OPTIMAL RISK TRANSFERS IN INSURANCE GROUPS

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Abstract.

Optimal risk transfers are derived within an insurance group consisting of two separate legal entities, operating under potentially different regulatory capital requirements and capital costs. Consistent with regulatory practice, capital requirements for each entity are computed by either a Value-at-Risk or an Expected Shortfall risk measure. The optimality criterion consists of minimising the risk-adjusted value of the total group liabilities, with valuation carried out using a cost-of-capital approach. The optimisation problems are analytically solved and it is seen that optimal risk transfers often involve the transfer of tail risk (unlimited reinsurance layers) to the more weakly regulated entity. We show that, in the absence of a capital requirement for the credit risk that specifically arises from the risk transfer, optimal risk transfers achieve capital efficiency at the cost of increasing policyholder deficit. However, when credit risk is properly reflected in the capital requirement, incentives for tail-risk transfers vanish and policyholder welfare is restored.

Keywords and phrases: Cost of Capital, Expected Shortfall, Insurance Groups, Optimal Reinsurance, Value-at-Risk.

1. Introduction

Insurance groups often comprise a number of distinct legal entities, operating in different territories. Diversification across an insurance group is no trivial matter and the way it operates depends on the group's legal structure. On the one hand, the risk exposures of different entities will in general not be perfectly correlated, and thus some *group level diversification* (Keller, 2007) is observed (e.g. by a parent company). On the other hand, risks and assets in the group portfolio are not pooled across entities, hence there are limits to the cross-subsidy, as well as the capital fungibility, within the group. Nonetheless, the risk and capital requirements of individual entities can be reduced, through a web of capital and risk transfer arrangements across entities. The capital efficiency thus produced can be seen as a result of *down-streaming of diversification* (Keller, 2007).

The complexity of group legal structures and intra-group risk transfers, with entities being potentially subject to different regulatory regimes, poses a major challenge for regulators; it is not surprising that a substantial part of the European Solvency II Directive (European Commission,

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2009) is dedicated to group supervision. Studying such complexity has motivated a lively academic literature. Filipović and Kupper (2008) discuss optimal risk transfers, in a framework where a finite set of risk transfer instruments is available and the capital requirements of individual entities are calculated using convex risk measures. Gatzert and Schmeiser (2011) study the impact of group diversification on shareholder value, considering a variety of group structures and capital and risk transfer instruments, while also offering a thorough literature review of diversification in financial conglomerates. Schlütter and Gründl (2011) assess the impact of group building on policyholder welfare. In their analysis, it is assumed that a particular type of rational risk transfer arrangement is enforced, while the group sets premium and equity targets in order to maximise shareholder value, allowing for the impact of entities' default risk on insurance demand.

The above literature generally analyzes the impact of risk transfers of pre-specified type. In contrast, the focus here is on deriving optimal functional forms of risk transfers. For this purpose, we use a formal setting with two legal entities, subject to potentially discrepant regulatory requirements. Hence our work is also related to the literature on optimal reinsurance contract design. The first attempts are attributed to Borch (1960) and Arrow (1963) where maximising the expected utility defines the optimality criterion. Further extensions have been developed for various decision criteria that depend on the risk measure choice (for example, see Heerwaarden *et al.*, 1989; Young, 1999; Kaluska, 2001 and 2005, Verlaak and Beirlant, 2003; Kaluszka and Okolewski, 2008; Ludkovski and Young, 2009; Bernard and Tian, 2010). Decisions based on two particular risk measures, *Value-at-risk (VaR)* and *Expected Shortfall (ES)*, are considered by Cai *et al.* (2008), Cheung (2010) and Chi and Tan (2011). All of the afore mentioned papers deal with the one-period model. The classical risk model setting has been successfully studied in the literature by Centeno and Guerra (2010) and Guerra and Centeno (2008 and 2010), via maximisation of the adjustment coefficient. Alternatively, there is a rich literature on dynamic risk allocation, where the objective is the ruin probability (see for example, Schmidli, 2001 and 2002, Hipp and Vogt, 2003). An excellent review paper regarding the latter approach, and not only, is given by Albrecher and Thonhauser (2009).

In this paper, optimal risk transfers are chosen such that the risk adjusted value of the group liabilities is minimised, when valuation takes place under a cost-of-capital methodology, similar in principle to the ones discussed in Wüthrich *et al.* (2010) and lying at the heart of regulatory valuation approaches (e.g. under the Swiss Solvency Test and Solvency II (Federal Office of Private Insurance, 2006, and European Commission, 2009).

Analytical solutions are provided for the corresponding optimisation problems, when the capital requirement for each entity is given either by VaR, the risk measure used under Solvency II, or ES, used in the Swiss Solvency Test. In addition, each entity is subject to a different cost of capital, due to potential differences in taxation or other operating costs. The results bear out the properties of the risk measures used, specifically the VaR measure's insensitivity to tail risk beyond the given confidence level. In particular, when one entity is subject to a lighter (VaR-based) regulatory requirement than the other, it ends up being allocated most of the group's extreme risk exposure, in the form of high (usually infinite) layers.

Such incentives ought to trouble regulators, since, beyond direct implications for policyholder welfare, transferring tail risk is arguably also associated with the transfer of operational and model

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risks. To further investigate these incentives, we focus on the case where the first entity is subject to an ES-based capital requirement, while the second entity, acting as subsidiary solely set up to reinsure the first, holds capital according to VaR. We then show that, in the absence of an allowance for credit risk in capital requirements, the transfer of tail risk to the VaR-regulated entity is detrimental to policyholder welfare as it increases the expected policyholder deficit. This is equivalent to saying that the value of the insurer's default option (calculated under a physical measure) increases (Phillips *et al.*, 1998, and Myers and Read, 2001).

Motivated by these findings, we then consider a situation where the counter-party credit risk arising from the risk transfer is reflected in the ES-based capital requirement of the first entity. A corresponding optimisation problem is formulated and its solution shows that incentives for transferring tail risk to the second entity vanish. Moreover, policyholder welfare is restored to pre-transfer levels.

We conclude from our analysis that discrepant regulatory regimes can produce risk transfers that trade off group capital efficiency against policyholder welfare. However, the group incentives for such action vanish as long as credit risk is fully allowed for, both in terms of quantifying the dependence between entities' exposures and assuming a reduced recovery given default. While our results are produced in a rather formal and simplified setting, we believe that they are informative in relation to the potentially damaging direction of incentives, which inconsistent capital requirements can produce.

In Section 2, some background on the risk measures used is offered. Then the optimisation problem is formally introduced and solutions are given for different combinations of risk measures. Section 3 deals with the impact of credit risk on policyholder deficit. Revised optimal risk transfers are obtained when the credit risk is reflected in the capital requirement, and the impact of the change is demonstrated by a numerical example. Finally, in the light of the results obtained, some of the key assumptions of our setting are discussed. Brief conclusions are stated in Section 4. Proofs tend to be technical, and therefore they are collected in the Appendix.

2. Optimal risk transfers

2.1. **Value-at-Risk and Expected Shortfall.** We start this section by briefly discussing the risk measures that will be extensively used throughout the paper. Motivated by standard regulatory requirements developed in the insurance industry, the risk measures considered are VaR and ES. The VaR of a generic loss variable Z at confidence level α, $VaR_\alpha(Z)$, represents the minimum amount of capital that will not be exceeded by the loss Z with probability α. Mathematically,

$$VaR_\alpha(Z) := \inf\{z \in \Re : \Pr(Z \leq z) \geq \alpha\},$$

where $\inf \emptyset = \infty$.

VaR has been criticized for its incomplete allowance for the risk of extreme events beyond the confidence level α (Dowd and Blake, 2006), which also leads to a violation of the commonly required subadditivity property (Artzner *et al.*, 1999). To correct such shortcomings, ES has been proposed (Artzner *et al.*, 1999) as an alternative risk measure. While VaR focuses on a particular point of the loss distribution, the ES at confidence level α, $ES_\alpha(Z)$, evaluates the expected loss amount incurred under the worst $100 \times (1-\alpha)\%$ loss scenarios of Z. The ES_α has multiple formulations in the literature (Acerbi and Tasche, 2002, and Hürlimann, 2003). In the present paper, we only refer to the following two representations:

$$ES_\alpha(Z) := \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(Z)\, ds = VaR_\alpha(Z) + \frac{1}{1-\alpha}E\big(Z - VaR_\alpha(Z)\big)_+, \tag{2.1}$$

where the notation $(z)_+ = \max\{z, 0\}$ is used.

While it is clearly the case that $VaR_\alpha(Z) \leq ES_\alpha(Z)$, it is sometimes desirable to derive a calibration of ES such that the two risk measures are comparable. One possibility is to set $\beta = 1-2(1-\alpha)$. Then, loosely speaking, $ES_\beta(Z)$ is the conditional expected value of Z given $Z > VaR_\beta$, while $VaR_\alpha(Z)$ is the median of the corresponding conditional distribution. Examples of such risk measures considered broadly consistent, are the $VaR_{0.995}$ used under Solvency II and the $ES_{0.99}$ used in the Swiss Solvency Test (EIOPA, 2011). Empirical evidence shows skewness of tails of loss distributions, and suggests that $ES_\beta(Z) > VaR_\alpha(Z)$ is expected to hold for such confidence level choices.

2.2. **General form of the optimisation problem.** Here, the formal setting is given for the optimisation problems solved throughout Section 2. We denote by $X \geq 0$ the total insurance liabilities that an insurance group consisting of two separate legal entities is exposed to. The distribution function of X is given by F and the survival function is $\bar{F} = 1 - F$. The right end-point $x_F := \inf\{z \in \Re : F(z) = 1\}$ of the loss distribution can be either finite or infinite.

The insurance group allocates the total risk X between its two entities, using appropriate risk transfer agreements. In particular, after risk transfers take place, the liabilities of the two entities are $I_1[X]$ and $I_2[X]$ respectively, such that $I_1[X] + I_2[X] = X$.

Each of the two entities is assumed to be subject to potentially different regulatory requirements, in each case quantified by a risk measure. We denote the risk measure used to regulate the first entity by φ_1, where $\varphi_1 \equiv VaR_{\alpha_1}$ or $\varphi_1 \equiv ES_{\alpha_1}$ and the risk measure used by the second entity by φ_2, where $\varphi_2 \equiv VaR_{\alpha_2}$ or $\varphi_2 \equiv ES_{\alpha_2}$. The total capital requirements of the first and second entity are thus $\varphi_1\big(I_1[X]\big)$ and $\varphi_2\big(I_2[X]\big)$, respectively. In addition, for each of the entities, one can define the *risk adjusted value* of its liabilities, by

$$RAV(I_k[X]; \varphi_k, \lambda_k) := E\big(I_k[X]\big) + \lambda_k\Big(\varphi_k\big(I_k[X]\big) - E\big(I_k[X]\big)\Big), \quad k \in \{1, 2\}, \tag{2.2}$$

where $\lambda_k \in (0, 1)$ is the cost of capital as a percentage of the pure risk capital $\varphi_k\big(I_k[X]\big) - E\big(I_k[X]\big)$. Such a valuation principle is used commonly in practice and is embedded in regulatory requirements under the Swiss Solvency Test and Solvency II. The idea behind it is that the taker of a liability in an arm's length transaction must be compensated by receiving a) the expected value of future claims and b) funds equal to the cost of raising the necessary regulatory capital to support the liability. Typically, the cost of capital in excess of the expected loss is considered, as assets equal to the expected loss are matched to liabilities and are thus not considered as shareholder equity. A detailed analysis of the cost-of-capital approach to actuarial valuation is given in Wüthrich *et al.* (2010).

Our purpose is now to derive risk transfers that minimise the risk adjusted value of liabilities across the group, reflecting the potentially different risk measures and capital costs pertaining to each of the two entities. Hence, we seek to minimise the quantity:

$$\begin{aligned} &RAV(I_1[X];\varphi_1,\lambda_1)+RAV(I_2[X];\varphi_2,\lambda_2) \\ &\quad = E\big(X\big)+\lambda_1\Big(\varphi_1\big(I_1[X]\big)-E\big(I_1[X]\big)\Big)+\lambda_2\Big(\varphi_2\big(I_2[X]\big)-E\big(I_2[X]\big)\Big), \end{aligned} \tag{2.3}$$

among the feasible set of risk allocations

$$\big\{I_1[x]+I_2[x]=x : I_1[x] \text{ and } I_2[x] \text{are non-decreasing functions with } I_1[0]=I_2[0]=0\big\}.$$

Since $I_1[X]$ and $I_2[X]$ are co-monotone and the risk measures considered are additive for such random variables (for details, see Denuit *et al.*, 2005, sections 2.3.2.5 and 2.4.3.4), it follows that $\varphi_1\big(I_1[X]\big)=\varphi_1\big(X\big)-\varphi_1\big(I_2[X]\big)$. Thus, minimising the risk adjusted value of (2.3) is equivalent to:

$$\min_{I_2\in\mathcal{F}}\big(\lambda_1-\lambda_2\big)E\big(I_2[X]\big)+\lambda_2\varphi_2\big(I_2[X]\big)-\lambda_1\varphi_1\big(I_2[X]\big), \tag{2.4}$$

where $\mathcal{F}$ is the set of all non-decreasing Lipschitz continuous functions that goes through the origin.

The rationale by which we solve (2.4) rests on a number of key assumptions, including:

a) Insurance liabilities are the only risk that the group is exposed to; in particular, the regulatory capital held is invested with no risk.
b) The optimisation problem (2.4) remains meaningful even when capital held by the group is higher than the regulatory minimum.
c) The capital requirements do not explicitly allow for counter-party credit risk arising from the risk transfers considered.
d) The optimal risk allocations $I_1[X],\ I_2[X]$ are co-monotone.
e) The risk transfers have no impact on the market value of insurance policies sold by the group and, hence, the group's profitability.

