ON PICARD GROUPS OF BLOCKS OF FINITE GROUPS

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Abstract. We show that the subgroup of the Picard group of a p-block of a finite group given by bimodules with endopermutation sources modulo the automorphism group of a source algebra is determined locally in terms of the fusion system on a defect group. We show that the Picard group of a block over a complete discrete valuation ring $\mathcal{O}$ of characteristic zero with an algebraic closure k of $\mathbb{F}_p$ as residue field is a colimit of finite Picard groups of blocks over p-adic subrings of $\mathcal{O}$. We apply the results to blocks with an abelian defect group and Frobenius inertial quotient, and specialise this further to blocks with cyclic or Klein four defect groups.

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1. Introduction

Throughout the paper, p is a prime number. Let $\mathcal{O}$ be a complete local principal ideal domain with residue field k of characteristic p. Assume that either $\mathcal{O} = k$ or that $\mathcal{O}$ has characteristic zero. By a *block* of $\mathcal{O}G$ for G a finite group we mean a primitive idempotent b of the center of the group algebra $\mathcal{O}G$, and we call $\mathcal{O}Gb$ a *block algebra* of $\mathcal{O}G$. For B an $\mathcal{O}$-algebra, we denote by $\mathrm{Pic}(B)$ the *Picard group of* B; that is, $\mathrm{Pic}(B)$ is the group of isomorphism classes of (B,B)-bimodules inducing a Morita equivalence, with group product induced by the tensor product over B. If B is symmetric (that is, B is free of finite rank as an $\mathcal{O}$-module and B is isomorphic to its $\mathcal{O}$-dual B^* as a (B,B)-bimodule) and if M is a (B,B)-bimodule inducing a Morita equivalence, then the inverse of its isomorphism class $[M]$ in $\mathrm{Pic}(B)$ is the class $[M^*]$ of the $\mathcal{O}$-dual M^* of M. If G is a finite group, then $\mathcal{O}G$ and the block algebras of $\mathcal{O}G$ are symmetric.

Let G, H be finite groups, and let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$, respectively. We say that a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ *has endopermutation source* if it is given by an $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule M which has an endopermutation $\mathcal{O}R$-module as a source for some vertex R of M, where M is regarded as an $\mathcal{O}(G\times H)$-module. By [28, 7.4, 7.6], a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ with endopermutation source induces an isomorphism $\alpha : P \cong Q$ between defect groups P and Q of b and c, respectively, such that the twisted diagonal subgroup $\Delta\alpha = \{(x, \alpha(x)) \mid x \in P\}$ is a vertex of M. Moreover, α induces an isomorphism between fusion systems of b and of c on P and Q, respectively (for some suitable choice of maximal Brauer

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pairs). Set $B = \mathcal{O}Gb$. By [36, Proposition 9], the Morita equivalences on B with endopermutation source form a subgroup of $\mathrm{Pic}(B)$, which we will denote by $\mathcal{E}(B)$. We denote by $\mathcal{L}(B)$ the subgroup of $\mathcal{E}(B)$ of Morita equivalences given by bimodules with linear source, and by $\mathcal{T}(B)$ the subgroup of $\mathcal{L}(B)$ of Morita equivalences given by bimodules with trivial source (the fact that these are subgroups follows from 2.6 below). If A is a source algebra of a block with defect group P, we denote by $\mathrm{Aut}_P(A)$ the group of algebra automorphisms of A which fix the image of P in A elementwise, and by $\mathrm{Out}_P(A)$ the quotient of $\mathrm{Aut}_P(A)$ by the subgroup of inner automorphisms induced by conjugation with elements in $(A^P)^\times$.

Let P be a finite p-group and V an endopermutation $\mathcal{O}P$-module having an indecomposable direct summand with vertex P. By results of Dade [9], the indecomposable direct summands of V with vertex P are all isomorphic. For any subgroup Q of P, denote by V_Q an indecomposable direct summand of $\mathrm{Res}^P_Q(V)$ with vertex Q. The tensor product of two indecomposable endopermutation $\mathcal{O}P$-modules with vertex P has an indecomposable direct summand with vertex P; this induces an abelian group structure on the set of isomorphism classes of indecomposable endopermutation $\mathcal{O}P$-modules. The resulting group is denoted $D_\mathcal{O}(P)$, called the *Dade group of P over $\mathcal{O}$*.

Let $\mathcal{F}$ be a saturated fusion system on P. We denote by $\mathfrak{foc}(\mathcal{F})$ the subgroup of P generated by elements of the form $\varphi(x)x^{-1}$, where $x \in P$ and $\varphi \in \mathrm{Hom}_\mathcal{F}(\langle x\rangle, P)$. Slightly modifying the terminology in [21, 3.3] we say that an endopermutation $\mathcal{O}P$-module V having an indecomposable direct summand with vertex P is *$\mathcal{F}$-stable* if for every isomorphism $\varphi : Q \to R$ in $\mathcal{F}$ between two subgroups Q, R of P we have $V_Q \cong {}_\varphi V_R$. (In [21, 3.3] this would be the definition of $\mathcal{F}$-stability for the class of V in $D_\mathcal{O}(P)$). Here ${}_\varphi V_R$ is the $\mathcal{O}Q$-module which is equal to V_R as an $\mathcal{O}$-module, with $x \in Q$ acting as $\varphi(x)$ on V_R. The isomorphism classes of $\mathcal{F}$-stable indecomposable endopermutation $\mathcal{O}P$-modules with vertex P form a subgroup of $D_\mathcal{O}(P)$, denoted $D_\mathcal{O}(P, \mathcal{F})$. See [21] for more details. The fusion stable linear characters of P form a subgroup of $D_\mathcal{O}(P, \mathcal{F})$ which we identify with the group $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$. We set $\mathrm{Out}_\mathcal{F}(P) = \mathrm{Aut}_\mathcal{F}(P)/\mathrm{Inn}(P)$. Following the notation introduced in [1, 1.13], we denote by $\mathrm{Aut}(P, \mathcal{F})$ the group of automorphisms of P which stabilise $\mathcal{F}$ and set $\mathrm{Out}(P, \mathcal{F}) = \mathrm{Aut}(P, \mathcal{F})/\mathrm{Aut}_\mathcal{F}(P)$. The group $\mathrm{Aut}(P)$ acts on $D_\mathcal{O}(P)$, with $\mathrm{Inn}(P)$ acting trivially. This action restricts to an action of $\mathrm{Aut}(P, \mathcal{F})$ on the subgroup $D_\mathcal{O}(P, \mathcal{F})$ of $D_\mathcal{O}(P)$, with $\mathrm{Aut}_\mathcal{F}(P)$ acting trivially on that subgroup, and hence inducing an action of $\mathrm{Out}(P, \mathcal{F})$ on $D_\mathcal{O}(P, \mathcal{F})$ and its torsion subgroup $D^t_\mathcal{O}(P, \mathcal{F})$. Again by [28, 7.6], Morita equivalences with an endopermutation source have sources which are stable with respect to the involved fusion systems. The assumption that the residue field k is a splitting field for the finite groups in the statements below implies that the fusion systems of the blocks are saturated.

Theorem 1.1. *Let G be a finite group and $\mathcal{O}$ be a complete local principal ideal domain with residue field k of characteristic p such that k is a splitting field for all subgroups of G. Let b a block of $\mathcal{O}G$ with defect group P and source idempotent i in $(\mathcal{O}Gb)^P$. Set $A = i\mathcal{O}Gi$ and $B = \mathcal{O}Gb$. Denote by $\mathcal{F}$ the fusion system on P determined by A.*

(i) *Let M be a (B, B)-bimodule which induces a Morita equivalence and which has an endopermutation module as a source. Then M is isomorphic to a* *direct summand of*

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times P}(V) \otimes_{\mathcal{O}P} i\mathcal{O}G$$

for some $\varphi \in \mathrm{Aut}(P, \mathcal{F})$*, and some* $\mathcal{F}$*-stable indecomposable endopermutation* $\mathcal{O}P$*-module* V*, regarded as an* $\mathcal{O}\Delta\varphi$*-module via the isomorphism* $P \cong \Delta\varphi$ *sending* $x \in P$ *to* $(x, \varphi(x))$*.*

(ii) *The correspondence sending* M *to the pair* (V, φ) *induces a group homomorphism* Φ *making the following diagram of groups commutative with exact rows:*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Out}_P(A) \arrow[r] \arrow[d, equal] & \mathcal{E}(B) \arrow[r, "\Phi"] & D_{\mathcal{O}}(P,\mathcal{F}) \rtimes \mathrm{Out}(P,\mathcal{F}) \\
1 \arrow[r] & \mathrm{Out}_P(A) \arrow[r] \arrow[d, equal] & \mathcal{L}(B) \arrow[r, "\Phi"] \arrow[u] & \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \rtimes \mathrm{Out}(P,\mathcal{F}) \arrow[u] \\
1 \arrow[r] & \mathrm{Out}_P(A) \arrow[r] & \mathcal{T}(B) \arrow[r, "\Phi"] \arrow[u] & \mathrm{Out}(P,\mathcal{F}) \arrow[u]
\end{tikzcd}$$

where the upwards arrows are the inclusions.

(iii) *The group* $\mathcal{E}(B)$ *is finite. Moreover, if* Φ *maps* $\mathcal{T}(B)$ *onto* $\mathrm{Out}(P, \mathcal{F})$*, then* Φ *maps* $\mathcal{E}(B)$ *to the finite group* $D^t_{\mathcal{O}}(P, \mathcal{F}) \rtimes \mathrm{Out}(P, \mathcal{F})$*.*

This will be proved in Section 3.

Remark 1.2.

(a) By a result of Puig [26, 14.9], the group $\mathrm{Out}_P(A)$ is canonically isomorphic to a subgroup of the finite abelian p'-group $\mathrm{Hom}(E, k^\times)$, where $E \cong \mathrm{Out}_{\mathcal{F}}(P)$ is the inertial quotient of b. The right column in the diagram in Theorem 1.1 depends only on the fusion system $\mathcal{F}$ on P, while the column in the middle depends on the source algebra structure of the block B. This is ultimately the reason for why in general the map Φ does not necessarily map $\mathcal{T}(B)$ onto $\mathrm{Out}(P, \mathcal{F})$, not even if B is nilpotent. See the examples in Section 7.

(b) We have a canonical embedding $\mathrm{Out}_P(A) \to \mathrm{Pic}(B)$ given by the correspondence sending $\alpha \in \mathrm{Aut}_P(A)$ to the (B, B)-bimodule $\mathcal{O}Gi_\alpha \otimes_A i\mathcal{O}G$. As a consequence of a result due independently to Scott [31] and Puig [28], the image of the group $\mathrm{Out}_P(A)$ in $\mathrm{Pic}(B)$ under this embedding is the subgroup $\mathcal{T}^\Delta(B)$ of $\mathcal{T}(B)$ given by bimodules which induce a Morita equivalence on $\mathrm{Mod}(B)$ and which are summands of $\mathcal{O}Gi \otimes_{\mathcal{O}P} i\mathcal{O}G$; that is, trivial source bimodules with diagonal vertex ΔP arising in statement (i) with $V = \mathcal{O}$ and $\varphi = \mathrm{Id}_P$. More generally, an isomorphism between source algebras of two blocks of (possibly different) finite groups induces isomorphisms between the associated diagrams of these blocks in Theorem 1.1 (ii). See for instance [17, 4.1] for a proof of the aforementioned result of Scott and Puig (which we will use repeatedly).

(c) All groups except possibly $D_{\mathcal{O}}(P, \mathcal{F})$ in the above diagram are finite. The image of $\mathcal{E}(B)$ under Φ is not known in general, except in some special cases, including blocks with cyclic or Klein four defect groups; see the theorems 1.4 and 1.5 below.

(d) Theorem 1.1 applies to $\mathcal{O} = k$. In general, the canonical map $\mathcal{E}(B) \to \mathcal{E}(k \otimes_{\mathcal{O}} B)$ is surjective (by [12, Theorem 1.13]), and its kernel is $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$.

(e) If $\mathcal{O} = k$, then the group $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ is trivial, and hence the second and third row in the diagram in Theorem 1.1 (ii) are equal in that case. By [19, Theorem 1.1, Lemma 3.15], if $\mathcal{O}$ has characteristic zero and contains a primitive $|G|$-th root of unity, then $\mathcal{L}(B)$ contains a canonical copy of the abelian p-group $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$, and hence we have a canonical isomorphism

$$\mathcal{L}(B) \cong \mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \rtimes \mathcal{T}(B)$$

compatible with the inclusion $\mathcal{T}(B) \to \mathcal{L}(B)$ and Φ. The subgroup of $\mathcal{L}(B)$ corresponding to $\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times)$ under this isomorphism is equal to the intersection $\mathcal{L}(B) \cap \ker(\mathrm{Pic}(B) \to \mathrm{Pic}(k \otimes_{\mathcal{O}} B))$, and Φ restricts to the identity on this subgroup. It follows from the constructions in [19, Theorems 1.1, 3.1] that this subgroup commutes with the image of $\mathrm{Out}_P(A)$; in other words, $\mathcal{L}(B)$ contains a canonical normal abelian subgroup isomorphic to

$$\mathrm{Hom}(P/\mathfrak{foc}(\mathcal{F}), \mathcal{O}^\times) \times \mathrm{Out}_P(A) .$$

(f) If P is abelian, then $\mathrm{Out}(P, \mathcal{F}) = N_{\mathrm{Aut}(P)}(E)/E$, where $E = \mathrm{Aut}_{\mathcal{F}}(P)$. We will use this identification in the statement of the next result.

Combining results of Puig [27], Zhou [36], Hertweck and Kimmerle [11], and Carlson and Rouquier [4] yields the following result. Unlike Theorem 1.1, the remaining results in this section require $\mathcal{O}$ to have characteristic zero.

