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Abstract. Suppose that all nontrivial subsections of a $p$-block $B$ are conjugate (where $p$ is a prime). By using the classification of the finite simple groups, we prove that the defect groups of $B$ are either extraspecial of order $p^3$ with $p \in \{3, 5\}$ or elementary abelian.

1. Introduction

Let $p$ be a prime, and let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$ (cf. [1] and [8]). We call $\mathcal{F}$ transitive if any two nontrivial elements in $P$ are $\mathcal{F}$-conjugate. In this case, $P$ has exponent $\exp(P) \leq p$, and $\text{Aut}_\mathcal{F}(P)$ acts transitively on $Z(P) \setminus \{1\}$. This paper is motivated by the following:

Conjecture 1.1. (cf. [23]) Let $\mathcal{F}$ be a transitive fusion system on a finite $p$-group $P$ where $p$ is a prime. Then $P$ is either extraspecial of order $p^3$ or elementary abelian.

Moreover, if $P$ is extraspecial of order $p^3$ then results by Ruiz and Viruel [26] imply that $p \in \{3, 5, 7\}$. Note that the conjecture is trivially true for $p = 2$ since groups of exponent 2 are abelian. Thus Conjecture 1.1 is only of interest for $p > 2$. The aim of this paper is to prove the conjecture above for saturated fusion systems coming from blocks.

Theorem 1.2. Let $p$ be a prime, and let $B$ be a $p$-block of a finite group $G$ with defect group $P$. If the fusion system $\mathcal{F} = \mathcal{F}_P(B)$ of $B$ on $P$ is transitive then $P$ is either extraspecial of order $p^3$ or elementary abelian.

If $P$ is extraspecial of order $p^3$ then the results in [26] and [20] imply that $p \in \{3, 5\}$. We call a block $B$ with defect group $P$ and transitive fusion system $\mathcal{F}_P(B)$ fusion-transitive. Whenever $B$ has full defect then the theorem is a consequence of the results in [23]. In our proof of the theorem above, we will make use of the classification of the finite simple groups.

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2. Saturated fusion systems

We begin with some results on arbitrary saturated fusion systems.

**Proposition 2.1.** Let $p$ be a prime, and let $\mathcal{F}$ be a transitive fusion system on a finite $p$-group $P$ where $|P| \geq p^4$. Suppose that $P$ contains an abelian subgroup of index $p$. Then $P$ is abelian.

**Proof.** We assume the contrary. Then $p > 2$.

Suppose first that $P$ contains two distinct abelian subgroups $A, B$ of index $p$. Then $AB = P$, $A \cap B \subseteq Z(P)$ and $|P : A \cap B| = p^2$. Since $P$ is nonabelian we conclude that $|P : Z(P)| = p^2$. Thus $1 \neq P' \subseteq Z(P)$. Since $\text{Aut}_\mathcal{F}(P)$ acts transitively on $Z(P) \setminus \{1\}$, we conclude that $P' = Z(P)$. Hence there are $x, y \in P$ such that $P = \langle x, y \rangle$. Then $P' = \langle [x, y] \rangle$ (cf. III.1.11 in [17]); in particular, we have $|P'| = p$ and $|P| = p^3$, a contradiction.

It remains to consider the case where $P$ contains a unique abelian subgroup $A$ of index $p$. Let $Z$ be a subgroup of order $p$ in $Z(P)$, and let $B$ be an arbitrary subgroup of order $p$ in $A$. By transitivity, there is an isomorphism $\phi : B \to Z$ in $\mathcal{F}$. By definition, $Z$ is fully $\mathcal{F}$-normalised. Thus, by Proposition 4.20 in [8], $Z$ is also fully $\mathcal{F}$-automatised and receptive. Hence $\phi$ extends to a morphism $\psi : N_\phi \to P$ in $\mathcal{F}$. Since $|B| = p$, we have

$$A \subseteq N_P(B) = C_P(B) \subseteq N_\phi$$

(cf. p. 99 in [8]). Since $\psi(A)$ is also an abelian subgroup of index $p$ in $A$ we conclude that $\psi(A) = A$. Thus $\psi|A \in \text{Aut}_\mathcal{F}(A)$, and $\psi|A$ maps $B$ to $Z$. This shows that $\text{Aut}_\mathcal{F}(A)$ acts transitively on the set of subgroups of order $p$ in $A$.

In the following, we view $A$ as a vector space over $\mathbb{F}_p$ and $G := \text{Aut}_\mathcal{F}(A)$ as a subgroup of $\text{GL}(A)$. If $S$ denotes the group of scalar matrices in $\text{GL}(A)$ then $H := GS$ is a transitive subgroup of $\text{GL}(A)$. The transitive linear groups were classified by Herig (cf. [16] or Remark XII.7.5 in [18]). We are going to use the list in Theorem 15.1 of [27].

Before we do this, we observe the following. By the uniqueness of $A$, $A$ is fully $\mathcal{F}$-automatised, i.e. $P/A = N_P(A)/C_P(A) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(A))$. Thus $G = \text{Aut}_\mathcal{F}(A)$ and $H = GS$ both have a Sylow $p$-subgroup of order $p$.

Now we write $|A| = p^n$ and go through the list in Theorem 15.1 of [27]:

(i) $H \subseteq \Gamma \Gamma_1(p^n)$; in particular, $|H|$ divides $|\Gamma \Gamma_1(p^n)| = n(p^n - 1)$.

In this case we can identify $A$ with the finite field $L := \mathbb{F}_{p^n}$. Moreover, $P$ is the semidirect product of $L$ with $B = \langle \beta \rangle$ where $\beta$ is a field automorphism of $L$. For $x \in L$, we have $x, \beta \in P$ and

$$1 = (x \beta)^p = x \beta x \beta \ldots x \beta = x \beta(x) \beta^2(x) \ldots \beta^{p-1}(x) = N^L_K(x)$$

where $K$ is the fixed field of $\beta$. However, it is known that $N^L_K(L) = K$, a contradiction.
(ii) \( n = km \) where \( k \geq 2 \) and \( \text{SL}_k(p^m) \leq H \).
Since the Sylow \( p \)-subgroups of \( H \) have order \( p \), we conclude that \( m = 1 \) and \( k = 2 \).
Then \( n = 2 \) and \( |P| = p^3 \), a contradiction.
(iii) \( n = km \) where \( k \geq 4 \) is even and \( \text{Sp}_k(p^m) \leq H \).
Since \( p > 2 \) we have \( \text{Sp}_k(p^m) = \text{Sp}_k(p^m) \). Thus \( \text{Sp}_k(p^m) \) has a Sylow \( p \)-subgroup of order \( p^{k^2/4} \geq p^4 \), a contradiction.
(iv) \( n = 6m, p = 2 \) and \( G_2(2^m) \leq H \).
This case is impossible as \( p > 2 \).
(v) \( n = 2 \) and \( p \in \{5, 7, 11, 19, 23, 29, 59\} \).
Then \( |P| = p^3 \) which is again a contradiction.
(vi) \( n = 4, p = 2 \) and \( H \cong \mathfrak{S}_2 \).
This case is also impossible as \( p > 2 \).
(vii) \( n = 4, p = 3 \) and \( H \) is one of the groups in Table 15.1 of [27].
In this case we have \( |P| = 3^5 = 243 \). Then Proposition 15.12 in [27] leads to a contradiction.
(viii) \( n = 6, p = 3 \) and \( H \cong \text{SL}_2(13) \).
In this case we have \( |P| = 3^7 = 2187 \). However, one can check that \( P \) has exponent 9 in this case, a contradiction. \( \square \)

**Proposition 2.2.** Let \( P \) be a nonabelian \( p \)-group with a transitive fusion system.
Then \( P \) is indecomposable (as a direct product).