Assumption a) is not particularly realistic, but is one made for reasons of formal clarity; in principle, some non-insurance risks could be absorbed into the total risk X.

Assumption b) can be justified as follows. While assets in excess of regulatory capital may be available to the group, regulatory minima retain their significance since they relate to assets that are not fungible across entities in different territories. Furthermore, if the risk adjusted value (2.2) represents the necessary payment to a third party for it to accept the risk, there is no reason why such a price should be dependent on the total assets of the bearer of the original risk.

The other three assumptions are potentially contentious and deserve further explanations, which are best provided when the ideas of the present section are further developed. Section 3 deals at length with c), while assumptions d) and e) are discussed in Section 3.5.

The remaining part of this section is devoted to solving the four optimisation problems arising from different risk measure combinations.

2.3. **VaR / VaR setting.** In the scenario where both entities are subject to VaR-based capital requirements, i.e. $\varphi_1\equiv VaR_{\alpha_1},\ \varphi_2\equiv VaR_{\alpha_2}$, the optimisation problem at hand is:

$$\min_{I_2\in\mathcal{F}}\big(\lambda_1-\lambda_2\big)E\big(I_2[X]\big)+\lambda_2 VaR_{\alpha_2}\big(I_2[X]\big)-\lambda_1 VaR_{\alpha_1}\big(I_2[X]\big). \tag{2.5}$$

Theorem 2.1 states the optimal risk transfer arrangement under this setting.

Theorem 2.1. *The optimal solution of (2.5) is:*

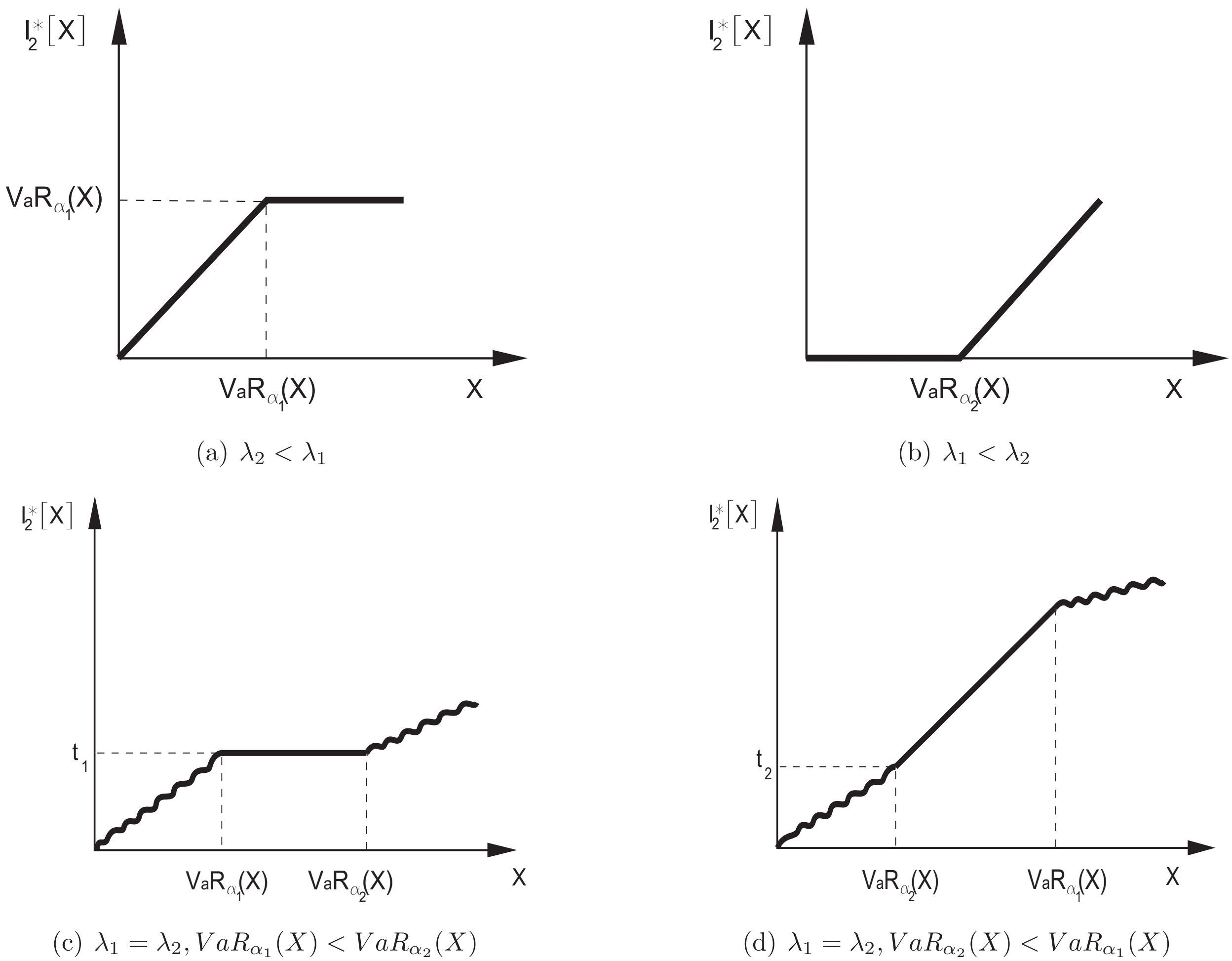
i) *If* $\lambda_1 \neq \lambda_2$,

$$I_2^*[X] = \begin{cases} \min\big\{X, VaR_{\alpha_1}(X)\big\}, & \lambda_1 > \lambda_2, \\ \big(X - VaR_{\alpha_2}(X)\big)_+, & \lambda_1 < \lambda_2. \end{cases}$$

ii) *If* $\lambda_1 = \lambda_2$ *and* $VaR_{\alpha_1}(X) \leq VaR_{\alpha_2}(X)$,

$$I_2^*[X] = \begin{cases} f_2(X), & X > VaR_{\alpha_2}(X), \\ \min\big\{f_1(X), t_1\big\}, & \textit{otherwise}, \end{cases}$$

where $f_1(\cdot)$ *and* $f_2(\cdot)$ *are non-decreasing Lipschitz continuous functions with unit constants such that* $f_1(0) = 0, f_1\big(VaR_{\alpha_1}(X)\big) = f_2\big(VaR_{\alpha_2}(X)\big) = t_1, t_1 \in [0, VaR_{\alpha_1}(X)]$.



(a) $\lambda_2 < \lambda_1$

(b) $\lambda_1 < \lambda_2$

(c) $\lambda_1 = \lambda_2, VaR_{\alpha_1}(X) < VaR_{\alpha_2}(X)$

(d) $\lambda_1 = \lambda_2, VaR_{\alpha_2}(X) < VaR_{\alpha_1}(X)$

FIGURE 2.1. Risk allocations arising from Theorem 2.1.

The optimal risk allocations $I_2^*[x]$ arising from Theorem 2.1 are presented in Figure 2.1, noting that $I_1^*[x] = X - I_2^*[x]$. To interpret the result, it is assumed, without loss of generality, that $\lambda_1 > \lambda_2$, meaning that it is more expensive for the first entity to hold capital. The optimal allocation of the risk X is then $I_1^*[X] = \big(X - VaR_{\alpha_1}(X)\big)_+,\ I_2^*[X] = \min\{X, VaR_{\alpha_1}(X)\}$, implying that extreme risk is retained by the first entity, while less extreme risk is transferred to the second. The effectiveness of such action follows from

$$VaR_{\alpha_1}(I_1^*[X]) = VaR_{\alpha_1}\Big(\big(X - VaR_{\alpha_1}(X)\big)_+\Big) = 0$$
$$VaR_{\alpha_2}(I_2^*[X]) = VaR_{\alpha_2}\Big(\min\big\{X, VaR_{\alpha_1}(X)\big\}\Big) = VaR_{\min\{\alpha_1,\alpha_2\}}(X).$$

Thus, the blindness of the VaR measure to extreme risk nullifies the capital requirement of the first entity, while capital held by the second entity at a lower cost, is lower or equal to the capital arising from holding all risk X in either of the two entities. The way that the risk allocation affects individual risk profiles is further illustrated in Figure 2.2, where the percentiles (VaRs) of the random variables X, $I_1^*[X]$, $I_2^*[X]$ are plotted against the confidence level β. It can be seen that the percentiles of $I_1^*[X]$ remain at zero until $\beta > \alpha_1$, while the percentiles of $I_2^*[X]$ increase with those of X up to confidence level α_1 and remain constant thereafter.

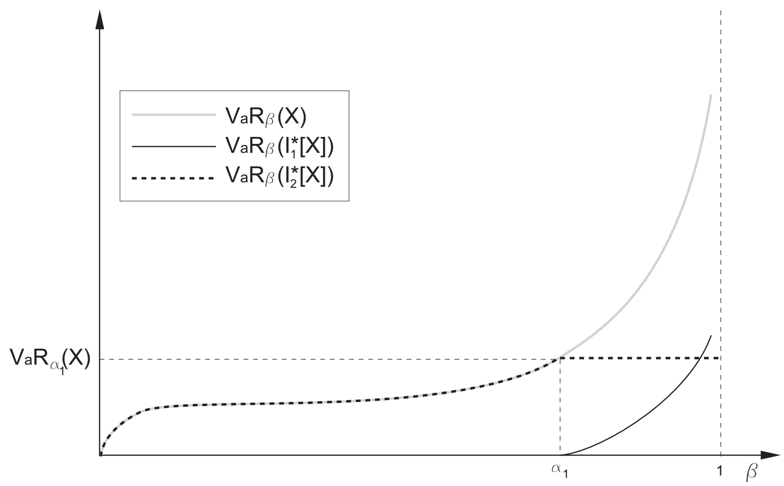


FIGURE 2.2. VaR allocations of $I_1^*[X]$ and $I_2^*[X]$ for various confidence levels β (Theorem 2.1 for $\lambda_1 > \lambda_2$).

The case $\lambda_1 = \lambda_2$ is somewhat different, as the equality of the cost-of-capital parameters makes solutions non-unique, making in turn the notation more complex. If $VaR_{\alpha_1}(X) \le VaR_{\alpha_2}(X)$, then Theorem 2.1 essentially states that the function $I_2^*[\cdot]$ should be constant on $[VaR_{\alpha_1}(X), VaR_{\alpha_2}(X)]$ and non-decreasing outside this interval, as seen in Figure 2.1. This means that the first entity retains a layer of the total risk between $VaR_{\alpha_1}(X)$ and $VaR_{\alpha_2}(X)$, which is "undetected" by the VaR_{α_1} measure used to calculate the first entity's capital, while losses smaller than $VaR_{\alpha_1}(X)$ or greater than $VaR_{\alpha_2}(X)$ are arbitrarily split between the entities. Finally, the mirror case $VaR_{\alpha_1}(X) \ge VaR_{\alpha_2}(X)$ follows by symmetry, and it is depicted in Figure 2.1, but not explicitly stated in Theorem 2.1.

2.4. **VaR / ES setting.** We now turn our attention to the optimal risk transfer for the insurance group, assuming that $\varphi_1 \equiv VaR_{\alpha_1}$, $\varphi_2 \equiv ES_{\alpha_2}$. This is perhaps the most interesting of the optimisation problems we consider, as it addresses inconsistencies between not only confidence levels and capital costs, but also the risk measures themselves.

In view of (2.4), the optimisation problem becomes

$$\min_{I_2 \in \mathcal{F}} (\lambda_1 - \lambda_2) E\big(I_2[X]\big) + \lambda_2 ES_{\alpha_2}\big(I_2[X]\big) - \lambda_1 VaR_{\alpha_1}\big(I_2[X]\big), \tag{2.6}$$

and its solution is stated in Theorem 2.2 below.

Theorem 2.2. *The optimal solution of (2.6) is:*

i) *If* $VaR_{\alpha_1}(X) \leq VaR_{\alpha_2}(X)$ *then*

$$I_2^*[X] = \begin{cases} \min\{X, VaR_{\alpha_1}(X)\}, & \lambda_1 > \lambda_2, \\ 0, & \lambda_1 < \lambda_2, \end{cases}$$

ii) *If* $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ *then*

$$I_2^*[X] = \begin{cases} \min\{X, VaR_{\alpha_1}(X)\}, & \lambda_1 > \lambda_2, \\ \min\Big\{\big(X - VaR_{\alpha_2^{**}}(X)\big)_+, VaR_{\alpha_1}(X) - VaR_{\alpha_2^{**}}(X)\Big\}, & \lambda_1 < \lambda_2, \end{cases}$$

where $\alpha_2^{**} = \min\{\alpha_1, \alpha_2^*\}$ *and* $\alpha_2^* = \frac{\lambda_2\alpha_2}{\lambda_1(1-\alpha_2)+\lambda_2\alpha_2}$.

iii) *If* $\lambda_1 = \lambda_2$ *and* $VaR_{\alpha_1}(X) \leq VaR_{\alpha_2}(X)$ *then* $I_2^*[X] = \min\{f_1(X), t_1\}$, *while if* $\lambda_1 = \lambda_2$ *and* $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ *it is*

$$I_2^*[X] = \begin{cases} \min\{X, VaR_{\alpha_1}(X)\} - VaR_{\alpha_2}(X) + t_2, & X > VaR_{\alpha_2}(X), \\ f_2(X), & \text{otherwise}, \end{cases}$$

where $f_1(\cdot)$ *and* $f_2(\cdot)$ *are non-decreasing Lipschitz continuous functions with unit constants such that* $f_1(0) = f_2(0) = 0$, $f_1\big(VaR_{\alpha_1}(X)\big) = t_1$, $f_2\big(VaR_{\alpha_2}(X)\big) = t_2$, *with parameters* $t_1 \in [0, VaR_{\alpha_1}(X)]$ *and* $t_2 \in [0, VaR_{\alpha_2}(X)]$.

