S^b}\mu^a(t^a)$ (Lemma D.2 in BFK). Because r^a is admissible we have that $(r^a, t^a) \in \overline{R}_1^a$. For each $m \geq 2$, $(s^a, t^a) \in \overline{R}_m^a$ implies that t^a has a validated belief in R_{m-1}^b. Because $\text{proj}_{S^b} R_{m-1}^b \subseteq A^b$ and $r^a \in \text{su}_{A^b}(s^a)$, we have that $(r^a, t^a) \in \overline{R}_m^a$ and $r^a \in Q^a$.

For part (ii) fix an $\text{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to Q^b, there is a v with supp $v = Q^b$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q^b}(s^a)$ (Lemma D.4 in BFK). Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that supp $\mu^a(s^a) = \{(s^b, s^b) : s^b \in Q^b\}$ and supp $\mu^b(s^b) = \{(s^a, s^a) : s^a \in Q^a\}$; and $\ell^a(s^a) = \{S^b\}$ and $\ell^b(s^b) = \{S^a\}$ for all s^a and s^b.

We first show that $Q^a = \text{proj}_{S^a}\overline{R}_1^a$ and $Q^b = \text{proj}_{S^b}\overline{R}_1^b$. By construction, for each $s^a \in Q^a$, s^a is optimal under $v = \text{marg}_{S^b}\mu^a(s^a)$ and admissible. Hence, (s^a, s^a) is event-rational and $Q^a \subseteq \text{proj}_{S^a}\overline{R}_1^a$. Suppose $(r^a, t^a) \in \overline{R}_1^a$, where $t^a = s^a$. Then, $r^a \in \text{su}_{Q^b}(s^a)$ and r^a is admissible with respect to both $S^a \times Q^b$ and $S^a \times S^b$. From the definition of an $\text{SAS}_{Q^a \times Q^b}$ this implies that $r^a \in Q^a$ and $Q^a = \text{proj}_{S^a}\overline{R}_1^a$. Applying similar arguments we have that $Q^b = \text{proj}_{S^b}\overline{R}_1^b$.

Moreover, each type $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b), which consists of event-rational strategy-type pairs, hence t^a has a validated belief in $\overline{R}_1^b$. Since all types only consider the list $\{S^b\}$ as possible, we have that $\overline{R}_m^a = \overline{R}_1^a$ and $\overline{R}_m^b = \overline{R}_1^b$ for all m, by induction. Since $\text{proj}_{S^a}\overline{R}_1^a \times \text{proj}_{S^b}\overline{R}_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$. $\square$

In words, for a given type structure, the strategies compatible with RCvBER form a subclass of all of the SAS, and there is a class of SAS (the $Q^a \times Q^b$ sets that are $\text{SAS}_{Q^a \times Q^b}$) whose strategies are compatible with RCvBER for some type structure. Because an $\text{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is an $\text{SAS}_{A^a \times A^b}$ but the converse is not true, Proposition 3 does not provide a characterization of RCvBER. It does show, however, that RCAR, which is characterized by SAS (BFK, Proposition 8.1), is less restrictive than RCvBER.

In fact, the following game provides an example of an SAS that is not an $\text{SAS}_{A^a \times A^b}$ and cannot be generated by RCvBER for any type structure. Hence, RCvBER generates a strict subclass of SAS.

	L	C	R
U	$1,1$	$2,1$	$1,1$
M	$2,2$	$0,1$	$1,0$
D	$0,1$	$4,2$	$0,0$

Note that all strategies except for R are admissible and that $\{\text{U}\} \times \{\text{L}, \text{C}\}$ is an SAS but not an $\text{SAS}_{A^a \times A^b}$. The reason is that D and M are in the support of a mixed strategy (assigning weight 1/2 to each) that is equivalent to U given that Bob plays his admissible strategies L and C, but not given the set of all strategies S^b. Since D and M are not included in $\{\text{U}\} \times \{\text{L}, \text{C}\}$, this is not an $\text{SAS}_{A^a \times A^b}$.

We now argue that $\{\text{U}\} \times \{\text{L}, \text{C}\}$ cannot be the outcome of RCvBER. First, note that if this were the case, the types of Ann included in RCvBER should assign zero probability to Bob playing R. Note also that U is a best response only when $Pr(\text{L}) = \frac{2}{3}$ and $Pr(\text{C}) = \frac{1}{3}$ and, for these conjectures, also M and D are best responses. Is it possible that M and D are excluded because types playing these strategies are not $\{\text{L}, \text{C}\}$-rational or S^b-rational? No, because M and D are admissible with respect to both $\{\text{L}, \text{C}\}$ and S^b. Hence, under RCvBER, for any type structure, whenever U is included, M and D are included as well.

In the following game all strategies are admissible, hence an SAS is equivalent to an $\text{SAS}_{A^a \times A^b}$.

	L	C	R
U	$1,1$	$2,1$	$1,1$
M	$2,2$	$0,1$	$1,5$
D	$0,1$	$4,2$	$0,0$

The same arguments show that RCvBER cannot produce $\{\text{U}\} \times \{\text{L}, \text{C}\}$ which is both an SAS and an $\text{SAS}_{A^a \times A^b}$ but not an $\text{SAS}_{Q^a \times Q^b}$. Hence, we cannot have a tighter characterization in terms of Proposition 3.

As a last comparison note that, from the proof of Proposition 2, a type of Ann that is event-rational and has $(m+1)$th order validated belief of event-rationality in a complete type structure, necessarily has the sets S_0^b, S_1^b, ..., S_m^b in the type's tie-breaking list. This gives the intuition behind how RCvBER generates the IA set. In comparison, in BFK a type t^a of Ann that is rational and satisfies $(m+1)$th order assumption of rationality in a complete type structure, necessarily satisfies

$$\forall k \leq m, \exists j, \bigcup_{i \leq j} \text{supp } \mu_i = S_k^b$$

where $(\mu_0, ..., \mu_{n-1})$ is the list of marginals over S^b associated with type t^a.

