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Goldstone bosons in different PT-regimes of non-Hermitian scalar quantum field theories

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Abstract

We study the interplay between spontaneously breaking global continuous and discrete antilinear symmetries in a newly proposed general class of non-Hermitian quantum field theories containing a mixture of complex and real scalar fields. We analyse the model for different types of global symmetry preserving and breaking vacua. In addition, the models are symmetric under various types of discrete antilinear symmetries composed out of nonstandard simultaneous charge conjugations, time-reversals and parity transformations; CPT. While the global symmetry governs the existence of massless Goldstone bosons, the discrete one controls the precise expression of the Goldstone bosons in terms of the original fields in the model and its physical regimes. We show that even when the CPT-symmetries are broken on the level of the action expanded around different types of vacua, the mass spectra might still be real when the symmetry is preserved at the tree approximation and the breaking only occurs at higher order. We discuss the parameter space of some of the models in the proposed class and identify physical regimes in which massless Goldstone bosons emerge when the vacuum spontaneously breaks the global symmetry or equivalently when the corresponding Noether currents are conserved. The physical regions are bounded by exceptional points in different ways. There exist special points in parameter space for which massless bosons may occur already before breaking the global symmetry. However, when the global symmetry is broken at these points they can no longer be distinguished from genuine Goldstone bosons.

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1. Introduction

It is quite well understood how to extend the conventional framework of Hermitian classical and quantum mechanics [1–3] to allow for the inclusion of non-Hermitian systems. When the latter systems admit an antilinear symmetry [4], such as for instance being invariant under a simultaneous reflection in time and space, referred to as $\mathcal{PT}$-symmetry, this can be achieved in a self-consistent manner. In these circumstances one encounters three types of regimes with qualitatively different behaviour, a $\mathcal{PT}$-symmetric phase, a spontaneously broken $\mathcal{PT}$-symmetric phase and a completely $\mathcal{PT}$-symmetry broken phase. Based on the formal analogy between the Schrödinger equation and the propagation of light in the paraxial approximation described by the Helmholtz equation many of the findings obtained in the quantum mechanical description have been confirmed experimentally and further developed in classical optical settings with the refractive index playing the role of a complex potential [5–9].

When implementing and extending these idea and principles to quantum field theories there is less consensus, and for some aspects alternative resolutions have been proposed. Naturally, as a direct extension of the well studied purely complex cubic potential in quantum mechanics the scalar field theory with imaginary cubic self-interaction term $i\phi^3$ has been investigated at first [10,11] and also the more generally deformed harmonic oscillator has been generalised to a field theoretical interaction term $\phi^2(i\phi)^\varepsilon$ more recently [12]. Non-Hermitian versions with a field theoretic Yukawa interaction [13–16] have been investigated in regard to Higgs boson decay. Besides bosonic theories also generalizations to non-Hermitian fermion theories such as a free fermion theory with a γ_5-mass term or the massive Thirring model have been proposed [17]. $\mathcal{PT}$-symmetric versions of quantum electrodynamics have been studied [18,19] as well.

Here we will focus on a feature that is very central to standard Hermitian quantum field theory, the Goldstone theorem, and investigate further how it extends to non-Hermitian theories. We recall that in the Hermitian case the theorem states that the number of massless Goldstone bosons in a quantum field theory is equal to the dimension of the coset G/H, with G denoting a global continuous symmetry group of the action and H the symmetry group that is left when the theory is expanded around a specific vacuum [20,21]. The question of extension was recently addressed by Alexandre, Ellis, Millington and Seynaeve [22] and separately by Mannheim [23]. Interestingly, both groups found that the theorem appears to hold for non-Hermitian theories as well, but they proposed two alternative variants for it to be implemented. In addition, Mannheim suggests that the non-Hermitian theory possess the new feature of an unobservable Goldstone boson at a special point. Here we find that the Goldstone bosons takes on different forms depending on whether the theory is in the $\mathcal{CPT}$-symmetric regime, at standard exceptional point or what we refer to as the *zero-exceptional point*. We distinguish here between a standard exceptional point, corresponding to two nonzero eigenvalues coalescing, and a zero-exceptional point defined as the point when a zero eigenvalue coalesces with a nonzero eigenvalue.

The problem that both groups have tried to overcome at first is the feature that the equations of motion obtained from functionally varying the action with respect to the scalar fields on one hand and on the other separately with respect to its complex conjugate field are not compatible. This is a well known conundrum for non-Hermitian quantum field theories and has for instance been pointed out previously and elaborated on in [24,25] for a non-Hermitian fermionic theory. Hence, without any modifications the proposed non-Hermitian quantum field theories appear to be inconsistent. To resolve this problem the authors of [22] proposed to use a non-standard variational principle by keeping some non-vanishing surface terms. In contrast, Mannheim [23] utilizes the fact that the action of a theory can be altered without changing the content of the theory as long as the equal time commutation relations are preserved, see e.g. [17]. Utilizing that principle he investigates a model based on a similarity transformed action of the previous one in which the entire set of equations of motion have consistent properties. Remarkably, it was found for both versions that the theory expanded around the global $U(1)$-symmetry breaking vacuum contains a massless Goldstone boson. Moreover, while in the approach that only validates half of the standard set of equations of motion non-standard currents are conserved and Noether's theorem seems to be evaded, the approach proposed in [23] is based on the standard variational principle leading to standard Noether currents.

Here we largely adopt the latter approach and analyse theories expanded about different types of vacua, global symmetry breaking and also preserving ones, for a class of models containing a mixture of several types of complex scalar of fields and also real self-conjugate fields. In particular, we identify the physical regions in parameter space by demanding the masses to be non-negative real-valued in order to be physically meaningful. This has not been considered previously, but is in fact quite essential as potentially the theory might be entirely unphysical. As is turns out, in many scenarios we are able to identify some physical regimes that are, however, quite isolated in parameter space. We find some vacua that break the $\mathcal{CPT}$-symmetries on the level of the action, but still possess physically meaningful mass spectra, as the symmetry breaking occurs at higher order couplings than at the tree approximation. Moreover, we derive the explicit forms of the Goldstone boson in all three $\mathcal{PT}$-regimes, the symmetric and spontaneously broken phases, as well as at the exceptional point.

Our manuscript is organised as follows: In section 2 we introduce a general model with n scalar field that might be genuinely complex but in some versions also contain real self-conjugate fields. In section 3 and 4 we investigate two specific examples of this general class of models in more detail and identify the physical regions in which Goldstone bosons may or may not occur. We investigate different types of vacua that may break the global $U(1)$-symmetry and also several variants of discrete $\mathcal{CPT}$-symmetries that might be broken separately. Starting from a complex squared mass matrix we construct the $\mathcal{P}$-operator that together with $\mathcal{T}$-operator can be used to identify the real eigenvalue regime and show how these operators, that can be thought off as quantum mechanical analogues, are related to the quantum field theoretical $\mathcal{CPT}$-operator. We identify the explicit form of the Goldstone boson in terms of the original fields in the action in different $\mathcal{PT}$-regimes. In section 5 we investigate how the interaction term may be generalised so that the action still respects a discrete $\mathcal{CPT}$-symmetry and a continuous global $U(1)$-symmetry. We state our conclusions and present an outlook in section 6.

2. A non-Hermitian model with *n* complex scalar fields

We consider here generalizations of the model originally proposed in [22] and further studied in [23]. To be a suitable candidate for the investigation of the non-Hermitian version of Goldstone's theorem the model should be not invariant under complex conjugation, possess a discrete $\mathcal{CPT}$-transformation symmetry and crucially be invariant under a global continuous symmetry. The actions $\mathcal{I}_n = \int d^4x \mathcal{L}_n$ involving the Lagrangian densities functional of the general form

$$\mathcal{L}_n = \sum_{i=1}^{n} \left(\partial_\mu \phi_i \partial^\mu \phi_i^* + c_i m_i^2 \phi_i \phi_i^* \right) + \sum_{i=1}^{n-1} \kappa_i \mu_i^2 \left(\phi_i^* \phi_{i+1} - \phi_{i+1}^* \phi_i \right) - \sum_{i=1}^{n} \frac{g_i}{4} (\phi_i \phi_i^*)^2 \tag{2.1}$$

possess all of these three properties. The parameter space is spanned by the real parameters $m_i, g_i, \mu_i \in \mathbb{R}$ and $c_i, \kappa_i = \pm 1$. The latter constants might be absorbed into the mass and the couplings μ_i when allowing them to be purely imaginary or real. However, we keep these constants separately since their values distinguish between different types of qualitative behaviour as we shall see below. When fixing those constants to specific values the action $\mathcal{I}_2$ reduces to the model discussed in [22,23]. In order to keep matters as simple as possible in our detailed analysis, we will set here $g_i = 0$ for $i \neq 1$, but in section 5 we argue that the interaction term may be chosen in a more complicated way with all three properties still preserved.

Functionally varying the action $\mathcal{I}_n$ separately with respect to ϕ_i and ϕ_i^* gives rise to the two sets of equations of motion

$$\frac{\delta \mathcal{I}_n}{\delta \phi_i} = \frac{\partial \mathcal{L}_n}{\partial \phi_i} - \partial_\mu \left[\frac{\partial \mathcal{L}_n}{\partial \left(\partial_\mu \phi_i \right)} \right] = 0, \qquad \frac{\delta \mathcal{I}_n}{\delta \phi_i^*} = \frac{\partial \mathcal{L}_n}{\partial \phi_i^*} - \partial_\mu \left[\frac{\partial \mathcal{L}_n}{\partial \left(\partial_\mu \phi_i^* \right)} \right] = 0. \tag{2.2}$$

We comment below on the compatibility of these equations. Evidently, the action $\mathcal{I}_n$ is not Hermitian when $\phi_i^* \neq \phi_i$ for some i. However, it is invariant under two types of $\mathcal{CPT}$-transformations

$$\begin{aligned} &\mathcal{CPT}_1 : \phi_i(x_\mu) \to (-1)^{i+1} \phi_i^*(-x_\mu), \\ &\mathcal{CPT}_2 : \phi_i(x_\mu) \to (-1)^{i} \phi_i^*(-x_\mu), \quad i = 1, \ldots, n. \end{aligned} \tag{2.3}$$

As pointed out in [26] these types of symmetries are not the standard $\mathcal{CPT}$ transformations as some of the fields are not simply conjugated and $\mathcal{P}$ does not simply act on the argument of the fields, but also acquire an additional minus sign as a factor under the transformation. Such type of symmetries were studied in the quantum field theory context in more detail in [26] and as argued therein make the non-Hermitian versions good candidates for meaningful and self-consistent quantum field theories, in analogy to their quantum mechanical versions, despite being non-Hermitian.

