LANDMARKS IN THE HISTORY OF ACTUARIAL SCIENCE (UP TO 1919)

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PREFACE

The objective of this essay is to provide a guide to the history of actuarial science until the early part of the twentieth century by identifying and commenting on over 100 landmark works. A difficulty immediately arises in defining the boundaries of a subject, especially when that subject is in its embryonic state. Given the constraints of time and space, we have decided to exclude items that specifically and exclusively deal with contiguous subject areas like demography and compound interest. However, the reader will find some comments on topics other than actuarial science. The range of material available, even in a specialist area like actuarial science, is very wide but this collection aims at being representative and capturing the major contributions to the subject, the major issues, developments and debates.

In deciding on which contributions to include in the listing of landmark works we have attempted to apply the following criteria consistently and have sought works that satisfy at least one of these criteria. Firstly, a work should be intellectually important. Secondly, it should have changed an aspect of theory or practice significantly. Thirdly, a work should be new and should have made a significant impact on theory or practice. We have also attempted to ensure that the items chosen are illustrative of the broad sweep of actuarial theory and practice.

In choosing a cut off date for this project, we have been mindful of the need to stand back from the development of actuarial science. A date too close to the modern day would make the judgement of significance difficult not least because of the short span of time that would have elapsed. A date at the start of the century allows us to see the direct descendants of much of what today is accepted and commonplace.

Another problem has been the allocation of works to themes when many contributions clearly straddle a number of different aspects of actuarial science. To avoid repetition we have allocated a work to the theme that seems to be most appropriate and in the text we have attempted to draw attention to the scope of each work.

In a project of this kind there are bound to be differences of view and judgement on what should be included. Each expert will doubtless find fault when it comes to his/her own area, but it is hoped that overall we have captured the major contributions.

There is a preponderance of works that appeared originally in English or were translated into English: in making this selection we have striven against the possibility of bias that might arise from our familiarity with English ideas and methodologies. The reader will be able to judge how successful we have been in attaining this particular objective.

I have been fortunate in being able to work alongside Trevor Sibbett who is a respected "amateur" historian of matters actuarial and I should like to acknowledge his assistance, guidance and forbearance. However, any errors remaining in this essay are entirely my responsibility.

Trevor and I should like also to acknowledge the sterling support and considerable assistance we received from the staff of the Institute of Actuaries' Library, in particular Sally Grover and Roy Park.

The essay is a modified version of the introdutory essay included in Haberman and Sibbett (1995) which includes reproductions of the landmark works, and complete translations into English, as well as accompanying text notes. It is hoped that this will prove to be a valuable primary source and a reference collection for research workers in the future which will provide easy access to the most important contributions to this particular body of literature.

LANDMARKS IN THE HISTORY OF ACTUARIAL SCIENCE

INTRODUCTION

The origins of actuarial science lie in the seventeenth century. During this period, commercial needs gave rise to transactions involving compound interest, marine insurance was commonplace and the algebra of life annuities came into existence. Indeed, valuation techniques for financial transactions were known even in ancient Roman times and became of increasing importance as world trade developed after 1500.

During the 16th Century, some of the European writers on arithmetic such as Simon Stevin and Jan Trenchant devoted some space to elementary problems in compound interest. There were no actuaries in the formal sense at that time. The nearest equivalent was the mathematical practitioner, a consultant who would tackle all kinds of problems on request from commercial matters to navigation. Richard Witt was one such; he practised in London in the early 1600's and wrote the first comprehensive book in English on compound interest. But Witt's book did not venture into life insurance mathematics and it was not until later in the 17th Century that the necessary tools became available. One such was the developing science of probability; another was the concept of the life table based on mortality investigations, the first published example being that of John Graunt in 1662.

Thus, by 1670 two of the main foundations of actuarial science were firmly in place: compound interest, probability theory and the life table. These tools were employed almost immediately by the Dutch prime minister, Jan de Witt, to investigate the value of Government life annuities. But his treatise, although of considerable merit, remained unnoticed for many years and did not influence the development of actuarial science. Then, in 1693 Edmund Halley, the British mathematician, published his famous Royal Society paper, describing the construction of the life table from observations. He also set down the method for valuing life annuities, which is, in essence, the same as that used today.

Actuarial science had thus arguably been born, and by 1700 the way was open for the techniques to be applied for commercial purposes. At this time, life annuities were bought and sold freely, although in the main their values were not properly calculated. Life assurance consisted mainly of short term risk-only contracts purchased by single premium from an individual underwriter, the premiums being assessed on the basis of the underwriter's experience and judgement, but with no formal scientific basis. Early in the 18th Century the Amicable Society established life assurance on a basis which involved some build-up of funds or reserves.

Some of the pioneers in the 1700s and 1800s were eminent scientists and mathematicians who became interested in actuarial problems. Thus, we find Leonhard Euler, James and Daniel Bernoulli, Carl Friedrich Gauss, Abraham de Moivre, Benjamin Gompertz becoming involved in the science and making significant contributions. In the UK, many papers on actuarial matters were read before the Royal Society, there being no other formal theatre for discussion.

The 1800s saw the establishment of many life assurance companies in the UK and elsewhere, operating on a scientific basis following the pioneering work of James Dodson and Richard Price. A wide range of level annual premium contracts was available to the public. Annuities were calculated by proper methods and the dangers of inappropriate mortality tables became known. Life insurance mathematics, compound interest and probability theory were then at an advanced state of development, and mortality tables had developed from crude tabulations of deaths to properly calculated and considered works. The first table of life assurance mortality from pooled data was published in 1843, as also was English Life Table No. 1, for the national population. The collection and analysis of sickness statistics were under way, but the level of attainment was still fairly crude.

By the mid nineteenth century, a considerable amount of fundamental work had been completed. There then followed a rapid expansion in the refinement and practical application of actuarial theory. Life insurance business was expanding rapidly and many actuaries were employed in the industry on a full-time basis. The Institute of Actuaries was established in 1848 and the Faculty of Actuaries in 1856, providing the first dedicated actuarial publication media, respectively the Journal and the Transactions. The first text books began to appear for assisting students to pass professional examinations. A steady stream of significant papers on actuarial theory and practice was published in these journals, together with translations of significant contributions from continental Europe.

A statutory system of actuarial supervision of life insurance began with Massachusetts legislative session of 1858 and the Life Assurance Companies Act of 1870 in the UK. This latter piece of legislation was the first detailed statutory framework regulating the life assurance business, Parliament having chosen the "freedom with publicity" principle; this led to debates on the relative merits of the bonus reserve and the net premium valuation methods. The 1860s had earlier seen the invention of the contribution method of the distribution of surplus, almost universally used in life assurance practice in North America. In the US, the Actuarial Society of America was founded in 1880, the forerunner of the Society of Actuaries and the Casualty Actuarial Society in 1914.

Among the significant contributions of the period up to the early part of our current century is the appearance of Makeham's formula for the force of mortality which was found able to represent faithfully much published mortality data for the next 100 years. We also find the development of different approaches to graduation; the publication of premium conversion tables which eased the burden of calculating life insurance premiums in the days before calculating machines became generally available; the development of analysis of surplus, whereby the causes of the surplus or deficiency emerging in any type of valuation can be revealed and further analysed; the development of risk and credibility theory; the development of the mathematical theory of multiple decrement and multiple state models; the systematic presentation of pension fund valuation ideas and formulae; the formulation of the principles of institutional investment; the agreement on a standard actuarial notation which provided a language to facilitate international dialogue. We also note the important creation of the International Actuarial Association in 1895 which over the last 100 years has fostered and encouraged the development of actuarial ideas and the actuarial profession around the world, and has provided a forum for international meetings and the cross-fertilisation of ideas.

To simplify the presentation, the discussion is now split up according to the following themes:

1. Life Tables and Survival Model
2. Life Insurance Mathematics
3. Life Insurance
4. Pensions
5. Investment

6. Risk Theory and Non-Life Insurance
7. Multiple Decrement and Multiple State Models
8. Health and Sickness Insurance
9. Experience Studies and Estimation of Rates
10. Graduations of Decremental Rates

and these are now considered in turn. The landmark works themselves are listed in the Appendix, categorised by theme and chronology.

Life Tables and Survival Model

The first noteworthy writings on the survival model and what came to be called the life table is Ulpian's Table, dating from around 220. The calculations were intended for the valuation of annuities as relating to legacies. The interpretation of the numbers is unclear and it is possible that Ulpianus may have derived the figures from actual observations rather than guesswork alone, although Greenwood (1940) presents strong doubts with the conviction that Ulpianus merely interpolated between legal maximum and minimum estimates for the expectation of life. Still the Table was authorised for the valuation of life annuities by the Government of Tuscany in 1814. Mays (1971) provides some further analysis of this work.

The period before the nineteenth century was characterised by high levels of mortality in Europe. One of the main contributing factors was the incidence of epidemic diseases, among which the plague was the worst. After the Black Death in 1348-1350, the plague recurred frequently for nearly four centuries. As Hald describes, an early "warning system" was set up in London in the 1530's by requesting the parish clerks to submit weekly reports on the number of plague deaths and all other deaths in case of an incipient outbreak. These weekly "bills" of mortality served to warn the authorities when measures should be taken against the epidemic and to warn the wealthy echelons of society when they should escape to the country. Starting in 1604, weekly bills of mortality for the parishes of London were published by the Company of Parish Clerks, and a bill for the whole year was published at the end of each year, in printed form from 1625. The ages of the deceased were not recorded before 1728. The weekly bills were published regularly until 1842 when they were superseded by publications from the Registrar General.

As Hald (1990) notes, "this large amount of data had not been analysed statistically before John Graunt published his book in 1662", a remarkable work which "is widely regarded as a landmark in the descriptive and statistical analysis of demographic data". Graunt's critical appraisal of the rather unreliable data that had been collected, his development of concepts and techniques relevant for the analysis of that information, his consideration of errors and ambiguities in the data, his study of mortality by cause of death, his estimation of the same quantity by several different methods, his demonstration of the stability of statistical ratios and his creation of the life table all set new standards for reasoning. Graunt's work led to further investigations along three different avenues: "political arithmetic" or demography; testing the stability of ratios; the calculation of survivorship probabilities and expectations of life: Hald (1990). We are concerned with the third of these avenues. However, it is worth noting in passing that Graunt's work on political arithmetic was immediately taken up by William Petty who first coined this particular phrase (see Hull (1899)).

As some commentators have mentioned, there are some errors in Graunt's work on the life table (see Westergaard (1932) and Glass (1950, 1964)). For example, he did not fully appreciate the relationship between the life table and the age distribution and size of the corresponding stationary population and indeed confuses these concepts. We should, however, recognise the pioneering nature of this work, which had immense influence. Thus, bills of mortality were introduced in other cities (for example, Paris in 1667). His methods of statistical and demographic analysis were adopted in England, France, the Netherlands and Germany and led ultimately to the creation of government statistical offices. Some would argue that, although Graunt was the first to describe the dying out of a population cohort, he did not compile the first life table in the modern actuarial sense (Sutherland (1963)). This concept was, nevertheless, picked up by the Huygens brothers, improved upon by de Witt and Halley and subsequently became a key tool in medical statistics and demography as well as actuarial science.

There was speculation for some time that Graunt's book was, in fact, written by Petty but this argument has been effectively disposed of by Glass (1964) and Sutherland (1963), inter alia.

As is clear from the discussion, there exists a large literature about Graunt's work. Most recently, Kreager (1988) casts new light on Graunt's book and connects his techniques with common bookkeeping.

The first authors to have used Graunt's life table were Christiaan and Lodewijk Huygens who corresponded in 1669 on a probabilistic interpretation of the life table. They considered the calculation of the average age at death, the corresponding expectation of life and the median remaining lifetime, carefully noting the distinction between these concepts. They also considered joint-life expectations, including the expectation of the longest and shortest lifetime of a group. The details are discussed further by David (1962), Hald (1987, 1990), Schevichaven et al (1898) and Seal (1980).

Then followed applications of the life table to the problems of valuing annuities. In the sixteenth and seventeenth centuries, states and cities often raised money for public purposes by selling life annuities to their citizens. In 1671, Jan de Witt, the distinguished and long-serving Prime Minister, submitted a report to the States of Holland showing how to calculate the value of annuities by means of a piecewise linear life table combined with the age of the nominee and the rate of interest. De Witt's life table was hypothetical, although his report refers to some investigations of the mortality of annuitants. De Witt's approach to annuity calculations was through the distribution of the number of deaths in the underlying life table, that is via the formula in modern notation

$$a_x = \sum_{t=1}^{\infty} a_{\overline{t|}} \frac{d_{x+t}}{l_x}$$

although Hald (1990) believes that de Witt was familiar with the alternative approach of Halley. De Witt also demonstrated some appreciation of the effects of self selection and changes in mortality levels on the value of an annuity. De Witt's report was forgotten until Hendriks (1852) rediscovered it and provided an English translation and commentary.

Edmund Halley's paper of 1693 was seminal and of great importance to actuarial science. He constructed a life table from observations of the yearly number of deaths in Breslau (where the parish registers were among the first to contain age at death) and used this in seven different ways (Hald (1990)), including considering the "proportion of men able to bear arms" (as did Graunt) and the median remaining lifetime (as did Huygens). He used the odds ${}_tp_x/{}_tq_x$ as a measure of the "differing degrees of ... vitality in all ages" and he commented on the relationship between the price of a term assurance and the odds function. He then calculated the first table of values of annuities as a function of the nominee's age and developed formulae for calculating the value of joint life annuities (for two and three lives; with geometrical diagrams by way of explanation) and emphasised the benefit of using logarithms to reduce the volume of calculation. His approach to calculating the present value of annuities was through the distribution of the number of survivors, that is via the formula in modern notation

$$a_x = \sum_{t=1}^{\infty} v^t \, \frac{l_{x+t}}{l_x}$$

- this is algebraically equivalent to de Witt's method although computationally more straightforward. However, we should note that, in a modern perspective, these calculations of present values actually relate to expected present values (of an annuity with a random term). De Witt's method can be easily adapted to considering higher moments, for example, the variance, whereas Halley's method cannot be progressed in this manner. Thus, de Witt's approach is also of lasting theoretical (and practical) importance.

Halley remarked that the government was selling annuities too cheaply and at a price independent of the age of the annuitant: his advice was ignored! As many commentators have noted, the life table function tabulated by Halley was what we would call L_{x-1} rather than l_x. It is noteworthy that the Breslau life table was reproduced in the updated version of 1737 of the abstract of the Amicable Society's charter and by-laws.

