"An Empirical Analysis and Stochastic Modelling of Aggregate Demand Behaviour in a Spare Parts Inventory System."

by

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Volume one

Dedication

This thesis is dedicated to the memory of my only son Darren, who died on March the 13th 1989 after a long and courageous battle against cancer. The work here is very humble in its task and pales in comparison with the determination Darren showed against the relentless progress of an awful disease. His endurance, and suffering; and finally his cheerful acceptance of the inevitable was beyond comprehension. He was a remarkable young man for whom life offered so much, only to be deceived by a cruel twist of fate.

This work was put aside during the two years of Darren's illness and I had long since given up any thoughts of completion; it was left to gather dust and was far from my mind. After Darren died life seemed to have little to offer. Slowly and painfully in the Autumn of 1989 I drew strength from what Darren taught me; 'one must never give up in the face of adversity'. He showed me that even though the final battle may not be won it is better to have fought well with dignity and courage than not to have fought at all.

Rest peacefully my son.

'de profundis factum est'

Gordon Wright, February 1991.

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The work of several authors in particular proved of immeasurable value to this work. Without the concepts and knowledge they each made available in their various publications this work could not have been produced because it rested on the underpinning theory provided by these authors. I am indebted to Professors Aitchison and Brown for providing the essential input on the underlying theory of the lognormal distribution. To Robert Goodell Brown for his pioneering work on the lognormal distribution in inventory applications. To Professor Andrew Ehrenberg for all his pioneering concepts and theory in the consumer purchase field. To Uji Ijiri and Herbert Simon for making me think very carefully about the law testing process and the retroductive testing of theories.

I am so very grateful for the continuing patient support of my wife Margaret, who must have thought this work would never be finished, and who shares my grief over the loss of our son.

Declaration

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Abstract

The focus of the work here was an empirical analysis of the aggregate independent demand behaviour for spare parts inventories, principally in the automotive industry. In particular, using the pioneering work of RG Brown (1959), who showed that inventory usage values are often lognormally distributed, we set out and developed models that go some considerable way to explaining the underlying stochastic basis for this phenomena, why it occurs and some limiting conditions. The justification for this approach was on the grounds that by providing a more fundamental understanding of the underlying stochastic processes that explain the emergent aggregate demand behaviour, a sound starting point would be provided for developing more sophisticated analytical ways to view an inventory range, as a total entity, for planning and control purposes. The analysis was based on extensive data collected from the DAF Trucks (GB) Ltd. spare parts systems spanning the period 1975 to 1986, together with supporting studies from a number of other systems.

The analysis showed that in the systems studied spare parts prices are lognormally distributed and this is most likely to be the result of a stochastic process known as the 'theory of breakage'. Analysis also showed that in the DAF Trucks case aggregate demand volumes in very short time periods are distributed as a combined Log Series /Negative Binomial distribution (LSD/NBD). The combined LSD/NBD model of aggregate demand volumes is itself fully explained by a stochastic model known as the Afwedson model, which in turn is derived from more elementary conditions based on the Poisson process. We then demonstrated that if these short period aggregate demand distributions are cumulated period by period they converge to a lognormal distribution as the stable long run model of aggregate demand volumes. As a result of the lognormality of prices and volumes the resultant inventory usage values are also lognormal. Furthermore from insight into the underlying factors that explain the lognormality we have identified the factors and variables that govern the values of the parameters of the particular lognormal models of usage values.

The research protocol used in this work incorporated the law verifying process know as 'retroduction' after work and discussions of Uji Ijiri and Herbert Simon (1977); and to a lesser extent we utilised simulation for validation and verification of the derived models. From the proven lognormality of demand volumes and usage values we have demonstrated that a number of related key inventory factors are also lognormal, in particular inventory item turnover rates. Furthermore our conclusions show that some standard inventory performance measures, such as the inventory wide 'stock turnover rate' and the 'stock to sales' ratio, are poor measures to use in the case of highly skewed inventory variables. Finally we have suggested several potentially fruitful areas for developing improved methods of monitoring inventory performance in a variety of circumstances.

Principal Abbreviations Used

LSD	The Log Series distribution
NBD	The Negative Binomial distribution
sP	The Stuttering Poisson distribution

Probability notations

$P(x)$ or $\mathrm{P}(x)$	The probability function of the variate 'x'
$\mathrm{P}(x>m)$	The probability that x exceeds a value m
$\mathrm{P}(x=n)$	The probability that the variate $x = n$
$f(x)$	The probability density function of x
$F(x)$ *or* $\mathrm{F}(x)$	The distribution function of x or the cumulative probability density function of x
$R(x)$ or $\mathrm{R}(x)$	The reliability function of x defined as $[1\text{-}\mathrm{F}(x)]$
$Z(x)$ or $\mathrm{Z}(x)$	The hazard function of x defined as $f(x)/R(x)$
$N(x)$ or $\mathrm{N}(x)$	Normal probability scale
$\sum_{n=1}^{n=j} x_n$	The sum of all x values from $n = 1$ to $n=j$
$\prod_{n=1}^{n=j} x_n$	The product of all x values from $n = 1$ to $n=j$
$N(\mu, \sigma)$	A normal distribution with mean μ and standard deviation σ.
$\Lambda(\mu, \sigma)$	A lognormal function with location parameter μ and shape parameter σ
χ^2	Chi Squared variate
$\bar{x}$ and s^2	Mean and variance of a sample
μ and σ^2	Mean and variance of a population
MAD	Mean Absolute Deviation
RMAD	Relative Mean Absolute Deviation
EOQ	Economic order Quantity

Cs	Stockout cost function
Ch	Holding cost function
Co	Ordering cost function
Fss	Safety stock factor
Fcs	Cycle stock factor
ABC	Inventory categorisation into groups ABC by turnover value
P_i	Price item *i*
S_i	Sales of item *i*
R_i	Rate of return on item *i*
M_i	margin of item *i* as proportion of P_i
V_{hi}	Volume of stock held of item *i*
V_{ai}	Annual volume sold of item *i*
r	The correlation coefficient
R^2	The coefficient of determination
DW	The Durbin Watson test
Dn	The Durbin Watson test statistic
SE	The standard error of a sampling distribution
z	The normal ordinate where $z = (\bar{x} - \mu) / SE$
ξ_q	Quantiles of order 'q' of the lognormal distribution
ν_q	Quantiles of order 'q' of the normal distribution

Introduction to the Research

1.0 Origin of the research

This research began with the author's interest in the application of certain areas of inventory theory to the practical problems of managing large independent spare parts demand inventories. In particular the author was very interested in achieving a more fundamental understanding of the seemingly universal applicability of the so called 'ABC' analysis of inventory usage values; a technique also variously referred to as Usage Value or Pareto analysis. This methodology gives management the opportunity to see where the major investment in inventory occurs and in this context it is a valuable tool for 'Aggregate Inventory Management'. We use the term aggregate here to mean the concern with the characteristics, analysis, planning and control of an inventory range as a whole entity. It is the holistic view of an inventory range that makes the Pareto methodology particularly valuable for planning, and control. This tried and trusted tool is widely used by managers, and in its basic form can be found in most texts covering the fundamentals of inventory theory and practice. Interest in this technique, as a tool for modelling aggregate inventories, led this author to the literature describing the use of the lognormal distribution as a statistical distribution that can be used to model inventory item usage values.