In addition, the action related to (2.1) is left invariant under the continuous global $U(1)$-symmetry

$$\phi_i \to e^{i\alpha} \phi_i, \qquad \phi_i^* \to e^{-i\alpha} \phi_i^*, \qquad i = 1, \ldots, n, \alpha \in \mathbb{R}, \tag{2.4}$$

when none of the fields in the theory is real, that is when $\phi_i^* \neq \phi_i$ for all i. Applying Noether's theorem and using the standard variational principle for this symmetry one obtains

$$\delta \mathcal{L}_n = \partial_\mu \left[\sum_{i=1}^{n} \frac{\partial \mathcal{L}_n}{\partial \left(\partial_\mu \phi_i \right)} \delta \phi_i + \frac{\partial \mathcal{L}_n}{\partial \left(\partial_\mu \phi_i^* \right)} \delta \phi_i^* \right] + \sum_{i=1}^{n} \left[\frac{\delta \mathcal{I}_n}{\delta \phi_i} \delta \phi_i + \frac{\delta \mathcal{I}_n}{\delta \phi_i^*} \delta \phi_i^* \right]. \tag{2.5}$$

Thus provided the equations of motion in (2.2) hold, and $\delta \mathcal{L}_n = 0$ when using the global $U(1)$-symmetry in the variation with $\delta \phi_j = i\alpha \phi_j$ and $\delta \phi_j^* = -i\alpha \phi_j^*$, we derive the Noether current associated to this symmetry as

$$j_\mu = i\alpha \sum\nolimits_i \left(\phi_i \partial_\mu \phi_i^* - \phi_i^* \partial_\mu \phi_i \right). \tag{2.6}$$

Below we discuss in more detail under which circumstances this current is conserved. We will argue that Noether's theorem holds in its standard form and is not evaded as concluded by some authors. Next we are mainly interested in the study of mass spectra resulting by expanding the potentials around different vacua as this probes the Goldstone theorem.

3. Discrete antilinear and continuous global symmetry

We now discuss the model $\mathcal{I}_3$ in more detail with all fields being genuinely complex scalar fields, i.e. $\phi_i \neq \phi_i^*$, $i = 1, 2, 3$. Then the action for (2.1) takes on the form

$$\mathcal{I}_3(\phi_i, \phi_i^*, \partial_\mu \phi_i, \partial_\mu \phi_i^*) = \int d^4x \mathcal{L}_3, \tag{3.1}$$

with Lagrangian density functional

$$\mathcal{L}_3 = \sum_{i=1}^{3} \partial_\mu \phi_i \partial^\mu \phi_i^* - V_3, \tag{3.2}$$

and potential

$$V_3 = -\sum_{i=1}^{3} c_i m_i^2 \phi_i \phi_i^* + c_\mu \mu^2 \left(\phi_1^* \phi_2 - \phi_2^* \phi_1\right) + c_\nu \nu^2 \left(\phi_2 \phi_3^* - \phi_3 \phi_2^*\right) + \frac{g}{4} (\phi_1 \phi_1^*)^2. \tag{3.3}$$

Compared to (2.1) we have simplified here the interaction term by taking $g_1 = g$ and $g_1 = g_2 = 0$. The model contains the real parameters $m_i, \mu, \nu, g \in \mathbb{R}$ and $c_i, c_\mu, c_\nu = \pm 1$. While this action $\mathcal{I}_3$ is not Hermitian, that is invariant under complex conjugation, it respects various discrete and continuous symmetries. It is invariant under two types of $\mathcal{CPT}$-transformations (2.3)

$$\mathcal{CPT}_{1/2} : \phi_1(x_\mu) \to \pm \phi_1^*(-x_\mu), \quad \phi_2(x_\mu) \to \mp \phi_2^*(-x_\mu), \quad \phi_3(x_\mu) \to \pm \phi_3^*(-x_\mu), \tag{3.4}$$

which are both discrete antilinear transformations. Moreover, the action (3.1) is left invariant under the continuous global $U(1)$-symmetry (2.4), which gives rise to the Noether current (2.6)

$$j_\mu = i\alpha \sum\nolimits_{i=1}^{3} \left(\phi_i \partial_\mu \phi_i^* - \phi_i^* \partial_\mu \phi_i\right). \tag{3.5}$$

With the dimension of the global symmetry group $G = U(1)$ being just 1, we may only encounter two possibilities for the Hermitian case, that is the model contains one or no massless Goldstone boson when the symmetry group for the expanded theory is $H = \mathbb{I}$ or $H = U(1)$, respectively, after a specific vacuum has been selected [20,21]. As we shall see, breaking in our model the global $U(1)$-symmetry for the vacuum will give rise to the massless Goldstone bosons in the standard fashion, albeit with some modifications and novel features for a non-Hermitian setting. The six equations of motion in (2.2) read in this case

$$\Box \phi_1 - c_1 m_1^2 \phi_1 - c_\mu \mu^2 \phi_2 + \frac{g}{2} \phi_1^2 \phi_1^* = 0, \tag{3.6}$$

$$\Box \phi_2 - c_2 m_2^2 \phi_2 + c_\mu \mu^2 \phi_1 + c_\nu \nu^2 \phi_3 = 0, \tag{3.7}$$

$$\Box \phi_3 - c_3 m_3^2 \phi_3 - c_\nu \nu^2 \phi_2 = 0, \tag{3.8}$$

$$\Box \phi_1^* - c_1 m_1^2 \phi_1^* + c_\mu \mu^2 \phi_2^* + \frac{g}{2} \phi_1 (\phi_1^*)^2 = 0, \tag{3.9}$$

$$\Box \phi_2^* - c_2 m_2^2 \phi_2^* - c_\mu \mu^2 \phi_1^* - c_\nu \nu^2 \phi_3^* = 0, \tag{3.10}$$

$$\Box \phi_3^* - c_3 m_3^2 \phi_3^* + c_\nu \nu^2 \phi_2^* = 0, \tag{3.11}$$

with d'Alembert operator $\Box := \partial_\mu \partial^\mu$ and metric $\operatorname{diag} \eta = (1, -1, -1, -1)$. We encounter here the same problem as pointed out for $\mathcal{I}_2$ with four scalar fields investigated in [22,23], namely that as a consequence of the non-Hermiticity of the action the equations of motions obtained from the variation with regard to the fields ϕ_i^*, (3.6)-(3.8), are not the complex conjugates of the equations obtained from the variation with respect to the fields ϕ_i, (3.9)-(3.11). Hence, the two sets of equations appear to be incompatible and therefore the quantum field theory related to the action (3.1) seems to be inconsistent.

An unconventional solution to this conundrum was proposed in [22], by suggesting to omit the variation with respect to one set of fields and also taking non-vanishing surface terms into account. Even though this proposal appears to lead to a consistent model, it remains somewhat unclear as to why one should abandon a well established principle from standard complex scalar field theory. Here we adopt the proposal made by Mannheim [23], which is more elegant and, from the point of view of extending the well established framework of non-Hermitian quantum mechanics to quantum field theory, also more natural. It consists of seeking a similarity transformation for the action that achieves compatibility between the two sets of equations of motion. It is easy to see that any transformation of the form $\phi_2 \to \pm i\phi_2$, $\phi_2^* \to \pm i\phi_2^*$ that leaves all the other fields invariant will achieve compatibility between the two sets of equations (3.6)-(3.8) and (3.9)-(3.11).

The analysis to achieve this is most conveniently carried out when reparameterising the complex fields in terms of real component fields. Parameterising therefore the complex scalar field as $\phi_i = 1/\sqrt{2}(\varphi_i + i\chi_i)$ with $\varphi_i, \chi_i \in \mathbb{R}$ the action $\mathcal{I}_3$ in (3.1) acquires the form

$$\begin{aligned}\mathcal{I}_3 = \int d^4x \Bigg\{ & \sum_{i=1}^{3} \frac{1}{2}\left[\partial_\mu\varphi_i\partial^\mu\varphi_i + \partial_\mu\chi_i\partial^\mu\chi_i + c_i m_i^2\left(\varphi_i^2+\chi_i^2\right)\right] \\ & + i c_\mu \mu^2 (\varphi_1\chi_2 - \varphi_2\chi_1) \\ & + i c_\nu \nu^2 (\varphi_3\chi_2 - \varphi_2\chi_3) - \frac{g}{16}(\varphi_1^2+\chi_1^2)^2 \Bigg\}.\end{aligned} \tag{3.12}$$

This approach differs slightly from Mannheim's, who took the component fields to be complex as well. The continuous global $U(1)$-symmetry (2.4) of the action is realised for the real fields as $\varphi_i \to \varphi_i\cos\alpha - \chi_i\sin\alpha$, $\chi_i \to \varphi_i\sin\alpha + \chi_i\cos\alpha$, that is $\delta\varphi_i = -\alpha\chi_i$ and $\delta\chi_i = \alpha\varphi_i$ for α small. The $\mathcal{CPT}_{1/2}$ symmetries in (3.4) manifests on these fields as

$$\begin{aligned}\mathcal{CPT}_{1/2}: & \varphi_{1,3}(x_\mu) \to \pm\varphi_{1,3}(-x_\mu), \quad \varphi_2(x_\mu) \to \mp\varphi_2(-x_\mu), \\ & \chi_{1,3}(x_\mu) \to \pm\chi_{1,3}(-x_\mu), \quad \chi_2(x_\mu) \to \mp\chi_2(-x_\mu), \quad i \to -i.\end{aligned} \tag{3.13}$$

In this form also the antilinear symmetry

$$\mathcal{CPT}_{3/4}: \varphi_{1,2,3}(x_\mu) \to \pm\chi_{1,2,3}(-x_\mu), \quad \chi_{1,2,3}(x_\mu) \to \pm\varphi_{1,2,3}(-x_\mu), \quad i \to -i,$$

leaves the action invariant. Let us now transform the action $\mathcal{I}_3$ in the form (3.12) to an equivalent Hermitian one.

3.1. A $\mathcal{CPT}$ equivalent action, different types of vacua

We define now the analogue to the Dyson map [27] in quantum mechanics as

$$\eta = \exp\left[\frac{\pi}{2}\int d^3x\,\Pi_2^{\varphi}(\mathbf{x},t)\varphi_2(\mathbf{x},t)\right]\exp\left[\frac{\pi}{2}\int d^3x\,\Pi_2^{\chi}(\mathbf{x},t)\chi_2(\mathbf{x},t)\right], \tag{3.14}$$

involving the canonical momenta $\Pi_i^{\varphi} = \partial_t \varphi_i$ and $\Pi_i^{\chi} = \partial_t \chi_i$, $i = 1, 2, 3$. Using the Baker-Campbell-Haussdorf formula we compute the adjoint actions of η on the scalar fields as

$$\eta\varphi_i\eta^{-1} = (-i)^{\delta_{2i}}\varphi_i, \quad \eta\chi_i\eta^{-1} = (-i)^{\delta_{2i}}\chi_i,$$
$$\eta\phi_i\eta^{-1} = (-i)^{\delta_{2i}}\phi_i, \quad \eta\phi_i^*\eta^{-1} = (-i)^{\delta_{2i}}\phi_i^*. \tag{3.15}$$

The equal time commutation relations $\left[\psi_j(\mathbf{x},t), \Pi_j^{\psi_j}(\mathbf{y},t)\right] = i\delta(\mathbf{x}-\mathbf{y})$, $i = 1, 2, 3$, for $\psi = \varphi, \chi$ are preserved under these transformations. Applying them to $\mathcal{I}_3$ in (3.12), we obtain the new equivalent action

$$\hat{\mathcal{I}}_3 = \eta\mathcal{I}_3\eta^{-1} = \int d^4x \sum_{i=1}^{3} \frac{1}{2}(-1)^{\delta_{2i}} \left[\partial_\mu\varphi_i\partial^\mu\varphi_i + \partial_\mu\chi_i\partial^\mu\chi_i + c_i m_i^2\left(\varphi_i^2 + \chi_i^2\right)\right] \tag{3.16}$$
$$+c_\mu\mu^2(\varphi_1\chi_2 - \varphi_2\chi_1) + c_\nu\nu^2(\varphi_3\chi_2 - \varphi_2\chi_3) - \frac{g}{16}(\varphi_1^2 + \chi_1^2)^2.$$