**Proof.** Let \( P = N_1 \times \cdots \times N_k \) be a decomposition into indecomposable factors \( N_i \neq 1 \).
Assume by way of contradiction that \( k \geq 2 \). Since \( P \) carries a transitive fusion system we have
\[
Z(N_1) \times \cdots \times Z(N_k) = Z(P) \subseteq P' = N'_1 \times \cdots \times N'_k.
\]
Let \( 1 \neq x \in Z(N_1) \). By hypothesis there exists \( \alpha \in \text{Aut}(P) \) such that \( \alpha(x) \in Z(P) \setminus (Z(N_1) \cup \ldots \cup Z(N_k)) \). By the Krull-Remak-Schmidt Theorem (see Satz I.12.5 in [17]) there is a normal automorphism \( \beta \) of \( P \) such that \( \beta(N_i) = \alpha(N_i) \) for some \( i \in \{1, \ldots, k\} \). In particular, there is \( g \in Z(N_i) \) such that \( \beta(g) = \alpha(x) \). By Hilfssatz I.10.3 in [17], for every \( g \in P \) there is a \( z_g \in Z(P) \) such that \( \beta(g) = gz_g \). Obviously the map \( P \to Z(P), g \to z_g \), is a homomorphism. Since \( Z(N_i) \subseteq N'_i \), we obtain \( z_g = 1 \). This gives the contradiction \( \alpha(x) = \beta(y) = y \in Z(N_i) \). \( \square \)

**Proposition 2.3.** Let \( P = \prod_{i=1}^{\infty} P_i^{a_i} \) where \( P_i = C_{p^{r_i}} \wr \ldots \wr C_{p^1} \) (i factors in the wreath product) and \( a_i \in \mathbb{N}_0, r_i \in \mathbb{N} \) for \( i \in \mathbb{N} \). Moreover, let \( U \) be a normal subgroup of \( P \) such that \( P/U \) is cyclic, and let \( Z \) be a cyclic subgroup of \( Z(U) \). Suppose that \( R := U/Z \) supports a transitive fusion system. Then \( R \) has order \( p^3 \) or is elementary abelian.

**Proof.** We assume the contrary. Then \( |R| \geq p^4 \) and \( p > 2 \).
Suppose first that \( r_j > 1 \) for some \( j > 1 \). Since \( p > 2 \), \( P' \) contains a subgroup isomorphic to \( C_{p^{r_j}} \times C_{p^{r_j}} \). Since \( P' \subseteq U \) we conclude that \( \exp(R) \geq p^2 \), a contradiction.
Thus \( r_j = 1 \) for \( j > 1 \), and \( P_j \) is the iterated wreath product of \( j \) copies of \( C_p \) in this case.

Suppose next that \( a_j > 0 \) for some \( j \geq 3 \). Since \( p > 2 \), \( P' \) contains a subgroup isomorphic to \( P_{j-1} \times P_{j-1} \). By Satz III.15.3 in [17], \( P_{j-1} \) has exponent \( p^{j-1} \geq p^2 \). Since \( P' \subseteq U \) we conclude that \( \exp(R) \geq p^2 \), a contradiction again.

Thus \( P = P_1^{a_1} \times P_2^{a_2} \) where \( P_1 = C_p^{a_1} \) and \( P_2 = C_p \). If \( a_2 \leq 1 \) then \( P \) and \( R \) contain abelian subgroups of index \( p \). In this case Proposition 2.2 gives a contradiction.

Hence we may assume that \( a_2 \geq 2 \). Let \( \pi : P \longrightarrow P_2^{a_2} \) be the relevant projection. Since \( \exp(P_2) = p^2 \) we cannot have \( \pi(U) = P_2^{a_2} \). On the other hand, \( P_2/P_2' \) is elementary abelian. Since \( P_2^{a_2}/\pi(U) \) is cyclic, \( \pi(U) \) is a maximal subgroup of \( P_2^{a_2} \). Let \( \pi_1 : P_2^{a_2} \longrightarrow P_2^{a_2-1} \) be the projection onto the direct product of the first \( a_2 - 1 \) copies of \( P_2 \), and let \( \pi_2 : P_2^{a_2} \longrightarrow P_2^{a_2} \) be the projection onto the direct product of the last \( a_2 - 1 \) copies of \( P_2 \).

Now suppose that \( a_2 \geq 3 \). Then an argument similar to the one above shows that \( \pi_1(\pi(U)) \) is a maximal subgroup of \( P_2^{a_2-1} = \pi_1(P_2^{a_2}) \). Thus \( \text{Ker}(\pi_1) \subseteq \pi(U) \) and, similarly, \( \text{Ker}(\pi_2) \subseteq \pi(U) \). Thus \( \pi(U) \) contains a subgroup isomorphic to \( P_2^{a_2} \). Hence \( \exp(R) \geq p^2 \), a contradiction.

We are left with the case \( a_2 = 2 \), i.e. \( P = A \times P_2 \times P_2 \) where \( A = P_1^{a_1} \cong C_p^{a_1} \) is abelian. Since \( \pi(U) \) is a maximal subgroup of \( P_2 \times P_2 \), we see that \( A \times \pi(U) \) is a maximal subgroup of \( P \). Let \( x \in P \) such that \( P = U \langle x \rangle \). Then \( U \langle x^p \rangle \subseteq A \times \pi(U) \). Since \( |P : U \langle x \rangle| \leq p \) we conclude that \( U \langle x^p \rangle = A \times \pi(U) \). Note that \( x^p \in \mathcal{U}(P) \subseteq Z(P) \).

Suppose that \( \exp(A) > p \), and choose an element \( a \in A \) of maximal order. We write \( x = x_1x_2 \) with \( x_1 \in A \) and \( x_2 \in P_2 \), we write \( a = ux^{pi} \) with \( u \in U \) and \( i \in \mathbb{Z} \), and we write \( u = u_1u_2 \) with \( u_1 \in A \) and \( u_2 \in P_2 \). Then \( a^p = u^px^{pi}u_1^px^{pi}u_2^px^{pi}u_1^px^{pi}u_2^p \). We conclude that \( u_2^p = 1 \) and \( a^p = u_1^px_1^{pi} \). Thus \( p < \exp(A) = |\langle a \rangle| = |\langle u_1 \rangle| = |\langle u \rangle| \), and \( 1 \neq u^p \in \mathcal{U}(U) \cap A \).

By Aufgabe III.15.36 in [17], the elements of order 1 or \( p \) form a union of two maximal subgroups. Thus \( P_2 \) contains \( p^{2p-2}(2p-1)^2 < p^{2p+1} \) elements of order 1 or \( p \). Hence \( \pi(U) \) contains elements of order \( p^2 \); in particular, \( \mathcal{U}(U) \) is noncyclic. Since \( \mathcal{U}(U) \subseteq Z \), this is a contradiction.

This contradiction shows that \( \exp(A) \leq p \), i.e. \( P = A \times P_2 \times P_2 \) where \( A \) is elementary abelian. Hence \( P/P' \) is elementary abelian. Since \( P/U \) is cyclic we conclude that \( U \) is a maximal subgroup of \( P \). Thus \( U = A \times \pi(U) \) and \( \mathcal{U}(U) \subseteq \pi(U) \). Since \( \pi(U) \) contains elements of order \( p^2 \), we have \( 1 \neq \mathcal{U}(U) \subseteq Z \). On the other hand, Satz III.15.4 in [17] implies that \( Z(U) \) is elementary abelian. Thus \( |Z| = p \) and \( Z = \mathcal{U}(U) \subseteq \pi(U) \). Since \( R \) supports a transitive fusion system we have

\[
AZ/Z \subseteq Z(U)/Z \subseteq Z(R) \subseteq R' = U'Z/Z = \pi(U)'Z/Z \subseteq \pi(U)/Z.
\]
Therefore \( A = 1 \), i.e. \( P = P_2 \times P_2 \). Recall that \( U \) is a maximal subgroup of \( P \) and that \( \pi_1, \pi_2 : P \to P_2 \) denote the two projections. Without loss of generality we have \( \pi_1(U) = P_2 \). Since \( \mathcal{U}(U) \) is cyclic, \( K_1 := \text{Ker}(\pi_1) \) has order \( p^q \) and exponent \( p \).

If \( \pi_2(U) \neq P_2 \) then \( U = P_2 \times \pi_2(U) \) and \( \exp(\pi_2(U)) = p \). Thus \( Z = \mathcal{U}(U) \subseteq P_2 \times 1 \) and \( R \cong P_2 / Z \rtimes \pi_2(U) \), a contradiction to Proposition 2.2.

Thus we must also have \( \pi_2(U) = P_2 \). Then also \( K_2 := U \cap \text{Ker}(\pi_2) \) has order \( p^q \) and exponent \( p \). Moreover, we have \( K_1 \times K_2 \subseteq U \).