Theorem 1.3. *Let G be a finite group. Suppose that $\mathrm{char}(\mathcal{O}) = 0$ and that the residue field k is a splitting field for the subgroups of G. Let b a block of $\mathcal{O}G$ with a nontrivial abelian defect group P. Set $B = \mathcal{O}Gb$, let i be a source idempotent in B^P, and set $A = iBi$. Denote by E the inertial quotient of B associated with the choice of i, regarded as a subgroup of $\mathrm{Aut}(P)$, and suppose that E is nontrivial cyclic and acts freely on $P \setminus \{1\}$. Set $N_E = N_{\mathrm{Aut}(P)}(E)/E$. We have $\mathrm{Pic}(B) = \mathcal{E}(B)$, and there is an injective group homomorphism $\Psi : \mathrm{Pic}(B) \to \mathrm{Hom}(E, k^\times) \rtimes N_E$ which makes the following diagram with exact rows commutative:*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Out}_P(A) \arrow[r] \arrow[d] & \mathrm{Pic}(B) \arrow[r, "\Phi"] \arrow[d, "\Psi"] & D_{\mathcal{O}}(P,\mathcal{F}) \rtimes N_E \arrow[d] & \\
1 \arrow[r] & \mathrm{Hom}(E, k^\times) \arrow[r] & \mathrm{Hom}(E, k^\times) \rtimes N_E \arrow[r] & N_E \arrow[r] & 1
\end{tikzcd}$$

where the top row is from Theorem 1.1, the bottom row is the canonical exact sequence, the left vertical arrow is the canonical embedding, and the right vertical map is the canonical surjection with kernel $D_{\mathcal{O}}(P, \mathcal{F})$.

If E is trivial, then B is nilpotent; see the Example 7.1 below for details in that case. Note that the image of Φ need not be contained in the group $N_E = N_{\mathrm{Aut}(P)}(E)/E$ since $\mathcal{E}(B)$ need not be equal to $\mathcal{T}(B)$; see the Example 7.2. Theorem 1.3 will be proved in Section 4.

For the sake of completeness, we describe the Picard groups for blocks with a cyclic or Klein four defect group over $\mathcal{O}$ with $\mathrm{char}(\mathcal{O}) = 0$, since these two cases have some additional properties. This is for the most part well-known and a combination of various results in the literature, such as [30], [34], [15, 4.3, 5.6, 5.8], [14, 1.1], [6, 1.1], [16, §11.4]. In particular, in both of these cases, if B is not nilpotent, then $\mathrm{Pic}(B)$ is equal to $\mathcal{T}(B)$, and hence $\mathrm{Im}(\Phi)$ is contained in N_E, where the notation is as in Theorem 1.3. Furthermore, if P is cyclic, then $\mathrm{Aut}(P)$ is abelian, and hence the semidirect product in the last statement of Theorem 1.3 becomes a direct product.

Theorem 1.4. *Let G be a finite group. Suppose that* $\mathrm{char}(\mathcal{O}) = 0$ *and that the residue field k is a splitting field for the subgroups of G. Let b be a block of $\mathcal{O}G$ with a nontrivial cyclic defect group P. Set $B = \mathcal{O}Gb$, let i be a source idempotent in B^P, and set $A = iBi$. Denote by E the associated inertial quotient of B, and suppose that E is nontrivial. Let M be a (B, B)-bimodule inducing a Morita equivalence. Then M is a trivial source module. We have*

$$\mathrm{Pic}(B) = \mathcal{T}(B) \cong \mathrm{Out}_P(A) \times \mathrm{Aut}(P)/E \ ,$$

the group $\mathrm{Out}_P(A)$ *is cyclic of order dividing $|E|$, with generator the isomorphism class $\Omega^n_{B\otimes_{\mathcal{O}}B^{\mathrm{op}}}(B)$, where n is the smallest positive integer such that this bimodule induces a Morita equivalence.*

The fusion system $\mathcal{F}$ on P determined by i in the above theorem is equal to that of the group $P \rtimes E$. Since the automorphism group of P is abelian, it follows that $\mathrm{Out}(P, \mathcal{F}) = \mathrm{Aut}(P)/E$. If $\mathcal{O}$ has characteristic zero and if two blocks with cyclic defect groups are Morita equivalent, then there is a Morita equivalence with endopermutation source, and hence Theorem 1.4 implies that any Morita equivalence between blocks over $\mathcal{O}$ with cyclic defect groups has endopermutation source. We have similar results for Klein four defect groups.

Theorem 1.5. *Let G be a finite group. Suppose that $p = 2$, that* $\mathrm{char}(\mathcal{O}) = 0$*, and that the residue field k is a splitting field for the subgroups of G. Let b be a block of $\mathcal{O}G$ with a Klein four defect group P. Set $B = \mathcal{O}Gb$, and suppose that B is not nilpotent. Let M be a (B, B)-bimodule inducing a Morita equivalence. Then M is a trivial source module. If B is Morita equivalent to $\mathcal{O}A_4$, then*

$$\mathrm{Pic}(B) = \mathcal{T}(B) \cong S_3 \ ,$$

if B is Morita equivalent to the principal block of $\mathcal{O}A_5$, then

$$\mathrm{Pic}(B) = \mathcal{T}(B) \cong C_2 \ .$$

In Theorem 1.5, since B is not nilpotent, the fusion system $\mathcal{F}$ of B on P is that of the group $A_4 \cong P \rtimes C_3$, and $\mathrm{Out}(P, \mathcal{F}) \cong S_3/C_3 \cong C_2$. Using the classification of finite simple groups via [6], it follows that every Morita equivalence between two non-nilpotent blocks over $\mathcal{O}$ with Klein four defect groups has trivial source, and any Morita equivalence between two nilpotent blocks with Klein four defect groups has linear source (and there cannot be a Morita equivalence between a nilpotent and non-nilpotent block with Klein four defect groups). The theorems 1.4 and 1.5 will be proved in Section 4.

We say that $\mathcal{O}$ is a *p-adic ring* if $\mathcal{O}$ has characteristic zero and k is finite, or equivalently, if $\mathcal{O}$ is a finite extension of the ring of p-adic integers. If $\mathcal{O}$ is p-adic and if B is a block algebra of a finite group algebra over $\mathcal{O}$, then $\mathrm{Pic}(B)$ is known to be finite, (see [8, Theorems (55.19), (55.25)]). We give an alternative proof of this fact as part of Theorem 6.2 below. If k is an algebraic closure of $\mathbb{F}_p$ then we do not know whether $\mathrm{Pic}(B)$ is finite, but the following result shows that if $\mathcal{O}$ has characteristic zero, then $\mathrm{Pic}(B)$ is the colimit of the Picard groups of blocks over some p-adic subrings of $\mathcal{O}$; in particular, $\mathrm{Pic}(B)$ is a torsion group. In general, the p-adic subrings of $\mathcal{O}$ form a directed system; the colimit of this system is the union in $\mathcal{O}$ of the p-adic subrings.

Theorem 1.6. *Suppose that k is an algebraic closure of $\mathbb{F}_p$ and that* $\mathrm{char}(\mathcal{O}) = 0$. *Let G be a finite group and b a central idempotent of $\mathcal{O}G$. Set $B = \mathcal{O}Gb$.*

(i) Let M be a (B, B)-bimodule inducing a Morita equivalence on B. There exists a p-adic subring $\mathcal{O}_0$ of $\mathcal{O}$ such that $b \in \mathcal{O}_0G$ and such that, setting $B_0 = \mathcal{O}_0Gb$, we have $M \cong \mathcal{O}\otimes_{\mathcal{O}_0} M_0$ for some (B_0, B_0)-bimodule M_0 inducing a Morita equivalence on B_0.

(ii) The group $\mathrm{Pic}(B)$ *is the colimit of the finite groups* $\mathrm{Pic}(\mathcal{O}_0Gb)$*, with $\mathcal{O}_0$ running over the p-adic subrings of $\mathcal{O}$ such that $b \in \mathcal{O}_0G$. In particular, all elements in* $\mathrm{Pic}(B)$ *have finite order.*

We prove this theorem and related facts in Section 5. It is well known that a result analogous to Theorem 1.6 holds over k; see Lemma 6.5 below. On the other hand, unlike the situation in characteristic zero, if k is algebraically closed then it is known that Picard groups of block algebras over k are not in general finite.

Remark 1.7. Two Morita equivalent blocks of finite groups via a bimodule with endopermutation source have isomorphic defect groups. It is an open problem at present whether Morita equivalent blocks always have isomorphic defect groups.

Remark 1.8. Let P be a finite p-group and Q a normal subgroup of P. Suppose that $\mathrm{char}(\mathcal{O}) = 0$. Weiss' criterion states that if M is a finitely generated $\mathcal{O}P$-module such that $\mathrm{Res}^P_Q(M)$ is a free $\mathcal{O}Q$-module and such that M^Q is a permutation $\mathcal{O}P/Q$-module, then M is a permutation $\mathcal{O}P$-module. This was proved by Weiss in [34, Theorem 2], [35, §6] for $\mathcal{O}$ the ring of p-adic integers $\mathbb{Z}_p$, extended to finite extensions of $\mathbb{Z}_p$ by Roggenkamp in [29, Theorem II], and further extended to $\mathcal{O}$ with perfect residue field in work of Puig [28, Theorem A.1.2], and for general $\mathcal{O}$ in work of McQuarrie, Symonds, and Zalesskii [22, Theorem 1.2]. Weiss' criterion is a key ingredient for calculating Picard groups of block algebras of finite groups. In applications below, we will make use of the fact that Q-fixed points in the above sitation are isomorphic to cofixed points. More precisely, with the notation and hypotheses above, one verifies that there is an $\mathcal{O}P/Q$-module isomorphism $\mathcal{O}\otimes_{\mathcal{O}Q} M \cong M^Q$ sending $1\otimes m$ to $\mathrm{Tr}^Q_1(m)$, where $m \in M$ and $\mathrm{Tr}^Q_1(m) = \sum_{y\in Q} ym$.

2. Tensoring bimodules with endopermutation source

Some calculations of Picard groups will involve stable equivalences of Morita type and stable Picard groups. We briefly review these notions. Let B, C be symmetric $\mathcal{O}$-algebras such that $B/J(B)$ and $C/J(C)$ are separable k-algebras. Following terminology introduced by Broué, we say that a (B, C)-bimodule M induces a stable equivalence of Morita type between B and C, if M is finitely generated projective as a left B-module, as a right C-module, and if we have bimodule isomorphisms $M \otimes_C M^* \cong B \oplus X$ for some projective $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module and $M^* \otimes_B M \cong C \oplus Y$ for some projective $C \otimes_{\mathcal{O}} C^{\mathrm{op}}$-module.

The *stable Picard group of B* is the group $\underline{\mathrm{Pic}}(B)$ of isomorphism classes in the stable category of (B, B)-bimodules inducing a stable equivalence of Morita type on B. The group structure on $\underline{\mathrm{Pic}}(B)$ is induced by taking tensor products over B. Any Morita equivalence is a stable equivalence of Morita type, and hence if B has no nonzero projective summand as a $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module, then we have an inclusion of groups $\mathrm{Pic}(B) \subseteq \underline{\mathrm{Pic}}(B)$.

Denoting by $\Omega(B)$ the kernel of a projective cover of B as a $B\otimes_{\mathcal{O}} B^{\mathrm{op}}$-module, it is well-known that if B has no nonzero projective summand as a $B\otimes_{\mathcal{O}} B^{\mathrm{op}}$-module, then $\Omega(B)$ induces a stable equivalence of Morita type on B, and the image of $\Omega(B)$ in $\underline{\mathrm{Pic}}(B)$ generates a cyclic central subgroup of $\underline{\mathrm{Pic}}(B)$, which we will denote by $\langle\Omega(B)\rangle$. See e. g. [13, Proposition 2.9].

By [13, Theorem 2.1], if B and C are in addition indecomposable as algebras and not projective as modules over $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$ and $C \otimes_{\mathcal{O}} C^{\mathrm{op}}$, respectively, and if M induces a stable equivalence of Morita type between B and C, then M has a unique indecomposable nonprojective direct summand in any decomposition as a direct sum of indecomposable $B \otimes_{\mathcal{O}} C^{\mathrm{op}}$-modules.

For P, Q finite groups and $\varphi : P \to Q$ a group isomorphism set

$$\Delta\varphi = \{(x, \varphi(x)) \mid x \in P\} \ .$$

We write ΔP instead of $\Delta\mathrm{Id}_P$. We regard an $\mathcal{O}P$-module U as an $\mathcal{O}\Delta\varphi$-module with $(x, \varphi(x))$ acting on U as x, where $x \in P$. That is, the action of $\Delta\varphi$ on U is determined by the action of the first component of an element in $\Delta\varphi$; this accounts for the slight asymmetry in the statements (i) and (ii) in the next Lemma. If V is an $\mathcal{O}Q$-module, we denote by ${}_{\varphi}V$ the $\mathcal{O}P$-module which is equal to V as an $\mathcal{O}$-module, such that $x \in P$ acts as $\varphi(x)$ on V. We use similar notation for right modules and bimodules. We regard an $\mathcal{O}(P \times Q)$-module M as an $(\mathcal{O}P, \mathcal{O}Q)$-bimodule via $xmy^{-1} = (x, y) \cdot m$, where $x \in P$, $y \in Q$ and $m \in M$.

Lemma 2.1. *Let P, Q, R be finite groups. Let $\varphi : P \to Q$ and $\psi : Q \to R$ be group isomorphisms. Let V be an $\mathcal{O}P$-module and W an $\mathcal{O}Q$-module.*

(i) *We have an $(\mathcal{O}P, \mathcal{O}Q)$-bimodule isomorphism*

$$\mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V) \cong (\mathrm{Ind}_{\Delta P}^{P\times P}(V))_{\varphi^{-1}}$$

sending $(x, y) \otimes v$ to $(x, \varphi^{-1}(y)) \otimes v$, for all $x \in P$, $y \in Q$ and $v \in V$.

(ii) *We have an $(\mathcal{O}P, \mathcal{O}Q)$-bimodule isomorphism*

$$\mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V) \cong {}_{\varphi}(\mathrm{Ind}_{\Delta Q}^{Q\times Q}({}_{\varphi^{-1}}V))$$

sending $(x, y) \otimes v$ to $(\varphi(x), y) \otimes v$, for all $x \in P$, $y \in Q$ and $v \in V$.