The first published work on life expectancy is due to Nicholas Bernoulli (1709), given that the correspondence between the Huygens brothers had not been published. In this work, Graunt's life table is used to provide illustrations of the calculation of the expectation of life and the median remaining lifetime. Bernoulli noted that expectations may be obtained by backward recursion and illustrates this process. There is no reference to the life tables of de Witt or of Halley, or to the methods described by Halley. An interesting problem solved by Bernoulli is the derivation of a formula for the expected lifetime of the last survivor of b lives dying within an interval of length a (under a uniform distribution of deaths assumption). In Chapter 6, Bernoulli considers and solves some simple problems from marine and life insurance by calculation of expectations.

The first graphical representation of a life table function is attributed to Isaac de Graaf (1729): the graph seems to represent the number alive in a life table, l_x, calculated on some hypothetical basis.

John Smart's life table of 1738 was the first specific to London and was based on the numbers of deaths occurring in London in the period 1728-37. At his request in 1726, the bills of mortality had included the numbers dying at each age from 1728 onwards on a weekly basis.

The earliest life tables developed for males and females are due to Nicholas Struyck (1740), based on registers of annuitants (as were, for example, those of de Witt and Deparcieux). He also noted, in passing, the effects of self selection and calculated annuity values for the two sexes and compared his results with those of Halley. Like Halley, he pointed out that the government was selling annuities too cheaply. He also gave examples of increasing, decreasing and deferred annuities. Struyck was also a writer on probability and demography.

Among the other great investigations of annuitant mortality are those of William Kersseboom (1742) and Antoine Deparcieux who produced the first French life table in 1746, based on the data from the operation of two tontines. He constructed the life table directly from observed deaths - methodological errors were reduced because the tontine populations were not generally subject to loss of observations due to withdrawal or migration. This life table was used extensively by French life insurance companies for many years well into the next century. Thus, at about the same time, Struyck, Kersseboom and Deparcieux appreciated the significant of constructing a life table from the registers of annuitants.

The concept of the annual rate of mortality was introduced by Thomas Watkins in 1761 (writing as TW). Watkins commented critically on Halley's work, noting problems with extrapolation of the life table to the oldest ages and with the use of a small radix. He pointed out that Halley's and Brackenridge's life tables appeared smooth from the progression of d_x values but not when the values of the annual rate of mortality were examined. Watkins drew attention to the need for smoothing (i.e. graduating) crude data and advocated considering the annual rate of mortality and the probable expectation of life and their first differences in the context of the life table. The annual rate of mortality was not widely adopted until the next century, with the work of Emmanuel Duvillard, John Finlaison inter alia. It is noteworthy that Johann Lambert (in volume III of his 1772 book) followed Halley's use of the odds function and used the reciprocal of q_x as a measure of *"vitality"* (Daw (1980)).

An interesting, although virtually unknown, article of 1760 from the great mathematician Leonhard Euler adapted the life table to solve a number of problems requiring inference from incomplete data. The work anticipated "important parts of modern stable population theory for a one-sex population closed to migration. Its ideas have been published many times during the subsequent two centuries by writers who independently rediscovered them" (Nathan Keyfitz writing in Smith and Keyfitz (1977)). Euler's aim was to use the life table model to study real populations through the progression of cohorts, allowing for population increase or decrease. He thereby considered the age structure of a stationary population and a stable population (in the demographic sense - see Keyfitz (1985)), thus generalising the usual life table formulae.

In 1749, the Swedish General Register Office was established and the first national set of population statistics started to be collected from that date. Per Wargentin in his paper of 1766 presented an analysis of mortality and population data for Sweden using these statistics. Wargentin combined the death registration data for 1755-1763 with the triennial *"censuses"* of 1757, 1760 and 1763 to calculate the inverse death rate for the two sexes for quinquennial age groups (except at the youngest ages). He noted the lower levels of mortality for females and commented on the likely errors in the data (in terms of individuals not counted and ages mis-stated). He criticised Halley's approach on the grounds that the underlying population would need to be stationary, noting that the population of Sweden was not stationary.

Wargentin's groundwork was taken up by Richard Price in his construction of a life table for Sweden using an arbitrary radix and a smoothed set of inverse mortality rates. Price followed Wargentin's methodology and used an extended version of his data, i.e. death registration data for 1755-1776 with the seven triennial *"censuses"* from 1757 to 1775. The construction of this life table represents an important breakthrough. It appears in the fourth edition (1783) of

Price's masterpiece "Observations on Reversionary Payments". Price prepared life tables for Sweden, for Stockholm on its own, for males and females separately. He also constructed a persons life table using the incorrect approach of taking a simple average of the constituent male and female life tables. This book also contains the second Northampton life table based on the experience from 1735-1780 (see later).

The first life table for the United States was due to Edward Wigglesworth (1789), based on deaths in the states of Massachusetts and New Hampshire. His methodology used observed deaths only, ignoring the developments that had been made by Wargentin and Price.

Joshua Milne's textbook of 1815 dealt with a number of aspects of life insurance mathematics. It also considered the construction of life tables. In particular, Milne described the construction of the Carlisle life table based on the 1779-1787 experience in two parishes of that town, following the methods that Price used for his Swedish life table. The data were sparse, (for example, only 406 deaths at ages 20-59) and the attempts at smoothing the grouped numbers of deaths and population counts were unsatisfactory. The life table was little heeded for some years after its publication but was later adopted enthusiastically by actuaries and became a standard table. Further discussion on the Carlisle table can be found in King (1884).

John Finlaison was the first actuary to be described as Government Actuary although the government had received advice prior to this, for example, from William Morgan. (Finlaison was also the first President of the Institute of Actuaries in 1848). Finlaison was Actuary of the National Debt Office from 1822-1851 and carried out a number of important duties. His most important contribution to actuarial science was his work on the life tables for government annuities. William Morgan had been consulted when the National Debt office began the sale of annuities in 1800. Morgan made the error of adopting a life table with suitably prudent margins for life insurance premiums (Richard Price's Northampton Table): this meant that the government was significantly undercharging for its annuities for about 20 years. (We recall that earlier de Witt and Halley had faced similar problems). In 1819, Finlaison pointed out the error. He was then commissioned by the Chancellor of the Exchequer to carry out an investigation into annuitants' mortality and to produce a new set of annuity tables. He carried out a major mortality study, which included the records of various tontines from 1695 to 1789 and which was published in 1829. In this work, Finlaison criticised earlier writers who had been dependent on others for the data used, for example, Deparcieux and Kersseboom, and he recognised the importance of treating the sexes separately. He took painstaking care in the collection and preliminary tabulation of the data to eliminate various types of errors. In particular, with tontines and government annuities, it was common for a subscriber to possess many shares via nominees. Finlaison was careful to avoid the duplication caused by counting the same person more than once. He thus produced a complete set of tables for single lives (by 1823) and for joint lives, which formed the basis for the pricing of government life annuities until 1884.

Benjamin Gompertz's paper of 1825 marked the "beginning of a new era" for actuarial science (Hooker (1965)). The well-known "law" of mortality that he proposed was an enormous improvement on previous attempts to represent life table functions by a mathematical formula and it thereby opened up a new approach to the life table. Hitherto, the table had been regarded as a record of observed numbers surviving from an initial cohort - Gompertz now introduced the idea that l_x was a continuous, mathematical function, connected by (what we would call) the underlying force of mortality. At this stage, the force of mortality had not been identified: Gompertz worked in terms of l_x and its fluxion (out of respect for Newton, he persisted in using the language of fluxions rather than differentials throughout his life). His objective in doing the background research was to find a general form to facilitate interpolation. He analysed actual experience before proposing his hypothesis and he demonstrated how his formula could be applied with a good deal of accuracy to the Carlisle and Northampton life tables and to Deparcieux's observations. He explained that his "law" could be interpreted in terms of the average exhaustion's of an individual's resistance to death. As Hooker (1965) points out, Gompertz does mention "two generally co-existing causes" of death; "the one, chance, without previous deposition to death or deterioration; the other, a deterioration, or an increased inability to withstand destruction". It now seems strange that his notion of two causes of death did not lead Gompertz to Makeham's later modification: $\mu_x = A+Bc^x$. However, Gompertz's presentation of ideas and train of thought were not completely clear (as noted by Makeham (1890)) and indeed his paper did not receive the wide recognition it merited, although it was subsequently championed by many eminent thinkers of the time, including de Morgan, Herschel, Sprague and Woolhouse. (The paper's reception may also be explained by Gompertz's use of the obsolete ideas of fluxions and the number of errata).

The curve of deaths, or a graphical presentation of the d_x function was introduced in 1826 by Thomas Young, the eminent physicist. He considered a number of well-known life tables and averaged the numbers of deaths and then introduced an obscure combination of polynomials of higher order (involving for example x^{40}) for different parts of the age range to represent this "average" curve of deaths.

Johann Lambert (volume III of his 1772 work) discussed the force of vitality, recognisable as the reciprocal of the force of mortality (Daw (1980)). It was TR Edmonds who introduced in 1832 the term "force of mortality" and showed its algebraic form. The regular use of the force of mortality (or hazard rate) in actuarial mathematics and statistics (for example, survival analysis) dates from this book. Edmonds used three Gompertz type curves to represent the force of mortality over different age ranges: up to age 9, from 9 to 55, and 55 and over.

Francis Corbaux was the first to argue that population life tables are constructed from an aggregate of different life tables for lives of separate subgroups (1833). He identified a number of factors (or co-variables) including sex and occupation that should be allowed for to avoid class selection. This work covered a number of topics. Thus, Corbaux recommended graduation of third or fourth differences of log q_x in order "that a rectification of any irregularities, incident to the data supplied by experience may thus be arrived at". He also discussed increasing, stationary and decreasing populations, expectation of life, initial and class selection of assured lives, fertility rates specific for age, and other demographic issues.

The idea of variability of life table calculations was first mentioned by Augustus de Morgan in his 1838 essay which, inter alia, surveyed life insurance mathematics. He derived an expression for the "probable error" of the expectation of life at high ages.

Gompertz's Law was given a more general structure by William Makeham in 1860, in what became known as Makeham's Law. However, Makeham regarded his own contribution only as a modification to Gompertz's Law. (Indeed, his final published paper of 1890 was entitled "Further Improvements of Gompertz's Law"). Though, as we have seen, Gompertz had considered causes of death as being of two kinds, chance and deterioration, he did not finally link them to mathematical expressions in his work. Makeham proposed that causes could be approximately divided between what would be roughly independent of age and what would be increasing with age. We can write his modification in the well-known form: $\mu_x = A + Bc^x$. He then demonstrated how convenient this assumption was in the calculation of joint life annuity values, where a version of de Morgan's Law of Uniform Seniority applies. At the time it was proposed, Makeham's Law appeared to fit existing data well, for example, the Seventeen Offices' Experience quoted in his paper, and numerous life tables have been graduated on the basis of Makeham's Law: the most recent major table was probably the CSO 1941 (US) table, which follows this "law" from ages 15 to 95. It is also noteworthy that Makeham presented this paper before he had passed any of the examinations of the Institute of Actuaries. In later work he proposed further modifications to Gompertz's Law involving polynomial terms in attempts to represent assured lives' experience: see Makeham (1890).

The fact that the Gompertz and Makeham laws could not be expected to represent the mortality experience throughout life led to the investigation of formulae which might be expected to do so. So, Gompertz (1860 paper reprinted in 1872) himself suggested a formula based on an amalgamation of several of his curves with different constants. The Danish mathematician, Thorvald Thiele, proposed a combination of three terms for interpolation purposes: a decreasing Gompertz curve to represent the mortality of infancy, a normal curve of error to represent mortality at young adult ages and an increasing Gompertz curve to represent old-age mortality viz $\mu_x = a_1 \exp(-b_1 x) + a_2 \exp\left[-\frac{1}{2}\left(\frac{x-c}{b_2}\right)^2\right] + a_3 \exp(b_3 x)$.

A translated version of this work appears in 1872 in the Journal of the Institute of Actuaries. Thiele's senior colleague, Ludvig Oppermann had proposed a law in 1870 to represent the force of mortality up to its "first point of inflection (or to the age of about 20)":
$\mu_x = ax^{-\frac{1}{2}} + b + cx^{\frac{1}{3}}$. Oppermann was the first to consider that the rate of decline of the force of mortality in infancy may not be exponential with respect to age, and that a transformation of age (say the square root) might be helpful. Thiele commented that "Oppermann has gained for himself lasting credit by this formula".

A different perspective on the life table and survival model was offered by two contributions from the eminent statistician Wilhelm Lexis. His first contribution (1875) was to devise the well-known demographic diagram that displays the population by age and time: each individual at any moment is represented by a point; the collection of points for any single individual represents his life-line through time; the end of the line represents the moment and age of his/her death. His second contribution (1877) was to represent the empirical distribution of deaths by age by a normal curve, noting an observed surplus of early deaths, which, after excluding deaths at childhood ages, he classed as "premature deaths". Those deaths that are represented by the normal curve, he described as "normal deaths". This approach was rediscovered by Clarke (1950) in his separation of deaths into "anticipated" and "senescent", the ages at death for the latter being measures of the natural lifespan.

These particular contributions from Makeham, Thiele, Oppermann and Lexis were not progressed further within the time frame of this study. It was not until the suggestions by Perks (1932) that the logistic family of curves be used and then by Heligman and Pollard

(1980) that a combination of double-exponential and lognormal curves be used to represent the odds q_x/p_x that progress towards a parametric mortality curve for the full age range was achieved.

Life Insurance Mathematics

The first textbook on life insurance mathematics was Abraham de Moivre's "Annuities on Lives" published in 1725. As Hald (1990) explains, at this time, "economic contracts that depended on the lifetimes of the parties involved were important parts of everyday life" in Europe, particularly in the UK. "Besides life annuities sold by the government, there were pensions granted by the government, the Church, municipalities, parishes and so on; life interests and reversions specified by wills and marriage settlements"; and complex contracts involving property. Such contracts were difficult to evaluate and the need for a more thorough mathematical analysis of these problems than that provided by Halley was clear. This is the "challenge taken up by de Moivre" (Hald (1990)).

As Hald notes, de Moivre had a "genius for developing mathematical approximations". He suggested approximating Halley's life table for Breslau by a piecewise linear function and proved that the value of an annuity under this hypothesis would be a linear function of an annuity-certain. So it was not necessary to tabulate the value of single life annuities since these could be derived directly from existing tables of annuities-certain. This avoided the difficult computation of the many products and sums that had bothered Halley. Although this hypothesis is sometimes quoted as de Moivre's Law, he realised that it was defective as a representation of human mortality over all ages. But the point was that de Moivre considered the assumption to be adequate for the purpose intended, that is the evaluation of annuities within the range of ages then commonly required in practice.