Naturally, a mirror image of Theorem 2.2 is obtained when we reverse the situation to $\varphi_1 \equiv ES_{\alpha_1},\ \varphi_2 \equiv VaR_{\alpha_2}$. In view of (2.4) we now aim to solve

$$\min_{I_2 \in \mathcal{F}} (\lambda_1 - \lambda_2)E\big(I_2[X]\big) + \lambda_2 VaR_{\alpha_2}\big(I_2[X]\big) - \lambda_1 ES_{\alpha_1}\big(I_2[X]\big). \tag{2.7}$$

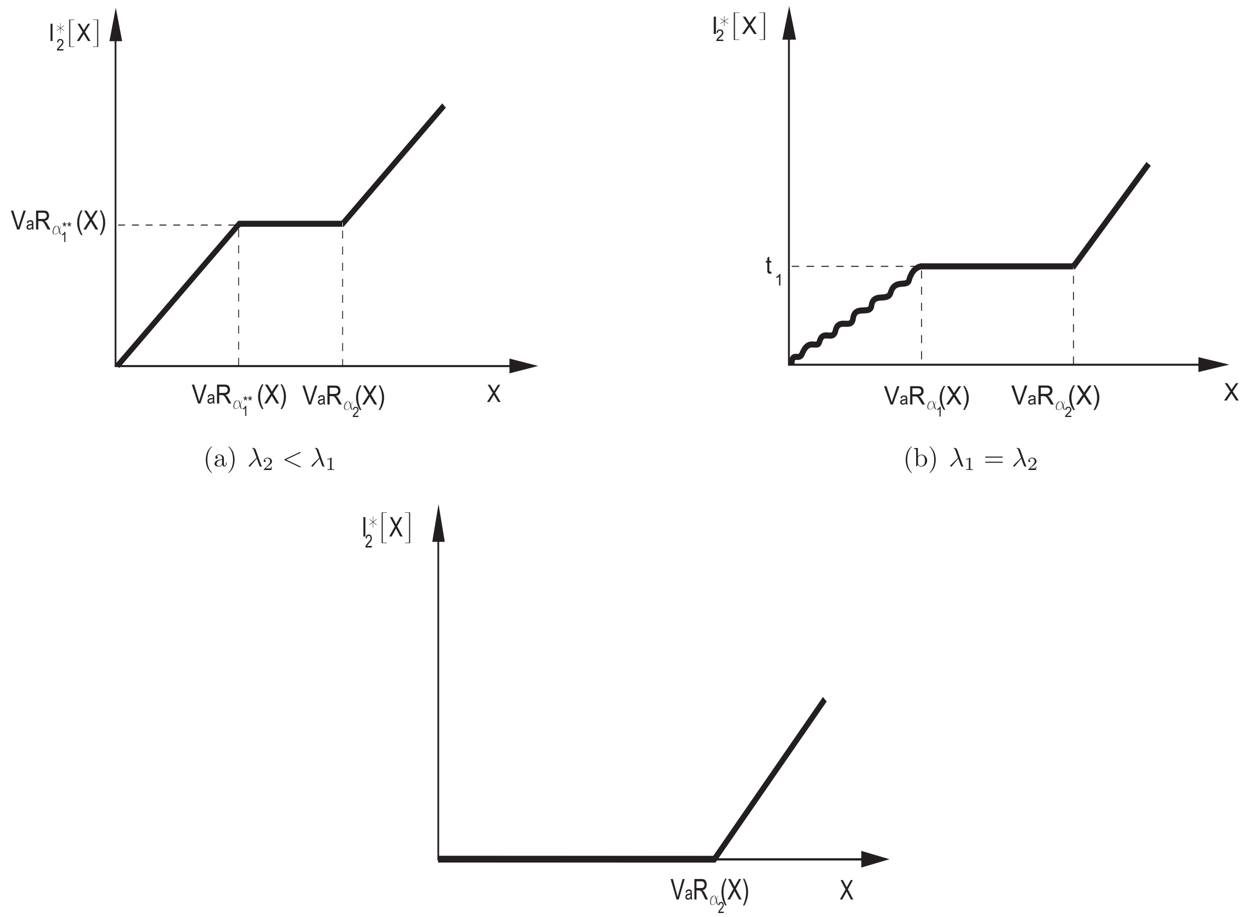
The full solution to problem (2.7) follows, by symmetry, directly from Theorem 2.2. Hence, we only state in Corollary 2.1 the solution for the scenario that is of most interest to us, where $VaR_{\alpha_1}(X) < VaR_{\alpha_2}(X)$. In that case, $\alpha_1,\ \alpha_2$ may be chosen such that ES_{α_1} and VaR_{α_2} are in a sense comparable, as in the case of regulatory requirements under the Swiss Solvency Test ($ES_{0.99}$) and Solvency II ($VaR_{0.995}$) (EIOPA, 2011). Moreover, we focus our discussion of the result around Corollary 2.1.

Corollary 2.1. *Let* $VaR_{\alpha_1}(X) \leq VaR_{\alpha_2}(X)$. *Then the optimal solution of (2.7) is:*

$$I_2^*[X] = \begin{cases} \big(X - VaR_{\alpha_2}(X)\big)_+ + \min\{X, VaR_{\alpha_1^{**}}(X)\}, & \lambda_1 > \lambda_2, \\ \big(X - VaR_{\alpha_2}(X)\big)_+ + \min\{f_1(X), t_1\}, & \lambda_1 = \lambda_2, \\ \big(X - VaR_{\alpha_2}(X)\big)_+, & \lambda_1 < \lambda_2 \end{cases}$$

where $\alpha_1^{**} = \min\{\alpha_1^*, \alpha_2\}$, $\alpha_1^* = \frac{\lambda_1\alpha_1}{\lambda_1\alpha_1+\lambda_2(1-\alpha_1)}$, *and* $f_1(\cdot)$ *is a non-decreasing Lipschitz continuous function with unit constant such that* $f_1(0) = 0$ *and* $f_1\big(VaR_{\alpha_1}(X)\big) = t_1$ *with* $t_1 \in [0, VaR_{\alpha_1}(X)]$.

The risk allocations arising from Corollary 2.1 are depicted in Figure 2.3.



(a) $\lambda_2 < \lambda_1$

(b) $\lambda_1 = \lambda_2$

(c) $\lambda_1 < \lambda_2$

FIGURE 2.3. Risk allocations arising from Corollary 2.1.

We first consider the case that $\lambda_1 > \lambda_2$, implying $\alpha_1 < \alpha_1^*$. Further, if $\alpha_1^* < \alpha_2$, then

$$\begin{aligned} I_1^*[X] &= \min\left\{(X - VaR_{\alpha_1^*}(X))_+, VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X)\right\}, \\ I_2^*[X] &= \left(X - VaR_{\alpha_2}(X)\right)_+ + \min\left\{X, VaR_{\alpha_1^*}(X)\right\}, \end{aligned}$$

The risk sharing arrangement is then such that a thin layer of $VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X)$ in excess of $VaR_{\alpha_1^*}(X)$ is retained by the first entity, while the rest of the risk is transferred to the second. The fact that most of the risk is optimally held by the second entity, is justified by its lower cost of capital. By the definition of α_1^* in Corollary 2.1, the more different λ_1 and λ_2 are, the further α_1 and α_1^* are from each other, hence the thinner the layer retained is. Moreover, the fact that the layer retained by the first entity is limited, essentially takes the "bite" out of the tail-sensitive risk measure, i.e. ES_{α_1}. At the same time, we note that $VaR_{\alpha_2}\big(I_2^*[X]\big) = VaR_{\alpha_1^*}(X) \leq VaR_{\alpha_2}(X)$. Thus, capital efficiency for the second entity arises again from the "blindness" property of the VaR_{α_2} measure to the tail component $\big(X - VaR_{\alpha_2}(X)\big)_+$ of $I_2^*[X]$, in the sense that $VaR_{\alpha_2}\big(X - VaR_{\alpha_2}(X)\big)_+ = 0$. This is further illustrated by Figure 2.4, where the percentiles of the random variables X, $I_1^*[X]$ and $I_2^*[X]$ are plotted. By the co-monotonicity of $I_1^*[X]$ and $I_2^*[X]$, the percentile of X equals the sum of the corresponding percentiles of $I_1^*[X]$ and $I_2^*[X]$, which holds at any given level. On the one hand, it is seen that the percentiles of $I_1^*[X]$ are equal to zero for confidence levels smaller than

α_1^*, increase in line with $VaR_\beta(X)$ for $\beta \in [\alpha_1^*, \alpha_2]$ and then remain constant for confidence levels higher than α_2. The fact that the high percentiles of $I_1^*[X]$ are bounded by $VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X)$ explains the reduction of $ES_{\alpha_1}\big(I_1^*[X]\big)$. On the other hand, constancy of the percentiles of $I_2^*[X]$ for $\beta \in [\alpha_1^*, \alpha_2]$ produces capital efficiency for the second entity, by forcing $VaR_{\alpha_2}\big(I_2^*[X]\big) = VaR_{\alpha_1^*}(X)$.

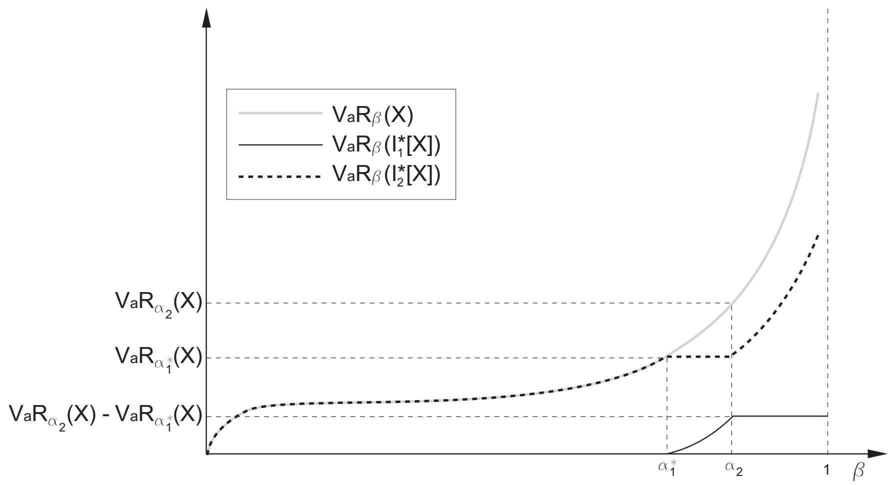


FIGURE 2.4. VaR allocations of $I_1^*[X]$ and $I_2^*[X]$ for various confidence levels β (Corollary 2.1 for $\lambda_1 > \lambda_2$).

The boundary solution, $\lambda_1 = \lambda_2$, is no longer unique. The same tail risk $\big(X - VaR_{\alpha_2}(X)\big)_+$ is transferred to the second entity and a layer between $VaR_{\alpha_1}(X)$ and $VaR_{\alpha_2}(X)$ is retained by the first. However, the allocation of losses lower than $VaR_{\alpha_1}(X)$ between the two entities is now arbitrary.

Finally, if $\lambda_1 < \lambda_2$, then more risk is retained by the first entity, which has a lower cost of capital, while the only risk transferred to the second entity is the extreme tail risk $\big(X - VaR_{\alpha_2}(X)\big)_+$. In addition, recall that VaR_{α_2} measure fails to reflect the tail risk, as argued earlier.

2.5. **ES / ES setting.** The last setting of the section assumes that both entities operate in markets subject to ES based capital requirements. Thus, the mathematical formulation of our problem becomes:

$$\min_{I_2 \in \mathcal{F}} (\lambda_1 - \lambda_2) E\big(I_2[X]\big) + \lambda_2 ES_{\alpha_2}\big(I_2[X]\big) - \lambda_1 ES_{\alpha_1}\big(I_2[X]\big). \tag{2.8}$$

The solution to this problem is provided below.

Theorem 2.3. *Let C be a constant given by:*

$$C = \lambda_2 \frac{\alpha_2}{1-\alpha_2} - \lambda_1 \frac{\alpha_1}{1-\alpha_1}.$$

Then, the optimal solution (2.8) is:

i) *If* $VaR_{\alpha_1}(X) = VaR_{\alpha_2}(X)$ *then*

$$I_2^*[X] = \begin{cases} X, & \lambda_1 > \lambda_2, \quad C < 0 \\ 0, & \lambda_1 < \lambda_2, \quad C > 0, \\ \min\big\{X, VaR_{\alpha_1}(X)\big\}, & \lambda_1 > \lambda_2, \quad C > 0, \\ \big(X - VaR_{\alpha_1}(X)\big)_+, & \lambda_1 < \lambda_2, \quad C < 0, \\ \big(X - VaR_{\alpha_1}(X)\big)_+ + \min\big\{f_1(X), t_1\big\}, & \lambda_1 = \lambda_2, \quad C < 0, \\ \min\big\{f_2(X), t_2\big\}, & \lambda_1 = \lambda_2, \quad C > 0, \end{cases}$$

where $f_1(\cdot)$ *and* $f_2(\cdot)$ *are non-decreasing Lipschitz continuous functions with unit constants such that* $f_1(0) = f_2(0) = 0, f_1\big(VaR_{\alpha_1}(X)\big) = t_1, f_2\big(VaR_{\alpha_1}(X)\big) = t_2$, *with parameters* $t_1, t_2 \in [0, VaR_{\alpha_1}(X)]$.

ii) *If* $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ *then*

$$I_2^*[X] = \begin{cases} X, & \lambda_1 > \lambda_2, \\ \big(X - VaR_{\alpha_2^*}(X)\big)_+, & \lambda_1 < \lambda_2, \quad C < 0, \\ 0, & \lambda_1 < \lambda_2, \quad C > 0, \\ \big(X - VaR_{\alpha_2}(X)\big)_+ + \min\big\{f_3(X), t_3\big\}, & \lambda_1 = \lambda_2, \end{cases}$$

where $\alpha_2^* = \frac{\lambda_2\alpha_2}{\lambda_1(1-\alpha_2)+\lambda_2\alpha_2}$ *and* $f_3(\cdot)$ *is a non-decreasing Lipschitz continuous function with unit constant such that* $f_3(0) = 0, f_3\big(VaR_{\alpha_2}(X)\big) = t_3 \in \big[0, VaR_{\alpha_2}(X)\big]$.