(iii) *We have an $(\mathcal{O}P, \mathcal{O}R)$-bimodule isomorphism*

$$\mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V) \otimes_{\mathcal{O}Q} \mathrm{Ind}_{\Delta\psi}^{Q\times R}(W) \cong \mathrm{Ind}_{\Delta(\psi\circ\varphi)}^{P\times R}(V \otimes_{\mathcal{O}} ({}_{\varphi}W)) \ .$$

Proof. The statements (i) and (ii) are straightforward verifications. Statement (iii) is a special case of a more general result of Bouc [2, Theorem 1.1]. One can prove (iii) also by first showing this for $P = Q = R$, $\varphi = \psi = \mathrm{Id}_P$ (see e. g. [20, Corollary 2.4.13]), and then using (i), (ii) to obtain the general case. □

Lemma 2.2. *Let P, Q be finite groups and $\varphi : P \to Q$ a group isomorphism, and V a finitely generated $\mathcal{O}$-free $\mathcal{O}P$-module. We have an $(\mathcal{O}P, \mathcal{O}P)$-bimodule isomorphism*

$$\mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V)^* \cong \mathrm{Ind}_{\Delta\varphi^{-1}}^{Q\times P}({}_{\varphi^{-1}}(V^*)) \ .$$

Proof. Induction and duality commute, and hence $\mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V)^* \cong \mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V^*)$ as $\mathcal{O}(P \times Q)$-modules. The isomorphism $P \times Q \cong Q \times P$ exchanging the two factors sends $(x, \varphi(x)) \in \Delta\varphi$ to $(\varphi(x), x) = (\varphi(x), \varphi^{-1}(\varphi(x))) \in \Delta\varphi^{-1}$, which accounts for the subscript φ^{-1} in the last term in the statement of the Lemma. □

Lemma 2.3. *Let P be a finite group and $\zeta : P \to \mathcal{O}^\times$ a group homomorphism. Denote by $\mathcal{O}_\zeta$ the $\mathcal{O}P$-module $\mathcal{O}$ with $x \in P$ acting by multiplication with $\zeta(x)$. Denote by τ the $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ satisfying $\tau(x) = \zeta^{-1}(x)x$ for all $x \in P$. There is an $\mathcal{O}(P \times P)$-module isomorphism*

$$\mathrm{Ind}_{\Delta P}^{P\times P}(\mathcal{O}_\zeta) \cong \mathcal{O}P_\tau$$

sending $(x, y) \otimes 1$ to $\zeta(y)xy^{-1}$ for x, $y \in P$.

Proof. This is a straightforward verification. □

For P, Q finite p-groups, $\mathcal{F}$ a fusion system on P and $\varphi : P \to Q$ a group isomorphism, we denote by ${}^\varphi\mathcal{F}$ the fusion system on Q induced by $\mathcal{F}$ via the isomorphism φ. That is, for R, S subgroups of P, we have $\mathrm{Hom}_{{}^\varphi\mathcal{F}}(\varphi(R), \varphi(S)) = \varphi \circ \mathrm{Hom}_\mathcal{F}(R, S) \circ \varphi^{-1}$, where we use the same notation φ, φ^{-1} for their restrictions to S, $\varphi(R)$, respectively.

Theorem 2.4 ([28, 7.6])**.** *Let G, H be finite groups, b, c blocks of $\mathcal{O}G$, $\mathcal{O}H$ with nontrivial defect groups P, Q, respectively, and let $i \in (\mathcal{O}Gb)^P$ and $j \in (\mathcal{O}Hc)^Q$ be source idempotents. Denote by $\mathcal{F}$ the fusion system on P of b determined by i, and denote by $\mathcal{G}$ the fusion system on Q determined by j. Let M be an indecomposable $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule with endopermutation source inducing a stable equivalence of Morita type.*

Then there is an isomorphism $\varphi : P \to Q$ and an indecomposable endopermutation $\mathcal{O}P$-module V such that M is isomorphic to a direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V) \otimes_{\mathcal{O}Q} j\mathcal{O}H$$

as an $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule. For any such φ and V, the following hold.

(i) *$\Delta\varphi$ is a vertex of M and V, regarded as an $\mathcal{O}\Delta\varphi$-module, is a source of M.*
(ii) *We have ${}^\varphi\mathcal{F} = \mathcal{G}$, and the endopermutation $\mathcal{O}P$-module V is $\mathcal{F}$-stable.*

See [20, Theorem 9.11.2] for a proof of the above Theorem using the terminology of the present paper.

Lemma 2.5 (cf. [24])**.** *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b, and $i \in (\mathcal{O}Gb)^P$ a source idempotent. Denote by $\mathcal{F}$ the fusion system on P of the block b determined by the choice of i. Let $\varphi \in \mathrm{Aut}(P)$. The following are equivalent.*

(i) *We have $\varphi \in \mathrm{Aut}_\mathcal{F}(P)$.*
(ii) *We have $i\mathcal{O}G \cong {}_\varphi(i\mathcal{O}G)$ as $(\mathcal{O}P, \mathcal{O}G)$-bimodules.*
(iii) *As an $\mathcal{O}(P \times P)$-module, $i\mathcal{O}Gi$ has an indecomposable direct summand isomorphic to $\mathrm{Ind}_{\Delta\varphi}^{P\times P}(\mathcal{O})$.*

Proof. The equivalence of (i) and (ii) is proved as part of [20, Theorem 8.7.4], and the equivalence of (i) and (ii) is proved as part of [20, Theorem 8.7.1]. □

Lemma 2.6. *Let G, H, L be finite groups, b, c, d blocks of $\mathcal{O}G$, $\mathcal{O}H$, $\mathcal{O}L$ with nontrivial defect groups P, Q, R, respectively. Let i, j, l be source idempotents in $(\mathcal{O}Gb)^P$, $(\mathcal{O}Hc)^Q$, $(\mathcal{O}Ld)^R$, respectively. Let V be an indecomposable endopermutation $\mathcal{O}P$-module with vertex P and W an indecomposable endopermutation $\mathcal{O}Q$-module with vertex Q. Let $\varphi : P \to Q$ and $\psi : Q \to R$ be group isomorphisms.*

Let M be an indecomposable direct summand of the $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule

$$X = \mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V) \otimes_{\mathcal{O}Q} j\mathcal{O}H$$

and let N be an indecomposable direct summand of the $(\mathcal{O}Hc, \mathcal{O}Ld)$-bimodule

$$Y = \mathcal{O}Hj \otimes_{\mathcal{O}Q} \mathrm{Ind}_{\Delta\psi}^{Q\times R}(W) \otimes_{\mathcal{O}R} l\mathcal{O}L\ .$$

Suppose that M induces a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$, and suppose that N induces a stable equivalence of Morita type between $\mathcal{O}Hc$ and $\mathcal{O}Ld$. Then the indecomposable nonprojective direct summand of the $(\mathcal{O}Gb, \mathcal{O}Ld)$-bimodule $M \otimes_{\mathcal{O}Hc} N$ is isomorphic to an indecomposable direct summand with vertex $\Delta(\psi\circ\varphi)$ of the $\mathcal{O}(G\times L)$-module

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta(\psi\circ\varphi)}^{P\times R}(U) \otimes_{\mathcal{O}R} l\mathcal{O}L$$

where U is an indecomposable direct summand with vertex P of $V \otimes_{\mathcal{O}}\, {}_{\varphi}W$.

Proof. Identify P, Q, R via φ, ψ; that is, we may assume that φ, ψ are the identity maps. With this identification, the groups $\Delta\varphi$ and $\Delta\psi$ are both equal to ΔP. It follows from Theorem 2.4 that the source idempotents i, j, l of b, c, d, respectively, determine the same fusion system $\mathcal{F}$ on P, and that V, W are $\mathcal{F}$-stable.

Clearly $M \otimes_{\mathcal{O}Hc} N$ induces a stable equivalence of Morita type, and hence its unique (up to isomorphism) indecomposable nonprojective summand Z induces a stable equivalence of Morita type, and moreover, Z is isomorphic to an indecomposable summand of

$$X \otimes_{\mathcal{O}Hc} Y =$$
$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}Hj \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(W) \otimes_{\mathcal{O}P} l\mathcal{O}L$$

Since Z is indecomposable, we may replace the term $j\mathcal{O}Hj$ in the middle by some indecomposable $(\mathcal{O}P, \mathcal{O}P)$-bimodule summand. Since Z must have a vertex of order at least $|P|$, it follows from Lemma 2.5 that we may choose that bimodule summand of $j\mathcal{O}Hj$ to be isomorphic to $\mathrm{Ind}_{\Delta\epsilon}^{P\times P}(\mathcal{O})$ for some $\epsilon \in \mathrm{Aut}_{\mathcal{F}}(P)$. Thus Z is isomorphic to a direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\epsilon}^{P\times P}(\mathcal{O}) \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(W) \otimes_{\mathcal{O}P} l\mathcal{O}L$$

By Lemma 2.1, this is isomorphic to

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\epsilon}^{P\times P}(V \otimes_{\mathcal{O}} ({}_{\epsilon}W)) \otimes_{\mathcal{O}P} l\mathcal{O}L$$

The fusion stability of W implies that this is isomorphic to

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\epsilon}^{P\times P}(V \otimes_{\mathcal{O}} W) \otimes_{\mathcal{O}P} l\mathcal{O}L$$

By Lemma 2.1 again, this is isomorphic to

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V \otimes_{\mathcal{O}} W)_{\epsilon^{-1}} \otimes_{\mathcal{O}P} l\mathcal{O}L$$

hence to

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V \otimes_{\mathcal{O}} W) \otimes_{\mathcal{O}P} {}_{\epsilon}l\mathcal{O}L$$

Now $\epsilon \in \mathrm{Aut}_{\mathcal{F}}(P)$, and hence Lemma 2.5 implies that ${}_{\epsilon}l\mathcal{O}L \cong l\mathcal{O}L$ as $(\mathcal{O}P, \mathcal{O}L)$-bimodules. Together it follows that Z is isomorphic to a direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V \otimes_{\mathcal{O}} W) \otimes_{\mathcal{O}P} l\mathcal{O}L$$

Since V, W are indecomposable endopermutation $\mathcal{O}P$-modules, it follows that U is up to isomorphism the unique indecomposable direct summand with vertex P of

$V \otimes_{\mathcal{O}} W$. The fact that Z has a vertex of order $|P|$ implies that Z is isomorphic to a direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(U) \otimes_{\mathcal{O}P} l\mathcal{O}L$$

as claimed. □

The vertex-source pairs of an indecomposable module over a finite group algebra are unique up to conjugation. For the situation in Theorem 2.4, this translates to the following statement.

Lemma 2.7. *Let G, H be finite groups, b, c blocks of $\mathcal{O}G$, $\mathcal{O}H$ with nontrivial defect groups P, Q, respectively, and let $i \in (\mathcal{O}Gb)^P$ and $j \in (\mathcal{O}Hc)^Q$ be source idempotents. Denote by $\mathcal{F}$ the fusion system on P of b determined by i, and denote by $\mathcal{G}$ the fusion system on Q determined by j. Let M be an indecomposable $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule inducing a stable equivalence of Morita type. Let φ, $\psi : P \cong Q$ be group isomorphisms, let V, W be indecomposable endopermutation $\mathcal{O}P$-modules. Suppose that M is isomorphic to a direct summand of both*

$$X = \mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times Q}(V) \otimes_{\mathcal{O}Q} j\mathcal{O}H$$

and

$$Y = \mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\psi}^{P\times Q}(W) \otimes_{\mathcal{O}Q} j\mathcal{O}H\ .$$

Then $\varphi^{-1} \circ \psi \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $W \cong V$ as $\mathcal{O}P$-modules.

Proof. Since $\mathcal{O}G$, $\mathcal{O}H$ are symmetric algebras, we have $(\mathcal{O}Gi)^* \cong i\mathcal{O}G$ as $(\mathcal{O}P, \mathcal{O}G)$-bimodules and $(j\mathcal{O}H)^* \cong \mathcal{O}Hj$ as $(\mathcal{O}H, \mathcal{O}Q)$-bimodules. Since the terms in Y are all finitely generated projective as one-sided modules, duality anticommutes with tensor products. We therefore have

$$Y^* \cong \mathcal{O}Hj \otimes_{\mathcal{O}Q} \mathrm{Ind}_{\Delta\psi^{-1}}^{Q\times P}({}_{\psi^{-1}}W^*) \otimes_{\mathcal{O}P} i\mathcal{O}G\ .$$

By Theorem 2.4 we have ${}^{\varphi}\mathcal{F} = \mathcal{G} = {}^{\psi}\mathcal{F}$, and the modules V, W are $\mathcal{F}$-stable. Then W^* is clearly $\mathcal{F}$-stable as well. By the assumptions, $\mathcal{O}Gb$ is a summand of the $(\mathcal{O}Gb, \mathcal{O}Gb)$-bimodule $M \otimes_{\mathcal{O}Hc} M^*$. Thus $\mathcal{O}Gb$ is isomorphic to a direct summand of the bimodule

$$X \otimes_{\mathcal{O}Hc} Y^*\ .$$

By Lemma 2.1 and Lemma 2.6, it follows that $\mathcal{O}Gb$ is isomorphic to a direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta(\psi^{-1}\circ\varphi)}^{P\times P}(U) \otimes_{\mathcal{O}P} i\mathcal{O}G$$

where U is an indecomposable direct summand of

$$V \otimes_{\mathcal{O}} ({}_{\psi^{-1}\circ\varphi}(W^*))$$

with vertex P. As an $\mathcal{O}(G \times G)$-module, $\mathcal{O}Gb$ has ΔP as a vertex and a trivial source (for any vertex), and hence $U = \mathcal{O}$. Thus the $\mathcal{O}(P \times P)$-module $i\mathcal{O}Gi$, and hence also the $\mathcal{O}(P \times P)$-module

$$i\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta(\psi^{-1}\circ\varphi)}^{P\times P}(\mathcal{O}) \otimes_{\mathcal{O}P} i\mathcal{O}Gi$$

has an indecomposable direct $\mathcal{O}(P\times P)$-summand isomorphic to $\mathrm{Ind}_{\Delta P}^{P\times P}(\mathcal{O})$. Lemma 2.5 implies that there are α, $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that

$$\mathrm{Ind}_{\Delta P}^{P\times P}(\mathcal{O}) \cong \mathrm{Ind}_{\Delta\alpha}^{P\times P}(\mathcal{O}) \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta(\psi^{-1}\circ\varphi)}^{P\times P}(\mathcal{O}) \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\beta}^{P\times P}(\mathcal{O})$$

and hence by Lemma 2.1 we have an isomorphism of $\mathcal{O}(P \times P)$-modules

$$\mathrm{Ind}_{\Delta P}^{P\times P}(\mathcal{O}) \cong \mathrm{Ind}_{\Delta\tau}^{P\times P}(\mathcal{O})$$

where

$$\tau = \beta \circ \psi^{-1} \circ \varphi \circ \alpha\ .$$

Thus ΔP and $\Delta\tau$ are both vertices of this $\mathcal{O}(P \times P)$-module, hence they are conjugate in $P \times P$, and this implies that τ is an inner automorphism of P. But then τ, α, β are all in $\mathrm{Aut}_{\mathcal{F}}(P)$, and hence so is $\psi^{-1} \circ \varphi = \beta^{-1} \circ \tau \circ \alpha^{-1}$.