6 Possibility Results for RCBER and RCvBER

Since the games are assumed to be finite, Propositions 1 and 2 suggest that RmBER and RmvBER generate the $S^\infty W$ and IA sets, respectively, for m large enough. However, an epistemic criterion for $S^\infty W$ and IA has to be the same across all games and therefore independent of m. Below, we show that RCBER and RCvBER are nonempty whenever the type structure is complete, continuous and compact. Recall that the universal type structure (Mertens and Zamir (1985) and Appendix B) satisfies these properties. Hence, we provide an epistemic criterion for $S^\infty W$ and IA.

Proposition 4. *Fix a complete, continuous and compact type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, RCBER and RCvBER are nonempty.*

Proof. First, note that from Propositions 1 and 2, the sets $R_m^a \times R_m^b$ and $\overline{R}_m^a \times \overline{R}_m^b$ are nonempty for each $m \geq 1$.

We first show that R_1^a is closed. Note that T^a is compact. For any sequence (s_n^a, t_n^a) in R_1^a, we have $s_n^a \in BR(v_n^a)$, where $v_n^a = \text{marg}_{S^b} \mu^a(t_n^a)$. If $(s_n^a, t_n^a) \to (s^a, t^a)$, then $v_n^a \to v^a = \text{marg}_{S^b} \mu^a(t^a)$, implying that $s^a \in BR(v^a)$. Also, because S^a is finite, we have $s^a = s_n^a$ for large n, so $s^a \in BR^a(v_n^a)$. Further, because S^b is finite, we can choose a subsequence with supp $v_n^a =$ supp v_k^a for all indices n, k and a fortiori supp $v^a \subset$ supp v_n^a. Let σ^a satisfy $\sigma^a \sim_{\text{supp } v^a} s^a$. If supp $v^a =$ supp v_n^a we have $\sigma^a \sim_{\text{supp } v_n^a} s^a$. Hence, for each $F_i \in \ell^a(t^a)$, there exists v_i with support equal to $F_i \setminus \text{supp } v^a$, such that $\pi^a(s^a, v_i) \geq \pi^a(\sigma^a, v_i)$. If supp $v^a \neq$ supp v_n^a, then because $s^a \in BR^a(v_n^a)$ and $\sigma^a \sim_{\text{supp } v^a} s^a$, it must be that there exists $\eta \in \Delta(S^b)$ with $\pi^a(s^a, \mu) \geq \pi^a(\sigma^a, \eta)$ and supp $\eta =$ supp $v_n^a \setminus$ supp v^a (η can be taken as the conditional of v_n^a on supp $v_n^a \setminus$supp v^a). Now put $\eta' = \alpha\eta + (1-\alpha)v_i$ for some $\alpha \in (0, 1)$, note that supp $\eta' = F_i \setminus \text{supp } v^a$ and that $\pi^a(s^a, \eta') \geq \pi^a(\sigma^a, \eta')$. That is, $(s^a, t^a) \in R_1^a$, so it is a closed subset of the compact space $S^a \times T^a$.

Consider $R_2^a = R_1^a \cap [S^a \times B^a(R_1^b)]$, and pick a convergent sequence (s_n^a, t_n^a) therein, with limit (s^a, t^a). Because R_1^b is closed and λ^a is continuous, we have $\limsup_{t_n^a \to t^a} \mu^a(t_n^a)(R_1^b) \leq \mu^a(t^a)(R_1^b)$. Hence $\mu^a(t^a)(R_1^b) = 1$ because $\mu^a(t_n^a)(R_1^b) = 1$ for every n. Also, event-rationality follows from an argument similar to the argument above, and we conclude that R^a_2 is compact. Inductively, R^a_m is compact for all m. It follows that $\bigcap_{m\geq 1} R^a_m \neq \emptyset$ because the family $\{R^a_m\}_{m\geq 1}$ has the finite intersection property: for any finite list $\{m_1, \dots, m_K\}$ of positive numbers, let $m_{\overline{k}}$ be the largest. Then we know that $R^a_{m_{\overline{k}}} \neq \emptyset$ and it is included in $\bigcap_{k=1}^{K} R^a_{m_k}$.

We also have compactness of the sets $\overline{R}^a_m$. Pick a sequence (s^a_n, t^a_n) in $\overline{R}^a_m$ converging to (s^a, t^a), and without loss of generality focus on a subsequence with $\ell^a(t^a_n) = \ell^a(t^a_k)$ for all n, k. It must then be that $\ell^a(t^a_n) = \ell^a(t^a)$. Repeat the argument in the first paragraph of the proof to conclude that (s^a, t^a) is event-rational because (s^a_n, t^a_n) is event-rational for each n, and $\text{proj}_{S^b}\overline{R}^b_{m-1} \in \ell^a(t^a)$, so $(s^a, t^a) \in \overline{R}^a_m$. Hence we have a nested sequence of nonempty compact spaces, so by the finite intersection property, we have $\bigcap_{m\geq 1} R^a_m \neq \emptyset$.

The same arguments apply to b.

□

7 Conclusion

Let us summarize the contributions of the paper. (1) We define a new notion of rationality, named *event-rationality*, and provide preference basis for it. The preferences of event-rational players are represented by a pair (v, ℓ), where v is a probability measure and ℓ is a set of events, used for breaking ties. We require that the set of all strategies of the opponent is a member of ℓ, obtaining as a result that event-rational agents play admissible strategies. (2) We define and provide decision theoretic foundations for a new notion of "believing", named *validated belief*, which relates to the preference representation of event-rationality. (3) We provide epistemic conditions for two well-known solution concepts in game theory, $S^\infty W$ and IA. We do so by constructing the set of states where "rationality and common belief of rationality" obtain, using event-rationality as the notion of rationality and (for the IA case) validated belief as the notion of belief. The epistemic characterization of IA solves a well-known and much-studied problem in a novel way *without* requiring the use of incomplete or discontinuous type structures. (4) We show that RCvBER can be used to justify a strictly smaller class of solutions than BFK's RCAR, thus showing that RCvBER and RCAR are not merely isomorphic conditions written in two different languages. (5) Finally, let us note that Appendix C provides two new solution concepts, HAS and HIA, that characterize RCBER and RCvBER respectively, when type spaces are not necessarily complete but satisfy an alternative richness condition.

