PURE-JUMP SEMIMARTINGALES

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ABSTRACT. A taxonomic hierarchy of pure-jump semimartingales is introduced. This hierarchy contains, in particular, the class of sigma-locally finite variation pure-jump processes. The members of this family can be explicitly characterized in terms of the predictable compensators of their jump measures. This family is also closed under stochastic integration and smooth transformations.

1. Introduction

Denote by $\mathscr{V}^d$ the set of finite variation pure-jump semimartingales, i.e., processes X whose jumps are absolutely summable on any finite time interval and such that $X = X_0 + \sum_{t\leq \cdot} \Delta X_t$. Equivalently, $X \in \mathscr{V}^d$ if $X = X_0 + x * \mu^X$ where μ^X is the jump measure of X and $x * \mu^X$ represents the standard jump measure integral (Jacod and Shiryaev 2003, II.1.5).

Consider now the $\mathbb{R}$–valued stochastic process X defined by the following properties:

$X_0 = 0$;

X has independent increments;

jumps of X occur only at fixed times $2 - 1/n$, for each $n \in \mathbb{N}$;

the process jumps by $\pm 1/n$ with equal probability, for each $n \in \mathbb{N}$;

X remains constant outside the fixed jump times.

This process is a well-defined semimartingale, in fact a uniformly integrable martingale, on the whole time line $[0,\infty)$. Moreover, for every $n \in \mathbb{N}$ the stopped process $X^{2-1/n}$ is in $\mathscr{V}^d$ so X is a limit of elements in $\mathscr{V}^d$ in this case. Yet X itself is not equal to the sum of its jumps in the conventional sense because the jumps of X are not absolutely summable. In particular, the standard integral $x * \mu^X$ diverges. Furthermore, if we denote the predictable compensator of μ^X by ν^X, the integral $x * \nu^X$ also diverges even though the drift of X (the predictable part in its Doob-Meyer decomposition) is zero.

In this paper we propose to view the process X in two novel, complementary ways, both of which involve an approximation by elements in $\mathscr{V}^d$. The first approach regards X as an element of $\mathscr{V}^d_\sigma$, i.e., as a process that belongs sigma-locally to $\mathscr{V}^d$. This leads to the convenient formula $X = X_0 + x \star \mu^X$, where $\star$ is the sigma-localized version of the standard jump measure integral $*$. The new integral, unlike $*$, will be associative so that $\zeta \cdot (\psi \star \mu^X) = (\zeta\psi) \star \mu^X$ for predictable processes ζ if the left-hand side is well defined. Furthermore, the drift of X, provided it exists, will be given by $x \star \nu^X$, also defined by sigma localization. Section 3 collects the precise statements and proofs concerning the new integrals and their connections to $\mathscr{V}^d_\sigma$.

The second approach views X as a sum of its jumps at a sequence of stopping times with convergence in the Émery semimartingale topology. Here it is in principle possible to encounter two different processes that share the same jump measure (by choosing different exhausting sequence of stopping times in each case). However, we show that such a situation cannot occur in $\mathscr{V}^d_\sigma$. This offers a wider sense in which all processes in $\mathscr{V}^d_\sigma$ are uniquely determined by their

Date: September 9, 2019.

2010 *Mathematics Subject Classification.* Primary: 60H05; 60G07; 60G48; 60G51; 60H05.

Key words and phrases. Jump measure; Lévy process; predictable compensator; semimartingale topology; stochastic calculus.

jump measure. In contrast, the family of quadratic pure-jump processes (see Definition 1.1) lacks such uniqueness property as adding a continuous deterministic finite variation process to X yields a different quadratic pure-jump process with the same jump measure.

The following definition and theorem provide a precise formulation of the relationships among the various families of pure-jump processes. For notation and setup, see Section 2. For a review of the semimartingale topology on the space of semimartingales $\mathscr{S}$, see Section 4. Theorem 1.2 is proved in Section 5.

Definition 1.1. Consider the following subsets of $\mathscr{S}$.

- $\mathscr{J}^1$: the class of quadratic pure-jump processes, i.e., those semimartingales X that satisfy $[X,X]^c = [X,X] - x^2 * \mu^X = 0$ (see Protter 1990, p. 63).
- $\mathscr{J}^2$: the class of pure-jump processes, i.e., those semimartingales X that satisfy
$$X = X_0 + \sum_{k=1}^{\infty} \Delta X_{\tau_k} \mathbf{1}_{[\![\tau_k,\infty[\![}$$
in the semimartingale topology for a family $(\tau_k)_{k\in\mathbb{N}}$ of stopping times.
- $\mathscr{J}^3$: the class of strong pure-jump processes, i.e., those semimartingales $X \in \mathscr{J}^2$ that satisfy $X = Y$ for all $Y \in \mathscr{J}^2$ with $\mu^Y = \mu^X$ and $Y_0 = X_0$.
- $\mathscr{J}^4 = \mathscr{V}^d_\sigma$: the sigma-localized class of finite variation pure-jump processes.
- $\mathscr{J}^5 = \mathscr{V}^d$: the class of finite variation pure-jump processes, i.e., those semimartingales X that satisfy $X = X_0 + x * \mu^X$.
- $\mathscr{J}^6$: the class of piecewise constant processes with finitely many jumps on each finite time interval. □

Theorem 1.2. *We always have*
$$\mathscr{J}^1 \supsetneq \mathscr{J}^2 \supset \mathscr{J}^3 \supset \mathscr{J}^4 \supset \mathscr{J}^5 \supsetneq \mathscr{J}^6. \tag{1.1}$$
In general, these set inclusions are strict. More precisely, there exists a filtered probability space such that simultaneously we have
$$\mathscr{J}^2 \supsetneq \mathscr{J}^3 \supsetneq \mathscr{J}^4 \supsetneq \mathscr{J}^5. \tag{1.2}$$
Moreover, the following statements hold.

(i) *For all $i \in \{1,2,3,4\}$, the families $\mathscr{J}^i$ equal their sigma-localized class $\mathscr{J}^i_\sigma$; that is, $\mathscr{J}^i = \mathscr{J}^i_\sigma$. Furthermore, by definition, $\mathscr{J}^4 = \mathscr{J}^5_\sigma$.*
(ii) *For all $i \in \{1,2,3,4,6\}$, the families $\mathscr{J}^i$ are closed under stochastic integration.*
(iii) *For all $i \in \{1,\dots,6\}$, the families $\mathscr{J}^i$ are invariant under equivalent measure changes. More precisely, with the obvious notation, if Q is locally absolutely continuous with respect to P we have $\mathscr{J}^i(\mathsf{P}) \subset \mathscr{J}^i(\mathsf{Q})$ for all $i \in \{1,\dots,6\}$.*

The strictness of the inclusion $\mathscr{J}^4 \subsetneq \mathscr{J}^3$ is of interest. It says that there exist strong pure-jump processes, i.e., pure-jump processes uniquely determined by their jump measure, that are not sigma-locally of finite variation. To prove the strictness of this inclusion, Subsection 5.6 contains a specific example of such a process X. This example relies on a jump measure μ^X with predictable compensator ν^X that supports a countable set of jump sizes.

To gain insight, consider the disintegrated form $\nu^X(\mathrm{d}t,\mathrm{d}x) = F^X_t(\mathrm{d}x)\mathrm{d}A^X_t$, where F^X is a transition kernel (see Section 2 for more details). The jump measure μ^X in this example relies on a kernel F^X that has large atoms in a neighbourhood of zero. As it turns out, this example is canonical. Indeed, Corollary 1.3 below states if X is a strong pure-jump process whose associated jump size kernel does not allow for too many large atoms, then X must be sigma-locally of finite variation.

We have already observed that a process $X \in \mathscr{J}^4 \subset \mathscr{J}^3$ is uniquely described by its jump measure. The following corollary of the proof of Theorem 1.2 provides explicit characterizations of the processes in $\mathscr{J}^4$ in relation to the bigger classes $\mathscr{J}^1$, $\mathscr{J}^2$, and $\mathscr{J}^3$. A further analytic representation for such processes is provided in Proposition 3.12 below.

Corollary 1.3. *Let X denote a process. Then the following statements are equivalent.*

(i) $X \in \mathscr{J}^4 = \mathscr{V}^d_\sigma$.

(ii) $X \in \mathscr{J}^3$ *and*

$$\left(\limsup_{x\downarrow 0} xF^X(\{x\})\right) \wedge \left(\limsup_{x\uparrow 0} |x|F^X(\{x\})\right) = 0, \qquad (\mathsf{P}\times \mathrm{d}A^X)\text{–}a.e. \tag{1.3}$$

(iii) $X \in \mathscr{J}^2$ *and*

$$\int |x|\mathbf{1}_{|x|\leq 1}F^X(\mathrm{d}x) < \infty, \qquad (\mathsf{P}\times \mathrm{d}A^X)\text{–}a.e. \tag{1.4}$$

(iv) $X \in \mathscr{J}^1$, (1.4) *holds,* $\int_0^\cdot |\int x\mathbf{1}_{|x|\leq 1}F^X_t(\mathrm{d}x)|\mathrm{d}A^X_t < \infty$, *and*

$$B^{X[1]} = \int_0^\cdot \left(\int x\mathbf{1}_{|x|\leq 1}F^X_t(\mathrm{d}x)\right)\mathrm{d}A^X_t.$$

Here $B^{X[1]}$ denotes the drift of $X - x\mathbf{1}_{|x|>1} * \mu^X$; see also Section 2.

Corollary 1.3 is proved in Subsection 5.5. Note that the condition in (1.3) is satisfied, for example, if F^X is atomless $(\mathsf{P}\times \mathrm{d}A^X)$–a.e.

Here now the outline of this paper. Section 2 introduces the notation used and the setup of this paper. Section 3 provides the definition and important properties of the extended integral with respect to an integer-valued random measure. This section also contains an analytic representation of the elements in $\mathscr{J}^4 = \mathscr{V}^d_\sigma$, the class of sigma-locally finite variation pure-jump processes. Section 4 collects relevant results concerning Émery's semimartingale topology and Section 5 provides the proof of Theorem 1.2. Finally, Section 6 briefly discusses the consequences of choosing ucp convergence in place of the semimartingale topology.

2. Notation and setup

We fix a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with a right-continuous filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$. Recall from Jacod and Shiryaev (2003, II.1.4) that a function $\eta : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is called predictable if it is $\mathcal{P}\times\mathcal{B}(\mathbb{R})$–measurable, where $\mathcal{P}$ denotes the predictable sigma field and $\mathcal{B}(\mathbb{R})$ the Borel sigma field on $\mathbb{R}$. If $\psi : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ denotes another (predictable) function we shall write $\psi(\eta)$ to denote the (predictable) function $(\omega, t, x) \mapsto \psi(\omega, t, \eta(\omega, t, x))$.

We shall consider an integer-valued random measure μ on $[0,\infty)\times\mathbb{R}$ with predictable compensator ν. A predictable function η with $\eta(0) = 0$ is integrable with respect to μ, i.e., $\eta * \mu$ exists if $|\eta| * \mu < \infty$. Recall from Jacod and Shiryaev (2003, II.2.9) that ν can be written in disintegrated form as

$$\nu(\mathrm{d}t, \mathrm{d}x) = F_t(\mathrm{d}x)\mathrm{d}A_t, \qquad t \geq 0,\ x \in \mathbb{R}, \tag{2.1}$$

where A is a predictable process, and F is a transition kernel from $(\Omega\times[0,\infty), \mathcal{P})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If we want to emphasize the probability measure under which ν is the predictable compensator of μ we shall write $\nu(\mathsf{P})$.

