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Optimal Retention Levels, Given the Joint Survival of Cedent and Reinsurer

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Abstract

A certain volume of risks is insured and there is a reinsurance contract, according to which claims and total premium income are shared between a direct insurer and a reinsurer in such a way, that the finite horizon probability of their joint survival is maximized. An explicit expression for the latter probability, under an excess of loss (XL) treaty is derived, using the improved version of the Ignatov and Kaishev's ruin probability formula (see Ignatov Kaishev and Krachunov, IME 29, (2001), 375-386) and assuming, Poisson claims arrivals, any discrete, joint distribution of the claims and any increasing, real premium income function. An explicit expression for the probability of survival of the cedent only, under an XL contract is also derived and used to determine the probability of survival of the reinsurer, given survival of the cedent. The absolute value of the difference between the probability of survival of the cedent and the probability of survival of the reinsurer, given survival of the cedent is used for the choice of optimal retention level. Formulae for the expected profits of correspondingly, the cedent and the reinsurer, given their joint survival up to the finite time horizon, are also given. The quota share contract is also briefly considered under the same model. Extensive, numerical comparisons, illustrating the performance of the proposed reinsurance optimality criteria are presented.

Keywords: excess of loss, optimal reinsurance, optimal retention levels, probability of joint survival of cedent and reinsurer

1.Introduction

Optimal reinsurance has been the subject of a large number of risk-theoretical papers and monographs. We will restrict ourselves to mentioning the works of De Finetti (1940), Verbeek (1966), Buhlmann (1970), Gerber (1979), Straub(1980) , Waters (1979), Centeno (1986, 1988, 1991, 1997), Centeno and Simoes (1991), Bowers, Gerber, Hickman Jones and Nesbitt(1997), Hesselager (1990), Taylor (1992), Dickson and Waters (1996, 1997), Kaluszka (2001), Krvavych (2001), Gajek and Zagrodny (2000). Although reinsurance is a risk sharing arrangement between a primary insurer (cedent) and a reinsurer, the quoted authors ignore the reinsurer's interests and consider reinsurance conditions, optimal with respect to the interest of the ceding company only. Thus, in the case when the probability of ruin (survival) was assumed as the optimality criterion it concerned the primary insurer only and was approximated by the upper Lundberg bound in a compound Poisson model. This was the approach of Gerber (1979) who showed that the cedent will minimize the probability of his eventual ruin if he chooses the excess of loss reinsurance, among all individual reinsurance treaties provided. Other works, considering optimal reinsurance, with respect to approximations of the ruin probability of the primary insurer, are those of Andersen (2000), Krvavych (2001), Waters(1979), Centeno (1986, 1997), Dickson and Waters (1996, 1997) and earlier the monographs of Bhulmann (1970) and Straub (1980).

A stochastic dynamic control approach to optimal reinsurance was recently demonstrated in a series of papers by Schmidli (2001, 2002), Hipp and Vogt (2001), Taksar and Markussen (2002), Asmussen, Hojgaard and Taksar (2000), Mnif and Sulem (2001) and Bauerle (2002). The authors consider dynamic, controlled diffusion models of proportional or excess of loss reinsurance, optimal from the point of view of solely the direct insurer, minimizing his probability of ruin (see Schmidli (2000, 2002), Hipp and Vogt (2001),Taksar and Markussen (2002)), or maximizing the expected discounted dividend pay-out (see Asmussen, Hojgaard and Taksar (2000), Bauerle (2002).

Borch seems to have been the first to note the importance of the obvious fact (see Borch 1969, p. 295) "... that there are two parties to a reinsurance contract, and that these parties have conflicting interests. The optimal contract must then appear as a reasonable compromise between these interests". Inspired by the fundamental work of von Neumann and Morgenstern (1944), Borch (1960, 1969)) looked at the problem of optimal reinsurance econometrically, applying the game-theoretic approach within the (re)insurance market framework. He considered (see Borch 1960) the so called reciprocal reinsurance treaty, according to which two companies share the total of their aggregate claim amounts in such a way that the product of their gains in utility due to the treaty is maximized. This was what Borch called the Nash solution to the optimal reinsurance problem. He considered quota share and stop loss risk sharing arrangements and provided some concrete examples of optimal retention levels in these cases. Buhlman (1980, 1984) has also developed general economic models of the (re)insurance market, considering the interests of all agents on the market, insurers, reinsurers and buyers of direct insurance. More recent papers in which (re)insurance problems are modelled as cooperative games with stochastic payoffs are due to Suijs, Borm, and De Waegenaere (1998) and Aase(2002). For further references on the subject we refer the interested reader to Aase (2002), who presents a detailed up to date overview of the current risk exchange models of a (re)insurance market, and considers competitive equilibrium, Pareto optimality and representative pricing of risk-sharing contracts, based on the game-theoretic approach.

Not surprisingly, optimal reinsurance has also been at the focus of the attention of practicing actuaries and reinsurance experts, working in general insurance and reinsurance companies. An example of how some of them see the problem of determining optimal retention levels and designing whole reinsurance programs can be found in the recent publication of the Swiss Reinsurance Company, authored by H. Schmitter (2001).-

The author is not concerned about retention levels affecting the optimal amount of risk the reinsurer would be prepared to take, to ensure his own safety and financial stability. He has rather concentrated on the interests of the primary insurer, demonstrating simple methods of how the retentions can be set optimally, to effectively maintain fluctuations in the results of the primary insurer at an acceptable level. The paper is based on the early work of De Finetti (1940) where the variance of the aggregate claims amount (the loss burden) is used as a measure of the risk, associated with a reinsurance portfolio. Optimal retentions are then described as the retentions minimizing the total variance over all the portfolios, supported by the primary insurer, keeping the reinsurance price (i.e. the fluctuation and expenses loading of the risk premium) at a certain fixed level.

In the present paper, we will take an approach, coherent with the remark of Borch (1969) quoted above. We will consider optimal reinsurance from the point of view of both the interests of the primary insurer and the reinsurer, as two parties jointly liable for the risk they share. We assume that a mass of risks is insured, thus generating individual claims and producing corresponding flow of premiums to an insurance company. In what follows, we will some times call these two flows, correspondingly, the flows of originally occurring claims and premiums. We will also assume that the two flows, "adequately" correspond to each other in the sense, that the company has taken into consideration actuarial methodology (premium rating principles, estimates of claims distributions, based on historical claim data. etc.), market constraints and other economic factors and business rules of thumb in determining its premiums. We will further assume that the insurance company wishes to insure its portfolio of risks, seeking an excess of loss (XL) reinsurance contract. Then, we will concentrate on defining the conditions of such a contract, according to which, the total premium income and the aggregated claims are shared between the ceding company and the reinsurer in an way, optimal with respect to the interests of both parties. This seems to be an approach, more natural than focusing only at the interest of the direct insurer, as has been the case in most of the past research quoted above. This is because the insurer and the reinsurer can be considered as partners, having common objectives in managing the risk they share. For example, maximizing the survival probability is one of the important conditions for achieving solvency and financial stability, objectives of top priority for both insurance and reinsurance companies.

The paper is organized as follows. In the next section, we have considered first the simple case when the primary insurer seeks no reinsurance, i.e., covers each claim by himself and fully retains the corresponding flow of premiums. His probability of ruin (survival) up to a finite horizon is then given by the improved version of the Ignatov and Kaishev's formula (see Ignatov Kaishev and Krachunov 2001), assuming the individual claims to him have Poisson arrivals and their severities have any discrete joint distribution. This result, together with the ruin probability formulae of Picard and Lefevre (1997), which applies to the same model but for independent, individual claims only, are given in Section 2 for further use in proving the main results of Section 3. In Section 3 we have considered the simple, unlimited XL reinsurance contract, and have defined two risk processes, of the cedent and of the reinsurer. With respect to these two risk processes, we have further introduced two optimality criteria for setting the XL retention level. They both can be viewed as measures of the risk, taken by the cedent and the reinsurer and are explored in Subsection 3.1.Thus, retention levels may be considered optimal if they minimize the these risk measures. The first such measure is the probability that both the primary insurer and the reinsurer will survive up to a finite time horizon . As we have shown, this joint survival probability is a function of the retention level. The latter is said to be optimal if it maximizes the probability of joint survival up to a finite time horizon of the cedent and the reinsurer. An alternative optimality criterion, introduced in Section 3, is the absolute value of the difference between the probability of survival to a finite moment in time of the cedent and the probability of survival of the reinsurer, given survival of the cedent up to that moment. The optimal retention level is the value which minimizes this difference. Theorems 1 and 2, proved in Section 3, give explicit expressions for these two criteria of optimality of the retention level, assuming the originally occurring claims to the primary insurer, i.e., claims before reinsurance, have any discrete joint distribution and their arrivals form a Poisson point process. Section 3 is further devoted to the extensive numerical investigation and sensitivity analysis of our two optimality criteria, given by formula (7), established in Theorem 1and by formula (18), established in Theorem 2. Both formulae are shown to be numerically efficient and convenient for practical evaluations. This is confirmed by the numerical results and graphical illustrations of optimal retention levels, obtained on their bases, for the case of logarithmically and multinomially distributed claims. In Subsection 3.2, the expected profits of correspondingly, the cedent and the reinsurer at the end of the time horizon, given their joint survival to that moment are considered, as measures of performance of the two parties. Explicit expressions for these expected profits are established by Theorems 4 and 5, correspondingly for the cedent and the reinsurer, and their numerical performance is tested. The numerical example and 3D graphs of the expected profits (see Fig. 7) illustrate the choice of an optimal pare of values of the retention and the proportion in which the original premium income is split between the two parties. Borch's point that the interests of the two parties are contradictory is clearly illustrated on the 3D plot of the two profits given in Fig 7 . Section 4 is devoted to the quota share contract. Theorem 6 therein establishes, that the choice of the quota share retention level does not affect our first optimality criterion, i.e., the probability of the joint survival of the cedent and the reinsurer, which coincides with the probability of survival of the primary insurer without a QS contract. Finally, Section 5 contains conclusions, comments about the practical applicability of the results, and some questions for future research.