We shall not discuss all of the numerous cases from Theorem 2.3. Indicatively, let us first consider the case that $\alpha_1 = \alpha_2$. Since, $C < 0$ and $C > 0$ respectively hold if and only if $\lambda_1 > \lambda_2$ and $\lambda_1 < \lambda_2$, then only the first two cases of Theorem 2.3 i) are of interest. If $C < 0$, the optimal risk share of the second entity is $I_2^*[X] = X$. Unsurprisingly, when the risk measures for the two entities are the same, all the risk is transferred to the one with the lowest cost of capital. A similar situation arises when $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ and $\lambda_1 > \lambda_2$, i.e. the capital for the first entity is both more costly and subject to a stronger capital requirement.

More interesting risk transfers arise when $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ and $\lambda_1 < \lambda_2$, where the entity with stronger capital requirement has a lower capital cost rate. Here, the constant C reflects the trade-off between capital costs and conservativeness of the risk measure. So, when $C < 0$, the effect of the more conservative risk measure of the first entity dominates the effect of its lower cost of capital. Thus, an unlimited layer in excess of $VaR_{\alpha_2^*}(X)$ is transferred to the second entity. On the other hand, when $C > 0$, the effect of capital costs is more important. Therefore, the entire risk is retained by the first entity, which has access to less expensive capital.

3. The impact of credit risk on policyholder welfare

3.1. **Expected policyholder deficit.** The analysis of Section 2 does not consider the impact of potential default on policyholder welfare. Given that the optimal risk transfers proposed so far typically involve the transfer of some extreme tail risk from one entity to another, it is necessary to consider whether the resulting saving in capital costs is achieved at the detriment of policyholder safety. This is of course related to the potential discrepancy of risk measures used to regulate different entities. In particular, as seen in Corollary 2.1, when tail risk is transferred from an ES-regulated entity to another regulated by VaR, capital savings occur due to VaR's "blindness" to the magnitude of extreme losses, even though it is exactly these sort of losses that policyholders are exposed to in the case of insurer default.

In this section, we quantify the impact of risk transfer on policyholder welfare by studying the resulting expected policyholder deficit, and comparing it to the case where all risk is retained by the first entity. The policyholder deficit is equal to the difference between nominal liabilities to policyholders and liabilities that will actually be paid, thus reflecting the reduction in the payoff received due to potential default. The policyholder deficit can also be seen as an asset transferred from policyholders to shareholders, reflecting the option of the latter to default on their obligations (Phillips *et al.*, 1998, and Myers and Read, 2001). Formally, the expected policyholder deficit for a random liability Z and available assets c (a fixed number in the current setting where asset risk is not considered) is defined by

$$EPD(Z;c) = E\left(Z-c\right)_{+} = \int_{c}^{x_F} P(Z>z)\,dz.$$

For the purposes of the present section we focus on the case where capital requirements for the first entity are given by ES_{α_1} and for the second are given by VaR_{α_2}, with $VaR_{\alpha_1}(X) < VaR_{\alpha_2}(X)$. Moreover, let us also assume that all risk X is initially held by the first entity (which thus acts as a primary insurer) and the second entity is a subsidiary whose sole business is to reinsure the first by providing a contingent payment X_2. Consequently, X_1 is the risk retained by the first entity and $X = X_1 + X_2$. Before the risk transfer, the expected policyholder deficit is given by

$$EPD\big(X; ES_{\alpha_1}(X)\big) = E\big(X - ES_{\alpha_1}(X)\big)_{+}. \tag{3.1}$$

There are several ways to examine the impact of risk transfer on policyholder deficit. First, one may consider the two entities as separately regulated, the deficit arising from each being the concern of a different regulator. In this case, the expected policyholder deficit is equal to the sum of those arising from the different entities:

$$\begin{aligned} &EPD\big(X_1; ES_{\alpha_1}(X_1)\big) + EPD\big(X_2; VaR_{\alpha_2}(X_2)\big) \\ &\quad = E\big(X_1 - ES_{\alpha_1}(X_1)\big)_{+} + E\big(X_2 - VaR_{\alpha_2}(X_2)\big)_{+}. \end{aligned} \tag{3.2}$$

However, equation (3.2) does not properly reflect the impact of credit risk arising from the risk transfer, since it does not reflect the direction of the risk transfer. If we assume, as we do in this section, that all risk is transferred from the first entity to the second, a default of the second entity does not necessarily have a direct impact on the policyholders, who have bought their policies from the first. The default of the second entity only matters to policyholders to the extent that it is related to the scenario of the first entity (primary insurer) defaulting. Following these considerations, the risk exposure of the first entity, allowing for the possible default of its subsidiary (and assuming no recoveries given default), is given by $\tilde{X}_1 = X_1 + \big(X_2 - VaR_{\alpha_2}(X_2)\big)_{+}$.

There are now two possibilities. The first is that the capital requirement applied to the first entity does not actually reflect the credit risk arising from the risk transfer. In that case the capital held by the first entity is still $ES_{\alpha_1}(X_1)$, and the corresponding expected policyholder deficit is:

$$EPD\left(\tilde{X}_1; ES_{\alpha_1}(X_1)\right) = E\left(\tilde{X}_1 - ES_{\alpha_1}(X_1)\right)_{+}. \tag{3.3}$$

However, such an approach would not be consistent with the spirit of current regulatory requirements, where "risk concentrations" within groups need to be closely monitored (EIOPA, 2009). A second and more appropriate approach is to include all credit risk in the capital requirement, which will thus become $ES_{\alpha_1}(\tilde{X}_1)$. The resulting expected policyholder deficit is

$$EPD\left(\tilde{X}_1; ES_{\alpha_1}(\tilde{X}_1)\right) = E\left(\tilde{X}_1 - ES_{\alpha_1}(\tilde{X}_1)\right)_+. \tag{3.4}$$

Finally, we remark that while in this paper we deal with the case of two entities, policyholder deficit in the case of a group with more entities could be dealt with in a similar way. For example, consider that the risk X originally held by the first entity is reinsured by a second and a third entity, such that the allocated risks are $(X_1,\ X_2,\ X_3)$, with $X = X_1 + X_2 + X_3$. Let the risk measures for the three entities be $\varphi_1,\ \varphi_2,\ \varphi_3$. Then we can define $\tilde{X}_1 = X_1 + \big(X_2 - \varphi_2(X_2)\big)_+ + \big(X_3 - \varphi_3(X_3)\big)_+$, such that the allocated risks, accounting for credit risk, are $(\tilde{X}_1,\ X_2,\ X_3)$. The resulting expected policyholder deficit would again be given by $EPD\big(\tilde{X}_1; \varphi_1(\tilde{X}_1)\big) = E\big(\tilde{X}_1 - \varphi_1(\tilde{X}_1)\big)_+$.

3.2. **Policyholder deficit arising from optimal risk transfers.** We now compare the expected policyholder deficit amounts, as calculated by equations (3.1), (3.2), (3.3), and (3.4). For those calculations it is assumed that the risk allocation $(X_1,\ X_2)$ is given respectively by the optimal solutions $I_1^*[X],\ I_2^*[X]$ of Corollary 2.1 for $\lambda_1 \neq \lambda_2$. If $\lambda_1 > \lambda_2$, then the optimal risk allocations are given by:

$$\begin{aligned} X_1 &= \min\left\{\big(X - VaR_{\alpha_1^*}(X)\big)_+, VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X)\right\}, \\ X_2 &= \min\left\{X, VaR_{\alpha_1^*}(X)\right\} + \big(X - VaR_{\alpha_2}(X)\big)_+, \end{aligned} \tag{3.5}$$

where $\alpha_1^* = \frac{\lambda_1\alpha_1}{\lambda_1\alpha_1 + \lambda_2(1-\alpha_1)}$ and λ_1, λ_2 are such that $\alpha_1^* < \alpha_2$. The next lemma provides the expected policyholder deficits corresponding to this setting.

Lemma 3.1. *For $X_1,\ X_2$ as defined in (3.5), the following hold:*

i) $EPD\big(X_1; ES_{\alpha_1}(X_1)\big) + EPD\big(X_2; VaR_{\alpha_2}(X_2)\big) = \displaystyle\int_{\min\left\{VaR_{\alpha_2}(X), VaR_{\alpha_1^*}(X) + ES_{\alpha_1}(X_1)\right\}}^{x_F} \bar{F}(x)\,dx,$

ii) $EPD\left(\tilde{X}_1; ES_{\alpha_1}(X_1)\right) = \displaystyle\int_{VaR_{\alpha_1^*}(X) + ES_{\alpha_1}(X_1)}^{x_F} \bar{F}(x)\,dx,$

iii) $EPD\left(\tilde{X}_1; ES_{\alpha_1}(\tilde{X}_1)\right) = \displaystyle\int_{VaR_{\alpha_1^*}(X) + ES_{\alpha_1}(\tilde{X}_1)}^{x_F} \bar{F}(x)\,dx,$ *where*

$$ES_{\alpha_1}(X_1) = \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1^*}(X)}^{VaR_{\alpha_2}(X)} \bar{F}(x)\,dx \quad \textit{and} \quad ES_{\alpha_1}(\tilde{X}_1) = \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1^*}(X)}^{x_F} \bar{F}(x)\,dx.$$

In particular,

$$EPD\big(\tilde{X}_1; ES_{\alpha_1}(X_1)\big) \geq EPD\big(X; VaR_{\alpha_2}(X)\big) \quad \textit{and} \quad EPD\big(\tilde{X}_1; ES_{\alpha_1}(\tilde{X}_1)\big) \leq EPD\big(X; ES_{\alpha_1}(X)\big).$$

Besides giving formulas for expected policyholder deficits, Lemma 3.1 provides information via the two stated inequalities. To interpret the first inequality, note again that for the confidence levels used in the regulatory practice of insurance (for example, $\alpha_1 = 0.99$ and $\alpha_2 = 0.995$), the condition $VaR_{\alpha_2}(X) \leq ES_{\alpha_1}(X)$ is typically satisfied, which in turn implies that

$$EPD\big(X; ES_{\alpha_1}(X)\big) \leq EPD\big(X; VaR_{\alpha_2}(X)\big) \leq EPD\big(\tilde{X}_1; ES_{\alpha_1}(X_1)\big).$$

Therefore, the expected policyholder deficit *increases* with the risk transfer, if credit risk is not properly accounted for in capital setting (case ii)). The second inequality shows that allowing for credit risk in the capital requirement of the first entity (case iii)) increases its capital sufficiently, so that the expected policyholder deficit is actually *reduced* in relation to the situation before the risk transfer. Of course, the risk transfer is no longer optimal in relation to this stronger regulatory requirement, which is an issue that will be picked up in the next section.

Whenever $\lambda_1 < \lambda_2$, the optimal risk allocations are given by:

$$X_1 = \min\big\{X, VaR_{\alpha_2}(X)\big\}, \quad X_2 = \big(X - VaR_{\alpha_2}(X)\big)_+. \tag{3.6}$$

We can now find the expected policyholder deficits corresponding to this setting, which are given in Lemma 3.2.

Lemma 3.2. *For $X_1,\ X_2$ defined as in (3.6), the following hold:*

i) $EPD\big(X_1; ES_{\alpha_1}(X_1)\big) + EPD\big(X_2; VaR_{\alpha_2}(X_2)\big) = \displaystyle\int_{\min\left\{VaR_{\alpha_2}(X), ES_{\alpha_1}(X_1)\right\}}^{x_F} \bar{F}(x)dx.$

ii) $EPD\left(\tilde{X}_1; ES_{\alpha_1}(X_1)\right) = \displaystyle\int_{ES_{\alpha_1}(X_1)}^{x_F} \bar{F}(x)dx.$

iii) $EPD\left(\tilde{X}_1; ES_{\alpha_1}(\tilde{X}_1)\right) = \displaystyle\int_{ES_{\alpha_1}(X)}^{x_F} \bar{F}(x)dx.$

where $ES_{\alpha_1}(X_1) = VaR_{\alpha_1}(X) + \dfrac{1}{1-\alpha_1}\displaystyle\int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_2}(X)} \bar{F}(x)dx.$ *In particular,*

$$EPD\big(\tilde{X}_1; ES_{\alpha_1}(X_1)\big) \geq EPD\big(\tilde{X}_1; ES_{\alpha_1}(\tilde{X}_1)\big) = EPD\big(X; ES_{\alpha_1}(X)\big).$$

Once more, the inequality shows that the expected policyholder deficit *increases* with the risk transfer, if credit risk is not properly accounted for in the capital setting (case ii)). On the contrary, when allowing for credit risk in the capital requirement of the first entity (case iii)), the expected policyholder deficit *remains unchanged* in relation to the situation before the risk transfer.