Finally, the fusion stability of W implies that

$$V \otimes_{\mathcal{O}} ({}_{\psi^{-1}\circ\varphi}(W^*)) \cong V \otimes_{\mathcal{O}} W^*$$

and this module has by the above a trivial summand, forcing $W \cong V$. $\square$

Given a block algebra B of a finite group algebra over $\mathcal{O}$, we denote by $\underline{\mathcal{E}}(B)$, $\underline{\mathcal{L}}(B)$, $\underline{\mathcal{T}}(B)$ the subgroups of $\underline{\mathrm{Pic}}(B)$ of isomorphism classes of bimodules inducing a stable equivalence of Morita type, having endopermutation, linear, trivial source, respectively. The canonical group homomorphism $\mathrm{Pic}(B) \to \underline{\mathrm{Pic}}(B)$ sends the subgroups $\mathcal{E}(B)$, $\mathcal{L}(B)$, $\mathcal{T}(B)$ to $\underline{\mathcal{E}}(B)$, $\underline{\mathcal{L}}(B)$, $\underline{\mathcal{T}}(B)$, respectively.

Lemma 2.8. *Let G, H, be finite groups, B, C block algebras of $\mathcal{O}G$, $\mathcal{O}H$, with a common nontrivial defect group P. Let i, j, be source idempotents in B^P, C^P, respectively. Suppose that B and C have the same fusion system $\mathcal{F}$ on P with respect to the choice of i and j. Let V be an indecomposable endopermutation $\mathcal{O}P$-module with vertex P, and let M be an indecomposable (B, C)-bimodule summand of*

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}H$$

such that M induces a Morita equivalence (resp. a stable equivalence of Morita type) between B and C. The following hold.

(i) *The functor $M^* \otimes_B - \otimes_B M$ induces group isomorphisms $\mathrm{Pic}(B) \cong \mathrm{Pic}(C)$ and $\mathcal{E}(B) \cong \mathcal{E}(C)$ (resp. group isomorphisms $\underline{\mathrm{Pic}}(B) \cong \underline{\mathrm{Pic}}(C)$ and $\underline{\mathcal{E}}(B) \cong \underline{\mathcal{E}}(C)$).*

(ii) *Suppose that $V \cong \Omega_P^n(\mathcal{O})$ for some integer n. The functor $M^* \otimes_B - \otimes_B M$ induces group isomorphisms $\mathcal{L}(B) \cong \mathcal{L}(C)$ and $\mathcal{T}(B) \cong \mathcal{T}(C)$ (resp. $\underline{\mathcal{L}}(B) \cong \underline{\mathcal{L}}(C)$ and $\underline{\mathcal{T}}(B) \cong \underline{\mathcal{T}}(C)$).*

Proof. Since M and its dual M^* induce a Morita equivalence, it follows that the functor $M^* \otimes_B - \otimes_B M$ induces a group isomorphism $\mathrm{Pic}(B) \cong \mathrm{Pic}(C)$. It follows from the hypotheses on M and from Lemma 2.6 that this isomorphism preserves the subgroups determined by Morita equivalences with endopermutation sources, and hence induces a group isomorphism $\mathcal{E}(B) \cong \mathcal{E}(C)$. If V is a Heller translate of $\mathcal{O}$, then V is stable under any automorphism of P, and hence by the Lemmas 2.3 and 2.6, the isomorphism $\mathcal{E}(B) \cong \mathcal{E}(C)$ restricts to isomorphisms $\mathcal{L}(B) \cong \mathcal{L}(C)$ and $\mathcal{T}(B) \cong \mathcal{T}(C)$. The argument for stable equivalences of Morita type is strictly analogous. $\square$

Remark 2.9. With the notation of Lemma 2.8, if V is not stable under $\mathrm{Aut}(P)$, then the functor $M^* \otimes_B - \otimes_B M$ need not induce isomorphisms $\mathcal{L}(B) \cong \mathcal{L}(C)$ or $\mathcal{T}(B) \cong \mathcal{T}(C)$; see the examples 7.1 and 7.2 below.

3. Proof of Theorem 1.1

We use the notation and hypotheses as in the statement of Theorem 1.1. If $P = \{1\}$, then B is a matrix algebra over $\mathcal{O}$. Thus $\mathrm{Pic}(B)$ is trivial, hence all groups in the diagram in (ii) are trivial, and the result is clear in that case. Assume that P is nontrivial. The fact that M is isomorphic to a summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times P}(V) \otimes_{\mathcal{O}P} i\mathcal{O}G$$

with V and φ as stated in (i) follows from Puig's result stated as Theorem 2.4 above. This proves (i).

Lemma 2.7 implies that there is a well-defined map Φ from $\mathcal{E}(\mathcal{O}Gb)$ to $D_{\mathcal{O}}(\mathcal{F}, P) \rtimes \mathrm{Out}(P, \mathcal{F})$ as stated. Lemma 2.6 shows that this map is a group homomorphism. The kernel of Φ consists of all isomorphism classes of bimodules M such that the corresponding pair (V, φ) satisfies $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $V \cong \mathcal{O}$. Since ${}_{\varphi}i\mathcal{O}G \cong i\mathcal{O}G$ as $(\mathcal{O}P, \mathcal{O}G)$-bimodule whenever $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$, it follows that we may choose $\varphi = \mathrm{Id}_P$. Note that $\mathrm{Ind}_{\Delta P}^{P\times P}(\mathcal{O}) \cong \mathcal{O}P$ as $(\mathcal{O}P, \mathcal{O}P)$-bimodules. Thus the isomorphism classes of bimodules M in the kernel of Φ correspond to trivial source bimodule summands of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} i\mathcal{O}G \ .$$

As noted in the Remark 1.2 (b), these in turn correspond bijectively to the elements of $\mathrm{Out}_P(A)$, via the map $\mathrm{Aut}_P(A) \to \mathcal{T}(B)$ sending $\alpha \in \mathrm{Aut}_P(A)$ to the isomorphism class of the (B, B)-bimodule $\mathcal{O}Gi_\alpha \otimes_A i\mathcal{O}G$. Any bimodule of this form induces a Morita equivalence on B because $\mathcal{O}Gi$ induces a Morita equivalence between B and A. Moreover, A is isomorphic to a direct summand of $A \otimes_{\mathcal{O}P} A$ (cf. [20, Theorem 6.4.7]), hence the previous bimodule is isomorphic to a direct summand of $\mathcal{O}Gi \otimes_{\mathcal{O}P} i\mathcal{O}G$, where we make use of the fact that α restricts to the identity on the image of $\mathcal{O}P$ in A. This shows the exactness of the first row of the diagram in the statement. Since by the previous argument, the image of $\mathrm{Aut}_P(A)$ in $\mathcal{E}(B)$ is contained in $\mathcal{T}(B)$, the remaining rows of the diagram are exact as well. This proves (ii).

For the finiteness of $\mathcal{E}(B)$, note first that a (B, B)-bimodule M inducing a Morita equivalence permutes the projective indecomposable B-modules. Thus for any primitive idempotent i in B, the B-module Mi is projective indecomposable, hence of $\mathcal{O}$-rank at most that of B. Since the $\mathcal{O}$-rank of B also bounds the cardinality of a primitive decomposition of 1 in B, it follows that the $\mathcal{O}$-rank of M is bounded in terms of the $\mathcal{O}$-rank of B. Thus in particular a source of M has an $\mathcal{O}$-rank bounded in terms of B. Moreover, endopermutation modules over a fixed p-group over a field of characteristic p are defined over a fixed finite field and lift to the ring of Witt vectors of that finite field (this is a consequence of the classification of endopermutation modules; see [33, Theorem 13.2, Theorem 14.2]). This implies that only finitely many isomorphism classes of indecomposable endopermutation modules arise as sources of bimodules inducing Morita equivalences of B. Since there are only finitely many isomorphism classes of indecomposable modules with a fixed vertex-source pair, the finiteness of $\mathcal{E}(B)$ follows. Alternatively, again using the above mentioned facts on endopermutation modules, one obtains the finiteness of $\mathcal{E}(B)$ as a consequence of the finiteness of $\mathrm{Pic}(B)$ whenever $\mathcal{O}$ is a p-adic ring (cf. [8, Theorems (55.19, (55.25)]).

Suppose that Φ maps $\mathcal{T}(B)$ onto $\mathrm{Out}(P,\mathcal{F})$. Let M be an indecomposable direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times P}(V) \otimes_{\mathcal{O}P} i\mathcal{O}G$$

inducing a Morita equivalence, for some $\varphi \in \mathrm{Aut}(P,\mathcal{F})$ and some indecomposable endopermutation $\mathcal{O}\Delta\varphi$-module V (which is then $\mathcal{F}$-stable, when regarded as an $\mathcal{O}P$-module, by Theorem 2.4). Since Φ is assumed to map $\mathcal{T}(B)$ onto $\mathrm{Out}(P,\mathcal{F})$, there is an indecomposable direct bimodule summand N of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\varphi}^{P\times P}(\mathcal{O}) \otimes_{\mathcal{O}P} i\mathcal{O}G$$

inducing a Morita equivalence. Since Φ is a group homomorphism, it follows that $U = M \otimes_B N^*$ is a bimodule inducing a Morita equivalence with vertex ΔP and source V, regarded as an $\mathcal{O}P$-module. That is, the image of the class of U under Φ is the class of the pair (V, Id_P), hence contained in the subgroup $D_{\mathcal{O}}(P,\mathcal{F})$. In other words, Φ sends the class of U to the class of V in $D_{\mathcal{O}}(P,\mathcal{F})$. By the above, the group $\mathcal{E}(B)$ is finite, and hence the image of V in $D_{\mathcal{O}}(P,\mathcal{F})$ has finite order. This proves (iii), which concludes the proof of Theorem 1.1.

4. Picard groups of blocks with abelian defect and cyclic Frobenius inertial quotient

In this section we make the blanket assumption that the residue field k is large enough for the finite groups and their subgroups which arise here. This hypothesis is in particular needed for the Theorems 1.3, 1.4, 1.5, whose proofs are given at the end of this section. We need the following result of Puig.

Theorem 4.1 ([27, 6.8]). *Let G be a finite group, b a block of $\mathcal{O}G$, and set $B = \mathcal{O}Gb$. Let P be a defect group of b, $i \in B^P$ a source idempotent of B and e the block of $kC_G(P)$ satisfying $\mathrm{Br}_P(i)e \neq 0$. Suppose that P is nontrivial abelian. Set $E = N_G(P,e)/C_G(P)$ and suppose that E acts freely on $P \setminus \{1\}$. Set $L = P \rtimes E$. Then there is an indecomposable E-stable endopermutation $\mathcal{O}P$-module V with vertex P and an indecomposable $(B,\mathcal{O}L)$-bimodule M with the following properties.*

(i) The bimodule M is a direct summand of $\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} \mathcal{O}L$.

(ii) As an $\mathcal{O}(G\times L)$-module, M has vertex ΔP and source V.

(ii) The bimodule M and its dual M^* induce a stable equivalence of Morita type between B and $\mathcal{O}L$.

For a proof of this theorem using the notation and terminology above, see [20, Theorem 10.5.1]. Note that in the situation of Theorem 4.1, if the acting group E is abelian, then E is in fact cyclic (cf. [10, Theorem 10.3.1]). The next result we need is due to Carlson and Rouquier.

Proposition 4.2 ([4, Corollary 3.3]). *Let P be a nontrivial abelian p-group and E an abelian p'-subgroup of $\mathrm{Aut}(P)$ acting freely on $P \setminus \{1\}$. Set $L = P \rtimes E$. Denote by Ω the Heller operator on the category of $(\mathcal{O}L,\mathcal{O}L)$-bimodules. Any indecomposable $(\mathcal{O}L,\mathcal{O}L)$-bimodule inducing a stable equivalence of Morita type on $\mathcal{O}L$ is isomorphic to $\Omega^n(M)$ for some integer n and some $(\mathcal{O}L,\mathcal{O}L)$-bimodule M inducing a Morita equivalence on $\mathcal{O}L$. Equivalently, we have*

$$\underline{\mathrm{Pic}}(\mathcal{O}L) = \mathrm{Pic}(\mathcal{O}L) \cdot \langle \Omega(\mathcal{O}L) \rangle \ ,$$

where we identify $\Omega(\mathcal{O}L)$ with its image in $\underline{\mathrm{Pic}}(\mathcal{O}L)$.

As mentioned at the beginning of Section 2, the cyclic subgroup $\langle\Omega(\mathcal{O}L)\rangle$ is in the center of $\underline{\mathrm{Pic}}(\mathcal{O}L)$. We need to determine the structure of $\mathrm{Pic}(\mathcal{O}L)$. The following result is a slight refinement of results of Y. Zhou [36, Theorem 14] and Hertweck and Kimmerle [11, Theorem 4.6] in the special case where the focal subgroup $[P, E]$ of P in $P \rtimes E$ is equal to P. The proof follows in part that of [15, Lemma 4.2]. As in the aforementioned papers [36], [11], [15], the key step is an application of Weiss' criterion.