De Moivre also gave a simple method for tabulating annuity values, a_x, by means of the backward recursion formula

$$a_x = vp_x(1+a_{x+1})$$

which is a generalization of Nicholas Bernoulli's formula for the calculation of the expectation of life (Hald (1990)). It seems clear that de Moivre appreciated that this formula had general application in the calculation of complete annuity tables; however, Young (1908) puts forward the claims of Euler as the first writer to appreciate the significance of this formula. Similarly, de Moivre considered the value of a temporary annuity using both his approximate method and a backward recursion method.

Further, de Moivre showed how the value of a joint-life annuity (for a group of independent lives) could be expressed approximately by means of the values of the corresponding single-life annuities so that joint-life annuities could be easily evaluated and manipulated. He began this investigation with his linear hypothesis but when the results for three lives became unwieldy he switched to a different hypothesis, viz assuming that the lives have geometrically decreasing probabilities of survival i.e. ${}_tp_x = p^t$ which corresponds to assuming a constant force of mortality. He applied the same approach to joint life assurances (on three lives). However, he did not present any systematic investigation of the error involved in using this approximation.

Importantly, he also gave a systematic exposition of formulae for the value of last survivor annuities for any number of lives, reversionary annuities and annuities on successive lives which were used in leasehold property contracts (where, for example, on the death of the annuitant a successor enjoys the annuity for the duration of his/her subsequent lifetime). He also considered joint life survival and contingent probabilities under his linear mortality hypothesis. He discussed single life and joint life expectations of life under his linear mortality hypothesis, using the trapezoidal rule to approximate the integrals. He also considered the general formula for the expectation of life for the longest of two lives. Here, he was obviously unaware of the earlier work of the Huygens brothers. De Moivre also considered the present value of contingent (or survivorship) assurances but his approach was in error; this was identified and corrected subsequently by Simpson.

Overall, the 1725 book laid the foundations of modern life insurance mathematics and can be regarded as one of the major landmarks in the development of the subject.

The book appeared in four separate editions. De Moivre also wrote the important text on probability "The Doctrine of Chances", which appeared in three editions: 1718, 1738 and a posthumous one in 1756. There is some overlap with material from "Annuities on Lives" appearing in the 1738 edition of the Doctrine with the 1756 edition effectively containing much of his work on life insurance mathematics.

The 1738 edition of the Doctrine contained de Moivre's first full full table of a_x values and some new material on successive lives (correcting an earlier error - we note that de Moivre's derivation of the formulae is rather artificial and is improved upon by the more direct proof given later by Simpson in his 1742 book) and on the value of annuities for children. Importantly, de Moivre included formulae for reversionary annuities involving up to four lives with complex last survivor statuses. He derived the formula for the present value of (or single premium for) a temporary life insurance benefit and calculated the value for a particular age and term using Halley's life table. This important result was ignored by the market which persisted with single premiums for one year policies that were not age dependent until 1762.

In a subsequent letter to William Jones, de Moivre gave an approximation to the value of a complete life annuity and provided an improved explanation (compared to the 1725 version) of the use of his piecewise linear life table hypothesis for calculating the value of annuities.

Following de Moivre, several books and tables on annuities were published in the decades after 1725, the most significant being Richard Hayes' book and the major contributions of Thomas Simpson.

Richard Hayes published the first book (1727) devoted solely to tables of annuities on lives, although he did not give any details of his methods of computation. The book offered no original ideas but one of the examples referred to what we would recognise as a whole life insurance policy paid for by level annual premiums. This idea was perfected later by James Dodson (1755) and it is not clear whether Dodson was aware of Hayes' earlier contribution.

Thomas Simpson's book of 1742 began with a favourable comment on de Moivre's "Annuities on Lives" and was largely built on de Moivre's. The structure and the expository problems were similar. But his presentation and explanations were clearer and more concise, he corrected the errors he had discovered in de Moivre's work, many of his proofs were more general and he made some important new contributions. Firstly, he constructed a life table based on the London bills of mortality in which he modified Smart's life table at ages under 25 because of Smart's failure to take migration into account: but the description of his methodology is too vague and unclear to follow (there are some helpful suggestions in Westergaard (1932)). Secondly, he used his life table for calculating tables of values of single life and joint life annuities for lives of the same age, adapting de Moivre's backward recursion formula to the case of joint life annuities. Thirdly, he devised computational rules (by trial and error) for calculating joint life annuities for different ages from the tabulated joint life annuity values. Fourthly, he demonstrated that de Moivre went too far in some of his simplifications and his formula for valuing joint life annuities was not sufficiently accurate. Thus, he provided the first satisfactory solution to the problem of calculating values of annuities for two and three lives. He also proved four general theorems on reversionary annuities involving two groups of three lives (involving the statuses $\overline{xyz}|\overline{abc}$, $xyz|abc$, $\overline{xyz}|abc$, $xyz|\overline{abc}$): from these formulae, all of de Moivre's results are easily found as well as some new results. He indicated how these results might be extended to apply to any number of lives and gives some examples of applications. Lastly, he considered the adjustments to be made to life annuity values if payments are to be made every half year or quarterly and derived the first order approximations in common use.

Simpson followed de Moivre closely in his discussion of contingent probabilities; but for the evaluation of the integrals involved he used the Newton-Cotes formulae for numerical integration rather than the trapezoidal rule. Instead of discussing expectations of life, he solved the related problem of defining the stationary population corresponding to a given life table and determining its size as the annual number of births multiplied by the expectation of life at age zero: Hald (1990).

Simpson's 1752 work dealt with a number of subjects in an accessible style. Part 6 considered annuities, providing some notable additions to his 1742 book and many new examples. He considered the single premium for a deferred annuity and the value of a continuous single life annuity (under de Moivre's linear hypothesis); he also provided a table of the value of annuities on two lives of different ages, a computational rule connecting the value of an annuity on three lives to the values of annuities on two lives and rules for calculating contingent assurances (under de Moivre's linear hypothesis) in which he corrected one of de Moivre's mistakes.

Simpson's contributions as a whole represent an important step forward. In particular, his clearly presented method of mathematical argument and proof became of great importance for the following generation of actuaries (Hald (1990)).

De Moivre accused Simpson of plagiarism. There is a close similarity between the texts and further, according to Hald (1990), Simpson had previously plagiarised de Moivre's Doctrine of Chances in his 1740 book on the Law of Chance. On both occasions, Simpson referred to de Moivre in the preface but not in the text even though he used all of de Moivre's results. In his later editions, de Moivre gave Simpson the same treatment of omission. Simpson reacted to the accusation with a vigorous defence. It is worth noting Hald's comments that Simpson's various writings were controversial and "aroused many accusations of plagiarism". However, he was a successful author of elementary textbooks and it is clear that he did make important contributions to actuarial science in his own right. Simpson also made some important contributions to statistical theory, recognising the importance of the idea of observational error. To quote Stigler (1986), he took the "crucial step towards error".

Hald (1990) notes that de Moivre advocated using an approximation to the life table to facilitate calculations whereas Simpson based his calculations on an observed life table. The next generation of actuaries used Simpson's approach although they abandoned his life table (which referred to the general population of London rather than to annuitants or to insured lives) and based their calculations on better life tables derived from the observed experience of life insurance companies. At the same time, many attempts were made to follow up de Moivre's approach and, as we have seen, it was not until 1825 that Benjamin Gompertz succeeded in formulating a mortality hypothesis that would be widely accepted.

In the strong competition between de Moivre and Simpson, a comprehensive theory of insurance mathematics for valuing annuities was forged, and the necessary tools for practical application were thereby provided (Hald (1990)). It is noteworthy that much of the theory developed by de Moivre and Simpson referred to annuities. It is strange that neither took the step of adapting these ideas to set up a theory of life assurances. Perhaps, as Hald comments, "they did not feel the need for such a theory because they usually expressed the benefits in terms of annuities".

James Dodson's work of 1755 demonstrated how permanent life insurance could be operated with level annual premiums calculated on the basis of age at entry and how this level charge for an increasing risk would lead to the build up of a fund. This work is a major landmark and marks the birth of the ideas of scientific life insurance which underpin the subsequent development of this industry to the present day. Indeed, Dodson is often described as the "father of modern life insurance".

The translation of these principles into practice introduced new problems for which solutions had to be found if the life office was to remain financially sound. Formerly, it was necessary only to consider the office's finances one year at a time with short term contracts being issued. With premiums being fixed at the outset, and hence not adjustable during the term of the policy in the light of subsequent experience, it was essential to take a long view in estimating experience. Thus, these principles profoundly affected not only the development of life insurance contracts available to the public but also the need to ensure that the life office fully understood the long term financial consequences of issuing these new contracts. This led to the development of the actuarial theory and practice of life insurance, discussed in the next section.

Richard Price's book, *"Observations of Reversionary Payments"* first published in 1771, and eventually running to seven editions, was a further major work in this area. Price dealt with a number of important subjects. He constructed the Northampton life table based on deaths in 1735-70 and then in 1735-80 and used this as the basis of premium calculations (Dodson's scale of premiums was based on a life table constructed from the deaths in London in 1728-50. This period included two years with mortality rates higher than the average by about 25% leading to premium rates that were too high in commercial terms). In his methodology, Price followed Halley and assumed that the underlying population was stationary, making adjustments for migration and the deaths of immigrants. The Northampton life table was in practical use for many years, having been adopted by the Equitable for its premium basis. Price's book also discussed problems in life insurance mathematics involving life insurance policies paid for by single and annual premiums dependent on age at entry, contingent probabilities and reversionary annuities. He considered insurance and annuity portfolios and then analysed the financially unsound position of some recently established annuity societies selling reversionary annuities: as a result some of these societies were closed. He provided a commentary on the status of the Equitable Life Assurance Society and the Amicable Society. He also considered a scheme providing for old age pensions and set down proposals for a sickness insurance scheme (see later) and commented on the National Debt, as well as on many other matters.

The fourth edition (1783) of Richard Price's masterpiece is important because it contains his second Northampton life table based on the experience from 1735-1780. With this life table, a set of tables of life contingency functions including assurance, annuity and annual premiums and expectations of life for the single life and joint life cases was also included. This life table and the derived financial and functions were to form the basis of much life insurance practice for the next century. Importantly, Price also constructed a life table for Sweden using a radix and a graduated set of inverse mortality rates following Wargentin's methodology: this represented an important breakthrough.

William Dale was the first to develop the ideas behind commutation functions as a device for facilitating arithmetic calculation. In his 1772 work, he tabulated and summed values of $l_x v^{x-50}$ at half yearly intervals from age $x = 93.5$ to 50.5 and then explained how to use these totals to calculate the value of a half-yearly annuity issued at any age over 50. Their use was illustrated for a number of life tables including a table calculated for London for the years 1728-1771 as well as Halley's and Simpson's life tables.

The comparison of actual and expected deaths has a long history: see, for example, Ogborn (1950) and Keiding (1987). The earliest published description can be traced to William Dale who in 1777 strove to highlight the inadequacies of the plans of the Laudable Society of Annuitants which provided annuities for its members and/or their widows. His calculations indicated a lighter mortality experience than expected with serious financial consequences for the writing of annuity business. In fact, Richard Price put in writing the idea of comparing actual and expected deaths in 1775 and this practice was subsequently adopted by William Morgan, (Elderton (1933)). There is evidence that the same methodology was being used by Johannes Tetens in the mid 1780s (Westergaard (1932)).

The origin of commutation functions has been a subject of some debate in the literature: see, for example, Hendriks (1851) and Seal (1950). Dale, as noted above, began this line of reasoning but did not explicitly develop commutation functions. Johannes Tetens' book of 1785 (which was the first German textbook on life insurance mathematics) introduced the single and double summations that we would recognise as D_x, N_x and S_x. Subsequently, George Barrett (see the appendix to the second edition of Baily (1813)) independently constructed the columnar functions and considered their application to increasing whole life and temporary annuities (with an error that was subsequently corrected by Griffith Davies) and joint life annuities. Tetens' book also tackled a number of other problems including annuities paid continuously.

Francis Baily was one of the leading figures of nineteenth century actuarial science and is notable for producing the first comprehensive textbook on modern life insurance mathematics, providing a systematic, algebraic treatment of the subject. The book contains 100 pages of tables and the basic data from 14 mortality investigations. We highlight here the first published presentation of policy values and surrender values in mathematical form. He emphasised the critical importance of allowing for reserves in determining the *"true profit"* of a life insurance company. Baily also successfully applied premium conversion relations to problems involving more than one life and obtained an approximate formula for the evaluation of contingent assurances (pages 183-185: "the formula that Simpson was looking for but did not find": Hald

(1990)). Shortly afterwards, Joshua Milne wrote a text-book on life contingencies and related problems (see earlier), taking further Baily's attempts to improve and standardize the notation used. These textbooks made actuarial mathematics accessible to those with a reasonable mathematical background and so represent an important stage in opening up the subject.

Griffith Davies' book of 1825 contained the first practical application of commutation functions to the life insurance field. The book contained Griffith Davies' Equitable Life mortality table (based on sparse data supplied by William Morgan) and tables of D_x, N_x, S_x, M_x and R_x functions for both single and joint lives in modern form, as well as other monetary functions.

One of the difficulties facing practising actuaries, as well as those with a theoretical interest, was the absence of a standard notation. Each textbook presented a different view. David Jones, in extensive volumes of life insurance mathematics and tabulations publised in 1843, also invented his own - the difference is that his notation was accepted by later writers and became the basis for the international standard notation agreed and introduced in 1898 at the Second International Congress of Actuaries by George King's committee. The use of a central symbol, annotated by upper and lower prefix and suffix, with each position having a unique meaning, was the key feature.

With the introduction of Gompertz's Law, the calculation of annuity and assurance functions was facilitated. This was taken further by Augustus de Morgan in 1859 who introduced the law of uniform seniority which enabled a joint life annuity (or assurance) to be exactly replaced by a single life annuity (or assurance) calculated at a modified age. This principle was effectively put into practical use through the construction of Uniform Seniority Tables by a number of actuaries: if the two lives are aged x and y ($y>x$), then there is a cross tabulation of (y-x) and n where $y+n$ is the modified age for the equivalent single life annuity. This principle was later extended to apply to annuities (and assurances) based on Makeham's Law (with m lives of different ages replaced by m lives of equal ages or replaced by 1 life and a modified rate of interest): this is discussed in the Appendix to William Makeham's 1867 paper in a contribution attributed to Thomas Sprague.

Another labour saving device were the tables of single and annual premiums published by William Orchard in 1850. These made use of the fundamental relationships: $P = \frac{1}{\ddot{a}} - d$ and $A = 1 - d\ddot{a}$ and permitted the calculation of P and A values from the appropriate annuity values. In the days before calculating machines were available, the saving in labour and time occasioned by these premium conversion tables was considerable - it was only necessary to calculate the ä function while the tables could be used for the remaining functions. A fuller discussion is provided by Gray (1857). As far as the broad sweep of the subject is concerned, this innovation is now seen as being of transient benefit only.