The $U(1)$-symmetry is still realised in the same way as for $\mathcal{I}_3$, but the $\mathcal{CPT}$-symmetries for $\hat{\mathcal{I}}_3$ are now modified to

$$\widehat{\mathcal{CPT}}_{1/2}: \varphi_{1,3}(x_\mu) \to \pm\varphi_{1,3}(-x_\mu), \quad \varphi_2(x_\mu) \to \mp\varphi_2(-x_\mu), \tag{3.17}$$
$$\chi_{1,3}(x_\mu) \to \mp\chi_{1,3}(-x_\mu), \quad \chi_2(x_\mu) \to \pm\chi_2(-x_\mu),$$
$$\widehat{\mathcal{CPT}}_{3/4}: \varphi_{1,2,3}(x_\mu) \to \pm\chi_{1,2,3}(-x_\mu), \tag{3.18}$$

accommodating the fact that no explicit imaginary unit i is left in the action. Notice that these symmetries are, however, no longer antilinear and therefore lack the constraining power of predicting the reality of non-Hermitian quantities. The equations of motion resulting from functionally varying $\hat{\mathcal{I}}_3$ with respect to the real fields are

$$-\Box\varphi_1 = \frac{\partial V}{\partial\varphi_1} = -c_1 m_1^2\varphi_1 - c_\mu\mu^2\chi_2 + \frac{g}{4}\varphi_1(\varphi_1^2 + \chi_1^2), \tag{3.19}$$
$$-\Box\chi_2 = -\frac{\partial V}{\partial\chi_2} = -c_2 m_2^2\chi_2 + c_\mu\mu^2\varphi_1 + c_\nu\nu^2\varphi_3, \tag{3.20}$$
$$-\Box\varphi_3 = \frac{\partial V}{\partial\varphi_3} = -c_3 m_3^2\varphi_3 - c_\nu\nu^2\chi_2, \tag{3.21}$$
$$-\Box\chi_1 = \frac{\partial V}{\partial\chi_1} = -c_1 m_1^2\chi_1 + c_\mu\mu^2\varphi_2 + \frac{g}{4}\chi_1(\varphi_1^2 + \chi_1^2), \tag{3.22}$$
$$-\Box\varphi_2 = -\frac{\partial V}{\partial\varphi_2} = -c_2 m_2^2\varphi_2 - c_\mu\mu^2\chi_1 - c_\nu\nu^2\chi_3, \tag{3.23}$$
$$-\Box\chi_3 = \frac{\partial V}{\partial\chi_3} = -c_3 m_3^2\chi_3 + c_\nu\nu^2\varphi_2. \tag{3.24}$$

We may write the action $\hat{\mathcal{I}}_3$ and the corresponding equation of motions more compactly. Introducing the column vector field $\Phi = (\varphi_1, \chi_2, \varphi_3, \chi_1, \varphi_2, \chi_3)^T$, the action acquires the concise form

$$\hat{\mathcal{I}}_3 = \frac{1}{2}\int d^4x \left[\partial_\mu\Phi^T I \partial^\mu\Phi - \Phi^T H_t \Phi - \frac{g}{8}\left(\Phi^T E \Phi\right)^2\right]. \tag{3.25}$$

Here we employed the Hessian matrix $H_{ij}(\Phi) = \left.\frac{\partial^2 V}{\partial\Phi_i \partial\Phi_j}\right|_\Phi$ which for our potential V_3 reads

$$H(\Phi) = \begin{pmatrix} \frac{g}{4}(3\varphi_1^2+\chi_1^2) - c_1 m_1^2 & -c_\mu \mu^2 & 0 & \frac{g}{2}\varphi_1\chi_1 & 0 & 0 \\ -c_\mu\mu^2 & c_2 m_2^2 & -c_\nu \nu^2 & 0 & 0 & 0 \\ 0 & -c_\nu\nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\ \frac{g}{2}\varphi_1\chi_1 & 0 & 0 & \frac{g}{4}(\varphi_1^2+3\chi_1^2) - c_1 m_1^2 & c_\mu\mu^2 & 0 \\ 0 & 0 & 0 & c_\mu\mu^2 & c_2 m_2^2 & c_\nu\nu^2 \\ 0 & 0 & 0 & 0 & c_\nu\nu^2 & -c_3 m_3^2 \end{pmatrix}. \tag{3.26}$$

In (3.25) we use $H_t = H\left(\Phi_1^0\right)$, $\Phi_1^0 = (0,0,0,0,0,0)$ and the 6×6-matrices I, E with $\operatorname{diag} I = (1,-1,1,1,-1,1)$ and $\operatorname{diag} E = (1,0,0,1,0,0)$. The equation of motion resulting from (3.25) reads

$$-\Box\Phi - IH_t\Phi - \frac{g}{4} I\left(\Phi^T E\Phi\right) E\Phi = 0. \tag{3.27}$$

We find different types of vacua by solving $\delta V = 0$, amounting to setting simultaneously the right hand sides of the equations (3.19)-(3.24) to zero and solving for the fields φ_i, χ_i. Denoting the solutions by $\Phi^0 = (\varphi_1^0, \chi_2^0, \varphi_3^0, \chi_1^0, \varphi_2^0, \chi_3^0)^T$, we find the vacua

$$\Phi_1^0 = (0,0,0,0,0,0), \tag{3.28}$$

$$\Phi_2^0 = K(0)\left(1, \frac{c_3 c_\mu m_3^2 \mu^2}{\kappa}, -\frac{c_3 c_\mu m_3^2\mu^2}{\kappa}, 0, 0, 0\right), \tag{3.29}$$

$$\Phi_3^0 = K(0)\left(0,0,0,-1, \frac{c_3 c_\mu m_3^2\mu^2}{\kappa}, \frac{c_\nu c_\mu \nu^2\mu^2}{\kappa}\right), \tag{3.30}$$

$$\begin{aligned}\Phi_4^0 = \Bigg(&\varphi_1^0, \frac{c_3 c_\mu m_3^2\mu^2\varphi_1^0}{\kappa}, -\frac{c_\nu c_\mu \nu^2\mu^2\varphi_1^0}{\kappa}, -K(\varphi_1^0), \\ &\frac{c_3 c_\mu m_3^2\mu^2 K(\varphi_1^0)}{\kappa}, \frac{c_\nu c_\mu\nu^2\mu^2 K(\varphi_1^0)}{\kappa}\Bigg),\end{aligned} \tag{3.31}$$

where for convenience we introduced the function and constant

$$K(x) := \pm\sqrt{\frac{4c_3 m_3^2\mu^4}{g\kappa} + \frac{4c_1 m_1^2}{g} - x^2}, \qquad \kappa := c_2 c_3 m_2^2 m_3^2 + \nu^4. \tag{3.32}$$

Notice, that in the vacuum Φ_4^0 the field φ_1^0 is generic and not fixed. When varied it interpolates between the vacua Φ_2^0 and Φ_3^0. For $(\varphi_1^0)^2 \to 4(c_1 m_1^2\kappa + c_3 m_3^2\mu^4)/g\kappa$ and $\varphi_1^0 \to 0$ we obtain $\Phi_4^0 \to \Phi_2^0$ and $\Phi_4^0 \to \Phi_3^0$, respectively. We also note that $K(0) = 0$ at the special value of the coupling $\mu = \mu_s^4 = -c_1 m_1^2\kappa/c_3 m_3^2$ so that $\Phi_2^0(\mu_s) = \Phi_1^0$. Unlike as in [23], where the vacuum is taken to be complex, our vacua are real. Next we probe Goldstone's theorem by computing the masses resulting by expanding around the different vacua in the tree approximation.

3.2. The mass spectra, $\mathcal{PT}$-symmetries

Defining the column vector field $\Phi = \Phi^0 + \hat{\Phi}$ with vacuum component Φ^0 as defined above and $\hat{\Phi} = (\hat{\varphi}_1, \hat{\chi}_2, \hat{\varphi}_3, \hat{\chi}_1, \hat{\varphi}_2, \hat{\chi}_3)^T$, we expand the potential about the vacua (3.28)-(3.31) as

$$V(\Phi) = V\left(\Phi^0 + \hat{\Phi}\right) = V\left(\Phi^0\right) + \nabla V\left(\Phi^0\right)^T \hat{\Phi} + \frac{1}{2}\hat{\Phi}^T H\left(\Phi^0\right)\hat{\Phi} + \ldots. \tag{3.33}$$

The linear term is of course vanishing, as by design $\nabla V\left(\Phi^0\right) = 0$. The squared mass matrix M^2 is read off from (3.27) as

$$\left(M^2\right)_{ij} = [IH\left(\Phi^0\right)]_{ij}. \tag{3.34}$$

The somewhat unusual emergence of the matrix I is due to the fact that as a consequence of the similarity transformation we now have negative signs in front of some of the kinetic energy terms, see also (3.20) and (3.23).

In general this matrix is not diagonal, but in the $\mathcal{CPT}$-symmetric regime we may diagonalise it and express the fields related to these masses in terms of the original fields in the action. Denoting the eigenvectors of the squared mass matrix by v_i, $i = 1, \ldots, 6$, the matrix $U = (v_1, \ldots, v_6)$, containing the eigenvectors as column vectors, diagonalizes M^2 as $U^{-1}M^2U = D$ with diag $D = (\lambda_1, \ldots, \lambda_6)$ as long as U is invertible. The latter property holds in general only in the $\mathcal{CPT}$-symmetric regime. Rewriting

$$\hat{\Phi}^T M^2 \hat{\Phi} = \sum\nolimits_i m_i^2 \psi_i^2 = \sum\nolimits_i m_i^2 \left(\hat{\Phi}^T IU\right)_i (U^{-1}\hat{\Phi})_i, \tag{3.35}$$

we may therefore introduce the masses m_i for the fields

$$\psi_i := \sqrt{\left(\hat{\Phi}^T IU\right)_i (U^{-1}\hat{\Phi})_i} \tag{3.36}$$

as the positive square roots of the eigenvalues of the squared mass matrix M^2, that is $m_i = \sqrt{\lambda_i}$. Naturally this means the fields ψ_i in the specific form (3.36) are absent when U is not invertible and since physical masses m_i are non-negative we must also discard scenarios in which $\lambda_i < 0$ or $\mathrm{Im}\,\lambda_i \neq 0$ as unphysical.