Proposition 4.3. *Let P be a nontrivial abelian p-group and E an abelian p'-subgroup of $\mathrm{Aut}(P)$ such that $[P, E] = P$. Set $L = P \rtimes E$. Denote by $\mathcal{F}$ the fusion system of L on P. Assume that $\mathrm{char}(\mathcal{O}) = 0$. Any class in $\mathrm{Out}(\mathcal{O}L)$ has a representative in $\mathrm{Aut}(\mathcal{O}L)$ which stabilises the image of P in $\mathcal{O}L$ as a set, and any $(\mathcal{O}L, \mathcal{O}L)$-bimodule which induces a Morita equivalence has trivial source. We have canonical group isomorphisms*

$$\begin{aligned}
\mathrm{Pic}(\mathcal{O}L) &= \mathcal{T}(\mathcal{O}L)\\
&\cong \mathrm{Out}(\mathcal{O}L)\\
&\cong \mathrm{Out}_P(\mathcal{O}L) \rtimes N_{\mathrm{Aut}(P)}(E)/E\ ,\\
\mathrm{Out}_P(\mathcal{O}L) &\cong \mathrm{Hom}(E, k^\times)\ ,\\
\mathrm{Out}(P, \mathcal{F}) &\cong N_{\mathrm{Aut}(P)}(E)/E\ .
\end{aligned}$$

Moreover, the inverse image of $N_{\mathrm{Aut}(P)}(E)/E$ in $\mathrm{Out}(\mathcal{O}L)$ consists of all classes of automorphisms of $\mathcal{O}L$ which stabilise the trivial $\mathcal{O}L$-module.

Proof. Since E is abelian, it follows that the algebra $\mathcal{O}L$ is basic, and hence the canonical map $\mathrm{Out}(\mathcal{O}L) \to \mathrm{Pic}(\mathcal{O}L)$ is a group isomorphism (cf. [20, 4.9.7]). Since p'-roots of unity in $k^\times$ lift uniquely to roots of unity of the same order in $\mathcal{O}^\times$, it follows that the canonical map $\mathcal{O}^\times \to k^\times$ induces a group isomorphism $\mathrm{Hom}(E, \mathcal{O}^\times) \to \mathrm{Hom}(E, k^\times)$. Any group homomorphism $\zeta : E \to \mathcal{O}^\times$ yields an algebra automorphism η of $\mathcal{O}(P \rtimes E)$ defined by $\eta(uy) = \zeta(y)uy$ for all $u \in P$ and all $y \in E$; in particular, $\eta \in \mathrm{Aut}_P(\mathcal{O}L)$. If ζ is nontrivial, then the $\mathcal{O}L$-module ${}_\eta\mathcal{O}$ is nontrivial, and hence η is not inner. Thus the correspondence $\zeta \to \eta$ induces an injective group homomorphism $\mathrm{Hom}(E, k^\times) \to \mathrm{Out}_P(\mathcal{O}L)$, and by [26, 14.9] this is an isomorphism. Note that the above arguments show that nontrivial classes in $\mathrm{Out}_P(\mathcal{O}L)$ do not stabilise the trivial $\mathcal{O}L$-module; we will make use of this fact later.

If α is any algebra automorphism of $\mathcal{O}L$, then ${}_\alpha\mathcal{O}$ is a module of rank one, hence corresponds to a group homomorphism $\zeta : E \to \mathcal{O}^\times$. Denoting by η the algebra homomorphism as above, it follows that $\eta^{-1} \circ \alpha$ stabilises the trivial $\mathcal{O}(P \rtimes E)$-module. Thus $\alpha = \eta \circ (\eta^{-1} \circ \alpha)$ is a product of an automorphism in $\mathrm{Aut}_P(\mathcal{O}L)$ and an automorphism which stabilises the trivial module.

It remains to show that the outer automorphism group of $\mathcal{O}L$ of automorphisms which stabilise the trivial module is canonically isomorphic to $N_{\mathrm{Aut}(P)}(E)/E$. If ψ is an automorphism of P which normalises E, then by elementary group theory, ψ extends to a group automorphism of $P \rtimes E$, hence to an algebra automorphism β of $\mathcal{O}L$ which stabilises the trivial module. Moreover, β normalises $\mathrm{Out}_P(\mathcal{O}L)$. The group of outer automorphisms obtained in this way intersects $\mathrm{Out}_P(\mathcal{O}L)$ trivially, as remarked earlier.

By a result of Coleman [5] (or by results of Puig on fusion in block source algebras in [24, Theorem 3.1]), any inner algebra automorphism of $\mathcal{O}(P \rtimes E)$ which stabilises P acts on P as some automorphism in E. Therefore β is inner if and only if $\psi \in E$, so the above correspondence yields an injective group homomorphism $N_{\mathrm{Aut}(P)}(E)/E \to \mathrm{Out}(\mathcal{O}L)$, and the image of this homomorphism in $\mathrm{Out}(\mathcal{O}L)$ normalises $\mathrm{Out}_P(\mathcal{O}L)$ and intersects $\mathrm{Out}_P(\mathcal{O}L)$ trivially.

We need to show that the image of this group homomorphism consists of all classes of automorphisms which stabilise the trivial module, and this is done using Weiss' criterion (see Remark 1.8 above).

Since $[P, E] = P$, it follows that no nontrivial linear character of P extends to L. Thus every simple kL-module lifts to a unique irreducible character, and the characters arising as lifts of simple kL-modules are precisely the characters having P in their kernel. Again since E is abelian, the left kL-module kE (with P acting trivially) is a direct sum of a set of representatives of the isomorphism classes of simple kL-modules, and hence the $\mathcal{O}L$-module $\mathcal{O}E$, with P acting trivially, is the unique lift (up to isomorphism) of kE to an $\mathcal{O}$-free $\mathcal{O}L$-module. Thus the isomorphism class of $\mathcal{O}E$ is stable under any algebra automorphism of $\mathcal{O}L$. Note that $\mathcal{O}E \cong \mathcal{O}L \otimes_{\mathcal{O}P} \mathcal{O}$. Let $\alpha \in \mathrm{Out}(\mathcal{O}L)$. Then the $(\mathcal{O}P, \mathcal{O}P)$-bimodule ${}_\alpha\mathcal{O}L$ is free as a right $\mathcal{O}P$-module, and we have isomorphisms ${}_\alpha\mathcal{O}L \otimes_{\mathcal{O}P} \mathcal{O} \cong {}_\alpha\mathcal{O}E \cong \mathcal{O}E$. This is a permutation module as a left $\mathcal{O}P$-module (since P acts in fact trivially on this module). Weiss' criterion implies that ${}_\alpha\mathcal{O}L$ is a permutation $\mathcal{O}(P \times P)$-module. As an $\mathcal{O}(P \times L)$-module, $\mathcal{O}L$ is indecomposable, and hence ${}_\alpha\mathcal{O}L$ is indecomposable as well. Thus ${}_\alpha\mathcal{O}L$ is a trivial source $\mathcal{O}(P \times L)$-module, with a 'twisted diagonal' vertex of the form $\Delta\varphi$ for some automorphism φ of P. Thus ${}_\alpha\mathcal{O}L$ is isomorphic to a direct summand of $\mathrm{Ind}_{\Delta\varphi}^{P \times L}(\mathcal{O}) \cong {}_\varphi\mathcal{O}L$, where the last isomorphism sends $(u, x) \otimes 1$ to $\varphi(u)x^{-1}$ for $u \in P$ and $x \in L$. Comparing ranks yields an isomorphism ${}_\alpha\mathcal{O}L \cong {}_\varphi\mathcal{O}L$ as $(\mathcal{O}P, \mathcal{O}L)$-bimodules. Any such isomorphism is in particular an isomorphism of right $\mathcal{O}L$-modules, hence is induced by left multiplication with an element c in $\mathcal{O}L^\times$. The fact that this is also a homomorphism of left $\mathcal{O}P$-modules implies that $c\alpha(u) = \varphi(u)c$ for all $u \in P$. Thus after replacing α by its conjugate by c, we may assume that α extends φ. Let $y \in E$, and denote by c_y the inner automorphism of $\mathcal{O}(P \rtimes E)$ given by conjugation with y. Then $\alpha \circ c_y \circ \alpha^{-1}$ acts on P as $\varphi \circ c_y \circ \varphi^{-1}$. But $\alpha \circ c_y \circ \alpha^{-1} = c_{\alpha(y)}$ acts on P also as the automorphism given by conjugation with $\alpha(y)$. As before, any inner algebra automorphism of $\mathcal{O}L$ which stabilises P acts on P as some automorphism in E. Thus $c_{\alpha(y)}$ acts on P as some element in E, or equivalently, φ normalises E. It follows from the above that φ extends to a group automorphism of L, which in turn extends to an algebra automorphism β, whose class is by construction in the image of the map $N_{\mathrm{Aut}(P)}(E)/E \to \mathrm{Out}(P)$. Since α and β induce the same action as φ, it follows that $\beta^{-1} \circ \alpha \in \mathrm{Aut}_P(\mathcal{O}L)$. Therefore, if α stabilises the trivial $\mathcal{O}L$-module, so does $\beta^{-1} \circ \alpha$, and hence $\beta^{-1} \circ \alpha$ is inner. Equivalently, α and β have in that case the same image in $\mathrm{Out}(\mathcal{O}L)$. The result follows. $\square$

This Proposition shows that the last sequence in Theorem 1.1 applied to $B = \mathcal{O}(P \rtimes E)$ as above is a split short exact sequence. The hypothesis $[P, E] = P$ is in particular satisfied if E is nontrivial and acts freely on $P \setminus \{1\}$.

Corollary 4.4. *Let P be a nontrivial abelian p-group and E a nontrivial cyclic p'-subgroup of* $\mathrm{Aut}(P)$ *acting freely on* $P\setminus\{1\}$. *Set* $L = P \rtimes E$. *Assume that* $\mathrm{char}(\mathcal{O}) = 0$. *Let M be an $(\mathcal{O}L, \mathcal{O}L)$-bimodule inducing a stable equivalence of Morita type.*

Then M has an endotrivial source as an $\mathcal{O}(L \times L)$-module. In particular, we have $\underline{\mathrm{Pic}}(\mathcal{O}L) = \underline{\mathcal{E}}(\mathcal{O}L)$.

Proof. If M induces a Morita equivalence, then M has a trivial source by Proposition 4.3. Thus the Heller translates of M have as source Heller translates of the trivial module (for a vertex), and these are endotrivial. The result follows from Proposition 4.2. $\square$

Proof of Theorem 1.3. We use the notation of Theorem 1.3, and we set $L = P \rtimes E$. With the notation of Theorem 4.1, since M induces a stable equivalence of Morita type, it follows that the functor $M^* \otimes_B - \otimes_B M$ induces a group isomorphism $\underline{\mathrm{Pic}}(B) \cong \underline{\mathrm{Pic}}(\mathcal{O}L)$. By Corollary 4.4, we have $\underline{\mathrm{Pic}}(\mathcal{O}L) = \underline{\mathcal{E}}(\mathcal{O}L)$. Lemma 2.8 implies that $\underline{\mathrm{Pic}}(B) = \underline{\mathcal{E}}(B)$; in particular, we have $\mathrm{Pic}(B) = \mathcal{E}(B)$. The above isomorphism $\underline{\mathrm{Pic}}(B) \cong \underline{\mathrm{Pic}}(\mathcal{O}L)$ restricts to an injective group homomorphism $\mathrm{Pic}(B) \to \underline{\mathrm{Pic}}(\mathcal{O}L)$. Since $\mathrm{Pic}(B) = \mathcal{E}(B)$, the group $\mathrm{Pic}(B)$ is a finite subgroup of $\underline{\mathrm{Pic}}(\mathcal{O}L) = \mathrm{Pic}(\mathcal{O}L) \cdot \langle \Omega(\mathcal{O}L) \rangle$, where the last equality uses Proposition 4.2. We show next that the image of the group homomorphism

$$\mathrm{Pic}(B) \to \underline{\mathrm{Pic}}(\mathcal{O}L)$$

induced by $M^* \otimes_B - \otimes_B M$ is contained in $\mathrm{Pic}(\mathcal{O}L)$, hence induces an injective group homomorphism $\Psi : \mathrm{Pic}(B) \to \mathrm{Pic}(\mathcal{O}L) \cong \mathrm{Hom}(E, k^\times) \rtimes N_E$, where the second isomorphism is from Proposition 4.3.

If P is not cyclic, then $\langle \Omega(\mathcal{O}L) \rangle$ is an infinite cyclic central subgroup of $\underline{\mathrm{Pic}}(\mathcal{O}L)$. Thus any finite subgroup of $\underline{\mathrm{Pic}}(\mathcal{O}L)$ is contained in $\mathrm{Pic}(\mathcal{O}L)$. Therefore, if P is not cyclic, then the functor $M^* \otimes_B - \otimes_B M$ induces an injective group homomorphism $\mathrm{Pic}(B) \to \mathrm{Pic}(\mathcal{O}L)$.

If P is cyclic, this is true as well, but requires a slightly different argument since in that case $\mathcal{O}L$ is periodic as a bimodule (see e. g. [15, Lemma 4.1]), and hence $\underline{\mathrm{Pic}}(\mathcal{O}L)$ is already a finite group. If P is cyclic, then the assumption $E \neq \{1\}$ implies that $|P| \geq 3$. The algebra kL is a symmetric Nakayama algebra, and thus even powers of the Heller operator on kL permute the isomorphism classes of simple modules. It follows that every even power of the Heller operator on $\mathcal{O}L$ is induced by a Morita equivalence. Hence, in order to show that the $(\mathcal{O}L, \mathcal{O}L)$-bimodule $M^* \otimes_B N \otimes_B M$ induces a Morita equivalence, we may replace N by any even Heller translate as a (B, B)-bimodule. Corollary 4.6 below implies we may assume that $N \otimes_B -$ stabilises the isomorphism classes of all finitely generated $k \otimes_{\mathcal{O}} B$-modules. It follows that the stable equivalence of Morita type given by $M^* \otimes_B N \otimes_B M$ stabilises the isomorphism classes of all finitely generated kL-modules, hence is a Morita equivalence by [13, Theorem 2.1].