3.3. **Optimal risk transfer when credit risk is reflected in the capital requirement.** It was seen in the previous section that reflecting credit risk in the capital requirement of the first entity leads to an expected policyholder deficit that is not higher than the one before the risk transfer. However, changing the capital setting criterion means that the risk transfers (3.5) and (3.6) discussed above will no longer be optimal and therefore, it would not necessarily be the choice of the group.

To address this issue, we derive optimal risk transfers, when the first entity incorporates the counter-party (reinsurance) default risk in its capital requirement. Slightly generalizing the arguments of the previous section, let us denote the *recovery rate* by $1-\gamma$, which is the percentage of the exposure to the second entity that will be recovered by the first in the case of the second entity's default. Given that the only business of the second entity is to reinsure the first, and that in our setting the assets available to pay for reinsurance claims will mainly consist of regulatory requirements on the second entity, it is reasonable to assume that γ is close to 1.

The mathematical formulation of the related problem is as follows. The risk to the first entity, including credit risk, becomes

$$g[X;\gamma] := I_1[X] + \gamma\Big(I_2[X] - I_2\big(VaR_{\alpha_2}(X)\big)\Big)_+,$$

and the corresponding optimisation problem can be written as

$$\min_{I_2 \in \mathcal{F}} E\big(g[X;\gamma]+I_2[X]\big)+\lambda_1\Big(ES_{\alpha_1}\big(g[X;\gamma]\big)-E\big(g[X;\gamma]\big)\Big)+\lambda_2\Big(VaR_{\alpha_2}\big(I_2[X]\big)-E\big(I_2[X]\big)\Big). \quad (3.7)$$

The solution to this problem is now given below.

Theorem 3.1. *Denote*

$$\gamma_1 = \frac{\lambda_2 + \lambda_1\alpha_1/(1-\alpha_1)}{1+\lambda_1\alpha_1/(1-\alpha_1)},$$

and let $VaR_{\alpha_1}(X) \le VaR_{\alpha_2}(X)$. The optimal solution of (3.7) is then given by:

i) *If $\lambda_1 > \lambda_2$, then*

$$I_2^*[X] = \begin{cases} \big(X - VaR_{\alpha_2}(X)\big)_+ + \min\big\{X, VaR_{\alpha_1^*}(X)\big\}, & 0 \le \gamma < \gamma_1, \\ \min\big\{X, VaR_{\alpha_1^*}(X)\big\}, & \gamma_1 < \gamma \le 1, \end{cases}$$

where α_1^{**} is as in Corollary 2.1.

ii) *If $\lambda_1 < \lambda_2$, then*

$$I_2^*[X] = \begin{cases} \Big(X - VaR_{\alpha_2}(X)\Big)_+, & 0 \le \gamma < \gamma_1, \\ 0, & \gamma_1 < \gamma \le 1. \end{cases}$$

In each case it is $I_1^*[X] = X - I_2^*[X]$.

It can be seen that the value of γ is crucial for the incentives produced. For a small value $\gamma < \gamma_1$, the optimal risk transfers are the same as in Corollary 2.1. On the other hand, for $\gamma > \gamma_1$, reflecting a lower recovery given default, the optimal risk transfer changes substantially, in that tail risk is no more transferred from the first (ES_{α_1}-regulated) to the second (VaR_{α_2}-regulated) entity. We would argue that the case $\gamma > \gamma_1$ is actually the more realistic one. For example, when $\lambda_1 = 0.10,\ \lambda_2 = 0.15,\ \alpha_1 = 0.99$, it follows that $\gamma_1 = 0.922$, while γ should be very close to 1.

This disincentive for transferring tail risk also ensures that the optimal risk transfer will not lead to increases in policyholder deficit, as will be shown in the numerical example that follows. However, the importance of producing disincentives for tail risk transfer to an entity subject to a weaker (eg VaR-based) regulatory requirement goes substantially beyond the consideration of formal welfare measures such as policyholder deficit. In particular, given the uncertainties and sensitivities involved in quantifying extreme risk, it may be argued that tail risk transfers are also associated with transfers of operational and model risks, which are now avoided.

3.4. **Numerical example.** We consider the two cases $\lambda_1 > \lambda_2$ and $\lambda_1 < \lambda_2$ separately. It is assumed that $\alpha_1 = 0.99$, $\alpha_2 = 0.995$ and $\gamma = 1$, as well as that the total risk X is Log-Normal distributed with mean of 1000 and standard deviation of 200. We consider four scenarios:

No Transfer: No risk is transferred, such that all risk is retained by the first entity.

Transfer 1 (no CR): A risk transfer as in Corollary 2.1 takes place, and the capital requirements do not account for credit risk (as in case ii) of Lemmas 3.1 and 3.2).

Transfer 2: A risk transfer as in Corollary 2.1 takes place, and the capital requirements do account for credit risk (as in case iii) of Lemmas 3.1 and 3.2).

Transfer 3: A risk transfer as in Theorem 3.1 takes place, such that capital requirements do account for credit risk.

For each scenario, the following quantities are calculated:

Total Capital: The total capital held by the group.

RAV-E(X): The risk adjusted value of the liabilities, in excess of the expected loss $E(X)$. In scenarios "No Transfer" and "Transfer 1" this is just the cost of capital; in scenarios "Transfer 2" and "Transfer 3", the "bad-debt reserve" $g[X;1] - I_1[X]$ is added to the cost of capital.

EPD: The expected policyholder deficit.

Case 1: $0.14 = \lambda_1 > \lambda_2 = 0.1$. The results for this case are summarized in Table 3.1 below. The

TABLE 3.1. Total capital, risk-adjusted value minus mean, and expected policyholder deficit for different risk transfer scenarios ($0.14 = \lambda_1 > \lambda_2 = 0.1$).

Scenario	Total Capital	RAV-E(X)	EPD
No Transfer	1666	93.2	0.402
Transfer 1	1617	62.7	0.629
Transfer 2	1671	70.7	0.381
Transfer 3	1671	70.2	0.381

example shows that when moving optimally from the "No Transfer" to the "Transfer 1" scenario, a substantial saving in capital and its cost is observed. However, the price of this is a big increase in expected policyholder deficit, arising from the transfer of tail risk to the more weakly regulated second entity. This anomaly is rectified once the capital requirement of the first entity is adjusted for credit risk (scenario "Transfer 2"), with the expected policyholder deficit returning to a value below its original level. In relation to the situation before the risk transfer, the total capital slightly increases (from 1666 to 1671); however there is still a notable reduction in the risk adjusted value (from 93.2 to 70.7) exploiting the lower cost of capital that the second entity is subject to. Finally, the scenario "Transfer 3", where risk is transferred optimally given that credit risk is included in the capital calculation, produces no further change apart from a very marginal reduction of risk adjusted value.

Case 2: $0.1 = \lambda_1 < \lambda_2 = 0.14$. The results for this case are summarized in Table 3.2 below.

As before, the example shows that when moving from the "No Transfer" to the "Transfer 1" scenario, a substantial saving in capital and its cost is observed. Again, this is at the expense of an increase in expected policyholder deficit, arising from the transfer of the tail risk to the second entity. This anomaly is rectified once the capital requirement of the first entity is adjusted for credit risk (scenario "Transfer 2"), with the expected policyholder deficit and total capital returning to its original level (1666) and only a marginal increase in the risk adjusted value (from 66.6 to 67). Finally, under scenario "Transfer 3", we return exactly to the situation before the risk transfer. This

TABLE 3.2. Total capital, risk adjusted value minus mean, and expected policyholder deficit for different risk transfer scenarios ($0.1 = \lambda_1 < \lambda_2 = 0.14$).

Scenario	Total Capital	RAV-E(X)	EPD
No Transfer	1666	66.6	0.402
Transfer 1	1611	61.1	0.663
Transfer 2	1666	67.0	0.402
Transfer 3	1666	66.6	0.402

arises directly upon observing from Theorem 3.1 that $I_2[X] = 0$, that is, it is optimal to transfer no risk at all to the second entity.

3.5. **Revisiting two assumptions.** Two key assumptions in this paper, co-monotonicity of risk allocations and non-impact of risk transfer on insurance prices, are now discussed.

3.5.1. *Co-monotone risk allocations.* In the optimisation problems solved in previous sections, it has always been assumed that the allocations of total risk to entities $I_1[X]$ and $I_2[X]$ are co-monotonic, i.e. non-decreasing functions of the total risk X. In the framework of Section 3, where the second entity acts as a subsidiary with sole business to reinsure the first, this assumption is fairly unproblematic, given the typical non-decreasing behaviour of reinsurance contracts.

Recall that each entity has a non-negative risk exposure before the risk transfer and all risk is first pooled and then shared by the two entities. However, it may not be immediately clear why the risk transfer should be structured such that the resulting risk allocations are co-monotone. To justify such an assumption, first note that a standard result in economics is that Pareto efficient risk allocations are co-monotone (see for example, Landsberger and Meilijson, 1994, and Ludkovski and Young, 2009). Therefore, it is reasonable to assume that the group will allocate risk to the two entities such that, at least approximately, the allocations are co-monotonic.

The second reason for making the co-monotonicity assumption is quite practical. We derive in this paper the optimal functional forms of risk transfers, rather than postulating a particular reasonable form for the contracts and then optimising other quantities (see Schlütter and Gründl, 2011). Consequently, certain limitations on admissible risk transfers need to be placed in order to avoid situations where moral hazard/regulatory arbitrage appear in blatant form. While plain cash transfers between group entities may be acceptable to a regulator (eg in the context of a parent bailing out a subsidiary), risk transfer contracts between distinct legal entities need to comply with the usual principles of insurance. Regulators do not allow risk-exchanges between group members that create problems of moral hazard, as this would undermine risk management by producing perverse incentives for the management of individual entities. This can be demonstrated through the following example, where the risk measures are $ES_{\alpha_1}, VaR_{\alpha_2}$. A particular form of admissible risk transfer (arising for example in Corollary 2.1 when $\lambda_1 < \lambda_2$) is one resulting into risk allocations $X_1 = \min\{X, VaR_{\alpha_2}(X)\}$, $X_2 = \big(X - VaR_{\alpha_2}(X)\big)_+$. In that case, the transfer of tail risk to the second (VaR_{α_2}-regulated) entity ensures that $VaR_{\alpha_2}(X_2) = 0$, which generates capital efficiency at the expense of policyholder welfare as seen in Lemma 3.2, as long as credit risk is not incorporated.

An alternative risk transfer is further considered:

$$\hat{X}_1 = X\mathbf{1}_{\{X \le VaR_{\alpha_2}(X)\}} = \begin{cases} X, & \text{if } X \le VaR_{\alpha_2}(X) \\ 0, & \text{otherwise,} \end{cases}$$

$$\hat{X}_2 = X\mathbf{1}_{\{X > VaR_{\alpha_2}(X)\}} = \begin{cases} X, & \text{if } X > VaR_{\alpha_2}(X) \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{1}_A$ is the indicator function of event A. Clearly, $\hat{X}_1$ and $\hat{X}_2$ are not co-monotone, as $\hat{X}_1$ may decrease in X. Compared to the previous risk allocation, one can write

$$\hat{X}_1 = X_1 - VaR_{\alpha_2}(X)\mathbf{1}_{\{X > VaR_{\alpha_2}(X)\}} \text{ and } \hat{X}_2 = X_2 + VaR_{\alpha_2}(X)\mathbf{1}_{\{X > VaR_{\alpha_2}(X)\}},$$

such that the liability $VaR_{\alpha_2}(X)\mathbf{1}_{\{X > VaR_{\alpha_2}(X)\}}$ is shifted from the first entity to the second. Consequently, $ES_{\alpha_1}(\hat{X}_1) \le ES_{\alpha_1}(X_1)$ and $VaR_{\alpha_2}(\hat{X}_2) = VaR_{\alpha_2}(X_2) = 0$. Hence, in the absence of a capital requirement in respect to credit risk, the capital available is now substantially reduced, leading to a further increase in policyholder deficit.

The implications for risk management of allowing such risk transfers are more wide-ranging. The non-increasing nature of the risk transfer $(\hat{X}_1,\hat{X}_2)$ creates moral hazard problems in the sense that the management of the first entity may have an incentive to preside over a high group loss $\{X > VaR_{\alpha_2}(X)\}$, leading to payment of $\{\hat{X}_1 = 0\}$, rather than a lower group loss $\{X \le VaR_{\alpha_2}(X)\}$, leading to payment of $\{\hat{X}_1 = X\}$.

3.5.2. *Impact of policyholder deficit on insurance prices.* Throughout this paper, a key idea has been that the insurance group seeks to perform risk transfers in order to minimise the risk adjusted value of its liabilities. This is sensible, as long as we assume that such risk transfers have no negative impact on the group's profitability. This is a potentially contentious assumption and we would like to justify it at this stage.

If a risk transfer takes place that damages policyholder welfare, then economic arguments suggest that the market consistent value of the insurance policies sold will reduce by the market consistent value of the resulting increase in policyholder deficit. If this reduction in the value of the pay-off is reflected in market insurance premiums (a strong assumption for some markets like personal lines insurance), the result is a reduction in profitability, which may outweigh the benefits of capital savings arising from the risk transfer.

While this is a plausible scenario, we do not consider it of great relevance to our setting, particularly in view of the numerical example presented above. First, we note that changes in the expected policyholder deficit (reflecting the reduction in policyholders' expected pay-offs) are an order of magnitude smaller than changes in the cost of capital. The changes in the value of the policyholder deficit would be increased if valued under a risk neutral measure (as in Myers and Read, 2001, and Schlütter and Gründl, 2011), however the upwards adjustment to the expected policyholder deficit would have to be very substantial in order for this to make any difference.