Since the squared mass matrix M^2 is not Hermitian, but may have real eigenvalues λ_i in some regime, we can employ the standard framework from $\mathcal{PT}$-symmetric quantum mechanics with M^2 playing the role of the non-Hermitian Hamiltonian [1,3]. We can then identify the antilinear $\mathcal{PT}$-operator that ensures the reality of the spectrum in that particular regime. The time-reversal operator $\mathcal{T}$ simply corresponds to a complex conjugation, but one needs to establish that the $\mathcal{P}$-operator obtained from the quantum mechanical description is the same as the one employed at the level of the action. In order to identify that connection let us first see which properties the $\mathcal{P}$-operator must satisfy at the level of the action. Expressing $\mathcal{I}_3$ in the form

$$\mathcal{I}_3[\Phi] = \mathcal{I}_3^{\mathrm{M}}[\Phi] + \mathcal{I}_3^{\mathrm{int}}[\Phi] = \frac{1}{2}\int d^4x \left[\Phi^T\left(\Box + M^2\right)\Phi\right] + \mathcal{I}_3^{\mathrm{int}}[\Phi], \tag{3.37}$$

with real field vector Φ, the action of the $\mathcal{CPT}$-operator on $\mathcal{I}_3^{\mathrm{M}}[\Phi]$ is

$$\mathcal{CPT}: \mathcal{I}_3^{\mathrm{M}}[\Phi] \to \frac{1}{2}\int d^4x \left[\Phi^T\left[\mathcal{P}^T\mathcal{P}\Box + \mathcal{P}^T\left(M^2\right)^*\mathcal{P}\right]\Phi\right]. \tag{3.38}$$

Hence for this part of the action to be invariant we require the $\mathcal{P}$-operator to obey the two relations

$$\mathcal{P}^T\mathcal{P} = \mathbb{I}, \quad \text{and} \quad \left(M^2\right)^*\mathcal{P} = \mathcal{P}M^2. \tag{3.39}$$

This is in fact the same property $\mathcal{P}$ needs to satisfy in the $\mathcal{PT}$-quantum mechanical framework. Let us see how to construct $\mathcal{P}$ when given the non-Hermitian matrix M^2. We start by constructing a biorthonormal basis from the left and right eigenvectors u_n and v_n, respectively, of M^2

$$M^2 v_n = \varepsilon_n v_n, \qquad \left(M^2\right)^\dagger u_n = \varepsilon_n u_n \tag{3.40}$$

satisfying

$$\langle u_n | v_n \rangle = \delta_{nm}, \qquad \sum\nolimits_n |u_n\rangle \langle v_n| = \sum\nolimits_n |v_n\rangle \langle u_n| = \mathbb{I}. \tag{3.41}$$

The left and right eigenvectors are related by the $\mathcal{P}$-operator as

$$|u_n\rangle = s_n \mathcal{P} |v_n\rangle, \tag{3.42}$$

with $s_n = \pm 1$ defining the signature. Combining (3.42), (3.41) and the first relation in (3.39) we can express the $\mathcal{P}$-operator and its transpose in terms of the left and right eigenvectors as

$$\mathcal{P} = \sum\nolimits_n s_n |u_n\rangle \langle u_n|, \qquad \text{and} \qquad \mathcal{P}^T = \sum\nolimits_n s_n |v_n\rangle \langle v_n|. \tag{3.43}$$

The biorthonormal basis can also be used to construct an operator, often denoted with the symbol C, that is closely related to the metric ρ used in non-Hermitian quantum mechanics

$$C = \mathcal{P}^T \rho = \sum\nolimits_n s_n |v_n\rangle \langle u_n|. \tag{3.44}$$

Despite its notation, this operator is not to be confused with the charge conjugation operator $\mathcal{C}$ employed on the level of the action. The operator C satisfies the algebraic properties [28]

$$\left[C, M^2\right] = 0, \qquad [C, \mathcal{PT}] = 0, \qquad C^2 = \mathbb{I}. \tag{3.45}$$

When compared to the quantum mechanical setting the operator U^{-1} plays here the analogue to the Dyson map η and the combination $\left(U^{-1}\right)^\dagger U^{-1}$ is the analogue to the metric operator ρ. However, constructing $\mathcal{P}$ with M^2 as a starting point does of course not guarantee that also $\mathcal{I}_3^{\text{int}}[\Phi]$ will be invariant under $\mathcal{CPT}$ when using this particular $\mathcal{P}$-operator. In fact, we shall see below that there are many solutions to the two relations in (3.39) that do not leave $\mathcal{I}_3^{\text{int}}[\Phi]$ invariant. Thus for these $\mathcal{CPT}$-operators the symmetry is broken on the level of the action, but the mass spectra would still be real as the symmetry is preserved at the tree approximation and the breaking only occurs at higher order.

3.3. $U(1)$ and $\mathcal{CPT}$ invariant vacuum, absence of Goldstone bosons

We investigate now in more detail the theory expanded about the vacuum Φ_1^0 in (3.28). According to our discussion at the end of the last section the theory expanded about this vacuum is invariant under the global $U(1)$-symmetry and all four $\mathcal{CPT}$-symmetries. As the dimension of the coset G/H equals 0 the standard field theoretical arguments on Goldstone's theorem suggest that we do not expect a Goldstone boson to emerge when expanding around this vacuum. We confirm this by considering the squared mass matrix as defined in (3.34), which for this vacuum decomposes into Jordan block form as

$$M_1^2 = \begin{pmatrix} -c_1 m_1^2 & -c_\mu \mu^2 & 0 & 0 & 0 & 0 \\ c_\mu \mu^2 & -c_2 m_2^2 & c_\nu \nu^2 & 0 & 0 & 0 \\ 0 & -c_\nu \nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_1 m_1^2 & c_\mu \mu^2 & 0 \\ 0 & 0 & 0 & -c_\mu \mu^2 & -c_2 m_2^2 & -c_\nu \nu^2 \\ 0 & 0 & 0 & 0 & c_\nu \nu^2 & -c_3 m_3^2 \end{pmatrix}, \tag{3.46}$$

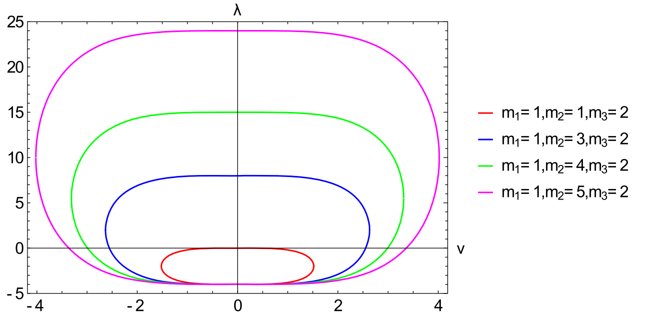

Fig. 1. Nonvanishing eigenvalues λ of M_1^2 as a function of ν for $c_1 = -c_2 = c_3 = 1$, at the special point $\mu = \mu_s$, fixed values for m_1 and m_3 at different values of m_2. The physical regions are $\nu \in (-\nu_{\rm ex}, -\nu_0)$ and $\nu \in (\nu_0, \nu_{\rm ex})$. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

where we label the entries of the matrix by the fields in the order as defined for the vector field Φ. The two blocks are simply related as $c_{\nu/\mu} \to -c_{\nu/\mu}$. We find that the eigenvalues of each block only depend on the combination $c^2_{\nu/\mu} = 1$, so that we have three degenerate eigenvalues with linear independent eigenvectors and it therefore suffices to consider one block only and subsequently implement the degeneracy. Evaluating the constant term of the third order characteristic equations we obtain $-c_3 m_3^2 \mu^4 - c_1 m_1^2 \nu^4 - c_1 c_2 c_3 m_1^2 m_2^2 m_3^2$ for each block. In general, this is not equal to zero indicating the absence of a massless Goldstone boson as expected or any other type of massless particle. The two choices $c_1 = c_2 = c_3 = \pm 1$ exclude the possibility for this term to vanish for *any* values in parameter space $(m_1, m_2, m_3, \mu, \nu)$. Alternatively this is also seen from $\det M_1^2 = (c_3 m_3^2 \mu^4 + c_1 m_1^2 \kappa)^2$ with κ as defined in (3.32).

All other choices for the constants c_i may lead to zero masses for specific values in the parameters space. For instance, when $c_1 = -c_2 = c_3 = 1$, the linear term vanishes for the special choice $\mu_s = (m_1^2 m_2^2 - \nu^4 m_1^2/m_3^2)^{1/4}$, so that we obtain two zero mass particles in the spectrum, of which, however, none is a Goldstone boson. As in the general case with unrestricted μ, the eigenvalues λ of M_1^2 indicate some unphysical regions, with λ being either negative or complex. However, the model has also a physical region in which two degenerate eigenvalues of the squared mass matrix are positive and, somewhat unexpectedly from the symmetry argument, there are also two massless particles present in the spectrum. The behaviour of the remaining two degenerate eigenvalues is depicted in Fig. 1.

The region $\nu \in (-\nu_0, \nu_0)$ with $\nu_0 = m_3(m_1^2 - m_2^2)^{1/4}/(m_1^2 - m_3^2)^{1/4}$ is therefore discarded as unphysical because one of the eigenvalues of M_1^2 is negative. At $\pm\nu_{\rm ex}$, with $\nu_{\rm ex} = [m_3^2(m_2^2 + m_3^2 - m_1^2)^2/4/(m_3^2 - m_1^2)]^{1/4}$ for $m_3^2 > m_1^2$, the two eigenvalues coalesce and become a complex conjugate pair, a scenario that for the energy spectrum in the quantum mechanical context is usually referred to as an exceptional point. Hence, also the regions $\nu < -\nu_{\rm ex}$ and $\nu > \nu_{\rm ex}$ are excluded as being unphysical. Crucially, however, the model is not empty and possess a physical region in parameter space.

3.4. $U(1)$ broken and $\mathcal{CPT}$-invariant vacua, presence of Goldstone bosons

Let us next choose another vacuum that breaks the global $U(1)$-symmetry. In this case we expect one massless Goldstone boson to appear. However, as in the previous case there are some regions in the parameter space for which the model may possess a second massless particle. We choose now the vacuum Φ_2^0. Notice that for $c_1 = -c_2 = c_3 = 1$ and $\mu \to \mu_s$, as defined above, the global symmetry breaking and symmetry preserving vacua coincide $\Phi_2^0 \to \Phi_1^0$, and therefore the previous discussion applies in that case. Expanding the action around this $U(1)$-symmetry breaking vacuum for $\mu \neq \mu_s$, the corresponding squared mass matrix becomes

$$M_2^2 = \begin{pmatrix} \frac{3c_3 m_3^2 \mu^4}{\kappa} + 2c_1 m_1^2 & -c_\mu \mu^2 & 0 & 0 & 0 & 0 \\ c_\mu \mu^2 & -c_2 m_2^2 & c_\nu \nu^2 & 0 & 0 & 0 \\ 0 & -c_\nu \nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c_3 m_3^2 \mu^4}{\kappa} & c_\mu \mu^2 & 0 \\ 0 & 0 & 0 & -c_\mu \mu^2 & -c_2 m_2^2 & -c_\nu \nu^2 \\ 0 & 0 & 0 & 0 & c_\nu \nu^2 & -c_3 m_3^2 \end{pmatrix}, \tag{3.47}$$

with $\det M_2^2 = 0$, hence indicating a zero eigenvalue. Let us now comment on where this Goldstone boson originates from. Both blocks in M_2^2 are of the following general 3×3-matrix form

$$\begin{pmatrix} A & W & 0 \\ -W & B & -V \\ 0 & V & -C \end{pmatrix}, \tag{3.48}$$

whose eigenvalues are solutions to the cubic characteristic equation $\lambda^3 + r\lambda^2 + s\lambda + t = 0$ with

$$r = C - A - B, \quad s = V^2 + W^2 + AB - C(A + B), \quad t = ABC + CW^2 - AV^2. \tag{3.49}$$