Back to the general case, it follows from Remark 1.2 (b) that the canonical image of $\mathrm{Out}_P(A)$ in $\mathrm{Pic}(B)$ is mapped to a subgroup of $\mathrm{Hom}(E, k^\times)$ under the injective group homomorphism $\mathrm{Pic}(B) \to \mathrm{Pic}(\mathcal{O}L)$. This shows the commutativity of the left rectangle in the statement of Theorem 1.3. For the commutativity of the right rectangle in that diagram, let N be a (B, B)-bimodule inducing a Morita equivalence on B. Let U be an indecomposable endopermutation $\mathcal{O}P$-module with vertex P and $\varphi \in N_{\mathrm{Aut}(P)}(E)$ such that the image of $[N]$ in $D_{\mathcal{O}}(P, \mathcal{F}) \rtimes N_E$ under the map Φ is equal to $[U][\varphi]$, where $[U]$ is the class of U in the Dade group and $[\varphi]$ is the image of φ in N_E. The map Ψ sends $[N]$ to the class of the $(\mathcal{O}L, \mathcal{O}L)$-bimodule

$N' = M^* \otimes_B N \otimes_B M$. By Lemma 2.6 the image of $[N']$ in $D_{\mathcal{O}}(P, \mathcal{F}) \rtimes N_E$ is

$$[V^*] \cdot [U] \cdot [\varphi] \cdot [V] = [V^* \otimes_{\mathcal{O}} U \otimes_{\mathcal{O}} {}^{\varphi}V] \cdot [\varphi] = [\varphi]$$

where the last equality uses the fact that $\mathrm{Pic}(\mathcal{O}L) = \mathcal{T}(\mathcal{O}L)$ from Proposition 4.3. This shows the commutativity of the right rectangle in the diagram and concludes the proof of Theorem 1.3. □

Let G be a finite group and b a block of kG with a nontrivial cyclic defect group P and inertial quotient E. We collect some well-known results on cyclic blocks, going back to Brauer, Dade, and Green. Our notation follows [15] (where proofs and further references can be found).

Set $B = kGb$, and denote by I a set of representatives of the conjugacy classes of primitive idempotents in B. We have $\ell(B) = |E| = |I|$. For any $i \in I$, the projective indecomposable B-module Bi has uniserial submodules U_i, V_i such that $U_i + V_i = J(B)i$ and $U_i \cap V_i = \mathrm{soc}(Bi)$. One may choose notation such that $\Omega(U_i) \cong V_{\rho(i)}$ for some $\rho(i) \in I$ and $\Omega(V_i) \cong U_{\sigma(i)}$ for some $\sigma(i) \in I$. The maps ρ, σ defined in this way are then permutations of I, and $\rho \circ \sigma$ is a transitive cycle on I. The Brauer tree of B is defined as follows. Each vertex corresponds to exactly one ρ-orbit, or one σ-orbit. The edges are labelled by the elements in I, in such a way that the edge i connects the ρ-orbit i^ρ of i with the σ-orbit i^σ of i. The permutations ρ and σ induce cyclic permutations of the set of edges emanating from a fixed vertex. If $|P| \geq 3$, then the $2|E|$ modules U_i, V_i, $i \in I$, are pairwise nonisomorphic, because they correspond to the simple $k(P \rtimes E)$-modules and their Heller translates through a stable equivalence of Morita type between B and $k(P \rtimes E)$. In particular, if $|P| \geq 3$, then the modules U_i, V_i all have period $2|E|$, for any even integer n and any $i \in I$ there is $j \in I$ such that $\Omega^n(U_i) \cong U_j$, and for any odd integer n and any $i \in I$ there is $j \in I$ such that $\Omega^n(U_i) \cong V_j$.

Proposition 4.5. *Let G be a finite group and b a block of kG with a cyclic defect group P of order at least 3. Set $B = kGb$. Let n be an integer such that the (B, B)-bimodule $M = \Omega^n_{B \otimes_k B^{\mathrm{op}}}(B)$ induces a Morita equivalence. Then n is even. As a $k(G \times G)$-module, M has vertex $\Delta P = \{(u, u) \mid u \in P\}$ and trivial source.*

Proof. Since the functor $M \otimes_B -$ is an equivalence, it permutes the isomorphism classes of simple modules in such a way that the induced permutation of the edges of the Brauer tree is a tree automorphism. By [12, 7.2] this automorphism stabilises a vertex (this is where we use that $|P| \geq 3$). Thus there is a ρ-orbit or a σ-orbit which is stabilised by this automorphism. Suppose that the ρ-orbit i^ρ of an edge i is stabilised. Thus $M \otimes_B U_i \cong U_{i'}$ for some i' belonging to the ρ-orbit of i. Note that $M \otimes_B U_i \cong \Omega^n(U_i)$. By the above, we have $U_{i'} \cong \Omega^m(U_i)$ for some even integer m. Thus $\Omega^{n-m}(U_i) \cong U_i$. Since U_i has period $2|E|$, it follows that $2|E|$ divides $n - m$; in particular, $n - m$ is even. Since also m is even, so is n. A similar argument applies if the Brauer tree automorphism stabilises a σ-orbit. The $k(G \times G)$-module B has vertex ΔP and trivial source. Since the trivial kP-module k has period 2 and since n is even, it follows that M has vertex ΔP and trivial source. □

Corollary 4.6. *Let G be a finite group and b a block of $\mathcal{O}G$ with a nontrivial cyclic defect group P. Set $B = \mathcal{O}Gb$. Let N be a (B, B)-bimodule inducing a Morita equivalence. Then there exists an integer n such that the bimodule $N' = \Omega^n_{B \otimes_{\mathcal{O}} B^{\mathrm{op}}}(N)$ induces a Morita equivalence with the property that the functor $N' \otimes_B -$ stabilises* *the isomorphism class of every finitely generated $k\otimes_{\mathcal{O}} B$-module. Moreover, if $|P| \geq 3$, then any integer n with this property is even.*

Proof. It follows from [15, 5.1] that there is an integer n such that $N' = \Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(N)$ induces a Morita equivalence with the property that the functor $N' \otimes_B -$ stabilises the isomorphism class of every finitely generated $k \otimes_{\mathcal{O}} B$-module. Then the bimodule $N' \otimes_B N^* \cong \Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(B)$ induces a Morita equivalence as well. Therefore, if $|P| \geq 3$, then Proposition 4.5 implies that n is even. □

Proof of Theorem 1.4. Let G be a finite group and b a block of $\mathcal{O}G$ with a nontrivial cyclic defect group P and nontrivial inertial quotient E. Set $B = \mathcal{O}Gb$. Since E is nontrivial, the block B is not nilpotent, and hence $|P| \geq 3$. Let M be a (B, B)-bimodule inducing a Morita equivalence. Since $\mathrm{Pic}(B) = \mathcal{E}(B)$, the vertices of M are twisted diagonal subgroups of $P \times P$ isomorphic to P. We need to show that M has a trivial source, for some vertex. Since the trivial $\mathcal{O}P$-module has period 2, it follows that M has a trivial source if and only $\Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(M)$ has a trivial source for some (and then necessarily any) even integer n. Therefore, by Corollary 4.6 we may assume that the functor $M \otimes_B -$ stabilises the isomorphism classes of all finitely generated $k \otimes_{\mathcal{O}} B$-modules. By [15, 4.3, 5.6], the subgroup of $\mathrm{Pic}(B)$ of bimodules with this property is canonically isomorphic to $\mathrm{Aut}(P)/E$, and by [15, 5.8], the bimodules with this property correspond to algebra automorphisms of A extending group automorphisms of P, and hence these bimodules have trivial source. By the Remark 1.2, the group $\mathrm{Out}_P(A)$ corresponds to the subgroup of $\mathrm{Pic}(B)$ of isomorphism classes of bimodules of the form $\Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(B)$, where n is an integer such that this bimodule induces a Morita equivalence. Note that in that case we have $\Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(M) \cong \Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(B) \otimes_B M \cong M \otimes_B \Omega^n_{B\otimes_{\mathcal{O}} B^{\mathrm{op}}}(B)$. Thus $\mathrm{Out}_P(A)$ is indeed a direct factor of $\mathrm{Pic}(B)$. □

Proof of Theorem 1.5. Let G be a finite group and b a non-nilpotent block of $\mathcal{O}G$ with a Klein four defect group P. Set $B = \mathcal{O}Gb$. Every endopermutation $\mathcal{O}P$-module is a Heller translate of a rank one module. By [14, Theorem 1.1], B is Morita equivalent to either $\mathcal{O}A_4$ or the principal block algebra of $\mathcal{O}A_5$, via a Morita equivalence with source a Heller translate of the trivial module. (Using the classification of finite simple groups it is shown in [6] that there is even a Morita equivalence with trivial source in these cases, but this is not needed for the present proof). By Lemma 2.8, we may assume that B is equal to one of these two algebras. Note that then B is its own source algebra.

If $B = \mathcal{O}A_4$, the result follows from Proposition 4.3; indeed, we have $\mathrm{Out}_P(B) \cong C_3$, and $\mathrm{Aut}(P) \cong S_3$, hence $\mathrm{Aut}(P)/E \cong S_3/C_3 \cong C_2$, which acts nontrivially on $\mathrm{Out}_P(B)$ and hence yields $\mathrm{Pic}(B) \cong S_3$.

Suppose that B is the principal block algebra $\mathcal{O}A_5b_0$. As a very special case of Theorem 4.1, identifying $\mathcal{O}A_4$ to its image in this algebra, induction and restriction yields a splendid stable equivalence of Morita type between B and $\mathcal{O}A_4$ (this is well-known and easy to verify directly). That is, we are in a situation of Lemma 2.8 in which $V = \mathcal{O}$. By Theorem 1.3 we have an embedding $\mathrm{Pic}(B) \to \mathrm{Pic}(\mathcal{O}A_4) \cong C_3 \rtimes C_2$ which maps $\mathrm{Out}_P(B)$ to the subgroup C_3.

We show next that $\mathrm{Out}_P(B)$ is trivial. Write $A_4 = P \rtimes E$ with $E \cong C_3$. We need to show that the nontrivial automorphisms of the form $uy \mapsto \zeta(y)uy$ of $\mathcal{O}A_4$ (with $\zeta : E \to k^\times$ a group homomorphism, $u \in P$, $y \in E$ as above) do not extend to B. Indeed, if they did, then the three trivial source B-modules corresponding (through the stable equivalence between $\mathcal{O}A_4$ and B) to the three linear characters of $\mathcal{O}A_4$ would have to have the same rank because they would be transitively permuted by this automorphism group. This is not possible: the trivial character of $\mathcal{O}A_4$ corresponds to the trivial character of B, while the two nontrivial characters of $\mathcal{O}A_4$ correspond to B-modules of $\mathcal{O}$-rank greater than 1 (because A_5 has no nontrivial character of rank 1). Thus $\mathrm{Out}_P(B)$ is trivial.

It follows that $\mathrm{Pic}(B)$ embeds into C_2. An automorphism of B given by conjugation with an involution in S_5 yields a nontrivial element in $\mathrm{Pic}(B)$, whence $\mathrm{Pic}(B) \cong C_2$. □

5. Proof of Theorem 1.6

We assume in this Section that $\mathcal{O}$ has characteristic zero. For the description of $\mathrm{Pic}(B)$ as colimit of finite groups in Theorem 1.6, we will need the following lemmas. One of the key ingredients is Brauer's Lemma 1 from [3], implying that if K is a field of characteristic 0 containing all p'-order roots of unity in an algebraic closure, then the Schur indices over K of absolutely irreducible characters of finite groups are all equal to 1.

Lemma 5.1. *Let K be a field of characteristic 0 containing all roots of unity of order prime to p in an algebraic closure of K and let K_0 be a subfield of K containing all algebraic elements in K. Let G be a finite group and X a finite-dimensional KG-module. Then there is a K_0G-module X_0 such that $X \cong K \otimes_{K_0} X_0$.*

Proof. Since KG is semisimple, we may assume that X is a simple KG-module. Let K' be a splitting field for X containing K. Let Y' be a simple $K'G$-module which is isomorphic to a direct summand of $K' \otimes_K X$, and denote by ψ the character of Y'. The field $K(\psi)$ generated by the values $\psi(g)$, $g \in G$, is contained in $K(\zeta)$ for some root of unity of p-power order ζ, since K contains all roots of unity of order prime to p. The minimal polynomial of ζ over K has algebraic numbers as coefficients, hence is also the minimal polynomial of ζ over K_0. Thus the restriction of field automorphisms of $K(\zeta)$ to $K_0(\zeta)$ induces an isomorphism of Galois groups $\mathrm{Gal}(K(\zeta) : K) \cong \mathrm{Gal}(K_0(\zeta) : K_0)$, hence an isomorphism of Galois groups $\mathrm{Gal}(K(\psi) : K) \cong \mathrm{Gal}(K_0(\psi) : K_0)$. By [3, Lemma 1], $K_0(\psi)$ is a splitting field for ψ; that is, $Y' \cong K' \otimes_{K_0(\psi)} Y_0$ for some simple $K_0(\psi)G$-module Y_0; in particular, $Y = K(\psi) \otimes_{K_0(\psi)} Y_0$ is a simple $K(\psi)$-module which appears in $K(\psi) \otimes_K X$. It follows from [23, Ch. 3 Theorem (1.30)] that we have an isomorphism

$$K(\psi) \otimes_K X \cong \bigoplus_{\sigma \in \mathrm{Gal}(K(\psi):K)} {}^{\sigma}Y$$

Combining the above isomorphisms yields that the K_0G-module

$$X_0 = \bigoplus_{\sigma \in \mathrm{Gal}(K_0(\psi):K_0)} {}^{\sigma}Y_0$$

satisfies

$$K(\psi) \otimes_{K_0} X_0 \cong K(\psi) \otimes_K X$$

and hence $K \otimes_{K_0} X_0 \cong X$. □

The following Lemma is well-known; we sketch a proof for convenience. As before, by a p-adic subring of $\mathcal{O}$ we mean a finite extension $\mathcal{O}_0$ of the p-adic integers contained in $\mathcal{O}$. Note that $J(\mathcal{O}_0) \subseteq J(\mathcal{O})$.

Lemma 5.2. *Suppose that k is an algebraic closure of $\mathbb{F}_p$. Denote by K the field of fractions of $\mathcal{O}$. Let R be the union of all p-adic subrings of $\mathcal{O}$, and let E be the field of fractions of R identified to its image in K. Denote by $\nu : K^\times \to \mathbb{Z}$ the π-adic valuation, where $\pi \in \mathcal{O}$ such that $J(\mathcal{O}) = \pi\mathcal{O}$.*

(1) The field K is the completion of E and $\mathcal{O}$ is the completion of R with respect to the topology induced by ν.
(2) The restriction $\nu|_E$ of ν to $E^\times$ is a discrete valuation with valuation ring R and valuation ideal $R \cap \pi\mathcal{O}$.
(3) The field E is the algebraic closure of $\mathbb{Q}_p$ in K.