It was R.G. Brown (1959) who first showed that usage values for an entire inventory range could very often be modelled by the lognormal distribution function. Brown also showed, by using the properties of this distribution, that aggregate inventory calculations such as investment in cycle stocks and safety stocks, could be determined by just a knowledge of the parameters of the particular lognormal distribution and the average usage value of the inventory range. This was a valuable finding by Brown and it opened access to a powerful methodology for aggregate analysis of inventories. It is particularly valuable in very large inventories, which may run to tens of thousands of items, for which the complete enumeration of some measure of interest might be both very time consuming and very expensive unless standard software procedures are available. Our early interest in this phenomena centred on questions such as 'why does the lognormal distribution fit inventory data in the way claimed, and why the lognormal as opposed to other highly skewed distributions; how truly universal is the process and what is its fundamental basis'. Much of the work in this thesis has been aimed at such questions, and with deducing the nature of the genesis of such models as applied to large spare parts inventories.

Commencing in 1979 and during the years that followed this author had the opportunity to examine many inventory management issues and problems with DAF Trucks (GB) Ltd., the wholly owned UK marketing and distribution subsidiary of DAF Trucks Holland. [In June 1987 DAF BV (Holland) merged with Leyland Trucks with the Dutch company effectively taking a 60% share of the British truck company in the UK. DAF Trucks GB Ltd., the wholly owned DAF sales and marketing operation merged with the Leyland Trucks sales operation to become known as Leyland DAF (GB) Ltd.]. The DAF UK sales company is from now on in this work referred to as DAF Trucks, or just DAF and this is the UK sales company before it merged with Leyland. No research was carried out on the company's inventory operations after the merger took effect.

DAF sold and distributed a range of heavy commercial vehicles in the UK market together with a range of some 25,000 plus spare parts to support and service the vehicles sold. It was initially in the context of the DAF spare parts inventory investigations that this author took the opportunity to test out the claims of R. G. Brown that usage values are lognormally distributed, and that use of this statistical model could form the basis of a sophisticated approach to setting aggregate inventory standards. Also the various researches quoted in the literature that related the lognormal distribution to inventory usage values were largely based on American studies and US data. Therefore it was considered important to validate the US work on the inventories of UK or European companies. Preliminary empirical and literature research in this area by the author also served to confirm a previously held view that most of the traditional methods to measure and monitor inventories in aggregate, such as inventory turnover ratio, inventory to sales ratios, and day sales in stock, are limited measures giving in effect only the picture for the average item. This does not seem to present a particular problem for inventories of modest size, or where the measure of interest is symmetrically distributed about the mean. We show here that for very large inventories these traditional measures of activity are really just the simple mean from highly skewed distributions, such as the lognormal. For example, we show in later chapters that the turnover ratios, item by item, are also lognormally distributed. Under such circumstances we argue that these simple average based ratios are of limited value in summarising the 'true' behaviour of the inventory as a whole entity in a manner of full value to management. This will be discussed in greater depth in later chapters.

Much of the research by this author was pursued on the premise that seeking a more fundamental understanding of the nature of the lognormal distribution, as applied to large inventories, would lead to a better understanding of the aggregate demand behaviour for such items. It was felt the such knowledge and insights might also lead to improved methods of measuring aggregate inventory performance and behaviour for management planning and control. Also we argue that whilst the analysis, planning and control of any individual inventory item is important, and can be performed with great sophistication using contemporary analytical techniques, it is the aggregate characteristics of an entire inventory range that are of most importance to management. Furthermore we need to measure such characteristics in more effective ways than by recourse to simple average summary measures.

Similar views have be made by several authors and Wharton (1975) has expressed it succinctly as follows -

> "*Although inventory control problems can be formulated in terms of the characteristics of individual stock items, it is the aggregate characteristics of any proposed system which are of over-riding importance to management. This is perhaps fortunate in view of the difficulties experienced by OR scientists in defining objectives and estimating relevant costs parameters for individual items.*
> *Whilst the OR scientist may seek to optimise the individual characteristics, he usually works under the constraint that management will be concerned about the aggregate characteristics. Those most likely to be of interest are average investment in cycle stock, expected number of orders/set ups per annum, average service level, and average investment in safety stock since these determine rate of turnover and customer satisfaction.*"

We argue that if we can begin to approach aggregate inventory management using new tools that will allow analysis, planning and control to be carried out to the same degree of sophistication now possible for single inventory items, then we will have made a great stride forward.

The research reported here has made an important contribution in reaching a fundamental understanding of the stochastic nature of the genesis of spare parts demand. We show by theoretical development, empirical analysis, model testing and simulation that a stochastic scheme can be developed that explains the observation of lognormality in spare parts demand volumes and usage values. The analysis, development and proof of the derived schema is entirely original and has not been reported previously. We also show those factors that govern the particular value of the parameters of the lognormal distribution for both demand volumes and usage values. So far the various discussions in the literature concerning the phenomena of lognormality of inventory usage values has been based entirely on empirical evidence of the statistical fit of such distributions.

1.1 The focus of the research

At the outset this author decided that the research would be focused on independent demand inventory items with special reference to spare parts in particular. Furthermore the emphasis was on the aggregate characteristics of the inventories held by individual companies for sales in their after markets, to reach a fundamental understanding of the aggregate behaviour of such inventories. The primary objective of this was to see if any lawlike relationships exist that govern or explain the underlying nature and characteristics of typical spare parts inventories, such as aggregate demand, as opposed to individual item characteristics. We initially postulated that if a model, such as the lognormal distribution, could be fitted to inventory usage values over the entire range of an inventory, then this would be strong evidence that more fundamental processes might well be at work. We additionally postulated that such processes will in turn govern the aggregate characteristics of an inventory as a whole entity, but are not evident from the nature of the individual items in the inventory.

The rational for this research focus was for several main reasons. First and probably foremost was because of the opportunity afforded to the author by DAF Trucks (GB) to investigate the nature of the spare parts inventory maintained by the company. This gave the author access to a considerable amount of data generated from the various systems in operation at the GB headquarters. Some of the data sets extended back to 1975, which provided a valuable opportunity to observe the nature and behaviour of the aggregate demand for inventory items effectively over some ten years of operation. Secondly, spare parts inventories are of fundamental importance to the profitability and competitive advantage of any company selling and distributing capital equipment. Hence it is a potentially valuable area for study in terms of improved operating decision rules and performance measures. In the DAF Trucks case maintaining an inventory of some £5 million at cost to support annual spare parts sales of some £20 million in 1986 at a 92% to 94% service level was paramount to their success and survival in the highly competitive heavy commercial vehicle market. Indeed DAF increased its share of the UK market for heavy commercial vehicles every year since the company started UK operations in 1972 until 1987 (when DAF and Leyland merged). It was the generally held view of DAF executives that a major part of the success had been due to the high level of service assured to potential truck operators from the HQ spare parts Operation and support to the dealer network. This success has been in the face of fierce and formidable competition from other truck companies such as Volvo, Saab, and Ford.

