Properties and Classification of Generalized Resultants and Polynomial Combinants

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Abstract—**Polynomial combinants define the linear part of the Dynamic Determinantal Assignment Problems, which provides the unifying description of the frequency assignment problems in Linear Systems. The theory of dynamic polynomial combinants have been recently developed by examining issues of their representation, parameterization of dynamic polynomial combinants according to the notions of order and degree and spectral assignment. Dynamic combinants are linked to the theory of "Generalised Resultants", which provide the matrix representation of polynomial combinants. We consider coprime set polynomials for which assignability is always feasible and provides a complete characterisation of all assignable combinants with order above and below the Sylvester order. The complete parameterization of combinants and coresponding Generalised Resultants is prerequisite to the characterisation of the minimal degree and order combinant for which spectrum assignability may be achieved.**

I. INTRODUCTION

The study of determinantal type problems (such as pole zero assignment, stabilisation) has been unified by the development of a framework referred to as Determinantal Assignment Problem (DAP) [8]. DAP is a multi-linear nature problem and thus may be naturally split into a linear and multi-linear problem (decomposability of multivectors). The final solution is thus reduced to the solvability of a set of linear equations coming from the spectrum assignability of polynomial combinants [7], characterising the linear problem, together with quadratics characterising the multi-linear problem of decomposability, which in turn define some appropriate Grassmann variety [3].

Dynamic compensation problems may also be studied within the DAP framework, but their linear sub-problem depends on dynamic polynomial combinants which have much richer properties and they have been studied recently [6]. Amongst the open issues in the area of dynamic frequency assignment problems, is defining the least complexity compensator, for which we may have solvability of the arbitrary spectrum assignment of the corresponding DAP. This is referred to as the minimal design problem of DAP. The fundamental aspects of the theory of dynamic polynomial combinants have been examined in [6], where their representation in terms of Generalized Resultants and Toeplitz matrices has been established [2]. Dynamic polynomial combinants have been parameterized in terms of order and degree [6] and this has introduced the foundations for the investigation of a number of properties of the family of dynamic combinants, where the most prominent is that of spectrum assignability for some value of the degree and order of the dynamic combinant. Under the conditions of coprimeness of polynomials defining the combinant, there is always an order and degree such that the corresponding combinant has its spectrum assignable. Parametrising all dynamic combinants according to order and degree is a problem that is considered here. We show that all combinants of degree greater than the Sylvester degree have elements which are assignable, and there is a set of degrees less than the Sylvester degree for which we have assignable combinants for some appropriate order. The latter property motivates the study for finding the least degree and order combinant that is spectral assignable. The paper provides an overview of the theory of dynamic combinants and examines the solution of the Minimal Design Problem.

Throughout the paper we use the notation: $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from the set $\tilde{n} \triangleq \{1,2,...,n\}$. If $\mathcal{V}$ is a vector space and $\{\underline{v}_{i_1},...,\underline{v}_{i_k}\}$ are vectors of $\mathcal{V}$ then $\underline{v}_{i1}\wedge...\wedge\underline{v}_{ik}=\underline{v}_{\omega}\wedge$, $\omega=(i_1,...,i_k)$ denotes their exterior product and $\wedge^r\mathcal{V}$ the $r-$th exterior power of $\mathcal{V}$. If $H\in\mathcal{F}^{m\times n}$ and $r\le\min\{m,n\}$, then $C_r(H)$ denotes the $r-$th compound matrix of H [11].

II. Basic Definitions and Properties of Combinants

A. *The Determinantal Assignment and Polynomial Combinants*

A large family of problems for Linear Systems involving Dynamic Compensation [4], may be reduced to a common formulation represented by the determinantal assignment problem (DAP) [8]. This deals with the study of the following equation with respect to polynomial matrix H(s):

$$\det(H(s)\,M(s)) = f(s) \qquad (1)$$

where f(s) is a polynomial of an appropriate degree d. If $M(s)\in\mathbb{R}^{p\times m}[s]$, $r\le p$, such that rank($M(s)$) = r and let $\mathcal{H}$ be a family of full rank $r\times p$ constant matrices having a certain structure. Solve with respect to $H\in\mathcal{H}$ the equation:

$$f_M(s,H) = \det(H\,M(s)) = f(s) \qquad (2)$$

where f(s) $\in\mathbb{R}[s]$ with a degree d. If $\underline{h}_i(s)^t$, $\underline{m}_i(s)$, $i\in\tilde{r}$, are the rows of $H(s)$, columns of $M(s)$ respectively, then

$$C_r(H(s)) = \underline{h}_1(s)^t\wedge\ ...\ \wedge\underline{h}_r(s)^t = \underline{h}(s)^t\wedge\in\mathbb{R}^{1\times\sigma}$$

$$C_r(M(s)) = \underline{m}_1(s)\wedge\ ...\ \wedge\underline{m}_r(s) = \underline{m}(s)\wedge\in\mathbb{R}^{\sigma}[s]$$

$\sigma=\binom{p}{r}$, then by the Binet-Cauchy theorem[11] we have that [7]

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$$f_M(s,H) = <\underline{h}(s)\wedge, \underline{m}(s)\wedge> = \sum_{\omega\in Q_{r,p}} h_\omega(s)m_\omega(s) \quad (3)$$

$\omega = (i_1, ..., i_r)\in Q_{r,p}$, and $h_\omega(s)$, $m_\omega(s)$ are the coordinates of $\underline{h}(s)\wedge$, $\underline{m}(s)\wedge$ respectively. Note that $h_\omega(s)$ is the $r\times r$ minor of $H(s)$, which corresponds to the ω set of columns of $H(s)$ and thus $h_\omega(s)$, is a multilinear alternating function of the entries $h_{ij}(s)$ of $H(s)$. The study of the zero assignment of $f_M(s,H)$ may thus be reduced to a linear subproblem and a standard multilinear algebra problem as it is shown below.

(i) Linear subproblem of DAP: Set $\underline{m}(s)\wedge = \underline{p}(s)\in\mathbb{R}^\sigma[s]$. Determine whether there exists a $\underline{k}(s)\in\mathbb{R}^\sigma[s]$, $\underline{k}(s) \neq \underline{0}$, :

$$f_M(s, \underline{k}(s)) = \underline{k}(s)^t\,\underline{p}(s) = \sum k_i(s)p_i(s) = f(s)\in\mathbb{R}[s] \quad (3)$$

(ii) Multilinear subproblem of DAP: Assume that $\mathcal{K}$ is the family of solution vectors $\underline{k}(s)$ of (3). Determine whether there exists $H(s)^t = [\underline{h}_1(s), ..., \underline{h}_r(s)]$, $H(s)^t\in\mathbb{R}^{p\times r}[s]$:

$$\underline{h}_1(s)\wedge...\wedge\underline{h}_r(s) = \underline{h}(s)\wedge = \underline{k}(s)\in\mathcal{K} \quad (4)$$

The representation problem of a given order and degree dynamic combinant is summarised here [6] and this involves the parameterization of all sets leading to a polynomial combinant of a given degree p. We assume that the maximal degree polynomial in $\mathcal{K}$, $k_1(s)\neq 0$. If we define $\mathcal{P}$ as

$$\mathcal{P} = \{p_i(s)\in[s], i\in\tilde{m},\ q = max\{deg\{p_{i(s)}\},\ i=2,...,m\}$$
$$n = deg\{p_1(s)\}\geq deg\{p_i(s)\},\ \ i=2,...,m, \quad (5a)$$

$$p_1(s) = s^n + a_{n-1}s^{n-1}+...+a_1 s + a_0, \quad (5.b)$$

$$p_i(s) = b_{i,q}s^q+...+b_{i,1}s+b_{1,0},\ i=2,...,m$$

$$\underline{p}(s) = \begin{bmatrix} p_1(s)\\ p_2(s)\\ \vdots\\ p_m(s)\end{bmatrix} = [\underline{p}_0, \underline{p}_1, ..., \underline{p}_n]\begin{bmatrix}1\\ s\\ \vdots\\ s^n\end{bmatrix} = P\underline{e}_n(s) \quad (5.c)$$

Then the set $\mathcal{P}$ will be referred to as an *(m;n(q))-ordered set* of R[s]. Consider now the $\mathcal{K} = \{k_i(s)\in[s], i\in\tilde{m}\}$, set $deg\{k_i(s)\}\leq d$ with the resulting *d-order polynomial combinant* of $\mathcal{P}$, defined as

$$f_d(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s) = \underline{k}^t(s)\underline{p}(s)$$

where $\underline{k}^t(s) = [k_1(s),...,k_m(s)]^t = \underline{k}_0^t+...+s^d\underline{k}_d^t$ (6)

The matrix $P\in\mathbb{R}^{m\times(n+1)}$ is the *basis matrix* of $\mathcal{P}$ and generates the representative $\underline{p}(s)\in\mathbb{R}^m[s]$ of $\mathcal{P}$.