August Zillmer's well-known 1863 pamphlet showed how initial expenses (and commission) can be properly allowed for in the calculation of gross premium policy values. Zillmer recognised that the presence of renewal expenses caused the policy value to equal the net premium policy value but the excess level of initial expenses created a problem. Tables were provided of a net premium with the addition needed to allow for different levels (1% and 1¼%) of the "Zillmer loading". Another table gave, by age at entry, the maximum initial expenses as a percentage of the sum assured (and of the first year's premium) if the policy value at the end of the first year is not negative. Zillmer discussed the problems caused by negative values and the consequences of a policy lapsing while the reserve is negative. He also noted the significant effects of a high rate of business growth combined with high acquisition costs on the financial condition of a life insurer. Zillmer's work was controversial at the time but the technique has been implemented world wide by practising actuaries and also by life insurance supervisory authorities.

Sprague (1870) entered this discussion with the suggestion that the whole of the first year's premium should be regarded as having been absorbed by the costs of insurance (expenses and mortality risk) and that the insurance should be regarded as having been effected one year later with the policyholder then one year older than at entry. This method obviates the danger of negative values but is somewhat artificial in not relating the allowance for initial expenses to the actual strain and may be excessive in its effect on certain types of policy, for example, endowment assurances: Coe and Ogborn (1952). It has been successfully used in Canada, however, in modified form.

A theoretically important early result which brings together the theories of statistics and life contingencies is Hattendorff's theorem (1868). This states that the losses in different years on a life insurance policy have zero means and are uncorrelated, so that the variance of the total loss is the sum of the variances of the annual losses. The loss in a year is defined as the net outgoings, insurance benefit less premiums, during the year, plus the reserve that has to be provided at the end of the year, and minus the reserve released at the start of the year, with all quantities being discounted back to the inception of the policy. Hattendorff's theorem was originally stated only as an approximation based on the assumption of a very large number of policies and the use of the Normal distribution. The first rigorous proofs are provided by Steffensen (1929). In recent years, the application of modern stochastic processes techniques to life contingencies has led to a resurgence of theoretical and practical interest in this result: see Bowers et al (1986) for an introductory discussion and Norberg (1992) for a review of recent results. We draw attention to Hickman (1964)'s consideration of the theorem for the multiple decrement model and to Wolthuis (1987) for the multiple state continuous time Markov model. Hattendorff also considered the amount of risk loading needed, in excess of the net or mathematical premium, to protect the insurer against fluctuations and possible insolvency. This work is an early example of what came to be known as risk theory.

Wesley Woolhouse's paper of 1869 is the first complete presentation of a theory of life insurance mathematics on a continuous basis, that is assurances are assumed to be payable at the moment of death and annuities are assumed to be payable continuously. He defined the corresponding values, $\bar{A}$ and $\bar{a}$, and proves the connecting conversion relationship: $\bar{A} = 1-\delta\bar{a}$. He derived Woolhouse's formula for approximating an integral in terms of a series of differential coefficients and used this tool to obtain approximate expressions for the values of annuities payable continuously and m times per year in terms of the corresponding annual annuities. Joint life annuities and (complete) annuities with a proportionate payment at the time of death were also discussed and approximations similarly developed. Woolhouse also discussed approximations for single and joint life expectations of life (the case $i = 0$) and for continuous single life, joint life and contingent assurances. This is an important work for providing the tools for practitioners to perform the approximate calculations needed for evaluating continuous and "m'thly" functions.

Thorvald Thiele's most important contribution to the field of life insurance mathematics is the differential equation for the net premium policy value for a life policy. He showed it to colleagues in 1875 (Hoem (1982)) but it was printed only in his obituary by Jorgen Gram in 1910. This equation

$$\frac{d}{dt}\,{}_tV_x = P_x + \delta\,{}_tV_x - \mu_{x+t}\,(1-{}_tV_x)$$

(in standard notation) is an important theoretical tool in the analysis of life insurance policies, both in deterministic and stochastic terms. The consideration of the change in the policy value in continuous time was an advance on the known discrete time relationship between policy values at integer durations.

Jorgen Gram also provides us (in a paper published in 1889) with one of the first examples of a stochastic approach to life contingencies. If T represents the future random lifetime of a life now aged x at time 0, then v^T is the discounted value at time 0 of a unit sum to be payable at the moment of death. If $\delta(>0)$ is the constant force of interest then Gram obtained the result

$$Var(v^T) = {}^{2\delta}\bar{A}_x - ({}^{\delta}\bar{A}_x)^2$$

and showed that similar, simple results hold for other types of policy. Hoem (1982) argues that Gram realized the usefulness of such formulae but not their general importance. He published the results some ten years after their derivation and then in Danish in a local journal with a narrow readership. As a consequence, the results were lost to the wider actuarial world until their rediscovery by Czuber (1910).

We close this section with mention of George Lidstone's important 1905 paper on changes in pure premium policy-values, which he considered to be "his best piece of work" (Elderton (1952)). Here he introduced the equation of equilibrium and used it to determine the effect on reserves of changes in the technical basis and contract terms, for example the sensitivity to changes in the rates of mortality or rate of interest. The results obtained have a general application and confirm the validity of some of the special cases that were known at that time. The theory has been subsequently extended: see, for example, Baillie (1951), Gershenson (1951) and Promislow (1981).

Life Insurance

Policies of life insurance have been available for many centuries, with the earliest recorded UK contract dated from 1583 (on the life of one William Gibbons). As a phenomenon, life insurance was reasonably common during the sixteenth century but until the middle of the seventeenth century its distribution was limited to short term temporary insurance policies on business partners and merchants' affairs. By the early 1700s, the direction was changing and schemes of annuities became important, in particular for governments attempting to raise revenues (see earlier sections).

This brings us to the early tontine schemes. The term "tontine" derives its name from Laurens Tonti (a Neapolitan banker living most of his life in Paris) who, around 1650, put forward proposals for raising monies to Cardinal Mazarin, the Chief Minister of France. In this plan, a fund was raised by subscriptions, the interest on which was to be divided each year among the surviving subscribers, the last survivor receiving the interest on the whole of the fund. The state thereby gains instant access to the subscriptions and in return the state guarantees the payment of a constant yearly annuity to be divided equally among the surviving members of the group and terminating with the death of the last survivor. The proposal was not accepted for some years in France. As Hald (1990) notes, "it is clear that the advertisement of such a project must contain a table of l_x as an essential element". Tonti also became involved with a tontine scheme in Denmark advertised in 1653. This project (jointly organised with Poul Klingenberg) is of interest because it contains the "first published life table": (Hald (1990)).

The first proposed tontine in the UK is due to Thomas Wagstaffe in a scheme laid before the City of London authorities in 1674 but which does not appear to have matured. As Raynes (1964) notes, there are three important features of the scheme:

i. a permanent contract was involved;
ii. members were classified according to age;
iii. it was essentially a method of raising money for the City.

Innumerable variations of the tontine idea have appeared from time to time. But the "temptations to fraud, and, even worse, to murder, grew so great, that the various governments had to interfere and suppress by law the use of it in any form" (Manly (1887)).

The first insurance office to offer long term contracts on the basis that funds would be built up systematically was the Amicable Society for a Perpetual Assurance, founded in 1706. The publicity material (attributable to John Hartley) is included here as a landmark work - and makes clear that the build up of reserves was an important aim. The premiums collected each year, after deducting expenses and amounts prescribed for contributing towards the growth of reserves, were added to the interest earned in the year and the total sum was divided up among the claimants. Thus, we see that long-term funded life insurance was available for the first time in 1706 and this marks a turning point in the history of insurance and of actuarial science.

In the first instance, the number of contributors was fixed and there was no variation in the annual subscription according to age. The amount payable at death was clearly uncertain, depending on the number of deaths during the year, and the contributions and interest earnings received during the year. This uncertainty (arising from the underlying tontine principle) in the level of benefits became recognised as a disadvantage and guaranteed minima were introduced in 1757 and then in 1770. In 1807, the Society changed its strategy, dropped the fixed maximum number of contributors and introduced life insurance in the modern sense with premiums based on age at entry for a guaranteed sum at death (Raynes (1964)).

In any scheme involving the build up of funds it would be essential to examine the financial progress of the funds on a long-term basis. The 1715 prospectus for The Friendly Society "Land Security for Establishing Perpetual Insurance on Lives of Men, Women and Children" contains perhaps the earliest known examples of emerging cost calculations in connection with life assurance.

We have noted already the significance of James Dodson's contributions to actuarial science. Refused admission to the Amicable Society on account of his age, he sought to form a more "equitable" society based on the scientific principles laid down earlier by Edmund Halley. He felt that the Amicable's approach to mutuality had two main disadvantages. Firstly, most such schemes had contribution rates set irrespective of age so that they were unfair on the younger subscribers. Secondly, in a year with fewer contributions and more deaths there would be little to pay to the beneficiaries. So he proposed a scheme which would offer level premiums for long-term contracts. The scheme was refused a Royal Charter in 1761 after protests by the Amicable and others. Dodson died in 1757, but his co-promoters entered into a voluntary partnership and adopted his ideas to form the Society for Equitable Assurances on Lives and Survivorships in 1762, which was the first life insurer to function on sound actuarial principles and is still operating today as a mutual company.

Dodson's 1756 lecture is thought to have been prepared for the committee to petition the Privy Council for a Charter. (The original manuscript seems not to have survived). The lecture presents Dodson's table of whole life insurance annual premiums, adopted by the Equitable in 1762, and also gives an investigation of the financial position of a hypothetical insurance company over a period of 20 years, under particular assumptions regarding mortality, interest and expenses experience. The first prospectus of the Equitable (1762) appears in a document attributable to Edward Mores.

In the early years, the Equitable received much assistance from Richard Price. In 1775, his nephew, William Morgan was appointed actuary, a post that he held until 1830. Raynes (1964) argues that Price and Morgan are responsible for laying the foundations of a strong actuarial profession in the UK. Under their guidance, the Equitable pioneered the scientific selection of lives, rating and valuation for different types of life insurance (Ogborn (1962)). Two works from Morgan are included here. The first (1776) represents the results of the first actuarial valuation of the individual policies of a life insurance company. Morgan compared the mortality experienced with that expected according to three standard tables (Dodson's, Simpson's and Halley's) and then compared the present value of the sums assured with that of the future premiums (on the same assumptions as in the premium basis) and the assets (at market value at the date of valuation). Of the surplus shown (£25,143), a sum of £11,000 was distributed in cash to the policyholders (the only cash distribution ever made by the Society).

The second item from Morgan (1779) sets down three methods due to Price for determining the progress of a life insurance company. These were

(a) "a comparison of actual with expected deaths" (as mentioned in the previous section),
(b) "a comparison of the actual claims with the whole of the premiums for temporary assurances for terms of less than 7 years and two-thirds of the premiums for assurances for the whole of life", and
(c) "a detailed valuation of the assets and liabilities" (Coe and Ogborn (1962)).

The comparison (b) was in effect a rough analysis of surplus, although not now used in this crude format. We thus see that the full importance of regular actuarial valuations was now recognised.

As Coe and Ogborn (1952) note, a major problem tackled by Morgan was the equitable distribution of surplus. He adapted the crude methods then being used into "a simple method of distribution of surplus which fitted the scale of premiums and the method of valuation, and dealt equitably with the problem having regard to the large carry-forward of surplus which was then necessary for the general security of the Society. The fit was approximate, but he stated that it was not possible to distribute the exact share of surplus to each policyholder in a society which was continually open for new members". He also justified his method by reference to the large profits from lapses in the early years which could be used to support the bonuses.

In the eighteenth century, speculators were ready to sell and buy insurance on any life who was considered to be in poor health or to be subject to additional risks. Public reaction to this type of wagering affected the development of legitimate life insurance and, as a form of moral hazard, increased the probability of insolvency for any insurer. The Life Insurance Act of 1774 was a turning point, by introducing the concept of insurable interest. Subsequently, although wagering policies continued for many years, the successful rejection of claims on the grounds of lack of insurable interest slowly led to the extinction of such abuses.

Having started in the UK, the progress of scientific life insurance throughout the world was slow. The first French life insurance company, the Compagnie Royal d'Assurance, was founded in 1787. We note that its prospectus includes a tribute to the profound contribution made by Richard Price to the underlying mathematical theory. We here include the first prospectus (1814) issued by the first life insurance company transacting the usual forms of life business with the general public and established outside of Europe - the Pennsylvania Company in the USA. Besso (1887) charts the international spread of life insurance in the 25 years up to 1883, with companies being established throughout Europe, in North America and Australia.

Early moves to establish life insurance supervision can be traced to the 1840s in both the UK and USA. Until this time there was no legislation to control the operation of life insurance companies.

In the US, actuarial supervison began with the 1858 Massachusetts legislative session and the passing of a law (the first designed specifically to regulate insurance companies) requiring the Commissioners of Insurance to calculate the reserves on all the policies of all the licensed companies. The Commissioners had the discretion to choose the valuation method and mortality and interest assumptions. As one of the Commissioners, Elizur Wright chose the net level premium method: "a controversial decision of immense inportance in actuarial history" (Moorhead (1989)). In 1861, the Massachusetts legislature, on Wright's recommendation, became the first jurisdiction to require guaranteed withdrawal values.

In the UK, many of the life insurance companies established in the late eighteenth and early nineteenth centuries had prospered and built up sound reputations, but the first half of the nineteenth century saw a rush of new life companies being established and a large number of failures or mergers, which have been attributed, inter alia, to questionable selling practices and unscrupulous management. (Daykin (1992)).

In the UK, the Select Committee on Joint Stock Companies was appointed to enquire into the state of supervision of such companies with a view to protecting the public. Among those who gave evidence were four leading actuaries and we include here the minutes relating to Arthur Morgan's evidence. The principle of "freedom with publicity" was advocated, with publication of the results of an insurer's operations on a regular basis so that observers, including members of the public, could detect evidence of problems. The origins of the statutory returns currently required of life insurance companies in the UK can be traced back to these tabulations - in the UK, they are published while in other jurisdictions the corresponding returns remain confidential to the supervisor.

