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BLOCK ALGEBRAS WITH HH^1 A SIMPLE LIE ALGEBRA

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ABSTRACT. The purpose of this note is to add to the evidence that the algebra structure of a p -block of a finite group is closely related to the Lie algebra structure of its first Hochschild cohomology group. We show that if B is a block of a finite group algebra kG over an algebraically closed field k of prime characteristic p such that $HH^1(B)$ is a simple Lie algebra and such that B has a unique isomorphism class of simple modules, then B is nilpotent with an elementary abelian defect group P of order at least 3, and $HH^1(B)$ is in that case isomorphic to the Witt algebra $HH^1(kP)$. In particular, no other simple modular Lie algebras arise as $HH^1(B)$ of a block B with a single isomorphism class of simple modules.

1. INTRODUCTION

Let p be a prime and k an algebraically closed field of characteristic p . For G a finite group, a *block of kG* is an indecomposable direct factor of the group algebra kG . There is an abundance of stable equivalences in block theory, but it is notoriously difficult to pin down even the most basic numerical invariants - such as the number of isomorphism classes of simple modules - through stable equivalences.

The main motivation for the present note is that on the one hand, the Lie algebra structure of $HH^1(B)$ of a block B of a finite group algebra kG is invariant under stable equivalences of Morita type (cf. [4, Theorem 10.7]), and on the other hand, there is evidence for close structural connections between the algebra structure of B and the Lie algebra structure of $HH^1(B)$ (cf. [1]). Understanding those connections might therefore ultimately contribute towards determining block invariants in some cases.

We describe some of the structural connections between B and $HH^1(B)$ in two extreme cases for blocks with a single isomorphism class of simple modules.

Theorem 1.1. *Let G be a finite group and let B be a block algebra of kG having a unique isomorphism class of simple modules. Then $HH^1(B)$ is a simple Lie algebra if and only if B is nilpotent with an elementary abelian defect group P of order at least 3. In that case, we have a Lie algebra isomorphism $HH^1(B) \cong HH^1(kP)$.*

Theorem 1.1 implies in particular that none of the other simple modular Lie algebras occur as $HH^1(B)$ of some block algebra of a finite group with the property that B has a single isomorphism class of simple modules. See [7], [8] for details and further references on the classification of simple Lie algebras in positive characteristic. We do not know whether the hypothesis on B to have a single isomorphism class of simple modules is necessary in Theorem 1.1.

Theorem 1.2. *Let G be a finite group and let B be a block algebra of kG having a nontrivial defect group and a unique isomorphism class of simple modules. Then $\dim_k(HH^1(B)) \geq 2$.*

The hypothesis that B has a single isomorphism class of simple modules is necessary in Theorem 1.2. For instance, if P is cyclic of order $p \geq 3$ and if E is the cyclic automorphism group of order $p-1$ of P , then $HH^1(k(P \rtimes E))$ has dimension one. This follows immediately from the centraliser decomposition of Hochschild cohomology; see [3, Theorem 1.4] for a more general result.

2. QUOTED RESULTS

We collect in this section results needed for the proof of Theorem 1.1.

Theorem 2.1 (Okuyama and Tsushima [5]). *Let G be a finite group and B a block algebra of kG . Then B is a nilpotent block with an abelian defect group if and only if $J(B) = J(Z(B))B$.*

Let A be a finite-dimensional (associative and unital) k -algebra. A derivation on A is a k -linear map $f : A \rightarrow A$ satisfying $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. The set $\text{Der}(A)$ of derivations on A is a Lie subalgebra of $\text{End}_k(A)$, with respect to the Lie bracket $[f, g] = f \circ g - g \circ f$, for any $f, g \in \text{End}_k(A)$. For $c \in A$, the map sending $a \in A$ to the additive commutator $[c, a] = ca - ac$ is a derivation on A ; any derivation arising this way is called an *inner derivation on A* . The set $\text{IDer}(A)$ of inner derivations is a Lie ideal in $\text{Der}(A)$, and we have a canonical identification $HH^1(A) \cong \text{Der}(A)/\text{IDer}(A)$. See [9, Chapter 9] for more details on Hochschild cohomology. If A is commutative, then $HH^1(A) \cong \text{Der}(A)$. A k -algebra A is *symmetric* if A is isomorphic to its k -dual A^* as an A - A -bimodule; this implies that A is finite-dimensional.