Reading off the entries for the block in the lower right corner of M_2^2 as $A = c_3 m_3^2 \mu^4/\kappa$, $B = -c_2 m_2^2$, $C = c_3 m_3^2$, $W = c_\mu \mu^2$, $V = c_\nu \nu^2$, we find that the constant term in the characteristic equation is zero, i.e. $t = 0$. Hence at least one eigenvalue becomes zero. The remaining equation is simply quadratic with solutions

$$\lambda_\pm = \frac{c_3 m_3^2 \mu^4}{2\kappa} - \frac{c_2 m_2^2 + c_3 m_3^2}{2} \pm \frac{1}{2\kappa}\sqrt{m_3^4(\mu^4 - \mu_e^4)^2 + 4c_\nu \nu^2 \kappa^{3/2}(\mu^4 - \mu_e^4)}. \tag{3.50}$$

We introduced here the quantity $\mu_{\mathrm{e}}^\pm = [\kappa(\kappa - m_3^4 + \nu^4 \pm 2c_\nu \nu^2 \sqrt{\kappa})]^{1/4}/m_3$, that signifies the value for μ at which the eigenvalues λ_+ and λ_- coincide, which is referred to as the exceptional point. For the block in the top left corner we identify $A = 3c_3 m_3^2 \mu^4/\kappa + 2c_1 m_1^2$, $B = -c_2 m_2^2$, $C = c_3 m_3^2$, $W = -c_\mu \mu^2$ and $V = -c_\nu \nu^2$. The linear term becomes $t = -2(c_3 m_3^2 \mu^4 + c_1 m_1^2 \nu^4 + c_1 c_2 c_3 m_1^2 m_2^2 m_3^2)$, which is exactly twice the value of t obtained previously for the vacuum Φ_1^0. For $t \neq 0$ we define with (3.49) the quantities

$$\rho = \sqrt{-\frac{p^3}{27}}, \quad \cos\theta = -\frac{q}{2\rho}, \quad p = \frac{3s - r^2}{3},$$
$$q = \frac{2r^3}{27} - \frac{rs}{3} + t, \quad \Delta = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \tag{3.51}$$

Then, provided that $p < 0$ and $\Delta \leq 0$, the remaining three eigenvalues are real and according to Cardano's formula of the form

$$\lambda_i = 2\rho^{1/3} \cos\left[\frac{\theta}{3} + \frac{2\pi}{3}(i - 1)\right], \qquad i = 1, 2, 3. \tag{3.52}$$

Similarly as for the vacuum Φ_1^0 the values of c_μ and c_ν are not relevant for the computation of the eigenvalues. Naturally, for these eigenvalues to be interpretable as squared masses to

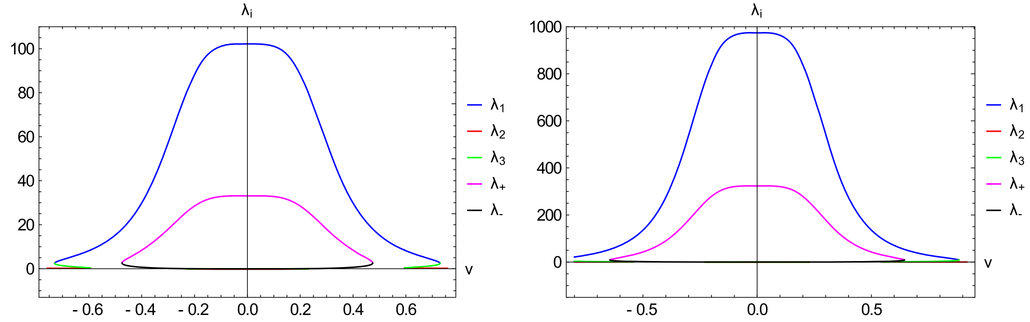

Fig. 2. Nonvanishing eigenvalues λ_i of M_2^2 as functions of ν for $c_1 = c_2 = c_3 = 1$, $m_1 = 1$, $m_2 = 1/2$ and $m_3 = 1/5$. In the left panel we choose $\mu = 1.7$ observing that there is no physical region for which all eigenvalues are non-negative. In the right panel we choose $\mu = 3$ and have two physical regions for $\nu \in (-0.64468, -0.54490)$ and $\nu \in (0.54490, 0.64468)$.

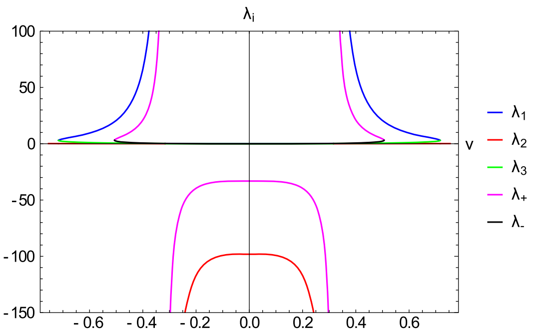

Fig. 3. Nonvanishing eigenvalues λ_i of M_2^2 as a function of ν for $c_1 = -c_2 = c_3 = 1$, $m_1 = 1$, $m_2 = 1/2$, $m_3 = 1/5$ and $\mu = 1.7$. Singularities occur at $\nu = \nu_{\text{sing}}^{\pm} \approx \pm 0.31623$. The regimes $\nu \in (-0.50608, \nu_{\text{sing}}^{-})$, $\nu \in (\nu_{\text{sing}}^{+}, 0.50608)$ are physical.

tree order they need to be non-negative. There are indeed some regions in the parameter space for which this holds, taking for instance $c_1 = c_3 = -c_2 = 1$, $m_1 = 1$, $m_2 = 1/2$, $m_3 = 1/5$, $\mu = 2$ and $\nu = 1/2$ we compute the six non-negative eigenvalues $(\lambda_1, \lambda_3, \lambda_2, \lambda_+, \lambda_-, 0) = (38.1493, 0.5683, 0.0639, 10.6534, 1.7471, 0)$. However, as seen in Fig. 2 these physical regions are quite isolated in the parameter space.

For the choice $c_1 = -c_3 = \pm 1$ we may also find a value for $\nu = \nu_{\text{sing}}^{\pm} = \pm\sqrt{m_2 m_3}$, for which $\kappa \to 0$ leading to singularities in the eigenvalues. Fig. 3 depicts such a situation.

As for the case with $U(1)$-invariant vacuum, for some specific choices of μ we can apparently generate an additional massless particle. Since the linear term of the characteristic equation for the upper right corner is simply twice the one of the previous section, this scenario occurs for $\mu = \mu_s$. However, as we pointed out above for this value of μ the two vacua Φ_1^0 and Φ_2^0 coincide, so that the discussion of the previous section applies. In addition, as the two blocks are different in this case there is a second choice $\bar{\mu}_s^4 = \kappa^2/(m_3^4 - \nu^4)$ for which $\lambda_- = 0$ and the non-zero eigenvalue coalesces with the zero eigenvalue at the zero-exceptional point. Hence, in this case it appears that besides the Goldstone boson there is a second massless, non-Goldstone, particle present in the model. We shall see below that this is actually not the case.

Choosing instead the vacuum Φ_3^0, the resulting mass matrix M_3^2 is similar to M_2^2 with the block in the top left corner and lower right corner exchanged accompanied by the transformation $c_{\nu/\mu} \to -c_{\nu/\mu}$, hence the previous discussion applied in this case.

Expanding instead around the vacuum Φ_4^0 the resulting mass matrix reads

$$M_4^2 = \begin{pmatrix} \frac{c_3 m_3^2 \mu^4}{\kappa} + (\varphi_1^0)^2 & -\frac{g}{2} c_\mu \mu^2 & 0 & \frac{g}{2}\varphi_1^0 \chi_1^0 & 0 & 0 \\ c_\mu \mu^2 & -c_2 m_2^2 & c_\nu \nu^2 & 0 & 0 & 0 \\ 0 & -c_\nu \nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\ \frac{g}{2}\varphi_1^0 \chi_1^0 & 0 & 0 & 2c_1 m_1^2 + \frac{3c_3 m_3^2 \mu^4}{\kappa} - \frac{g}{2}(\varphi_1^0)^2 & c_\mu \mu^2 & 0 \\ 0 & 0 & 0 & -c_\mu \mu^2 & -c_2 m_2^2 & -c_\nu \nu^2 \\ 0 & 0 & 0 & 0 & c_\nu \nu^2 & -c_3 m_3^2 \end{pmatrix}. \tag{3.53}$$

Computing the sixth order characteristic polynomial for M_4^2 we find that the dependence on the free field φ_1^0 drops out entirely. We also note that the linear term always vanishes and that therefore a Goldstone boson is present for this vacuum. We will not present here a more detailed discussion as the qualitative behaviour of the model is similar to the one discussed in detail in the previous section. The model posses various well defined physical regions. For instance, for $c_1 = -1$, $c_2 = c_3 = c_\mu = c_\nu = 1$, $m_1 = 2$, $m_2 = 1/2$, $m_3 = 1/10$, $\mu = 3/2$ and $\nu = 0.28$ we find the eigenvalues $(0, 0.0130, 0.2731, 0.7294, 4.8655, 9.0186)$ for M_4^2. Let us now see how to explain the reality of the mass spectrum.

3.5. *From quantum mechanical to field theoretical $\mathcal{P}$-operators*

We consider now the lower right block of the squared mass matrix in (3.47) and construct a $\mathcal{P}$-operator in a manner as describes in section 3.2, i.e. taking the mass matrix as a starting point. Subsequently we verify whether the operator constructed in the manner is a parity operator that can be used in the $\mathcal{CPT}$-symmetry transformations that leave the quantum field theoretical actions invariant. Including the remaining part of the squared mass matrix is straightforward.

We consider the version of M_2^2 resulting from the action before carrying out the similarity transformation, with the lower right block in (3.47) given as

$$\mathcal{M} = \begin{pmatrix} \frac{c_3 m_3^2 \mu^4}{\kappa} & i c_\mu \mu^2 & 0 \\ i c_\mu \mu^2 & -c_2 m_2^2 & -i c_\nu \nu^2 \\ 0 & -i c_\nu \nu^2 & -c_3 m_3^2 \end{pmatrix}. \tag{3.54}$$

The standard argument that explains the reality of the spectrum for this non-Hermitian matrix is simply stated: Iff there exists an antilinear operator $\mathcal{PT}$, satisfying

$$[\mathcal{M}, \mathcal{PT}] = 0, \qquad \text{and} \qquad \mathcal{PT} v_n = v_n \tag{3.55}$$

with v_n denoting the eigenvectors of $\mathcal{M}$, the eigenvalues λ_n of $\mathcal{M}$ are real. When in (3.55) only the first relation holds and $\mathcal{PT} v_n \neq v_n$, the $\mathcal{PT}$-symmetry is spontaneously broken and some of the eigenvalues emerge in complex conjugate pairs.