Proof. Let $W(k) \subseteq \mathcal{O}$ be the ring of Witt vectors of k in $\mathcal{O}$. By the structure theory of complete discrete valuation rings, we have $\mathcal{O} = W(k)[\pi]$ for an element π satisfying a monic polynomial $f(x)$ in $W(k)[x]$ such that $J(\mathcal{O}) = \pi\mathcal{O}$ (see [32, Chapter 2, Theorems 3, 4 and Chapter 1, Propositions 17, 18]). Every element of $W(k)$ is a limit of some sequence $(\sum_{0\leq i\leq n} \zeta_i p^i)_{n\geq 0}$, where $\zeta_n \in W(k)^\times$ is a p'-root of unity, for all $n \in \mathbb{N}$ (see proof of Theorem 3, Chapter 2 of [32]). Since every p'-root of unity in $W(k)$ lies in a finite extension of $\mathbb{Z}_p$ contained in $W(k)$, it follows that $W(k)$ is the completion of the union of the finite extensions of $\mathbb{Z}_p$ contained in $W(k)$. Hence, by an application of Krasner's lemma, we may assume that $f(x) \in A[x]$, for some finite extension A of $\mathbb{Z}_p$ contained in $W(k)$ (see [32, Chapter 2, Exercises 1,2]). Consequently, π lies in a finite extension of A and hence in R. Since every element of $\mathcal{O}$ is a polynomial in π of degree at most the degree of f and with coefficients in $W(k)$, it follows by the same reasoning as above that every element of $\mathcal{O}$ is a limit of a sequence of elements of R implying (1).

As shown above, $\pi \in R$. Thus the restriction of ν to $E^\times$ has valuation group $\mathbb{Z}$ and $\nu|_E$ is a discrete valuation. In order to show (2) we need to observe that $R = E \cap \mathcal{O}$. The inclusion $\subseteq$ is trivial. For the reverse inclusion, any element in $E \cap \mathcal{O}$ belongs to the fraction field L of some p-adic subring $\mathcal{O}_0$ of $\mathcal{O}$ with nonnegative valuation, hence to the valuation ring of L, and that is precisely $\mathcal{O}_0$.

If L is a subfield of K containing $\mathbb{Q}_p$ such that the degree of $L/\mathbb{Q}_p$ is finite, then the valuation ring of L is a p-adic subring of $\mathcal{O}$, and hence L is contained in E. Since E is the union of finite extensions of $\mathbb{Q}_p$, all elements in E are algebraic over $\mathbb{Q}_p$, whence statement (3). $\square$

Proposition 5.3. *Suppose that k is an algebraic closure of $\mathbb{F}_p$. Denote by K the field of fractions of $\mathcal{O}$. Let R be the union of all p-adic subrings of $\mathcal{O}$, and let E be the field of fractions of R identified to its image in K. Let G be a finite group, b a central idempotent of $\mathcal{O}G$, and set $B = \mathcal{O}Gb$. Let M be a (B, B)-bimodule inducing a Morita equivalence on B. There exist a p-adic subring $\mathcal{O}_0$ of $\mathcal{O}$ such that $b \in \mathcal{O}_0 G$ and an $(\mathcal{O}_0 Gb, \mathcal{O}_0 Gb)$-bimodule inducing a Morita equivalence on $\mathcal{O}_0 Gb$ such that $M \cong \mathcal{O} \otimes_{\mathcal{O}_0} M_0$.*

Proof. Let $X = K \otimes_{\mathcal{O}} M$. So, M is a full $\mathcal{O}(G\times G)$-lattice in X. By Lemma 5.2, E contains all algebraic numbers in K. Since k is an algebraic closure of $\mathbb{F}_p$, it follows that E contains all p'-order roots of unity in an algebraic closure of K. Hence by Lemma 5.1, there exists an $E(G \times G)$-module X_E such that $X = K \otimes_E X_E$. Let $M_R = X_E \cap M$. By Lemma 5.2, $\mathcal{O}$ is the completion of R and K is the completion of E. Hence, by [7, Corollary 30.10], we have that $M \cong \mathcal{O} \otimes_R M_R$. Let $\mathcal{B}$ be an R-basis of M_R. For $u \in G \times G$ and x, $y \in \mathcal{B}$, let $\alpha^u_{x,y}$ denote the coefficient of y when ux is written as an R-linear combination of elements of $\mathcal{B}$. Let $b = \sum_{g\in G} \beta_g g$,

$\beta_g \in \mathcal{O}$. Then $\beta_g \in R$ for all $g \in G$. Since every finite extension of $\mathbb{Z}_p$ in $\mathcal{O}$ is contained in a p-adic subring of $\mathcal{O}$, there exists a p-adic subring $\mathcal{O}_0$ of $\mathcal{O}$ containing $\alpha^u_{x,y}$ and containing β_g for all $u \in G \times G$, $x, y \in \mathcal{B}$ and all $g \in G$. Let M_0 be the $\mathcal{O}_0$-submodule of M_R generated by $\mathcal{B}$. Then M_0 is an $(\mathcal{O}_0 Gb, \mathcal{O}_0 Gb)$-bimodule, finitely generated and free as $\mathcal{O}_0$-module, satisfying $M_R \cong R \otimes_{\mathcal{O}_0} M_0$. It follows that $M \cong \mathcal{O} \otimes_{\mathcal{O}_0} M_0$. By [12, Lemma 4.4, Prop. 4.5] we have that M_0 induces a Morita equivalence on $\mathcal{O}_0 Gb$. The result follows. □

Proof of Theorem 1.6. Statement (i) follows from Proposition 5.3. Thus $\mathrm{Pic}(B)$ is the colimit of the groups $\mathrm{Pic}(\mathcal{O}_0 Gb)$ as stated. By [8, Theorems (55.19), (55.25)] (see also Theorem 6.2 below for an alternative proof) the groups $\mathrm{Pic}(\mathcal{O}_0 Gb)$ are finite, whence (ii). □

6. On the finiteness and structure of Picard groups over p-adic rings

In this section we provide a proof for the finiteness of $\mathrm{Pic}(B)$ for a block algebra B of a group algebra over a p-adic ring. We start out assuming that $\mathcal{O}$ has characteristic zero, and specialize later to the case that $\mathcal{O}$ is a p-adic ring. Write $J(\mathcal{O}) = \pi\mathcal{O}$ for some prime element π of $\mathcal{O}$. Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. We use without further comment some standard facts relating automorphisms to Picard groups (see [8, §55 A] or [20, 2.8.16] for more details). If $\alpha \in \mathrm{Aut}(A)$, then the (A, A)-bimodule A_α induces a Morita equivalence on $\mathrm{mod}(A)$, and we have $A_\alpha \cong A$ as (A, A)-bimodules if and only if α is inner. The map sending α to A_α induces an injective group homomorphism $\mathrm{Out}(A) \to \mathrm{Pic}(A)$. If A is basic, this is an isomorphism (see e. g. [20, 4.9.7]). In general, if M is an (A, A)-bimodule inducing a Morita equivalence, then $M \cong A_\alpha$ as (A, A)-bimodules for some $\alpha \in \mathrm{Aut}(A)$ if and only if $M \cong A$ as left A-modules. In particular, if the functor $M \otimes_A -$ stabilises the isomorphism classes of all simple modules, hence of all finitely generated projective modules, then $M \cong A_\alpha$ for some $\alpha \in \mathrm{Aut}(A)$. It follows that the image of $\mathrm{Out}(A)$ in $\mathrm{Pic}(A)$ has finite index because it contains the subgroup of $\mathrm{Pic}(A)$ of Morita equivalences which stabilise the isomorphism classes of all simple modules.

Let r be a positive integer. We denote by $\mathrm{Aut}_r(A)$ the set of $\mathcal{O}$-algebra automorphisms of A which induce the identity on $A/\pi^r A$. A trivial verification shows that $\mathrm{Aut}_r(A)$ is a subgroup of $\mathrm{Aut}(A)$. We denote by $\mathrm{Out}_r(A)$ the image of $\mathrm{Aut}_r(A)$ in $\mathrm{Out}(A)$. The map sending $\alpha \in \mathrm{Aut}_r(A)$ to the (A, A)-bimodule A_α induces a group isomorphism

$$\mathrm{Out}_r(A) \cong \ker(\mathrm{Pic}(A) \to \mathrm{Pic}(A/\pi^r A)) ;$$

see e. g. [18, 3.1]. In particular, $\mathrm{Out}_1(A)$ is isomorphic to the kernel of the canonical map $\mathrm{Pic}(A) \to \mathrm{Pic}(k \otimes_{\mathcal{O}} A)$, and $\mathrm{Out}_r(A)$ is a normal subgroup of $\mathrm{Out}_1(A)$ for all $r \geq 1$.

If $\mathcal{O}$ is a p-adic ring, G a finite group and B a block algebra of $\mathcal{O}G$, then by a result of Hertweck and Kimmerle [11, 3.13], the group $\mathrm{Out}_r(B)$ is trivial, for r large enough. The proof of [11, 3.13] uses a theorem of Weiss in [35] (restated as Theorem 3.2 in [11]). As a consequence of Maranda's theorem [7, (30.14)], this remains true if $\mathcal{O}$ is not necessarily p-adic, but Maranda's theorem leads to a larger bound for r. More precisely, the smallest positive integer r for which $\mathrm{Out}_r(B)$ is trivial in [11, 3.13] depends only on the ring $\mathcal{O}$, while the lowest bound obtained from Maranda's theorem, applied to the $\mathcal{O}(G \times G)$-modules B and B_α, with $\alpha \in \mathrm{Out}_r(B)$, would be an integer r such that $\pi^r\mathcal{O} = \pi|G|^2\mathcal{O}$. We give an elementary proof of the fact that $\mathrm{Out}_r(B)$ is trivial for r large enough which does not require Weiss' results and which slightly improves on Maranda's bound; that is, we obtain a lower bound for r which depends on $\mathcal{O}$ and the size of a defect group of B.

Proposition 6.1. *Let G be a finite group, B a block algebra of $\mathcal{O}G$ and P a defect group of B. Let d be the positive integer such that $\pi^d\mathcal{O} = |P|\mathcal{O}$. For any integer $r > d$ the group $\mathrm{Out}_r(B)$ is trivial.*

Proof. It suffices to show this for $r = d + 1$. Let $\alpha \in \mathrm{Aut}_r(B)$. In order to show that α is inner, we need to show that the (B, B)-bimodule B_α is isomorphic to B. Equivalently, we need to show that B_α and B are isomorphic as $\mathcal{O}(G\times G)$-modules. By the assumptions on α, we have a bimodule isomorphism $B_\alpha/\pi^r B_\alpha \cong B/\pi^r B$; in particular, $B_\alpha/\pi B_\alpha \cong B/\pi B$ as $k(G \times G)$-modules. Since p-permutation modules over k lift uniquely, up to isomorphism, to p-permutation modules over $\mathcal{O}$, it suffices to show that B_α is a p-permutation $\mathcal{O}(G\times G)$-module. Since B is relatively $P\times P$-projective, it suffices to show that B_α is a permutation $\mathcal{O}(P \times P)$-module.

The hypotheses on α imply that for any $x \in B$ we have $\alpha(x) = x + \pi|P|c_x$, for some $c_x \in B$. Identify the defect group P to its image in B. Set $v = \sum_{x\in P} \alpha(x)x^{-1}$. We have

$$\alpha(x)x^{-1} = 1 + \pi|P|c_x x^{-1}$$

for all $x \in P$. Thus

$$v = |P| \cdot 1 + \pi|P|a$$

for some $a \in B$. Set $u = 1 + \pi a$; this element is clearly invertible in B and satisfies $|P|u = v$. We are going to show that $uyu^{-1} = \alpha(y)$ for all $y \in P$. This is equivalent to showing that $uy = \alpha(y)u$, hence to $vy = \alpha(y)v$. In order to show this, fix $y \in P$. Then

$$vy = \sum_{x\in P} \alpha(x)x^{-1}y = \sum_{x\in P} \alpha(yy^{-1}x)x^{-1}y = \sum_{x\in P} \alpha(y)\alpha(y^{-1}x)x^{-1}y \ .$$

If x runs over all elements of P, then so does $y^{-1}x$, and therefore this element is equal to $\alpha(y)v$ as required. This implies that composing α with the inner automorphism given by conjugation with u^{-1} yields an automorphism β which is the identity on P and which belongs to the same class as α in $\mathrm{Out}(B)$. The fact that $u \in 1 + \pi B$ implies that $\beta \in \mathrm{Aut}_1(B)$. Since the images of α and β in $\mathrm{Out}(B)$ are equal, it follows that $B_\alpha \cong B_\beta$ as (B, B)-bimodules, or equivalently, as $\mathcal{O}(G \times G)$-modules. Since β is the identity on P, it follows that B_β, and therefore also B_α, is a permutation $\mathcal{O}(P \times P)$-module (isomorphic to B as an $\mathcal{O}(P \times P)$-module). The result follows. □

Combining Proposition 6.1 with [18, 3.5] yields a proof of the following result, including the finiteness of $\mathrm{Pic}(B)$ in Part (ii) in the case that $\mathcal{O}$ is a p-adic ring.

Theorem 6.2. *Let G be a finite group and B a block algebra of $\mathcal{O}G$ with a defect group P. Set $\bar{B} = k \otimes_{\mathcal{O}} B$. We have a canonical exact sequence of groups*

$$1 \longrightarrow \mathrm{Out}_1(B) \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(\bar{B})$$

with the following properties.

(i) The group $\mathrm{Out}_1(B)$ has a finite p-power exponent dividing p^d, where d is the positive integer satisfying $\pi^d\mathcal{O} = |P|\mathcal{O}$.

(ii) If k is finite, then $\mathrm{Pic}(B)$ *is finite, and* $\mathrm{Out}_1(B)$ *is a finite p-group.*

(iii) If k is an algebraic closure of $\mathbb{F}_p$*, then every element in* $\mathrm{Pic}(B)$ *has finite order.*

We restate the part of [18, 3.5] required for the proof of Theorem 6.2.