Theorem 2.2 ([1, Theorem 3.1]). *Let A be a symmetric k -algebra and let E be a maximal semisimple subalgebra. Let $f : A \rightarrow A$ be an E - E -bimodule homomorphism satisfying $E + J(A)^2 \subseteq \ker(f)$ and $\text{Im}(f) \subseteq \text{soc}(A)$. Then f is a derivation on A in $\text{soc}_{Z(A)}(\text{Der}(A))$, and if $f \neq 0$, then f is an outer derivation of A . In particular, we have*

$$\sum_S \dim_k(\text{Ext}_A^1(S, S)) \leq \dim_k(\text{soc}_{Z(A)}(HH^1(A)))$$

where in the sum S runs over a set of representatives of the isomorphism classes of simple A -modules.

Corollary 2.3 ([1, Corollary 3.2]). *Let A be a local symmetric k -algebra. Let $f : A \rightarrow A$ be a k -linear map satisfying $k \cdot 1 + J(A)^2 \subseteq \ker(f)$ and $\text{Im}(f) \subseteq \text{soc}(A)$. Then f is a derivation on A in $\text{soc}_{Z(A)}(\text{Der}(A))$, and if $f \neq 0$, then f is an outer derivation of A . In particular, we have*

$$\dim_k(J(A)/J(A)^2) \leq \dim_k(\text{soc}_{Z(A)}(HH^1(A))) .$$

Theorem 2.4 (Jacobson [2, Theorem 1]). *Let P be a finite elementary abelian p -group of order at least 3. Then $HH^1(kP)$ is a simple Lie algebra.*

The converse to this theorem holds as well.

Proposition 2.5. *Let P be a finite abelian p -group. If $HH^1(kP)$ is a simple Lie algebra, then P is elementary abelian of order at least 3.*

Proof. Suppose that P is not elementary abelian; that is, its Frattini subgroup $Q = \Phi(P)$ is nontrivial. Since P is abelian, we have $HH^1(kP) = \text{Der}(kP)$. We will show that the set of derivations with image contained in $I(kQ)kP = \ker(kP \rightarrow kP/Q)$ is a nonzero Lie ideal in $\text{Der}(kP)$, where $I(kQ)$ is the augmentation ideal of kQ . Indeed, every element in Q is equal to x^p for some $x \in P$, and hence every element in $I(kQ)$ is a linear combination of elements of the form

$(x-1)^p$, where $x \in P$. Every derivation on kP annihilates all elements of this form (using the fact that k has characteristic p), and hence every derivation on kP preserves $I(kQ)kP$. Thus there is a canonical Lie algebra homomorphism $\text{Der}(kP) \rightarrow \text{Der}(kP/Q)$. This homomorphism is nonzero; indeed, it is an isomorphism on the components of Hochschild cohomology corresponding to $H^1(P; k) \cong H^1(P/Q; k)$ under the centraliser decomposition. The kernel of this canonical Lie algebra homomorphism contains all derivations with image in $\text{soc}(kP)$, so this kernel is nonzero by Corollary 2.3. Thus $HH^1(kP)$ is not simple as a Lie algebra, whence the result. \square

Remark 2.6. Theorem 1.1 implies that the hypothesis on P being abelian is not necessary in the statement of 2.5.

3. AUXILIARY RESULTS

In order to exploit the hypothesis on HH^1 being simple in the statement of Theorem 1.1, we consider Lie algebra homomorphisms into the HH^1 of subalgebras and quotients.