We let $\mathscr{S}$ denote the space of $\mathbb{R}$–valued semimartingales. For a semimartingale $X \in \mathscr{S}$, we let X_- denote its left-limit process with the convention $X_{0-} = X_0$ and we let $\Delta X = X - X_-$ denote its jump process. Next, we let μ^X denote the jump measure of X and ν^X its predictable compensator. For a predictable function η with $\eta(0) = 0$ we then have $\eta * \mu^X = \sum_{0<t\leq\cdot} \eta(\Delta X_t)$ if $|\eta| * \mu^X < \infty$. The corresponding quantities in (2.1) shall be written with a superscript X. If X is special, we write B^X for its drift, i.e., the predictable finite variation part of X, always assumed to start in zero, i.e., $B_0^X = 0$. If Y denotes another semimartingale then $[X,Y]$ denotes the quadratic covariation of X and Y. Moreover, we write $X[1] = X - x\mathbf{1}_{|x|>1} * \mu^X$ and note that $X[1]$ is special. Next, $L(X)$ denotes the family of X–integrable predictable processes.

If $\mathscr{J} \subset \mathscr{S}$ is a family of semimartingales, we say that a semimartingale Y belongs to $\mathscr{J}_\sigma$, the sigma–localized class of $\mathscr{J}$, if there is a sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets increasing to $\Omega\times[0,\infty)$ such that $\mathbf{1}_{D_n}\cdot Y \in \mathscr{J}$ for each $n \in \mathbb{N}$. We say that $\mathscr{J}$ is stable under sigma-stopping (see Kallsen 2004, Definition 2.1) if for every $X \in \mathscr{J}$ and every predictable set D the process $\mathbf{1}_D \cdot X$ belongs to $\mathscr{J}$. Finally, we shall say that Q is a probability measure that is locally absolutely continuous with respect to P if Q is absolutely continuous with respect to P on $\mathscr{F}_t$ for each $t \geq 0$.

Remark 2.1. Throughout this paper, we only consider the scalar case, which helps in reducing notation. The careful reader can convince themselves that quite a few results (in particular those of Section 3) generalize to the higher-dimensional case, for example when X takes values in $\mathbb{R}^d$ and the predictable functions below map into $\mathbb{R}^n$, etc., for some $d, n \in \mathbb{N}$. A notable exemption is statement (ii) in Theorem 1.2, where we do not know whether the one-dimensional situation generalizes. Indeed, we do not know whether $\mathscr{J}^2$ is a vector space — the lack of such structure would seem to imply that such a result does not hold in higher dimensions. □

3. Extended integral with respect to a random measure

We start by extending the standard definition of integral with respect to a random measure and derive some basic properties in Subsection 3.1. Then, in Subsection 3.2, we prove some associativity properties of this integral. In Subsection 3.3 we connect the integral to the representation of sigma-locally finite variation pure-jump processes. In particular, this will enable us to write the process X of the introduction as $X = x \star \mu^X$ and its drift under any locally absolutely continuous measure Q as $B^X(\mathsf{Q}) = x \star \nu^X(\mathsf{Q})$.

3.1. Definition and basic properties of the extended integral.

Definition 3.1 (Extended integral with respect to random measure)**.**

(i) Denote by $L(\mu)$ the set of predictable processes that are absolutely integrable with respect to μ. We say that a predictable function η belongs to $L_\sigma(\mu)$, the sigma–localized class of $L(\mu)$, if there is a sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets increasing to $\Omega\times[0,\infty)$ and a semimartingale Y such that $\mathbf{1}_{D_n}\eta \in L(\mu)$ for each $n \in \mathbb{N}$ and
$$(\mathbf{1}_{D_n}\eta) * \mu = \mathbf{1}_{D_n}\cdot Y, \qquad n \in \mathbb{N}.$$
In such case the semimartingale Y is denoted by $\eta \star \mu$.

(ii) Denote by $L(\nu)$ the set of predictable processes that are absolutely integrable with respect to ν. We say that a predictable function η belongs to $L_\sigma(\nu)$, the sigma–localized class of $L(\nu)$, if there is a sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets increasing to $\Omega\times[0,\infty)$ and a semimartingale Y such that $\mathbf{1}_{D_n}\eta \in L(\nu)$ for each $n \in \mathbb{N}$ and
$$(\mathbf{1}_{D_n}\eta) * \nu = \mathbf{1}_{D_n}\cdot Y, \qquad n \in \mathbb{N}.$$
In such case the semimartingale Y is denoted by $\eta \star \nu$. □

Note that if $\mu = \nu$ is a predictable random measure then the two definitions above agree; hence $L_\sigma(\mu)$ and $L_\sigma(\nu)$ are well-defined and we have $L_\sigma(\mu) = L_\sigma(\nu)$. Note also that $\eta\star\mu$ (resp., $\eta \star \nu$) is uniquely defined provided that $\eta \in L_\sigma(\mu)$ (resp., $\eta \in L_\sigma(\nu)$).

Remark 3.2. Let Q denote a probability measure locally absolutely continuous with respect to P. With the obvious notation, we then have $L_\sigma^{\mathsf{P}}(\mu) \subset L_\sigma^{\mathsf{Q}}(\mu)$. For $L_\sigma^{\mathsf{P}}(\nu(\mathsf{P}))$ and $L_\sigma^{\mathsf{Q}}(\nu(\mathsf{Q}))$, no such inclusions hold in general. However, refer also to the positive statement in Remark 3.5. □

The following characterization of $L_\sigma(\nu)$ appears in the literature.

Lemma 3.3 (Kallsen 2004, Definition 4.1 and Lemma 4.1)**.** *For a predictable function η the following statements are equivalent.*

(i) $\eta \in L_\sigma(\nu)$.
(ii) *The following two conditions hold.*
  (a) $\int |\eta_t(x)| F_t(\mathrm{d}x) < \infty \quad (\mathsf{P} \times \mathrm{d}A)$*-a.e.*
  (b) $\int |\int \eta_t(x) F_t(\mathrm{d}x)| \,\mathrm{d}A_t < \infty$.

Moreover, for $\eta \in L_\sigma(\nu)$ one has

$$\eta \star \nu = \int_0^\cdot \left(\int \eta_t(x) F_t(\mathrm{d}x) \right) \mathrm{d}A_t.$$

To the best of our knowledge, the class $L_\sigma(\mu)$ has not been studied previously. The following characterization seems to be new.

Proposition 3.4. *For a predictable function η the following statements are equivalent.*

(i) $\eta \in L_\sigma(\mu)$.
(ii) *The following two conditions hold.*
  (a) $|\eta|^2 * \mu < \infty$.
  (b) $\eta \mathbf{1}_{\{|\eta| \leq 1\}} \in L_\sigma(\nu)$.

Furthermore, for $\eta \in L_\sigma(\mu)$ one has

$$\eta \star \mu = \eta \mathbf{1}_{\{|\eta|>1\}} * \mu + \eta \mathbf{1}_{\{|\eta| \leq 1\}} * (\mu - \nu) + \eta \mathbf{1}_{\{|\eta| \leq 1\}} \star \nu, \tag{3.1}$$

where the integral with respect to the compensated measure $\mu - \nu$ is defined in Jacod and Shiryaev (2003, II.1.27(b)).

Remark 3.5. In the setup of Remark 3.2, choose a predictable function η with $|\eta|^2 * \mu < \infty$. Proposition 3.4 now yields that if $\eta \mathbf{1}_{\{|\eta| \leq 1\}} \in L_\sigma^{\mathsf{P}}(\nu(\mathsf{P}))$ then also $\eta \mathbf{1}_{\{|\eta| \leq 1\}} \in L_\sigma^{\mathsf{Q}}(\nu(\mathsf{Q}))$. □

Proof of Proposition 3.4. In the following we argue both inclusions and (3.1).

(i)⇒(ii): Let $(D_n)_{n \in \mathbb{N}}$ be as in Definition 3.1(i). Then $\mathbf{1}_{D_n} |\eta|^2 * \mu = \mathbf{1}_{D_n} \cdot [\eta \star \mu, \eta \star \mu]$ for all $n \in \mathbb{N}$, and a monotone convergence argument yields $|\eta|^2 * \mu = [\eta \star \mu, \eta \star \mu] < \infty$. Let us now set $\overline{\eta} = \eta \mathbf{1}_{\{|\eta| \leq 1\}}$. Then $\overline{\eta} \in L_\sigma(\mu)$ and we directly get

$$\int_0^\cdot \mathbf{1}_{D_n}(t) \left(\int |\overline{\eta}_t(x)| F_t(\mathrm{d}x) \right) \mathrm{d}A_t = \mathbf{1}_{D_n} |\overline{\eta}| * \nu < \infty.$$

Thanks to Lemma 3.3 we now only need to argue that $\int |\int \overline{\eta}_t(x) F_t(\mathrm{d}x)| \,\mathrm{d}A_t < \infty$. We note that $|\Delta(\overline{\eta} \star \mu)| \leq 1$, hence $\overline{\eta} \star \mu$ is special, say with predictable finite variation drift $\overline{B}$. By monotone convergence, we now get

$$\begin{aligned} \int_0^\cdot \left| \int \overline{\eta}_t(x) F_t(\mathrm{d}x) \right| \mathrm{d}A_t &= \lim_{n \uparrow \infty} \int_0^\cdot \mathbf{1}_{D_n}(t) \left| \int \overline{\eta}_t(x) F_t(\mathrm{d}x) \right| \mathrm{d}A_t = \lim_{n \uparrow \infty} \int_0^\cdot \mathbf{1}_{D_n}(t) |\mathrm{d}\overline{B}_t| \\ &= \int_0^\cdot |\mathrm{d}\overline{B}_t| < \infty. \end{aligned}$$

This yields $\overline{\eta} \in L_\sigma(\nu)$, hence the implication (i)⇒(ii) is shown.

(ii)⇒(i) and (3.1): Let $(D_n)_{n \in \mathbb{N}}$ be as in Definition 3.1(ii). Note that all terms on the right-hand side of (3.1) are well defined and yield a semimartingale Y provided that (ii) holds. Thanks to the uniqueness of $\eta \star \mu$ we only need to observe that $\mathbf{1}_{D_n} \cdot Y = (\mathbf{1}_{D_n} \eta) * \mu$ for all $n \in \mathbb{N}$. However, this is straightforward, which concludes the proof of the proposition. □

Remark 3.6. Note that $L_{\mathrm{loc}}(\mu) = L(\mu)$, that is $L(\mu)$ is closed under standard localization. However, we have $L_\sigma(\mu) \supsetneq L(\mu)$ on sufficiently large probability spaces; see Example 3.7. □

Example 3.7. We now provide an example for a jump measure μ such that $x \in L_\sigma(\mu) \setminus L(\mu)$. Thanks to Jacod and Shiryaev (2003, III.1.14) a jump measure μ with compensator $\nu(\mathrm{d}t, \mathrm{d}x) = |x|^{-2+t}\mathbf{1}_{|x|<1}\mathrm{d}x\mathrm{d}t$ exists. Note that

$$x^2 * \nu = \int_0^\cdot \left(\int_{-1}^1 |x|^t \mathrm{d}x \right) \mathrm{d}t = \int_0^\cdot \frac{2}{1+t} \mathrm{d}t < \infty;$$

hence $x^2 * \mu < \infty$. Moreover, $\int_{-1}^1 |x|^{-1+t}\mathrm{d}x < \infty$ for all $t > 0$. Hence $x \in L_\sigma(\nu)$ by Lemma 3.3 and $x \in L_\sigma(\mu)$ by Proposition 3.4. However, since $|x| * \nu = \infty$ we have $|x| * \mu = \infty$ and $x \notin L(\mu)$. □

3.2. **Associativity properties of the extended integral.** We remind the reader that μ without a superscript refers to a given integer-valued random measure, while μ^X refers to the jump measure of a semimartingale X; see Section 2.