To check this statement for our concrete matrix and in particular to construct an explicit expression for the $\mathcal{P}$-operator we compute first the normalised left and right eigenvectors for this non-Hermitian matrix as defined in (3.40)

$$v_j = (-1)^{\delta_{-,j}} u_j^* = \frac{1}{N_j}\{-\lambda_j \Lambda_j - \kappa, -i\Lambda_j^3 c_\mu \mu^2, -c_\mu c_\nu \mu^2 \nu^2\}, \quad j = 0, \pm, \tag{3.56}$$

with normalisation constants

$$N_\pm^2 = (\kappa + \lambda_\pm \Lambda_\pm)\lambda_\pm (\lambda_+ - \lambda_-), \tag{3.57}$$

$$N_0^2 = \kappa \lambda_- \lambda_+, \tag{3.58}$$

where we abbreviated $\Lambda_j := \lambda_j + c_2 m_2^2 + c_3 m_3^2$ and $\Lambda_j^k := \lambda_j + c_k m_k^2$. We confirm that the set of vectors $\{v_j, u_j\}$ with $j = 0, \pm$ form indeed a biorthonormal basis by verifying (3.41).

Next we use relation (3.43) to compute the $\mathcal{P}$-operator

$$\mathcal{P} = \sum_{j=0,\pm} \frac{s_j}{N_j^2} \begin{pmatrix} \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right)^2 & i\mu^2 \Lambda_j^3 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) & \mu^2 \nu^2 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) \\ -i\mu^2 \Lambda_j^3 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) & \mu^4 \left(\Lambda_j^3\right)^2 & -i\nu^2 \mu^4 \Lambda_j^3 \\ \mu^2 \nu^2 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) & i\nu^2 \mu^4 \Lambda_j^3 & \mu^4 \nu^4 \end{pmatrix} \tag{3.59}$$

Given all possibilities for the signatures s_n, we have found eight different $\mathcal{P}$-operators. All of them satisfy the two relations in (3.39). However, two signatures are very special as for them the expressions simplify considerably

$$\mathcal{P}(s_0 = \pm 1, s_- = \mp 1, s_+ = \pm 1) = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}. \tag{3.60}$$

Moreover, in this case the $\mathcal{P}$-operators are indeed the operators involved in the $\mathcal{CPT}_{1/2}$-symmetry transformation that is respected by the entire action. Notice that at the exceptional point, $\lambda_- = \lambda_+$, the normalisation factors $N_\pm$ becomes zero so that the eigenvector $v_\pm$ and $u_\pm$ are no longer defined. Passing this point corresponds to breaking the $\mathcal{PT}$-symmetry spontaneously and the second relation in (3.55) no longer holds.

Next we calculate the operator C as defined in equation (3.44) in two alternative ways to

$$C = \sum_{j=0,\pm} \frac{(-1)^{\delta_{-,j}} s_j}{N_j^2} \begin{pmatrix} \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right)^2 & i\mu^2 \Lambda_j^3 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) & \mu^2 \nu^2 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) \\ i\mu^2 \Lambda_j^3 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) & -\mu^4 \left(\Lambda_j^3\right)^2 & i\nu^2 \mu^4 \Lambda_j^3 \\ \mu^2 \nu^2 \left(\Lambda_j^2 \Lambda_j^3 + \nu^4\right) & i\nu^2 \mu^4 \Lambda_j^3 & \mu^4 \nu^4 \end{pmatrix}. \tag{3.61}$$

We verify that C does indeed satisfy all the relations in (3.45). The Dyson operator is identified as $\eta = U^{-1}$ with $U = (v_0, v_+, v_-)$ and the metric operator as $\rho = \eta^\dagger \eta$. Since $\det U = i\lambda_- \lambda_+ (\lambda_- - \lambda_+)\mu^4 \nu^2 / \, N_0 N_- N_+$ both operators exist in the $\mathcal{PT}$-symmetric regime. The fact that the C-operator is not unique [29] is a well known fact, similarly as for the metric operator.

3.6. The Goldstone boson in the $\mathcal{PT}$-symmetric regime

Let us now compute the explicit expression for the Goldstone boson. As we have seen in section 3.5, the Goldstone boson emerges from the lower right block so that it suffices to consider that part of the squared mass matrix. Denoting the quantities related to the lower right block by a subscript r and the upper left block by ℓ, we decompose the Lagrangian into $\mathcal{L}_3 = \mathcal{L}_{3,\ell} + \mathcal{L}_{3,r}$ and define the quantities

$$\hat{\Phi}_r := (\hat{\chi}_1, \hat{\varphi}_2, \hat{\chi}_3), \qquad (M_2^2)_r v_i = \lambda_i v_i, \qquad U := (v_0, v_+, v_-), \quad i = 0, \pm. \tag{3.62}$$

Similarly for $\mathcal{L}_{3,\ell}$, which we, however, do not analyse here as it does not contain a Goldstone boson. Thus, as long as the spectrum of M_2^2 is not degenerate, and hence all the eigenvectors v_i are linearly independent, the matrix U diagonalizes the lower right block of the squared mass matrix $U^{-1}(M_2^2)_r U = D$ with $\operatorname{diag} D = (\lambda_0, \lambda_+, \lambda_-) = (m_0^2, m_+^2, m_-^2)$. As argued in general in (3.35)-(3.36), we may therefore defined the fields ψ_k, $k = 0, \pm$, with masses m_i by re-writing the mass term

$$\hat{\Phi}_r^T (M_2^2)_r \hat{\Phi}_r = \sum\nolimits_{k=0,\pm} m_k^2 \psi_k^2 = \sum\nolimits_{k=0,\pm} m_k^2 (\hat{\Phi}_r^T I U)_k (U^{-1}\Phi_r)_k. \tag{3.63}$$

Hence, the Goldstone field corresponding to ψ_0 is expressible as

$$\psi_{\mathrm{Gb}} := \sqrt{\left(\hat{\Phi}_r^T I U\right)_0 (U^{-1}\hat{\Phi}_r)_0}. \tag{3.64}$$

The unnormalised right eigenvectors for M_2^2 are computed to

$$v_i = \{-\lambda_i \Lambda_i - \kappa, \Lambda_i^3 c_\mu \mu^2, c_\mu c_\nu \mu^2 \nu^2\}, \quad i = 0, \pm, \tag{3.65}$$

so that the explicit form of the Goldstone boson field in the original fields becomes

$$\psi_{\mathrm{Gb}} := \frac{1}{\sqrt{N}} \left(-\kappa \hat{\chi}_1 - c_3 c_\mu m_3^2 \mu^2 \hat{\varphi}_2 + c_\mu c_\nu \mu^2 \nu^2 \hat{\chi}_3\right), \tag{3.66}$$

with

$$N = m_3^4(m_2^4 - \mu^4) + (2c_2c_3 m_2^2 m_3^2 + \mu^4)\nu^4 + \nu^8 = \kappa^2 \left(1 - \frac{\mu^2}{\bar{\mu}_s^2}\right), \tag{3.67}$$

where $\bar{\mu}_s$ is defined as above being the special value of μ for which $\lambda_- = 0$, that is the zero-exceptional point. Computing the determinant of U to $\det U = c_\nu \lambda_- \lambda_p (\lambda_- - \lambda_p)\nu^2 \mu^4$, the origin of this singularity is clear, as U is not invertible for vanishing for $\lambda_- = 0$ and at the standard exceptional points when $\lambda_- = \lambda_p$. The former scenario occurs for $\mu = \bar{\mu}_s$ and the latter for $\mu_{\mathrm{e}}^{\pm} = [\kappa(\kappa - m_3^4 + \nu^4 \pm 2c_\nu \nu^2 \sqrt{\kappa})]^{1/4}/m_3$. So that in these circumstances the Goldstone boson of the form (3.66) does not exist. We discuss these two scenarios separately in the next two sections. However, for $\mu = \mu_s$, that is the value for which the other sector develops a massless particle, all terms in ψ_{Gb} are regular. This means at this point we have two massless particles in the model. One is tempted to interpret one as a genuine Goldstone boson and the other as simple massless particle. However, recalling that at $\mu = \mu_s$ one is actually expanding around the $U(1)$-symmetry preserving vacuum the emergence of none of them can be attributed to a global symmetry breaking and the discussion in section 3.3 applies.

3.7. The Goldstone boson at the exceptional point

As pointed out in the previous section, at the exceptional point when $\lambda_- = \lambda_p =: \lambda_e$ the matrix U is no longer invertible so that ψ_{Gb} in (3.64) becomes ill-defined. However, when $\mu = \mu_{\mathrm{e}}^+ = \mu_{\mathrm{e}}$ we may transform the lower right block of M_2^2 into Jordan normal form as

$$T^{-1}\left[M_2^2(\mu=\mu_{\rm e})\right]_r T=\begin{pmatrix}0&0&0\\0&\lambda_{\rm e}&a\\0&0&\lambda_{\rm e}\end{pmatrix}=J, \tag{3.68}$$

for some as yet unspecified constant $a\in\mathbb{R}$. For simplicity we select here the upper sign of the two possibilities $\mu_{\rm e}^{\pm}$. We can then express the transformed action expanded around the vacuum Φ_2^0 and formulate the Goldstone boson in terms of the original fields

$$\hat{\mathcal{I}}_3=-\frac{1}{2}\int d^4x\left[\hat{\Phi}^T I(\Box+M_2^2)\hat{\Phi}+\mathcal{L}_{\rm int}(\hat{\Phi})+\mathcal{L}_{3,\ell}\right], \tag{3.69}$$

$$=-\frac{1}{2}\int d^4x\left[\hat{\Phi}^T IT(\Box+J)T^{-1}\hat{\Phi}+\mathcal{L}_{\rm int}(\hat{\Phi})+\mathcal{L}_{3,\ell}\right], \tag{3.70}$$

$$=-\frac{1}{2}\int d^4x\left[\sum_{i=1}^{3}\psi_i\Box\psi_i+\lambda_e(\psi_2^2+\psi_3^2)+a\psi_2^L\psi_3^R+\mathcal{L}_{\rm int}(\psi_i)+\mathcal{L}_{3,\ell}\right]. \tag{3.71}$$

We have introduced here the fields

$$\psi_i:=\sqrt{\psi_i^L\psi_i^R},\qquad \psi_i^L:=(\hat{\Phi}_r^T IT)_i,\qquad \psi_i^R:=(T^{-1}\hat{\Phi}_r)_i, \tag{3.72}$$

with the Goldstone boson at the exceptional point being identified as $\psi_{\rm Gb}^{\rm e}:=\psi_1$. Notice that when $T^T IT=\mathbb{I}$, the field coincide, i.e. we have $\psi_i^L=\psi_i^R=\psi_i$. Let us now determine the matrix T and demonstrate that it is well-defined. We take $\mu=\mu_{\rm e}$ so that the nonzero eigenvalue for $M_2^2(\mu_{\rm e})$ becomes

$$\lambda_{\rm e}=\frac{\nu^4-m_3^4+c_\nu\nu^2\sqrt{\kappa}}{c_3m_3^2}. \tag{3.73}$$