2. Preliminaries

We will consider a portfolio of insurance risks with claim severities, modeled by the integer valued r.v.s. $W_1, W_2, \ldots$, occurring with inter-occurrence times $\tau_1, \tau_2, \ldots$, assumed identically, exponentially distributed r.v.s with parameter, λ. The joint distribution of $W_1, W_2, \ldots$, is denoted by $P(W_1 = w_1, \ldots, W_i = w_i) = P_{w_1,\ldots,w_i}$, where $w_1 \geq 1$, $w_2 \geq 1$, $\ldots w_i \geq 1$, i=1,2,....and the number of claims is represented by a counting process $N_t = \#\{i : \tau_i + \ldots + \tau_i \leq t\}$, where # denotes the number of elements in the set $\{.\}$. The r.v.s $W_1, W_2, \ldots$, are assumed to be independent of N_t.Then, the risk (surplus) process R_t, at time t, is given by

$$R_t = h(t) - \sum_{i=1}^{N_t} W_i ,$$

where $h(t)$ is a nonnegative, increasing, real function, defined on R_+ , representing the aggregated premium income up to time t, to be received for carrying the risk associated with the entire portfolio. We will be concerned with the probability of non ruin (survival) $P(T > x)$ in a finite time interval $[0, x]$, $x > 0$, where the time T of ruin is defined as

$$T := \inf\{t : t > 0,\ R_t \leq 0\} . \qquad (1)$$

The premium income function $h(t)$ is defined aggregating, up to time t, the premiums on individual contracts, rated using the well known actuarial (net) premium rating principles (see e.g. Gerber 1979) and/or other rating techniques, used in practice. Thus, it will be assumed that the function $h(t)$ "adequately corresponds" to the total amount of claims, generated by the insurance portfolio up to time t. If the primary insurer is not buying reinsurance then, as shown in Ignatov, Kaishev and Krachunov (2001), his survival probability $P(T > x)$ may be expressed as

$$P(T > x) = e^{-x\lambda} \sum_{k=1}^{n} \left(\sum_{\substack{w_1 \geq 1, \ldots, w_{k-1} \geq 1 \\ w_1 + \ldots + w_{k-1} \leq n-1}} P(W_1 = w_1, \ldots, W_{k-1} = w_{k-1}; W_k \geq n - w_1 - \ldots - w_{k-1}) \sum_{j=0}^{k-1} (-1)^j b_j(z_1, \ldots, z_j) \lambda^j \sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!} \right), \tag{2}$$

where,

$n = [h(x)] + 1$, $[h(x)]$ is the integer part of $h(x)$,

$v_{n-1} \leq x < v_n$, $v_i = h^{-1}(i)$, for $i = 0, 1, 2, \ldots$, noting that $0 = v_0 \leq v_1 \leq v_2 \ldots$, and k is such that $w_1 + \ldots + w_{k-1} \leq n-1$, $w_1 + \ldots + w_k \geq n$, $(1 \leq k \leq n)$, $z_l = v_{w_1 + \ldots + w_l}$, $l = 1, 2, \ldots$ and $b_j(z_1, \ldots, z_j)$ is defined recurrently as

$$b_j(z_1, \ldots, z_j) = (-1)^{j+1} \frac{z_j^j}{j!} + (-1)^{j+2} \frac{z_j^{j-1}}{(j-1)!} b_1(z_1) + \ldots + (-1)^{j+j} \frac{z_j^1}{1!} b_{j-1}(z_1, \ldots, z_{j-1}), \tag{3}$$

with $b_0 \equiv 1$, $b_1(z_1) = z_1$.

In the special case of independent, identically distributed claims $W_1, W_2, \ldots$ with common distribution $\{P_j, j \in N^*\}$(where $N^*$is the set of natural numbers) i.e., when $P(W_i = j) = P_j$, $i = 1, 2, \ldots, j = 1, 2, \ldots$with mean μ , one can express the probability of survival of the direct insurer, without reinsurance, by applying the Picard and Lefevre (1997) formula

$$P(T > x) = e^{-\lambda x} \sum_{i=0}^{\infty} A_i(x) I_{\{x \geq v_i\}}, \tag{4}$$

where $I_{\{.\}}$ is the indicator of the event $\{.\}$ and $A_i(x)$ $i = 1,2,\ldots.$ are the Appell polynomials, defined as

$$A_0(x) = 1,$$

$$A_i'(x) = \sum_{j=0}^{k-1} \lambda P_j A_{i-j}(x),$$

with

$$A_i(v_i) = 0, \; i > 0.$$

Appell polynomials $A_i(x)$, $i = 1, 2, \ldots$ are expressed as $A_i(x) = \sum_{r=0}^{i} b_r e_{i-r}(x)$, where

$e_i(x) = \sum_{k=0}^{i} \frac{(\lambda x)^k}{k!} q_i^{*k}$, $\quad i \geq 0$, $\quad e_0 = 1$, $\quad q_j^{*k} = P(W_1 + \ldots + W_k = j)$, $\quad k > 0$, $q_j^{*0} = \delta_{j0}$ and b_r, $r = 0, 1, \ldots, i$ are unknown coefficients, obtained by solving the system

$$\sum_{r=0}^{i} b_r e_{i-r}(v_i) = \delta_{i0}.$$

Now, let us assume the direct insurer is seeking reinsurance cover for part of the risk, related to his portfolio of insurance polices. Thus, it will be assumed throughout the sequel that the originally occurring, individual claims, modeled by the r.v.s W_i, $i = 1, 2, \ldots.$ are shared between the ceding insurer and a reinsurer, according to the clauses of a reinsurance contract between them. In the next sections we will consider excess of loss and proportional reinsurance contracts.

3. Optimal excess of loss

Recall, that according to the excess of loss (XL) treaty, the direct insurer covers each individual claim up to a certain retention level M and what exceeds M is covered by the reinsurer. Since here claims are integer valued, the retention level will also take integer values, i.e., $M = 1, 2, \ldots$ .Let us denote by $W_i^c$and by $W_i^r$the parts of the total claim W_i, $i = 1, 2, \ldots$covered correspondingly by the cedent and by the reinsurer. Obviously, $W_i^c = \min(M, W_i)$, $W_i^r = (W_i - M)_+$, and $W_i = W_i^c + W_i^r$, $i = 1, 2, \ldots.$ As is natural with reinsurance contracts, the cedent and the reinsurer share not only the risk but also the total premium income for covering that risk. Thus, we will assume that $h(t) = h_c(t) + h_r(t)$, where $h_c(t)$, $h_r(t)$ and $h(t)$ are increasing (not necessarily strictly) and possibly discontinuous, non-negative, real functions, representing the premium income up to time t of correspondingly, the cedent, the reinsurer and the total original portfolio of risks. We will consider two risk processes, that of the direct insurer and of the reinsurer, which we will correspondingly define as

$$R_t^c = h_c(t) - \sum_{i=1}^{N_x} \min(M, W_i), \tag{5}$$

and

$$R_t^r = h_r(t) - \sum_{i=1}^{N_x} (W_i - M)_+ . \tag{6}$$

The type of optimality problem one may pose with respect to $R_t^c$and $R_t^r$is to find the value of the retention level Mwhich maximizes (minimizes) a certain optimality criterion, assuming $h_c(t)$, $h_r(t)$ and $h(t)$ are fixed and such that $h(t) = h_c(t) + h_r(t)$. The optimality criterion can either be related to the risk, accepted by the cedent and by the reinsurer or to their business performance. One may choose different risk and performance measures and specify the optimality criterion, based on the particular choice. Thus, Section 3.1 is devoted to the "probability of survival" based optimality, as a measure of the risk and in Section 3.2 the optimal retention levels are considered, which maximize the expected profit at time x, given that both the cedent and the reinsurer survive up to that time.

▪ 3.1.1 Retention levels optimal with respect to $P((T^c > x) \cap (T^r > x))$

Let us first consider the probability $P((T^c > x) \cap (T^r > x))$ of joint survival of the cedent and the reinsurer as an optimality criterion. To address the two optimal reinsurance problems 1) and 3) we need an explicit expression for the probability $P\big((T^c > x) \cap (T^r > x)\big)$, which is given by the following

Theorem 1. The probability of joint survival in finite time x of cedent and reinsurer under an excess of loss reinsurance treaty

$$P((T^c > x) \cap (T^r > x)) =$$
$$e^{-x\lambda} \sum_{k=1}^{n} \sum_{\substack{w_1 \geq 1, \ldots, w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n-1}} P(W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1}) \tag{7}$$
$$\sum_{j=0}^{l-1} (-1)^j \, b_j(\tilde{z}_1, \ \ldots, \ \tilde{z}_j) \, \lambda^j \sum_{m=1}^{l-j-1} \frac{(x\lambda)^m}{m!} \quad ,$$

where n is defined as in (2), $\tilde{z}_j := \max(v_{w_1{}^c + \ldots + w_j{}^c}, \, v_{w_1{}^r + \ldots + w_j{}^r})$, $v_{w_1{}^c + \ldots + w_j{}^c} := h_c^{-1}(w_1{}^c + \ldots + w_j{}^c), w_j{}^c = \min(M, \, w_j), v_{w_1{}^r + \ldots + w_j{}^r} := h_r^{-1}(w_1{}^r + \ldots + w_j{}^r), w_j{}^r := \left(w_j - M\right)_+$, and l is such that, $\tilde{z}_1 \leq \ldots \leq \tilde{z}_{l-1} \leq x < \tilde{z}_l$.

Proof. Consider the probability of survival of the cedent, without reinsurance, given by formula (2) and recall (see Ignatov, Kaishev and Krachunov 2001) that the expression

$$e^{-x\lambda} \sum_{j=0}^{k-1} (-1)^j \, b_j(z_1, \ \ldots, \ z_j) \, \lambda^j \sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!}$$

coincides with the conditional probability $P(T > x \mid W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1})$, i.e.,

$$P(T > x \mid W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1})$$
$$= e^{-x\lambda} \sum_{j=0}^{k-1} (-1)^j \, b_j(z_1, \ \ldots, \ z_j) \, \lambda^j \sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!} \tag{8}$$

provided that $v_{w_1 + \ldots + w_{k-1}} \leq x < v_{w_1 + \ldots + w_k}$.