Proposition 3.8. *Let $\eta \in L_\sigma(\mu)$ and $\psi : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a predictable function. Then the following statements are equivalent.*

(i) $\psi \in L_\sigma(\mu^{\eta\star\mu})$.
(ii) $\psi(\eta) \in L_\sigma(\mu)$.

Furthermore, if either condition holds then $\psi \star (\eta \star \mu) = \psi(\eta) \star \mu$. Moreover, the same assertions hold with μ replaced by ν.

Proof. Let us first prove the statement with μ replaced by ν. To this end, note that

$$\nu^{\eta\star\nu}(\mathrm{d}t, \mathrm{d}x) = \overline{F}_t(\mathrm{d}x)\mathrm{d}A_t, \qquad t \geq 0,\, x \in \mathbb{R},$$

where $\overline{F}_t$ is the image of measure F_t under η_t. Then the equivalence follows from Lemma 3.3. The statement for μ follows exactly in the same manner, now using Proposition 3.4. □

Next, we prove a composition property for stochastic integrals.

Proposition 3.9. *Let $\eta \in L_\sigma(\mu)$ and $\zeta : \Omega \times [0, \infty) \to \mathbb{R}^n$ be a predictable process. Then the following statements are equivalent.*

(i) $\zeta \in L(\eta \star \mu)$.
(ii) $\zeta\eta \in L_\sigma(\mu)$.

Furthermore, if either condition holds then $\zeta \cdot (\eta \star \mu) = (\zeta\eta) \star \mu$. Moreover, the same assertions hold with μ replaced by ν.

Proof. We shall prove the statement only for μ as the same argument works if μ is replaced by ν. Note that there is a sigma–localizing sequence $(D_n)_{n\in\mathbb{N}}$ such that

$$\{(\omega, t) \in \Omega \times [0, \infty) : |\zeta(\omega, t)| \leq n\} \subset D_n$$

and $\mathbf{1}_{D_n}\eta \in L(\mu)$ with $\mathbf{1}_{D_n} \cdot (\eta \star \mu) = (\mathbf{1}_{D_n}\eta) * \mu$. Thanks to Kallsen (2004, Lemma 2.2) and Definition 3.1(i) it suffices to observe that the statement holds with ζ replaced by $\zeta\mathbf{1}_{D_n}$ for each $n \in \mathbb{N}$. □

Proposition 3.10. *Let $\eta \in L_\sigma(\mu)$ and $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Then for $Y = Y_0 + \eta \star \mu$ we have $\xi = f(Y_- + \eta) - f(Y_-) \in L_\sigma(\mu)$ and $f(Y) = f(Y_0) + \xi \star \mu$.*

Proof. Assume for the moment that we have argued $\xi \in L_\sigma(\mu)$. Then $\Delta f(Y) = \Delta(\xi \star \mu)$ and the statement follows by sigma localization. Let us now argue that $\xi \in L_\sigma(\mu)$. First, note that $\xi\mathbf{1}_{|\eta|>1} \in L(\mu) \subset L_\sigma(\mu)$. Hence we may and shall assume that $|\eta| \leq 1$ from now on. By localization we may also assume that $\partial/(\partial y)f(Y_-)$ and $\sup_{z\in[-1,1]} \partial^2/(\partial^2 y)f(Y_- + z)$ are bounded. The statement follows now from Proposition 3.4 and Taylor's theorem. □

Example 3.11. As a counterpoint to Proposition 3.9, we will now exhibit an integer-valued random measure μ with the following properties.

(1) $x \in L_\sigma(\mu)$.
(2) $x \star \mu$ is quasi-left-continuous with bounded jumps.
(3) there is a predictable process $\zeta \in L(x * (\mu - \nu))$ such that $\zeta \notin L(x \star \mu)$.

To this end, let N denote a standard Poisson process. That is, N jumps up by one with standard exponentially distributed waiting times and $B^N_t = t$ for all $t \geq 0$. Let now $\varphi_t = {}^1\!/\!_k$ for all $t \in [k-1, k)$ and all $k \in \mathbb{N}$, and fix $n \in \{1, 2\}$. Then $\varphi^n \in L(N) \cap L(B^N)$ and $\varphi^n \cdot N = \varphi^n \cdot (N - B^N) + \varphi^n \cdot B^N$ is the sum of a uniformly integrable martingale and an increasing process (of bounded variation in the case $n = 2$). Indeed, Kolmogorov's two-series theorem, applied to the sequence $(\varphi^n \cdot (N - B^N)_k)_{k \in \mathbb{N}}$, and an application of the Borel-Cantelli lemma, or Larsson and Ruf (2018, Corollary 4.4), yield the existence of the random variable $\varphi^n \cdot (N - B^N)_\infty = \lim_{t \uparrow \infty} \varphi^n \cdot (N - B^N)_t$. The Burkholder-Davis-Gundy inequality yields that $\varphi^n \cdot (N - B^N)$ is a uniformly integrable martingale as claimed.

In particular, it follows that the process $Y_t = \varphi^2 \cdot N_{\tan(t \wedge \pi/2)}$ is a special semimartingale on the whole time line with

$$B^Y_t = \int_0^{\tan(t \wedge \pi/2)} \varphi^2_u \mathrm{d}u, \qquad t \geq 0.$$

Statements (1)–(3) now follow by taking $\mu = \mu^Y$, $\zeta_t = 1/\varphi_{\tan(t)} \mathbf{1}_{t < \pi/2}$ and observing that $x \star \mu = Y$ and

$$\zeta \cdot (x * (\mu - \nu))_t = \zeta \cdot (Y - B^Y)_t = \varphi \cdot (N - B^N)_{\tan(t \wedge \pi/2)}, \qquad t \geq 0.$$

From $\lim_{t \uparrow \infty} \varphi \cdot B^N_t = \infty$ we obtain $\zeta \notin L(B^Y)$, whereby $\zeta \in L(Y - B^Y)$ yields $\zeta \notin L(Y) = L(x \star \mu)$. □

3.3. **Sigma-locally finite variation pure-jump processes and the extended integral.** The statements in the previous subsections can also be expressed in terms of the class $\mathscr{V}^{\mathrm{d}}_\sigma$.

Proposition 3.12. *If $\eta \in L_\sigma(\mu)$ then $\eta \star \mu \in \mathscr{V}^{\mathrm{d}}_\sigma$. Conversely, if $X \in \mathscr{V}^{\mathrm{d}}_\sigma$ then $x \in L_\sigma(\mu^X)$ and*

$$\begin{aligned} X &= X_0 + x \star \mu^X \\ &= X_0 + x\mathbf{1}_{|x|>1} * \mu^X + x\mathbf{1}_{|x|\leq 1} * (\mu^X - \nu^X) + \int_0^\cdot \left(\int x \mathbf{1}_{|x| \leq 1} F^X_t(\mathrm{d}x) \right) \mathrm{d}A^X_t. \end{aligned} \tag{3.2}$$

Proof. The first part of the assertion follows directly from the definitions of $L_\sigma(\mu^X)$ and $\mathscr{V}^{\mathrm{d}}_\sigma$. The second equality in (3.2) is the consequence of Lemma 3.3 and Proposition 3.4. □

Corollary 3.13. *Let $X \in \mathscr{V}^{\mathrm{d}}_\sigma$, $\zeta \in L(X)$, and f be a twice continuously differentiable function. Then $\zeta \cdot X, f(X) \in \mathscr{V}^{\mathrm{d}}_\sigma$ with*

$$\begin{aligned} \zeta \cdot X &= (\zeta x) \star \mu^X; \\ f(X) &= f(X_0) + (f(X_- + x) - f(X_-)) \star \mu^X. \end{aligned}$$

Proof. This follows from Proposition 3.12 in conjunction with Propositions 3.9 and 3.10. □

Yoeurp (1976) has shown that every local martingale can be uniquely decomposed into two components, one quasi-left-continuous and the other with jumps only at predictable times, such that the quadratic covariation of the two components is zero. This motivates the following result.

Proposition 3.14. *Every semimartingale X has the unique decomposition*

$$X = X_0 + X^{\mathrm{qc}} + X^{\mathrm{dp}}, \tag{3.3}$$

where $X^{\mathrm{qc}}_0 = X^{\mathrm{dp}}_0 = 0$, X^{qc} is a quasi-left-continuous semimartingale, X^{dp} jumps only at predictable times, and $X^{\mathrm{dp}} \in \mathscr{V}^{\mathrm{d}}_\sigma$. We then have $[X^{\mathrm{qc}}, X^{\mathrm{dp}}] = 0$.

Proof. Let τ denote any predictable time. Note that $\Delta X_\tau = \Delta X^{\mathrm{dp}}_\tau$ for any decomposition of X by the quasi-left-continuity of X^{qc}. This proves the uniqueness of the decomposition. Consider now the predictable process $(x^2 \wedge 1) * \nu^X$. Applying Jacod and Shiryaev (2003, I.2.24) yields a family $(\tau_k)_{k\in\mathbb{N}}$ of predictable times that exhausts its jumps. Define next the bounded predictable process

$$\zeta = \mathbf{1}_{\{\nu^X(\{\cdot\})>0\}} = \sum_{k=1}^{\infty} \mathbf{1}_{[\![\tau_k]\!]}.$$

Setting $X^{\mathrm{qc}} = (1-\zeta)\cdot X$ and $X^{\mathrm{dp}} = \zeta \cdot X$ then yields the decomposition in (3.3), the quasi-left-continuity of X^{qc}, and $[X^{\mathrm{qc}}, X^{\mathrm{dp}}] = 0$. Finally, setting $D_n = (\Omega \times [0,\infty)) \setminus \bigcup_{k=n}^{\infty} [\![\tau_k]\!]$ in Definition 3.1(i) for each $n \in \mathbb{N}$ yields $X^{\mathrm{dp}} \in \mathscr{V}^{\mathrm{d}}_\sigma$. □

Observe that the family of predictable stopping times $\mathcal{T} = (\tau_k)_{k\in\mathbb{N}}$ from the previous proof exhausts the jumps of X^{dp}. Simultaneously, Theorem 1.2 yields $X^{\mathrm{dp}} \in \mathscr{J}^2$. A priori, it is not clear that $\mathcal{T}$ is good enough to approximate X^{dp} in $\mathscr{J}^2$ because the membership of $\mathscr{J}^2$ only ever guarantees one exhausting sequence of stopping times (with the desired convergence property) and that sequence is not even predictable in principle. The next result therefore appears to be rather strong.

Proposition 3.15. *For an arbitrary semimartingale X consider the process X^{dp} from Proposition 3.14. Let $\mathcal{T}$ be any sequence of predictable stopping times that exhausts the jumps of X^{dp}. Then $\mathcal{T}$ also approximates X^{dp} in $\mathscr{J}^2$, i.e., we have*

$$X^{\mathrm{dp}} = \sum_{\tau\in\mathcal{T}} \Delta X_\tau \mathbf{1}_{[\![\tau,\infty[\![}$$

in the semimartingale topology.

Proof. Apply Lemma 4.3(iii) below with the same sequence $(D_n)_{n\in\mathbb{N}}$ as in the proof of Proposition 3.14. □

For another statement about the summability of jumps of a (semi)martingale X at predictable times, see Galtchouk (1980).

4. Émery's semimartingale topology

We now briefly review the definition and basic facts of the semimartingale topology (in short, $\mathscr{S}$–topology), introduced by Émery (1979).

Definition 4.1. Let $(X^{(k)})_{k\in\mathbb{N}} \subset \mathscr{S}$ denote a sequence of semimartingales. We say that this sequence converges to $X \in \mathscr{S}$ in the semimartingale topology (in short, $\mathscr{S}$–topology) if

$$\lim_{k\uparrow\infty}\left(\sup_{\zeta:|\zeta|\le 1} \mathsf{E}\left[\left|\zeta_0 X^{(k)}_0 + \zeta\cdot X^{(k)}_t - \zeta_0 X_0 - \zeta\cdot X_t\right| \wedge 1\right]\right) = 0 \tag{4.1}$$

for all $t \ge 0$, where the supremum is taken over all predictable processes ζ with $|\zeta| \le 1$. □

The space $\mathscr{S}$ equipped with this topology is a complete metric space (Émery 1979, Theoreme 1), say with distance $d_{\mathscr{S}}$. Note that if a sequence $(X^{(k)})_{k\in\mathbb{N}} \subset \mathscr{S}$ converges in the $\mathscr{S}$–topology it also converges in the sense of uniform convergence on compacts in probability.