It appears that the government had originally intended to introduce separate legislation for insurance companies to deal with their special features (as was done for banks) but at a late stage it was decided to include them in the more general legislation. This led to anomalies as to which legislation governed companies established before and after the 1844 Act (Raynes

(1964)). Despite the objectives of the 1844 legislation, it proved to be a device for registration rather than regulation and therefore became a convenient means of establishing new companies. The 1844 Act failed to prevent serious failures by life offices, many of which resulted from the mushrooming of new companies that were "churning" the same business from one to another - for example, seven life offices were transferred from company to company **four** times! (Raynes (1964)). As a result, the later Select Committee on Assurance Associations was formed in 1853 to investigate life insurance matters further. Again, they took evidence from a number of leading actuaries. In their recommendations, the Committee supported a relatively free development of new life insurance companies as being in the public interest but proposed a minimum level of capital to be invested in public funds, as evidence of the bona fide intentions of the promoters and to act as additional security for the liabilities. They recommended a requirement for a complete valuation of the assets and liabilities at least once every 5 years, and the publication of the results through depositing them with the Registrar of Joint Stock Companies, who would make them available to the wider public on request. They also recommended publication of an annual statement of new business, premium income and sums assured (for new and current policies), expenses, assets (subdivided by type), average rate of interest earned (by type of asset) together with disclosure of the mortality and interest assumptions underlying the current premium rates.

The Joint Stock Companies Act 1856 unintentionally exempted life offices from the requirements of the 1844 Act without replacing them with any new rules. For various reasons, efforts to introduce life insurance legislation lapsed (Walford (1887)). The demise of the Albert Life Assurance Company in 1869 led to calls for a formal supervisory regime and this was enacted in 1870 broadly under the terms devised earlier: the Life Assurance Companies Act 1870 (see Daykin (1992)).

The new Act's requirements were of far reaching important - they were divided into five main headings: deposits; separation of the life fund; accounts, returns and other information for policyholders; amalgamation; and winding up. The separation of the accounts for life insurance and annuity business to form a separate Fund was intended to provide security for these policyholders. Amalgamation (or transfer) of life insurance companies (where there had been so much abuse in the recent past), could only take place on confirmation by the Court. The procedures laid down required a statement of the material facts to be sent to the policyholders of the transferred company - if policyholders with 10% or more of the total sums assured objected, then the Court had no power to sanction the amalgamation or transfer. The Act went further than earlier legislation and required the standardization of the revenue accounts and balance sheets as well as the publication of statistics on the actuarial valuation. At this stage of the development of the accountancy profession, there was considerable conceptual confusion about the nature of "income" and "expenditure", inter alia, and there was much loose terminology (Edwards (1986)). The first returns under the new legislation revealed that those compiling the accounts were not able to produce them as required. Thomas Sprague's classic treatise of 1874 sets down the principles that should be adopted to comply with the Act's requirements, explaining the differences between a revenue account and a cash account (as identified earlier by William Morgan). He noted that before the passing of the Act "the accounts published by most of the insurance companies were mere cash accounts of receipts and expenditure" and devoted several pages to explaining the difference between cash payments and "the true income and true charge of the year".

The requirement of the 1870 Act for publication of these accounts and statistics was intended to enable information to be furnished to the regulatory authorities and the public, (and especially to existing and potential policyholders and shareholders) to enable them to form the basis of a reliable estimate of the financial position of the company (Tapp (1986)). Although there was a requirement for the quinquennial actuarial valuation, no special auditing requirements were imposed on insurance companies, so that they remained in the same position as the generality of limited companies at this time.

Bonus systems were discussed by many actuaries over the years, with the first contributions (as noted earlier) coming from Richard Price and William Morgan. Thomas Sprague's paper of 1858 discussed different methods of distributing surplus in relation to the then common methods of loading the premiums. The importance of consistency between the bonus system and loading in the premium formula was emphasised. The closing three paragraphs are of interest, as they effectively anticipated the bonus reserve valuation method (first described by that name by Coutts in 1908 - see later).

Shephard Homans' paper of 1863 contained the first published description of the "contribution" method for distributing surplus, which the author had developed together with his assistant, David Fackler (then too young to be a Fellow in the US). The principle of the plan was that the divisable surplus should be allotted to the various assurances in proportion to the individual contribution of each assurance to the surplus. The share of divisible surplus so computed was, in fact, usually paid in cash as a "dividend" though this, of course, is not a necessary feature of the method. The surplus could also be converted to a reversionary bonus. This proposal arose from the experience of the Mutual Life Insurance Company of New York (where Homans was the Actuary): the mortality had been much lower than that assumed in the premium basis, and interest earnings had been high. The surplus was consequently high and Homans believed that any of the usual methods of surplus distribution would have been inequitable (and apparently in violation of his company's charter). The method was largely ignored in the UK but was rapidly and universally taken up in North America (where it has been further developed).

James Meikle's work of 1865 (effectively four papers) introduced the scientific principles of analysis of surplus to life insurance, formalising the earlier ideas of Price, Morgan and others. Although introduced to this specific area of actuarial practice, the principles were later applied elsewhere, for example occupational pension schemes and non-life insurance companies. Meikle identified three principal sources of "profit" relative to the valuation basis (taken to be equivalent to the premium basis): interest, mortality and "margins" and considered (both algebraically and numerically) the identification, calculation and distribution of these three types of profit. He did not, however, go on to show how this process could be used as an independent check on the valuation of a life insurer's liabilities.

The granting of life insurance to lives who were not in perfect health became commonplace, with the establishment of some life insurance companies in the early nineteenth century having this facility as an important feature (for example, the Clerical, Medical and General in the UK). George Humphreys' report of 1874 represents the first detailed published account of an investigation into the experience of a group of lives accepted for life insurance at higher premium rates because of medical impairments identified when the policies were issued. The study covers 3147 lives accepted between 1808 and 1871, with impairments classified into eight groupings. At this time the higher premium was calculated by assuming that the policyholder's age was higher than reality, that is the age at entry was "rated up". Humphreys analysed the extent to which the impaired lives were rated up and constructed a life table based on this mortality experience, expressing the rate of mortality as a function of the life's age plus the number of years he was rated up.

The first large scale study of impaired insured lives' mortality experience was conducted in North America, based on the experience of various classes of lives between 1870 and 1899: the Specialised Mortality Investigation (1903). Classes of risks investigated included different countries of origin, different ethnic backgrounds, different occupations and medical impairments. Mortality ratios were used in reporting the results and inferences were made on a comparison of actual with expected deaths, following on from Richard Price's idea. This study is the direct forerunner of modern investigations of impaired lives' experience.

Inspired by the manner in which these results had been presented as mortality ratios, Oscar Rogers and Arthur Hunter, Medical Director and Actuary respectively of the New York Life Insurance Company, devised a system of risk evaluation based on mortality ratios expressed as a percentage in their 1919 paper. This system is known as the *numerical rating system*. The principle of the system assumes that average mortality is represented as 100%, and each factor influencing mortality is expressed numerically in terms of percentage mortality. Debits or credits are allotted to each factor in multiples of five according to whether it has an unfavourable or favourable influence. The total sum is then computed and the result expressed with reference to the standard of 100%. Rogers and Hunter presented a full critique of the system and published the basic ratings in use at the time. Over 80 years later, the numerical rating system is now used almost universally for the underwriting of lives and has completely replaced older and more empirical methods of risk evaluation (Brackenridge (1985)). The magnitude of the debits and credits used in the numerical rating system is estimated using a combination of medico-statistical studies and perceived wisdom of the medical profession. The credits and debits have been compiled into underwriting manuals by the major life insurance and reinsurance companies around the world.

As we have noted, the origins of the bonus reserve valuation method can be dated to Thomas Sprague's paper of 1858. The method was used by some life offices towards the end of the nineteenth century for their statutory valuations. Henry Manly, in contributing to a discussion at the Institute of Actuaries in 1902, advocated such an approach as an alternative to the net premium valuation method, with the present value of the bonus liability at the rate to be declared being treated as a prospective liability. The full implications of the bonus reserve method are however drawn out by Charles Coutts in this paper of 1907. The debate between the relative merits and demerits became at times "hostile" (Coe and Ogborn (1952)). The bonus reserve method was "strongly attacked because it treats future bonus as a liability, in complete contradiction of the true nature of bonus; but on the other hand it has been stoutly defended for the way in which it keeps the true facts in view, in contrast to its chief competitor for favour, the net premium method, which keeps none of the facts in view, although making indirect provision for future bonus by the use of an artificially low rate of interest". On the other hand, the net premium method has been criticised for its deliberate use of unrealistic assumptions, with expenses and future bonuses being met implicitly out of margins in the interest rate and valuation premium. As the modern literature indicates, this debate has not been resolved although the emphasis has switched to the differences between "active" and "passive" valuation bases (Redington (1952)).

Pensions

Widows' funds were among the first pension type arrangement to appear, for example, Duke Ernest the Pious of Gotha founded a widows' fund for clergy in 1645 and another for teachers in 1662. "Various schemes of provision for ministers' widows were then established throughout Europe at about the start of the eighteenth century, some based on a single premium others based on yearly premiums to be distributed as benefits in the same year" (Hald (1990)).

The Scottish Ministers' Widows Fund was set up in 1743 and is the earliest formal widows' pension scheme (providing annuity benefits to widows and children) funded by annual contributions. It has functioned ever since. The scheme was promoted by Robert Wallace, Andrew Webster and George Wishart, and the calculations were checked by the mathematician, Colin Maclaurin. The technicalities of the scheme represented a major advance over the earlier suspect methods that had been used and its construction "has served as a model for many late pension funds" (Hald (1990)). The terms were revised in 1748 (as described by Andrew Webster's report included here) because the finances had become unsound: the original data used were defective and membership was then made compulsory thereby preventing adverse selection by the members. The income during the first years of operation greatly exceeded the expenses, so a fund was built up so that, when the stationary state arrived, contributions and interest income from the fund would balance the benefits and administrative expenses (Hald (1990)). The actuarial calculations were based on statistics on the average number of ministers, widows and orphans for the period 1722-1741, on Halley's life table, and on an interest rate of 4% as shown in Andrew Webster's memorandum. A full history of this fund has been written by Dow (1975) and Dunlop (1993).

The construction of this fund provides a model taken up by some subsequent widows'pension funds. The Trustees report of 1759 provides the first known emerging cost calculations with an allowance for new contracts. The future projections of the Fund proved particularly accurate, for example, the accumulated fund for 1765 amounted to £58,347 compared with a forecast of £58,348!

Francis Maseres with Richard Price put forward to the UK Parliament in 1772 the first proposals for the setting up of a national system of retirement pensions for the entire population. The method of computing the deferred life annuities was based on Thomas Simpson's 1752 book and flat and increasing payments were considered. The scheme was to be managed by church wardens through local parishes. The bill was accepted by the House of Commons but was rejected by the House of Lords. A similar fate awaited a second plan proposed by Thomas Ackland in 1789 and based on the friendly society movement, again with the technical support of Richard Price; the calculations and tables appear in the fifth edition of Price's *"Observations on Reversionary Payments"* (1792).

We have commented earlier on the contribution of Johannes Tetens. In Chapter 5 of Volume 2 of his book (1786), he discussed the surplus/deficit that can arise from a widows' fund and how it should be calculated, with examples drawn from the experience of the Calenburg Widows' Fund. Inter alia, he considered how the financial status of a fund should be measured, how surplus arises and its definition in terms of the assets and the present value of the prospective liabilities less the contributions due to be paid.

The UK Civil Service pension scheme began in the eighteenth century and became an important model for employees in other sectors, because of the generous level of benefit provided

(Rhodes (1965)). The anonymous extract from an 1821 report, that is included here, represents a protest on behalf of civil servants at government proposals to make the Civil Service pension scheme contributory. The author presented calculations to demonstrate the inequity of the proposals and the profit that would be made at the expense of different groups of members. Approximate allowance was made for early ill health retirements and their subsequent levels of increased mortality. Importantly, the calculations presented contain the first mention and use of a promotional salary scale. Despite their early appearance and their later subsequent use by William Farr, salary scale calculations were not fashionable until later pioneered by Henry Manly (see below).

The great mathematician Carl Friedrich Gauss also contributed to the development of these ideas with his valuations of the pension fund for widows and orphans of the professors of Göttingen University in 1851: see Reichel (1978) for a full discussion. There are similarities between Gauss's work and the earlier report of Webster. Both the Scottish Ministers and the Göttingen schemes were only open to a restricted class of members and so both were protected against sudden influxes of new entrants and also forecasts of future population size would be subject to less error than otherwise. These valuations and reports of Gauss are considered to be one of the milestones in the development of actuarial science in Germany. It is interesting to note that Gauss calculated the contribution rate for new entrants and found it to be inadequate but did not recommend an increase in the contribution rate.

In the late nineteenth century in the UK, private pensions tended to be granted on an ex-gratia and personal basis to favoured employees of a company often as a reward for long service. Often there was no formal scheme, only statements of intent or merely expectations based on custom. Informal arrangements of this character, without legal protection, offered opportunities for abuse. In particular, preferential treatment of individuals often soured employer - employee relations, since the arbitrary nature of the pension award could tempt an employer to use the threat of withholding a pension as a means of placing pressure on an employee. More formal and legally protected arrangements were clearly advisable and there was soon pressure to introduce them generally. Attempts to do so, by establishing pre-funded schemes, met fiscal difficulties and special tax concessions for the private provision of retirement and related benefits were required. Section 32 of the Finance Act 1921 was to set the pattern for pension funds as they are now known.

The earliest private schemes (in the nineteenth century) were for large groups of employees and arranged by means of a trust fund with an actuary recommending the necessary contributions. In the USA the situation was similar but in the 1920s life assurance companies entered the field, offering guarantees and administrative services (to be copied a decade later in the UK). Although funded occupational schemes continued in Anglo Saxon countries and, for example the Netherlands, they had disappeared in France by the 1940s and were replaced by schemes operating by répartition or assessmentism.

Regarding the mathematics of operating funded occupation pension schemes, Henry Manly's paper of 1901 was the first paper in English to present a systematic treatment of pension funds. Manly discussed and solved a series of increasingly complex illustrative problems, giving the algebraic development of the formulae (in text book style) and using a consistent notation. Numerical examples were provided as well as examples of full valuations of a number of funds with different benefit patterns. There was a parallel development in the German literature: thus, Johannes Karup's classic 1893 report produces a full discussion of the valuation of widows' and orphans' benefits (see later). James M'lauchlan's paper of 1908 aimed at explaining the characteristics of a pension fund in simple terms, providing a careful analysis of the evolution of a typical fund at various points in its lifetime. He showed (as did Webster and others previously) that large reserves were a necessary feature of the actuarial approach to valuation. M'Lauchlan's paper provides a clear exposition of the subject although it is less significant in terms of conceptual content than Manly's work.

Investment

During the seventeenth century, the UK Government discovered that it was impossible to raise sufficient funds through taxes to pay for its wars; therefore, money was borrowed from various sources, including wealthy traders to cover these expenses. Payment of interest on the money borrowed was a natural development and so the National Debt came into being. When the Government borrowed sums of money, fixed, clearly defined, rates of interest were payable to the lenders.