Proposition 6.3 (cf. [18, 3.5])**.** *Let A be an $\mathcal{O}$-algebra which is finitely generated free as an $\mathcal{O}$-module. Let r be a positive integer. Suppose that the canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective. We have an exact sequence of groups*

$$1 \longrightarrow \mathrm{Out}_{2r}(A) \longrightarrow \mathrm{Out}_r(A) \longrightarrow HH^1(A/\pi^r A) \quad .$$

For any integer $a \geq 0$, the map $\mathrm{Out}_r(A) \to HH^1(A/\pi^r A)$ *in this sequence sends the subgroup* $\mathrm{Out}_{r+a}(A)$ *of* $\mathrm{Out}_r(A)$ *to* $\pi^a HH^1(A/\pi^r A)$*.*

We briefly review the construction of the map from $\mathrm{Out}_r(A) \to HH^1(A/\pi^r A)$ in Proposition 6.3. Let $\alpha \in \mathrm{Aut}_r(A)$. For all $a \in A$ we have $\alpha(a) = a + \pi^r \tau(a)$ for some $\tau(a) \in A$. By comparing the expressions $\alpha(ab)$ and $\alpha(a)\alpha(b)$ for any two a, $b \in A$, one sees that the linear map τ on A induces a derivation $\bar{\tau}$ on $A/\pi^r A$. (For the ring $k[[t]]$ instead of $\mathcal{O}$ and $r = 1$, this construction is due to Gerstenhaber; the derivations which arise in this way are called *integrable*.) It is shown in [18, 3.5] that the map $\alpha \mapsto \bar{\tau}$ sends inner automorphisms to inner derivations, hence induces a map $\mathrm{Out}_r(A) \to HH^1(A/\pi^r A)$. This is shown to be a group homomorphism. By construction, this map sends $\mathrm{Out}_{r+a}(A)$ to $\pi^a HH^1(A/\pi^r A)$, hence has $\mathrm{Out}_{2r}(A)$ in its kernel. It is further shown in [18, 3.5] that $\mathrm{Out}_{2r}(A)$ is indeed equal to that kernel.

Block algebras satisfy the hypothesis on the surjectivity of the canonical map $Z(A) \to Z(A/\pi^r A)$. We note the following consequences of Proposition 6.3.

Corollary 6.4. *Let A be an $\mathcal{O}$-algebra which is finitely generated free as an $\mathcal{O}$-module. Let r be a positive integer. Suppose that the canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective. The following hold.*

(i) The quotient $\mathrm{Out}_r(A)/\mathrm{Out}_{2r}(A)$ *is abelian of exponent dividing p^r. If k is finite, then* $\mathrm{Out}_r(A)/\mathrm{Out}_{2r}(A)$ *is a finite abelian p-group of exponent dividing p^r.*

(ii) The quotient $\mathrm{Out}_r(A)/\mathrm{Out}_{r+1}(A)$ *is abelian of exponent dividing p. If k is finite, then* $\mathrm{Out}_r(A)/\mathrm{Out}_{r+1}(A)$ *is a finite elementary abelian p-group.*

Proof. The exact sequence in 6.3 implies that the quotient $\mathrm{Out}_r(A)/\mathrm{Out}_{2r}(A)$ is isomorphic to a subgroup of the additive abelian group $HH^1(A/\pi^r A)$. This group is annihilated by π^r, hence by p^r. If k is finite, then $A/\pi^r A$ is a finite set, and hence $HH^1(A/\pi^r A)$ is a finite abelian p-group of exponent dividing p^r. This proves (i). It follows from (i) that the quotient $\mathrm{Out}_r(A)/\mathrm{Out}_{r+1}(A)$ is abelian of exponent dividing p^r. We need to show that its exponent is at most p. Let $\alpha \in \mathrm{Aut}_r(A)$. For $a \in A$, write $\alpha(a) = a + \pi^r \tau(a)$ for some $\tau(a) \in A$. An easy induction shows that for $n \geq 1$ we have $\alpha^n(a) \equiv a + n\pi^r \tau(a)$ modulo $\pi^{r+1}A$, and thus $\alpha^p \in \mathrm{Aut}_{r+1}(A)$. The rest of (ii) follows as in (i). □

One can improve the statement on the exponent in statement (i) of Corollary 6.4 by taking ramification into account; we ignored this in the above proof and simply argued that if π^r annihilates an $\mathcal{O}$-module, then the underlying abelian group has exponent dividing p^r. The following statement is well-known; we include a proof for convenience.

Lemma 6.5. *Let k be an algebraic closure of $\mathbb{F}_p$, let A be a finite-dimensional k-algebra, and let X be a k-basis of A. Then $\mathrm{Pic}(A)$ is the colimit of the finite groups $\mathrm{Pic}(A_{\mathbb{F}})$, with $\mathbb{F}$ running over the finite subfields of k which contain the multiplicative structure constants of X, where $A_{\mathbb{F}}$ is the $\mathbb{F}$-algebra spanned by X. In particular, every element in $\mathrm{Pic}(A)$ has finite order.*

Proof. We may assume that A is basic, and hence that $\mathrm{Out}(A) \cong \mathrm{Pic}(A)$. Since k is an algebraic closure of $\mathbb{F}_p$, all elements in k are algebraic, and hence any finite subset of k generates a finite subfield. In particular, the multiplicative constants of the basis X generate a finite subfield k_0 of A. Let $\alpha \in \mathrm{Aut}(A)$. For any $x \in X$, write $\alpha(x) = \sum_{y \in X} \lambda(x,y)y$ for some coefficients $\lambda(x,y)$ in k. By the above, these coefficients are contained in a finite subfield k_1 of k, which we may choose to contain k_0. Thus α is the extension to A of a k_1-algebra automorphism α_1 of the k_1-subalgebra A_1 of A with k_1-basis X, and hence $\mathrm{Pic}(A)$ is a colimit as stated. Since A_1 is a finite set, it follows that α_1 has finite order, implying that α has finite order. This completes the proof. □

Proof of Theorem 6.2. Set $\bar{B} = k \otimes_{\mathcal{O}} B$. As mentioned above, the kernel of the canonical map $\mathrm{Pic}(B) \to \mathrm{Pic}(\bar{B})$ is isomorphic to $\mathrm{Out}_1(B)$, whence the canonical exact sequence as stated.

By 6.4, the group $\mathrm{Out}_1(B)$ is filtered by the normal subgroups $\mathrm{Out}_i(B)$ such that the quotient of each two consecutive groups has exponent at most p. By Maranda's theorem [7, (30.14)] or by Proposition 6.1 above, there exists a positive integer r such that $\mathrm{Out}_r(B)$ is trivial, and then $\mathrm{Out}_1(B)$ has exponent dividing p^{r-1}. The statement on the exponent in (i) follows from Proposition 6.1.

By 6.4 again, if k is finite, then the quotients of subsequent subgroups in this filtration are finite elementary abelian p-groups, and hence $\mathrm{Out}_1(B)$ is a finite p-group in that case. If k is finite, then $k \otimes_{\mathcal{O}} B$ is a finite set; in particular, its automorphism group as a k-algebra is finite. Since $\mathrm{Out}(\bar{B})$ is isomorphic to a subgroup of finite index in $\mathrm{Pic}(\bar{B})$, it follows that $\mathrm{Pic}(\bar{B})$ is finite, and hence so is $\mathrm{Pic}(B)$, proving (ii).

For statement (iii), assume that k is an algebraic closure of $\mathbb{F}_p$. By (i), every element in $\mathrm{Out}_1(B)$ has finite order, and hence it suffices to show that every element in $\mathrm{Pic}(\bar{B})$ has finite order. The result follows from Lemma 6.5. □

7. Examples

Example 7.1. By results of Roggenkamp and Scott [30] (see also Weiss [34], [35]), we have a canonical group isomorphism

$$\mathrm{Pic}(\mathcal{O}P) = \mathcal{L}(\mathcal{O}P) \cong \mathrm{Hom}(P, \mathcal{O}^\times) \rtimes \mathrm{Out}(P)$$

This isomorphism is induced by the map Φ in Theorem 1.1 applied to $\mathcal{O}P$. (The source algebra A in 1.1 is in that case $\mathcal{O}P$, and $\mathrm{Out}_P(A)$ is trivial). This isomorphism restricts to an isomorphism

$$\mathcal{T}(\mathcal{O}P) \cong \mathrm{Out}(P) \ ,$$

so in particular, the last horizontal map in the diagram in Theorem 1.1 is surjective. This need not be the case, however, for an arbitrary nilpotent block.

Let G be a finite group and let b be a nilpotent block of $\mathcal{O}G$ with a nontrivial defect group P. That is, the fusion system $\mathcal{F}$ on P determined by any choice of a maximal B-Brauer pair (P, e) is equal to the trivial fusion system $\mathcal{F} = \mathcal{F}_P(P)$.

Note that we have $\mathrm{Out}(P, \mathcal{F}) = \mathrm{Out}(P)$ in that case. By the structure theory of nilpotent blocks from Puig [25] there is a Morita equivalence between $\mathcal{O}P$ and the block algebra $B = \mathcal{O}Gb$ given by a $(B, \mathcal{O}P)$-bimodule M with vertex ΔP and endopermutation source V. Then the $\mathcal{O}$-dual M^* has vertex ΔP and endopermutation source V^*. The map sending an $(\mathcal{O}P, \mathcal{O}P)$-bimodule N to $M \otimes_{\mathcal{O}P} N \otimes_{\mathcal{O}P} M^*$ induces an isomorphism

$$\mathrm{Pic}(\mathcal{O}P) \cong \mathrm{Pic}(B)$$

Thus the map Φ in Theorem 1.1 applied to the nilpotent block b sends $\mathrm{Pic}(B)$ isomorphically to the subgroup

$$[V] \cdot (\mathrm{Hom}(P, \mathcal{O}^\times) \rtimes \mathrm{Out}(P)) \cdot [V^*]$$

of $D_{\mathcal{O}}(P) \rtimes \mathrm{Out}(P)$. The source algebra A is in this case isomorphic to $\mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathcal{O}P$, and $\mathrm{Out}_P(A)$ is again trivial (because the inertial quotient of B is trivial). It follows that the elements of $\mathrm{Pic}(B)$ correspond under Φ to elements in $D_{\mathcal{O}}(P) \rtimes \mathrm{Out}(P)$ of the form

$$[V] \cdot [\zeta] \cdot [\varphi] \cdot [V^*] = [\zeta] \cdot [V \otimes {}^{\varphi}(V^*)] \cdot [\varphi]$$

where $[V]$, $[\zeta]$ are the classes in $D_{\mathcal{O}}(P)$ of V and of the rank 1 module determined by $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$, respectively, and where $[\varphi]$ is the class of an automorphism φ of P in $\mathrm{Out}(P)$. If V is not stable under φ, then the commutator

$$[V \otimes {}^{\varphi}(V^*)] = [[V], [\varphi]]$$

in $D_{\mathcal{O}}(P) \rtimes \mathrm{Out}(P)$ is not trivial. In that case, the corresponding element of $\mathrm{Pic}(B)$ is given by a bimodule with $\Delta\varphi$ as a vertex and a nontrivial, and possibly nonlinear, endopermutation source, so the class of φ in $\mathrm{Out}(P)$ is not in the image of the map $\mathcal{T}(B) \to \mathrm{Out}(P)$ induced by Φ. This scenario does arise; see the next example.

Example 7.2. Any Morita equivalence between two $\mathcal{O}$-algebras A, B given by an (A, B)-bimodule M and a (B, A)-bimodule N can be interpreted as a self Morita equivalence of the algebra $A \otimes_{\mathcal{O}} B$ given by the bimodules $(M \otimes_{\mathcal{O}} N)_\tau$ and ${}_\tau(N \otimes_{\mathcal{O}} M)$, where $\tau : A \otimes_{\mathcal{O}} B \to B \otimes_{\mathcal{O}} A$ is the isomorphism satisfying $\tau(a \otimes b) = b \otimes a$. Applied to block algebras of finite groups, one can use this to construct self Morita equivalences with endopermutation sources which need not be linear.

Let G, H be finite groups, b, c blocks of $\mathcal{O}G$, $\mathcal{O}H$, respectively, with a common defect group P. Set $B = \mathcal{O}Gb$ and $C = \mathcal{O}Hc$. Then $B \otimes_{\mathcal{O}} C$ is a block of $G \times H$ with defect group $P \times P$.

Let M be a (B, C)-bimodule with vertex ΔP and endopermutation source V inducing a Morita equivalence between B and C. Then $M \otimes_{\mathcal{O}} C$, regarded as a $(B \otimes_{\mathcal{O}} C, C \otimes_{\mathcal{O}} C)$-bimodule, induces a Morita equivalence between $B \otimes_{\mathcal{O}} C$ and $C \otimes_{\mathcal{O}} C$ with vertex $\Delta(P \times P)$ and source $V \otimes_{\mathcal{O}} \mathcal{O}$. Let τ be the automorphism of $C \otimes_{\mathcal{O}} C$ defined by $\tau(c \otimes c') = c' \otimes c$. Then τ restricts to an automorphism of $P \times P$ exchanging the two copies of P, inducing an automorphism of $\Delta(P \times P)$ in the obvious way (and all those restrictions of τ are again denoted by τ).

Thus the $(C \otimes_{\mathcal{O}} C, C \otimes_{\mathcal{O}} C)$-bimodule $T = (C \otimes_{\mathcal{O}} C)_\tau$ induces a self Morita equivalence with vertex

$$\Delta\tau = \{((u, v), (v, u)) \mid u, v \in P\}$$

and trivial source. The $(B \otimes_{\mathcal{O}} C, B \otimes_{\mathcal{O}} C)$-bimodule

$$(M \otimes_{\mathcal{O}} C) \otimes_{C \otimes_{\mathcal{O}} C} T \otimes_{C \otimes_{\mathcal{O}} C} (M^* \otimes_{\mathcal{O}} C)$$

induces a self Morita equivalence of $B \otimes_{\mathcal{O}} C$ with vertex $\Delta\tau$ and the $\Delta\tau$-module $V \otimes_{\mathcal{O}} V^*$ as a source (and this is nontrivial if V is nontrivial, and nonlinear if V is nonlinear).

Applied to a nilpotent block B with a source algebra of the form $\mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathcal{O}P$ for some endopermutation $\mathcal{O}P$-module V of rank strictly bigger than 1, the block $C = \mathcal{O}P$ (which is its own source algebra) and a Morita equivalence between B and C given by a bimodule with vertex ΔP and source V yields a nilpotent block $B \otimes_{\mathcal{O}} \mathcal{O}P$ of $\mathcal{O}(G \times P)$ with defect group $P \times P$ with the property that the class in $\mathrm{Out}(P \times P)$ of the automorphism of $P \times P$ exchanging the two copies of P is not in the image of $\mathcal{T}(B \otimes_{\mathcal{O}} \mathcal{O}P)$ under the map Φ from Theorem 1.1.

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