Remark 4.2. In contrast to Émery (1979), we have not assumed (nor excluded) that the underlying filtration $\mathfrak{F}$ be augmented by the P–null sets. Nevertheless, the cited results by Émery (1979) below can be applied by choosing appropriate process modifications. For example, $\mathscr{S}$ equipped with the $\mathscr{S}$–topology is a complete metric space as any limit (in the augmented filtration) can be identified with an $\mathfrak{F}$–semimartingale by taking appropriate modifications. See, for example, Perkowski and Ruf (2015, Appendix A) for a summary of these techniques. □

We now collect some well known facts concerning the $\mathscr{S}$–topology.

Lemma 4.3. *Let $(X^{(k)})_{k\in\mathbb{N}} \subset \mathscr{S}$ denote a sequence of semimartingales with $X_0^{(k)} = 0$. Then the following statements hold.*

(i) *If the sequence $(X^{(k)})_{k\in\mathbb{N}}$ converges locally in the $\mathscr{S}$–topology then it also converges in the $\mathscr{S}$–topology.*
(ii) *If $\lim_{k\uparrow\infty} X^{(k)} = X$ in the $\mathscr{S}$–topology for some semimartingale $X \in \mathscr{S}$ and if D is a predictable set then $\lim_{k\uparrow\infty}(\mathbf{1}_D \cdot X^{(k)}) = \mathbf{1}_D \cdot X$ in the $\mathscr{S}$–topology.*
(iii) *If $(D_k)_{k\in\mathbb{N}}$ is a nondecreasing sequence of predictable sets such that $\bigcup_{k\in\mathbb{N}} D_k = \Omega\times[0,\infty)$ and X is a semimartingale with $X_0 = 0$ then $\lim_{k\uparrow\infty}(\mathbf{1}_{D_k} \cdot X) = X$ in the $\mathscr{S}$–topology.*
(iv) *If $\lim_{k\uparrow\infty} X^{(k)} = X$ in the $\mathscr{S}$–topology for some semimartingale $X \in \mathscr{S}$ we have*
$$\lim_{k\uparrow\infty}[X^{(k)}, X^{(k)}] = [X, X] \qquad \text{and} \qquad \lim_{k\uparrow\infty}[X^{(k)}, X^{(k)}]^c = [X, X]^c$$
in the $\mathscr{S}$–topology.
(v) *If $\lim_{k\uparrow\infty} X^{(k)} = X$ in the $\mathscr{S}$–topology for some semimartingale $X \in \mathscr{S}$ and if $X^{(k)}$ is predictable for each $k \in \mathbb{N}$ then X has a predictable version.*
(vi) *Assume that the probability measure Q is locally absolutely continuous with respect to P. If $\lim_{k\uparrow\infty} X^{(k)} = X$ in the $\mathscr{S}$–topology for some semimartingale $X \in \mathscr{S}$ under P then also $\lim_{k\uparrow\infty} X^{(k)} = X$ in the $\mathscr{S}$–topology under Q.*

Proof. First, (i) and (ii) follow from the definition of $\mathscr{S}$–topology and (iii) and (iv) are argued in Émery (1979, Proposition 3 and Remarque 1 on p. 276). To see (v), recall that also $\lim_{k\uparrow\infty} X^{(k)} = X$ (in the sense of uniform convergence on compacts); hence also almost surely along a subsequence. In conjunction with Remark 4.2 this yields the claim. Finally, (vi) is proved by applying Émery (1979, Proposition 6) in conjunction with (i). □

Next, we consider sums of semimartingales and their convergence in the $\mathscr{S}$–topology.

Lemma 4.4. *Let $(X^{(k)})_{k\in\mathbb{N}} \subset \mathscr{S}$ denote a sequence of semimartingales with $X_0^{(k)} = 0$. Then the following statements hold.*

(i) *If there exists $C > 0$ such that $|\Delta X^{(k)}| \le C$ for each $k \in \mathbb{N}$, and if $\sum_{k=1}^\infty [X^{(k)}, X^{(k)}] < \infty$, then $\sum_{k=1}^\infty (X^{(k)} - B^{X^{(k)}})$ converges in the $\mathscr{S}$–topology to a local martingale.*
(ii) *If $X^{(k)}$ has finite variation on compacts for each $k \in \mathbb{N}$ and if $\sum_{k=1}^\infty \int_0^\cdot |\mathrm{d}X^{(k)}| < \infty$, then $\sum_{k=1}^\infty X^{(k)}$ converges in the $\mathscr{S}$–topology to a finite variation process.*
(iii) *Assume that $\sum_{k,l=1}^\infty [X^{(k)}, X^{(l)}]^- < \infty$. Then the following two statements are equivalent.*
  (I) *$\sum_{k=1}^\infty [X^{(k)}, X^{(k)}] < \infty$ and $\sum_{k=1}^\infty B^{X^{(k)}[1]}$ converges in the $\mathscr{S}$–topology to a process B.*
  (II) *$\sum_{k=1}^\infty X^{(k)}$ converges in the $\mathscr{S}$–topology to a process X.*
If one (hence both) of these conditions hold then $B^{X[1]} = B$. If additionally $[X^{(k)}, X^{(l)}] = 0$ for all $k, l \in \mathbb{N}$ with $k \neq l$ then we also have $\sum_{k=1}^\infty \Delta X^{(k)} = \Delta X$.

Proof. We first argue (i). By localization and by Lemma 4.3(i) we may assume that there is a constant $\kappa \ge 0$ such that $\sum_{k=1}^\infty [X^{(k)}, X^{(k)}] \le \kappa$. Next, fix for the moment $k \in \mathbb{N}$ and define the local martingale $M^{(k)} = X^{(k)} - B^{X^{(k)}}$. Let $(\tau_m)_{m\in\mathbb{N}}$ be a nondecreasing sequence of stopping times such that $[M^{(k)}, B^{X^{(k)}}]^{\tau_m}$ is a uniformly integrable martingale for each $m \in \mathbb{N}$. Then we have
$$\mathsf{E}\left[\left[M^{(k)}, M^{(k)}\right]_\infty\right] = \lim_{m\uparrow\infty} \mathsf{E}\left[\left[M^{(k)}, M^{(k)}\right]_{\tau_m}\right] \le \lim_{m\uparrow\infty} \mathsf{E}\left[\left[X^{(k)}, X^{(k)}\right]_{\tau_m}\right] = \mathsf{E}\left[\left[X^{(k)}, X^{(k)}\right]_\infty\right].$$
The Burkholder-Davis-Gundy inequality now yields a constant $\kappa' > 0$ such that
$$\sum_{k=1}^\infty \mathsf{E}\left[\left(M_\infty^{(k)}\right)^2\right] \le \kappa' \sum_{k=1}^\infty \mathsf{E}\left[\left[M^{(k)}, M^{(k)}\right]_\infty\right] \le \kappa' \sum_{k=1}^\infty \mathsf{E}\left[\left[X^{(k)}, X^{(k)}\right]_\infty\right] \le \kappa'\kappa.$$

Hence, Doob's inequality yields that $\sum_{k=1}^\infty M^{(k)}$ converges locally in $\mathcal{H}_2$ to a martingale; see, for example, Kunita and Watanabe (1967, Proposition 4.1) or Doléans-Dade and Meyer (1970, Lemme 1). Hence by Émery (1979, Theoreme 2), (i) follows.

Let us now argue (ii). First, $\sum_{k=1}^\infty X^{(k)}$ converges to a finite variation process X in the sense of uniform convergence on compacts in probability. Next, note that

$$\zeta \cdot \sum_{k=1}^{n} X^{(k)} - \zeta \cdot X \leq \sum_{k=n+1}^{\infty} \left| \mathrm{d}X^{(k)} \right|$$

for all predictable processes ζ with $|\zeta| \leq 1$. Hence, (ii) follows.

To see the implication from (I) to (II) in (iii), apply (i) to the sequence $(X^{(k)}[1])_{k\in\mathbb{N}}$ and (ii) to $(x\mathbf{1}_{|x|>1} * \mu^{X^{(k)}})_{k\in\mathbb{N}}$. For the reverse direction (II) to (I) note that since X is a semimartingale, the assumption and Lemma 4.3(iv) yield directly that $\sum_{k=1}^\infty [X^{(k)}, X^{(k)}] < \infty$. Moreover, as above, the sums corresponding to $(X^{(k)}[1] - B^{X^{(k)}[1]})_{k\in\mathbb{N}}$ and $(x\mathbf{1}_{|x|>1} * \mu^{X^{(k)}})_{k\in\mathbb{N}}$ converge in the $\mathscr{S}$–topology; hence so must the sums corresponding to $(B^{X^{(k)}[1]})_{k\in\mathbb{N}}$. Finally, if (I) and (II) hold then

$$X[1] = \sum_{k=1}^{\infty} \left(X^{(k)}[1] - B^{X^{(k)}[1]} \right) + B$$

in the $\mathscr{S}$–topology, where the first term is a local martingale by (i) and B may be assumed to be predictable and of finite variation thanks to Lemma 4.3(v)&(iv).

Let us additionally assume that $[X^{(k)}, X^{(l)}] = 0$ for all $k, l \in \mathbb{N}$ with $k \neq l$. Then the sum $\sum_{k=1}^\infty \Delta X^{(k)}$ is well defined since at most one summand is nonzero, $(\mathsf{P} \times \mathrm{d}A^X)$–a.e. By Lemma 4.3(iv) and the fact that $\sum_{k=1}^\infty X^{(k)}$ converges to X also in the sense of uniform convergence on compacts in probability we may conclude. □

5. Proof of Theorem 1.2 (and of Corollary 1.3)

This section contains the proof of this paper's main theorem. It is split up in six subsections. Subsections 5.1, 5.2, and 5.3 provide the proofs of Theorem 1.2(i), (ii), and (iii), respectively. Subsection 5.4 yields the set inclusions in (1.1). Then Subsection 5.5 focuses on the proof of Corollary 1.3, while Subsection 5.6 concludes with a proof of (1.2), namely the strictness of the inclusions.

5.1. Proof of Theorem 1.2(i).

In this subsection we argue that $\mathscr{I}^i = \mathscr{I}^i_\sigma$ for all $i \in \{1,2,3,4\}$. Indeed, fix $X \in \mathscr{I}^1_\sigma$ and the corresponding sigma-localizing sequence $(D_k)_{k\in\mathbb{N}}$ of predictable sets. Then

$$[X,X]^c = \left(\lim_{k\uparrow\infty} \mathbf{1}_{D_k}\right) \cdot [X,X]^c = \lim_{k\uparrow\infty} (\mathbf{1}_{D_k} \cdot [X,X]^c) = \lim_{k\uparrow\infty} [\mathbf{1}_{D_k} \cdot X, \mathbf{1}_{D_k} \cdot X]^c = 0,$$

which yields $X \in \mathscr{I}^1$. As $\mathscr{I}^5 = \mathscr{V}^d$ is stable under sigma-stopping, Kallsen (2004, Proposition 2.1) yields the statement for $i = 4$.

The cases $i = 2$ and $i = 3$ follow from Lemmata 5.3 and 5.4. Before stating and proving them, we first present a useful tool for pure-jump processes in the next lemma.