By analogy with formula (2) we can write

$$P((T^c > x) \cap (T^r > x)) =$$
$$e^{-x\lambda} \sum_{k=1}^{n} \sum_{\substack{w_1 \geq 1, \ldots, w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n-1}} P(W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1}) \tag{9}$$
$$\times P((T^c > x) \cap (T^r > x) \mid W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1})$$

By analogy with equality (7) of Ignatov and Kaishev (2000) for the risk process R_t we can write for the risk processes $R_t{}^c$ and $R_t{}^r$

$$
\begin{aligned}
& P((T^c > x) \cap (T^r > x) \mid W_1 = w_1 \equiv w_1{}^c + w_1{}^r, \ ..., \\
& \quad W_{k-1} = w_{k-1} \equiv w_{k-1}{}^c + w_{k-1}{}^r; W_k \geq n - w_1 - ... - w_{k-1}) \\
& = P((T^c > x) \cap (T^r > x) \mid W_1{}^c = w_1{}^c, W_1{}^r = w_1{}^r, \ ..., W_{k-1}{}^c = w_{k-1}{}^c, \\
& \quad W_{k-1}{}^r = w_{k-1}{}^r; W_k{}^c + W_k{}^r \geq n - w_1{}^c - w_1{}^r - ... - w_{k-1}{}^c - w_{k-1}{}^r) \\
& = P\,((\tau_1 \geq v_{w_1{}^c}, \tau_1 + \tau_2 \geq v_{w_1{}^c + w_2{}^c}, \ ..., \tau_1 + ... + \tau_{k-1} \geq v_{w_1{}^c + ... + w_{k-1}{}^c}, \tau_1 + ...+ \tau_k \geq x) \\
& \qquad \cap (\tau_1 \geq v_{w_1{}^r}, \tau_1 + \tau_2 \geq v_{w_1{}^r + w_2{}^r}, \ ..., \tau_1 + ... + \tau_{k-1} \geq v_{w_1{}^r + ... + w_{k-1}{}^r}, \tau_1 + ...+ \tau_k \geq x)) \\
& = P(\tau_1 \geq \max(v_{w_1{}^c}, v_{w_1{}^r}), \tau_1 + \tau_2 \geq \max(v_{w_1{}^c + w_2{}^c}, v_{w_1{}^r + w_2{}^r}), \ ..., \\
& \quad \tau_1 + ... + \tau_{k-1} \geq \max(v_{w_1{}^c + ... + w_{k-1}{}^c}, v_{w_1{}^r + ... + w_{k-1}{}^r}), \tau_1 + ...+ \tau_k \geq x)
\end{aligned}
\tag{10}
$$

From (8) and (10) it is not difficult to deduce that

$$
\begin{aligned}
& P(\tau_1 \geq \max(v_{w_1{}^c}, v_{w_1{}^r}), \tau_1 + \tau_2 \geq \max(v_{w_1{}^c + w_2{}^c}, v_{w_1{}^r + w_2{}^r}), \ ..., \\
& \quad \tau_1 + ... + \tau_{k-1} \geq \max(v_{w_1{}^c + ... + w_{k-1}{}^c}, v_{w_1{}^r + ... + w_{k-1}{}^r}), \tau_1 + ...+ \tau_k \geq x) = \\
& e^{-x\lambda} \sum_{j=0}^{k-1} (-1)^j b_j(\bar{z}_1, \ ..., \bar{z}_j) \lambda^j \sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!}
\end{aligned}
\tag{11}
$$

Now, if l is an index, such that $\bar{z}_1 \leq ... \leq \bar{z}_{l-1} \leq x < \bar{z}_l$ then, we can sum up in (11), with respect to j from 0 to $l-1$, since we consider the events of ruin of the cedent and the reinsurer up to time x only. Hence, we can rewrite (11) as

$$
\begin{aligned}
& P(\tau_1 \geq \max(v_{w_1{}^c}, v_{w_1{}^r}), \tau_1 + \tau_2 \geq \max(v_{w_1{}^c + w_2{}^c}, v_{w_1{}^r + w_2{}^r}), \ ..., \\
& \quad \tau_1 + ... + \tau_{k-1} \geq \max(v_{w_1{}^c + ... + w_{k-1}{}^c}, v_{w_1{}^r + ... + w_{k-1}{}^r}), \tau_1 + ...+ \tau_k \geq x) = \\
& e^{-x\lambda} \sum_{j=0}^{l-1} (-1)^j b_j(\bar{z}_1, \ ..., \bar{z}_j) \lambda^j \sum_{m=1}^{l-j-1} \frac{(x\lambda)^m}{m!}
\end{aligned}
\tag{12}
$$

The assertion of the Theorem now follows from (10), (12) and (9).□

Numerical illustrations and sensitivity analysis

Now, we are in a position to solve the two optimal reinsurance problems 1) and 3). Deriving analytical expressions for the optimal retention level M in 1) and the optimal split of $h(t)$ into $h_c(t)$ and $h_r(t)$ in 2) is in general a formidable task, taking into account the complexity of (7) with respect to M , $h_c(t)$ and $h_r(t)$ and their inverse functions. Alternatively, one can find the optimal solutions of problems 1) and 3) directly using (7). For the purpose, formula (7) has been implemented as a *Mathematica* module, named `OpXlJo` and used in what follows for numerical investigations.

Logarithmically distributed individual claims

To illustrate the numerical solution of problems 1) and 3) we have chosen , without loss of generality, the following test model, which we will use for testing purposes throughout the sequel:

*Individual claims:*independent, logarithmically distributed with parameter α, i.e.,
$W_i \sim \text{Log}(\alpha)$, $P(W = i) = -\alpha^i / \ln(1-\alpha)\, i$;
Premium income functions: $h_c(t) = c_c\, t$, $h_r(t) = c_r\, t$ and $h(t) = c\, t = (c_c + c_r)\, t$, $(u = 0)$;
Claim inter-arrival times τ_i $i = 1, 2, \ ...$: exponentially distributed with parameter λ. (13)

We have computed $P((T^c > x) \cap (T^r > x))$, by means of **OpXlJo**, with respect to model (13) as a function of the retention level M, for different, fixed values of c_c and c_r, such that $c = c_c + c_r$. Fig. 1 illustrates graphically this function for $\alpha = 0.9$, size of the time interval $x = 8$, $\lambda = 0.4$, total premium rate $c = 1.75$, and values of the reinsurer's premium rate $c_r = 0.7, 0.5, 0.25, 0.15$. Dots on the graphs correspond to $P((T^c > x) \cap (T^r > x))$ evaluated at retentions $M = 1, 2, ..., 10$, whereas, maximum of the probability of joint survival is achieved at the bigger (red) dots, for optimal retention levels $M_{\text{opt}} =$ 10, 9, 4, 2 corresponding to $c_r =$ 0.15, 0.25, 0.5, 0.7, which means that optimal retention levels decrease with the increase of the proportion of the total premiums, transferred to the reinsurer, as is natural to expect, since the reinsurer will cover larger part of each claim for larger part of the total premium income passed on to him by the direct insurer.

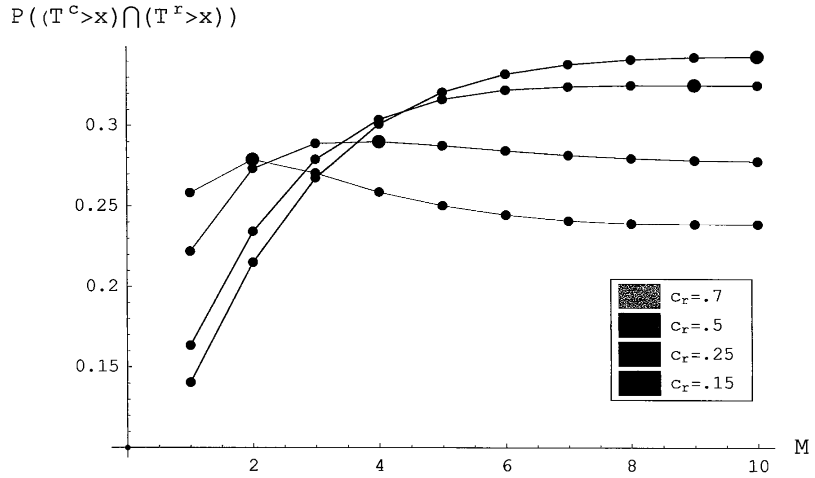

Fig.1. Optimal retentions with respect to the probability of joint survival of cedent and reinsurer, for fixed values of c_r (and c_c).

Optimal retention levels for the values of $c_r = 0.15, 0.25, 0.5, 1.0$, for increasing values of α are summarized in Table 1. As can be seen, for fixed value of c_r, optimal retentions increase as α increases, since the expected value of the claim severities increases with α, causing higher proportion of each claim to be retained by the cedent in order to keep the joint probability of survival maximal. Table 2 illustrates that for fixed c_r the optimal retention levels decrease as the Poisson intensity of the claim arrivals is increased. By increasing λ the amount of claims arriving on average is increased, causing higher risk of ruin for the cedent and hence lower optimal retention levels, necessary to keep his and the reinsurer's join probability of survival maximal.

Table 1. Optimal retention levels for increasing reinsurer's premium rate c_r
and parameter of the logarithmically distributed claims α

c\α	.1	.3	.5	.7	.9	.99	.999
.15	1	1	2	2	3	4	4
.25	1	1	1	2	3	3	4
.50	1	1	1	1	2	3	3
1.	1	1	1	1	1	1	2

Table 2. Optimal retention levels for increasing reinsurer's premium rate c_r and Poisson parameter λ

c\λ	.1	.2	.3	.4	.5	.6	.7	.8	.9
.15	4	3	3	3	3	3	2	2	2
.25	3	3	3	3	2	2	2	2	2
.5	2	2	2	2	2	2	2	2	1
1.	1	1	1	1	1	1	1	1	1

To investigate numerically the behavior of $P((T^c > x) \cap (T^r > x))$ in the optimal reinsurance problem 3) we have used **OpXlJo**, to compute and plot on Fig 2, the graph of $P((T^c > x) \cap (T^r > x))$ as a function of c_r, increased from 0.01 to 1.5 with a step 0.01, for values of the retention level $M = 1, 2, 3, 4, 5, 7$ and fixed values of $c = 1.75$, $(c_c = 1.75 - c_r)$, $\alpha = 0.9$, $x = 8$ and $\lambda = 0.4$. A straightforward maximization yields the optimal values $c_r^{\text{opt}} = .0$ .93, 0.59, 0.44, 0.29, 0.01, 0.01 of the reinsurer's premium rate c_r, (also the optimal cedent's rates , $c_c^{\text{opt}} = 1.75 - c_r^{\text{opt}}$) , at which, correspondingly, the maxima $P((T^c > x) \cap (T^r > x))^{\max} =$ 0.279, 0.280, 0.292, 0.306, 0.330, 0.355 were attained. They are marked by the (red) dots on Fig 2, for the retention levels $M = 1, 2, 3, 4, 5, 7$ respectively. As can be seen, looking at Fig 2 and at the values $P((T^c > x) \cap (T^r > x))^{\max}$, the maximum of each of the curves increases as M increases from 1 to 7, i.e., global maximum of the joint survival probability is achieved for $c_r^{\text{opt}} = .0$ .01 and $M = 7$ which is close to having no reinsurance arrangement. This is natural to expect, since by increasing the retention level M up to 7 and higher, the direct insurer tends to retain almost all the risk and hence the proportion covered by the reinsurer tends to vanish, bringing up to unity his probability of individual survival, which on its turn increases the joint probability of both cedent and reinsurer surviving. This extreme case is of course not realistic since existing insurance and reinsurance market conditions and regulatory requirements (such as statutory solvency requirements etc.) cause both insurers and reinsurer's to look for risk sharing contracts between each other.