B. *Generalised Resultant Representations of Dynamic Combinants*

For the general $(m;d)$ set $\mathcal{K}$ with a representative vector

$$\underline{k}(s)^t = \underline{k}_0^t+...+s^d\underline{k}_d^t = [k_1(s),...,k_m(s)] \quad (7)$$

then $f_d(s,\mathcal{K},\mathcal{P})$ may be expressed as

$$f_d(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m}[k_{i,d},...,k_{i,1},k_{1,0}]\begin{bmatrix} s^d p_i(s)\\ \vdots\\ sp_i(s)\\ p_i(s)\end{bmatrix} \quad (8)$$

The above leads to the following representation of dynamic combinants:

Proposition (1): Every dynamic combinant combinant $f_d(s,\mathcal{K},\mathcal{P}))$ defined by an $(m;d)$ set $\mathcal{K}$ is equivalent to a constant polynomial combinant defined by the $(m(d+1);0)$ set $\mathcal{K}^0$ and generated by the $(m(d+1);(n+d)(q+d))$ the d-th power of the $(m;n(q))$ set $\mathcal{P}$, defined by

$$\mathcal{P}^d = \{s^d p_1(s),...,p_1(s);...;s^d p_m(s),...,p_m(s)\} \quad (9)\ ■$$

If $\mu = n+d$, $\tilde{\underline{e}}_\mu(s)^t = [s^\mu,...,s,1]$, $\partial\,[\underline{p}_{1,d}(s)] = n+d$, $\partial\,[\underline{p}_{i,d}(s)]\leq q+d$ for all $i=2,...,m,$ then

$$\underline{p}_{1,d}(s) = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0\\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0\\ \vdots & & \ddots & & & & & & \vdots\\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0\end{bmatrix}\tilde{\underline{e}}_\mu(s)$$

$$= S_{n,d}(p_1)\tilde{\underline{e}}_\mu(s),\ S_{n,d}(p_1)\in\mathbb{R}^{(d+1)\times(\mu+1)},\ i=2,...,m$$

$$\underline{p}_{i,d}(s) = \begin{bmatrix} 0 & ... & 0 & b_{i,q} & ... & b_{i,1} & b_{i,0} & 0 & ... & ... & 0\\ 0 & ... & 0 & 0 & b_{i,q} & ... & b_{i,1} & b_{i,0} & 0 & ... & 0\\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots\\ 0 & ... & 0 & 0 & ... & 0 & b_{i,q} & ... & ... & b_{i,1} & b_{i,0}\end{bmatrix}\tilde{\underline{e}}_\mu(s) \quad (10)$$

$$= S_{n,d}(p_i)\tilde{\underline{e}}_\mu(s),\ S_{n,d}(p_i)\in\mathbb{R}^{(d+1)\times(\mu+1)}$$

The set $\mathcal{P}^d$ has then a basis matrix representation as shown in (11) where $S_{\mathcal{P},d}\in\mathbb{R}^{m(d+1)\times(\mu+1)}$ which is the *d-th Generalised Resultant representation representation* [1], [2] of the set $\mathcal{P}$ and $S_{\mathcal{P},d}$ is the basis matrix of the $\mathcal{P}^d$ set. An alternative expression for the dynamic combinant is obtained using the basis matrix description of the set $\mathcal{P}$ [6], referred to as the Toeplitz representation.

$$\underline{p}_d(s)=\begin{bmatrix}\underline{p}_{1,d}(s)\\ \underline{p}_{2,d}(s)\\ \vdots\\ \underline{p}_{m,d}(s)\end{bmatrix}=\begin{bmatrix}S_{n,d}(p_1)\\ S_{n,d}(p_2)\\ \vdots\\ S_{n,d}(p_m)\end{bmatrix}\tilde{\underline{e}}_\mu(s)=S_{\mathcal{P},d}\tilde{\underline{e}}_\mu(s) \quad (11)$$