Lemma 5.1. *Let $(X^{(k)})_{k\in\mathbb{N}} \subset \mathscr{I}^2$ be a sequence of pure-jump processes such that $X_0^{(k)} = 0$ and $[X^{(k)}, X^{(l)}] = 0$ for all $k, l \in \mathbb{N}$ with $k \neq l$. Then the following two statements are equivalent.*

(I) *$\sum_{k=1}^\infty [X^{(k)}, X^{(k)}] < \infty$ and $\sum_{k=1}^\infty B^{X^{(k)}[1]}$ converges in the $\mathscr{S}$–topology to a process B.*
(II) *$\sum_{k=1}^\infty X^{(k)}$ converges in the $\mathscr{S}$–topology to a process X.*

If one (hence both) of these conditions hold then $B^{X[1]} = B$, $\sum_{k=1}^\infty \Delta X^{(k)} = \Delta X$, and X is a pure-jump process.

Proof. Thanks to Lemma 4.4(iii) it suffices to argue that X is a pure-jump process provided the two statements hold. For each $k \in \mathbb{N}$ we have a sequence $(\tau_n^{(k)})_{n\in\mathbb{N}}$ of stopping times (by possibly setting $\tau_n^{(k)} = \infty$ for n large enough if $X^{(k)}$ has only finitely many jumps) such that $X^{(k)} = \sum_{n=1}^{\infty} \Delta X^{(k)}_{\tau_n^{(k)}} \mathbf{1}_{[\![\tau_n^{(k)},\infty[\![}$ in the $\mathscr{S}$–topology and $\Delta X^{(k)}_{\tau_n^{(k)}} \neq 0$ on $\{\tau_n^{(k)} < \infty\}$. Thanks to Lemma 4.4(iii) we have $\Delta X_{\tau_n^{(k)}} = \Delta X^{(k)}_{\tau_n^{(k)}}$ on $\{\tau_n^{(k)} < \infty\}$ for all $k, n \in \mathbb{N}$. Furthermore, $(\tau_n^{(k)})_{k,n\in\mathbb{N}}$ exhausts the jumps of X.

Next, for each $m \in \mathbb{N}$, let K_m and N_m be the smallest integers such that

$$d_{\mathscr{S}}\left(X, \sum_{k=1}^{K_m} X^{(k)}\right) \le \frac{1}{2m}$$

and

$$d_{\mathscr{S}}\left(X^{(k)}, \sum_{n=1}^{N_m} \Delta X^{(k)}_{\tau_n^{(k)}} \mathbf{1}_{[\![\tau_n^{(k)},\infty[\![}\right) \le \frac{1}{2mK_m} \qquad \text{for all } k \in \{1, \cdots, K_m\}.$$

By a standard diagonalization argument we can now construct a sequence of stopping times $(\tau_i)_{i\in\mathbb{N}}$ such that

$$\lim_{m\uparrow\infty} d_{\mathscr{S}}\left(X, \sum_{i=1}^{m} \Delta X_{\tau_i} \mathbf{1}_{[\![\tau_i,\infty[\![}\right) = 0,$$

yielding the statement. □

Corollary 5.2. *The sum of two pure-jump processes whose quadratic covariation is zero is again a pure-jump process.*

The next two lemmata exploit the fact that $\mathscr{J}^2$ is stable under sigma-stopping thanks to Lemma 4.3(ii).

Lemma 5.3. *If $X \in \mathscr{S}$ is sigma–locally a pure-jump process then it is a pure-jump process.*

Proof. By assumption there exists a nondecreasing sequence $(D_k)_{k\in\mathbb{N}}$ of predictable sets such that $\bigcup_{k\in\mathbb{N}} D_k = \Omega \times [0,\infty)$ and $\mathbf{1}_{D_k} \cdot X$ is a pure-jump process. With $D_0 = \emptyset$, define $\overline{D}_k = D_k \setminus D_{k-1}$ for all $k \in \mathbb{N}$ and note that $X^{(k)} = \mathbf{1}_{\overline{D}_k} \cdot X$ is also a pure-jump process as $\mathscr{J}^2$ is stable under sigma-stopping. Moreover, we have $[X^{(k)}, X^{(l)}] = 0$ for all $k, l \in \mathbb{N}$ with $k \neq l$ and

$$\sum_{k=1}^{n} X^{(k)} = \sum_{k=1}^{n} \left(\mathbf{1}_{\overline{D}_k} \cdot X\right) = \mathbf{1}_{D_n} \cdot X, \qquad n \in \mathbb{N},$$

which converges in the $\mathscr{S}$–topology to X (as $n \uparrow \infty$) thanks to Lemma 4.3(iii). Hence by Lemma 5.1, X is a pure-jump process. □

Lemma 5.4. *If $X \in \mathscr{S}$ is sigma–locally a strong pure-jump process then it is a strong pure-jump process.*

Proof. By assumption there exists a nondecreasing sequence $(D_k)_{k\in\mathbb{N}}$ of predictable sets such that $\bigcup_{k\in\mathbb{N}} D_k = \Omega \times [0,\infty)$ and $\mathbf{1}_{D_k} \cdot X$ is a strong pure-jump process. Assume there exists a pure-jump process Y with $\mu^Y = \mu^X$ and $Y_0 = X_0$ but $Y \neq X$. Since $\lim_{k\uparrow\infty}(\mathbf{1}_{D_k} \cdot X) = X - X_0$ and $\lim_{k\uparrow\infty}(\mathbf{1}_{D_k} \cdot Y) = Y - Y_0$ in the $\mathscr{S}$–topology thanks to Lemma 4.3(iii), there exists some $k \in \mathbb{N}$ such that $\mathbf{1}_{D_k} \cdot X \neq \mathbf{1}_{D_k} \cdot Y$. This, however, contradicts the assumption since $\mathbf{1}_{D_k} \cdot Y \in \mathscr{J}^2$ for each $k \in \mathbb{N}$ because $\mathscr{J}^2$ is stable under sigma-stopping. □

5.2. **Proof of Theorem 1.2(ii).** In this subsection we argue that $\mathscr{J}^i$ is closed under stochastic integration for all $i \in \{1,2,3,4,6\}$. First, the cases $i=1$ and $i=6$ are clear. The case $i=4$ follows from Corollary 3.13.

For the case $i=2$, assume that $X \in \mathscr{J}^2$ and fix $\zeta \in L(X)$. We need to argue that $\zeta \cdot X \in \mathscr{J}^2$. Thanks to Theorem 1.2(i) (see also Lemma 5.3), we may assume that $|\zeta|$ is bounded. The statement then follows directly from the definition of $\mathscr{S}$–topology.

The remaining case $i=3$ follows from the next lemma.

Lemma 5.5. *Let $X \in \mathscr{J}^3$ and $\zeta \in L(X)$. If $Y \in \mathscr{J}^2$ is a pure-jump process with $\mu^Y = \mu^{\zeta\cdot X}$ then $Y = \zeta \cdot X$.*

Proof. Note that
$$Z = \left(\mathbf{1}_{\{\zeta\neq 0\}}\frac{1}{\zeta}\right)\cdot Y + \mathbf{1}_{\{\zeta=0\}}\cdot X$$
satisfies $\mu^Z = \mu^X$. Moreover, Z is a pure-jump process thanks to Corollary 5.2 in conjunction with the closedness of $\mathscr{J}^2$ under stochastic integration. Since $X \in \mathscr{J}^3$ we get $Z = X$. This again yields that
$$Y = \mathbf{1}_{\{\zeta=0\}}\cdot Y + \mathbf{1}_{\{\zeta\neq 0\}}\cdot Y = \mathbf{1}_{\{\zeta=0\}}\cdot Y + \zeta\cdot Z = \mathbf{1}_{\{\zeta=0\}}\cdot Y + \zeta\cdot X.$$
We conclude after observing that $\mu^{\mathbf{1}_{\{\zeta=0\}}\cdot Y} = 0$ and $\mathbf{1}_{\{\zeta=0\}}\cdot Y \in \mathscr{J}^2$, hence $\mathbf{1}_{\{\zeta=0\}}\cdot Y = 0$. □

Remark 5.6. The last step of the previous proof relied on the fact that if $X \in \mathscr{J}^2$ and $\mu^X = 0$ then $X = 0$; i.e., a pure-jump process that has no jumps has to equal the zero process. □

5.3. **Proof of Theorem 1.2(iii).** In this subsection we argue that if Q is locally absolutely continuous with respect to P we have $\mathscr{J}^i(\mathsf{P}) \subset \mathscr{J}^i(\mathsf{Q})$ for all $i \in \{1,\dots,6\}$. The cases $i=1$, $i=5$, and $i=6$ are clear. The case $i=2$ follows from Lemma 4.3(vi). The case $i=4$ is a consequence of Proposition 3.12 and Remark 3.2.

The remaining case $i=3$ follows from Lemma 5.8. It requires the following result regarding the lift of a pure-jump process from $\mathscr{J}^2(\mathsf{Q})$ to $\mathscr{J}^2(\mathsf{P})$.

Lemma 5.7. *Let Z denote a nonnegative P–martingale, Q a probability measure that satisfies $\mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P}|_{\mathscr{F}_t} = Z_t$ for all $t \geq 0$, and Y an element of $\mathscr{J}^2(\mathsf{Q})$. Assume that there exists some stopping time σ such that $Y = Y^\sigma$, P–almost surely, and Z^σ does not hit zero continuously, P–almost surely, Then there exists a P–semimartingale $Y_\uparrow \in \mathscr{J}^2(\mathsf{P})$ with $Y_\uparrow = Y$, Q–almost surely, and $Y_\uparrow = Y_\uparrow^\sigma$, P–almost surely.*

Proof. Let $(\tau_k)_{k\in\mathbb{N}}$ denote a sequence of stopping times such that $Y = Y_0 + \sum_{k=1}^\infty \Delta Y_{\tau_k}\mathbf{1}_{[\![\tau_k,\infty[\![}$ in the $\mathscr{S}$–topology under Q. We may now assume that $Y_0 = 0$ and $\{\tau_k < \infty\}\cap\{Z_{\tau_k} = 0\} = \emptyset$ for all $k \in \mathbb{N}$. Considering the sum $B^{(n)} = \sum_{k=1}^n B^{\Delta Y_{\tau_k}\mathbf{1}_{[\![\tau_k,\infty[\![}[1]}(\mathsf{P})$ for each $n \in \mathbb{N}$ it now suffices to prove that these drifts converge in the $\mathscr{S}$–topology under P. Indeed, Lemma 5.1 then yields a limiting process $Y_\uparrow \in \mathscr{J}^2(\mathsf{P})$ and Lemma 4.3(vi) yields that $Y_\uparrow = Y$, Q–almost surely. Clearly, we then also have $Y_\uparrow = Y_\uparrow^\sigma$, P–almost surely.