Gradually, a market was created in what was referred to as 'stocks and shares'. If you lent money to the Government, you were then issued with Government stock paying a certain rate of interest. If you then decided that you wanted a capital sum before the agreed repayment date, you were able to sell your stock (and hence the right to future interest and the eventual capital repayment). Similarly, if you had taken shares in a trading venture and you required the return of your investment before the conclusion of the venture you were able to sell your shares in the market.

The advent of the Industrial Revolution meant that capital was required on a large scale to finance the new infrastructure projects, including the building of new canals, railways and factories. As a result, the existing methods for raising capital and for dealing in stocks and shares were expanded. This development was assisted by legislation passed in 1862 which allowed the formation of limited liability companies; previously shareholders had been personally liable for a company's debts if it had failed. At this time, in addition, many companies borrowed money on fixed terms, in much the same was as the Government were doing, (Lewin (1970)).

These developments were parallelled in other economically developing nations. Thus, today's sophisticated and complex capital markets are the result of development over at least two centuries.

The first two documents included here, due to John Castaing (1698) and Stephen Daubuz (1731), are illustrative of the workings of the early financial system against which actuarial science was developing. It is noteworthy that Stephen Daubuz was willing to quote for *"refusal and puts"* (that is options) although not on the day in question.

With the principles of scientific life insurance, laid down by Dodson and advanced by Price, Morgan and so on, life insurance companies necessarily built up large reserves through the charging of a level premium for an increasing risk. The premium calculations, of course, contained an assumption about the rate of interest that was expected to be earned on such funds and participating policyholders anticipated receiving bonus additions to the guaranteed life insurance benefits. Thus, the investment of the funds of life assurance societies became an extremely important issue. Arthur Bailey's seminal paper of 1862 discussed the actuarial principles on which the investment policy of life insurers should be founded. He proposed five principles (which came to be known as *"canons"* in the British actuarial literature) for the selection of investments. These canons remained orthodoxy, despite comments from Coutts (1933) until Pegler revisited the principles of investment selection and the basis of the valuation of investments in 1948.

The paper by Louis Bachelier of 1900 represents a remarkable advance with its stochastic modelling approach to investment problems in continuous time. This is the clear forerunner of the modern theory of option pricing, which has been developed for the valuation of complex derivative securities. *"These models have provided the springboard for a bewildering array of applications in investment and corporate finance. In particular, they provide the scientific framework for the measurement and control of different types of financial risk".* (Boyle (1992)). Bachelier deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion with zero drift: the significance of the paper is now acknowledged by those who have subsequently developed the modern theory, for example Merton (1973).

The letter by Dr D Moll of 1909 is significant in that he discussed the idea that liabilities and assets should be valued at the same rate of interest and that assets can be invested to reduce the fluctuations in surplus arising from interest rate changes. It would appear that at this early stage Moll appreciated the underlying point about immunization, without putting down a rigorous mathematical argument. These ideas were not picked up at the time by others (nor developed by Moll) and remained dormant for decades until the development of the parallel theories of duration (Macaulay (1938) and Hicks (1939)), matching (Haynes and Kirton (1952)) and immunization (Redington (1952)).

Risk Theory and Non-Life Insurance

As is well-known, non-life insurance largely began with marine insurance, probably in northern Italy about the end of the twelfth century. The first marine insurance policies involving the payment of premiums to specialised underwriters are thought to date from the first half of the fourteenth century (Daykin (1992)). By the late 1600s financiers were specialising in the underwriting of marine expeditions. Often, negotiations were conducted in coffee houses - one such was opened around 1680 by Edward Lloyd in London and became frequented by shipowners, merchants and mariners. It soon became the marine underwriting centre and eventually grew into the institution now colloquially known as Lloyd's.

Despite this long history of non-life insurance (now covering fire, liability, accident, motor, aviation, workmen's compensation and many other risks), the list of actuarial contributions to this area of practice is much briefer. The reasons why actuaries in many countries focused on other areas of activity, particularly life insurance, during the period up to 1900, are speculative. However, increasingly during the twentieth century there has been a growing appreciation among non-life insurance companies that there is room for improvement in the statistical and financial bases of their operations. It has been realized, especially in the more competitive classes such as motor insurance, that the company's experience needs to be examined and analysed thoroughly in order to quantify and manage the risks that the company faces; to estimate the reserves that the company should hold as protection against fluctuations in its claims experience; to price accurately the risks so that the premiums are competitive, allowing for existing and potential risk factors; and to avoid losing the *"better risks"* to other insurers; and so on.

We have noted that Nicholas Bernoulli considers simple probability problems from marine insurance in his thesis of 1709. An important early work is by Daniel Bernoulli (1738) on the problem now referred to as the *"St Petersburg Paradox"* in which Bernoulli suggested that mathematical expectation should be replaced by *"moral expectation"* a concept which we today know as expected utility.

Corbyn Morris's essay of 1747 represents one of the first attempts to use mathematics to solve non-life insurance problems; in this case, the examples are entirely from marine insurance, but are of wider application. The main objective of the work was to demonstrate that an insurer's *"probability of risk"* decreases as he spreads his available wealth between more and more policies at any one time. His tools were the binomial theorem and the various examples are presented with great clarity. Although the discussion may appear elementary, for its time this was clearly a pioneering work. We note in passing that Morris did not consider the determination of the risk premium from observational data; the case where the insured and insurer do not agree on the risk premium; the addition of a safety loading to the premium or the effects of competition. He does remark on the inadvisability of allowing the sum insured to include the amount of the premium - his argument recognises the existence of what we now call moral hazard. i.e. *"an inducement to very bad Practices"*.

The development of the theory of risk is described by Kanner (1869) and Böhlmann (1909) in two excellent review papers. It should be noted that the theory is concerned exclusively with life insurance and annuities at this stage and dealt mainly with deviations that were expected to be produced by random fluctuations in individual policies.

Independently of Morris, Johannes Tetens (in volume II of this work: 1786) addressed the theory of risk, which he defined as the expected loss to the insurer, if the contract leads to a loss. He demonstrated that *"for any amount of insurance, whatever may be the ages of the lives insured, and whatever the nature of the insurances the sums of the mathematical expectation of gain and loss for any interval of time, are equal to each other if only the premiums are calculated upon the supposition of the most probable case"* (Kanner (1869)). Tetens also showed that the risk to an insurance fund is larger when the number of members is larger and the risk in respect of each individual member reduces when the fund becomes larger and is proportionate to the square root of the number of members.

The ideas of Daniel Bernoulli were taken up by Barrois (1834) who used them to develop a complete and modern theory of fire insurance, in which he showed how a premium could be determined that is acceptable to both the person to be insured and the insurance company; the insurance contract increasing the expected utility of both parties.

The work of Barrois seems to have been ignored by the following generation, but Bernoulli's ideas reappeared in the insurance literature once the game theory of von Neumann and Morgenstern had made utility ideas fashionable and demonstrated that they should occupy a central position in any theory of decision making under risk and uncertainty. These ideas have been developed recently, in particular, by Borch in a series of studies beginning in 1961.

The next step forward comes from Carl Bremiker's work of 1859, which appeared translated into English (in two sections) in 1871. He considered the total claims distribution in life insurance and calculated risk measures for a single policy and for portfolios of life insurance policies, recognising that funds would be needed to cover the actual deviations from the mean level of mortality anticipated. He considered the separate cases of single and annual premium policies. The risk measures, R, are defined as the square root of the mean squared errors allowing the insured's survival time to be random i.e. for a single premium whole life policy, with single premium A, and unit amount insured:

$$R^2 = \sum_{t=0}^{\infty} \frac{d_{x+t}}{l_x} (A - v^{t+½})^2 .$$

Bremiker did not introduce the possibility of other sources of variability, for example interest rates. We have already noted that Hattendorff similarly considered risk loadings in net premiums in his 1868 paper.

These results were only of a limited practical value. The theory developed thus far by Tetens, Bremiker, Hattendorff and others did not consider the more important probabilistic questions (as in fact Morris did!) nor questions about within what limits an insurer's probable loss or gain would lie during different periods. The theory received little attention from practitioners and its applications to practical problems of insurance activities remained unimportant. Further, the focus of this work remained on life insurance and yet it is non-life insurance where the most rewarding applications would lie (Beard et al (1969)).

A new phase of development of risk theory began with the seminal work of Filip Lundberg in 1909 which thanks to Harold Cramér (1930) and other Swedish authors, has become known as the collective theory of risk. Lundberg used stochastic processes in continuous time to model the evolution of the liabilities of an insurance company. He dealt with the occurrence of claims collectively, without reference to the individual policies. Lundberg considered an insurance company as a container into which flows a continuous stream of premiums and from which is paid a sequence of claims. He considered the distribution of aggregate claims using a convolution procedure, the effect of reinsurance and initial reserves on an insurer, premium income and claim payments, together with stability measures. He considered also the probability of ruin and obtained the following elegant result for the probability of ruin in infinite time, known as Lundberg's formula,

$$\psi(u) = C(u)\, e^{-Ru}$$

where u is the initial reserve, of the insurer, C(u) is an auxiliary function with values in the interval (0,1) and R is the adjustment coefficient (or Lundberg's coefficient) which is dependent on some of the parameters of the underlying risk process. In his discussions, Lundberg ingeniously used the concept of operational time, closely related to the *"entropy time"* used in theoretical physics. The transformation of the time scale means that expected claim payments in any period are equal to the length of the period, and makes it possible both to simplify the underlying theory and extend its validity. Lundberg's work was overlooked for almost three decades until rediscovered by Cramér and this may be explained by his mathematical arguments being too deep and advanced for most contemporary and practical insurance people.

Because this theory essentially studied the progress of the insurance business from a probabilistic view, the theory had clear applications for non-life insurance as well as life insurance. This new way of expressing the problem has proved fruitful and the development of the theory has since been continued by several other authors in a principally Scandinavian school of academic and practising actuaries. (There have been numerous review articles, starting with Cramér (1955)). Jewell (1980) has argued that this work on risk theory and the dynamic risk model provided an important impetus to the development of stochastic processes and its applications. In recent years the fundamental assumptions of the theory, and thus the range of its applications, have been significantly enlarged by the use of more general probability models, which allow, for example, for certain types of fluctuation in the basic probabilities. Advances in the theory of stochastic processes have been reflected in the recent developments of both individual and collective risk theory. Advances in computing and simulation have similarly led to developments in the practical applications.

The importance of the actuarial contribution to workmen's compensation insurance was recognised by the establishment in 1914 of the Casualty Actuarial Society in the United States (see Moorhead (1989) for further discussion). The developing literature in North America contains some significant contributions two of which (in the area of experience rating) are highlighted here. It is noteworthy that both appeared in the new journal: the Proceedings of the Casualty Actuarial Society.

To calculate premiums or to estimate future claims for insurance policies in a large portfolio, the portfolio is usually divided into a number of subgroups on the basis of observable risk factors and these subportfolios are assumed to be homogenous (in the sense that all policies have identical claim distributions). Since not all of the risk characteristics are observable, for example the driving skill of the insured in a portfolio of personal automobile policies, the subportfolios will in practice not be as homogeneous as assumed. All the available information about the unobservable risk characteristics of a policy is contained in its own claim data, which is often rather scarce (and therefore not very reliable). For this reason, techniques were developed which looked at individual claims, as well as the reliability of the estimates for which the claims data of other policies in the subportfolio were used (Kling (1993)).

The tension between using policy data only or using all the subportfolio data has led to the introduction of the so-called *credibility theory*. The technique itself is based on weighted averages between policy and subportfolio estimates, where the weights are determined by the reliability of the experience data.

Credibility theory's development started with Albert Mowbray's 1914 paper which considered, using heuristic reasoning, the calculation of a (pure) premium for accident insurance and the size of experience needed for the premium to be *"dependable"*. A dependable pure premium is one for which the probability is high (at least equal to an assigned value) that it does not differ from the true pure premium by more than an arbitrary limit which may be selected with regard to the other factors present. Albert Whitney's paper of 1918 proposed that the true individual risk premium in workmen's compensation insurance be given by a weighted average of an estimate based on individual and another estimate based on collective experience data. As Kling (1993) explains, "the weight attached to the individual experience, also called the credibility factor, was assumed to be between zero and one. As a result it follows that the two extremes (zero and one) coincide with estimators solely based on collective or on individual experience data, respectively. The credibility factor itself should reflect the heterogeneity of the collective as well as the accuracy of the individual estimator measured by the amount and variation of the individual experience data".

Effectively, Whitney also introduced a Bayesian approach, which was followed up by Michelbacher (1918), but these ideas have only recently been formalised and developed: see Bailey (1950), Mayerson, (1964) Bühlmann (1967), Bühlmann and Straub (1970) and Norberg (1979) for an extensive review of the development of this theory.

Multiple Decrement and Multiple State Models

Multiple decrement and multiple state models are an important theoretical tool in the actuarial fields of pensions and sickness and disability insurance schemes. The history of the development of this subject has been described by Seal (1977) and Daw (1979). These models can be traced back to Daniel Bernoulli's memoir of 1766 which applied the methods of differential calculus to a particular problem and then solved the resulting differential equations under certain constraints.

What is the problem? *"Given two states A and B such that individuals in state A have mutually exclusive probabilities, possibly dependent on the time spent in state A, of leaving that state because of (i) death, or (ii) passage to state B, what is the probability of an individual passing to state B and dying there within a given period?"*. (Seal (1977)). Bernoulli's state A consisted of individuals who had never had the smallpox while state B comprised those who had contracted smallpox and would either die from it, almost immediately, or survive and no longer be suffering from that disease. In solving this problem, Bernoulli started with Edmund Halley's life table and effectively produced the first double decrement life table together with one of the related single decrement tables, as well as considering the efficacy of inoculation and deriving a mathematical model of the behaviour of smallpox in a community. The approach was based on setting up and solving a series of differential equations. (This work was the forerunner of considerable developments in the mathematical theory of infectious diseases: see Bailey (1975) for further discussion).

During the next 50 years, there were a number of contributions from other authors on the subject of smallpox and inoculation, including Jean d'Alembert and Jean Trembley (Daw (1979)).

Johann Lambert's 1772 work (from volume III of his book) explained how numerical data could be used to study Bernoulli's problem and laid the practical foundations for the double decrement model and life table. (Daw (1980)). He obtained an approximate formula for the rate of mortality when smallpox is excluded, thereby setting down a practical connection between the double decrement model and the underlying single decrement models. Daw (1979) argues that, with Lambert's contribution, *"the practical and theoretical foundations of double decrement tables had been laid down"*.