We may choose elements \( x, y \in U \) such that \( \pi_1(x) \) and \( \pi_2(x) \) have order \( p^2 \). Since \( \langle x^p \rangle = Z = \langle y^p \rangle \) we see that \( \pi_2(x) \) and \( \pi_1(y) \) have order \( p^2 \). However, we may choose \( y \) such that \( yK_1 \) contains an element \( y' \) such that \( \pi_2(y') \) has order \( p \). Since \( \pi_1(y) = \pi_1(y') \) still has order \( p^2 \), we have a final contradiction.

\[ \Box \]

3. Blocks

We now present the proof of Theorem 1.2.

Proof. Suppose that the result is false. Then \( P \) is nonabelian with \( |P| \geq p^4 \) and \( p > 2 \).

By [11, Proposition IV.6.3] we may assume that \( B \) is quasiprimitive. This means that, for any normal subgroup \( H \) of \( G \), \( B \) covers a unique \( p \)-block of \( H \).

Now let \( H \) be a normal subgroup of \( G \), and let \( b \) be the unique \( p \)-block of \( H \) covered by \( B \). Suppose that \( P \cap H = 1 \). (This is satisfied, for example, whenever \( H \) is a \( p' \)-subgroup.) Then \( b \) has defect zero. By Clifford theory, there exist a finite group \( G^* \), a central \( p' \)-subgroup \( H^* \) of \( G^* \), and a \( p \)-block \( B^* \) of \( G^* \) with defect group \( P^* \cong P \)

such that \( \mathcal{F}_{p'}(B^*) \) is equivalent to \( \mathcal{F} \). Thus we may replace \( G \) by \( G^* \) and \( B \) by \( B^* \).

Repeating the argument above we may therefore assume that every normal subgroup \( H \) of \( G \) with \( P \cap H = 1 \) is central. In particular, we have \( O_{p'}(G) \subseteq Z(G) \).

It is well-known that \( M := O_p(G) \subseteq P \). Suppose first that \( M \neq 1 \). Since \( \mathcal{F} \) is transitive this implies \( M = P \). Then \( \Phi(P) \) is a normal subgroup of \( G \) and properly contained in \( P \). Since \( \mathcal{F} \) is transitive, we must have \( \Phi(P) = 1 \). Thus \( P \) is elementary abelian in this case.

Hence, in the following, we may assume that \( O_p(G) = 1 \). Then \( F(G) = O_{p'}(G) = Z(G) \). Moreover, the layer \( E(G) \) is nontrivial. Let \( b \) be the unique \( p \)-block of \( E(G) \) covered by \( B \). Then \( b \) has defect group \( P \cap E(G) \neq 1 \). Since \( B \) is transitive, this implies that \( P \subseteq E(G) \).

Let \( L_1, \ldots, L_n \) denote the components of \( G \). Then \( E(G) = L_1 \ast \cdots \ast L_n \) is a central product. For \( i = 1, \ldots, n \), the unique \( p \)-block \( b_i \) of \( L_i \) covered by \( b \) has defect group \( P_i := P \cap L_i \neq 1 \). Moreover, we have \( P = P_1 \times \cdots \times P_n \). Since \( \mathcal{F} \) is transitive, this implies that \( n = 1 \). Thus \( E(G) = L_1 =: L \) is quasisimple, and \( G/Z(G) \) is isomorphic to a subgroup of \( \text{Aut}(L) \).

If \( |P| = p^4 \) then Proposition 15.14 in [27] gives a contradiction. Thus we may assume that \( |P| \geq p^5 \); in particular, \( |L| \) is divisible by \( p^5 \). If \( P \) is a Sylow \( p \)-subgroup
of $G$ then the results of [23] imply our theorem. Hence we may assume that $|G|$ is divisible by $p^6$.

We now make use of the classification of the finite simple groups and discuss the various possibilities for the simple group $F^* (G)/Z(G) \cong L/Z(L)$. Since $F$ is transitive we have $C_L(u) \cong C_L(v)$ for any $u, v \in P \setminus \{1\}$. This will be a very useful fact.

It can be checked with GAP [13] that $L/Z(L)$ cannot be a sporadic simple group. Similarly, $L/Z(L)$ cannot be a simple group with an exceptional Schur multiplier.

Suppose that $L = \mathfrak{A}_n$ is an alternating group. Then $P$ is a defect group of a $p$-block of $\mathfrak{A}_n$. Hence $P$ is also a defect group of a $p$-block of the symmetric group $\mathfrak{S}_n$. Thus $P$ is a direct product of (iterated) wreath products of groups of order $p$. Since $C_p \wr C_p$ has exponent $p^2$ we conclude that $P$ is a direct product of groups of order $p$, and the result follows in this case.

Suppose next that $L = \mathfrak{A}_n$ is the 2-fold cover of $\mathfrak{A}_n$. We may assume that $b$ is a faithful block of $\mathfrak{A}_n$. In this case the defect groups of $b$ have a similar structure as those in $\mathfrak{A}_n$ (cf. [24, Theorem 5.8.8]), so we are done here by the same argument.

Suppose now that $L/Z(L)$ is a group of Lie type in characteristic $p$. Then the $p$-block $b$ of $L$ has full defect, i.e. $P$ is a Sylow $p$-subgroup of $L$. Since $F$ is transitive, every nontrivial element $u \in P$ is conjugate in $G$ to an element $v \in Z(P)$. Thus $|L : C_L(u)| = |L : C_L(v)|$ is not divisible by $p$. Therefore the results in [25] imply that $P$ is abelian.

Finally suppose that $L/Z(L)$ is a group of Lie type in characteristic $r \neq p$. First we deal with the exceptional groups of Lie type. Let $S \in \text{Syl}_p(L)$. By §10.1 in [14], $S$ contains an abelian normal subgroup $N$ such that $S/N$ is isomorphic to a subgroup of the Weyl group of $L/Z(L)$. If $|S/N| \leq p$, then Proposition 2.1 gives a contradiction. This already implies the claim for $p \geq 7$. Now let $p = 5$. Then by the same argument we may assume that $L/Z(L) \cong E_6(q)$ where $q \equiv \pm 1 \pmod{5}$. This case will be handled in Section 6. Now let $p = 3$. Here we need to discuss the following groups: $F_4$, $E_6$, $E_7$, and $E_8$. For $L/Z(L) \cong F_4(q)$ we have $|P| \leq p^6$ and the result follows by Proposition 15.13 in [27]. The remaining cases will be discussed in Section 6.

We may therefore assume that $L/Z(L)$ is a classical group. In this case our theorem follows from the results of the next section.

4. Classical Groups in non-describing characteristic

We keep the notation of the previous section. We suppose in this section that $L/Z(L)$ is a simple group of Lie type in characteristic $r$, $r \neq p$. Let $q$ be a power of $r$. Suppose that $L = L^F/Z$, where $L$ is a simple simply connected algebraic group defined over an algebraic closure $\bar{F}_q$ of a field $F_q$ of $q$ elements, $F : L \rightarrow L$ a Frobenius morphism with respect to an $F_q$-structure on $L$ and $Z$ is a central subgroup of $L^F$. Note that by the classification of finite simple groups, we may assume that if $q$ is a
power of 2, then \( L \) is not of type \( C_n \). Let \( \bar{b} \) be the block of \( L^F \) dominating \( b \) and \( \bar{P} \) be a defect group of \( \bar{b} \) such that \( \bar{P}Z/Z = P \).

We define groups \( H \) as follows. If \( L/Z(L) = B_n(q) \), then \( H = \text{SO}_{2n+1}(\mathbb{F}_q) \). If \( L/Z(L) = C_n(q) \), then \( H = \text{Sp}_{2n}(\mathbb{F}_q) \). If \( L/Z(L) = D_n^\pm(q) \), then \( H = \text{SO}_{2n}(\mathbb{F}_q) \). Here, if \( q \) is a power of 2, and \( L \) is of type \( B_n \), then by \( \text{SO}_{2n}(\mathbb{F}_q) \) we mean the adjoint simple group of type \( B_n \). If \( q \) is a power of 2 and if \( L \) is of type \( D_n \), then by \( \text{SO}_{2n}(\mathbb{F}_q) \) we mean the simple algebraic group of type \( D_n \) corresponding to the root datum \((X, \Phi, Y, \Phi^\vee)\) for which the fundamental roots are \( e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n \) and \( X = \left\{ \sum_{i=1}^n a_\iota : a_\iota \in \mathbb{Z} \right\} \) for an orthonormal basis, \( e_1, e_2, \ldots, e_n \), of \( n \)-dimensional Euclidean space. We may and will assume that \( H \) is an \( F \)-stable quotient of \( L \).