We still need to argue the convergence of the P–drifts $(B^{(n)})_{n\in\mathbb{N}}$ in the $\mathscr{S}$–topology under P. First note that $\lim_{n\uparrow\infty} B^{(n)} = B$ in the $\mathscr{S}$–topology under Q for some Q–almost surely predictable finite variation process B by the assumption and Lemmata 4.4(i) and 4.3(vi)&(v). Since the first time σ' that B has infinite variation is predictable and hence $\mathsf{E}^{\mathsf{P}}_{\sigma'-}[\Delta Z_{\sigma'}] = 0$ on $\{\sigma' < \infty\}$ we may conclude that B is P–almost surely of finite variation, too. It suffices to argue now that the variations of $(B - B^{(n)})_{n\in\mathbb{N}}$ converge to zero in probability under P. As they do under Q and as the first time σ'' that the variations of $(B - B^{(n)})_{n\in\mathbb{N}}$ do not converge to zero is predictable, we may argue again that $\sigma'' = \infty$, P–almost surely, yielding the statement. □

Lemma 5.8. *Let Q be a probability measure that is locally absolutely continuous with respect to P and let $X \in \mathscr{J}^3(\mathsf{P})$. If $Y \in \mathscr{J}^2(\mathsf{Q})$ is a pure-jump process with $\mu^Y = \mu^X$, Q–almost surely, then $Y = X$, Q–almost surely.*

Proof. For each $m \in \mathbb{N}$, let σ_m be the first time that the nonnegative martingale $(\mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P}|_{\mathscr{F}_t})_{t\geq 0}$ crosses the level $1/m$. Then $X^{\sigma_m} \in \mathscr{J}^3(\mathsf{P})$ for each $m \in \mathbb{N}$ by Lemma 5.5 and $\lim_{m\uparrow\infty} \sigma_m = \infty$, Q–almost surely. Hence, thanks to Lemma 5.4 applied under Q, we may and shall assume from now on that $X = X^\sigma$, where $\sigma = \sigma_m$ for some $m \in \mathbb{N}$. Let $Y \in \mathscr{J}^2(\mathsf{Q})$ satisfy $\mu^Y = \mu^X$, Q–almost surely, and $Y_0 = X_0$. Then $Y = Y^\sigma$ and Lemma 5.7 yields $Y_\uparrow \in \mathscr{J}^2(\mathsf{P})$ with $Y_\uparrow = Y$, Q–almost surely, and $Y_\uparrow = Y_\uparrow^\sigma$, P–almost surely.

Define now the P–semimartingale $Y' = Y_\uparrow + (\Delta X_\sigma - \Delta Y_{\uparrow\sigma})\mathbf{1}_{[\![\sigma,\infty[\![}$. Then we have $\mu^{Y'} = \mu^X$ and $Y' = Y$, Q–almost surely. Since $X \in \mathscr{J}^3(\mathsf{P})$ we thus have $Y' = X$, P–almost surely, yielding $Y = Y' = X$, Q–almost surely, as required. □

5.4. **Proof of the set inclusions in (1.1).** Lemma 4.3(iv) yields the inclusion $\mathscr{J}^1 \supset \mathscr{J}^2$. The strictness of this inclusion follows from observing, for example, that $X_t = t$ for all $t \geq 0$ satisfies $X \in \mathscr{J}^1 \setminus \mathscr{J}^2$. The inclusions $\mathscr{J}^2 \supset \mathscr{J}^3$ and $\mathscr{J}^4 \supset \mathscr{J}^5 \supset \mathscr{J}^6$ are clear. Since the deterministic semimartingale $X = \sum_{k=1}^\infty k^{-2}\mathbf{1}_{[\![1/k,\infty[\![}$ satisfies $X \in \mathscr{J}^5 \setminus \mathscr{J}^6$, we also have the strictness of the last inclusion.

To see $\mathscr{J}^3 \supset \mathscr{J}^4$, consider now $X \in \mathscr{J}^5$. By definition of the $\mathscr{S}$–topology every exhausting sequence for X also yields an approximating sequence of stopping times for X in $\mathscr{J}^2$. This shows $\mathscr{J}^5 \subset \mathscr{J}^2$, and in fact $\mathscr{J}^5 \subset \mathscr{J}^3$. Hence Lemma 5.4 yields $\mathscr{J}^4 = \mathscr{J}^5_\sigma \subset \mathscr{J}^3_\sigma = \mathscr{J}^3$.

5.5. **Proof of Corollary 1.3 and the strictness of the inclusion $\mathscr{J}^2 \supset \mathscr{J}^3$.** First, note that Corollary 1.3(i) implies the remaining statements (ii)–(iv) thanks to the characterization of $x \in L_\sigma(\mu^X)$ in Propositions 3.12, 3.4 and Lemma 3.3. Since any quadratic pure-jump process X has the representation

$$X = X_0 + x\mathbf{1}_{|x|>1} * \mu^X + x\mathbf{1}_{|x|\leq 1} * (\mu^X - \nu^X) + B^{X[1]} \tag{5.1}$$

(Jacod and Shiryaev 2003, II.2.34), we also get the implication from (iv) to (i). The implication from (iii) to (i) is a direct consequence of (5.1) and Lemma 5.9(i) below. Lemma 5.12 below yields the implication from (ii) to (i). Finally, Lemma 5.13 shows that the inclusion $\mathscr{J}^2 \supset \mathscr{J}^3$ is usually strict.

Lemma 5.9. *Let $X \in \mathscr{J}^2$ be a pure-jump semimartingale and define the predictable set*

$$D = \left\{(\omega, t) : \int |x|\mathbf{1}_{|x|\leq 1} F_t^X(\mathrm{d}x) < \infty\right\}.$$

Then the following statements hold.

(i) *We have $\mathbf{1}_D \cdot X \in \mathscr{V}^d_\sigma$.*
(ii) *The following equalities hold.*

$$D = \left\{(\omega, t) : \int x^+\mathbf{1}_{x^+\leq 1} F_t^X(\mathrm{d}x) < \infty\right\} = \left\{(\omega, t) : \int x^-\mathbf{1}_{x^-\leq 1} F_t^X(\mathrm{d}x) < \infty\right\}, \ (\mathsf{P}\times \mathrm{d}A^X)\text{–a.e.}$$

(iii) *There exists a predictable process β^X with $|\beta^X| \cdot A^X < \infty$ such that $B^{X[1]} = \beta^X \cdot A^X$. Moreover, on D we have $\beta^X = \int x\mathbf{1}_{|x|\leq 1} F^X(\mathrm{d}x)$, $(\mathsf{P}\times \mathrm{d}A^X)$–a.e.*

Proof. For (i), thanks to Theorem 1.2(ii) (or the fact that $\mathscr{J}^2$ is stable under sigma-stopping), we may assume that $X = \mathbf{1}_D \cdot X$. Define now the predictable sets

$$D_k = \left\{(\omega, t) : \int |x|\mathbf{1}_{|x|\leq 1} F_t^X(\mathrm{d}x) \leq k\right\}, \qquad k \in \mathbb{N},$$

and $\overline{D}_k = D_k \setminus D_{k-1}$ with $D_0 = \emptyset$. Again $X^{(k)} = \mathbf{1}_{\overline{D}_k} \cdot X$ is a pure-jump process for each $k \in \mathbb{N}$. Moreover, by Proposition 3.4 and Lemma 3.3, $x \in L_\sigma(\mu^{X^{(k)}})$ for each $k \in \mathbb{N}$. Since by (1.1) $x \star \mu^{X^{(k)}} \in \mathscr{J}^4 \subset \mathscr{J}^3$ we have $X^{(k)} = x \star \mu^{X^{(k)}}$ for each $k \in \mathbb{N}$. Thanks to Proposition 3.12 this yields

$$B^{X^{(k)}[1]} = \int_0^\cdot \mathbf{1}_{\overline{D}_k}(t)\left(\int x\mathbf{1}_{|x|\leq 1} F_t^X(\mathrm{d}x)\right)\mathrm{d}A_t^X, \qquad k \in \mathbb{N}.$$

Hence by Lemmata 4.3(iii) and 5.1 we have

$$\int_0^\cdot \mathbf{1}_{D_n}(t)\left(\int x\mathbf{1}_{|x|\leq 1}F_t^X(\mathrm{d}x)\right)\mathrm{d}A_t^X$$

converges in the $\mathscr{S}$–topology (as $n\uparrow\infty$) to a finite variation process, yielding the result.

To see (ii), fix $\kappa>0$ and consider the predictable set

$$D'=\left\{(\omega,t):\int x^-\mathbf{1}_{x^-\leq 1}F_t^X(\mathrm{d}x)<\kappa\right\}.$$

By symmetry, it suffices now to argue that $\mathbf{1}_{D'}x^+\mathbf{1}_{x^+\leq 1}\nu^X<\infty$. As above, we may assume that $X=\mathbf{1}_{D'}\cdot X$. Then $x^-*\mu^X<\infty$ and (4.1) with $\zeta=1$ yield that also $x^+*\mu^X<\infty$; hence $x^+\mathbf{1}_{x^+\leq 1}*\nu^X<\infty$ as required.

For (iii) Lebesgue's decomposition of measures yields a predictable set $\widetilde{D}$ and a predictable process β^X such that $\mathbf{1}_{\widetilde{D}}\cdot A^X=0$, $|\beta^X|\cdot A^X<\infty$, and $B^{X[1]}=\mathbf{1}_{\widetilde{D}}\cdot B^{X[1]}+\beta^X\cdot A^X$. Next, note that $\mathbf{1}_{\widetilde{D}}\cdot X$ is also a pure-jump process and $x^2*\nu^{\mathbf{1}_{\widetilde{D}}\cdot X[1]}=\mathbf{1}_{\widetilde{D}}x^2*\nu^{X[1]}=0$, yielding $\mathbf{1}_{\widetilde{D}}\cdot X[1]=0$, hence $\mathbf{1}_{\widetilde{D}}\cdot B^{X[1]}=0$. The second assertion of (iii) follows directly from (i) and Proposition 3.12. □

Lemma 5.10. *Let μ be a jump measure with $x^2*\mu<\infty$. Moreover, let $(f^{(k)})_{k\in\mathbb{N}}$ and $(g^{(k)})_{k\in\mathbb{N}}$ denote two nonincreasing sequences of strictly positive predictable processes. For each $k\in\mathbb{N}$, define*

$$\beta^{(k)}=\int x\mathbf{1}_{\{x\in[-1,-g^{(k)}]\cup[f^{(k)},1]\}}F(\mathrm{d}x).$$

If $|\beta^{(k)}|\cdot A<\infty$ for each $k\in\mathbb{N}$ and $\beta^{(k)}\cdot A$ converges in the $\mathscr{S}$–topology (as $k\uparrow\infty$) to a process B, then there exists a pure-jump process X with $\mu^X=\mu$ and $B^{X[1]}=B$.

Proof. Note that

$$\int|x|\mathbf{1}_{\{x\in[-1,-g^{(k)}]\cup[f^{(k)},1]\}}F(\mathrm{d}x)<\infty,\qquad(\mathsf{P}\times\mathrm{d}A)\text{–a.e.},\qquad k\in\mathbb{N},$$

by the strict positivity of $g^{(k)}$ and $f^{(k)}$. Hence by assumption and Propositions 3.4 and 3.12, $x\mathbf{1}_{|x|=1}, x\mathbf{1}_{\{x\in(-g^{(k-1)},-g^{(k)}]\cup[f^{(k)},f^{(k-1)})\}}\in L_\sigma(\mu)$, $X^{(0)}=x\mathbf{1}_{|x|=1}\star\mu\in\mathscr{V}_\sigma^d\subset\mathscr{J}^2$, and

$$X^{(k)}=x\mathbf{1}_{\{x\in(-g^{(k-1)},-g^{(k)}]\cup[f^{(k)},f^{(k-1)})\}}\star\mu\in\mathscr{V}_\sigma^d\subset\mathscr{J}^2,\qquad k\in\mathbb{N},$$

where $g^{(0)}=f^{(0)}=1$. An application of Lemma 5.1 now concludes. □

The following lemma complements Lemma 5.9(ii)&(iii). Given a jump measure μ and a predictable process β, both satisfying technical conditions, it constructs a pure-jump process X with $\mu^X=\mu$ and drift rate β on the predictable set where the jump sizes do not integrate.