III. DEGREE AND ORDER PARAMETERISATION OF $\mathcal{K}$

The general representation of dynamic combinants based on the order may lead to combinants of varying degree. An alternative characterisation based on the fixed degree of $f_d(s,\mathcal{K},\mathcal{P})$ but with varying order $\mathcal{K}$ provides an explicit parameterisation of the $\mathcal{K}$ sets. The fixed degree parameterisation of combinants is summarised below [6]:

Theorem (1): Given the set $\mathcal{P}$ and a general proper *(m;d)* set $\mathcal{K}$. Then,

(i) For all proper (*m;d*) sets $\mathcal{K}$ $n\leq\partial[f_d(s,\mathcal{K},\mathcal{P})]\leq n+d$

(ii) If $p\in\mathbb{N}_{>0}$, $p\geq n$, then the family $\{\mathcal{K}_p\}$ for which $\partial\,[f_d(s,\mathcal{K},\mathcal{P})]=p$, satisfies the conditions $\partial\,[k_1(s)]\leq p-n,\ \partial\,[k_i(s)]\leq p-q,\ i=2,\ldots,m$ where at least one of the first two conditions holds as an equality.

(iii) The fixed degree p family $\{\mathcal{K}_p\}$ contains *n-q+1* subfamilies parameterised by a fixed order *d*. The possible values for the order are:

$$d_1=p-q>d_2=p-q-1>\ldots>d_{n-q+1}=p-n$$

and the corresponding subfamilies are

$$\{\mathcal{K}_p^{d_1}\}=\{k_i(s):\partial\,[k_1(s)]\leq p-n,\partial\,[k_2(s)]=d_1= \\ =p-q,\partial\,[k_i(s)]\leq d_1,i=3,\ldots,m\}$$

$$\{\mathcal{K}_p^{d_2}\}=\{k_i(s):\partial\,[k_1(s)]\leq p-n,\partial\,[k_2(s)]=d_2= \\ =p-q-1,\partial\,[k_i(s)]\leq d_2,i=3,\ldots,m\}$$

$$\vdots$$

$$\{\mathcal{K}_p^{d_{n-q+1}}\}=\{k_i(s):\partial\,[k_1(s)]=\partial\,[k_2(s)]=d_{n-q+1}=p-n, \\ \partial\,[k_i(s)]\leq d_{n-q+1}=p-n,\ i=3,\ldots,m\}$$

■

Clearly, the degree of the proper combinants satisfies $p\geq n$. The entire family of proper combinants of $\mathcal{P}$ may thus be parameterised by degree and orders. The set of all $\mathcal{K}$ vectors, is denoted as $<\mathcal{K}>$ and may be partitioned as

$$<\mathcal{K}>=\{\mathcal{K}_n\}\cup\{\mathcal{K}_{n+1}\}\cup\ldots\cup\{\mathcal{K}_{n+q-1}\} \quad (12)$$

whereas each subset $\{\mathcal{K}_p\}$ has the structure defined by the previous result. Thus, $\{\mathcal{K}_n\}$ class acts as a generator of all other classes derived simply by adding the corresponding increase in the degree. For a class $\{\mathcal{K}_p^d\}$, $<\mathcal{K}_p^d>$ denotes the ordered set of degrees of the $\{k_i(s),i\in\tilde{m}\}$ polynomials.

Corollary (1): Given an (*m;q*) set $\mathcal{P}$ and a general (*m;d*) set $\mathcal{K}$, then:

(i) The minimal degree family *p=n*, $\{\mathcal{K}\}_n$ is expressed as

$$\{\mathcal{K}_n\}=\begin{matrix}\{\{\mathcal{K}_n^0\}:<\mathcal{K}_n^0> &=(0,0,\ldots,0)\ ;\\ \{\mathcal{K}_n^1\}:<\mathcal{K}_n^1> &=(0,1,\ldots,1)\ ;\\ \vdots & \\ \{\mathcal{K}_n^{n-q}\}:<\mathcal{K}_n^{n-q}> &=(0,n-q,\ldots,n-q)\}\end{matrix} \quad (13)$$