1.2 The nature of spare parts inventories

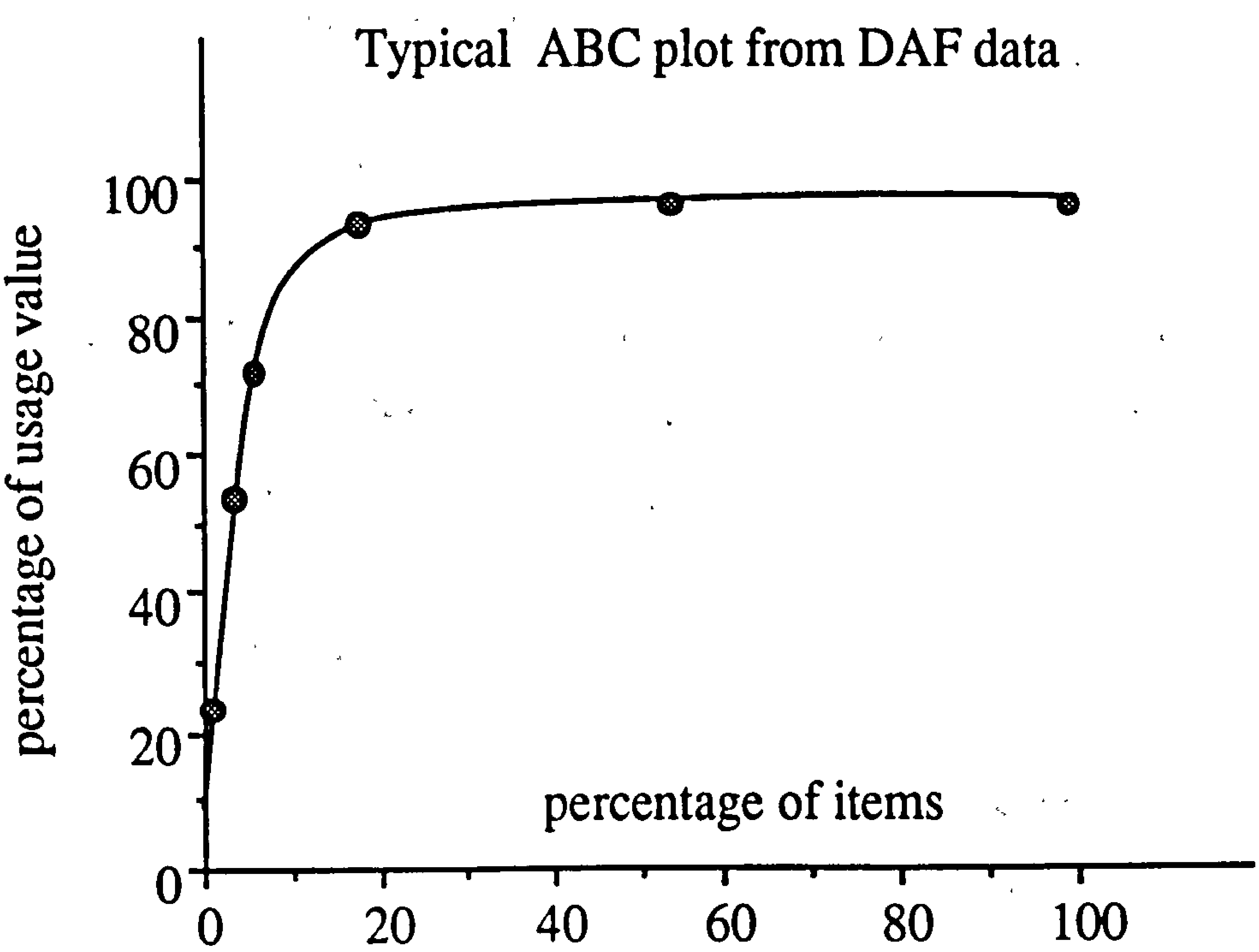
Spare parts inventories for complex capital equipment often exhibit dramatic variations in their basic characteristics of price, demand, and item variety. It is the magnitude of these variations that mark out spare parts inventories as very different to many other independent demand inventories such as wholesale and retail food inventories, electrical goods, clothes, books and newsprint, chemicals, and pharmaceuticals. If we consider in detail the component structure of capital items such as trucks, cars, tanks and aircraft, for example, we soon become aware that they are assembled from a bewildering array of component parts ranging from simple nuts and bolts to complex sub units such as gear boxes, engines and differential drive units. Additionally we can recognise major component groups such as electrical, mechanical, body components etc. Some parts exhibit gradual wear out patterns of demand whilst others are subject to random failure. Also the number of different components in an individual truck model can reach in excess of 8,000. In comparison the number of parts in a Boeing 747 or example will be many times this figure. The total number of items listed in the parts catalogue for potentially active items at DAF GB headquarters during 1986 was around 25,000 part numbers, although only some 12,000 of these showed a positive demand in the preceding year. The parent company in Holland maintained a parts range in excess of 60,000 items to support an even larger truck model range. Caterpillar Tractors at Coventry have an inventory of spare parts well in excess of 65,000 items to support their range of earth moving equipment. In contrast the total number of items held at the Daventry warehouse of Ford of Great Britain exceeds one million parts, whereas a local Ford car dealer is likely to supply around 120,000 parts. In 1984 Dan Air, the commercial airline company, held a stock of some 78,000 different consumable spares to keep their fleet of aircraft serviceable, together with a range of some 15,000 rotatable spares. The remarkable feature about these figures is that the numbers could have been significantly higher were it not for the fact that these companies generally had vigorous policies of variety reduction to keep item variety under some degree of control.

The demands patterns for individual spare items typically range from very regular to highly erratic. Some are also slow moving whilst others are very fast moving. In the DAF inventory the range of variation seen is very wide. Items such as fan belts, oil filters and tyres and gasket sets (classified as consumable or wear items by some authors) generally exhibit regular fast moving demand patterns. Whereas complex items such as gear boxes, engine units, differentials etc., (often classified as repair items) generally exhibit erratic slow moving demand patterns. Items such as body parts are usually subject to replacement through accident occurrences. Wing mirrors, for example, seem prone to breakage at a regular rate whilst other components in less prone positions show a more erratic or lumpy demand pattern. As will be shown in this work the one characteristic that seems common to all the observed demand behaviours for spare parts, whether for regular, erratic, slow or fast moving items, is that the underlying demand behaviour is Poisson in nature and demands for individual items can be generally modelled by various formulations of what we will refer to at this stage as modified Poisson models. This is a point that will be considered in depth in chapter five.