**Proposition 4.1.** Suppose that \( p \) is an odd prime and \( L/Z(L) \) is a classical group in non-describing characteristic different from triality \( D_4 \). Suppose that \( B \) is a fusion-transitive block with \( P \) of order at least \( p^5 \). Then \( P \) is abelian.

**Proof.** Suppose that \( L/Z(L) \) is the projective special linear group \( \text{PSL}_n(q) \), so \( L = \text{SL}_n(\mathbb{F}_q) \) and \( L = \text{SL}_n(q) \). Let \( D \) be a defect group of a block of \( \text{GL}_n(q) \) covering \( \bar{b} \) such that \( \bar{P} = D \cap \text{SL}_n(q) \). By the results of Fong and Srinivasan on blocks of finite general linear groups [12, Theorem (3C)], \( D \) is isomorphic to the Sylow \( p \)-subgroup of a direct product of general linear groups over finite extensions of \( \mathbb{F}_q \). Since \( Z(L) \) and \( D/P \) are cyclic, the claim follows from Proposition 2.3. The case that \( L/Z(L) \) is the projective special unitary group can be handled similarly.

Now consider the case that \( L/Z(L) \) is of type \( B, C \) or \( D \). Then \( \bar{P} \) is a defect group of \( L^F \). Let \( 1 \neq z \in Z(\bar{P}) \). Since \( p \) is odd, \( C_L(z) \) is a Levi subgroup of \( L \). For any subset \( A \) of \( L \), denote by \( \overline{A} \) the image of \( A \) under the isogeny from \( L \) onto \( H \) and denote by \( U \) the kernel of the isogeny. Since \( U \) is a central 2-subgroup of \( L \), \( C_L(z) = C_H(\overline{z}) \).

The group \( C_H(\overline{z}) \) is a direct product

\[
C_H(\overline{z}) = H_0 \times \cdots \times H_r,
\]

where \( H_0 \) is either the identity or a classical group and for \( i \geq 1 \), \( H_i \) is a direct product of general linear groups with \( F \) transitively permuting the factors. This follows easily from the standard description of the root datum of \( H \). So,

\[
C_H(\overline{z})^F = H_0^F \times \cdots \times H_r^F,
\]

where \( H_i^F \) is a finite general linear or unitary group for \( i \geq 1 \) and \( H_0^F \) is a finite classical group (possibly the identity).

Let \( L_i \) be the inverse image in \( C_L(z) \) of \( H_i \), \( 0 \leq i \leq r \). Then \( L_i \) is a normal \( F \)-stable subgroup of \( C_L(z) \), \( C_L(z) = L_0 \cdots L_r \) and

\[
[L_i, L_0 \cdots L_{i-1}L_{i+1} \cdots L_r] \leq L_i \cap (L_0 \cdots L_{i-1}L_{i+1} \cdots L_r) = U.
\]
We claim that \( L_i^F \) is a normal subgroup of \( H_i^F \) of 2-power index. Indeed, let \( M \) be the inverse image in \( L_i \) of \( H_i^F \). Then \( M \) is \( F \)-stable since \( U \) is \( F \)-stable. Further, \([M,F] \leq U\). Since \( U \) is central in \( M \), the map \( M \to U \) defined by \( x \to x^{-1}F(x) \) is a group homomorphism. The kernel of this map is \( L_i^F \) whence \( L_i^F \) is a normal subgroup of \( M \) and the index of \( L_i^F \) in \( M \) divides \( |U| \). The claim follows since \( U \) is a 2-group.

The claim implies that \( L_0^F \cdots L_r^F \) is a normal subgroup of 2-power index of \( C_L(z)^F \). So, \( \tilde{P} \) is a defect group of \( L_0^F \cdots L_r^F \). The commutator relationship given above then implies that \( \tilde{P} \) is a direct product \( P_0 \cdots P_r \), where \( P_i \) is a defect group of \( L_i^F \), \( 0 \leq i \leq r \). By Proposition 2.2 \( \tilde{P} = P_i \) for some \( i \), \( 1 \leq i \leq r \). Since \( z \) is central in \( C_L(z), i \geq 1 \) and \( H_i^F \) is a general linear or unitary group with a central \( p \)-element. Let \( R = \tilde{P} \cap [L_i, L_i]^F \), a defect group of \([L_i, L_i]^F \). By suitably replacing \( \tilde{P} \) by an \( L_i^F \)-conjugate, we may assume that the relevant block of \([L_i, L_i]^F \) is \( \tilde{P} \)-stable and hence that \( \tilde{P} \) is a defect group of \([L_i, L_i]^F \). The isogeny \( L_i \to H_i \) restricts to an isogeny \([L_i, L_i] \to [H_i, H_i] \) with kernel \( U \cap [L_i, L_i] \). However \([H_i, H_i] \) is a simply connected semisimple group, being the direct product of special linear groups. Thus, \( U \cap [L_i, L_i] = 1 \) and the restriction of the isogeny to \([L_i, L_i] \) is an abstract group isomorphism from \([L_i, L_i] \) to \([H_i, H_i] \) which commutes with \( F \). Consequently, \([L_i, L_i]^F \cong [H_i, H_i]^F \). Also, \( U \cap [L_i, L_i] = 1 \) and the induced map \([L_i, L_i]^F \to H_i^F \) is injective. Thus \( \overline{\tilde{P}} \cong \overline{\tilde{P}} \cong \overline{P} \) is a defect group of \([L_i, L_i]^F \) and \([H_i, H_i]^F \). Since \( H_i^F \) is a finite general linear or unitary group, the result now follows from [12] Theorem (3C) and Proposition 2.3 in the same way as for the case that \( L/Z(L) \) is a projective special linear or unitary group.

\[\Box\]

5. On \( A_{p-1} \)-COMPONENTS

**Lemma 5.1.** Suppose that \( p \) is an odd prime and let \( G \) be a finite group isomorphic to one of the groups \( \text{SL}_p(q) \) or \( \text{SU}_p(q) \) for some prime power \( q \) not divisible by \( p \). Let \( U \) be a non-abelian \( p \)-subgroup of \( G \). Then \( U \) contains a normal abelian subgroup \( U_0 \) of index \( p \) such that any element of \( U \setminus U_0 \) has order \( p \). If \( |U| \geq p^{p+1} \), then \( U_0 \) contains an element of order \( p^2 \).

**Proof.** First, consider the case that \( G \) is special linear or unitary. By replacing \( q \) if necessary by some power we may assume that \( U \leq \text{SL}_p(q) \) and \( p \) divides \( q-1 \). Let \( S_0 \) be the Sylow \( p \)-subgroup of the group of diagonal matrices of \( \text{SL}_p(q) \) and let \( \sigma \) be a non-diagonal, monomial matrix in \( \text{SL}_p(q) \) of order \( p \). Then \( S := \langle S_0, \sigma \rangle \) is a Sylow \( p \)-subgroup of \( \text{SL}_p(q) \), \( S_0 \) is normal in \( S \), abelian, of index \( p \) in \( S \), rank \( p-1 \) and any element of \( S \) not in \( S_0 \) has order \( p \). Let \( U_0 = U \cap S_0 \). Then \( U_0 \) has index at most \( p \) in \( U \). On the other hand, since \( U \) is non-abelian and \( S_0 \) is abelian, \( U \) is not contained in \( U_0 \). Thus \( U_0 \) has index \( p \) in \( U \), proving the first assertion. Now suppose that \( U \) has exponent \( p \). Then \( U_0 \) is elementary abelian. On the other hand, \( U_0 \leq S_0 \) and the \( p \)-rank of \( S_0 \) is \( p-1 \). Hence, \( |U| = p|U_0| \leq p^2 \).
In the rest of this section, $p$ will denote a fixed prime and $G$ will denote a connected reductive group in characteristic $r \neq p$ with a Frobenius morphism $F$ with respect to some $\mathbb{F}_{p^r}$ structure for some power $r'$ of $r$. In what follows, whenever we talk of a component of $G$, we will mean a simple component of $[G, G]$.

We need a slight variation of the previous lemma.