(ii) The general degree family *p=n+d*, $\{\mathcal{K}_p\}$ is then expressed as

$$\{\mathcal{K}_p\}=\begin{matrix}\{\{\mathcal{K}_p^d\}:<\mathcal{K}_p^d>=(0,\ldots,0)+(d,d,\ldots,d);\\ \{\mathcal{K}_p^{d+1}\}:<\mathcal{K}_p^{d+1}>=(0,1,\ldots,1)+(d,d,\ldots,d);\\ \vdots\\ \{\mathcal{K}_p^{d+n-q}\}:<\mathcal{K}_p^{d+n-q}>=(0,n-q,\ldots,n-q)+(d,d,\ldots,d)\}\end{matrix} \quad (14)$$

(iii) For the general degree *p* family, $p\geq n$, the values of possible orders in decreasing order are:

$$d_1=p-q>d_2=p-q-1>\ldots.>d_{n-q} \\ =p-n+1>d_{n-q+1}=p-n$$

and they are given as

$$d_i=p-q+1-i,\quad i=1,2,\ldots,n-q+1$$ ■

Amongst all (*m;d*) sets $\mathcal{K}$, the set defined by

$$\{\mathcal{K}_{n+q-1}^{n-1}\}=\{k_1(s):\partial\,[k_1(s)]=q-1, \\ k_i(s):\partial\,[k_i(s)]=n-1,\ i=2,\ldots,m\}$$

is referred to as the *Sylvester set* of $\mathcal{P}$. The general *p* degree family may be expressed as:

$$\{\mathcal{K}_p\}=\ \{\mathcal{K}_p^{\tilde{d}_i},\ \tilde{d}_i=p-n+i,\quad i=1,2,\ldots,n-q+1\}= \\ =\ \{\mathcal{K}_p^{p-n};\mathcal{K}_p^{p-n+1};\ldots.;\mathcal{K}_p^{p-q-1};\mathcal{K}_p^{p-q}\}$$

The set $\mathcal{K}_p^{p-q}$ with the highest order $d_1=p-q$ is the *generator* of the family and its degrees are

$$<\mathcal{K}_p^{p-q}>\ =(p-n,p-q,\ldots,p-q) \quad (15)$$

Similarly, the set $\mathcal{K}_p^{p-n}$ with the $d_{n-q+1}=p-n$ lowest order is the *co-generator* of the family and its degrees are

$$<\mathcal{K}_p^{p-n}>\ =(p-n,p-n,\ldots,p-n) \quad (16)$$

The above suggests that the entire family $<\mathcal{K}>$ may be expressed in "direct sum" form ($\dot{\cup}$) as

$$<\mathcal{K}> \;=\; \{\mathcal{K}_n\}\dot{\cup}\{\mathcal{K}_{n+1}\}\dot{\cup}\; ..... \;\dot{\cup}\{\mathcal{K}_{n+q-1}\}\dot{\cup}\; .....$$

$$\{\mathcal{K}_p\} = \{\mathcal{K}_p^{p-n}\}\dot{\cup}\{\mathcal{K}_p^{p-n+1}\}\dot{\cup}.....\dot{\cup}\{\mathcal{K}_p^{p-q}\} \quad (17)$$

IV. Generalised Resultants and Parametrisations

The parameterisation of the sets $\mathcal{K}$ induces a natural parameterisation of the corresponding Generalized Resultants. For the $(m;d)$ set $\mathcal{K}$ that leads to combinants of degree p its structure is explicitly defined by:

$$\{\mathcal{K}_p^d\} = \{k_1(s) : \partial\,[k_1(s)] = p-n = \tilde{d},$$
$$k_2(s) : \partial\,[k_2(s)] = d,\; \tilde{d} \le d \le d^* = p-q, ...,$$
$$k_i(s) : \partial\,[k_i(s)] \le d,\; i = 3,...,m\} \quad (18)$$

The set $\{\mathcal{K}_p^d\}$, $p \ge n$ and with d taking values as above, represents the general set generating dynamic combinants of a given degree d and order p. This representation leads to:

Proposition (2): The dynamic combinant $f_d(s, \mathcal{K}_d^d, \mathcal{P})$, generated by the set $\{\mathcal{K}_p^d\}$ is equivalent to a constant combinant of degree p that is generated by the polynomial set $\mathcal{P}_p^d$, $\tilde{d} = p-n,\;\; \tilde{d} \le d \le p-q = d^*$, as in (9). ■

The set $\mathcal{P}_p^d$ is the (p,d)- power of $\mathcal{P}$ and has degree p and its vector representative is

$$\underline{p}_d(s) = \begin{bmatrix} \underline{p}_{1,\tilde{d}}(s) \\ \underline{p}_{2,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} = \begin{bmatrix} S_{n,\tilde{d}}(p_1) \\ S_{q,d}(p_2) \\ \vdots \\ S_{q,d}(p_m) \end{bmatrix} \tilde{\underline{e}}_p(s) = S_{p,d}\,\tilde{\underline{e}}_p(s) \quad (19)$$

Proposition (3): The Generalised Resultants corresponding to the parameterized set $\{\mathcal{K}_p^d\}$ are defined by:

(i) Given that $\underline{p}_{1,\tilde{d}}(s)$ has degree $\tilde{d} + n = p-n+n = p$, then

$$S_{n,\tilde{d}}(p_1) = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{bmatrix} \quad (20a)$$

(ii) Given that $\underline{p}_{i,d}(s)$ has degree $d+q$ which satisfies the inequality $p-(n-q) \le d+q \le p$ and thus $d+q+1 \le p+1$, the structure of $S_{q,d}(p_i)$ is defined for all $i = 2,...,m$ and $\forall d : p-n \le d \le p-q$ by

$$S_{q,d}(p_i) = \begin{bmatrix} 0 & \dots & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} & 0 & \dots & 0 \\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & b_{i,q} & \dots & \dots & b_{i,1} & b_{i,0} \end{bmatrix} \quad (20b)$$

■

The matrix $S_{p,d}(\mathcal{P}) \in \mathbb{R}^{\sigma\times(p+1)}$, $\sigma = p-n-d+m(d+1)$ will be called the (p,d)- *Generalised Resultant* of the set $\mathcal{P}$ where the possible values of d are: $p-n \le d \le p-q$. Clearly the $S_{p,d}(\mathcal{P})$ matrix, or $S_{p,d}$, is the basis matrix of the (p,d) power of $\mathcal{P}$, $\mathcal{P}_p^d$.

Remark (1): For the set $\mathcal{P}$ we can parameterise all dynamic combinants by the degree p and the corresponding order d as:

(a) p=n: then $0 \le d \le n-q$

(b) p=n+1 : then $1 \le d \le n-q+1$

(c) p>n+1: then $p-n \le d \le p-q$

and their properties are defined by the properties of corresponding (p,d)- generalised resultants $S_{p,d}(\mathcal{P})$ ■

The properties of all dynamic combinants are described by the corresponding family of matrices

$$S(\mathcal{P}) = \{S_{p,d} \forall p \ge n \text{ and } \forall d : p-n \le d \le p-q\} \quad (21)$$

referred to as the *family of Generalised Resultants* of the set $\mathcal{P}$. We distinguish a special element that corresponds to $p=n+q-1$, $d=n-1$ $\partial\,[k_1(s)] = p-n = q-1$, $S_{n+q-1,n-1}(\mathcal{P})$, denoted by $\tilde{S}_{\mathcal{P}}$ which is the *Sylvester Resultant* of the set $\mathcal{P}$

$$\tilde{S}_{\mathcal{P}} = \begin{bmatrix} S_{n,\,q-1}(p_1) \\ S_{q,\,n-1}(p_2) \\ \vdots \\ S_{q,\,n-1}(p_m) \end{bmatrix} \in \mathbb{R}^{\tau\times(n+q)},\;\; \tau = [q+(m-1)n] \quad (22)$$

where $S_{n,q-1}(p_1) \in \mathbb{R}^{q\times(n+q)}$, $S_{q,n-1}(p_i) \in \mathbb{R}^{n\times(n+q)}$, $j = 2,...,m$ and $\tau = [q+(m-1)n]$.