The economic value of spare parts also typically exhibit very wide variation. The price of a component spare part will in general reflect the cost of the base materials plus the 'value added' increment to cover direct labour effort and appropriate overhead and profit margins. Therefore as items become larger and more complex in production requirement they tend to become proportionately more costly. In the DAF inventory the cost price range, is from one new pence for nuts and washers, to new engine units which cost in excess of £6,000 (at 1986 prices), with a vast number of items in the range £2 to £50. When prices are considered in conjunction with the demand volume we can compute usage values and we find these vary individually from pence to over £100,000 on an annual basis. When usage values are considered in terms of the typical ABC or Pareto curve we see another distinguishing feature of spares inventories. Inventory usage values are generally very concentrated such that the major part of the turnover in a period is concentrated in just a that the major part of the turnover in a period is concentrated in just a small proportion of the item range. A typical ABC curve for the DAF inventory is shown in figure 1.1.

The often quoted yardstick of 80% of the turnover being achieved by 20% of the items certainly does not hold for spare parts inventories for very complex equipments such as trucks or aircraft. In the DAF case the top 20% of items account for approximately 94% of the turnover, whilst 80% of the total turnover is achieved by only 7 % of the items. This high degree of concentration is due to several factors. The great range of parts prices, coupled with the great variation in the demand volume, and the existence of a very large number of very slow moving parts in the inventory range.

Figure 1.1



Various authors have attempted to classify spare parts, but these than the supplier. Such classifications involve a consideration of the nature of demand, whether random generated by failure processes or by continuous wear demand, the volume of demand, and usually a measure of value. These classifications are not considered here because they are not directly relevant to this research. The reader is referred to Mitchell (1962), and Tan (1984).

In the context of the DAF data the following broad classes of spare parts items were identified, after discussion with company management, and it is based on the nature of the process seen as most likely to cause a need for a replacement spare part.

(a) **continuous wear parts:**

Those items which are subject to continuous wear processes such as brake pads, oil filters, tyres etc.

(b) **accident failure parts:**

Items for which a demand occurs due to spasmodic random events of the accident failure type. Items such as body parts would be subject to this kind of demand.

(c) **wear out parts:**

This group would comprise items that exhibit a wear out pattern whereby they remain serviceable for long periods but then failure is accelerated during a so called wear out phase.The majority of mechanical moving parts fall in this category such as bearings, gear wheels, valves etc. The average time to failure of such parts can be estimated from reliability studies.

(d) **random failure items:**

This group will comprise those items for which the failure pattern is completely random. The mean time to failure may be predictable but items in this group generally have constant failure rate probability. The group is typified by electrical components such as bulbs, fuses, relays and the like.

(e) **consumable repair items:**

The demand for items in this group generally depends on the demand for other items as they are typified by the myriad of nuts, bolts, brackets clips etc. that are replaced when repairs and servicing is undertaken to replace parts from the other groups.

Items can also be further categorised into fast, moderate or slow moving in demand volume terms, and high, moderate and low unit value. However, the above categorisation has no special value in this research other than to add further insight into the great diversity in the characteristics of typical spare parts inventories.

1.3 Preliminary analysis and inferences

In this section we present some of the preliminary observations and initial analysis of the DAF Trucks data. As indicated earlier the empirical research started with the objective of verifying the claims of R.G. Brown that inventory period usage values could be represented by the lognormal distribution and that the parameters of the fitted distribution could be used to set aggregate inventory standards. The statistical nature and properties of the lognormal distribution are considered in some depth in chapter three so only an outline of this distribution is given here. It has a probability density function as given by equation 1.1 below-

$$f(x)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}\{\log_e x-\mu\}^2\right] \quad \text{------------------} \quad 1.1$$

It is a two parameter distribution where σ is the shape parameter and μ is the location parameter. If a lognormal variate 'x' is subjected to the logarithmic transformation $\log_e x = y$ then 'y' is normally distributed with mean μ and standard deviation σ. From this we then have a method for a first attempt to test if a particular set of data is likely to lognormally distributed because by taking the logarithms of the variate of interest we can check for normality of the transformed data.

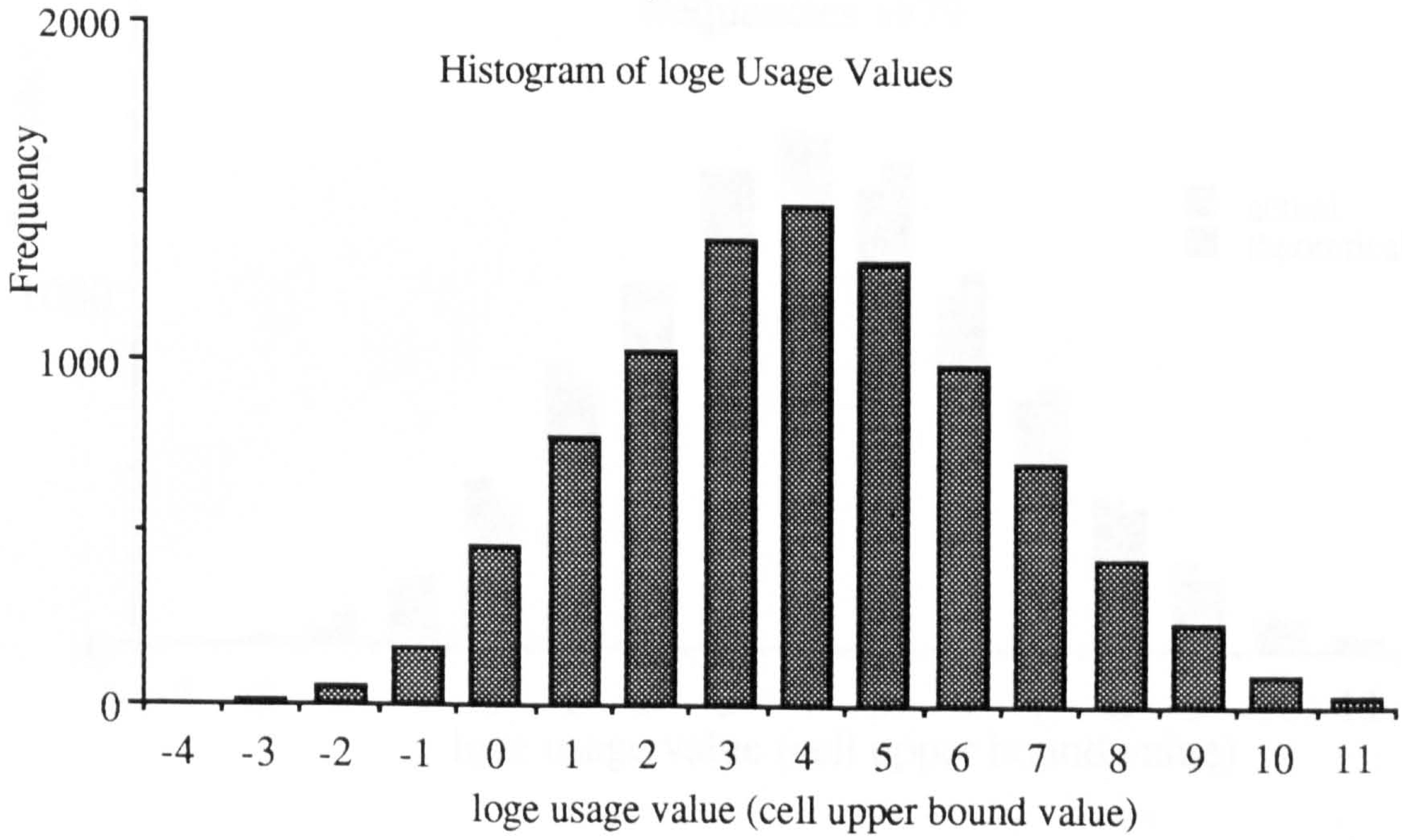
The data set initially analysed was usage values for the entire active inventory range for 1979 at the DAF UK headquarters; the distributor level. Fortunately the computer system in use at DAF at the time contained a programme that could produce an ABC listing of usage values in descending order of value. Although this document was not used on a regular basis by the Parts department personnel the system to produce it was run for the benefit of the author's work. Fortunately ABC listings were also available in the company for earlier years, because no one had bothered to clean out an old cabinet where the reports had been gathering dust for several years. Because of the availability of these reports there was no need to resort to any sampling at this stage as an ABC listing contains a complete picture for all active items. A copy of part of the ABC report for 1979 is given in appendix seven. The data from this report was divided into equal logarithmic bands as follows and the frequency of the number of usage values falling in each band counted to give the results shown below in table 1.1. Figure 1.2 also shows the same data in histogram form.