Lemma 5.11. *Let μ be a jump measure with $x^2*\mu<\infty$ and*

$$\left(\limsup_{x\downarrow 0}xF(\{x\})\right)\wedge\left(\limsup_{x\uparrow 0}|x|F(\{x\})\right)=0,\qquad(\mathsf{P}\times\mathrm{d}A)\text{–a.e.}\tag{5.2}$$

Assume that the predictable set

$$D=\left\{(\omega,t):\int|x|\mathbf{1}_{|x|\leq 1}F_t(\mathrm{d}x)<\infty\right\}$$

satisfies

$$D=\left\{(\omega,t):\int x^+\mathbf{1}_{x^+\leq 1}F_t(\mathrm{d}x)<\infty\right\}=\left\{(\omega,t):\int x^-\mathbf{1}_{x^-\leq 1}F_t(\mathrm{d}x)<\infty\right\},\qquad(\mathsf{P}\times\mathrm{d}A)\text{–a.e.}\tag{5.3}$$

and let β denote any nonnegative predictable process such that

$$\int_0^\cdot \mathbf{1}_{D^c}(t)|\beta_t|\mathrm{d}A_t < \infty.$$

Then there exists a pure-jump process $X \in \mathscr{J}^2$ such that $\mu^X = \mu$ and

$$B^{X[1]} = \int_0^\cdot \left(\mathbf{1}_D(t)\int x\mathbf{1}_{|x|\leq 1}F_t(\mathrm{d}x) + \mathbf{1}_{D^c}(t)\beta_t\right)\mathrm{d}A_t.$$

Proof. Consider the predictable sets

$$D' = D^c \cap \left\{(\omega,t) : \limsup_{x\downarrow 0} xF_t(\{x\}) = 0\right\}; \qquad D'' = D^c \setminus D'.$$

By Corollary 5.2, symmetry, and Subsection 3.3 we may assume that $D'' = \emptyset$, $\mathbf{1}_{|x|>1} * \mu = 0$, $\mu = \mathbf{1}_{D'}\mu$, and $\beta = \mathbf{1}_{D'}\beta$.

To make headway, consider the predictable process

$$c = \inf\{\varepsilon > 0 : \varepsilon F(\{\varepsilon\}) > 1\} \wedge 2$$

and note that $c > 0$ by assumption. Next, consider the process

$$d = \mathbf{1}_{D'^c} + \mathbf{1}_{D'}\sup\left\{\varepsilon > 0 : \beta + \int |x|\mathbf{1}_{x\in[-1,-\varepsilon]}F(\mathrm{d}x) \geq \int x\mathbf{1}_{\{x\in[c,1]\}}F(\mathrm{d}x)\right\}.$$

Then by (5.3), $d > 0$ and since

$$\{(\omega,t) : d_t \geq \varepsilon\} = \left\{(\omega,t) : \int |x|\mathbf{1}_{x\in[-1,-\varepsilon]}F_t(\mathrm{d}x) \geq \int x\mathbf{1}_{\{x\in[c,1]\}}F_t(\mathrm{d}x) - \beta\right\} \in \mathcal{P}, \qquad \varepsilon \in [0,1],$$

it is easy to see that d is predictable. Next, define the processes

$$g^{(k)} = d \wedge \frac{1}{k}, \qquad k \in \mathbb{N};$$

$$f^{(k)} = \mathbf{1}_{D'^c} + \mathbf{1}_{D'}\sup\left\{\varepsilon > 0 : \int x\mathbf{1}_{x\in[\varepsilon,1]}F(\mathrm{d}x) \geq \beta + \int |x|\mathbf{1}_{\{x\in[-1,-g^{(k)}]\}}F(\mathrm{d}x)\right\}, \qquad k \in \mathbb{N}.$$

Similarly as for d we may argue that $f^{(k)} > 0$ and $f^{(k)}$ is predictable for each $k \in \mathbb{N}$. Again by (5.3), we also have $\lim_{k\uparrow\infty} f^{(k)} = 0$ on D'. Note that we also have $f^{(k)} < c$ for each $k \in \mathbb{N}$.

Next, define

$$\beta^{(k)} = \int x\mathbf{1}_{\{x\in[-1,-g^{(k)}]\cup[f^{(k)},1]\}}F(\mathrm{d}x), \qquad k \in \mathbb{N}.$$

Since

$$\beta^{(k)} \in \left[\beta, \beta + f^{(k)}F\left(\left\{f^{(k)}\right\}\right)\right] \subset [\beta, \beta+1], \qquad k \in \mathbb{N},$$

we have $|\beta^{(k)}| \cdot A < \infty$ and

$$\int_0^\cdot \beta^{(k)}\mathrm{d}A_t \in \left[\int_0^\cdot \beta_t\mathrm{d}A_t, \int_0^\cdot \left(\beta_t + f_t^{(k)}F_t\left(\left\{f_t^{(k)}\right\}\right)\right)\mathrm{d}A_t\right], \qquad k \in \mathbb{N}.$$

Since $\lim_{k\uparrow\infty} f^{(k)} = 0$ on D' we also have

$$\lim_{k\uparrow\infty} f^{(k)}F\left(\left\{f^{(k)}\right\}\right) = 0$$

by assumption. Hence, dominated convergence yields $\lim_{k\uparrow\infty}\beta^{(k)} \cdot A = \beta \cdot A$ in the $\mathscr{S}$–topology. An application of Lemma 5.10 now concludes. □

Lemma 5.12. *Let $X \in \mathscr{J}^3$ denote a strong pure-jump process such that (1.3) holds. Then $X \in \mathscr{J}^4$.*

Proof. We only need to argue that (1.4) holds. Recall Lemma 5.9(ii) and apply Lemma 5.11 with $\mu = \mu^X$ and $\widetilde{\beta} = 1$ and $\overline{\beta} = -1$. If (1.4) did not hold then we would obtain two pure-jump processes $\widetilde{X}$ and $\overline{X}$ with $\widetilde{X} \neq \overline{X}$ but with the same jump measures, contradicting the fact that $X \in \mathscr{J}^3$. This concludes the proof. □

Lemma 5.13. *Assume that the filtered probability space is large enough so that it supports a probability measure* μ *that satisfies* (5.2), (5.3), *and*

$$\mathsf{P}\left[\int_0^\cdot \mathbf{1}_{\{\int |x|\mathbf{1}_{|x|\leq 1} F_t(\mathrm{d}x) = \infty\}} \mathrm{d}A_t > 0\right] > 0.$$

Then $\mathscr{J}^2 \neq \mathscr{J}^3$.

Proof. As in the proof of Lemma 5.12, consider the two predictable processes $\widetilde{\beta} = 1$ and $\overline{\beta} = -1$ and conclude by applying Lemma 5.11 twice. □

As an illustration of Lemma 5.13 and a preparation for the next subsection, let us now discuss the Lévy situation by means of the following example. When X is a Lévy process we abuse the notation to treat F^X as a deterministic measure over $\mathbb{R}$ rather than a stochastic process.

Example 5.14. Let X be an α–stable Lévy process without Brownian component. Specifically, take $F^X(\mathrm{d}x) = \mathbf{1}_{x \neq 0} |x|^{-1-\alpha} \mathrm{d}x$ for all $x \in \mathbb{R}$ with $0 < \alpha < 2$ and $A_t^X = t$ for all $t \geq 0$. Observe that $X \in \mathscr{J}^4$ is equivalent to $X \in \mathscr{J}^5$ since X is Lévy. Let us now write $\beta = B_t^{X[1]}/t$, where $t > 0$, for the drift rate of X.

- If $0 < \alpha < 1$ and $\beta = \int x \mathbf{1}_{|x|\leq 1} F^X(\mathrm{d}x) = 0$ then X belongs to $\mathscr{J}^5 \setminus \mathscr{J}^6$.
- If $0 < \alpha < 1$ and $\beta \neq \int x \mathbf{1}_{|x|\leq 1} F^X(\mathrm{d}x)$ then X belongs to $\mathscr{J}^1 \setminus \mathscr{J}^2$.
- If $1 \leq \alpha < 2$ then X belongs to $\mathscr{J}^2 \setminus \mathscr{J}^3$ for any value of β. □

5.6. **Proof of (1.2).** On finite probability spaces we have $\mathscr{J}^2 = \mathscr{J}^5$. However, in general, this is not true. Lemma 5.13 already asserts that $\mathscr{J}^2 \neq \mathscr{J}^3$ as long as the probability space is large enough. The process X of the introduction shows that usually we have $\mathscr{J}^4 \neq \mathscr{J}^5$. Example 5.18 below illustrates that $\mathscr{J}^3 \neq \mathscr{J}^4$ is also possible.

Theorem 1.2 asserts that all these inequalities may hold simultaneously for some probability space. To see that such a probability space exists it suffices to piece together these three examples. For example, take the product of a probability space that allows for a process as in Example 5.18 and another probability space that satisfies the assumptions of Lemma 5.13 and additionally allows for a process as in the introduction. As filtration consider the one of Example 5.18 between time 0 and 1 and afterwards allow the filtration to be large enough to allow for the other examples.

Example 5.18 requires a few technical prerequisites that we introduce now. Throughout this subsection, $\mathfrak{F}^X$ and $\mathfrak{F}_+^X$ shall denote the natural filtration of a process X and its right-continuous modification.

Lemma 5.15. *Let* g *denote a* $\{0,1\}$*–valued predictable function and* $X \in \mathscr{S}$ *a semimartingale. Then* $g(\Delta X)$ *is optional. Moreover, if* τ *is an* $\mathfrak{F}^X$*–stopping time then* $[\![\tau]\!] = \{g(\Delta X) = 1\}$ *for some* $\{0,1\}$*–valued* $\mathfrak{F}^X$*–predictable function* g.

Proof. Thanks to Dellacherie and Meyer (1982, Theorem IV.79a) $g(\Delta X)$ is optional as a composition of appropriately measurable functions. Let $\mathscr{O}^X$ now denote the $\mathfrak{F}^X$–optional sigma algebra. It suffices to prove that $\mathscr{O}^X \subset \overline{\mathscr{O}}$, where $\overline{\mathscr{O}} = \bigcup_g \{g(\Delta X) = 1\}$ with the union is taken over all $\{0,1\}$–valued $\mathfrak{F}^X$–predictable functions. First note that $\overline{\mathscr{O}}$ is a sigma algebra since the maximum of countably many predictable functions is again predictable. Next, taking $g = \mathbf{1}_E$, with a slight misuse of notation, for E either an event in the $\mathfrak{F}^X$–predictable sigma algebra or in the Borel sigma algebra on $\mathbb{R}$, shows that $\overline{\mathscr{O}}$ contains the predictable sigma algebra and the one generated by ΔX. Another application of Dellacherie and Meyer (1982, Theorem IV.79a) hence concludes. □

Lemma 5.16. *Assume that X is a Lévy process. For an $\mathfrak{F}^X_+$–stopping time τ we then have $[\![\tau]\!] = \{g(\Delta X) = 1\}$ for some $\{0,1\}$–valued $\mathfrak{F}^X$–predictable function g.*

Proof. Since X is Feller, $\mathfrak{F}^X_+$ can be obtained from $\mathfrak{F}^X$ by augmenting it with the null sets of $\mathscr{F}^X_\infty$. Hence there exists an $\mathfrak{F}^X$–stopping time $\overline{\tau}$ with $\overline{\tau} = \tau$, P–almost surely. Thus, an application of Lemma 5.15 concludes. □

Lemma 5.17. *Assume that X is a Lévy process with $|\Delta X| \leq 1$ and assume that Y is an $\mathfrak{F}^X_+$–pure-jump process with $\mu^Y = \mu^X$. Then there exists a nondecreasing sequence $(f^{(n)})_{n\in\mathbb{N}}$ of $\{0,1\}$–valued $\mathfrak{F}^X$–predictable functions such that*

$$\int_0^\cdot \left(\int \left(|x| f^{(n)}_t(x)\right) F^X(\mathrm{d}x)\right) \mathrm{d}t < \infty, \qquad n \in \mathbb{N}; \tag{5.4}$$

$$\lim_{n\uparrow\infty} \int_0^\cdot \left(\int \left(x f^{(n)}_t(x)\right) F^X(\mathrm{d}x)\right) \mathrm{d}t = B^Y \qquad \textit{in the } \mathscr{S}\textit{–topology.} \tag{5.5}$$