V. Spectrum Assignment of Dynamic Combinants and the Sylvester Resultant

We now consider the problem of arbitrary assignment of the spectrum of dynamic combinants for some appropriate order and degree. The results in this section follow from the equivalence of dynamic combinants to constant combinants We may summarise the results from [8] below:

Lemma (1) [1], [2]**:** Let $\mathcal{P}$ be an *(m,n(q))* set with Sylvester Resultant $\tilde{S}_{\mathcal{P}}$ .The set $\mathcal{P}$ is coprime, if and only if $\tilde{S}_{\mathcal{P}}$ has full rank.

■

Theorem (2): Let $\mathcal{P}$ be an *(m,n(q))* set. There exists a *d* such that $f_d(s,\mathcal{K},\mathcal{P})$ is completely assignable, if and only if the set $\mathcal{P}$ is coprime.

■

Corollary (2): For the *(m,n(q))* coprime set $\mathcal{P}$ the following properties hold true:

(i) There exists a combinant $\tilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ of degree *p=n+q-1* and order *d=n-1* which is completely assignable

(ii) All combinants $\tilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ of order *d=n-1* and degree $p: n+q-1 \le p \le 2n-1$ are completely assignable.

(iii) All combinants $\tilde{f}_p(s,\mathcal{K},\mathcal{P})$ of degree $p > p_s = n+q-1$ have an assignable element by selection of some appropriate order $p-n \le d \le p-q$.

■

The special combinant of order *d=n-1* and degree *p=n+q-1* is the *Sylvester combinant* of the set $\mathcal{P}$ denoted by $\tilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s) p_i(s)$ $\partial\,[k_1(s)] = q-1,$ and for $i = 2,...,m$, $\partial\,[k_i(s)] = n-1,$ and the zero assignment problem is expressed as making $\tilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ an arbitrary polynomial $\alpha(s)$ of degree *n+q-1*.

It is clear that two combinants of the same order d=n-1 and different degrees may be both assignable. In fact, both combinants $\tilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}),\ f_{n-1}(s,\mathcal{K},\mathcal{P})$ of degrees respectively *n+q-1* and *2n-1* are assignable. This raises the following important questions of investigating the assignability of all combinants $f_d(s,\mathcal{K},\mathcal{P})$ with *d<n-1* and parameterize all combinants $\hat{f}_d(s,\mathcal{K},\mathcal{P})$ of order $d, d \le n-1$ and degree $p \le n+q-1$ which are assignable. The family of all resultants of degree less or equal to p_s is referred to as *proper* subset and is defined as

$$\mathcal{S}_{pr}(\mathcal{P}) = \{S_{p,d}(\mathcal{P}) : n \le p \le n+q-1 = p_s,\ d = p-q-\rho,\ \rho = 0,1,...,n-q\} \tag{23}$$

This family is partitioned by the degrees and the orders:

Proposition (3): The family of proper generalised resultants of the (*m,n*(*q*)) set $\mathcal{P}$ is partitioned into q-1 sets as

$$\mathcal{S}_{pr}(\mathcal{P}) = \{\mathcal{S}_{p_s}\} \cup \{\mathcal{S}_{p_{s-1}}\} \cup ... \cup \{\mathcal{S}_n\} \tag{24}$$

where $p_s = n+q-1$ and each subset of a fixed degree is partitioned by the corresponding order has n-q+1 elements. ■

VI. Construction of the Family of the Proper Sylvester Resultants

The construction of the generalised resultants together with the paramaterisation of the $\mathcal{K}$ sets leads:

Proposition (4): The proper combinant of the (*m,n(q)*) set $\mathcal{P}$ that has $p_s = n+q-1$ degree and order $d = n-1-\rho,\ \rho = 1,2,...,n-q$ is defined by the generalised resultant $S_{p_s,n-1-\rho}$ defined as in (2.12) which is also expressed as

$$S_{p_s,n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \vdots S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q,n-1-\rho}(p_m) \end{bmatrix} \tag{25}$$

where $S_{n,q-1}(p_1),\ S_{n,q-1-\rho}(p_i),\ i = 2,...,m$ are the standard Sylvester blocks. Furthermore, any two successive combinants of degree p_s and order $d = n-1-\rho$ and $d' = n-\rho-2$ are related as

$$S_{p_s,n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \vdots S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q,n-1-\rho}(p_m) \end{bmatrix} \cong \begin{bmatrix} x...x \\ \vdots\ \vdots \\ x...x \\ S_{p_s,n-\rho-2} \end{bmatrix} \tag{26}$$

where $\cong$ denotes row equivalence on matrices.