A Preference Basis

We develop preference foundations for event-rationality and validated beliefs, using the idea that a decision maker is represented by a list of preferences. Let Ω be a state space and $\mathcal{A}$ the set of all measurable functions from Ω to $[0, 1]$. For simplicity, assume that Ω is finite (abstracting from technical details, the considerations below carry through in a more general state space). A decision maker has preferences over elements of $\mathcal{A}$. We assume that the outcome space $[0, 1]$ is in utils. That is, all preferences considered below agree on constant acts over an outcome space, so the Bernoulli indices are uniquely defined and omitted from the analysis that follows. For $x, y \in \mathcal{A}$, $0 \leq \alpha \leq 1$, $\alpha x + (1 - \alpha)y$ is the act that at ω gives payoff $\alpha x(\omega) + (1 - \alpha)y(\omega)$. Unless otherwise noted, we assume that a preference relation $\succsim$ satisfies completeness, transitivity, independence and has an expected utility representation.

Definition 14. *$x \succsim_E y$ if for some $z \in \mathcal{A}$, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$.*

Note that for preferences satisfying the aforementioned axioms, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$ holds for all z if it holds for some z. An event E is **Savage null** if $x \sim_E y$ for all $x, y \in \mathcal{A}$. For a given $\succsim$, the set $N(\succsim) \subset \Omega$ denotes the union of all non Savage null events according to $\succsim$.

Fix a game and the resulting set of available acts $\mathcal{B}$. An act $x \in \mathcal{B}$ is **event-rational** if there exist a preference $\succsim$ and a list $\ell = \{F_1, ..., F_k\}$, with $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = \Omega$ such that

- $x \succsim y$ for every $y \in \mathcal{B}$,
- for each $F_i \in \ell$ with $F_i \setminus N(\succsim) \neq \emptyset$ and act $y \in \mathcal{B}$ with $x(\omega) = y(\omega)$ for all $\omega \in N(\succsim)$, there exists a preference $\succsim'$ with $N(\succsim') = F_i \setminus N(\succsim)$ such that $x \succsim' y$.

Therefore, the definition of event-rationality is identical to that of the main text.

Consider a decision maker represented by a list of preferences $\{\succsim_i\}_{i=0}^k$ with $N(\succsim_i) \cap N(\succsim_0) = \varnothing$ for $i = 1, ..., k$ and $N(\succsim_1) \subsetneq N(\succsim_2) \subsetneq \cdots \subsetneq N(\succsim_k) = \Omega \setminus N(\succsim_0)$.[15] The interpretation is that $N(\succsim_0)$ is the theory of the decision maker, and the list $\{N(\succsim_i)\}_{i=1}^k$ represents the thought experiments used to break ties. Formally, given a list of preferences $\{\succsim_i\}_{i=0}^k$ satisfying the aforementioned two properties we define an induced preference relation over acts, $\succsim^c$, as follows:

[15] One can think of conditional preferences, as in Luce and Krantz (1971), Fishburn (1973) and Ghirardato (2002).

Definition 15. *$x \succsim^c y$ if and only if either*

- *$x \succsim_0 y$ and $x \neq y$ on $N(\succsim_0)$ or*
- *$x = y$ on $N(\succsim_0)$ and $x \succsim_i y$ for $i = 1, ..., k$.*

An act x is $\succsim^c$-rational if $x \succsim^c y$ for every $y \in \mathcal{B}$.

Proposition 5. *An act x is $\succsim^c$-rational if and only if it is event-rational.*

Proof. By definition, if x is $\succsim^c$-rational, then it is event-rational under $\succsim = \succsim_0$ and $\ell = \{F_1, ..., F_k\}$, with $F_i = N(\succsim_i) \cup N(\succsim_0)$ for $i = 1, ..., k$.

Conversely, let x be event-rational under $\hat{\succsim}$ and $\ell = \{F_1, ..., F_k\}$. If $x \neq y$ on $N(\hat{\succsim})$, then $x \succsim^c y$ using $\succsim_0 = \hat{\succsim}$. So let us focus on acts in $C = \{y \in \mathcal{B} : y = x \text{ on } N(\hat{\succsim})\}$. Let $m = \#\Omega \setminus N(\hat{\succsim})$, and note that the set C can be identified as a convex in $[0,1]^m$, with $x \in C$. For each $i = 1, ..., k$ where $E_i = F_i \setminus N(\hat{\succsim}) \neq \emptyset$, let $B_i = \{r \in \mathbb{R}^m_+ : r|_{E_i} \gg x|_{E_i}\}$, where $x|_{E_i}$ denotes the vector x restricted to states in E_i. Note that $B_i \cap C = \emptyset$, because otherwise there would exist an act y that is outcome-equivalent to x and strictly preferred to x for any preference $\succsim'$ with $N(\succsim') = E_i$, contradicting event-rationality of x. Because B_i is also convex, by the separating hyperplane theorem there exists $\alpha_i \in \mathbb{R}^m$ with $\alpha_i \cdot r > \alpha_i \cdot y$ for all $r \in B_i$ and $y \in C$. Take $r^\varepsilon \in \mathbb{R}^m_+$ with $r^\varepsilon(\omega) = x(\omega)$ for $\omega \notin E_i$ and $r^\varepsilon(\omega) = x(\omega) + \varepsilon$ for $\omega \in E_i$ and $\varepsilon > 0$. Then $r^\varepsilon \in B_i$. Letting $\varepsilon \to 0$, we have $r^\varepsilon \to x$ and we obtain $\alpha_i \cdot x \geq \alpha_i \cdot y$ for every $y \in C$.