Lemma 3.1. *Let A be a finite-dimensional k -algebra and f a derivation on A . Then f sends $Z(A)$ to $Z(A)$, and the map sending f to the induced derivation on $Z(A)$ induces a Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(Z(A))$.*

Proof. Let $z \in Z(A)$. For any $a \in A$ we have $az = za$, hence $f(az) = f(a)z + af(z) = f(z)a + zf(a) = f(za)$. Comparing the two expressions, using $zf(a) = f(a)z$, yields $af(z) = f(z)a$, and hence $f(z) \in Z(A)$. The result follows. \square

Lemma 3.2. *Let A be a local symmetric k -algebra such that $J(Z(A))A \neq J(A)$. Then the canonical Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(Z(A))$ is not injective.*

Proof. Since $J(Z(A))A < J(A)$, it follows from Nakayama's lemma that $J(Z(A))A + J(A)^2 < J(A)$. Thus there is a nonzero linear endomorphism f of A which vanishes on $J(Z(A))A + J(A)^2$ and on $k \cdot 1_A$, with image contained in $\text{soc}(A)$. In particular, f vanishes on $Z(A) = k \cdot 1_A + J(Z(A))$. By 2.3, the map f is an outer derivation on A . Thus the class of f in $HH^1(A)$ is nonzero, and its image in $HH^1(Z(A))$ is zero, whence the result. \square

Lemma 3.3. *Let A be a local symmetric k -algebra and let f be a derivation on A such that $Z(A) \subseteq \ker(f)$. Then $f(J(A)) \subseteq J(A)$.*

Proof. Since A is local and symmetric, we have $\text{soc}(A) \subseteq Z(A)$, and $J(A)$ is the annihilator of $\text{soc}(A)$. Let $x \in J(A)$ and $y \in \text{soc}(A)$. Then $xy = 0$, hence $0 = f(xy) = f(x)y + xf(y)$. Since $y \in \text{soc}(A) \subseteq Z(A)$, it follows that $f(y) = 0$, hence $f(x)y = 0$. This shows that $f(x)$ annihilates $\text{soc}(A)$, and hence that $f(x) \in J(A)$. \square

Lemma 3.4. *Let A be a finite-dimensional k -algebra and J an ideal in A .*

- (i) *Let f be a derivation on A such that $f(J) \subseteq J$. Then $f(J^n) \subseteq J^n$ for any positive integer n .*
- (ii) *Let f, g be derivations on A and let m, n be positive integers such that $f(J) \subseteq J^m$ and $g(J) \subseteq J^n$. Then $[f, g](J) \subseteq J^{m+n-1}$.*

Proof. In order to prove (i), we argue by induction over n . For $n = 1$ there is nothing to prove. If $n > 1$, then $f(J^n) \subseteq f(J)J^{n-1} + Jf(J^{n-1})$. Both terms are in J^n , the first by the assumptions, and the second by the induction hypothesis $f(J^{n-1}) \subseteq J^{n-1}$. Let $y \in J$. Then $[f, g](y) = f(g(y)) - g(f(y))$. We have $g(y) \in J^n$; that is, $g(y)$ is a sum of products of n elements in J .

Applying f to any such product shows that the image is in J^{m+n-1} . A similar argument applied to $g(f(y))$ implies (ii). \square

Proposition 3.5. *Let A be a finite-dimensional k -algebra. For any positive integer m denote by $\text{Der}_{(m)}(A)$ the k -subspace of derivations f on A satisfying $f(J(A)) \subseteq J(A)^m$.*

- (i) *For any two positive integers m and n we have $[\text{Der}_{(m)}(A), \text{Der}_{(n)}(A)] \subseteq \text{Der}_{(m+n-1)}(A)$.*
- (ii) *The space $\text{Der}_{(1)}(A)$ is a Lie subalgebra of $\text{Der}(A)$.*
- (iii) *For any positive integer m , the space $\text{Der}_{(m)}(A)$ is an ideal in $\text{Der}_{(1)}(A)$.*
- (iv) *Suppose that A is local. The space $\text{Der}_{(2)}(A)$ is a nilpotent Lie subalgebra of $\text{Der}(A)$.*

Proof. Statement (i) follows from 3.4 (ii). The statements (ii) and (iii) are immediate consequences of (i). Since A is local and since 1 is annihilated by any derivation on A , statement (iii) follows from (i) and the fact that $J(A)$ is nilpotent. \square

4. PROOF OF THEOREMS 1.1 AND 1.2

Let G be a finite group and B a block of kG . Suppose that B has a single isomorphism class of simple modules. If B is nilpotent and P a defect group of B , then by [6], B is Morita equivalent to kP , and hence there is a Lie algebra isomorphism $HH^1(B) \cong HH^1(kP)$. Thus if B is nilpotent with an elementary abelian defect group P of order at least 3, then $HH^1(B)$ is a simple Lie algebra by 2.4.