Lemma 5.2. Suppose that $p$ is odd. If $[G, G] = SL_p$, then any $p$-subgroup of $G^F$ has an abelian subgroup of index $p$.

Proof. Since $G = Z^0(G)[G, G]$ any element and hence any subgroup of $G^F$ is contained in $Z^0(G)^{F^d}[G, G]^{F^d}$ for some $d \geq 1$. This can be seen as follows. Since $G = Z^0(G)[G, G]$, any element $u$ of $G$ can be written in the form $u = xy$, where $x \in Z^0(G)$ and $y \in [G, G]$. Let $\iota : G \to GL_n$ be an embedding. Then for some power, say $F^t$ of $F$, some power, say $s$ of $r$, and for all $g \in G$, $F^t \circ \iota(g) = F^s(\iota(g))$ where $F^s$ is the standard Frobenius morphism of $GL_n$, raising every matrix entry to the $s$-th power. The claim follows since for any $h \in GL_n$, $F^m(h) = h$ for some natural number $m$. Since any Sylow $p$-subgroup of $Z^0(G)^{F^d}[G, G]^{F^d}$ is of the form $R_1 R_2$, where $R_1$ is a Sylow $p$-subgroup of $Z^0(G)^{F^d}$ and $R_2$ is a Sylow $p$-subgroup of $[G, G]^{F^d}$, the result follows from the previous Lemma and the fact that $R_1$ is central in $R_1 R_2$. □

Lemma 5.3. Suppose that $p$ is odd. Let $X = SL_p$ be an $F$-stable component of $G$ such that $X^F$ has a central element of order $p$ and let $Y$ be the product of all other components of $G$ and $Z^0(G)$. Let $P$ be a $p$-subgroup of $G^F$ such that $P \cap X^F$ is non-abelian of order at least $p^s$ and $P$ is not contained in $X^F Y^F$. Then there exists an element of order $p^2$ in $P$. Further, if $Z$ is a central subgroup of $G^F$ of order $p$ such that $P/Z$ has exponent $p$, then $Z \leq X^F$.

Proof. Let $\bar{P}$ be the inverse image of $P$ under the surjective group homomorphism $X \times Y \to G$ induced by multiplication. The kernel of the multiplication map is isomorphic to $X \cap Y = Z(X) \cap Z(Y)$. Since $X$ is a simple group of type $A_{p-1}$, the kernel of the multiplication map is a group of order $p$ and in particular, $\bar{P}$ is a finite $p$-group. Let $P_1 \leq X$ be the image of $\bar{P}$ under the projection of $X \times Y \to X$. Clearly $P_1$ contains $P \cap X^F$. We claim that $P \cap X^F$ is proper in $P_1$. Indeed, otherwise $\bar{P} \leq (P \cap X^F) \times Y$, whence $\bar{P} \leq (P \cap X^F) Y$. This implies that $P \leq (P \cap X^F)(P \cap Y^F) \leq P \cap X^F Y^F$, a contradiction. Since $P \cap X^F$ is assumed to have order at least $p^s$, the claim implies that $|P_1| \geq p^{s+1}$.

Now $P_1$ is a finite subgroup of $X$, thus of some finite special linear (or unitary) group. Hence, by Lemma 5.1 there exists an element $x \in P_1$ of order $p^2$. Let $y \in Y$ be such that $w = xy \in P$. Since $P \cap X^F$ is non-abelian again by Lemma 5.1, there exists $\sigma \in P \cap X^F$ such that $x \sigma$ has order $p$. Then $w$ and $w \sigma \in P$, $w^p = x^p y^p$ and $(w \sigma)^p = y^p$. Then either $w^p \neq 1$ or $(w \sigma)^p \neq 1$, proving the first part of the result.
Suppose that $P/Z$ has exponent $p$. Then, $w^p, (w\sigma)^p$ are in $Z$. Hence $x^p \in Z$. Since $1 \neq x^p$ and $Z$ has order $p$ the second assertion follows.

**Lemma 5.4.** Let $\mathcal{X}$ be an $F$-stable subset of components of $G$. Let $X$ be the product of all elements of $\mathcal{X}$ and let $Y$ be the product of $Z^\circ(G)$ and all the components of $[G,G]$ not in $\mathcal{X}$.

(i) Let $P$ be a defect group of a block $b$ of $G^F$. Then $P \cap X^FY^F$ is a defect group of a block of $X^FY^F$ covered by $b$ and is of the form $P_1P_2$, where $P_1$ is a defect group of a block of $X^F$ covered by $b$ and $P_2$ is a defect group of a block of $Y^F$ covered by $b$. If $Z(X)^F \cap Z(Y)^F$ has $p'$-order, then $P = P_1P_2$ and the product is direct.

(ii) Let $c$ be a $p$-block of $X^FY^F$. Then the index of the stabiliser of $c$ in $G^F$ is prime to $p$. Suppose further that $Z(X)^F \cap Z(Y)^F$ is a $p'$-group. Then $c$ is $G^F$-stable, $c$ is covered by a unique block of $G^F$ and if $P$ is a defect group of the block of $G^F$ covering $c$, then $P \cap X^FY^F$ is a defect group of $c$ and $P/(P \cap X^FY^F) \cong G^F/X^FY^F$.

**Proof.** The first statement of (i) follows from the theory of covering blocks as $X^FY^F$ is a normal subgroup of $G^F$, $X^F$ and $Y^F$ centralise each other and $X^F \cap Y^F = Z(X)^F \cap Z(Y)^F \subseteq Z(G)^F$ is central in $X^FY^F$. The second assertion of (i) follows from the first assertion, the fact that $|G^F| = |X^F||Y^F|$ and $X^F \cap Y^F = Z(X)^F \cap Z(Y)^F$.

We now prove (ii). Let $u \in G^F$ be a $p$-element. Then $u = xy$, with $x \in X$ and $y \in Y$ such that $x^{-1}F(x) = yF(y^{-1})$ is an element of $Z(X) \cap Z(Y)$. We may assume without loss of generality that $x$ and $y$ are $p$-elements. The block $c$ of $X^FY^F$ is a product $c_1c_2$ of blocks $c_1$ of $X^F$ and $c_2$ of $Y^F$. Thus, it suffices to prove that $x_{c_1} = c_1$ and $y_{c_2} = c_2$.

Now consider a regular embedding $X \leq \tilde{X}$, where $\tilde{X}$ is a connected reductive group with connected centre containing $X$ as a closed subgroup, such that $[\tilde{X}, \tilde{X}] = [X,X]$ and such that $F$ extends to a Frobenius morphism of $\tilde{X}$. Since $x^{-1}F(x) \in Z(X) \leq Z^\circ(\tilde{X})$, $x = x_1z$ for some $x_1 \in \tilde{X}^F$, and $z \in Z^\circ(\tilde{X})$. We may assume also that $x_1$ is a $p$-element. Then $x_{c_1} = x_{1c_1}$. On the other hand, $c_1$ contains an ordinary irreducible character $\chi$ in a Lusztig series corresponding to a semisimple element of order prime to $p$ in the dual group of $X$, hence the index in $\tilde{X}^F$ of the stabiliser in $\tilde{X}^F$ of $\chi$ has order prime to $p$ (see for instance [3 Corollaire 11.13]). This proves the first assertion. If $Z(X)^F \cap Z(Y)^F$ is a $p'$-group, then $|G^F/X^FY^F| = |Z(X)^F \cap Z(Y)^F|$ is a power of $p$. By the first assertion, $c$ is $G^F$-stable and by standard block theory, there is a unique block of $G^F$ covering $c$. The second assertion of (ii) now follows from (i).

**Lemma 5.5.** Suppose that $p$ is odd. Let $X$ be an $F$-stable component of $G$ of type $A_{p-1}$ and let $Y$ be the product of all other components of $G$ and $Z^\circ(G)$. Suppose that
$Z(X)^F \cap Z(Y)^F \neq 1$ and that $P$ is a defect group of $G^F$ such that $P \cap X^F$ is abelian. Then there exists an $F$-stable torus $T$ of $X$ such that $P$ is a defect group of $(YT)^F$.