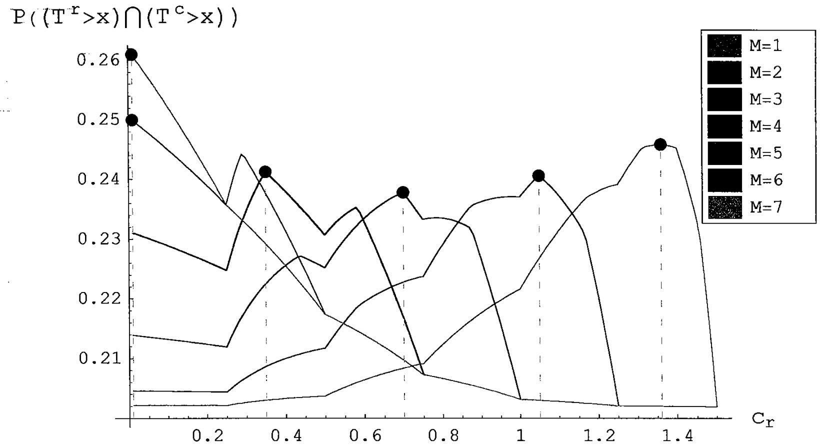

Fig. 3. Optimal values of the reinsurer's (direct insurer's) premium rate c_r ($c_c = 1.75 - c_r$) for fixed retention levels M.

The peculiar shape of the probability of joint survival of the cedent and the reinsurer as a function of the premium income rate c_r and the retention level M, for the same multinomially distributed claims example is illustrated by the 3D plots, given on Fig. 4 and Fig. 5 which differ only by the view point.

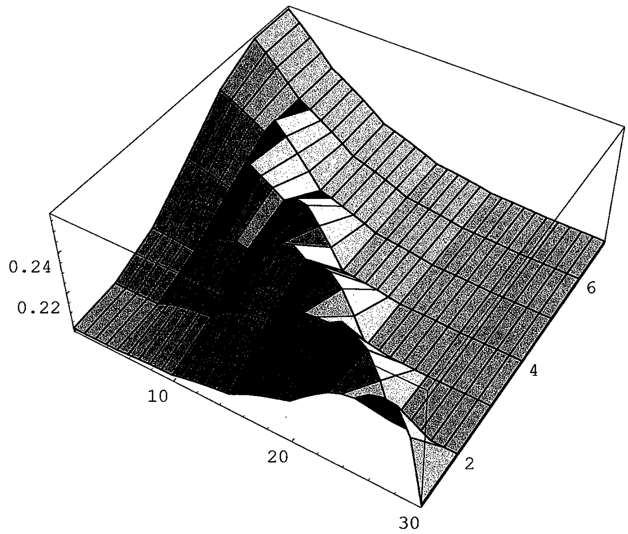

Fig. 4. A 3D plot of $P((T^c > x) \cap (T^r > x))$as a function of the premium rate c_r ($c_c = 1.75 - c_r$) and the retention level M.

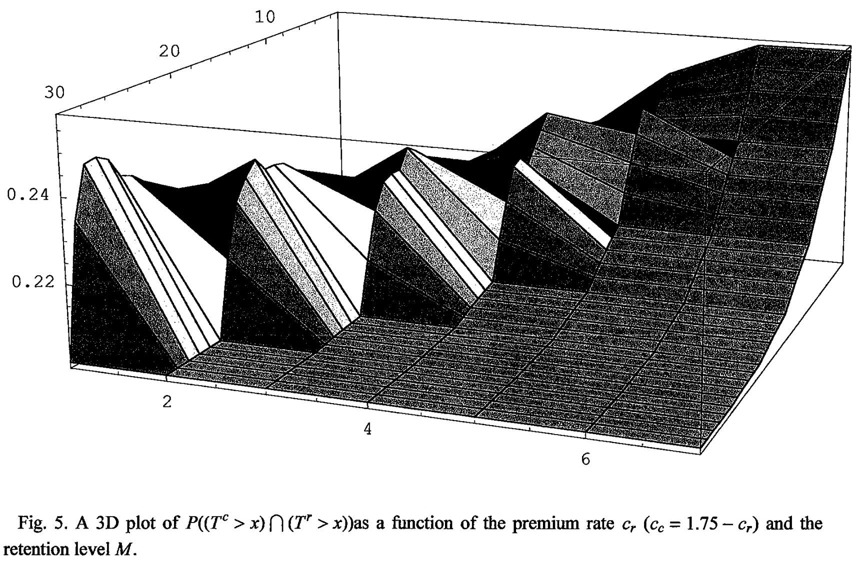

Fig. 5. A 3D plot of $P((T^c > x) \cap (T^r > x))$as a function of the premium rate c_r ($c_c = 1.75 - c_r$) and the retention level M.

▪ 3.1.2 Retention levels optimal with respect to $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$

Let us now turn our attention to the second definition of the optimality criterion, i.e., to $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$. Obviously, we have

$$P((T^r > x) \mid (T^c > x)) = \frac{P((T^c>x) \cap (T^r>x))}{P(T^c>x)},$$

hence, to evaluate $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$ we need to find a formula for the probability $P(T^c > x)$ and then use it, together with the expression (7) for the joint survival probability. To obtain such a formula we will consider the claims process to the ceding insurer. It is modeled by the r.v.s W_i^c, $i = 1, 2, \ldots$, which take values

$$\begin{array}{ll} w_i & \text{if } w_i \le M \\ M & \text{if } w_i > M \end{array}, \qquad (14)$$

i.e., $\min(w_i, M)$, $i = 1, 2, \ldots$ at the moments of arrival of the original claims.

We will further adopt the notation, $P(W_1{}^c = w_1, \ldots., W_n{}^c = w_n) = P^c{}_{w_1,\ldots,w_n}$ for the distribution of the individual claims to the ceding insurer. It is not difficult to conceive that we can express $P^c{}_{w_1,\ldots,w_n}$ in terms of the original claim amounts distribution $P_{w_1,\ldots,w_n}$ as

$$P^c{}_{w_1,\ldots,w_n} = 0, \qquad \text{if } w_i > M, \text{ at least for one value of } i = 1, \ldots, n$$

$$P^c{}_{w_1,\ldots,w_n} = P_{w_1,\ldots,w_n} \text{ if } w_i \le M, \text{ for all values of } i = 1, \ldots, n \qquad (15)$$

$$\begin{aligned} P^c{}_{w_1,\ldots,w_n} = {} & P^c{}_{w_1,\ldots,w_{l_1-1},M,w_{l_1+1},\ldots.,w_{l_2-1},M,w_{l_2+1},\ldots,w_{l_k-1},M,w_{l_k+1},\ldots,w_n} \\ & \equiv P(W_1 = w_1, \ldots, W_{l_1-1} = w_{l_1-1}, \\ & W_{l_1} \ge M, W_{l_1+1} = w_{l_1+1}, \ldots, W_{l_2-1} = w_{l_2-1}, W_{l_2} \ge M, W_{l_2+1} = w_{l_2+1}, \ldots, \\ & W_{l_k-1} = w_{l_k-1}, W_{l_k} \ge M, W_{l_k+1} = w_{l_k+1}, \ldots, W_n = w_n) \\ & = \sum_{\substack{M \le w_{l_1} < +\infty \\ M \le w_{l_2} < +\infty \\ \ldots. \\ M \le w_{l_k} < +\infty}} P_{w_1,\ldots,w_{l_1-1},w_{l_1},w_{l_1+1},\ldots.,w_{l_2-1},w_{l_2},w_{l_2+1},\ldots,w_{l_k-1},w_{l_k},w_{l_k+1},\ldots,w_n} \end{aligned} \qquad (16)$$

if $1 \le w_j \le M - 1$, $0 \le j \le n$, $j \ne l_1$, $j \ne l_2$, ..., $j \ne l_k$, $1 \le l_1 < l_2 < \ldots < l_k \le n$, $0 \le k \le n$, and there is no summation when $k = 0$, i.e., $P^c{}_{w_1,\ldots,w_n} = P_{w_1,\ldots,w_n}$.