Also, α_i can be chosen to satisfy $\alpha_i(\omega) > 0$ only if $\omega \in E_i$. Otherwise, say that $\alpha_i(\omega') > 0$ and $\omega' \notin E_i$. If $y(\omega') = 0$ for every act in $\mathcal{B}$, then $\alpha_i(\omega')$ can be set equal to zero without loss. If $x(\omega') = 0$ and there exists $y \in C$ with $y(\omega') > 0$, then it cannot be the case that $F_i = \{\omega'\}$ for any $i = 1, ..., k$. So set $y(\omega) = x(\omega)$ for every $\omega \neq \omega'$ and $y(\omega') > x(\omega')$, with $y \in C$. Such a y exists because $E_i \neq \Omega \setminus N(\hat{\succsim})$ (if it was equal, then ω' would not exist) and there is no F_i equal to $\{\omega'\}$. Then $\alpha_i \cdot r^\varepsilon > \alpha_i \cdot y$, for the r^ε constructed above. But as $\varepsilon \to 0$, $r^\varepsilon \to x$ and $\alpha_i \cdot x < \alpha_i \cdot y$ by construction. This contradicts $\alpha_i \cdot r^\varepsilon > \alpha_i \cdot y$ for all ε. In the case that $x(\omega') > 0$, change the r^ε above by having $r^\varepsilon(\omega') = 0$, while keeping the other values. Then as $\varepsilon \to 0$, we must get $\alpha_i \cdot r^\varepsilon < \alpha_i \cdot x$, another contradiction. So the support of α_i is contained in E_i.

Moreover, because for each $y \in C$ there exists $\succsim'$ with $N(\succsim') = E_i$ and $x \succsim' y$, it must be that $\alpha(\omega) > 0$ if $\omega \in E_i$. If not, then there is $\omega' \in E_i$ with $\alpha_i(\omega') = 0$, and there is no other α_i' with $\alpha_i'(\omega') > 0$ that would separate B_i and C. Now take the original r^ε and $y \in C$ with $y(\omega') > x(\omega')$. Such a y must exist, for otherwise there would exist the required α_i'. But there is no $\succsim'$ with $N(\succsim') = E_i$ and $x \succsim' y$, a contradiction. So it must be that $\alpha_i(\omega) > 0$ if and only if $\omega \in E_i$.

Normalizing α_i yields a probability distribution ν_i with supp $\nu_i = E_i$ for which x is a better response than any $y \in C$. Let $\succsim_i$ be the preference relation represented by the underlying Bernoulli index and ν_i. The construction above is true for every $i = 1, ..., k$. Setting $\succsim_0 = \hat{\succsim}$ and collecting the list $\{\succsim_0, \succsim_1, ..., \succsim_k\}$ it follows that x is $\succsim^c$-rational. $\square$

In what follows, for ease of notation, we use $N_i = N(\succsim_i)$ for $i = 0, ..., k$, $x \succ_{iE} y$ to denote that x is preferred to y according to $\succsim_i$ conditional on E (according to Definition 14), and $x =_{0E} y$ to denote that $x(\omega) = y(\omega)$ for all $\omega \in N_0 \cap E \neq \emptyset$. The notions of beliefs we use in the main text are as follows.

Definition 16. *Event E is* ***believed under*** $\succsim^c$ *if $N_0 \subset E$.*

Definition 17. *Event E has a* ***validated belief under*** $\succsim^c$ *and i if $E = N_0 \cup N_i$.*

In words, the decision maker believes an event E if she believes it according to her theory. She has a validated belief in it if it is equal to the union of N_0 and some N_i. Note that it may well be that $i = 0$, so the decision maker may have a validated belief in the event $E = N_0$. Note that in the text we "validated" a belief with events that describe strategies only. Here we do not make this distinction for ease of exposition. It is straightforward to consider a product state space $\Omega = \Omega_1 \times \Omega_2$ and define belief for events on Ω and validated beliefs as those that are validated by the projection of an N_i to Ω_1.

We now define a notion of conditional $\succsim^c$-preference that is consistent with tie-breaking ideas.

Definition 18. *Say that $x \succ^c_E y$ under i if*

- *$x \succ_{0E} y$ or*
- *$x =_{0E} y$, $x \succ_{iE} y$ and $x \succsim_j y$ for every $j \neq i$.*

Say that $x \succ^c_E y$ if $x \succ^c_E y$ for some i. Note that $x \succ^c_E y$ under i and $x =_{0E} y$ necessarily mean that $i > 0$.

Definition 19. *An event E is* ***nontrivial under*** $\succsim^c$ *and i if*

- *there is a pair x, y with $x \succ^c_E y$ under i, and*

- *if $\omega \in E$ is such that there is no pair x, y with $x \succ^c_\omega y$, then there is a pair x, y with $x = y$ on N_0 such that $x \succ^c_{E(\omega)} y$ under i, where $E(\omega) = E \cap (N_0 \cup \{\omega\})$.*

Definition 20. *An event E satisfies **strict determination under** $\succsim^c$ and i if for all x, y, $x \succ^c_E y$ under i implies $x \succ^c y$.*

The following Lemma characterizes validated belief with respect to nontriviality and strict determination.

Lemma 4. *There exists i such that E has a validated belief under $\succsim^c$ and i if and only if it is nontrivial and satisfies strict determination under $\succsim^c$ and i.*

Proof. By nontriviality, $E \cap N_0 \neq \varnothing$, for otherwise there would exist no pair x, y with $x \succ^c_E y$. Assume by way of contradiction that there exists $\hat{\omega} \in N_0 \setminus E$. Also, let $\omega' \in E \cap N_0$. Set $x(\omega') = 1$ and zero otherwise, and set

$$y(\omega) = \begin{cases} a & \text{if } \omega = \hat{\omega} \\ b & \text{if } \omega = \omega' \\ 0 & \text{otherwise} \end{cases}$$

where $a > \frac{v_0(\omega')(1-b)}{v_0(\hat{\omega})}$, $0 < b < 1$, and v_0 is the conjecture associated with $\succsim_0$. Then, conditional on E, the payoff of x is equal to 1 whereas the payoff of y is $b < 1$, so $x \succ^c_E y$; but the unconditional payoff of x is equal to $v_0(\omega')$ whereas the payoff of y is $av_0(\hat{\omega}) + bv_0(\omega')$, so $y \succ^c x$, contradicting strict determination. Hence $N_0 \subset E$. Therefore, if for all $\omega \in E$ there exists a pair x, y with $x \succ^c_\omega y$, then $E \subset N_0$, and we conclude that $E = N_0 \cup N_i$, with $i = 0$.