table 1.1

1979 Usage value data

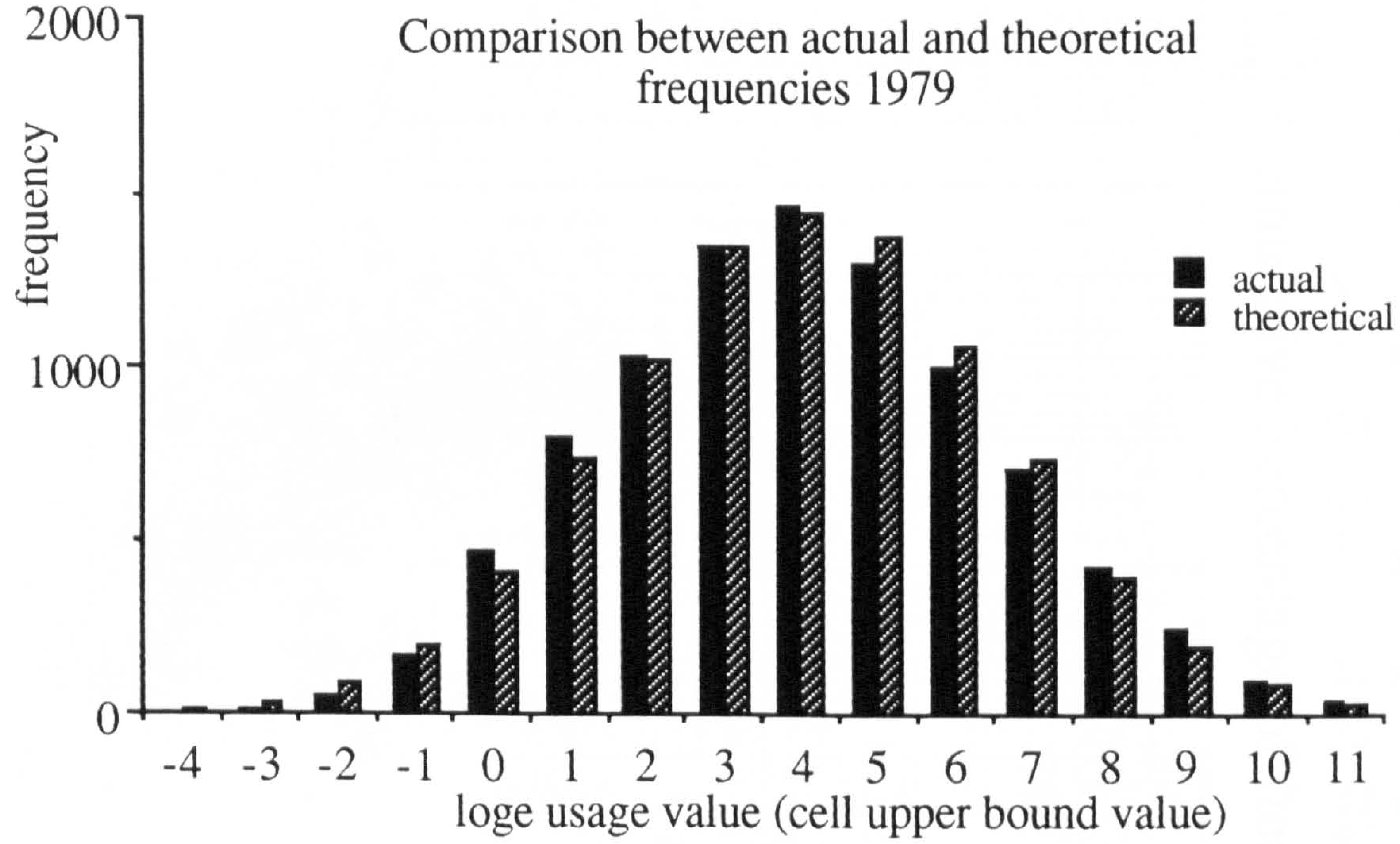
Usage value upper bound	loge Usage value	frequency of items	theoretical frequency	difference
442413.00	13	1	2	-1
162754.00	12	0	8	-8
59874.00	11	35	28	7
22026.00	10	94	83	11
8103.00	9	241	196	45
2980.90	8	427	401	26
1096.60	7	697	724	-27
403.43	6	987	1052	-65
148.41	5	1285	1371	-86
54.60	4	1452	1439	13
20.08	3	1344	1331	13
7.39	2	1072	1022	50
2.72	1	774	723	51
1.00	0	461	408	53
0.37	-1	171	192	-21
0.13	-2	50	80	-30
0.05	-3	7	28	-21
0.02	-4	2	8	-6
0.01	-5	0	2	-2
	SUM	9100	9098	15