To avoid infinite summation on the right-hand side of (16) and hence to facilitate its exact numerical evaluation we will rewrite it as

$$\begin{aligned} & P(W_1 = w_1, \ldots, W_{l_1-1} = w_{l_1-1}, W_{l_1} \ge M, W_{l_1+1} = w_{l_1+1}, \ldots, W_{l_2-1} = w_{l_2-1}, W_{l_2} \ge M, W_{l_2+1} = w_{l_2+1}, \ldots, \\ & \qquad W_{l_k-1} = w_{l_k-1}, W_{l_k} \ge M, W_{l_k+1} = w_{l_k+1}, \ldots, W_n = w_n) \\ & = P(W_1 = w_1, \ldots, W_{l_1-1} = w_{l_1-1}, W_{l_1+1} = w_{l_1+1}, \ldots, W_{l_2-1} = w_{l_2-1}, W_{l_2+1} = w_{l_2+1}, \ldots, \\ & \qquad W_{l_k-1} = w_{l_k-1}, W_{l_k+1} = w_{l_k+1}, \ldots, W_n = w_n, W_{l_1} \ge M, W_{l_2} \ge M, \ldots, W_{l_k} \ge M) \\ & = P\left(W_1 = w_1, \ldots, W_{l_1-1} = w_{l_1-1}, W_{l_1+1} = w_{l_1+1}, \ldots, W_{l_2-1} = w_{l_2-1}, W_{l_2+1} = w_{l_2+1}, \ldots, \right. \\ & \qquad \left. W_{l_k-1} = w_{l_k-1}, W_{l_k+1} = w_{l_k+1}, \ldots, W_n = w_n, \overline{\overline{W_{l_1} \ge M, W_{l_2} \ge M, \ldots, W_{l_k} \ge M}}\right) \\ & = P(W_1 = w_1, \ldots, W_{l_1-1} = w_{l_1-1}, W_{l_1+1} = w_{l_1+1}, \ldots, W_{l_2-1} = w_{l_2-1}, W_{l_2+1} = w_{l_2+1}, \ldots, \\ & \qquad W_{l_k-1} = w_{l_k-1}, W_{l_k+1} = w_{l_k+1}, \ldots, W_n = w_n, \overline{(W_{l_1} < M) \cup (W_{l_2} < M) \cup \ldots \cup (W_{l_k} < M)}) \end{aligned}$$

$$
\begin{aligned}
&= P(W_1 = w_1,\ \ldots,\ W_{l_1-1} = w_{l_1-1},\ W_{l_1+1} = w_{l_1+1},\ \ldots,\ W_{l_2-1} = w_{l_2-1},\ W_{l_2+1} = w_{l_2+1},\ \ldots, \\
&\qquad W_{l_k-1} = w_{l_k-1},\ W_{l_k+1} = w_{l_k+1},\ \ldots,\ W_n = w_n) \\
&- P(\textstyle\bigcup_{j=1}^{k} ((W_1 = w_1,\ \ldots,\ W_{l_1-1} = w_{l_1-1},\ W_{l_1+1} = w_{l_1+1},\ \ldots,\ W_{l_2-1} = w_{l_2-1},\ W_{l_2+1} = w_{l_2+1},\ \ldots, \\
&\qquad W_{l_k-1} = w_{l_k-1},\ W_{l_k+1} = w_{l_k+1},\ \ldots,\ W_n = w_n) \cap (W_{l_j} < M)))
\end{aligned} \tag{17}
$$

Now, applying the inclusion-exclusion theorem to the second term on the right-hand side of the last equality in (17) and introducing the notation

$$
\begin{aligned}
&P(W_1 = w_1,\ \ldots,\ W_{l_1-1} = w_{l_1-1},\ W_{l_1+1} = w_{l_1+1},\ \ldots,\ W_{l_2-1} = w_{l_2-1},\ W_{l_2+1} = w_{l_2+1},\ \ldots, \\
&\qquad W_{l_k-1} = w_{l_k-1},\ W_{l_k+1} = w_{l_k+1},\ \ldots,\ W_n = w_n) = \\
&P_{w_1,\ldots,w_{l_1}-1,\text{«}w_{l_1}\text{»},w_{l_1}+1,\ldots,w_{l_2}-1,\text{«}w_{l_1}\text{»},w_{l_2}+1,\ldots,w_{l_k}-1,\text{«}w_{l_k}\text{»},w_{l_k}+1,\ldots,w_n}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&P(W_1 = w_1,\ \ldots,\ W_{l_1-1} = w_{l_1-1},\ W_{l_1} \geq M,\ W_{l_1+1} = w_{l_1+1},\ \ldots,\ W_{l_2-1} = w_{l_2-1},\ W_{l_2} \geq M,\ W_{l_2+1} = w_{l_2+1},\ \ldots, \\
&\qquad W_{l_k-1} = w_{l_k-1},\ W_{l_k} \geq M,\ W_{l_k+1} = w_{l_k+1},\ \ldots,\ W_n = w_n)
\end{aligned}
$$

$$
\begin{aligned}
&= P_{w_1,\ldots,w_{l_1}-1,\text{«}w_{l_1}\text{»},w_{l_1}+1,\ldots,w_{l_2}-1,\text{«}w_{l_2}\text{»},w_{l_2}+1,\ldots,w_{l_k}-1,\text{«}w_{l_k}\text{»},w_{l_k}+1,\ldots,w_n} \\
&- \sum_{j=1}^{k} \sum_{w_{l_j}=1}^{M-1} P_{w_1,\ldots,w_{l_1}-1,\text{«}w_{l_1}\text{»},w_{l_1}+1,\ldots,\text{«}w_{l_{j-1}}\text{»},w_{l_{j-1}}+1,\ldots,w_{l_j}-1,w_{l_j},w_{l_j}+1,\ldots,\text{«}w_{l_{j+1}}\text{»},\ldots,\text{«}w_{l_k}\text{»},\ldots,w_n} \\
&+ \sum\nolimits_{1\leq j<s\leq k} \sum_{w_{l_j}=1}^{M-1} \\
&\qquad \sum_{w_{l_s}=1}^{M-1} P_{w_1,\ldots,w_{l_1}-1,\text{«}w_{l_1}\text{»},w_{l_1}+1,\ldots,\text{«}w_{l_{j-1}}\text{»},w_{l_{j-1}}+1,\ldots,w_{l_j}-1,w_{l_j},w_{l_j}+1,\ldots,w_{l_s}-1,w_{l_s},w_{l_s}+1,\ldots,\text{«}w_{l_k}\text{»},\ldots,w_n} \\
&\qquad + \ldots + (1)^k \sum_{w_{l_1}=1}^{M-1} \cdots \sum_{w_{l_k}=1}^{M-1} P_{w_1,\ldots,w_{l_1},\ldots,w_{l_k},\ldots,w_n},
\end{aligned} \tag{18}
$$

Let us note that the distributions $P_{w_1,\ldots,w_{l_1}-1,\text{«}w_{l_1}\text{»},w_{l_1}+1,\ldots,w_{l_2}-1,\text{«}w_{l_1}\text{»},w_{l_2}+1,\ldots,w_{l_k}-1,\text{«}w_{l_k}\text{»},w_{l_k}+1,\ldots,w_n}$, $1 \leq l_1 < l_2 < \ldots < l_k \leq n,\ 0 \leq k \leq n,$ are easily obtained from $P_{w_1,\ldots,w_{l_1}-1,w_{l_1},w_{l_1}+1,\ldots,w_{l_2}-1,w_{l_2},w_{l_2}+1,\ldots,w_{l_k}-1,w_{l_k},w_{l_k}+1,\ldots,w_n}$ by summing it out with respect to the corresponding, set of indexes. To apply formula (18), one needs to know these marginal distributions. Explicit formulae for the latter, for the multinomial and the 'negative binomial' distributions are given in the Appendix. Now, we are ready to state and prove

Theorem 2. The probability of survival in finite time x of the cedent, under an excess of loss reinsurance treaty is

$$P(T^c > x) = e^{-x\lambda} \sum_{k=1}^{n_c} \sum_{\substack{M \geq w_1 \geq 1, \ldots, M \geq w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n_c - 1}} \sum_{i=n_c-w_1-\ldots-w_{k-1}}^{M} P^c{}_{w_1, \ldots, w_{k-1}, i} \left(\sum_{j=0}^{k-1} (-1)^j b_j(z_1, \ldots, z_j) \lambda^j \sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!} \right), \tag{19}$$

where $n_c = [h_c(x)] + 1$, $z_l = v_{w_1 + \ldots + w_l} = h_c^{-1}(w_1 + \ldots + w_l)$ and $P^c{}_{w_1, \ldots, w_{n_c}}$ is given by (15) and (16) combined with (18).

Proof. Since we are interested in the probability of survival of solely the cedent we can directly apply formula (2), substituting in it, $h(t)$, n and $P(W_1 = w_1, \ldots, W_{k-1} = w_{k-1}; W_k \geq n - w_1 - \ldots - w_{k-1})$ correspondingly with $h_c(t)$, n_c and $P(W_1{}^c = w_1, \ldots, W_{k-1}{}^c = w_{k-1}; M \geq W^c{}_k \geq n_c - w_1 - \ldots - w_{k-1})$, taking into consideration that $W^c{}_i \leq M,\ i = 1, 2, \ldots$. Thus, we have

$$P(T^c > x) = e^{-x\lambda} \sum_{k=1}^{n_c} \sum_{\substack{M \geq w_1 \geq 1, \ldots, M \geq w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n_c - 1}} P(W^c{}_1 = w_1, \ldots, W^c{}_{k-1} = w_{k-1}; M \geq W^c{}_k \geq n_c - w_1 - \ldots - w_{k-1}) \sum_{j=0}^{k-1} (-1)^j b_j(z_1, \ldots, z_j) \lambda^j \sum_{m=0}^{k-j-1} \frac{(x\lambda)^m}{m!}$$

which leads to (19), noting that

$$P(W^c{}_1 = w_1, \ldots, W^c{}_{k-1} = w_{k-1}; M \geq W^c{}_k \geq n_c - w_1 - \ldots - w_{k-1}) = \sum_{i=n_c-w_1-\ldots-w_{k-1}}^{M} P^c{}_{w_1, \ldots, w_{k-1}, i}.$$

We can substitute in (19) the expression of $P^c{}_{w_1, \ldots, w_{n_c}}$ via the distribution of the original claims, as given by (15) and (16), combined with (18), which completes the proof of the theorem. ∎

Numerical illustrations and sensitivity analysis of $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$.

Here, we will restrict our numerical illustrations of how one can apply Theorem 2 to the case of the test model (13), i.e., to independent claims only. Further numerical investigations of the criterion $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$for the case of multinomial, and 'multivariate negative binomial' distributions of the claims is straightforward, applying (19), combined correspondingly with the formulae for the marginal multinomial and 'multivariate negative binomial' distributions, given in the Appendix. Such numerical illustrations, on the example of the multinomially distributed claims are provided in Subsection 3.1.4. Next, we will consider $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$, for the case of model (13), with model parameters $\lambda = .4, \alpha = .9, c_c + c_r = 1.75$, $u1 = u2 = 0$, $x = 8$.Our purpose will be to illustrate the solution of optimality problem 4), in which we assume the retention level M is varied from 1 to 5 and for each fixed value of M we solve problem 4), i.e., find the optimal split of the total premium rate c into c_c and c_r, such that $|P(T^c > x) - P((T^r > x) \mid (T^c > x))| \longrightarrow \min$. For $M = 1, 2, 3$ the optimal values of c_r, for which $|P(T^c > x) - P((T^r > x) \mid (T^c > x))| = 0$ are respectively, equal to 0.913, 0.439 and 0.126 with values of the probability $P(T^c > x) = P((T^r > x) \mid (T^c > x))$ respectively, 0.528, 0.518 and 0.510. This is illustrated graphically on Fig. 6, on which the optimal solutions are marked by dots on the curves, obtained for each fixed value of M, by computing $P(T^c > x)$and $P((T^r > x) \mid (T^c > x))$ for values of c_r, varied with a sufficiently small step . As seen on Fig.6 the dots are closely situated, since their corresponding probabilities 0.528, 0.518 and 0.510 are close, which means that the cedent will prefer to choose the pair of optimal values (M, c_r), correspondingly equal to (3, 0.126) since in this case he will pass the lowest possible proportion, 0.126 of the premium income to the reinsurer. Note, that increasing M does not decrease significantly his probability of survival.