If there is $\omega \in E$ for which there is no pair x, y with $x \succ^c_\omega y$, then $\omega \notin N_0$. By nontriviality, there is a pair x, y with $x = y$ on N_0 with $x \succ^c_{E(\omega)} y$ under i, meaning that $x \succ_{iE(\omega)} y$, which in turn means that $\omega \in N_i$ and $i \neq 0$. Hence we must have $E \subset N_0 \cup N_i$. Similarly to above, assume by way of contradiction that there exists $\hat{\omega} \in N_i \setminus E$. Also, let $\omega' \in E \cap N_i$. Construct x and y as follows: $x = y$ on N_0, and on $\Omega \setminus N_0$ x and y are as above, with $a > \frac{v_i(\omega')(1-b)}{v_i(\hat{\omega})}$. Strict determination is again violated, so we must have $N_0 \cup N_i \subset E$, and we conclude that $E = N_0 \cup N_i$ with $i > 0$.

Conversely, assume that $E = N_0 \cup N_i$ for some i. Let $x = 1$ on N_0, 0 otherwise and $y(\omega) = 0$ for every ω. Then $x \succ^c_0 y$ and $x \succ^c_E y$ under i. For the second condition, if $i = 0$, then $E = N_0$ and there does not exist $\omega \in E$ such that there is no pair x, y with $x \succ^c_\omega y$. If $i \neq 0$, pick $\omega \in N_i$ (so $\omega \notin N_0$). Set $x = y$ on N_0, $x(\omega) = 1$, $y(\omega) = 0$ and $x = y = 0$ elsewhere. Then $x \succ^c_{E(\omega)} y$, so nontriviality is satisfied.

Finally, let $x \succ_E^c y$ under i. If $x \succ_{0E} y$ then $x \succ_0 y$, implying that $x \succ^c y$. If $x =_{0E} y$, $x \succ_{iE} y$ and $x \succsim_j y$ for every $j \neq i$, then $x = y$ on N_0, $x \succ_i y$ and $x \succsim_j y$ for every $j \neq i$, which again means that $x \succ^c y$. So strict determination is satisfied.

□

Corollary 1. *An event E is believed under $\succsim^c$ if and only if it satisfies strict determination under $\succsim^c$ and $i = 0$ and there exists a pair x, y with $x \succ_E^c y$ under $i = 0$.*

B Type Spaces

We show that the standard construction of all hierarchies of beliefs about beliefs generates a complete and continuous type structure. Because the types consistent with event-rationality are mapped to both probability measures and lists, we need to adapt the standard construction. One route is to follow Epstein and Wang (1995) and work with more general beliefs about beliefs. Another route, followed bellow, is to construct an complete, continuous and compact auxiliary type structure, using the standard construction, and then use it to derive the desired type structure.

Let $\Delta^*(X \times L^i)$ be the space of all probability measures over $X \times L^i$ (endowed with the weak* topology) for which the marginal on L^i is a mass point, for $i = a, b$.

Let $\Omega_1^a = S^b \times L^b$ and $T_1^a = \Delta^*(S^b \times L^b)$. Inductively, set $\Omega_{k+1}^a = S^b \times L^b \times T_k^b$ where

$$T_{k+1}^a = \{(\mu_1^a, ..., \mu_k^a, \mu_{k+1}^a) \in T_k^a \times \Delta^*(\Omega_{k+1}^a) : \mathrm{marg}_{\Omega_k^a} \mu_{k+1}^a = \mu_k^a\}.$$

Likewise for b. Then, the standard arguments in the literature show the existence of compact spaces T_*^a and T_*^b, with T_*^a homeomorphic to $\Delta^*(S^b \times T_*^b \times L^b)$ and T_*^b homeomorphic to $\Delta^*(S^a \times T_*^a \times L^a)$.[16] In fact, let T_*^a be the projective limit of the spaces $(T_k^a)_{k=1}^\infty$. T_*^a is compact as it is a product of compact spaces. Construct T_*^b similarly. Then, Theorem 8 in Heifetz (1993) shows that, for each tower $(\mu_k^a)_{k=1}^\infty$, there exists $\mu^a \in \Delta(S^b \times L^b \times T_*^b)$ with $\mathrm{marg}_{\Omega_k^a} \mu^a = \mu_k^a$, for all $k \geq 1$. In particular, the marginal of μ^a on L^b is a mass point, so $\mu^a \in \Delta^*(S^b \times L^b \times T_*^b)$. Conversely, each $\mu^a \in \Delta^*(S^b \times L^b \times T_*^b)$ gives rise to a tower $(\mu_k^a)_{k=1}^\infty$, given by the list of marginals. Hence, there is a bijection $\lambda_*^a : T_*^a \to \Delta^*(S^b \times L^b \times T_*^b)$. Theorem 9 in Heifetz (1993) ensures that λ_*^a is a homeomorphism, likewise for b. Therefore, we have constructed a complete, continuous and compact auxiliary type structure

$$\langle S^i, L^i, T_*^i, \lambda_*^i \rangle_{i \in \{a,b\}}$$

[16]See for instance Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Heifetz (1993).

with $\lambda^i_* : T^i_* \to \Delta^*(S^j \times T^j_* \times L^j)$ for $j \neq i = a, b$. Note that $\lambda^i_*(t^i_*) = \mu(t^i_*) \otimes \delta_{\ell(t^i_*)}$ where δ_x is the point mass at x.