Proof. In the proof, we will identify blocks with the corresponding central primitive idempotents. Let $b$ be a block of $G^F$ with $P$ as defect group and let $P_0 := P \cap X^F Y^F$. The hypothesis implies that $|Z(X)^F \cap Z(Y)^F| = p$. So, by Lemma 5.3 $b$ is a block of $X^F Y^F$, $P_0$ is a defect group of $b$ as block of $X^F Y^F$ and $P/P_0$ is isomorphic to $G^F / X^F Y^F$. Let $b = b_1 b_2$, where $b_1$ is the block of $X^F$ covered by $b$ and $b_2$ is the block of $Y^F$ covered by $b$.

Let $u \in P$ generate $P$ modulo $P_0$ and write $u = xy$, $x \in X$, $y \in Y$. Since $u$ is a $p$-element, we may assume that both $x$ and $y$ are $p$-elements.

Now consider an $F$-compatible regular embedding of $X$ in $\tilde{X}$ such that $\tilde{X}^F$ is a finite general linear (or unitary) group. Since $Z(\tilde{X})$ is connected, there exists $z \in Z^0(\tilde{X})$ such that $g := xz^{-1} \in \tilde{X}^F$. Further, we may choose $z$ such that $g$ is a $p$-element.

Since $u = xy$ normalises $P_1$, $x$ normalises $P_1$ and therefore $g$ normalises $P_1$. Therefore $S = \langle P_1, g \rangle \leq \tilde{X}^F$ is a $p$-group. Since $u$ normalises $b_1$ it also follows that $b_1$ is $S$-stable.

We claim that there exists a block of $\tilde{X}^F$ covering $b_1$ with a defect group $D$ containing $S$. Indeed, in order to prove the claim, it suffices to prove that $Br_S(b_1) \neq 0$. Since $b_1$ and $b_2$ are both $G^F$-stable,

$$0 \neq Br_P(b) = Br_P(b_1) Br_P(b_2)$$

and consequently $Br_P(b_1) \neq 0 \neq Br_P(b_2)$. Hence writing $b_1 = \sum_{v \in X^F} \alpha_v v$ as an element of the modular group algebra of $X^F$ there exists $v \in X^F$ with $\alpha_v$ non-zero such that $v$ centralises $P$ and in particular $v$ centralises $P_1$ and $u$. Since $z$ is central, and $y$ centralises $X$, we have that $v$ also commutes with $g$. Hence $v$ centralises $S$ and it follows that $Br_S(b_1) \neq 0$, proving the claim.

By the block theory of finite general linear (or unitary) groups (see [12]; noting that $p$ divides $q - 1$ in the linear case and that $p$ divides $q + 1$ in the unitary case) $D$ is a Sylow $p$-subgroup of the centraliser of some semisimple element of $\tilde{X}^F$. Since by hypothesis $P_1 = D \cap X^F$ is abelian, we have that $D$ is abelian, hence $D$ is the Sylow $p$-subgroup of $T^F$ for some $F$-stable maximal torus $T$ of $\tilde{X}$. Set $T = X \cap T$, an $F$-stable maximal torus of $X$. Then $P_1 = D \cap X^F$ is a Sylow $p$-subgroup of $T^F$. Now $g = xz \in S \leq D \leq \tilde{T}$, and $z \in T$ (as $z$ is central), hence $x = gz^{-1} \in T \cap X = T$.

Set $G_0 = TY$. We have $u = xy \in G_0^F$. Since $X \cap Y \leq Z(X) \leq T$, we have that $G_0^F \cap X^F Y^F = T^F Y^F$ and $G_0^F / T^F Y^F$ is isomorphic to a subgroup of $G^F / X^F Y^F$ and in particular has order $p$. Hence $G_0^F = \langle T^F Y^F, u \rangle$. Let $e$ be a block of $T^F$ such that $eb_2 \neq 0$. Since $T^F$ and $Y^F$ commute, $eb_2$ is a block of $T^F Y^F$. Since $T$ is central in $G_0$, $e$ is $G_0^F$-stable. Further, $b_2$ is $P$-stable hence $b_2$ is $G_0^F$-stable. So $eb_2$ is a $G_0^F$-stable block of $T^F Y^F$ and therefore a block of $G_0^F$. Since $P_1$ is the Sylow $p$-subgroup of $T^F$ and $T^F$ is abelian, $P_1$ is the defect group of $e$ and $P_2$ is a defect group of $b_2$. Thus, $P_1 P_2$ is a defect group of $eb_2$ as block of $T^F Y^F$. Since
\[ \text{Br}_P(eb_2) = \text{Br}_P(e)\text{Br}_P(b_2) \] is non-zero, it follows by order considerations that \( P \) is a defect group of \( eb_2 \).

6. The case \( p = 3,5 \)

In this section we handle the remaining exceptional groups of Lie type for \( p \leq 5 \).

**Lemma 6.1.** Let \( G, H \) be finite groups, \( B \) a \( p \)-block of \( G \) and \( C \) a \( p \)-block of \( H \) such that \( B \) and \( C \) are Morita equivalent. Let \( P \) be a defect group of \( B \), and \( Q \) a defect group of \( C \). Suppose that \( P \) has exponent \( p \). Then \( P \) is abelian if and only if \( Q \) is abelian. Further, \( P \) has an abelian subgroup of index \( p \) if and only if \( Q \) has an abelian subgroup of index \( p \).

**Proof.** By [21, Satz J], the exponent of defect groups is an invariant of Morita equivalence, hence \( Q \) has exponent \( p \). In particular any abelian subgroup of \( P \) or of \( Q \) is elementary abelian. The remaining statements follow by the fact that Morita equivalence preserves the rank of the corresponding defect groups (see [2, Theorem 2.6]). □

**Lemma 6.2.** Let \( L \) be connected reductive, with Frobenius morphism \( F \), and let \( Z \) be a central \( p \)-subgroup of \( L \). Let \( b \) be a block of \( L^F \) and \( P \) a defect group of \( b \). Suppose that \( P/Z \) supports a transitive fusion system and \( |P/Z| \geq p^4 \). Let \( H \) be an \( F \)-stable Levi subgroup of \( L \), let \( c \) be a Bonnafé-Rouquier correspondent of \( b \) in \( H \) and let \( Q \) be a defect group of \( c \). Then \( Q/Z \) has exponent \( p \) and \( Q/Z \) does not have an abelian subgroup of index \( p \). In particular, a Sylow \( p \)-subgroup of \( H \) does not have an abelian subgroup of index \( p \).

**Proof.** Let \( \bar{b} \) be the block of \( L^F/Z \) dominated by \( b \) and let \( \bar{c} \) be the block of \( H^F/Z \) dominated by \( c \). By [10, Prop. 4.1], \( \bar{b} \) and \( \bar{c} \) are Morita equivalent. Further, \( P/Z \) is a defect group of \( \bar{b} \) and \( Q/Z \) is a defect group of \( \bar{c} \). The result now follows from Lemma 2.1 and Lemma 6.1. □

**Proposition 6.3.** Let \( L \) be connected reductive, in characteristic \( r \neq p = 3 \) with Frobenius morphism \( F \), and suppose that \([L,L]\) is simply connected of type \( E_6 \) in characteristic \( r \neq 3 \). Let \( Z \) be a cyclic subgroup of \( Z(L^F) \) of order 1 or 3 and let \( P \) be a defect group of \( L^F \). Suppose that \( P/Z \) supports a transitive fusion system and \( |P/Z| \geq 3^7 \). Suppose further that either \( Z = 1 \) or that \( L \) is simple. Then \( P/Z \) is abelian.

**Proof.** Suppose that \( P/Z \) is non-abelian. Let \( H \) be an \( F \)-stable Levi subgroup of \( L \) and \( c \) a block of \( H^F \) such that \( c \) is quasi-isolated and \( b \) and \( c \) are Bonnafé-Rouquier correspondents. Let \( s \in H^* \) be a semisimple label of \( c \) (and \( b \)). Since \( b \) and \( c \) are Bonnafé-Rouquier correspondents, \( C_{L^*}(s) = C_{H^*}(s) \). Let \( Q \) be a defect group of \( c \). By Lemma 6.2 we may assume that \( Q/Z \) has exponent 3 and does not have an abelian subgroup of index 3. Note that all components of \( L \) and hence of \( H \) are simply connected.
If $H^F$ has a component of type $D_4$ or $D_5$, then the only other possible components are of type $A_1$. We get a contradiction by Lemma 5.4(i), Lemma 5.2 and the fact that finite groups of type $D_4(q)$, $D_5(q)$, $2D_4(q)$, $2D_5(q)$ and $3D_4(q)$ have a Sylow 3-subgroup with an abelian subgroup of index 3.