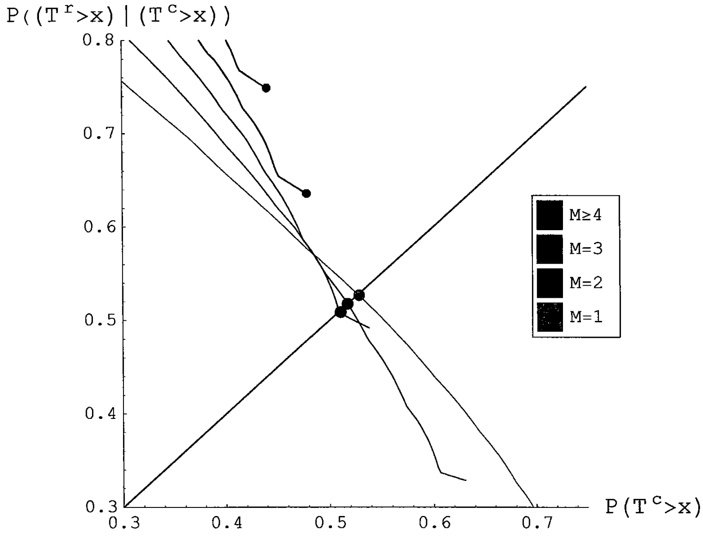

Fig.6. Optimal split of the premium income rate c for increasing values of the retention level M.

Now, let us perform some sensitivity analysis of our criteria $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$on the example of the test model (13). We will investigate how the change of the parameter α of the Log distributed claims, affects the optimal solutions of problem 2), i.e., the values of M minimizing $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$. The model parameters have been set as follows: $\lambda = .4$, $u1 = u2 = 0$; $c_c = 1.75 - .15$; $c_r = .15$; $x = 8$, and α has been given values 0.1, 0.3, 0.5, 0.7, 0.9, 0.99, 0.999 and for each value of α the retention level M has been varied from 1 to 10 with a step of 1, computing for each pare of values (α, M) the probabilities $P(T^c > x)$ and $P((T^r > x) \mid (T^c > x))$.The results of this investigation are illustrated on Fig. 7. As seen, the red (thick) dots, i.e., the dots situated closest to the straight line with a unit slope, represent the values of M which 'equalize' the two probabilities $P(T^c > x)$ and $P((T^r > x) \mid (T^c > x))$, i.e. bring them as close to being equal as possible. As can be seen, these values, for values of α equal to 0.1, 0.3, 0.5, 0.7, 0.9, 0.99, 0.999, are correspondingly equal to 1, 1, 2, 2, 3, 4, 4, since the dots on each curve, starting from the first dot lying on the vertical, dashed line, up the curve, correspond to values of M increasing from 1 to 10. The vertical dashed line at $P(T^c > x) = 0.75$ occurs since $P(T^c > x)$ does not change if we change α and keep M equal to unity, as is for the dots on the vertical dashed line. Note, that by increasing α we increase the mean value of the Log distributed claims, but since M is unity along the dashed line, this does not affect the probability of survival of the cedent, which is therefore constant, equal to 0.75. Obviously, the latter value will increase, causing the dashed line to move to the right, as we increase the intensity λ of the claim arrivals or the cedent's premium rate c_c. At the same time, increasing the mean of the claims, keeping $M = 1$, i.e., descending along the dashed line, decreases the probability of survival of the reinsurer, given survival of the cedent, which is natural to be expected. Another observation we can make, looking at Fig. 7, is that increasing α, i.e., increasing the mean of the claims, causes the cedent to retain a higher proportion of each claim, i.e., the optimal values of M increase which is to be expected, since both c_c and c_r are fixed. Another observation we can make, looking at Fig. 7, is that, for fixed α, i.e., if we go up along each curve, the dots, corresponding to increasing values of M come closer to one another, i.e with M going up its influence on the two probabilities $P(T^c > x)$ and $P((T^r > x) \mid (T^c > x))$ diminishes and also, each curve tends to the value of $P(T^c > x)$ reached in the case when the cedent covers each claim only by himself, without passing part of it to the reinsurer.

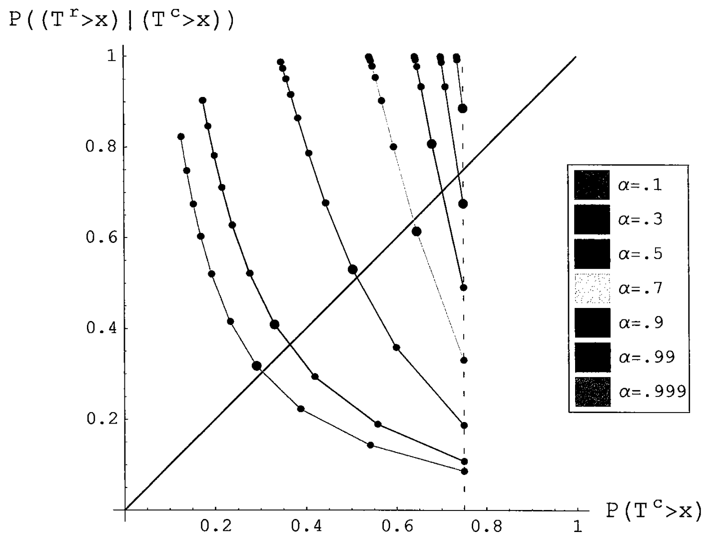

Fig.7. Optimal retention values for increasing values of the parameter α of the Log distributed claims.

Another sensitivity analysis of the criterion $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$, based on the same test example of model (13), with $\alpha = 0.9$, $c_c = 1.6$, $c_r = 0.15$, u1 = u2 = 0, $x = 8$ is to see how the optimal values of Mchange as the Poisson intensity λ of the claim arrivals is increased from 0.1 to 0.9 with an increment of 0.1. To obtain $M^{opt}$the value of $M$is varied from 1 to 10, for each fixed value of λ.The results are shown on Fig. 8, where one can see that as λ increases, the optimal values of M, marked by the red (thick) dots, decrease, taking values, correspondingly 4, 3, 3, 3, 3, 3, 2, 2, 2 since, the more claims arrive the higher the risk for the cedent and the less of it should he retain, the values of α, $c_c$and $c_r$being fixed. Another phenomena to be observed is that both $P(T^c > x)$and $P((T^r > x) \mid (T^c > x))$decrease as λ increases, which is also reasonable, since the risk to both parties increases as more claims arrive on average. As before the distance between consecutive values of M, the dots along the curves decrease, affecting less and less the two probabilities of survival.the probability of survival of the reinsurer reaching unity as Mapproaches 10. As is also normal to expect for fixed values of M, α, $c_c$and c_r both $P(T^c > x)$and $P((T^r > x) \mid (T^c > x))$decrease as λ increases, since the more claims arrive the higher is the risk for both of the parties. This is of course true for values of $M < 6$.(see the patterns of the dots on Fig. 8)

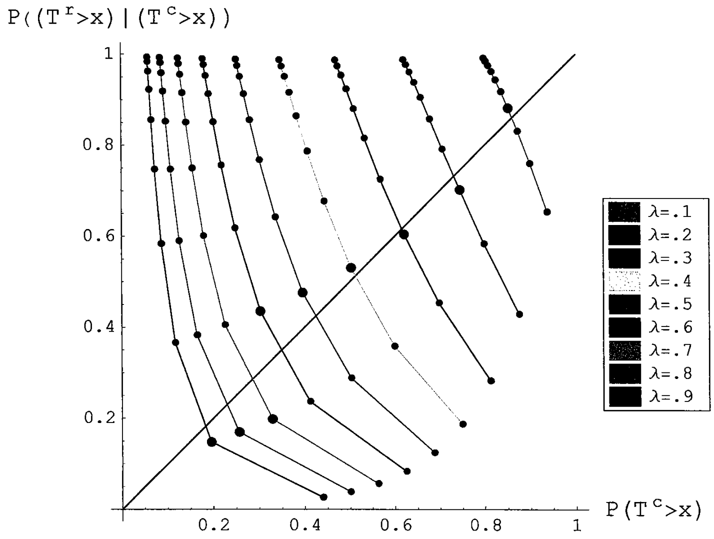

Fig.8. Optimal retention values for increasing values of the parameter λ of the Poisson intensity of the claim arrivals.

▪ 3.1.3 Independent individual claims

Let us now consider the special case of independent, identically distributed claims $W_1, W_2, \ldots$ with common distribution $\{P_j,\ j \in N^*\}$(where $N^*$is the set of natural numbers) i.e., when, $P(W_i = j) = P_j$, $i = 1, 2, \ldots,$ $j = 1, 2, \ldots$with mean μ . In this case one can express the probability of survival of the direct insurer and the reinsurer without conditioning on the survival of the cedent, applying the Picard and Lefevre formula (4), accordingly adjusting it. To see how this can be done we will first express the distribution of the claims to the cedent in terms of the distribution of the original claims as

$$
\begin{aligned}
&P^c{}_j = 0, && \text{if } j > M, \\
&P^c{}_M = \sum_{M \le j < +\infty} P_j && \text{if } j \equiv M, \qquad (20)\\
&P^c{}_j = P_j\ , && \text{if } 1 \le j \le M-1.
\end{aligned}
$$

As for the arrivals of claims to the reinsurer , one has to bare in mind that, since he receives claims only when the original claims exceed the retention M, hence the process of arrival of claims to the reinsurer will be formed from the arrivals of the original claims by dropping the moments of arrival of claims of size less then M, covered by the cedent. In other words, each point from the Poisson process of arrivals of the original claims with intensity λ will remain and form the process of arrivals of the claims to the reinsurer with probability $\sum_{j=M+1}^{\infty} P_j$. Hence the letter, thinned process will have intensity lower than λ, equal to $\lambda' = \lambda\,(\sum_{j=M+1}^{\infty} P_j)$. For the distribution of the claims severities to the reinsurer one can write

$$
P^r{}_j = \frac{P_{j+M}}{P_{m+1} + P_{M+2} + \ldots}\,, \qquad (21)
$$

where $P^r{}_j$ is the conditional probability $P(W^r = j) = P(W = M + j \,/\, W > M)$.