Now set $T^i = T^i_*$ (carrying the same topology, so T^i is compact Hausdorff) and $\lambda^i(t^i_*) = (\mu(t^i_*), \ell(t^i_*))$, for $i = a, b$. The assignment $\lambda^i_* \mapsto \lambda^i$ is a bijection and preserves continuity: λ^i is continuous if and only if λ^i_* is continuous. Indeed, let $t^i_\alpha \to t^i$ in T^i. This is a converging net in T^i_*, so $\lambda^i_*(t^i_\alpha) \to \lambda^i_*(t^i)$, or $\mu(t^i_\alpha) \otimes \delta_{\ell(t^i_\alpha)} \to \mu(t^i) \otimes \delta_{\ell(t^i)}$. But $\delta_{\ell(t^i_\alpha)} \to \delta_{\ell(t^i)}$ in the weak* topology if and only if $\ell(t^i_\alpha) \to \ell(t^i)$. So $(\mu(t^i_\alpha), \ell(t^i_\alpha)) \to (\mu(t^i), \ell(t^i))$, or $\lambda^i(t^i_\alpha) \to \lambda^i(t^i)$, for $i = a, b$. A similar argument establishes that λ^i_* is continuous if λ^i is continuous. Moreover, λ^i is injective and surjective. Hence, it is a homeomorphism, as a continuous bijection between compact Hausdorff spaces. Therefore, the type structure

$$\langle S^i, L^i, T^i, \lambda^i \rangle_{i \in \{a,b\}},$$

with $\lambda^i : T^i \to \Delta(S^j \times T^j) \times L^j$ for $j \neq i = a, b$ just constructed, is complete, continuous and compact.

It is important to emphasize a conceptual point here. The two players form beliefs about beliefs about what is relevant for rational choices. That is, Ann has beliefs over $S^b \times L^b$, and these beliefs are given by a conjecture over S^b and a list $\ell \in L^b$ (or, equivalently, a point mass over L^b.) What is relevant for event-rational choices is precisely the conjecture and the list. But Ann does not know what Bob's beliefs are, and the hierarchies of beliefs about beliefs constructed above yield a type structure as the one we use in the paper.

B.1 Lists over Types

We argued in the text that lists over strategies suffice for the analysis. Indeed, it is redundant to include subsets of types in the tie-breaking lists, as types do not play any role in breaking ties. Also, provided that we consider a rich list of subsets of types, such lists would not interfere in the constructions in the text that used validated beliefs. Let us now show how to obtain a type structure with rich lists over strategies and types from a given type structure.

Let the type structure $\langle S^i, L^i, T^i, \lambda^i \rangle_{i \in \{a,b\}}$ be given. For $i \neq j = a, b$, let $\mathcal{F}(T^i)$ denote the space of all closed subsets of T^i, endowed with the Fell topology.[17] Say $\ell^i(t^i) = \{E_1, ..., E_k\}$, with $E_r \subset S^j$ for $r = 1, ..., k$. Let $E_r = \{s^j_1, ..., s^j_m\}$ and construct $\mathcal{E}_r = \{(\{s^j_1\} \times K, ..., \{s^j_m\} \times K') : (K, ..., K') \in (\mathcal{F}(T^j))^m\}$, for $r = 1, ..., k$, where $(\mathcal{F}(T^j))^m$ denotes the product of m

[17]See, for instance, Molchanov (2005) for definitions of topologies on spaces of subsets. The nice feature of the Fell topology is that $\mathcal{F}(T^i)$ is compact whenever T^i is Hausdorff. When T^i is compact metric, the Fell topology coincides with the standard Hausdorff metric topology.

copies of $\mathcal{F}(T^j)$. Note that $\mathcal{E}_r$ is compact whenever T^j is Hausdorff. Finally, put $\hat{\ell}^i(t^i) = \{\mathcal{E}_1, ..., \mathcal{E}_k\}$ as the extended list. Repeat the procedure for all t^i and $i = a, b$, to construct the type structure

$$\langle S^i, \hat{L}^i, T^i, \hat{\lambda}^i \rangle_{i \in \{a,b\}}$$

where $\hat{\lambda}^i = (\mu^i, \hat{\ell}^i)$ and $\hat{L}^i$ is the space of extended lists (as the one constructed above) of subsets of $S^i \times T^i$.

Now, for any closed subset $F \subset S^j \times T^j$, we have

$$F \in \hat{\ell}^i(t^i) \Leftrightarrow \mathrm{proj}_{S^j} F \in \ell^i(t^i).$$

That is, extended lists do not interfere with statements about validated beliefs. Extended lists do not interfere with breaking ties either. So the arguments in the text apply to the corresponding type structure with extended lists with no change (other than notation).

C Other Solution Concepts

In this section we define two new solution concepts that characterize RCBER and RCvBER in all type structures that satisfy a richness condition. The first is Hypo-Admissible Sets (HAS) and we compare it with the solution concepts defined in the main body of the paper.

Definition 21. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an HAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$.*

  For each $s^a \in Q^a$ there is nonempty $Q_0 \subseteq Q^b$ such that

- *s^a is admissible with respect to $S^a \times Q_0$,*

- *for any $s^a \in Q^a$, if $r^a \in su_{Q_0}(s^a)$ and r^a is admissible with respect to $S^a \times S^b$ then $r^a \in Q^a$.*

Likewise for b.

Note that the first two properties for a WBRS are equivalent to the first two properties for an HAS and they are implied by the first two properties for an SAS. Hence, the SAS and the HAS are always WBRS but the opposite does not hold. Moreover, an SAS is not necessarily an HAS and an HAS is not necessarily an SAS.

Note that the $S^\infty W$ set is both an HAS and a WBRS (but not an SAS) and the IA set is an SAS and a WBRS (but not an HAS). The following game from Section 2 illustrates the various definitions:

	L	R
U	$1,0$	$1,3$
M	$0,2$	$2,2$
D	$0,4$	$1,1$

The IA set is $\{M\} \times \{R\}$. It is an SAS but not an HAS, because although $L \in \mathrm{su}_{\{M\}}(R)$ and L is admissible, it does not belong to the IA set. Moreover, $S^\infty W = \{U, M\} \times \{L, R\}$ is an HAS but not an SAS, because L is not admissible with respect to $\{U, M\}$. That is, in a sense the SAS captures IA whereas the HAS captures $S^\infty W$.