■

Corollary (3): If $S_{p_s,n-\rho-1}, S_{p_s,n-\rho-2}$ are two generalised Sylvester matrices corresponding to combinants of degree p_s and orders $d = n-\rho-1,\ d' = n-\rho-2$ respectively, then $rank(S_{p_s,n-\rho-1}) \ge rank(S_{p_s,n-\rho-2})$. Furthermore, if $S_{p_s,n-\rho-1}$ has full rank then all higher order generalised resultants are also full rank.

■

The investigation of links between generalised resultants of different degree is considered next. In the following we will use the notation $S^{\rho}_{q,n-1-\rho}(p_i) = \left[0_\rho \vdots S_{q,n-1-\rho}(p_i)\right]$. With this notation for the p_s and the $p_s - 1$ degrees we have

$$S_{p_s,n-\rho-1} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S^{\rho}_{q,n-\rho-2}(p_2) \\ \vdots \\ S^{\rho}_{q,n-\rho-2}(p_m) \end{bmatrix} \tag{25}$$

where $d = n-1-\rho,\ q-1 \le d \le n-1,\ \rho = 0,1,2,...,n-q$. For the $p_s - 1$ degree with $q-2 \le d' \le n-2$, $d' = n-2-\rho',\ \rho' = 0,1,2,...,n-q$ we have

$$S_{p_s-1,n-1-\rho'} = \begin{bmatrix} S_{n,q-2}(p_1) \\ S^{\rho'}_{q,n-2-\rho'}(p_2) \\ \vdots \\ S^{\rho'}_{q,n-2-\rho'}(p_m) \end{bmatrix} \qquad (26)$$

Remark (2): The definition of Generalised Resultants readily establishes the following relationship:

$$S_{p_s,n-1} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} = \begin{bmatrix} 1 & x \dots x \\ \underline{0} & S_{n,q-2}(p_1) \\ x & x \dots x \\ \underline{0} & S_{n,q-2}(p_2) \\ \vdots & \vdots \\ x & x \dots x \\ \underline{0} & S_{n,q-2}(p_m) \end{bmatrix} \cong \begin{bmatrix} \underline{0} & X \\ 1 & x \dots x \\ \underline{0} & S_{p_s-1,n-2} \end{bmatrix}$$

(27) ■

The above clearly leads to the following result:

Proposition (5): For the maximal order generalised resultants $S_{p_s,n-1}$ and $S_{p_s-1,n-2}$ of degrees p_s, p_{s-1}, p_{s-2} etc are related as

$$S_{p_s,n-1} \cong \begin{bmatrix} 1 & x \dots x \\ \underline{0} & X \\ \underline{0} & S_{p_s-1,n-2} \end{bmatrix} \cong \begin{bmatrix} I_1 & X \\ 0 & X \\ 0 & S_{p_s-2,n-3} \end{bmatrix}$$

$$\cong \begin{bmatrix} I_\mu & X \\ 0 & X \\ 0 & S_{p_s-\mu,n-\mu-1} \end{bmatrix}, \mu = 0,1,...,q-1 \qquad (28)$$

and thus

$$rank(S_{p_s,n-1}) \ge 1 + rank(S_{p_s-1,n-2}) \ge$$
$$\ge 2 + rank(S_{p_s-2,n-3}) \ge ... \ge q-1 + rank(S_{n,n-q}) \qquad (6.7)$$

■

VII. Conclusion

The fundamentals of the theory of dynamic polynomial combinants have been introduced and their representation in terms of Generalized Resultants has been established. The parameterization of combinants in terms of order and degree has been introduced and this lays the foundations for investigating the properties of the family of Generalised Resultants. The current framework allows the development of the theory of dynamic combinants that may answer questions related to zero distribution of combinants, and its links to the existence of a nontrivial GCD, as well as "approximate GCD". The parameterizations in terms of order and degree and the conditions for existence of spectrum assignable combinants provide the means for the investigation of the minimal design problem dealing with finding the least order and degree for which spectrum assignability may be guaranteed. The study of this problem and the proof of the results is given in [14].

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