Now, we can state the following theorem

Theorem 3. The probability $P(T^c > x)$and $P(T^r > x)$ of survival in finite time x of correspondingly the cedent and the reinsurer is given by formula (4) replacing $h^{-1}(i)$, $i = 0, 1, 2, \ldots$ in the definition of v_i with $h_c^{-1}(i)$, $i = 0, 1, 2, \ldots$ and $P_j$with $P^c{}_j$, $j = 1, 2, \ldots$given by (20) for the probability $P(T^c > x)$and $h^{-1}(i)$, $i = 0, 1, 2, \ldots$with $h_r^{-1}(i)$, $i = 0, 1, 2, \ldots,$ P_j with $P^r{}_j$, $j = 1, 2, \ldots$ given by (21) and the Poisson intensity $\lambda$with $\lambda' = \lambda\,(\sum_{j=M+1}^{\infty} P_j)$, for the probability $P(T^r > x)$.

Proof. Directly follows, applying formula (4) and taking into consideration the construction of the two risk processes $R_t^c$and R_t^r and the distributions $P^c{}_j$ and $P^r{}_j$, defined correspondingly by (20) and (21). □

Remark: Let us note that, as mentioned in Section 2, survival of the reinsurer, unconditional of the survival of the direct insurer is irrelevant, unless we assume that in case of ruin all his liabilities (with respect to settling current claims and paying reinsurance premiums) are automatically taken by another insurance company, in which case the probability $P(T^r > x)$ given by Theorem 3 may be of some interest.

▪ 3.1.4 Numerical comparisons of optimality criteria $P((T^c > x) \bigcap (T^r > x))$ and $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$.

We can now give numerical solutions to the two optimal reinsurance problems 2) and 4) with the second optimality criterion, i.e, $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$ and compare these solutions with results, obtained with the first optimality criterion $P((T^c > x) \bigcap (T^r > x))$. For the purpose, formula (19) was implemented as a *Mathematica* module, named **OpXlCe** and $P((T^r > x) \mid (T^c > x))$ and $P(T^c > x)$ were computed as functions of the retention level M for the test model (13) and for the same model parameters, used to investigate $P((T^c > x) \bigcap (T^r > x))$. Fig. 6 illustrates graphically the solution of problem 2), by plotting these two functions, together with the probability $P((T^c > x) \bigcap (T^r > x))$. As can be seen, the two optimality criteria give different optimal retentions, the first criterion producing considerably higher optimal retention levels (marked with dotted lines on Fig 6). The difference between the two optimal solutions decreases as the value of the reinsurer's premium rate c_r increases. It can also be noted, looking at the graphs on Fig. 6, that the second criterion is much less sensitive with respect to varying the premium rate c_r, i.e., it produces optimal retentions which do not change significantly, taking values 3 to 1 as the proportion of split of the total premium income, between the direct insurer and the reinsurer, varies in a wide range , i.e., c_r being varied from 0.15 to 0.7. This suggests that the second criterion produces retention levels more favorable for the primary insurer, since the optimal retention level it produces is significantly lower, than the one obtained under the first criterion, for the same fixes values of $c_r$and c_c, such that $c_c + c_r = c$.(see Fig 6.). The same conclusion is confirmed by Fig. 7. on which the two criteria are compared with respect to the inverse optimality problems 3) and 4), stated in Subsection 3.1, i.e., fixing the value of the retention level M,the optimal split of the premium rate c into $c_c$and $c_r$is calculated and indicated by the dashed vertical lines on Fig 7. As seen the case when both criteria give similar optimal solutions is when the retention level $M = 1$.

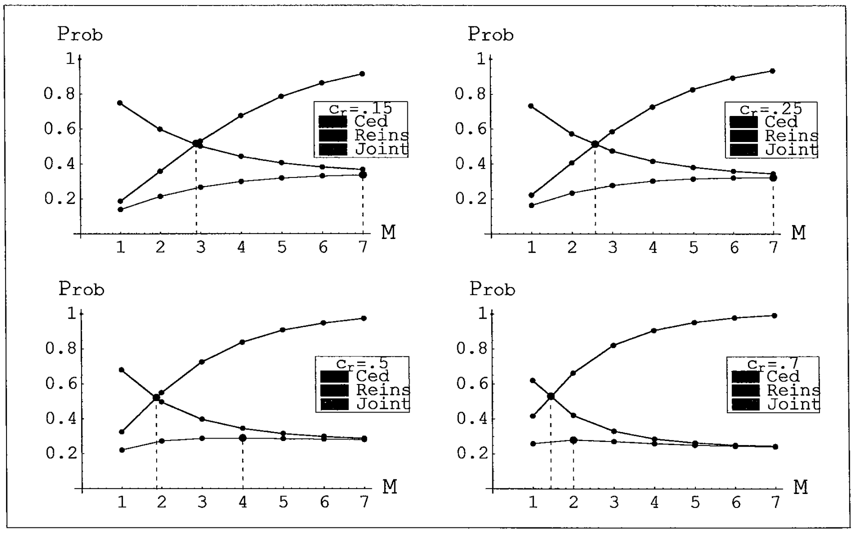

Fig.6. Optimal retention levels for Log Distributed claims, with respect to criteria $P((T^c > x) \bigcap (T^r > x))$and $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$

The two optimality criteria are further compared with respect to the problems 2) and 4).

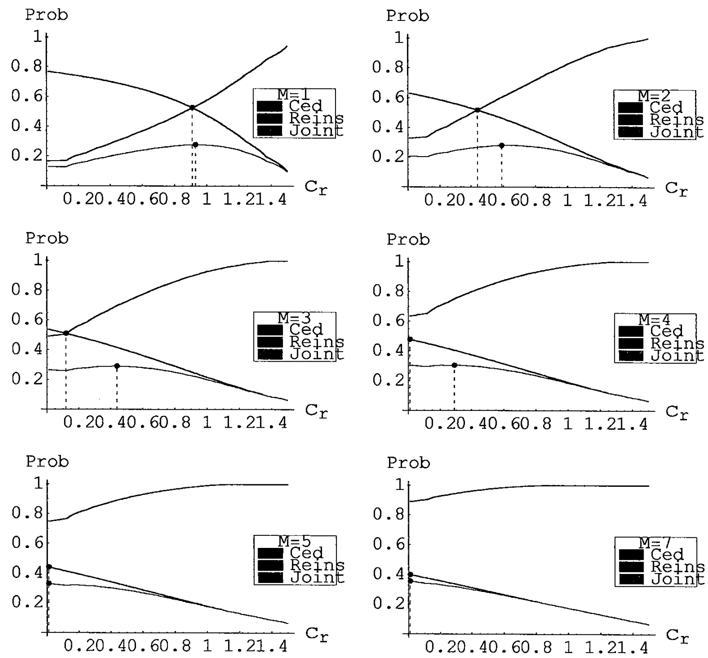

Fig.7 Optimal value of c_r for fixed retention level, for Log distributed claims, with respect to criteria $P((T^c > x) \cap (T^r > x))$and $|P(T^c > x) - P((T^r > x) \mid (T^c > x))|$

Similar comparison has been performed on the example of our second test model in which the distribution of the claims was taken to be the Multinomial with parameters $m = 10$, $d_i = 1/(i(i+1))i = 1, 2, ...,$ $x = 4$, $\lambda = 0.4$. As can be seen, the fact that the distribution is dependent and multinomial does not affect the conclusions made above, concerning the comparison of the two criteria.

3.2 'Expected profit' based optimality

We will consider our initial reinsurance problem of a primary insurer and a reinsurer sharing, under an XL contract, a mass of risks, generating claims and premium income, adequate to the potential losses. We will be interested here in an appropriate measure of performance of the two companies. A natural choice of such a measure can be based on the profit each of them is expected to make at the end x of the finite time interval, under an XL reinsurance contract, conditional on their joint survival up to the moment x. Indeed, no profit at any future moment can be expected from a company, ruined before that moment. As before, it is crucial to consider the joint survival of both of the parties since, there will be no flow of premiums (claims) to the reinsurer if the primary insurer gets ruined. We define the profits at time x of the cedent and the reinsurer, correspondingly as the value R_x^c and R_x^r of their risk processes, given by (5) and (6), at time x. Let us introduce the indicator random variables I_A and I_B of the events $A = \{T^c > x\}$ and $B = \{T^r > x\}$ respectively. For the conditional expectation $E(R_x^c \mid I_A, I_B)$ there exists a suitable function $\psi(u, v)$ such that $E(R_x^c \mid I_A, I_B) \overset{a.s.}{=} \psi(I_A, I_B)$. We will denote the value of $\psi(I_A, I_B)$ when $I_A \equiv 1$ and $I_B \equiv 1$, i.e., $\psi(1, 1)$ as $E[R_x^c \mid ((T^c > x) \cap (T^r > x))]$ and will call it, the expected profit of the cedent at time x, under the condition that he and the reinsurer have not been ruined up to time x. Alternatively, we will call this quantity the cedent's expected profit at time x, given his and the reinsurer's joint survival up to time x. Similarly, one can define $E[R_x^c \mid ((T^c > x) \cap (T^r > x))]$ and call it the reinsurer's expected profit at time x, given his and the insurer's joint survival up to time x.

Thus, the conditional on joint survival to x, expected profits of the cedent $E[R_x^c \mid ((T^c > x) \cap (T^r > x))]$ and of the reinsurer $E[R_x^r \mid ((T^c > x) \cap (T^r > x))]$ can serve to form yet another criterion for the optimal choice of the XL retention level M. One can formulate the same two types of problems as in 2) and 4) but with respect to the absolute value of the difference of the two expected profits, i.e.,

$$\left| E[R_x^c \mid ((T^c > x) \cap (T^r > x))] - E[R_x^r \mid ((T^c > x) \cap (T^r > x))] \right| \tag{22}$$

Another possibility is to look for a pair of optimal values i.e., values of the retention level and of the split of the function $h(t)$ into $h_c(t)$ and $h_r(t)$, $h(t) = h_c(t) + h_r(t)$, such that the two expected profits at time x, given joint survival up to x, be equal or in a preliminary agreed proportion. To be able to test criterion (22) we need explicit formulae for the corresponding expected profits. The following theorems present such formulae.