Using the null vector of $M_2^2(\mu_{\rm e})$ and the eigenvector corresponding to the eigenvalue $\lambda_{\rm e}$ in the first and second column of T, respectively, we solve equation (3.68) for T as

$$T=\begin{pmatrix}-\kappa c_3m_3^2 & -c_3m_3^2\mu_{\rm e}^2 & t\\ m_3^4\mu_{\rm e}^2 & \kappa+c_\nu\nu^2\sqrt{\kappa} & s\\ c_3c_\nu\nu^2m_3^2\mu_{\rm e}^2 & c_3m_3^2\sqrt{\kappa} & \frac{s-\sqrt{\kappa}}{c_3m_3^2+\lambda_{\rm e}}\nu^2\end{pmatrix}, \tag{3.74}$$

with abbreviations $t:=(1-m_3^4-\nu^4)\mu_{\rm e}^2/(\lambda_{\rm e}\sqrt{\kappa})$, $s:=t\left(\lambda_{\rm e}/\mu_{\rm e}^2-c_3m_3^2\mu_{\rm e}^2/\kappa\right)-\nu^2$ and a as defined in (3.68) taken to $a=\nu^2/m_3^6$. We compute $\det T=\kappa m_3^4\lambda_{\rm e}^2$. We have imposed here $\psi_1^L=\psi_1^R=\psi_1$. Using these expressions we obtain from (3.72) the Goldstone boson at the exceptional point as

$$\psi_{\rm Gb}^{\rm e}=\frac{1}{\kappa c_3m_3^2\lambda_{\rm e}^2}\left(-\kappa\hat{\chi}_1-m_3\mu_{\rm e}^2\hat{\varphi}_2+\nu^2\mu_{\rm e}^2\hat{\chi}_3\right). \tag{3.75}$$

Thus at the exceptional point the Goldstone boson $\psi_{\rm Gb}^{\rm e}$ is well-defined unless $\lambda_{\rm e}=0$, $\kappa=0$ or $m_3=0$, as in these cases the matrix T is not invertible.

3.8. The Goldstone boson at the zero-exceptional point

Another interesting point at which the general expression for the Goldstone boson in (3.64) is not valid occurs for $\mu=\bar{\mu}_s$, that is when $\lambda_-=0$, i.e. at the zero-exceptional point. In this case we may transform the lower right block of M_2^2 into the form

$$S^{-1}\left[M_2^2(\mu=\bar{\mu}_s)\right]_r S=\begin{pmatrix} 0 & 0 & b \\ 0 & \lambda_s & 0 \\ 0 & 0 & 0 \end{pmatrix}=K, \tag{3.76}$$

for some as yet unspecified constant $b \in \mathbb{R}$. As before we can then express the transformed action expanded around the vacuum Φ_2^0 and formulate the Goldstone boson in terms of the original fields

$$\hat{\mathcal{I}}_3=-\frac{1}{2}\int d^4x\left[\hat{\Phi}^T I(\Box+M_2^2)\hat{\Phi}+\mathcal{L}_{\text{int}}(\hat{\Phi})+\mathcal{L}_{3,\ell}\right], \tag{3.77}$$

$$=-\frac{1}{2}\int d^4x\left[\hat{\Phi}^T I S(\Box+K)S^{-1}\hat{\Phi}+\mathcal{L}_{\text{int}}(\hat{\Phi})+\mathcal{L}_{3,\ell}\right], \tag{3.78}$$

$$=-\frac{1}{2}\int d^4x\left[\sum_{i=1}^{3}\psi_i\Box\psi_i+\lambda_s\psi_2^2+b\psi_1^L\psi_3^R+\mathcal{L}_{\text{int}}(\psi_i)+\mathcal{L}_{3,\ell}\right], \tag{3.79}$$

where we introduced

$$\psi_i:=\sqrt{\psi_i^L\psi_i^R}, \qquad \psi_i^L:=(\hat{\Phi}_r^T I S)_i, \qquad \psi_i^R:=(S^{-1}\hat{\Phi}_r)_i. \tag{3.80}$$

Taking $\mu=\bar{\mu}_s$, the only nonzero eigenvalue for $M_2^2(\bar{\mu}_s)$ becomes

$$\lambda_z=\frac{\left(c_2m_2^2+2c_3m_3^2\right)\nu^4-c_3m_3^6}{m_3^4-\nu^4}. \tag{3.81}$$

Using the null vector of $M_2^2(\bar{\mu}_s)$ and the eigenvector corresponding to the eigenvalue λ_e in the first and second column of S, respectively, we solve equation (3.76) for S to

$$S=\begin{pmatrix} -\sqrt{m_3^4-\nu^4} & -\nu^2\kappa & 0 \\ c_3m_3^2 & \left(c_2m_2^2+c_3m_3^2\right)\nu^2\sqrt{m_3^4-\nu^4} & \frac{b}{\kappa}(\nu^4-m_3^4) \\ \nu^2 & (m_3^4-\nu^4)^{3/2} & -\frac{b}{\kappa}\left(c_2m_2^2+c_3m_3^2\right)\nu^2 \end{pmatrix}. \tag{3.82}$$

We compute $\det S=-b\lambda_z^2(m_3^4-\nu^4)^2/\kappa$. The massive field ψ_2 can be identified easily for any value of b as

$$\psi_2=\frac{1}{N_2}\psi_2^L \tag{3.83}$$

when noting that

$$\psi_2^L=N_2^2\psi_2^R=-\kappa\nu^2\hat{\chi}_1-\left(c_2m_2^2+c_3m_3^2\right)\nu^2\sqrt{m_3^4-\nu^4}\hat{\varphi}_2+(m_3^4-\nu^4)^{3/2}\hat{\chi}_3, \tag{3.84}$$

with $N_2=(m_3^4-\nu^4)\lambda_z$. However, we can not identify the Goldstone boson simply as ψ_1, since we can no longer achieve $\psi_1^L\propto\psi_1^R\propto\psi_1$. Given the eigenvalue spectrum we have now two massless particles that interact with each other and it is impossible to distinguish the Goldstone boson from the massless particle. However, we can identify a combination of the two fields as a massless particle

$$\psi_{\text{Gb}}^z=\psi_1^L+\alpha\psi_3^L=\psi_1^R+\alpha\psi_3^R \tag{3.85}$$

$$=-\sqrt{m_3^4-\nu^4}\hat{\chi}_1+\frac{\left(m_3^4-\nu^4\right)^2+\nu^4(1-\kappa)-m_3^4}{(m_3^4-\nu^4)\lambda_z}\hat{\varphi}_2+\nu^2\left[1+\frac{c_2m_2^2+c_3m_3^2}{(m_3^4-\nu^4)\lambda_z}\right]\hat{\chi}_3,$$

for $b = -\bar{\mu}_s^4/(\alpha\kappa\lambda_z)$ and $\alpha^2 = 1 + (\bar{\mu}_s^4 - m_2^4 - m_3^4 + 2\nu^4)/[\lambda_z^2(m_3^4 - \nu^4)]$. However, we can not avoid that constituents of the field, that is ψ_1^L and ψ_3^R, interact with each other. The peculiar behaviour at the zero-exceptional point was also discussed by Mannheim [23] in the context of the $\mathcal{I}_2$-model.

4. Discrete antilinear and broken continuous global symmetry

Next we study a non-Hermitian $\mathcal{CPT}$-invariant action but with broken continuous global $U(1)$-symmetry. This is achieved by keeping in the Lagrangian density functional (2.1) only the two complex scalar fields ϕ_1 and ϕ_2 genuinely complex and taking the field ϕ_3 to be real. Hence we consider the Lagrangian density functional

$$\mathcal{L}_3' = \sum_{i=1}^{2}\left(\partial_\mu\phi_i\partial^\mu\phi_i^* + c_i m_i^2\phi_i\phi_i^*\right) + \left(\partial_\mu\phi_3\partial^\mu\phi_3 + c_3 m_3^2\phi_3^2\right) \tag{4.1}$$
$$+c_\mu\mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) + c_\nu\nu^2\phi_3\left(\phi_2 - \phi_2^*\right) - \frac{g}{4}(\phi_1\phi_1^*)^2.$$

Clearly this model is still $\mathcal{CPT}_{1,2}$ -invariant, but due to the presence of the real scalar field the continuous global $U(1)$-symmetry is broken already at the level of the action. We parameterize $\phi_i = 1/\sqrt{2}(\varphi_i + i\chi_i)$ with $\varphi_i,\ \chi_i \in \mathbb{R}$ for $i = 1, 2$ and $\phi_3 = \varphi_3/\sqrt{2}$. Defining the vector field $\Phi := (\varphi_1, \chi_2, \varphi_3, \chi_1, \varphi_2)^T$ and the diagonal 5×5-matrix E with diag $E = (1, 0, 0, 1, 0)$, we can write $\mathcal{L}_3'$ with the real field content in the compact form

$$\mathcal{L}_3' = \frac{1}{2}\partial_\mu\Phi^T\partial^\mu\Phi - \frac{1}{2}\Phi^T M^2\Phi - \frac{g}{16}\left(\Phi^T E\Phi\right)^2, \tag{4.2}$$

with complex mass matrix

$$M^2 = \begin{pmatrix} -c_1m_1^2 & -ic_\mu\mu^2 & 0 & 0 & 0 \\ -ic_\mu\mu^2 & -c_2m_2^2 & -ic_\nu\nu^2 & 0 & 0 \\ 0 & -ic_\nu\nu^2 & -c_3m_3^2 & 0 & 0 \\ 0 & 0 & 0 & -c_1m_1^2 & ic_\mu\mu^2 \\ 0 & 0 & 0 & ic_\mu\mu^2 & -c_2m_2^2 \end{pmatrix}. \tag{4.3}$$

As in the previous section, we similarity transform the corresponding action using the same Dyson map (3.14), hence obtaining

$$\hat{\mathcal{I}}_3' = \eta\mathcal{I}_3'\eta^{-1} = \int d^4x\left[\frac{1}{2}\partial_\mu\Phi^T I\partial^\mu\Phi - \frac{1}{2}\Phi^T H\Phi - \frac{g}{16}\left(\Phi^T E\Phi\right)^2\right], \tag{4.4}$$

with H being identical to M^2 in (4.3) with all imaginary units i removed. The equation of motion resulting from (4.4) reads

$$-\Box I\Phi - H\Phi - \frac{g}{4}\left(\Phi^T E\Phi\right)E\Phi = 0, \tag{4.5}$$

from which we identify the mass matrix as $\hat{M}^2 = IH$ and by solving $\delta V = 0$ we obtain the five vacua

$$\Phi_0^{(0)} := (0,0,0,0,0)^T, \tag{4.6}$$

$$\Phi_0^{(1\pm)} := \frac{2}{m_2}\sqrt{\frac{\kappa}{c_2 g}}\left(0,0,0,\pm 1,\mp\frac{c_\mu \mu^2}{c_2 m_2^2}\right)^T, \tag{4.7}$$

$$\Phi_0^{(2\pm)} := 2\sqrt{\frac{c_3 c_\mu m_3^2 \mu^4 + c_1 m_1^2 \kappa}{g\kappa}}\left(\pm 1, \frac{c_3 c_\mu m_3^2 \mu^2}{\kappa}, \mp 1, \frac{c_\nu c_\mu \nu^2 \mu^2}{\kappa}, 0\right)^T. \tag{4.8}$$

Expanding around these vacua the corresponding squared mass matrices are

$$M_i^2 = \begin{pmatrix} A_i & -c_\mu \mu^2 & 0 & 0 & 0 \\ c_\mu \mu^2 & -c_2 m_2^2 & c_\nu \nu^2 & 0 & 0 \\ 0 & -c_\nu \nu^2 & -c_3 m_3^2 & 0 & 0 \\ 0 & 0 & 0 & B_i & c_\mu \mu^2 \\ 0 & 0 & 0 & -c_\mu \mu^2 & -c_2 m_2^2 \end{pmatrix}, \qquad i = 0,1,2, \tag{4.9}$$

with

$$A_0 = B_0 = -c_1 m_1^2,\ A_1 = \frac{\mu^4}{c_2 m_2^2},\ B_1 = 2c_1 m_1^2 + 3A_1,$$
$$A_2 = 2c_1 m_1^2 + 3B_2,\ B_2 = \frac{c_3 m_3^2 \mu^4}{\kappa}. \tag{4.10}$$

The different signs in $\Phi_0^{(1\pm)}$ and $\Phi_0^{(2\pm)}$ give rise to the same mass matrix so that we may ignore that distinction in what follows.