Suppose conversely that $HH^1(B)$ is a simple Lie algebra. If $J(B) = J(Z(B))B$, then B is nilpotent with an abelian defect group P by 2.1. As before, we have $HH^1(B) \cong HH^1(kP)$, and hence 2.5 implies that P is elementary abelian of order at least 3.

Suppose that $J(Z(B))B \neq J(B)$. Let A be a basic algebra of B . Then $J(Z(A))A \neq J(A)$. Moreover, A is local symmetric, since B has a single isomorphism class of simple modules. Thus $\text{soc}(A)$ is the unique minimal ideal of A . We have $J(A)^2 \neq \{0\}$. Indeed, if $J(A)^2 = \{0\}$, then $\text{soc}(A)$ contains $J(A)$, and hence $J(A)$ has dimension 1, implying that A has dimension 2. In that case B is a block with defect group of order 2. But then $HH^1(A) \cong HH^1(kC_2)$ is not simple, a contradiction. Thus $J(A)^2 \neq \{0\}$, and hence $\text{soc}(A) \subseteq J(A)^2$. By 3.2, the canonical Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(Z(A))$ is not injective. Since $HH^1(A)$ is a simple Lie algebra, it follows that this homomorphism is zero. In other words, every derivation on A has $Z(A)$ in its kernel. It follows from 3.3 that every derivation on A sends $J(A)$ to $J(A)$. Thus, by 3.4, every derivation on A sends $J(A)^2$ to $J(A)^2$. This implies that the canonical surjection $A \rightarrow A/J(A)^2$ induces a Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(A/J(A)^2)$. Note that the algebra $A/J(A)^2$ is commutative, and hence $HH^1(A/J(A)^2) = \text{Der}(A/J(A)^2)$. Since $J(A)^2$ contains $\text{soc}(A)$, it follows that the kernel of the canonical map $HH^1(A) \rightarrow HH^1(A/J(A)^2)$ contains the classes of all derivations with image in $\text{soc}(A)$. Since there are outer derivations with this property (cf. 2.3), it follows from the simplicity of $HH^1(A)$ that the canonical map $HH^1(A) \rightarrow HH^1(A/J(A)^2) = \text{Der}(A/J(A)^2)$ is zero. Thus every derivation on A has image in $J(A)^2$. But then 3.5 implies that $\text{Der}(A) = \text{Der}_{(2)}(A)$ is a nilpotent Lie algebra. Thus $HH^1(A)$ is nilpotent, contradicting the simplicity of $HH^1(A)$. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Denote by A a basic algebra of B . Since B has a unique isomorphism class of simple modules and a nontrivial defect group, it follows that A is a local symmetric algebra of dimension at least 2. By 2.3 we have $\dim_k(HH^1(A)) \geq \dim_k(J(A)/J(A)^2)$. Thus

$\dim_k(HH^1(A)) \geq 1$. Moreover, if $\dim_k(HH^1(A)) = 1$, then $\dim_k(J(A)/J(A)^2) = 1$, and hence A is a uniserial algebra. In that case B is a block with a cyclic defect group P and a unique isomorphism class of simple modules, and hence B is a nilpotent block. Thus $A \cong kP$. We have $\dim_k(HH^1(kP)) = |P|$, a contradiction. The result follows. \square

Remark 4.1. All finite-dimensional algebras in this paper are split thanks to the assumption that k is algebraically closed. It is not hard to see that one could replace this by an assumption requiring k to be a splitting field for the relevant algebras. The statements 3.1 and 3.4 do not require any hypothesis on k .

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