Theorem 4. The expected profit at time x of the cedent, given his and the reinsurer's joint survival up to x is

$$\begin{aligned}
&E[R_x^c \mid ((T^c > x) \cap (T^r > x))] = \\
&\Big\{\sum_{k=1}^{n} \sum_{\substack{w_1 \geq 1, \ldots, w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n-1}} P(W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1}) \\
&\qquad \sum_{i=1}^{l} \Big[\Big(h_c(x) - \sum_{s=1}^{i-1} w_s^c\Big)(B_i(\bar{z}_1, \ \ldots, \ \bar{z}_{i-1}, \ x) - B_{i-1}(\bar{z}_1, \ \ldots, \ \bar{z}_{i-2}, \ x))\Big]\Big\} \Big/ \\
&\Big\{\sum_{k=1}^{n} \sum_{\substack{w_1 \geq 1, \ldots, w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n-1}} P(W_1 = w_1, \ \ldots, \ W_{k-1} = w_{k-1}; \ W_k \geq n - w_1 - \ldots - w_{k-1}) \\
&\qquad B_l(\bar{z}_1, \ \ldots, \ \bar{z}_{l-1}, \ x)\Big\}
\end{aligned} \tag{23}$$

,

where n, l,and $\bar{z}_i$ are defined as in (7), $w_s{}^c = \min(M, w_s)$ and

$$B_i(\bar{z}_1, \ldots, \bar{z}_{i-1}, x) = \sum_{j=0}^{i-1} (-1)^j b_j(\bar{z}_1, \ldots, \bar{z}_j) \lambda^j \sum_{m=1}^{i-j-1} \frac{(x\lambda)^m}{m!} \quad , \tag{24}$$

$i = 0, 1, 2 \ldots, l\ B_0(\cdot) \equiv 0,\ B_1(\cdot) = 1.$

Proof. Taking into account the construction of the risk processes $R_t{}^c$and $R_t{}^r$ (see (5) and (6)) we can express the unconditional expectation $E[R_x{}^c \cdot I_A \cdot I_B]$ as

$$E(R_x{}^c \cdot I_A \cdot I_B) = e^{-x\lambda} \sum_{k=1}^{n} \sum_{\substack{w_1 \geq 1, \ldots, w_{k-1} \geq 1, \\ w_1 + \ldots + w_{k-1} \leq n-1}} P(W_1 = w_1, \ldots, W_{k-1} = w_{k-1}; W_k \geq n - w_1 - \ldots - w_{k-1})$$
$$\sum_{i=1}^{l} \left[\left(h_c(x) - \sum_{s=1}^{i-1} \min(M, w_s) \right) (B_i(\bar{z}_1, \ldots, \bar{z}_{i-1}, x) - B_{i-1}(\bar{z}_1, \ldots, \bar{z}_{i-2}, x)) \right] \tag{25}$$

where n, l,and $\bar{z}_i$ are defined as in (7) and

$$B_i(\bar{z}_1, \ldots, \bar{z}_{i-1}, x) = \sum_{j=0}^{i-1} (-1)^j b_j(\bar{z}_1, \ldots, \bar{z}_j) \lambda^j \sum_{m=1}^{i-j-1} \frac{(x\lambda)^m}{m!} \quad ,$$

$i = 0, 1, 2 \ldots, l\ B_0(\cdot) \equiv 0,\ B_1(\cdot) = 1.$

In equality (25) we have used the fact that the difference $e^{-x\lambda}(B_i(\bar{z}_1, \ldots, \bar{z}_{i-1}, x) - B_{i-1}(\bar{z}_1, \ldots, \bar{z}_{i-2}, x))$, $i = 1, 2, \ldots$is equal to the probability that exactly $i-1$claims from the original risk process R_t have occurred up to time x, i.e., exactly $i-1$claims from the cedent's risk process $R_t{}^c$ have occurred and at the same time, neither the cedent, nor the reinsurer got ruined up to time x. In (25) we have also taken into consideration that if $i-1$claims up to time x have occurred the profit of the cedent is equal to

$$h_c(x) - \sum_{s=1}^{i-1} w_s{}^c \equiv h_c(x) - \sum_{s=1}^{i-1} \min(M, w_s). \tag{26}$$

Let us note, that obviously, when $i = 1$, i.e., when 0 claims have occurred up to time x the sums in (26) vanish and the profit is equal to the premium income, accumulated up to x. We can now write for the conditional expectation

$$E[R_x{}^c \mid ((T^c > x) \cap (T^r > x))] = \frac{E[R_x{}^c \cdot I_A \cdot I_B]}{P((T^c > x) \cap (T^r > x))} \tag{27}$$

Substituting (25) and (7) in (27), using in (7) the notation $B_l(\bar{z}_1, \ldots, \bar{z}_{l-1}, x)$for the sum with respect to $j$we get the assertion of the theorem.□

Similarly we have for the expected profit of the reinsurer

Theorem 5. The expected profit at time x of the reinsurer, given his and the cedent's joint survival up to x is

5. Conclusions

Our main conclusion, which was extensively illustrated through the graphs and the numerical results presented, is that considering the problem of optimal reinsurance, and in particular optimal XL, from the point of view of not only the primary insurer but also of the reinsurer, is quite natural and reasonable. We have done this by introducing measures of risk and performance, based on the joint survival of both parties and leading to well defined optimal retention levels. The latter can be efficiently computed, based on the explicit formulae (7), (18) () and (), using the corresponding *Mathematica* modules, developed for the purpose. We have demonstrated how, fixing the proportion of the premium income the primary insurer would like to retain, he can find the corresponding level of retention which applies to each occurring claim, so that his and the reinsurer's probability of survival is maximized . And vice versa, fixing the retention level, accordingly with how risk averse the primary insurer is he can find the optimal proportion in which the premium income should be divided between him and the reinsurer. We have further compared the two optimality criteria introduced as measures of risk, i.e., the probability of joint survival of the cedent and the reinsurer and the absolute value of the difference between the probability of survival of the cedent and the probability of survival of the reinsurer, given survival of the cedent. As illustrated numerically, the two criteria produce different optimal retention levels. The optimal retention level obtained if the joint survival probability is used is higher than the one obtained if the absolute value of the difference of the two survival probabilities is used, the proportion in which the original premium income is shared between the two parties being fixed (see Fig. 6).This suggests that using the second criterion is more favorable for the primary insurer since he retains smaller part of the risk while keeping the same proportion of the premium income.

What we have also established in Subsection 3.2 is that the expected profits of both parties, given their joint survival, also depend on the retention level and on how the premium income is divided between them. These measures are appropriate for measuring the performance of the two parties and can also serve to obtain a pair of optimal values of the retention level and the split of the premium income. Thus, having two measures, one for the risk and one for the performance, which can be evaluated for different values of the retention level, and choices of the premium income function, the claim amount distribution and their parameters, the cedent and the reinsurer can construct different scenarios and perform a DFA type of analysis. They can construct efficient frontier plots and choose the retention level which may not be the optimal one, but will provide the desired balance between the risk and the performance the two parties want to achieve.

Another conclusion, which deserves to be noted, is that the XL retention level affects the expected profits and the probabilities of survival of both the primary insurer and the reinsurer, as well as the probability of their joint survival and the optimal choice of this level is of crucial importance for both parties, not only for the primary insurer. This conclusion is confirmed by the consequences of the tragic events of the 11-th of September 2001 terrorist attack on the New York World Trade Center. These events showed that reinsurers, no matter how big they are, can also be vulnerable to substantial losses, which may lead to their insolvency and closure (see e.g,. Clark (2002)). At the same time, the primary insurers can manage to survive in such cases, having retained the "optimal" proportion of the risk, which result from considering optimality only with respect to their own financial interests. Their strategy may simply be to pass as large proportion of the risk to the reinsurer as possible, against as modest reinsurance premium as possible. Although since the 11-th of September, there is no official information confirming that, as a result of such a strategy some reinsurers have closed, while the primary insurers covered by them have survived, this could have very well been the case. So the possibility of this happening is sufficient to justify , further research on joint optimality criteria.

One question for further research, which can be directly approached, is to consider an XL contract under the same optimality criteria, introduced in Section 3 but with two levels, a retention level and a limiting level,

Kaluszka, M. (2001). Optimal reinsurance under mean-variance premium principles. *Insurance:Mathematics and Economics, 24, 3, 219-227.*

Krvavych, Y. (2001). On existence of insurer's optimal excess of loss reinsurance strategy. Paper presented at the 5-th International Congress on *Insurance:Mathematics and Economics.*

Mnif, M. and Sulem, A. (2001). Optimal risk control under excess of loss reinsurance. *INRIA Research Report No 4317.*

von Neumann J.and Morgenstern O.(1944).Theory of Games and Economic Behavior, Princeton .

Picard, P. and Lefevre, C. (1997).The probability of ruin in finite time with discrete claim size distribution. *Scand. Actuarial J. 1, 58-69*

Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Actuarial J. 1, 55-68.*

Schmidli, H. (2002). On minimizing the ruin probability by investment and reinsurance. Preprint, University of Aarhus.

Schmitter, H. (2001). Setting optimal reinsurance retentions. Swiss Reinsurance Company .

Straub E. (1988). Non-life insurance mathematics , Springer.

Suijs, J. , Borm, P., De Waegenaere, A. (1998).Stochastic cooperative games in insurance. *Insurance:Mathematics &Economics, 22, 209-228.*

Taksar M. and Markussen C. (2002). Optimal Dynamic Reinsurance Policies for Large Insurance Portfolios. Finance and Stochastics, forthcoming.

Taylor, G. (1992). Risk exchange II. Optimal reinsurance contracts. *Scand. Actuarial J. 1, 40-59.*

Verbeek, H. G. (1966). On optimal reinsurance.*ASTIN Bulletin, vol. IV, Part I, 29-38.*

Waters, H. (1979). Excess of Loss Reinsurance Limits. *Scand. Actuarial J. 37-43.*

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