figure 1.2

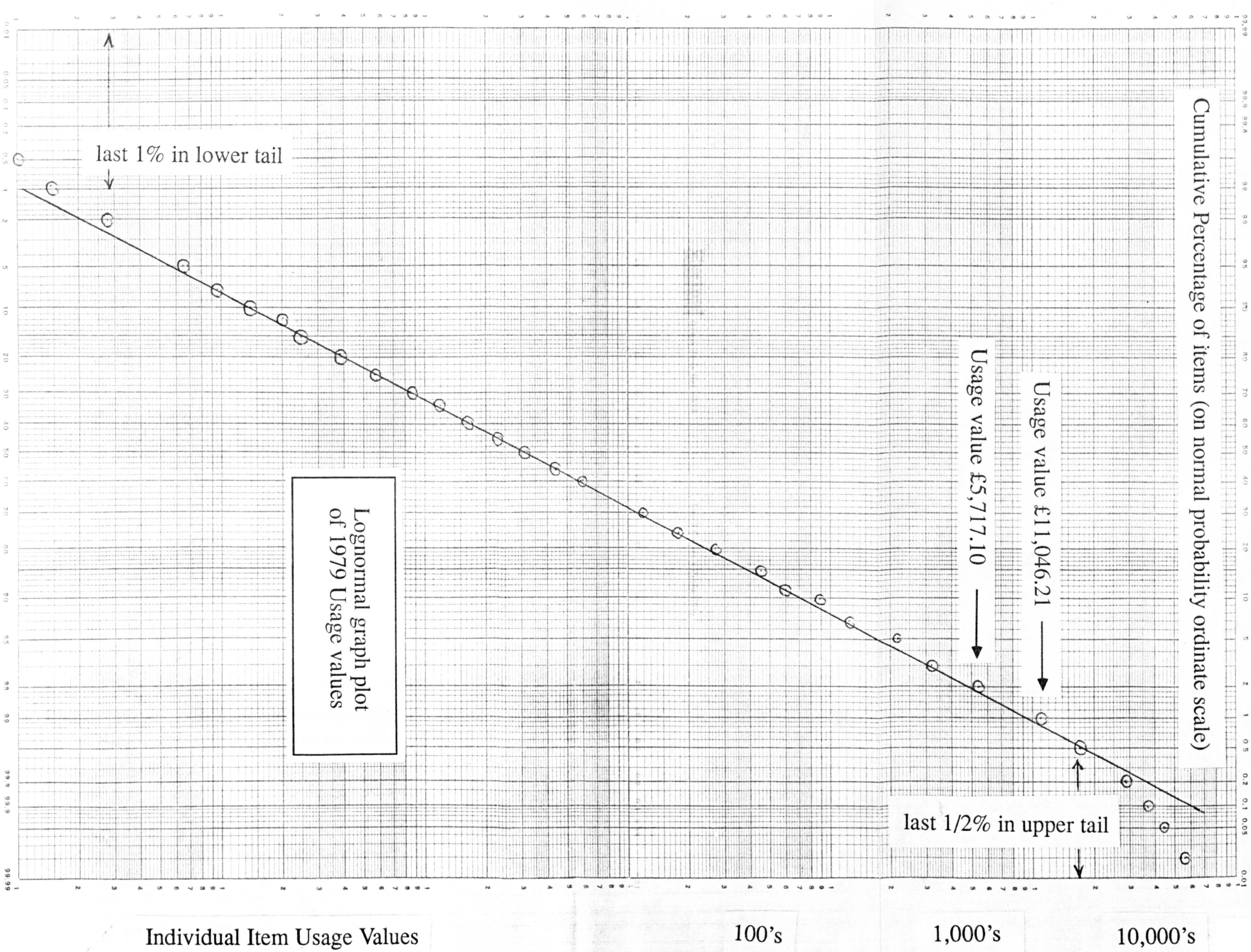


It can be seen from both table 1.1 and figure 1.2 that the empirical data of log usage value gives a remarkably symmetrical distribution which is characteristic of a normal curve. The theoretical frequencies in the table are those calculated for a theoretical normal curve with the same mean of 3.51 and standard deviation of 2.48 as the observed empirical distribution. The overall total deviation of the actual frequencies from the theoretical values is very small at 15 from 9100 items indicating a very close fit to the corresponding theoretical normal curve. The close correspondence between actual and theoretical frequencies can be readily seen from the histograms in figure 1.3.

Figure 1.3



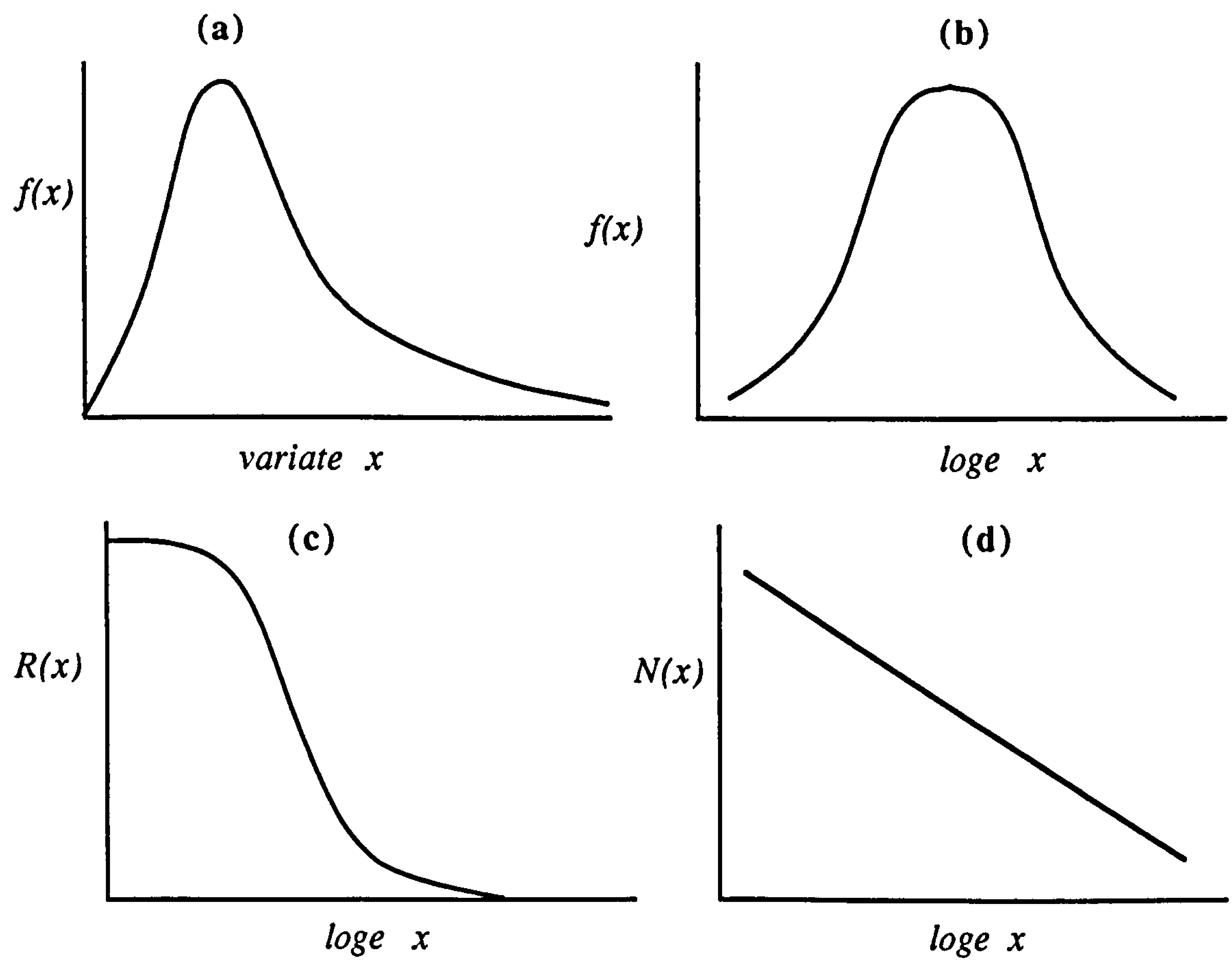
Perhaps an even more impressive fit of the empirical data to a lognormal curve is to plot usage values on lognormal probability paper as shown on the next page. This is achieved by plotting cumulative item percentage against the usage value of the same item from the 1979 ABC listing. (See example pages in appendix seven). For example, the item that represents 1% of all items is the 91st. in the list with a usage value of £11,046.21, whilst the 2% item is the 182nd. item in the list with a usage value of £5717.10 and so on. As can be readily seen the fit to the straight line is incredibly close throughout most of the entire range. Only in the extreme tails of the distribution does any significant departure from lognormality exist. This is the last 0.5% in the upper tail (large values) and the last 1% in the lower tail.