Furthermore, it can be seen that once credit risk is properly accounted for, optimal risk transfers do not actually lead to increases in policyholder deficit. Hence, there is no reason that a corresponding reduction in profit should be observed. In conclusion, and in light of the incentives produced by

Theorem 3.1, it seems that the most powerful mechanism to discourage risk transfers that damage policyholder welfare is regulatory action, rather than market forces.

4. Conclusion

This paper contributes to the literature on insurance groups, by discussing the implications of intra-group risk transfers for capital efficiency and policyholder welfare, and to the literature on insurance contract design, by deriving optimal risk transfers under preferences driven by regulatory risk measures. We find that optimal risk transfers under a cost-of-capital valuation criterion involve the transfer of extreme loss layers to the entity subject to a weaker regulatory regime. In particular, the discrepancy between the VaR and ES measures may be exploited, with tail risk transferred to a VaR-regulated entity.

Regulators should be wary of such incentives emerging. We show that, if the credit risk arising from risk transfers is not fully reflected in capital requirements, group capital efficiency is achieved at the cost of compromising policyholder security. If credit risk is fully reflected in the capital requirements, this problem does not emerge. Hence, it is crucial that regulators address the issue of intra-group credit risk when specifying quantitative capital requirements. Specifically, when an insurer transfers extreme risk to a subsidiary belonging to the same group, the high dependence between the insurer's gross loss and the subsidiary's default event, as well as the low recovery rate, need to be taken into account in the calculation of capital requirements.

Appendix A. Proofs

It is useful to note here that I_1 and I_2 are Lipschitz continuous functions with unit constants, i.e. $|I_k[y] - I_k[x]| \leq |y - x|$ holds for all $x, y \geq 0$ and $k \in \{1, 2\}$, due to the fact that I_1 and I_2 are non-decreasing functions (see also Carlier and Dana, 2003). In addition, $VaR_{\alpha_k}\big(I_k[X]\big) = I_k\big(VaR_{\alpha_k}(X)\big)$ is true for all $k \in \{1, 2\}$ (see for example, Denuit *et al.*, 2005, p.19, and Embrechts and Hofert, 2010).

The following notations,

$$\mathcal{A}_1 = \Big\{(\xi_1, \xi_2) : 0 \leq \xi_1 \leq VaR_{\alpha_1}(X), 0 \leq \xi_2 - \xi_1 \leq VaR_{\alpha_2}(X) - VaR_{\alpha_1}(X)\Big\}$$
$$\mathcal{A}_2 = \Big\{(\xi_1, \xi_2) : 0 \leq \xi_2 \leq VaR_{\alpha_2}(X), 0 \leq \xi_1 - \xi_2 \leq VaR_{\alpha_1}(X) - VaR_{\alpha_2}(X)\Big\}$$

are useful for explaining all the proofs. In addition, $\Pr(Z \leq z) < \alpha$ is equivalent to $z < VaR_\alpha(X)$.

A.1. **proof of Theorem 2.1.** The first part is now investigated assuming that $\lambda_1 > \lambda_2$. The mirror case, $\lambda_1 < \lambda_2$, results by the symmetry property of the objective function from equation (2.5). Now, for any arbitrary risk transfer I_2, we have

$$\begin{aligned}
&\big(\lambda_1-\lambda_2\big)E\big(I_2[X]\big)+\lambda_2 I_2\big(VaR_{\alpha_2}(X)\big)-\lambda_1 I_2\big(VaR_{\alpha_1}(X)\big)\\
&\quad=\big(\lambda_1-\lambda_2\big)E\Big(I_2[X]-I_2\big(VaR_{\alpha_1}(X)\big)\Big)+\lambda_2\Big(I_2\big(VaR_{\alpha_2}(X)\big)-I_2\big(VaR_{\alpha_1}(X)\big)\Big)\\
&\quad\geq\big(\lambda_1-\lambda_2\big)E\left(I_2\Big[\min\big\{X,VaR_{\alpha_1}(X)\big\}\Big]-I_2\big(VaR_{\alpha_1}(X)\big)\right)\\
&\qquad+\lambda_2\Big(I_2\big(VaR_{\min\{\alpha_1,\alpha_2\}}(X)\big)-I_2\big(VaR_{\alpha_1}(X)\big)\Big) \qquad\qquad (A.1)\\
&\quad\geq\big(\lambda_1-\lambda_2\big)E\Big(\min\big\{X,VaR_{\alpha_1}(X)\big\}-VaR_{\alpha_1}(X)\big)+\lambda_2\big(VaR_{\min\{\alpha_1,\alpha_2\}}(X)-VaR_{\alpha_1}(X)\big)\\
&\quad=\big(\lambda_1-\lambda_2\big)E\big(I_2^*[X]\big)+\lambda_2 I_2^*\big(VaR_{\alpha_2}(X)\big)-\lambda_1 I_2^*\big(VaR_{\alpha_1}(X)\big),
\end{aligned}$$

where the second last step is due to the Lipschitz property of function $I_2(\cdot)$. Note that the optimal solution stated in Theorem 2.1i) was $I_2^*[X]=\min\{X,VaR_{\alpha_1}(X)\}$, which concludes this case.

Finally, the $\lambda_1=\lambda_2$ case is discussed, which is reduced to minimising

$$I_2\big(VaR_{\alpha_2}(X)\big)-I_2\big(VaR_{\alpha_1}(X)\big). \qquad (A.2)$$

Recall that the Lipschitz property implies

$$|I_2\big(VaR_{\alpha_2}(X)\big)-I_2\big(VaR_{\alpha_1}(X)\big)|\leq|VaR_{\alpha_2}(X)-VaR_{\alpha_1}(X)|.$$

Thus, the solutions of (A.2) only require $I_2(\cdot)$ to be flat on $\big[VaR_{\alpha_1}(X),VaR_{\alpha_2}(X)\big]$ whenever $VaR_{\alpha_1}(X)<VaR_{\alpha_2}(X)$, and have a slope of 1 on $\big[VaR_{\alpha_2}(X),VaR_{\alpha_1}(X)\big]$ otherwise. The latter and Lipschitz property help in recovering the $\lambda_1=\lambda_2$ case. The proof is now complete.

A.2. **proof of Theorem 2.2.** The proofs are developed in a similar manner as in Theorem 2.1, and therefore we only outline the main steps. The case in which $\lambda_1>\lambda_2$ can be proved in the same manner as shown in equation (A.1). Specifically,

$$\begin{aligned}
&\big(\lambda_1-\lambda_2\big)E\big(I_2[X]\big)+\lambda_2 ES_{\alpha_2}\big(I_2[X]\big)-\lambda_1 I_2\big(VaR_{\alpha_1}(X)\big)\\
&\quad\geq\big(\lambda_1-\lambda_2\big)E\left(I_2\Big[\min\big\{X,VaR_{\alpha_1}(X)\big\}\Big]\right)+\lambda_2 ES_{\alpha_2}\left(I_2\Big[\min\big\{X,VaR_{\alpha_1}(X)\big\}\Big]\right)\\
&\qquad-\lambda_1 I_2\big(VaR_{\alpha_1}(X)\big)\\
&\quad=\big(\lambda_1-\lambda_2\big)E\left(I_2\Big[\min\big\{X,VaR_{\alpha_1}(X)\big\}\Big]-I_2\big(VaR_{\alpha_1}(X))\big)\right)\\
&\qquad+\lambda_2 ES_{\alpha_2}\left(I_2\Big[\min\big\{X,VaR_{\alpha_1}(X)\big\}\Big]-I_2\big(VaR_{\alpha_1}(X)\big)\right)\\
&\quad\geq\big(\lambda_1-\lambda_2\big)E\Big(\min\big\{X,VaR_{\alpha_1}(X)\big\}-VaR_{\alpha_1}(X)\Big)+\lambda_2 ES_{\alpha_2}\Big(\min\big\{X,VaR_{\alpha_1}(X)\big\}-VaR_{\alpha_1}(X)\Big)\\
&\quad=\big(\lambda_1-\lambda_2\big)E\big(I_2^*[X]\big)+\lambda_2 ES_{\alpha_2}\big(I_2^*[X]\big)-\lambda_1 I_2^*\big(VaR_{\alpha_1}(X)\big),
\end{aligned}$$

which implies that the optimisation problem from (2.6) is attained at $I_2^*[X]=\min\{X,VaR_{\alpha_1}(X)\}$.

Similarly, the $VaR_{\alpha_1}(X)\leq VaR_{\alpha_2}(X)$ case such that $\lambda_1<\lambda_2$ is examined. Clearly,

$$\big(\lambda_1-\lambda_2\big)E\big(I_2[X]\big)+\lambda_2 ES_{\alpha_2}\big(I_2[X]\big)-\lambda_1 I_2\big(VaR_{\alpha_1}(X)\big)\geq\big(\lambda_2-\lambda_1\big)\Big(ES_{\alpha_2}\big(I_2[X]\big)-E\big(I_2[X]\big)\Big)\geq 0.$$

The lower bound is attained whenever $I_2^*[X]=0$, which fully concludes part i).

The remaining of part ii), $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ with $\lambda_1 < \lambda_2$, needs to be elaborated in greater detail than the previous one. The optimisation problem (2.6) is solved via a two stage optimisation procedure. The mathematical formulation of the first stage optimisation problem becomes

$$\begin{cases} \min\limits_{I_2\in\mathcal{F}} \big(\lambda_1-\lambda_2\big)E\big(I_2[X]\big)+\lambda_2 ES_{\alpha_2}\big(I_2[X]\big) \;\text{ subject to} \\ I_2\big(VaR_{\alpha_1}(X)\big)=\xi_1,\; I_2\big(VaR_{\alpha_2}(X)\big)=\xi_2, \end{cases}$$

where $(\xi_1,\xi_2)\in\mathcal{A}_2$ are some constants. Keeping in mind that $\lambda_1-\lambda_2<0$ and relation (2.1), the function $I_2(\cdot)$ should increase as fast as possible on $[0, VaR_{\alpha_1}(X)]$, and remain flat on $[VaR_{\alpha_1}(X), x_F]$. The optimal solution is then pictured in Figure A.1.

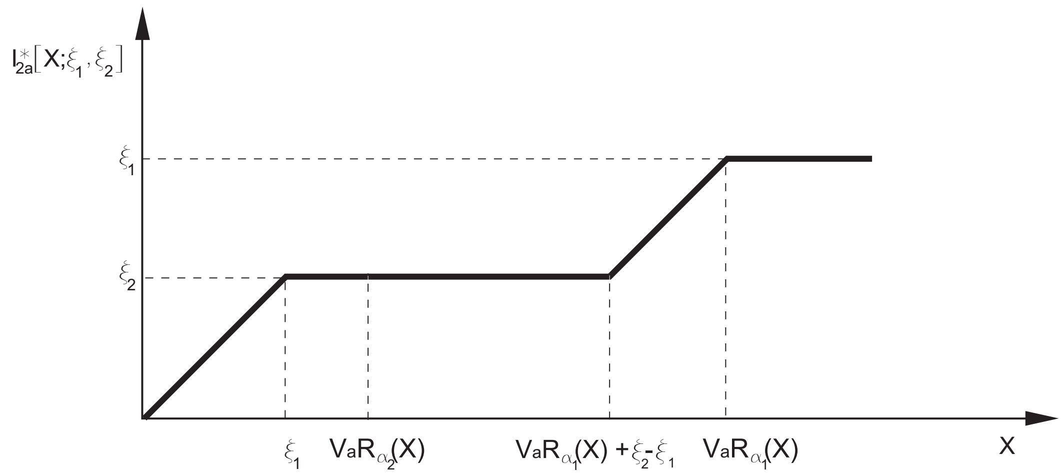


Figure A.1. The construction of $I_{2a}^*[X;\xi_1,\xi_2]$ in Theorem 2.2.

One may mathematically formulate the latter risk transfer as

$$I_{2a}^*[X;\xi_1,\xi_2]=\begin{cases} \min\big\{X-VaR_{\alpha_1}(X)+\xi_1,\xi_1\big\}, & X>VaR_{\alpha_1}(X)+\xi_2-\xi_1, \\ \min\big\{X,\xi_2\big\}, & \text{otherwise.} \end{cases}$$

Thus, the solution of our second stage optimisation problem

$$\min_{(\xi_1,\xi_2)\in\mathcal{A}_2} H(\xi_1,\xi_2)=(\lambda_1-\lambda_2)\left(\int_0^{\xi_2}\bar{F}(x)\,dx+\int_{VaR_{\alpha_1}(X)+\xi_2-\xi_1}^{VaR_{\alpha_1}(X)}\bar{F}(x)\,dx\right)+$$
$$\lambda_2\left(\xi_2+\frac{1}{1-\alpha_2}\int_{VaR_{\alpha_1}(X)+\xi_2-\xi_1}^{VaR_{\alpha_1}(X)}\bar{F}(x)\,dx\right)-\lambda_1\xi_1$$

replicates the global minimiser of (2.6). The derivative of the above-mentioned function with respect to ξ_1 becomes

$$\frac{dH}{d\xi_1}=\left(\lambda_1+\lambda_2\frac{\alpha_2}{1-\alpha_2}\right)\bar{F}\big(VaR_{\alpha_1}(X)+\xi_2-\xi_1\big)-\lambda_1, \tag{A.3}$$

which is non-positive if and only if $VaR_{\alpha_1}(X)+\xi_2-\xi_1\geq VaR_{\alpha_2^*}(X)$. Due to the fact that $\lambda_1<\lambda_2$, it is not difficult to find that $\alpha_2^*>\alpha_2$. Note that $VaR_{\alpha_2}(X)\leq VaR_{\alpha_1}(X)+\xi_2-\xi_1\leq VaR_{\alpha_1}(X)$ holds. Having all of these in mind, we may conclude

$$H(\xi_1,\xi_2)\geq H(\xi_2,\xi_2),\;\;\text{if } \alpha_2^*\geq\alpha_1$$

and

$$H(\xi_1,\xi_2) \geq H\big(\xi_2 + VaR_{\alpha_1}(X) - VaR_{\alpha_2^*}(X), \xi_2\big), \text{ if } \alpha_2^* < \alpha_1.$$

Finding the minimum values of the above-right hand side functions over the domain of ξ_2, namely $\big[0, VaR_{\alpha_2}(X)\big]$, the global solution of (2.6) is attained at $(0,0)$ and $\big(VaR_{\alpha_1}(X) - VaR_{\alpha_2^*}(X), 0\big)$, if $\alpha_2^* \geq \alpha_1$ and $\alpha_2^* < \alpha_1$, respectively. Thus, part ii) is concluded.