Thus, either all components of $H$ are of type $A$ or $H$ has a component of type $E_6$. Let us first consider the case that all components of $H$ are of type $A$. In particular, $C_{H^*}(s)$ is a Levi subgroup of $H^*$ and since $s$ has order prime to 3, $C_{L^*}(s) = C_{H^*}(s)$ is connected. It follows that $s$ is central in $H^*$, hence that $Q$ is a defect group of a unipotent block of $H^F$.

Suppose that $H$ has a component $X$ of type $A_5$. Then $X$ is $F$-stable and is the only component of $H$. If $X^F$ does not contain a central element of order 3, then by Lemma 5.4(i), a Sylow 3-subgroup of $H^F$ is a direct product of a Sylow 3-subgroup of $X^F$ with the Sylow 3-subgroup of $Z^s(H^F)$. Furthermore in this case a Sylow 3-subgroup of $X^F$ has an abelian subgroup of index 3. If $X^F$ contains a central element of order 3, then by [3] Proposition 3.3 and Theorem, the principal block is the only unipotent block of $X^F$, and it follows that $Q/Z$ has an element of order 9 since $PSL_6$ (respectively $PSU_6(q)$) has elements of order 9 if $3 \mid q - 1$ (respectively $3 \mid q + 1$).

Suppose that $H$ has a component of type $A_4$. Then the only other possible component is of type $A_1$ and it follows from Lemma 5.4(i) that a Sylow 3-subgroup of $H^F$ has an abelian subgroup of index 3.

Suppose that $H$ has a component $X$ of type $A_3$. If all other components are of type $A_1$, then the above argument applies. If $H$ has a component of type $A_2$, say $Y$, then this is the only other component of $H$. If the Sylow 3-subgroups of $X^F$ are abelian, then Lemma 5.4(i) and Lemma 5.2 give the result. Thus, we may assume that the Sylow 3-subgroups of $X^F$ are non-abelian. Then, $X^F$ is isomorphic to $SL_4(q)$ (respectively $SU_4(q)$) with $3 \mid q - 1$ (respectively $3 \mid q + 1$). Consequently, the principal block is the unique unipotent block of $X^F$. In particular, $Q$ contains a Sylow 3-subgroup of $X^F$ and $Q/Z$ has an element of order 9.

Thus, we may assume that all components of $H$ are of type $A_2$ or $A_1$. By rank considerations, there can be at most two components of type $A_2$. By Lemma 5.4(i) and Lemma 5.2 we may assume that there are two $F$-stable components $X$ and $Y$ of type $A_2$ such that both $X^F$ and $Y^F$ have central elements of order 3. Consequently, the principal block of $X^F$ is the only unipotent block of $X^F$ and similarly for $Y^F$. The only other component of $H$, if it exists is of type $A_1$, which also has a unique unipotent block. Hence $Q$ is a Sylow 3-subgroup of $H^F$.

Since $H$ is a Levi subgroup of $L$, there is surjective group homomorphism from $Z(G)/Z^s(G)$ to $Z(H)/Z^s(H)$ (see [3] Proposition 4.1) and by hypothesis, $[L, L]$ is simple of type $E_6$. Hence $Z(H)/Z^s(H)$ is cyclic of order 1 or 3. Since $X$ and $Y$ are the only components of $H$ with central elements of order 3, it follows that either $Z(X)$ or $Z(Y)$ covers $Z(H)/Z^s(H)$. Thus, either $Z(X) \leq Z(Y)Z^s(H)$ or $Z(Y) \leq Z(X)Z^s(H)$. 

Assume that \( Z(X) \leq Z(Y)Z^c(H) \). Let \( U \) be the product of all components of \( H \) other than \( X \) and \( Z^c(H) \). Then, \( Z(X)^F \leq (Z(Y)Z^c(H))^F \leq U^F \) and hence \( 3 \mid |X^F \cap U^F| \). Since \( Q \) is a Sylow 3-subgroup of \( H^F \) and \( |H^F| = |X^F|/|U^F| \), \( Q \) is not contained in \( X^F U^F \). Further, \( Q \cap X^F \) is a Sylow 3-subgroup of \( X^F \) and in particular is non-abelian of order at least 3. By Lemma 6.2, \( Q/Z \) has exponent 3. So, by Lemma 5.3, \( 1 \neq Z \leq Z(X) \) whence \( Z = Z(X) \). Since \( Z \neq 1 \), \( L \) is simple by hypothesis. In particular, \( Z = Z(X) \) covers \( Z(G)/Z^c(G) \). It follows that \( Z(Y) \leq Z(X)Z^c(H) \). By the same argument as above with \( Y \) replacing \( X \), we get that \( Z = Z(Y) \). In particular \( Z(X) = Z(Y) \), a contradiction since \( X \cap Y = 1 \).

Finally, consider the case that \( H \) has a component of type \( E_6 \). Then \( H = L \) and \( b = c \). Let \( b_0 \) be a block of \([L, L]^F\) covered by \( b \) and let \( P_0 = P \cap [L, L]^F \) be a defect group of \( b_0 \). Let \( R \) be the Sylow 3-subgroup of \( Z^c(L)^F \). By Lemma 5.4(i) applied with \( X = [L, L] \) and \( Y = Z^c(L), P = [L, L]^F Z^c(L)^F = P_0 R \). So, \( P/P_0 R \) is a subgroup of \( L^F/([L, L]^F Z^c(L)^F) \). Since \( L^F/([L, L]^F Z^c(L)^F) \) is either trivial or has order 3, we have that \( P_0 R \) has index at most 3 in \( P \). If \( P_0 \) is abelian, then \( P \) and hence \( P/Z \) has an abelian subgroup of index 3. Thus, \( P_0 \) is non-abelian. We claim that \( R \leq P_0 \). Indeed, by hypothesis, either \( Z = 1 \) or \([L, L] = L \). If \( L = [L, L] \), then \( R = 1 \) and the claim holds trivially. If \( Z = 1 \), then \( P \) supports a transitive fusion system. Hence \( R \leq Z(P) \leq [P, P] \leq [L, L]^F \) and the claim is proved. Thus, \( P_0 = PR \) has index at most 3 in \( P \).

Assume first that \( b_0 \) is unipotent. The unipotent 3-blocks of exceptional groups have been described in [11]. If \( b_0 \) is the principal block, then \( P/Z \) has exponent greater than 3. So, \( b_0 \) is non-principal and \( P_0 \) is non-abelian. By [11] (last part of the proofs for Tableau I), \( P_0 \) is the extension of a homocyclic group, say \( T \), of rank 2 by a group of order 3. If \( T \) is not elementary abelian, then \( TZ/Z \) has exponent at least 9 and hence so does \( P/Z \). Thus, we may assume that \( T \) is elementary abelian. So, \( |P_0| = 3^a \) and \( |P| \leq 3^a \), a contradiction.

So, we may assume that \( b_0 \) is quasi-isolated but not unipotent. Here the blocks are described in [19, Section 4.3]. In particular, \( b_0 \) corresponds to one of lines 13, 14, or 15 of Table 4 of [19] (and the corresponding Ennola duals; see the last remark of Section 4 of [19]). If \( b_0 \) corresponds to line 14, then \( P_0 \) is the extension of a homocyclic group, say \( T \), of rank 4 by a group of order 3. If \( T \) is not elementary abelian, then \( TZ/Z \) has exponent at least 9 and if \( T \) is elementary abelian, then \( |P_0| \leq 3^3 \), whence \( |P| \leq 3^6 \), a contradiction. If \( b_0 \) corresponds to line 13, then \( P_0 \) contains a subgroup isomorphic to a Sylow 3-subgroup of \( SL_6(q) \) with \( 3 \mid q - 1 \). In particular, \( U^1(P) \) is not cyclic. On the other hand, since \( P/Z \) has exponent 3, \( U^1(P) \leq Z \). This is a contradiction as \( Z \) is cyclic. \[ \square \]

**Proposition 6.4.** Suppose that either \( p = 3 \) and \( L \) is simple and simply connected of type \( E_7 \) or \( E_8 \) in characteristic \( r \neq 3 \) or that \( p = 5 \) and \( L \) is simple of type \( E_8 \) in characteristic \( r \neq 5 \). Let \( F \) be a Frobenius morphism on \( L \) and let \( P \) be a defect group.
of a $p$-block of $L^F$. Suppose that $P$ supports a transitive fusion system and $|P| \geq 3^7$ if $p = 3$. Then $P$ is abelian.