last 1% in lower tail
Lognormal graph plot of 1979 Usage values
Usage value £5,717.10
Usage value £11,046.21
Cumulative Percentage of items (on normal probability ordinate scale)
last 1/2% in upper tail
Individual Item Usage Values
100's
1,000's
10,000's

Lognormal probability graph paper effectively transforms the original data as shown in the following diagrams-

Figure 1.4
Lognormal transformation



In the above diagrams, from (a) through to (c), (a) represents the lognormal distribution, (b) is the transformation of x to $\log_e x$ giving a normal form, (c) is the inverse distribution function of $\log_e x$ and in (d) $N(x)$ is the normal probability ordinate plotted against $\log_e x$.

When this author first plotted DAF 1979 usage value data on lognormal probability graph paper he was very surprised by the high degree of closeness of the degree of fit to a straight line that was obtained.

A similar degree of fit was also seen for each year from 1975 to 1979, and then subsequently for 1980 onwards. There is no doubt even without conducting any goodness of fit tests, such as the Chi Squared or Kolmogorov-Smirnov, that usage value data from the DAF spare parts system do indeed fit very closely to lognormal distribution curves. The general issues relating to goodness of fit tests for such highly skewed distributions are considered in chapter two so we will not pursue any discussion at this stage. Suffice to say here it is a thorny issue and one that must be given serious consideration in research of the type reported in this work.

The author also made a further important observation at an early stage in this research. If usage values are lognormally distributed, then the nature of the distribution of prices and volumes must also be considered. Because the usage value of an inventory item is the product of its price (or cost) and the volume demanded in a chosen period. Analysis subsequently revealed that prices were lognormally distributed and so were demand volumes provided that the time period chosen was reasonably long (at least 9 months in the DAF situation). This subsequently turned out to be an important finding to the development of a model explaining the underlying processes at work. The phenomena of prices and volumes also being lognormal in nature is consistent with statistical theory because Aitchison and Brown [not R.G. Brown] (1957) have shown that if a variate 'x_1' is lognormal and so is a separate variate 'x_2' then the product '$x_1 x_2$' is also lognormally distributed with mean $(\mu_1+\mu_2)$ and variance $(\sigma^2{}_1 + \sigma^2{}_2)$, where μ_1 and μ_2 are the means of the individual variates, and $\sigma^2{}_1$ and $\sigma^2{}_2$ are the separate variances. This author was surprised to discover this finding had not been reported before in the context of inventory period usage values. Either it had not be recognised by previous authors, or alternatively they did not appreciate the significance of this finding.

The fact that usage volumes were only lognormal in form for periods longer than nine months, was discovered quite accidentally. When usage values for a particular three month period were determined during part of the general exploratory analysis the characteristic normal curve was not obtained when these usage values were subjected to the logarithmic transformation. Further examination showed that it was the volume part of these usage values that failed to be lognormal giving a very irregular shaped distribution. It was subsequently noted that if the usage volume data was cumulated period by period (a period being four weeks) then the volume data gradually converged to a lognormal distribution and once lognormality was achieved it appeared to be the stable long run distribution of usage values. We also discovered that the aggregate distribution of demand volumes in very short time periods (four weeks) was clearly not lognormal, but such demands did show a remarkable degree of statistical regularity from period to period that extended over a long time. This also proved to be a very important finding for the modelling work of chapter seven.

1.4 The existence of lawlike relationships

The very high degree of regularity shown in the DAF inventory period usage values did answer the initial question posed in this research namely- can such variates be explained and modelled by a lognormal distribution curve. Clearly even the limited analytical evidence presented so far strongly supported this. The extent to which the data fitted the lognormal model also posed several questions. Could such regularity arise by chance or was some fundamental process operating that could bring about such regularity? If some underlying mechanism was controlling the form of the empirical distribution then what was the nature of this mechanism. However, as will be discussed in chapter two, before a particular statistical model can be accepted as the appropriate form to explain the regularity seen in a particular system there should also be strong stochastic evidence to support the model. Such evidence could be of an empirical nature indicating some regular process following a consistent pattern or path over a period of time. Or it could be a theoretical process developed and proven from other fields of enquiry and shown that the same model is obtained as the long run equilibrium state of the system.

Given the very diverse nature of the data observed, for such regularity to have occurred as a chance event was seen as most improbable. As we have shown the inventory range in the DAF system comprised such diverse items as gear boxes, lorry cabs, oil filter elements, spark plugs, nuts and bolts, bulbs and fuses and so on. When the equally diverse unit price range and demand volumes were also considered then to find usage values with such statistical regularity and stability was really a staggering phenomena, and not one that could conceivably occur by chance. Furthermore the regularity seen was not confined to one data set. The same degree of behaviour of the system was observed every year at the DAF UK headquarters from 1975 to 1979 and then subsequently for the period 1980 to 1986.

This consistency from year to year must also be considered against the changing economic and commercial circumstances of the period. Throughout the time considered, effectively 1975 to 1986 the truck market in the UK went through two major recessions with the knock on effect on global parts sales. This period was also marked by changing inflation rates which were comparatively low in the early 70's, rising to very high figures around 18% to 20% in the late 70's and early 80's. Then inflation fell back to levels around 4% to 5% in the mid 80's. The effect of this on parts prices was quite dramatic with general price increases every year, sometimes two during the high inflation periods. Also the sales for some parts were put under strong commercial pressures during the mid

80's as parts 'copy cat' manufacturers and distributors began to move into the market to offer price competitive alternatives to genuine DAF parts. The company response to this was to price certain parts more competitively; hence parts price movements have not been uniform throughout the parts range. Indeed some parts prices were decreased to combat competition. Also over the period 1976 to 1986 new truck models were introduced, which in turn increased the number of parts to be stocked. Additionally many parts were superseded by new designs. In some cases one new part replacing several old ones because of design rationalisation programmes. Over the same period demand for some parts gradually declined; down to zero in some cases as old truck models became fewer in number and gradually declined from the active truck park in commercial use. Overall however, the active parts range within DAFs UK operations (parts with a positive demand in a one year period) gradually increased year by year from around 5,500 per annum in 1975 to almost 12,000 per annum in 1986, whilst the total number of parts on the stock list grew to around 25,000.