Proof. Let $(\tau_k)_{k\in\mathbb{N}}$ be an exhausting sequence of $\mathfrak{F}^X_+$–stopping times for the jumps of Y such that $Y = \sum_{k=1}^\infty \Delta X_{\tau_k} \mathbf{1}_{[\![\tau_k,\infty[\![}$ in the $\mathscr{S}$–topology. By Lemma 5.16, there exists an $\mathfrak{F}^X$–predictable $\{0,1\}$–valued function $g^{(k)}$ such that

$$\Delta X_{\tau_k} \mathbf{1}_{[\![\tau_k,\infty[\![} = \Delta X g^{(k)}(\Delta X) = x g^{(k)}(x) * \mu^X, \qquad k \in \mathbb{N}.$$

Observe also that

$$\int_0^\cdot \left(\int \left(|x| g^{(k)}_t(x)\right) F^X(\mathrm{d}x)\right) \mathrm{d}t = B^{|\Delta X_{\tau_k}| \mathbf{1}_{[\![\tau_k,\infty[\![}} < \infty, \qquad k \in \mathbb{N}.$$

Since the elements of $(\tau_k)_{k\in\mathbb{N}}$ have disjoint support we may assume that $f^{(n)} = \sum_{k=1}^n g^{(k)}$ is also $\{0,1\}$–valued for each $n \in \mathbb{N}$. Then clearly (5.4) holds and Lemma 5.1 yields that

$$\lim_{n\uparrow\infty} \left(\int_0^\cdot \left(\int x f^{(n)}_t(x) F^X(\mathrm{d}x)\right) \mathrm{d}t\right) = \lim_{n\uparrow\infty} \sum_{k=1}^n B^{\Delta X_{\tau_k} \mathbf{1}_{[\![\tau_k,\infty[\![}} = B^Y$$

in the $\mathscr{S}$–topology, yielding (5.5). □

Example 5.18. Let X be a Lévy process with Lévy measure

$$F^X(\mathrm{d}x) = \sum_{k=1}^\infty k^2 3^{2k} \delta_{1/(k^2 3^k)}(\mathrm{d}x) + \sum_{k=1}^\infty k^2 3^{2k} \delta_{-1/(k^2 3^k)}(\mathrm{d}x), \qquad x \in \mathbb{R},$$

and without a drift and Brownian motion component; in particular, $B^{X[1]} = 0$. Note that

$$\int (x^2 \wedge |x|) F^X(\mathrm{d}x) = 2 \sum_{k=1}^\infty k^2 3^{2k} \frac{1}{k^4 3^{2k}} = 2 \sum_{k=1}^\infty \frac{1}{k^2} < \infty.$$

Moreover, X is a pure-jump process by Lemma 5.1 with $X^{(k)} = x \mathbf{1}_{|x|\in(1/(k+1),1/k]} * \mu^X$ for each $k \in \mathbb{N}$.

Since $\int |x| F^X(\mathrm{d}x) = \infty$ it is clear that $X \notin V^d_\sigma$. However, we claim that X is a strong pure-jump process, i.e., $X \in \mathscr{J}^3$. Indeed, let Y denote any pure-jump process with $\mu^Y = \mu^X$. Thanks to the canonical representation of quadratic pure-jump processes in (5.1) it suffices to show that $B^Y = 0$.

To this end, thanks to Lemma 5.17, there exists a nondecreasing sequence $(f^{(n)})_{n\in\mathbb{N}}$ of $\{0,1\}$–valued $\mathfrak{F}^X$–predictable functions such that (5.4) and (5.5) hold. Assume that $B^Y \neq 0$. By Lemma 5.9(iii) there exist some $\kappa \in \mathbb{N}$ and a predictable set D such that $\int_0^\cdot \mathbf{1}_D(t)|\mathrm{d}B^Y_t| > 0$ and

$$\mathbf{1}_D \sup_{n\in\mathbb{N}} \left| \int \left(x f^{(n)}(x)\right) F^X(\mathrm{d}x) \right| < \frac{3^\kappa}{2}, \qquad (\mathsf{P} \times \mathrm{d}t)\text{–a.e.} \tag{5.6}$$

Consider now the predictable sets

$$\mathcal{A}_t^{(n),+} = \left\{k \in \mathbb{N} : f_t^{(n)}\left(\frac{1}{k^2 3^k}\right) = 1\right\}; \quad \mathcal{A}_t^{(n),-} = \left\{k \in \mathbb{N} : f_t^{(n)}\left(-\frac{1}{k^2 3^k}\right) = 1\right\}, \quad t \geq 0,\, n \in \mathbb{N},$$

along with their symmetric differences

$$\mathcal{A}_t^{(n)} = \left(\mathcal{A}_t^{(n),+} \setminus \mathcal{A}_t^{(n),-}\right) \cup \left(\mathcal{A}_t^{(n),-} \setminus \mathcal{A}_t^{(n),+}\right), \qquad t \geq 0,\, n \in \mathbb{N}.$$

Thanks to (5.4), $k_t^{(n)} = \max \mathcal{A}_t^{(n)} < \infty$ (with $\max \emptyset = 0$), $(\mathsf{P} \times \mathrm{d}t)$–a.e., for each $n \in \mathbb{N}$. If $k_t^{(n)} = 0$ then $\int (x f_t^{(n)}(x)) F^X(\mathrm{d}x) = 0$ for all $t \geq 0$ and $n \in \mathbb{N}$. If $k_t^{(n)} \in \mathbb{N}$ then

$$\left|\int \left(x f_t^{(n)}(x)\right) F^X(\mathrm{d}x)\right| \geq 3^{k_t^{(n)}} - \sum_{k=1}^{k_t^{(n)}-1} 3^k \geq 3^{k_t^{(n)}} - \frac{3^{k_t^{(n)}}}{2} = \frac{3^{k_t^{(n)}}}{2}, \qquad t \geq 0,\, n \in \mathbb{N}.$$

Hence by (5.6), on D, we have $\max \mathcal{A}^{(n)} < \kappa$ for all $n \in \mathbb{N}$. We have just argued that

$$\mathbf{1}_D \int \left(\mathbf{1}_{|x| \leq 1/(\kappa^2 3^\kappa)} x f^{(n)}(x)\right) F^X(\mathrm{d}x) = 0, \qquad (\mathsf{P} \times \mathrm{d}t)\text{–a.e.}$$

Thus

$$\mathbf{1}_D \cdot B^Y = \lim_{n \uparrow \infty} \int_0^\cdot \left(\mathbf{1}_D(t) \int \left(\mathbf{1}_{|x| > 1/(\kappa^2 3^\kappa)} x f_t^{(n)}(x)\right) F^X(\mathrm{d}x)\right) \mathrm{d}t = 0$$

in the $\mathscr{S}$–topology since $\mathbf{1}_D \mathbf{1}_{|x|>1/(\kappa^2 3^\kappa)} x * \mu^X \in \mathscr{V}^d$, hence a strong pure-jump process. This is a contradiction to the assumption that $\int_0^\cdot \mathbf{1}_D(t) |\mathrm{d}B_t^Y| > 0$. This shows that X is a strong pure-jump process. □

6. Replacing the $\mathscr{S}$–topology by UCP convergence

In this final section we briefly discuss the choice of the $\mathscr{S}$–topology in the definition of $\mathscr{J}^2$. Indeed, one may define an alternative class $\mathscr{J}^{2\dagger} \subset \mathscr{S}$, where the convergence in the $\mathscr{S}$–topology is replaced by uniform convergence on compacts in probability (ucp). That is, a semimartingale $X \in \mathscr{S}$ is in $\mathscr{J}^{2\dagger}$ if for a family $(\tau_k)_{k \in \mathbb{N}}$ of stopping times we have

$$\lim_{n \uparrow \infty} \mathsf{E}\left[\sup_{s \leq t} \left| X_s - X_0 - \sum_{k=1}^n \Delta X_{\tau_k} \mathbf{1}_{[\![\tau_k, \infty[\![}(s)\right| \wedge 1\right] = 0$$

for all $t \geq 0$.

Note that the equivalence of (I) and (II) in Lemma 5.1 holds with convergence in $\mathscr{S}$–topology replaced by convergence in the sense of ucp in its statement. However, $\mathscr{J}^{2\dagger}$ is not stable under σ–stopping, i.e., if $X \in \mathscr{J}^{2\dagger}$ and D is a predictable set then not necessarily $\mathbf{1}_D \cdot X \in \mathscr{J}^{2\dagger}$.

The new class $\mathscr{J}^{2\dagger}$ contains all elements of $\mathscr{J}^2$ because the semimartingale topology is stronger than ucp convergence. Proposition 6.1 below shows $\mathscr{J}^{2\dagger}$ is in fact too large for practical purposes or for representing 'pure-jump' processes.

Proposition 6.1. *Let X denote a Lévy process with $|\Delta X| \leq 1$, $\int x^+ F^X(\mathrm{d}x) = \infty$, and symmetric and atomless Lévy measure. Moreover, let W denote an independent Brownian motion stopped when its absolute value hits 1. Then $X + W \in \mathscr{J}^{2\dagger}$; hence, in particular $\mathscr{J}^{2\dagger} \setminus \mathscr{J}^1 \neq \emptyset$ for sufficiently rich probability spaces.*

Proof. Fix for the moment $n \in \mathbb{N}$ and let $W^{(n)}$ denote a piecewise constant approximation of W with $W_{k/n+t}^{(n)} = W_{k/n}$ for all $k \in \mathbb{N}$ and $t \in [0, 1/n)$. Next, let $B^{(n)} = \int_0^\cdot b_t^{(n)} \mathrm{d}t$ denote the trailing continuous piecewise linear predictable approximation of $W^{(n)}$. By this we mean the process $B^{(n)}$ such that $B_0^{(n)} = B_{1/n}^{(n)} = 0$, $B_{2/n}^{(n)} = W_{1/n}^{(n)}$, $B_{3/n}^{(n)} = W_{2/n}^{(n)}, \cdots$ and $b^{(n)}$ is constant on each interval $[k/n, (k+1)/n)$ for $k \in \mathbb{N}$. Then it is clear that $\lim_{n \uparrow \infty} B^{(n)} = W$ in the sense of ucp.

We now claim that there exist two nonincreasing sequences $(c^{(n)})_{n\in\mathbb{N}}$ and $(d^{(n)})_{n\in\mathbb{N}}$ of piecewise constant predictable processes with $c^{(n)}, d^{(n)} \in (0, 1/n]$ such that

$$\int |x|\mathbf{1}_{\{x\notin(-g^{(n)},f^{(n)})\}}F^X(\mathrm{d}x) < \infty \quad \text{and} \quad x\mathbf{1}_{\{x\notin(-g^{(n)},f^{(n)})\}} * \nu^X = B^{(n)}.$$

Then the statement follows by using the appropriate modification of Lemma 5.1.

To see the claim assume one has constructed $g^{(n)}$ and $f^{(n)}$ for some $n \in \mathbb{N}$ as required. Consider now the intermediate predictable processes $\overline{g} = g^{(n)} \wedge 1/(n+1)$ and $\overline{f} = f^{(n)} \wedge 1/(n+1)$ and the intermediate piecewise constant predictable process

$$\overline{b} = b^{(n+1)} - \int x\mathbf{1}_{\{x\notin(-\overline{g},\overline{f})\}}F^X(\mathrm{d}x).$$

Whenever $\overline{b} > 0$, one now sets $g^{(n+1)} = \overline{g}$ and sets $f^{(n+1)}$ so that $\int x\mathbf{1}_{\{x\in(f^{(n+1)},\overline{f})\}}F^X(\mathrm{d}x) = \overline{b}$. When $\overline{b} < 0$ one sets $g^{(n+1)}$ and $f^{(n+1)}$ in the opposite way. This construction satisfies the requirements, hence concluding the proof. □

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