In his 1806 book, Emmanuel Duvillard calculated the reduction in mortality rates in France if smallpox were eliminated, considering the same problem as Bernoulli but in relation to the less risky method of vaccination introduced by Edward Jenner in 1796 (rather than inoculation). The book contains extensive tabulations of observed data on smallpox and of life tables constructed on the application of Bernoulli's theory. As Daw (1979) points out, the method of construction of Duvillard's basic life table is unclear and it appears to have been prepared by adding the deaths by age (see earlier) with the assumptions that the population was stationary and that migration could be ignored. Despite this possible doubt, French insurance companies only abandoned the use of this life table in the 1890s (Quiquet 1934). Quiquet (1934)) describes Duvillard as *"the first French actuary"* and Greenwood (1948) describes his book as a *"classic of vital statistics"*.

Both the calculations of Lambert and Duvillard drew on different sources of data. Joshua Milne, in part of his treatise of 1815 (see earlier), estimated the effect of the elimination of deaths from smallpox on the Carlisle life table and hence on the expectation of life, using data exclusively from the Carlisle experience.

Despite this progress by the early 1800s there were two outstanding problems, namely (i) deriving accurate practical formulae for application to numerical data, linking the discrete and continuous cases; and (ii) obtaining exact results in a convenient form (d'Alembert had derived an exact result in terms of an integral that was difficult to evaluate).

These problems were attacked successfully and independently by Antoine-Augustin Cournot in 1843 (Chapter XIII of his book) and by William Makeham in 1867. They were the first to set down the fundamental relations of multiple decrement models: in modern notation

$$\mu_x^k = (a\mu)_x^k \text{ for } k=1, \ldots, m$$

$$(a\mu)_x = \sum_{k=1}^{m} \mu_x^k$$

from which it follows that ${}_n(ap)_x = \prod_{k=1}^{m} {}_np_x^k$.

Makeham's paper of 1867 also contains an analysis of the *"partial"* forces of mortality for different causes of death, suggesting a re-interpretation of his formula for the aggregate force of mortality $\mu_x = A+Bc^x = \left[\sum_{i=1}^{n} A_i\right] + \left[\sum_{j=1}^{m} B_i\right] c^x$ to represent the separate contributions from m+n causes of death.

Makeham went on to use this connection between the forces of decrement to interpret the prior development of the theory: he demonstrated that the earlier results of Bernoulli and d'Alembert satisfied this additive law for the forces of decrement and this multiplicative law for the probabilities (or corresponding l_x functions). He also identified an error committed by d'Alembert (Makeham 1875)).

Johannes Karup in a report in 1875, that was not publicly published (on the invalidity and widow's pensions scheme for railway officials), described the properties and use of single decrement probabilities and forces of decrement in the context of an illness-death model (with no recoveries permitted) i.e. the *"independent or pure"* probabilities of mortality and disablement. He also discussed the estimation of the independent probabilities from observed data. This approach was regarded as controversial by many German actuaries and let to a furious debate in the national literature: the arguments are reviewed by Du Pasquier in his 1913 paper (see below). Karup's classic report of 1893 provides a definitive review of this work and describes clearly and in full detail the valuation of liabilities of and contributions for widow's funds, dealing also with pensions payable on invalidity and pensions payable to orphans (using the "reversionary method"). Thomas Sprague's paper of 1879 is an important contribution as far as the UK literature is concerned, demonstrating how to estimate directly the forces of decrement and the (independent) single decrement probabilities. Sprague's arguments are intuitive, based on an assumption of a uniform distribution of decrements over each year of age, but nonetheless effective. Sprague's paper is a standard which has laid the foundations for subsequent practical applications in the English literature.

Gustav du Pasquier's two papers (1912/1913) took a dramatic step forward by providing a rigorous, mathematical discussion of the invalidity or sickness process with the introduction of a three state illness-death model in which recoveries were permitted. He derived the full differential equations for the transition probabilities and showed that these lead to a second order differential equation of Riccati type which he then solved for the case of constant forces of decrement. The three sections of these papers dealt respectively with a general presentation of the differential equations, the special case where recoveries are not permitted and then the more general case allowing for recoveries. Du Pasquier's work is very significant, presenting an early application of Markov Chains, and laying the foundations for modern actuarial applications to disability insurance, long term care insurance and critical illness, inter alia (Pitacco (1994)).

The work of Karup, du Pasquier and others in mainland Europe meant that the actuarial theory associated with sickness and invalidity insurance developed there more rapidly than in the UK where simple methods based on proportions sick (as advocated by Alfred Watson's 1903 book - see below) remained in use through the nineteenth century and until recently.

Despite the interest and importance of these problems to actuaries and the consistent contributions made to the actuarial literature since the mid nineteenth century, these have essentially been renamed and rediscovered (as the theory of computing risks) by Fix and Neyman (1951) who developed the formulae and results ab initio in the context of Markov processes, with little reference to their actuarial predecessors.

<u>Health and Sickness Insurance</u>

The actuarial contribution to health and sickness insurance is closely linked with the evolution of friendly societies in the UK and corresponding institutions in other countries. The traditional friendly society is a mutual association which gives financial assistance to its members in times of sickness and old age and meets the costs of burial. Its operations are based on insurance principles, the benefits being paid from a fund accumulated from the members' regular contributions.

The origins of friendly societies in the UK may be traced to the craft guilds of the thirteenth and fourteenth centuries. But the most significant factor in their later development was the Industrial Revolution, which was accompanied by widespread poverty and insecurity. At this time, a rapid population increase (attributable to a decline in mortality rates) provided a source of cheap and easily exploitable labour. In the absence of state provision through social insurance schemes and of occupational pension or sick pay schemes, friendly societies, established through the initiative of working men, were the only form of financial security available. The development of friendly societies in the UK in the nineteenth century was paralleled elsewhere, for example in the USA and France and by the miners mutual help societies of Germany and Austria which date from the eighteenth century.

In the early days, the financial arrangement of these societies was rudimentary. The fact that the average cost per member of payments on the contingencies covered tends to increase as age increases was generally ignored. The contributions were often fixed on the levy principle (i.e. the cost of current claims was met by a uniform charge on all active members, regardless of age) leading to classic instances of adverse selection and financial ruin (Magee (1958), Cummins et al (1982)).

The actuarial profession began to be concerned with friendly society finance during the nineteenth century. As their techniques developed, actuaries advised on the rates of contribution and on the accumulation of funds to meet the future liabilities being promised.

The first attempt to produce age-related sickness rates was made by Richard Price (published in 1792 in the fifth edition of his classic book) at the request of a House of Commons Committee. It is not clear whether the age-related rates are hypothetical or based on the collection and analysis of experience data. The rates were used to produce tables of contributions for given levels of benefit in respect of incapacity for work.

The first investigation into the observed sickness experience of an institution is the report of 1824 from the Highland Society of Scotland. At this time, UK friendly societies were in difficulties through a failure to understand their financial status; in some, capital funds increased regularly for a number of years and benefit payments were then increased without there being a full recognition of the prospective implications arising from benefit levels that might already have been too high. The Highland Society's analysis was crude but did include the average number of weeks sick in decennial age groups, interpolated to give rates for each year of age as well as an occupational analysis of the experience. The rates emerging were much lower than those subsequently experienced - one defect noted by many writers was the improper treatment of the many withdrawals.

John Finlaison's 1829 report on annuities (discussed earlier) contains sickness rates based on six year's data (based on the experience of the London Society up to age 60 and the Highland Society thereafter). Finlaison calculated the present value of a standard set of friendly society benefits at different ages and the equivalent age dependent contribution rates, noting the implications of charging the same contribution rate for all (see above). This work represents the first English investigation into sickness rates.

G Hubbard's book of 1852 contains the first investigation into the sickness experience of friendly societies in France with an analysis by sex and occupation. For example, we find the first published analysis of data on length of hospital stays for each spell of sickness for different types of occupation. (A review of the development of the subject in France at this time is provided by Brown (1855)). Hubbard's work was used subsequently by French friendly societies for the proper calculation of contribution rates.

In 1855, K F Heym published work on the organization of friendly societies, with special reference to Leipzig, in which he advocated the use of premiums dependent on age at entry. He also published age-related sickness rates and invalidity probabilties, based on data collected from local friendly societies, and sets of annual and single premiums for standard patterns of sickness benefit. (Lazarus (1860)). Heym is regarded by some commentators as the *"creator of invalidity insurance science"* (Seal (1977)).

In 1854, Prussia legislated for the compulsory membership of sick relief societies and this legislation was copied by the other Northern states of Germany. In 1871, the owners of railways, mines and factories were made liable by law for injuries (or deaths) caused to the employed by accidents where the sufferers were not themselves culpable. In this context, we include August Wiegand's report which deals with the reorganisation of the invalidity insurance scheme for the German railways. Wiegand based his numerical calculations on the probabilities and rates presented earlier by Heym. He also published an analysis of the mortality and invalidity experience of railway officials and used insurance company data for the costing of invalidity benefits (with adjustments to allow for the select nature of railway workers). In the previous section, we considered Johannes Karup's classic text of 1893 which also provides a clear presentation of the methodology for valuing invalidity benefits.

Alfred Watson's 1903 account of his investigations into the sickness and mortality experience for 1893-1897 of the Manchester Unity Friendly Society has become one of the standard works on sickness insurance. Its methodology dominated UK actuarial practice for almost nine decades. Although methods have now moved on from Watson's approach, it is still possible to appreciate his clear presentation and analysis of data cross-classified by several factors. Earlier investigations had revealed that occupation was an important variable (see above) but Watson's consideration of the combined effect of occupation and region was new.

The friendly society movement has declined in importance during the current century; however, there is now a resurgence of interest in many countries in long term permanent health insurance, long term care insurance for the elderly, critical illness protection and other types of related cover offered by life insurance companies.

Experience Studies and Estimation of Rates

We have already discussed the construction of life tables (for example, Halley's life table, the Northampton and Carlisle life tables, those of Wargentin and Deparcieux). We now consider the investigation of insured lives' mortality and the construction of life tables appropriate to such subsections of the population.

Charles Brand's 1780 edition of John Smart's compound interest tables included (in the Appendix) the first mortality data derived directly from the mortality experience of a life office, in this case the Amicable Society. There are tables of expectation of life and average policy duration at the date of death for different ages at entry.

In 1828, William Morgan reported on the construction of a life table from the mortality experience of the Equitable: this represents the first published life table based on the experience of a life office. The life table was then used to calculate values of annuities as well as single and annual premiums. Earlier attempts had been made by Griffith Davies in 1825 and Charles Babbage in 1826 based on the limited information that was then publicly available.

The concept of time exposed to risk, with proper allowance for new entrants and withdrawals, as a basis for the estimation of rates of decrement, is due to Wesley Woolhouse's important 1839 report on Mortality in the Indian Army. This effectively began the use and development of the theory of exposed to risk in the actuarial literature.

The Seventeen Life Assurance Offices' Table of 1843 was the first mortality experience to be derived from the combined data from a number of insurance companies and hence is the first in a long line of similar investigations that have been complied in the last 150 years worldwide. The Tables were prepared under the guidance of a Committee comprising, inter alia, Griffith Davies, Benjamin Gompertz, Joshua Milne and Wesley Woolhouse. The report included here describes the process of data collection from the contributing insurance companies and the forms used and the method of compilation of the number exposed to risk based on Woolhouse's 1839 method. A number of sub-divisions of the data were investigated. It is interesting that the assured females experienced higher mortality rates than the males. The investigation was based on policies rather than lives (as are most modern descendants) which makes interpretation more difficult because of the presence of duplicate policies. The average duration of policies included was 5½ years so that the final tables represented a lower level of mortality than could be expected to prevail in the long run. The report includes comparison with other standard tables in use at the time, including the Northampton and Carlisle tables and the Equitable's experience life tables as produced by Morgan and Davies (see above). The methods used in the construction of these tables are described further by Woolhouse (1867). These life tables were widely used, particularly overseas - for example the valuation standard adopted by Elizur Wright in Massachusetts following the law of 1858.

The original report and tables did not contain any monetary functions and before the Committee had reached a conclusion on what to calculate and publish, Jenkin Jones seized the initiative and published a *"series of tables of annuities and assurances calculated from a new rate of mortality amongst assured lives"*.

Charles Gill's report of 1851 represents the first report on the mortality experience of a life insurance company in the USA. He compared the actual experience with that expected according to the life table used in the premium basis (by implication, the Carlisle Life Table of Milne). He drew attention to the adverse experience for risks in California. In his report, there are comparisons of mortality experience between the geographical areas as well as with the combined experience of a group of 15 life insurance companies in England. Contemporary reviews of the mortality among American assured lives are provided by Brown (1860) and McKay (1870), both of whom mention the important contributions of Gill and Homans. We note here that in later work, Sheppard Homans reported on mortality differences between class of policy.

Similar investigations and life tables constructions were taking place in other countries with developing insurance industries. Thus, Hopf (1855) in his extensive review of the operations of the Gotha Life, reported on the mortality experience, in comparison with life tables from relevant population studies (England; France; Belgium; Hanover; Saxony) and insurance investigations (Morgan's Equitable; Seventeen Offices; Prussian Widows' Fund for the period 1776 to 1845 - the first life tables, for the 58 year period up to 1834 were published by Rechnungsrath Brune in 1837 in Berlin).

Theodor Wittstein's 1862 paper is an important theoretical contribution to the theory of exposed to risk and the estimation of mortality rates. He considered a year of age running from age x to age x+1 and assumed that A lives start the investigation at exact age x, that B lives enter the investigation during the year of age and that C lives *"escape from observation"* during the year of age. It is assumed that d deaths occur during the year of age to this changing group of lives. Deterministic and random exits from observation are aggregated together (Seal (1977)). The two components of the difference (B-C) are assumed to be uniformly distributed over the year of age. An expression for the expected number of lives remaining under observation at age x+1 is obtained and equated to the actual number of lives present at age x+1. Thus, as Puzey (1993) argues, Wittstein has determined expected deaths using observed new entrants and withdrawals, and equated this expression to actual deaths applying similar principles to those introduced subsequently by Cantelli (1914). To derive an expression for q_x, Wittstein then introduced two alternative assumptions: a uniform distribution of deaths over the year of age or a constant force of mortality over the year of age. Both lead to the following, familiar, albeit approximate result:

$$q_x \simeq \frac{d}{A+\frac{1}{2}(B-C)}.$$

Wittstien also identified the appropriate error term. (The expression becomes exact if the Balducci assumption is used but this implies that the force of mortality decreases throughout the year of age).

William Elderton's paper of 1906 on spurious selection dealt with the problems arising from the amalgamation of data over a long period of time and was a critique of the prevailing ideas in the UK on the construction of mortality tables. The possible correlation between mortality rates, years of entry and years of death was discussed as the source of this selection. He also discussed the difficulties of identifying the end of a select period in data where heterogeneity is present: difficulties which have persisted through the current century (for example, see Joint Mortality Investigation Committee (1974)).