The second solution concept is the Hypo-Iteratively Admissible (HIA) set.

Definition 22. *A set $Q^a \times Q^b$ is a hypo-iteratively admissible (HIA) set if there exist sequences of sets $\{W_i^a\}_{i=0}^\infty$, $\{W_i^b\}_{i=0}^\infty$, with $W_0^a = S^a$, $W_0^b = S^b$, such that for each $m \geq 0$,*

- *each $s^a \in W_{m+1}^a$ is admissible with respect to $S^a \times W_m^b$ and belongs to W_m^a,*
- *for any k, m, where $k \geq m$, if $s^a \in W_{k+1}^a$, $r^a \in su_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$, then $r^a \in W_{m+1}^a$,*
- *there is k such that for all $m \geq k$, $W_m^a = Q^a$.*

Likewise for b.

The HIA sets resemble the IA set, with the only difference that one starts with a subset of admissible strategies and always includes the strategies that are equivalent (in the sense of su_Q) to strategies that survive subsequent rounds. Moreover, the HIA can be thought of as an analogue of the best response set (BRS).[18] If we replace admissible with strongly undominated in the definition of HIA then we get a BRS. Conversely, each BRS $Q^a \times Q^b$ can be written as a modified HIA (just set $W_i^a = Q^a$ and $W_i^b = Q^b$ for all $i \geq 1$).

[18]Recall that $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

C.1 Characterizations

Proposition 6 below shows that RCBER is characterized by the HAS set in a rich type structure. We say that a type structure is **rich** if, for each type t^a with $\ell^a(t^a) = (E_1^b, ..., E_n^b)$ and any list ℓ' such that $S^b \in \ell' \subseteq \ell^a(t^a)$, there exists a type t_0^a with $\ell^a(t_0^a) = \ell'$, and $\mu^a(t^a) = \mu^a(t_0^a)$. Similarly for b. Recall our notation: RCBER is given by $\bigcap_{m=1}^{\infty} R_m^a \times \bigcap_{m=1}^{\infty} R_m^b$.

Proposition 6. *(i) Fix a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is an HAS.*

(ii) Fix an HAS $Q^a \times Q^b$. Then there is a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} R_m^b$.

Proof. Throughout we keep the convention that for any two sets, E and F, $E \times F = \emptyset$ implies $E = \emptyset$ and $F = \emptyset$. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is empty, then the conditions for HAS are satisfied, so suppose that it is nonempty and fix $s^a \in Q^a = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a$. Then, for some t^a, (s^a, t^a) is consistent with RCBER and s^a is admissible, by Lemma 3. Since t^a believes each R_m^b, for all m, it also believes $\bigcap_{m=1}^{\infty} R_m^b$. From the conjunction and marginalization properties of belief there is $v = \text{marg}_{S^b}\mu^a(t^a)$, with support contained in $\text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$, such that s^a is optimal under v.

Let $Q_0 = \text{supp } v$. We have that s^a is admissible with respect to $Q_0 = \text{supp } v$, which is a subset of $Q^b = \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. Suppose $s^a \in Q^a$, $r^a \in \text{su}_{\text{supp } v}(s^a)$ and r^a is admissible. From Lemma D.2 in BFK, r^a is optimal under v whenever $(s^a, t^a) \in R_1^a$.[19] Because the type structure is rich, there exists type t_0^a with $\mu^a(t_0^a) = \mu^a(t^a)$ and $\ell^a(t_0^a) = S^b$. Since r^a is admissible, we have that $(r^a, t_0^a) \in R_1^a$. The same is true for all R_m^a, hence the third property for an HAS is satisfied.

For part (ii) fix an HAS $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to $Q_{s^a} \subseteq Q^b$, there is a v with $\text{supp } v = Q_{s^a}$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q_{s^a}}(s^a)$ (Lemma D.4 in BFK).[20] Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that $\text{supp } \mu^a(s^a) = \{(s^b, s^b) : s^b \in Q_{s^a}\}$, $\ell^a(s^a) = \{S^b\}$ and $v = \text{marg}_{S^b}\mu^a(s^a)$ for the v found above. Similarly for b. Note that the type structure is rich.

[19] Lemma D.2 specifies that if F is a face of a polytope P and $x \in F$, then $\text{su}(x) \subseteq F$, where $\text{su}(x)$ is the set of points that support x. The geometry of polytopes is presented in Appendix D in BFK.

[20] Lemma D.4 specifies that if x belongs to a strictly positive face of a polytope P, then $\text{su}(x)$ is a strictly positive face of P.

First, we show that for each $s^a \in Q^a$, (s^a, s^a) is event-rational. By construction, s^a is optimal under $v = \mathrm{marg}_{S^b}\mu^a(s^a)$ and admissible. Hence, (s^a, s^a) is event-rational and $Q^a \subseteq \mathrm{proj}_{S^a} R_1^a$. Suppose $(r^a, t^a) \in R_1^a$, where $t^a = s^a$. Then, $r^a \in \mathrm{su}_{Q_{s^a}}(s^a)$ and r^a is admissible with respect to Q_{s^a}. From Lemma 3, r^a is admissible. From the definition of an HAS this implies that $r^a \in Q^a$ and $Q^a = \mathrm{proj}_{S^a} R_1^a$. Applying similar arguments we have that $Q^b = \mathrm{proj}_{S^b} R_1^b$.

By construction, each $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b) which consists of event-rational strategy-type pairs, hence t^a believes R_1^b and $(s^a, s^a) \in R_2^a$. This implies that $R_2^a = R_1^a$ and likewise for b. Thus, $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m, by induction. Since $\mathrm{proj}_{S^a} R_1^a \times \mathrm{proj}_{S^b} R_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. $\square$

Proposition 7 shows that RCvBER is characterized by the HIA set in a rich type structure. Recall our notation: RCvBER is given by $\bigcap_{m=1}^{\infty} \overline{R}_m^a \times \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proposition 7.