Finally, the $\lambda_1 = \lambda_2$ case is under investigation. Our optimisation problem becomes

$$\min_{I_2 \in \mathcal{F}} ES_{\alpha_2}\big(I_2[X]\big) - VaR_{\alpha_1} I_2[X].$$

Thus, the first stage optimisation problem is equivalent to minimising $ES_{\alpha_2}\big(I_2[X]\big)$.

The $VaR_{\alpha_1}(X) \leq VaR_{\alpha_2}(X)$ subcase makes $I_2(\cdot)$ to be flat for losses higher than $VaR_{\alpha_2}(X)$. Thus, the second stage problem is to minimise $\xi_2 - \xi_1$ on $\mathcal{A}_1$, and keeping in mind the Lipschitz property, one may recover our claim.

The last subcase, $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$, is developed in greater detail. The first part of A.2 only requires for the function $I_2(\cdot)$ to not increase on $\big[VaR_{\alpha_2}(X), VaR_{\alpha_1}(X) - \big(\xi_1 - \xi_2\big)\big]$, then to increase with a slope of 1 until it reaches the ξ_1 level at $VaR_{\alpha_1}(X)$, and remain flat on $\big[VaR_{\alpha_1}(X), x_F\big]$. Thus, the second stage problem of A.2 becomes

$$\min_{\mathcal{A}_2} \ \xi_2 + \frac{1}{1-\alpha_2} \int_{VaR_{\alpha_1}(X) - (\xi_1 - \xi_2)}^{VaR_{\alpha_1}(X)} \bar{F}(x)\, dx - \xi_1.$$

Therefore, the objective function of the above is non-increasing in ξ_1 for any fixed ξ_2, and straightforward calculations show that the set of optimal solutions is given by

$$\Big\{\big(x + VaR_{\alpha_1}(X) - VaR_{\alpha_2}(X), x\big) : x \in \big[0, VaR_{\alpha_2}(X)\big]\Big\},$$

which justifies in full the $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ subcase. This ends the proof.

A.3. **proof of Theorem 2.3.** i) If $VaR_{\alpha_1}(X) = VaR_{\alpha_2}(X)$, the first stage optimisation problem is given by

$$\min_{I_2 \in \mathcal{F}} \big(\lambda_1 - \lambda_2\big) E\big(I_2[X]\big) + \lambda_2 ES_{\alpha_2}\big(I_2[X]\big) - \lambda_1 ES_{\alpha_1}\big(I_2[X]\big) \quad \text{subject to } I_2\big(VaR_{\alpha_1}(X)\big) = \xi, \quad \text{(A.4)}$$

where $\xi \in \big[0, VaR_{\alpha_1}(X)\big]$ represents an arbitrarily chosen constant.

Let $\lambda_1 > \lambda_2$. It is not difficult to justify via some geometric argumentation that (A.4) has solutions given by $\big(X - VaR_{\alpha_1}(X) + \xi\big)_+$ and $\min\big\{\big(X - VaR_{\alpha_1}(X) + \xi\big)_+, \xi\big\}$ whenever $C < 0$ and $C > 0$, respectively.

Next, it is assumed that $C < 0$, and plugging the solution of (A.4) in its objective function, the second stage optimisation problem is reduced to minimising

$$\big(\lambda_1 - \lambda_2\big) \int_{VaR_{\alpha_1}(X) - \xi}^{x_F} \bar{F}(x)\, dx + \lambda_2\big(\xi + ES_{\alpha_2}(X) - VaR_{\alpha_1}(X)\big) - \lambda_1\big(\xi + ES_{\alpha_1}(X) - VaR_{\alpha_1}(X)\big),$$

over all possible values of ξ. The latter function is decreasing on $\big[0, VaR_{\alpha_1}(X)\big]$, which leads to full transfer of the risk to the second insurance company.

For positive values of $C > 0$, the second stage optimisation problem is retrieved by minimising

$$\big(\lambda_1 - \lambda_2\big) \left(\int_{VaR_{\alpha_1}(X) - \xi}^{VaR_{\alpha_1}(X)} \bar{F}(x)\, dx - \xi \right),$$

over $\xi \in \big[0, VaR_{\alpha_1}(X)\big]$ due to the fact that

$$ES_{\alpha_1}\Big(\min\big\{\big(X - VaR_{\alpha_1}(X) + \xi\big)_+, \xi\big\}\Big) = ES_{\alpha_2}\Big(\min\big\{\big(X - VaR_{\alpha_1}(X) + \xi\big)_+, \xi\big\}\Big) = \xi.$$

The decreasing property of the above-mentioned function implies that $I_2^*[X] = \min\big\{X, VaR_{\alpha_1}(X)\big\}$, which concludes the $\lambda_1 > \lambda_2$ case.

We now consider the situation in which $\lambda_1 < \lambda_2$. One may get that (A.4) is minimised by $\min\{X,\xi\} + \big(X - VaR_{\alpha_1}(X)\big)_+$ and $\min\big\{X,\xi\big\}$ whenever $C < 0$ and $C > 0$, respectively. Thus, the second stage problems are the minimisation in ξ of

$$\big(\lambda_1 - \lambda_2\big)\left(\int_0^{\xi} + \int_{VaR_{\alpha_1}(X)}^{x_F}\right)\bar{F}(x)\,dx + \big(\lambda_2 - \lambda_1\big)\big(\xi - VaR_{\alpha_1}(X)\big) + \lambda_2 ES_{\alpha_2}(X) - \lambda_1 ES_{\alpha_1}(X)$$

and

$$\big(\lambda_1 - \lambda_2\big)\left(\int_{VaR_{\alpha_1}(X)-\xi}^{VaR_{\alpha_1}(X)}\bar{F}(x)\,dx - \xi\right),$$

provided that $C < 0$ and $C > 0$, respectively, over the set $\big[0, VaR_{\alpha_1}(X)\big]$. Clearly, both functions are increasing on the whole domain, and therefore one may easily recover optimal contracts in our setting, i.e. if $\lambda_1 < \lambda_2$.

The $\lambda_1 = \lambda_2$ setting is investigated at the moment, for which

$$\min_{\mathcal{F}}\ ES_{\alpha_2} I_2[X] - ES_{\alpha_2} I_2[X] \tag{A.5}$$

needs to be solved. The first stage problem becomes

$$\min_{0\le\xi\le VaR_{\alpha_1}(X)}\ \frac{\alpha_2 - \alpha_1}{(1-\alpha_1)(1-\alpha_2)} E\big(I_2[X] - \xi\big)_+,\ \text{subject to } I_2\big(VaR_{\alpha_1}(X)\big) = \xi$$

as a result of relation (2.1). Note that $C < 0$ is the same as $\alpha_1 > \alpha_2$, which makes $I_2(\cdot)$ to increase with slope 1 from $VaR_{\alpha_1}(X)$ onwards in order to solve A.6. In turn, the $C > 0$ case requires a leveled $I_2(\cdot)$ function on $\big[VaR_{\alpha_1}(X), x_F\big]$, since $\alpha_1 < \alpha_2$. Thus, the justification of $VaR_{\alpha_1}(X) = VaR_{\alpha_2}(X)$ is now completed.

ii) The $VaR_{\alpha_1}(X) > VaR_{\alpha_2}(X)$ case is further examined. If $\lambda_1 > \lambda_2$, then

$$\big(\lambda_1 - \lambda_2\big)E\big(I_2[X]\big) + \lambda_2 ES_{\alpha_2}\big(I_2[X]\big) - \lambda_1 ES_{\alpha_1}\big(I_2[X]\big) \ge \big(\lambda_2 - \lambda_1\big)\Big(ES_{\alpha_1}\big(I_2[X]\big) - E\big(I_2[X]\big)\Big),$$

and therefore (2.8) is the same as to maximse $ES_{\alpha_1}\big(I_2[X]\big) - E\big(I_2[X]\big)$, which in turn is equivalent to minimising $ES_{\alpha_1}\big(I_1[X]\big) - E\big(I_1[X]\big)$. Thus, $I_1^*[X] = 0$, which concludes the $\lambda_1 > \lambda_2$ subcase.

Our main objective from (2.8) can be solved via a two stage optimisation, whenever $\lambda_1 \le \lambda_2$. Particularly, the first step can be written in the following fashion

$$\begin{cases} \min\limits_{I_2\in\mathcal{F}} \big(\lambda_1 - \lambda_2\big)E\big(I_2[X]\big) + \lambda_2 ES_{\alpha_2}\big(I_2[X]\big) - \lambda_1 ES_{\alpha_1}\big(I_2[X]\big)\ \text{ subject to} \\ I_2\big(VaR_{\alpha_1}(X)\big) = \xi_1,\ I_2\big(VaR_{\alpha_2}(X)\big) = \xi_2, \end{cases}$$

where $(\xi_1, \xi_2) \in \mathcal{A}_2$ is a fixed vector of constants. It can be shown that the solutions are

$$I_2^*[X;\xi_1,\xi_2] = \begin{cases} I_{2a}^*[X;\xi_1,\xi_2], & \lambda_1 < \lambda_2, \quad C > 0, \\ I_{2b}^*[X;\xi_1,\xi_2], & \lambda_1 < \lambda_2, \quad C < 0, \end{cases}$$

where

$$I_{2b}^*[X;\xi_1,\xi_2] = \begin{cases} X - VaR_{\alpha_1}(X) + \xi_1, & X > VaR_{\alpha_1}(X) + \xi_2 - \xi_1, \\ \min\{X, \xi_2\}, & \text{otherwise.} \end{cases}$$

In the case that $\lambda_1 < \lambda_2$ and $C < 0$, we only need to solve

$$\min_{(\xi_1,\xi_2)\in\mathcal{A}_2} H(\xi_1,\xi_2) := (\lambda_1 - \lambda_2)\left(\int_0^{\xi_2} + \int_{VaR_{\alpha_1}(X)+\xi_2-\xi_1}^{x_F}\right)\bar{F}(x)\,dx \tag{A.6}$$
$$+\lambda_2\left(\xi_2 + \frac{1}{1-\alpha_2}\int_{VaR_{\alpha_1}(X)+\xi_2-\xi_1}^{x_F}\bar{F}(x)\,dx\right) - \lambda_1\left(\xi_1 + \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1}(X)}^{x_F}\bar{F}(x)\,dx\right).$$

Note that $\alpha_2 < \alpha_2^* < \alpha_1$ is true under this setting. The derivative of the above with respect to ξ_1 is given by (A.3), and it is non-positive if and only if $\xi_2 \le \xi_1 \le \xi_2 + VaR_{\alpha_1}(X) - VaR_{\alpha_2^*}(X)$. Thus,

$$H(\xi_1,\xi_2) \ge H\big(\xi_2 + VaR_{\alpha_1}(X) - VaR_{\alpha_2^*}(X), \xi_2\big) = (\lambda_1 - \lambda_2)\left(\int_0^{\xi_2}\bar{F}(x)\,dx - \xi_2\right) + K,$$

where K is a constant with respect to ξ_2. The latter function is increasing in ξ_2 on $\big[0, VaR_{\alpha_2}(X)\big]$, and therefore $(A.6)$ is solved by $\big(\xi_1^*,\xi_2^*\big) = \big(VaR_{\alpha_1}(X) - VaR_{\alpha_2^*}(X), 0\big)$, which replicates the optimal risk transfer in this subcase.

The situation in which $\lambda_1 < \lambda_2$ and $C > 0$ requires minimising over $\mathcal{A}_2$

$$(\lambda_1 - \lambda_2)\left(\int_0^{\xi_2} + \int_{VaR_{\alpha_1}(X)+\xi_2-\xi_1}^{VaR_{\alpha_1}(X)}\right)\bar{F}(x)\,dx + \lambda_2\left(\xi_2 + \frac{1}{1-\alpha_2}\int_{VaR_{\alpha_1}(X)+\xi_2-\xi_1}^{VaR_{\alpha_1}(X)}\bar{F}(x)\,dx\right) - \lambda_1\xi_1.$$

Clearly, $\alpha_2 < \alpha_1 < \alpha_2^*$, and therefore the function from above is non-decreasing in ξ_1 for any fixed ξ_2, since again its derivative with respect with ξ_1 is given by equation (A.3). Similar derivations to the previous subcases lead to the global minimum to be attained at $\big(\xi_1^*,\xi_2^*\big) = (0,0)$.