This work led on to the joint paper of 1912 written by Elderton and Richard Fippard which showed how mortality rates could be found from assurance data by the methods used to find rates of mortality from censuses and deaths records in the general population. They also showed how this *"census method"* could be adapted in the case of a prolonged investigation to give a continuous (and hence up-to-date) mortality investigation. These ideas led to arrangements for the continuous collection of the experience of annuitants and assured lives which were agreed by the Institute and Faculty of Actuaries in 1914. The implementation was delayed by the Great War but in 1924 the Continuous Mortality Investigation was established.

Paul Böhmer's work of 1912 represents a remarkable breakthrough that was subsequently ignored for many years, until re-invented by Kaplan and Meier (1958). He developed the product-limit estimator for mortality rates and avoided some of the assumptions that Woolhouse and Wittstein and others had been forced (implicitly or explicitly) to make. Instead of introducing an external and deterministic rule for creating sub intervals of age, Böhmer divided his age interval (say a year) into sub intervals each of which contained one decrement or one accession. Thus, it is then not necessary to impose any mortality assumption on the behaviour of mortality within the interval being considered. Further details on this approach are provided by Seal (1981) and Puzey (1993).

Graduation of Decremental Rates

Graduation may be regarded as the principles and methods by which a set of observed (or crude) probabilities are adjusted in order to provide a suitable basis for inferences to be drawn and further practical computations to be made (e.g. life tables to be constructed from sets of probabilities of death by age).

The fundamental justification for the graduation of a set of observed probabilities like $\mathring{q}_x$ is the premise (suggested by experience of nature) that, if the number of individuals in the group on whose experience the data are based, n_x, had been considerable larger, the set of observed probabilities would have displayed a much more regular progression with x. In the limit, with n_x indefinitely large, the set of probabilities would thus have exhibited a smooth progression with x. Therefore the observed data may be regarded as a sample from a large population so that the observed probabilities, derived there from, are subject to sampling errors. Providing these errors are random in nature, they may be reduced by increasing the size of the sample and thereby extending the scope of the investigation. A simpler, cheaper and more practicable alternative is often to use graduation to remove these random errors (rather than increasing the sample size).

Taking the case where n_x is not large enough for the above estimate $\mathring{q}_x$ to be sufficiently reliable, a statistical method is needed which combines the information at related values of x.

The actuarial methods of graduation developed in the literature can be described broadly as (a) graphical methods, (b) summation and adjusted-average methods and (c) parametric methods.

We here draw attention to a few of the major contributions made to the extensive literature on this subject. There are many early examples of rudimentary examples of graduations including Joshua Milne (1815) for the Carlisle life table, Thomas Young (1826) and Francis Corbaux (1833).

We begin with Johann Lambert's work of 1765 (volume I) where he graduated the value of l_x, at decennial ages, which he had calculated from the deaths recorded in the London Bills of Mortality for 1753-8. Two methods of graduation and/or interpolation are discussed. The first was a geometric method for introduction *"osculating parabolas"* between any two points, given the two points on the curve and the tangents at those points. The second was a method of fitting a polynomial of fifth degree to represent a section of the curve which was then able to *"hang together"* with the corresponding polynomials for the immediately preceding and succeeding sections of the curve (Daw (1980)). This methodology is effectively what came to be known as "osculatory interpolation", and was re-invented by Thomas Sprague (see below).

John Finlaison's 1829 work on life annuities included the first life table consisting of graduated observations at individual ages. His eventual formula (its derivation is not explained) is based on overlapping piecewise linear arcs extending over nine successive values, with eight of the nine values being used in the next arc (Seal (1982)), and thus represents the first published example of a graduation by the adjusted-average method.

The history of graduation by piecewise cubic polynomials is discussed fully by Seal (1982) and readers are referred to this excellent review for a more detailed discussion of the issues and contributions.

Wesley Woolhouse's paper of 1865 (the third part of a major work on interpolation and graduation) presented a detailed exposition of graduation of mortality rates using summation formulae, stressing the conceptual differences between graduation and interpolation. He considered the case where the fourth differences of the corrections $v_x = \mathring{q}_x - q_x$ to an observed series of rates had small values and proposed to minimize $\sum v_x^2$ in terms of $\Delta^4 v_x$ and thus obtain estimates of v_x and hence of q_x. Seal (1982) demonstrates that the equations for $\hat{q}_x$ are equivalent to those which arise from fitting piecewise cubic polynomials by least squares to equidistant observations. In later work, Woolhouse (1870) adapted these methods for the graduation of the *"healthy lives"* life tables (H^{MF}) published by the Institute of Actuaries and based on the mortality experience of 20 British life offices. It was based on piecewise cubic polynomials extending over fifteen years of age.

The first least squares estimate of a mortality parameter was Moser (1839) who estimated the parameter a in his law

$$l_x = 1\text{-}ax^{1/4} \quad (0 \leq x \leq 25)$$

by unweighted least squares using values of l_x derived by Kersseboom from life annuitants in Holland and West-Friesland. (Seal (1980)).

Hoem (1982) has argued that the first use of the weighted least squares approach to estimating the parameters for a mortality rate, namely the force of mortality

$$\mu_x = (\alpha+\beta x)\, e^{-\delta x} + \gamma e^{\lambda x} \quad (10 \leq x \leq 95)$$

is attributable to Ludvig Oppermann. This work was kept secret but approximately dated as 1870 (by Jorgen Gram). The first published work on this subject is however due to Seth Chandler in 1872 who used this method with weights proportional to the reciprocals of the variances of the rates to fit a Makeham curve for the central death rate $m_x = A+Bc^{x+1/2}$ for fixed c to the published mortality experience of the Mutual Life of New York. The weights were calculated from a guessed value of c and rough values of A and B obtained from a section of the data. Chandler also showed that if c had to be estimated, then the first method could be used for the initial parameter values and an updating iterative procedure could be used to estimate the parameters. He also approximated the standard errors of the Makeham parameters. Seal (1980) describes this paper as the *"only such fitting described in English"* in the nineteenth century. The late nineteenth century saw a number of writers consider the fitting of an actuarial function by weighted least squares, including Karup (1884) and subsequently Galbrun (1906). As noted by Seal (1980) in his review of this subject, only Chandler and Galbrun approximated the standard errors of the Makeham parameters and the replacement of the initially calculated weights by updated corrections was not attempted by any author at this time, although Chandler does discuss how this might be accomplished.

Despite Chandler's paper re-appearing in the Journal of the Institute of Actuaries, the contemporary UK approach followed the method of moments, which nowadays would only be used as a first approximation, and was continued by British and American actuaries who adopted Hardy's (1909) version of the method of moments approach.

The use of symmetrical moving weighted average formulae to smooth equally spaced observations of a function u_i say of one variable (for example, a mortality rate), which generalized Woolhouse's summation formulae, was systematically investigated in a series of papers by the American statistician, Erastus de Forest, two of which appear here (1873, 1875). His work was published in obscure places (at this time there were no American actuarial journals and, for some reason, he chose not to look overseas) and was rescued from total oblivion by the efforts of Wolfenden (1925). De Forest's principal innovation was to introduce optimality criteria into the problem of estimating the coefficients. So he aimed to replace each u_i by a symmetric linear function of surrounding values, say

$$w_i = \alpha_0\, u_i + \alpha_1\, (u_{i-1} + u_{i+1}) + \ldots + \alpha_m\, (u_{i-m} + u_{i+m})$$

De Forest supposed that the observed u_i differed from the true values U_i by random errors which he assumed were independent and of equal variance. He then assumed that the U_i sequence was smooth in the sense that any 2m+1 U_is differed little from a polynomial of degree j in i (age). In the 1873 paper, he concentrated on m=2 and j=3 i.e. any 5 consecutive U_i's could be represented *"very nearly"* by a cubic in i. This approach retains considerable flexibility relative to assuming a particular parametric model or assuming that the U_i are cubic for all ages. He first considered a least squares approach to determine α_0, α_1 and α_2. He solved this problem and then noted that he had been anticipated to some extent by the 1867 work of the Italian astronomer Schiaparelli. But he went further and noticed that the minimum mean square error criterion need not produce a very smooth relation globally, notwithstanding the assumption of being a cubic locally. He then proposed a criterion based on smoothness: minimize the expected value of $(\Delta^4 w_i)$ subject to the constraint that

$$U_i = \alpha_0 U_i + \alpha_1 (U_{i-1} + U_{i+1}) + \alpha_2 (U_{i-2} + U_{i+2})$$

and he solved this problem for several different cases, taking pains to record the underlying assumptions needed (Stigler (1978)). Stigler notes that *"de Forest's introduction of this measure of smoothness as an optimality criterion was well ahead of its time"*, in statistical terms. De Forest showed that the equations used by Woolhouse (1870) in the graduation of the H^{MF} life tables fitted into his proposed general scheme and also that his own corresponding graduation was superior to Woolhouse's (as judged by various optimality criteria). De Forest also investigated the possible statistical tests of a *"good"* graduation.

These methods were developed further by Wolfenden (1925). Whittaker (1923) suggested an alternative method of graduation, which has been very popular in North America where it is widely called Whittaker-Henderson graduation (see also Henderson (1924)). This can be regarded as a Bayesian approach to graduation, resulting in the minimization of

$$\sum_{i=1}^{n} (u_i - w_i)^2 + \beta \sum_{i=1}^{n-3} (\Delta^3 w_i)^2$$

combining a measure of goodness of fit of the graduation to the observations and a measure of smoothness. These important methods, however, fall outside of our (self-inflicted) period of study. The problems caused by moving weighted average methods failing to give smoothed values of the first m and last m observations have recently been addressed by Greville (1981), among others. Graduation by summation formulae was further developed in the UK by Spencer (1907) and Lidstone (1907, 1908a).

The definitive paper on graphic graduation appears at this time: Sprague (1887). It is dismissive of all piecewise cubic polynomial curve fitting and sparks a controversial debate with Woolhouse and Higham (Seal (1982)). An important legacy was that Sprague's judgement of the successful fit of a graduation by comparing the succession of *"actual and expected"* deaths became one of the standard tests of a graduation (Seal (1982)), again following on from Richard Price's suggestion dating from the 1770's.

Thomas Sprague's paper of 1881 rediscovered (following Lambert) osculatory interpolation, showing how formulae could be devised to ensure continuity of the first derivatives of overlapping interpolation curves. It is arranged for these curves to join smoothly or *"kiss"*. Cubics were used to ensure the continuity of the first derivatives and fifth degree polynomials to ensure the continuity of both the first and second derivatives. Lidstone (1908b) provides an elegant alternative derivation of the osculatory interpolation formulae: this is discussed intuitively in his contribution to the discussion of King's paper in 1908. Osculatory interpolation was used as a method of graduation by George King for the English Life Tables No 7 and 8 which were based on the 1911 Census. (Registrar-General (1914)).

The history of the first three decades of the use of osculatory interpolation in the graduation of life tables has been informatively sketched by Buchanan (1929). This idea of considering the degree of contact desirable in a graduation leads on to the use of splines, in particular cubic splines, for interpolation and smoothing (see Schoenberg (1964) for theoretical developments and Greville (1967) for a review). Also the move away from formal parametric graduation has been more recently encouraged by the development of modern non-parametric techniques - for example the kernel methods of Copas and Haberman (1983).

Concluding Comments

So the twentieth century begins and our discipline sees a flurry of important contributions. It is interesting to note that some of these led to the establishment of new bodies of knowledge which have flourished in both theoretical and practical terms. Others were, for various reasons, lost or ignored so that subsequently a rediscovery of the ideas was needed before progress could be made.

In the former group, we see the development of life assurance reserving ideas following Coutts and the widespread use of the numerical rating system following Rogers and Hunter. The risk theory of Lundberg and the foundations of credibility theory as laid down by Mowbray and Whitney have led to widespread developments over the past 80-90 years. The work of Elderton and Fippard on mortality rate estimation and of Watson on sickness measures became standard at least in the UK, although through the perspective of modern statistical methods we see the flaws in both methodologies. In the pensions area, Manly and M'Lauchlan's works have had the foundations for modern practice in defined benefit provision.

Elsewhere, we see a more problematic progression. In investment, Bailey's ideas were not revisited until Pegler in 1948 and Bachelier's work was largely ignored. We now see the seminal work of Black and Scholes and Merton on option pricing as direct descendants of Bachelier. It is natural to compare Bachelier with Lundberg : both studied problems connected with stochastic processes in continuous time about thirty years before this concept had been rigorously defined, and the work of both was ignored for decades and practically forgotten. Similarly, Moll's short note suggests the later ideas of duration, matching and immunization but again was not immediately pursued. On the modelling side, Du Pasquier's work on Markov models and multiple state applications was not resurrected in the actuarial field until Amsler (1968) and Hoem (1969) and now these methodologies have universally superseded the Watson Manchester Unity approach to sickness rates. De Forest's work on graduation was published in somewhat obscure media and was lost until Wolfenden rediscovered it and from then on there have been significant developments, through Whittaker-Henderson methods, kernel methods and other non parametric approaches. Similarly, the use of splines in modern statistical modelling and graduation can be traced back to the idea of osculatory interpolation due to Sprague (and perhaps Lambert before him). We finally note the contribution of Böhmer on product-limit mortality estimators which was ignored for decades and was reinvented by Kaplan and Meir in 1958 in their seminal contribution to the field of medical statistics.

APPENDIX

LANDMARK WORKS IN THE HISTORY OF ACTUARIAL SCIENCE

LISTING BY THEME AND CHRONOLOGY

LIFE TABLES AND SURVIVAL MODEL

200 Domitius Ulpianus, *Ulpian's Table.*

1662 John Graunt, *Natural and Political Observations Upon the Bills of Mortality,* (London; John Martin).

1669 Christiaan and Ludwig Huygens, *Extracts from Letters.*

1671 Johan de Witt, *Value of Life Annuities,* (Amsterdam).

1693 Edmund Halley, *An Estimate of the Degrees of the Mortality of Mankind, Drawn from Curious Tables of the Births and Funerals of Breslaw; with an Attempt to Ascertain the Price of Annuities Upon Lives.* Philosophical Transactions of Royal Society, vol 17, pp 596-610.

1709 Nicholas Bernoulli, *De Usu Artis Conjuctandi in Jure,* (Basel; doctoral thesis).

1737 John Smart, *A letter to George Heathcote re enclosing tables extracted from Ye Bills of Mortality from the last ten years, and from them showing the Probabilities of Life, in order to estimate annuities, etc.*,(Presented to Royal Society in May 1738).

1740 Nicholas Struyck, *Appendix to Introduction to General Geography together with Astronomy and Other Matters*, (Amsterdam; Isaak Tirion).

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