The parameter study of all mass matrices M_i reveals that there are physical regions for all three models bounded by exceptional points similarly as in the previous section for the purely complex $\mathcal{I}_3$-model. Our crucial observation is here that the determinants

$$\det M_0^2 = -(c_1 c_2 m_1^2 m_2^2 + \mu^4)(c_1 m_1^2 \kappa + c_3 m_3^2 \mu^4), \tag{4.11}$$

$$\det M_1^2 = -\frac{2\mu^4 \nu^4}{c_2 m_2^2}(c_1 c_2 m_1^2 m_2^2 + \mu^4), \tag{4.12}$$

$$\det M_2^2 = \frac{2\mu^4 \nu^4}{\kappa}(c_1 m_1^2 \kappa + c_3 m_3^2 \mu^4), \tag{4.13}$$

are always nonvanishing when $m_i \neq 0$, $\mu \neq 0$ and $\nu \neq 0$. Hence in all sectors of the $\mathcal{PT}$-symmetries this model does not possess any Goldstone boson, which is expected in the absence of a global symmetry. There are of course special points as for the previous model, such as $\mu_s^4 = -c_1 c_2 m_1^2 m_2^2$ or $\bar{\mu}_s^4 = -c_1 m_1^2 \kappa / c_3 m_3^2$, for which massless bosons enter the model. However, these massless bosons are present in the model from the very beginning and not the result of the breaking of a continuous symmetry by expanding around particular vacua. Hence they are not interpreted as Goldstone bosons.

5. General interaction term

In our initial Lagrangian density functional (2.1) we chose a particularly simple interaction term and carried out our analysis for an even simpler version. In this section we explore the possibilities of allowing for more general interaction terms so that the action still respects the discrete $\mathcal{CPT}$-symmetries (2.3) and the continuous global $U(1)$-symmetry (2.4), while keeping the kinetic and mass term as previously. We present here explicitly the case for $\mathcal{I}_3$, after which it becomes evident how to generalize to all $\mathcal{I}_n$. We carry out our analysis for the equivalent action $\hat{\mathcal{I}}_n$.

We find that the action

$$\hat{\mathcal{I}}_3[\Phi] = \frac{1}{2}\int d^4x \left[\partial_\mu \Phi^T I \partial^\mu \Phi - \Phi^T H \Phi - \frac{g}{8}\left(\Phi^T E \Phi\right)^2 - \frac{g}{8}\left(\Phi^T F \Phi\right)^2\right], \tag{5.1}$$

is $\widehat{\mathcal{CPT}}$ and $U(1)$-invariant, where we recalled the field vector $\Phi := (\varphi_1, \chi_2, \varphi_3, \chi_1, \varphi_2, \chi_3)^T$ and introduced

$$H = \begin{pmatrix} -c_1 m_1^2 & c_\mu \mu^2 & 0 & 0 & 0 & 0 \\ c_\mu \mu^2 & c_2 m_2^2 & c_\nu \nu^2 & 0 & 0 & 0 \\ 0 & c_\nu \nu^2 & -c_3 m_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_1 m_1^2 & -c_\mu \mu^2 & 0 \\ 0 & 0 & 0 & -c_\mu \mu^2 & c_2 m_2^2 & -c_\nu \nu^2 \\ 0 & 0 & 0 & 0 & -c_\nu \nu^2 & -c_3 m_3^2 \end{pmatrix},$$

$$E = \begin{pmatrix} A & 0 \\ 0 & \Omega A \Omega \end{pmatrix}, F = \begin{pmatrix} 0 & B \\ \Omega B \Omega & 0 \end{pmatrix}. \tag{5.2}$$

Here A and B can be arbitrary 3×3-matrices and $\operatorname{diag}\Omega = (-1, 1, -1)$.

We briefly show how the form of this action is obtained. The respective symmetries (3.17) and (2.4) are realised as follows

$$\widehat{\mathcal{CPT}}_{1,2}: \hat{\mathcal{I}}_3[\Phi] = \hat{\mathcal{I}}_3\left[C_{1,2}\Phi\right] \tag{5.3}$$

$$U(1): \hat{\mathcal{I}}_3[\Phi] = \hat{\mathcal{I}}_3[U\Phi] \tag{5.4}$$

with

$$C_{1,2} = \pm\begin{pmatrix} \mathbb{I}_3 & 0 \\ 0 & -\mathbb{I}_3 \end{pmatrix}, \quad U = \mathbb{I}_6 + \alpha\hat{\Omega} = \mathbb{I}_6 + \alpha\begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}, \tag{5.5}$$

when α is taken to be small. Next we compute how these symmetries are implemented when taking the interaction term to be of the general form

$$\frac{g}{16}\left(\Phi^T \hat{E} \Phi\right)^2, \quad \hat{E} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.6}$$

with as yet unknown 3×3-matrices A, B, C and D. The transformed Noether current (2.6) resulting from the $U(1)$-symmetry (5.5)

$$j_\mu = \frac{\alpha}{2}\left(\partial_\mu \Phi^T \hat{\Omega} \Phi - \Phi^T \hat{\Omega} \partial_\mu \Phi\right) \tag{5.7}$$

is vanishing upon using the equation of motion for the action $\hat{\mathcal{I}}_3[\Phi]$ with interaction term (5.6)

$$-\Box\Phi - H\Phi - \frac{g}{4}\left(\Phi^T \hat{E} \Phi\right)\hat{E}\Phi = 0 \tag{5.8}$$

if

$$\partial_\mu j^\mu = \frac{\alpha}{2}\left(\Box\Phi^T\hat{\Omega}\Phi - \Phi^T\hat{\Omega}\Box\Phi\right) = \frac{\alpha}{2}\Phi^T\left(\left[\hat{\Omega}, H\right] - \frac{g}{4}\Phi^T\hat{E}\Phi\left[\hat{\Omega}, \hat{E}\right]\right)\Phi = 0. \tag{5.9}$$

Combining the constraints for the $\widehat{\mathcal{CPT}}$ and $U(1)$-symmetry we require therefore

$$\left[\hat{\Omega}, H\right] = 0, \quad \left[\hat{\Omega}, \hat{E}\right] = 0, \quad \left[C_{1,2}, H\right] = 0, \quad \left[C_{1,2}, \hat{E}\right] = 0, \tag{5.10}$$

or

$$\left[\hat{\Omega}, H\right] = 0, \quad \left[\hat{\Omega}, \hat{E}\right] = 0, \quad \left[C_{1,2}, H\right] = 0, \quad \left\{C_{1,2}, \hat{E}\right\} = 0, \tag{5.11}$$

with $\{\cdot, \cdot\}$ denoting the anti-commutator. The solutions to (5.10) for $\widehat{\mathcal{CPT}}_1$ and $\widehat{\mathcal{CPT}}_2$ are E and F, respectively, whereas the solutions to (5.11) for $\widehat{\mathcal{CPT}}_1$ and $\widehat{\mathcal{CPT}}_2$ are F and E, respectively. This mean the action (5.1) contains the most general $\widehat{\mathcal{CPT}}_{1,2}$ and $U(1)$ invariant interaction terms of the form (5.6). There is no distinction between a $\widehat{\mathcal{CPT}}_1$ or $\widehat{\mathcal{CPT}}_2$-invariant action as the solutions of (5.10) and (5.11) always combine to allow for both $\widehat{\mathcal{CPT}}$-symmetries to be implemented.

We carried out our analysis for the Goldstone boson for diag $A = (1, 0, 0)$ and $B = 0$, but from the above it is now evident how this structure of more complicated interaction terms generalises to $\hat{\mathcal{I}}_n$, and therefore $\mathcal{I}_n$, for $n > 3$. Similar computations can also be carried out for the symmetries $\mathcal{CPT}_{3/4}$ and $\mathcal{CP}'\mathcal{T}$, where $\mathcal{P}'$ is any of the six remaining operators constructed in section 3.5. We note here that while it is a uniquely well defined process to identify the $\widehat{\mathcal{CPT}}$-symmetries when given the $\mathcal{CPT}$-symmetries, that is going from $\mathcal{I}_n$ to $\hat{\mathcal{I}}_n$, care needs to be taken in the inverse procedure.

6. Conclusions and outlook

We proposed and analysed a new non-Hermitian model with n complex scalar fields that possess a global $U(1)$-symmetry when none of the scalar fields involved are self-conjugate. Making use of the general fact that actions can be similarity transformed without changing the content of the theory, as long as the equal time-commutation relations are preserved, we mapped the models to equivalent Hermitian systems. The models obtained in this manner possess different types of vacua that may either respect or break the global continuous symmetry. As expected from the Hermitian version of Goldstone's theorem the models do not possess any Goldstone bosons when the vacuum around which the theory is expanded preserves the $U(1)$ symmetry, see section 3.3, and when the symmetry is broken already on the level of the action by taking some of the complex fields to be real, see section 4. In both cases there are special points in the parameter space for which the model contains massless particles, which are, however, not identified as Goldstone bosons. In contrast, when expanding the action around a $U(1)$-symmetry breaking vacuum a Goldstone boson emerges. In the $\mathcal{PT}$-symmetric regime and at the standard exceptional point its explicit form in terms of the original fields in the model can be identified, although it takes on different forms in these two regimes. In contrast, at the zero-exceptional point one can not identify the Goldstone boson, but only a linear combination of it with another massless particle. Hence the general statement of the Goldstone theorem holds for Hermitian as well as for non-Hermitian actions, but the latter possesses special regimes with behaviour that have no analogue in the former. As the reality of the mass spectra and the explicit form of the Goldstone bosons are strictly governed by the $\mathcal{PT}$-symmetric at the tree approximation this leads to the interesting possibility that one may have models with broken $\mathcal{CPT}$-symmetry on the level of the action, but with real physical masses.

There are various issues that are worthy further investigation. First of all one may of course consider more complicated complex models by investigating those for larger values of n and also include more involved interaction terms as derived in section 5. In particular, one may construct those that remain $\mathcal{CPT}$-symmetric beyond the tree level when employing the remaining six $\mathcal{P}$-operators constructed in section 3.5. A richer structure is expected to be revealed by considering non-Hermitian models that possess global continuous non-Abelian symmetries so that more Goldstone bosons are generated via a symmetry breaking [30].

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