The final setting, under which $\lambda_1 = \lambda_2$, needs to be justified. The first stage optimisation problem of $(A.5)$ forces the risk transfer to remain flat on $\big[VaR_{\alpha_2}(X), VaR_{\alpha_1}(X) - (\xi_1 - \xi_2)\big]$, and increase as fast as possible onwards, since $\alpha_1 > \alpha_2$. The second stage problem becomes

$$\min_{\mathcal{A}_2}\ \xi_2 - \xi_1 + \frac{1}{1-\alpha_2}\int_{VaR_{\alpha_1}(X)-(\xi_1-\xi_2)}^{x_F}\bar{F}(x)\,dx - \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1}(X)}^{x_F}\bar{F}(x)\,dx. \tag{A.7}$$

Note that the above objective function is non-increasing in ξ_1 for any fixed ξ_2, and therefore the set of solutions for $(A.7)$ is given by

$$\Big\{\big(VaR_{\alpha_1}(X) - VaR_{\alpha_2}(X) + x, x\big) : 0 \le x \le VaR_{\alpha_2}(X)\Big\}.$$

Thus, we only need to impose a linear increasing with slope 1 for $I_2(\cdot)$ on $[VaR_{\alpha_2}(X), x_F]$ in order to solve the initial optimisation problem. This completes the proof.

A.4. **proof of Lemma 3.1.** Note that $VaR_{\alpha_1}(X_1) = 0$ and $VaR_{\alpha_2}(X_2) = VaR_{\alpha_1^*}(X)$, which together with (2.1) imply that

$$ES_{\alpha_1}(X_1) = \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1^*}(X)}^{VaR_{\alpha_2}(X)}\bar{F}(x)\,dx.$$

Now,

$$\begin{aligned}
EPD\big(X_1; ES_{\alpha_1}(X_1)\big) &= \int_{ES_{\alpha_1}(X_1)}^{x_F} P(X_1 > z)\, dz \\
&= \int_{\min\left\{VaR_{\alpha_2}(X), VaR_{\alpha_1^*}(X)+ES_{\alpha_1}(X_1)\right\}}^{VaR_{\alpha_2}(X)} \bar{F}(x)\, dx \\
EPD\big(X_2; VaR_{\alpha_2}(X_2)\big) &= \int_{VaR_{\alpha_2}(X_2)}^{x_F} P(X_2 > z) dz = \int_{VaR_{\alpha_2}(X)}^{x_F} \bar{F}(x)\, dx
\end{aligned}$$

from which part i) follows. Clearly, $\tilde{X}_1 = \big(X - VaR_{\alpha_1^*}(X)\big)_+$ and we further have that

$$ES_{\alpha_1}\big(\tilde{X}_1\big) = \frac{1}{1-\alpha_1} \int_{VaR_{\alpha_1^*}(X)}^{x_F} \bar{F}(x)\, dx,$$

which concludes parts ii) and iii).

The first inequality in the lemma holds as long as

$$VaR_{\alpha_1^*}(X) + \frac{1}{1-\alpha_1} \int_{VaR_{\alpha_1^*}(X)}^{VaR_{\alpha_2}(X)} \bar{F}(x)\, dx \leq VaR_{\alpha_2}(X).$$

The latter holds as a result of

$$\begin{aligned}
\frac{1}{1-\alpha_1} \int_{VaR_{\alpha_1^*}(X)}^{VaR_{\alpha_2}(X)} \bar{F}(x)\, dx &\leq \frac{\bar{F}\big(VaR_{\alpha_1^*}(X)\big)}{1-\alpha_1} \big(VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X)\big) \\
&\leq \frac{1-\alpha_1^*}{1-\alpha_1} \big(VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X)\big) \\
&\leq VaR_{\alpha_2}(X) - VaR_{\alpha_1^*}(X).
\end{aligned}$$

It only remains to show the very last inequality from Lemma 3.1, for which it is sufficient to show that

$$VaR_{\alpha_1^*}(X) + \frac{1}{1-\alpha_1} \int_{VaR_{\alpha_1^*}(X)}^{x_F} \bar{F}(x)\, dx \geq VaR_{\alpha_1}(X) + \frac{1}{1-\alpha_1} \int_{VaR_{\alpha_1}(X)}^{x_F} \bar{F}(x)\, dx.$$

The latter is true since $\bar{F}\big(VaR_{\alpha_1}(X)\big) \leq 1 - \alpha_1$ and the fact that

$$\begin{aligned}
\int_{VaR_{\alpha_1}(X)}^{x_F} \bar{F}(x)\, dx &= \int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_1^*}(X)} \bar{F}(x)\, dx + \int_{VaR_{\alpha_1^*}(X)}^{x_F} \bar{F}(x)\, dx \\
&\leq \big(VaR_{\alpha_1^*}(X) - VaR_{\alpha_1}(X)\big) \bar{F}\big(VaR_{\alpha_1}(X)\big) + \int_{VaR_{\alpha_1^*}(X)}^{x_F} \bar{F}(x)\, dx.
\end{aligned}$$

A.5. **proof of Lemma 3.2.** Note first that $VaR_{\alpha_1}(X_1) = VaR_{\alpha_1}(X)$ and $VaR_{\alpha_2}(X_2) = 0$, which together with (2.1) give that

$$ES_{\alpha_1}(X_1) = VaR_{\alpha_1}(X) + \frac{1}{1-\alpha_1} \int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_2}(X)} \bar{F}(x)\, dx.$$

Clearly, $\tilde{X}_1 = X_1 + \big(X_2 - VaR_{\alpha_2}(X_2)\big)_+ = X$. Now, part i) simply follows from

$$\begin{aligned} EPD\big(X_1; ES_{\alpha_1}(X_1)\big) &= \int_{ES_{\alpha_1}(X_1)}^{x_F} P(X_1 > z)dz = \int_{\min\left\{VaR_{\alpha_2}(X), ES_{\alpha_1}(X_1)\right\}}^{VaR_{\alpha_2}(X)} \bar{F}(x)\, dx \\ EPD\big(X_2; VaR_{\alpha_2}(X_2)\big) &= \int_{VaR_{\alpha_2}(X_2)}^{x_F} P(X_2 > z)dz = \int_{VaR_{\alpha_2}(X)}^{x_F} \bar{F}(x)\, dx. \end{aligned}$$

Parts ii) and iii), as well as the equality $EPD\big(\tilde{X}_1; ES_{\alpha_1}(\tilde{X}_1)\big) = EPD\big(X; ES_{\alpha_1}(X)\big)$ are true since $\tilde{X}_1 = X$. Finally, the inequality at the end of the lemma follows from $ES_{\alpha_1}(X_1) \le ES_{\alpha_1}(X)$.

A.6. **proof of Theorem 3.1.** It is easy to obtain

$$\begin{aligned} E\big(g[X;\gamma]\big) &= E\big(I_1[X]\big) + \gamma E\Big(I_2[X] - I_2\big(VaR_{\alpha_2}(X)\big)\Big)_+ \\ &= E\big(I_1[X]\big) + \gamma(1-\alpha_2)\Big(ES_{\alpha_2}\big(I_2[X]\big) - I_2\big(VaR_{\alpha_2}(X)\big)\Big), \end{aligned}$$

as a result of equation (2.1). Moreover,

$$\begin{aligned} ES_{\alpha_1}\big(g[X;\gamma]\big) &= ES_{\alpha_1}\big(I_1[X]\big) + \gamma ES_{\alpha_1}\Big(I_2[X] - I_2\big(VaR_{\alpha_2}(X)\big)\Big)_+ \\ &= ES_{\alpha_1}(X) - ES_{\alpha_1}\big(I_2[X]\big) + \gamma\frac{1-\alpha_2}{1-\alpha_1}\Big(ES_{\alpha_2}\big(I_2[X]\big) - I_2\big(VaR_{\alpha_2}(X)\big)\Big) \end{aligned}$$

due to the co-monotonicity property and relation (2.1). Combining the last two equations, one may reduce our main problem to minimising

$$h_1(\gamma)I_2\big(VaR_{\alpha_2}(X)\big) + (\lambda_1 - \lambda_2)E\big(I_2[X]\big) + h_2(\gamma)ES_{\alpha_2}\big(I_2[X]\big) - \lambda_1 ES_{\alpha_1}\big(I_2[X]\big),$$

where $h_1(\lambda) = \lambda_2 - h_2(\lambda)$ and $h_2(\lambda) = \gamma(1-\alpha_2)\big(1 + \lambda_1\alpha_1/(1-\alpha_1)\big)$. Note that

$$\gamma\left(1 + \frac{\lambda_1\alpha_1}{1-\alpha_1}\right) - \lambda_2 - \frac{\lambda_1\alpha_1}{1-\alpha_1}$$

is negative if $\gamma \in [0, \gamma_1)$ and positive if $\gamma \in (\gamma_1, 1]$.

Now, the solution for the first stage problem from part i) needs to increase as slowly as possible on $\big[0, VaR_{\alpha_1}(X)\big]$ and increase rapidly on $\big[VaR_{\alpha_1}(X), VaR_{\alpha_2}(X)\big]$. In addition, the optimal solution remains flat on $\big[VaR_{\alpha_2}(X), x_F\big]$ if $\gamma_1 < \gamma \le 1$ and increases with slope 1 whenever $0 \le \gamma < \gamma_1$. All of these facts generate the optimal solution

$$\begin{cases} \min\left\{\big(X - VaR_{\alpha_1}(X) + \xi_1\big)_+, \xi_2\right\} + \big(X - VaR_{\alpha_2}(X)\big)_+, & 0 \le \gamma < \gamma_1, \\ \min\left\{\big(X - VaR_{\alpha_1}(X) + \xi_1\big)_+, \xi_2\right\}, & \gamma_1 < \gamma \le 1, \end{cases}$$

where $(\xi_1, \xi_2) \in \mathcal{A}_1$. Straightforward computations yield that the difference between the objective functions for the second stage problem corresponding to the two scenarios, is a constant with respect to ξ_1 and ξ_2. Therefore, both cases lead to the same solution, which is $\big(VaR_{\alpha_1}(X), VaR_{\alpha_1^*}(X)\big)$. We only discuss the second case, $\gamma_1 < \gamma \le 1$, for which the second stage optimisation problem is equivalent to minimising over $\mathcal{A}_1$

$$H(\xi_1,\xi_2) := \lambda_2\xi_2 + (\lambda_1 - \lambda_2)\int_{VaR_{\alpha_1}(X)-\xi_1}^{VaR_{\alpha_1}(X)+\xi_2-\xi_1} \bar{F}(x)\, dx - \lambda_1\left(\xi_1 + \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_1}(X)+\xi_2-\xi_1} \bar{F}(x)\, dx\right).$$

Taking the derivative with respect to ξ_2, we get that

$$H(\xi_1,\xi_2) \geq H\big(\xi_1, \xi_1 + VaR_{\alpha_1^*}(X) - VaR_{\alpha_1}(X)\big), \text{ for all } (\xi_1,\xi_2) \in \mathcal{A}_1.$$

The right hand side function is increasing in ξ_1, which justifies the first part of Theorem 3.1.

Part ii) is developed in the same manner as the previous case. The solution for the first stage problem requires a rapid variation on $\big[0, VaR_{\alpha_2}(X)\big]$ since $\lambda_1 - \lambda_2 < 0$ and

$$\lambda_1 - \lambda_2 - \lambda_1/(1-\alpha_1) = -\lambda_1\alpha_1/(1-\alpha_1) - \lambda_2 < 0.$$

In addition, the optimal solution remains flat on $\big[VaR_{\alpha_2}(X), x_F\big]$, if $\gamma_1 < \gamma \leq 1$ and increases with slope 1 whenever $0 \leq \gamma < \gamma_1$. Thus, our solutions are

$$\begin{cases} \min\{X,\xi_1\} + \min\big\{\big(X - VaR_{\alpha_1}(X)\big)_+, \xi_2 - \xi_1\big\} + \big(X - VaR_{\alpha_2}(X)\big)_+, & 0 \leq \gamma < \gamma_1, \\ \min\{X,\xi_1\} + \min\big\{\big(X - VaR_{\alpha_1}(X)\big)_+, \xi_2 - \xi_1\big\}, & \gamma_1 < \gamma \leq 1, \end{cases}$$

where $(\xi_1,\xi_2) \in \mathcal{A}_1$. Similarly, both cases lead to the same solution, which is $(0,0)$. Let us justify this for the case in which $\gamma_1 < \gamma \leq 1$, where the second stage optimisation problem is equivalent to

$$\begin{aligned} \min_{\mathcal{A}_1} H(\xi_1,\xi_2) \ :=\ & \lambda_2\xi_2 + (\lambda_1 - \lambda_2)\left(\int_0^{\xi_1} + \int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_1}(X)+\xi_2-\xi_1}\right)\bar{F}(x)\,dx \\ & -\lambda_1\left(\xi_1 + \frac{1}{1-\alpha_1}\int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_1}(X)+\xi_2-\xi_1}\bar{F}(x)\,dx\right). \end{aligned}$$

Taking the derivative with respect to ξ_2, we get that $H(\xi_1,\xi_2) \geq H\big(\xi_1,\xi_1\big)$ for all $(\xi_1,\xi_2) \in \mathcal{A}_1$, due to the fact that

$$\bar{F}\big(VaR_{\alpha_1}(X) + \xi_2 - \xi_1\big) \leq \bar{F}\big(VaR_{\alpha_1}(X)\big) \leq 1 - \alpha_1 \leq \frac{\lambda_2}{\lambda_2 + \lambda_1\alpha_1/(1-\alpha_1)}.$$

Finally, $H\big(\xi_1,\xi_1\big)$ is increasing in ξ_1, which completes the proof of Theorem 3.1.

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