Proof. Suppose if possible that $P$ is not abelian. As before $P$ has exponent $p$, and is indecomposable and $P$ does not have an abelian subgroup of index $p$. Let $z \in Z(P)$. Since $L$ is simply connected, $H := C_L(z)$ is a connected reductive subgroup of $L$ of maximal rank and of semisimple rank at most 8 and by [24, Chapter 5, Theorem 9.6], $P$ is a defect group of $H^F$. The possible components of $H$ are of type $A$, $D$, $E_6$ or $E_7$.

Let $\mathcal{X}$ be an $F$-stable subset of components of $H$ and let $X$ be the product of the elements of $\mathcal{X}$. Suppose that $X^F$ does not have a central element of order $p$. By Lemma 5.4(i), $P = (P \cap X^F) \times (P \cap Y^F)$ where $Y$ is the product of $Z^c(H)$ and all components of $H$ other than those in $\mathcal{X}$. The indecomposability of $P$ implies that either $P \leq X^F$ or $P \leq Y^F$. Since $z$ is a central $p$-element of $H^F$, and $X^F$ does not have a central element of order $p$, it follows that $P \leq Y^F$. By replacing $H$ by $Y$, we may assume that the fixed points of every $F$-orbit of components of $H$ have central elements of order $p$ ($Y$ may have rank less than $H$). Thus, if $p = 5$ the only possible components are of type $A_4$ and if $p = 3$, then the only possible components are of type $A_2$, $A_5$, $A_8$ or $E_6$.

Suppose that $H$ has an $F$-stable component $X$ of type $A_{p-1}$. Let $Y$ be the product of all components of $H$ other than those in $X$ with $Z^c(H)$. By Lemma 5.4(i) and the indecomposability of $P$, we may assume that $Z(X)^F \cap Z(Y)^F$ and hence $H^F/X^FY^F$ has order $p$. So, by Lemma 5.4(ii), $P$ is not contained in $X^FY^F$. By Lemma 5.5 we may assume that $P \cap X^F$ is not abelian since otherwise we can replace $X$ by a torus. Since $X^F$ has a central element of order $p$, $X^F$ is a special linear (respectively unitary) group. The only non-abelian defect groups of a finite special linear (or unitary) group of degree $p$ in non-describing characteristic are Sylow $p$-subgroups and $P \cap X^F$ is a non-abelian defect group of $X^F$. Thus, $P \cap X^F$ is a Sylow $p$-subgroup of $X^F$ and consequently has order at least $p^6$. Since we have shown above that $P$ is not contained in $X^FY^F$, by Lemma 5.3 $P$ has an element of order $p^2$, a contradiction. Thus, we may assume that any component of $H$ of type $A_{p-1}$ lies in an $F$-orbit of size at least 2.

If $p = 5$, the only case left to consider is that $H$ has two components of type $A_4$ (and these are the only ones) transitively permuted by $F$. In this case, by rank considerations, $Z^c(H)$ is trivial, and hence $H^F$ is isomorphic to a special linear or unitary group. In particular the Sylow 5-subgroups of $H^F$ have an abelian subgroup of index 5, a contradiction. This completes the proof for the case that $p = 5$.

Now assume that $p = 3$. Let us first consider the case that there is a component $X$ of $H$ of type $A_8$. Then $H = X = SL_8$ and we may argue as in the first part of the proof of Proposition 4.1.
Let us next consider the case that there is a component $X$ of $H$ of type $A_5$. If $X$ also has a component of type $A_2$, then by rank consideration this is the unique component of type $A_2$ and we have ruled out this situation above. Thus $X$ is the unique component of $H$. Let $P_0$ be a defect group of a covered block of $X^F$. The Sylow $3$-subgroup of $Z^c(H)^F$ is contained in $Z(P)$ and $Z(P) \leq [P, P] \leq [X, X] \cap H^F \leq X^F$, hence we have that the Sylow $3$-subgroup of $Z^c(H)^F$ is contained in $X^F$ and in particular has order at most $3$. Thus, $P_0$ has index at most $3$ in $P$. In particular $P_0$ is non-abelian. Now $X = M/Z$, where $M$ is a special linear group of degree 6 (with a compatible $F$-action) and $Z$ is a central subgroup. Since $Z(M)$ is cyclic of order 6 (or 3 if $r = 2$) and since $X$ has a central element of order 3, $Z$ is either trivial or of order 2, $Z$ is $F$-stable and $Z^F = Z$. Further, $M^F/Z$ is a normal subgroup of $X^F = (M/Z)^F$ of index $|Z|$. Thus $P_0$ is a defect group of $M^F/Z$ and up to isomorphism a defect group of $M^F$ and $M^F = SL_6(q)$ (respectively $SU_6(q)$). Since $M^F/Z$ has index prime to 3, $M^F/Z$ contains the 3-part of the centre of $X^F$, hence $M^F$ has a central element of order 3. Thus, $P_0$ is the intersection with $X^F$ of a Sylow $3$-subgroup of the centraliser of a semisimple $3'$-element of $GL_6(q)$ (or $GU_6(q)$). Since $P_0$ has exponent 3 and is non-abelian, the possible structures of semisimple centralisers in $GL_6(q)$ (or $GU_6(q)$) force that the centraliser in $GL_6(q)$ (respectively $GU_6(q)$) has the form $GL_3(q^2)$. Hence $|P_0| \leq p^3$ and $|P| \leq p^4$ a contradiction.

Suppose $H$ has a component of type $E_6$. Arguing as in the previous case $H$ has no components of type $A_2$ and hence the $E_6$-component is the unique component of $H$. This component is of simply connected type since as explained in the beginning of the proof we may assume that the $F$-fixed point subgroup of every $F$-orbit of components of $H$ has central elements of order 3 and we are done by Proposition 6.3 (note that we apply Proposition 6.3 here in the case that $Z = 1$).

The only case left to consider is that all components of $H$ are of type $A_2$ and no component is $F$-stable. By rank considerations and the fact that groups of type $E_8$ do not have semisimple centralisers with component type $A_2$ (see the tables in [9]), we are left with two possibilities: either $H$ has exactly three components, all of type $A_2$ and in a single $F$-orbit or $H$ has exactly two components both of type $A_2$ and in a single $F$-orbit. In any case, $[H, H]^F$ has a quotient or subgroup $H_0$ isomorphic to $PSL_3(q)$ (respectively $PSU_3(q)$) for some $q$ such that $|H, H|^F/|H_0|$ equals 1 or 3. Let $P_0 = P \cap [H, H]$ and let $P_0'$ be either the intersection of $P_0$ with $H_0$ or the image of $P_0$ in $H_0$. Then $P_0'$ has exponent 3. Since any 3-subgroup of a finite projective special linear or unitary group of degree 3 has an abelian subgroup of index 3 and since the 3-rank of these groups is 2, it follows that $|P_0'| \leq 3^3$. Hence $|P_0| \leq 3^4$.

We claim that the index of $P_0$ in $P$ is at most 3. Indeed, let $R$ be the Sylow 3-subgroup of $Z^c(H)^F$. Then $R \leq Z(P) \leq [P, P] \leq [H, H]$, that is $R \leq P_0$. On the other hand, $|P/P_0'|$ divides $|Z([H, H]^F)|_3$ and we have seen from the structure of $[H, H]^F$ that $Z([H, H]^F)$ has order at most 3. This proves the claim. Hence $|P| \leq 3^5$, a contradiction. □
7. Consequences

We note some consequences of Theorem 1.2.

**Theorem 7.1.** Let $B$ be a block of a finite group such that $k(B) - l(B) = 1$ (e.g. a block with multiplicity 1). Then $B$ has elementary abelian defect groups.

**Proof.** See proof of Theorem 3.6 in [23]. □

**Corollary 7.2.** Let $B$ be a block of a finite group such that $k(B) = 3$. Then $B$ has elementary abelian defect groups.

**Proof.** We have $l(B) \in \{1, 2\}$. In case $l(B) = 1$ it was shown by Külshammer [22] that the defect groups of $B$ have order 3. The remaining case $l(B) = 2$ follows from Theorem 7.1. □

### References


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