(i) Fix a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $\mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is an HIA set.

(ii) Fix an HIA set $Q^a \times Q^b$. Then there is a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proof. For part (i), if $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is empty, then the conditions for an HIA set are satisfied, so suppose that it is nonempty.

Set $W_m^a = \mathrm{proj}_{S^a} \overline{R}_m^a$ for $m \geq 1$ and likewise for b. From Lemma 3, all strategies in $\mathrm{proj}_{S^b} \overline{R}_{m+1}^a$ are admissible with respect to $S^a \times W_m^b$ and, by construction, belong to $\mathrm{proj}_{S^b} \overline{R}_m^a$.

Suppose that for some k, m, where $k \geq m$, we have that $s^a \in W_{k+1}^a = \mathrm{proj}_{S^b} \overline{R}_{k+1}^a$, $r^a \in \mathrm{su}_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$. This implies that for some t^a, $(s^a, t^a) \in \overline{R}_{k+1}^a$, where $\mathrm{supp}\, \mathrm{marg}_{S^b}\mu^a(t^a) \subseteq W_k^b$ and list $\ell^a(t^a)$ contains at least all sets W_p^b, for $p \leq m$. Because the type structure is rich, there exists type t_0^a, with $\ell^a(t_0^a)$ that contains all sets W_p^b, for $p \leq m$, and nothing else. Moreover, t_0^a is identical to t^a in all other respects. Since $r^a \in \mathrm{su}_{W_k^b}(s^a)$, r^a is optimal given $\mathrm{marg}_{S^b}\mu^a(t_0^a)$. Moreover, r^a is admissible with respect to $S^a \times W_p^b$, for $p \leq m$.

All these imply that $(r^a, t_0^a) \in \overline{R}_{m+1}^a$. The third condition is satisfied because $\mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is nonempty and the strategies are finite.

For part (ii), fix an HIA set $Q^a \times Q^b$, with sequences of sets $\{W^a_m\}_{m=0}^{m=n'}$, $\{W^b_m\}_{m=0}^{m=n}$, where $W^a_{n'} = Q^a$ and $W^b_n = Q^b$. Construct the following type structure. For each $m \geq 1$, for each $s^a \in W^a_m$, find the measure $v(s^a, m)$ with support on W^b_{m-1} such that r^a is a best response to $v(s^a, m)$ if and only if $r^a \in \mathrm{su}_{W^b_{m-1}}(s^a)$. This is possible because of Lemma D.4 in BFK. Type $t^a(s^a, m)$ has a marginal $v(s^a, m)$ on S^b, the list $\ell^a(t^a(s^a, m)) = \{W^b_0, \dots, W^b_{m-1}\}$ on L^b (omitting W^b_{m-j} if it is equal to W^b_{m-j-1}) and assigns positive probability only to strategy-types $(s^b, t^b(s^b, m-1))$, for $s^b \in W^b_{m-1}$. Finally, assign to each $s^a \in S^a$ type $t^a(r^a, 0)$ which is equal to $t^a(r^a, k)$, for some $r^a \in W^a_k$, $k > 0$. Similarly for b.

We now show that RCvBER generates the HIA set. For $m = 1$, we show that $\mathrm{proj}_{S^a}\overline{R}^a_1 = W^a_1$. Suppose that $s^a \in W^a_1$. Because s^a is admissible and a best response to $v(s^a, 1)$, we have $(s^a, t^a(s^a, 1)) \in \overline{R}^a_1$ and $s^a \in \mathrm{proj}_{S^a}\overline{R}^a_1$. Suppose $r^a \in \mathrm{proj}_{S^a}\overline{R}^a_1$. Then, r^a is a best response to some measure $v(s^a, k+1)$, $k \geq 0$, for $s^a \in W^a_{k+1}$ and $r^a \in \mathrm{su}_{W^b_k}(s^a) \cap W^a_0$. Because $(r^a, t^a(s^a, k+1))$ is event-rational, r^a is admissible. Therefore, by the second property for an HIA set, $r^a \in W^a_1$. Moreover, by construction, for each $s^a \in W^a_1$, $(s^a, t^a(s^a, 1)) \in \overline{R}^a_1$, and similarly for b.

Assume that for up to m, $\mathrm{proj}_{S^a}\overline{R}^a_m = W^a_m$ and for each $s^a \in W^a_m$, $(s^a, t^a(s^a, m)) \in \overline{R}^a_m$. Similarly for b. Suppose that $s^a \in W^a_{m+1}$. By construction, s^a is a best response to $v(s^a, m+1)$, which has a support of $W^b_m = \mathrm{proj}_{S^b}\overline{R}^b_m$, and it is admissible with respect to $S^a \times W^b_m$. Moreover, $\ell^a(t^a(s^a, m+1)) = \{W^b_0, \dots, W^b_m\}$ and type $t^a(s^a, m+1)$ assigns positive probability only to types $(s^b, t^b(s^b, m)) \in \overline{R}^b_m$, for $s^b \in W^b_m$. This implies that $(s^a, t^a(s^a, m+1)) \in \overline{R}^a_{m+1}$ and $s^a \in \mathrm{proj}_{S^a}\overline{R}^a_{m+1}$. Suppose $r^a \in \mathrm{proj}_{S^a}\overline{R}^a_{m+1}$. By construction, the only measures that have support which is a subset of W^b_m are measures that are associated with strategies s^a that belong to W^a_{k+1}, where $k+1 > m$. Hence, $(r^a, t^a(s^a, k+1)) \in \overline{R}^a_{m+1}$ and r^a is a best response to some measure $v(s^a, k+1)$. By construction, $r^a \in \mathrm{su}_{W^b_k}(s^a)$. Moreover, r^a is admissible with respect to $S^a \times W^b_m$. Hence, by the second property for an HIA set we have that $r^a \in W^a_{m+1}$.

□

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