The 70's period brought forth the dramatic rise in fuel prices following the two major oil price increases of 1973 and 1978/79. The follow on recession in general trade after each oil price shock hit the UK commercial vehicle industry hard as fewer commercial goods in particular were transported by road haulage. Under such conditions many commercial vehicles would have been under utilised, whilst others would have been over utilised because of contractors attempting to bid more competitively by spreading operating costs. This behaviour would almost certainly have had an effect on the maintenance practice and policy of the users. Also subjecting vehicles to heavier duty would have taken its toll on component failure rates and vehicle breakdown occurrence. It can be reasoned that these circumstances would bring about changes in the pattern of demand for many spare parts in the inventories of companies such as DAF. Given the background of these changing macro economic and commercial factors one fact shone through concerning the DAF inventory. The usage rates remained consistently lognormal in nature and perhaps more importantly we observed that the values of the parameters of the fitted distributions exhibited a remarkable degree of stability over the entire period studied.

1.5 The need for theoretical development

Clearly from the foregoing discussion the direct empirical evidence and background circumstances strongly indicated that some underlying stochastic process was regulating DAF inventory usage values, and it appeared to be a process that exerted a remarkable degree of stability. There were many questions that needed to be explored. Of prime importance was the question concerning the underlying stochastic process. What was the form and nature of this process and what could it tell us about the practical application of lognormal theory to inventory management issues? We also needed to know how stable the process might be over time, and if the lognormal distribution was the natural long run stable model of the system. Furthermore we needed to see if a stochastic model could be developed that would satisfactorily explain the nature and behaviour of the processes at work and the ultimate occurrence of the lognormal distribution? We also needed to know if a more fundamental understanding of the processes at work would enable us to develop better tools for monitoring and controlling the behaviour of the DAF and similar inventory systems. The need to answer these foregoing questions ultimately formed the principal objectives we set for this research work.

The method of attack to achieve these aims is discussed in some depth in the next chapter concerning research methodology. Suffice to say at this stage the approach was broadly two pronged. First the literature was reviewed for processes that can lead to lognormal distributions as stable long run models of economic variates. The author regarded this as the 'top down' approach by starting with processes that can yield a lognormal model of a particular variate. Such evidence was drawn mainly from sources in the literature on economic models and from theoretical statistics. The Literature on inventory theory and practice was of very little value in this area. The second main attack was from the stand point of examining the basic nature of demand processes for individual items, and to see if these could possibly be combined into models of aggregate demand. We regarded this as the 'bottom up' approach and it was commenced by drawing on sources from the inventory literature, the buyer behaviour literature and the theoretical statistical literature. It subsequently transpired that certain stochastic processes explaining demand for individual items could be applied to the aggregate case. The initial clues and validity for this were confirmed by evidence obtained from the Biometrics literature. The final stage was to explore ways in which the two approaches could be drawn together to give an overall model that explains the basic nature of the aggregate demand processes and the occurrence of the lognormality of usage values.

1.6 The structure of the thesis

This thesis is structured into five main sections. The first covers this introduction and a chapter concerning methodological issues. The second section covering chapters three, four and five presents much of the core theoretical material reviewed from the literature. The third section covers chapter six, that presents much of the empirical analysis for DAF trucks and chapter seven that presents the author's theoretical development of new models of aggregate demand. Section four is primarily about testing and validating the theory using empirical data from the DAF system (chapter eight), simulation studies (chapter nine), and empirical data from other spare parts systems (chapter ten). Section five covers some novel applications, a consideration of the concentration principle applied to certain economic variates, and finally interpretations and conclusions, and suggestions for further research. Because this thesis was multidisciplinary in nature we drew on concepts, ideas, theories and previous empirical work from a wide variety of sources and different fields of investigation. In consequence to bring all these divergent factors and aspects together in a systematic framework there are more sections and chapters concerned with reviewing and summarising the essential prior research work than might be found in many theses.

In chapter two we consider aspects of the empirical approach to research and in particular we review some important aspects concerning the approach to the testing and verification of so called 'extreme hypotheses'. We also make a critical examination of the standard methods used for the statistical goodness of fit tests. The chapter then presents the organising framework for the overall research process adopted in this work.

Chapter three presents the underlying theory, history and properties of the lognormal distribution. We also summarise many of its applications in the broad field of business and economics. Chapter four reviews the important stochastic processes that are known to yield lognormal and similar functions in specified circumstances and some related controversies.

In chapter five we lay down the essential foundations concerning the stochastic nature of recurrent event processes. In particular we review the nature of Poisson processes and various important modified Poisson process models that have been used in inventory modelling and consumer purchase research. By drawing on prior research work and our early understanding of the empirical nature of DAF inventory usage values we develop our initial working hypotheses in this chapter.

The major part of the empirical analysis on DAF inventory data is presented in chapter six. From the work presented in this chapter we are able to demonstrate strong support for the working hypotheses of chapter five. Chapter seven develops much of the new theory and stochastic models that explain the underlying nature of lognormality of spare parts usage values and concludes with refined statements of our research hypotheses. Chapter eight applies the process of retroductive testing of the theoretical models of chapter seven, by applying them to DAF data. The results strongly support the acceptance of the hypotheses of chapter seven. The next chapters, nine (simulation) and ten (additional empirical studies) are concerned with validating the developed theory and hypotheses by providing additional supporting evidence.

In chapter eleven we present a review of the literature on the concentration principle of economic variates with particular reference to inventory usage values. We then discuss the relationships which underpin the so called Pareto principle and provide essential background to the work in chapter twelve. The literature work in chapter eleven was not included with the literature work of the early chapters because it would have made them far to large and logically this subsequent review underpinned the novel development work presented in chapter twelve. Therefore we have broken with tradition somewhat in our overall structure. Although we used the classical route of 'introduction, literature review, empirical analysis, theory development and testing', we followed this process with sections concerned with novel applications and supporting literature reviews. The justification for this is that we have used the results and outputs from the research to highlight and develop a number of potentially valuable areas for extending the derived theory, and developing new inventory management applications. We hope that by this process we have laid a foundation for considerable additional work.

Research Methodology and Methodological Considerations.

2.0 The empirical approach

In this chapter we consider some of the general characteristics of empirical research and discuss the views of several prominent authors regarding the law finding process. We then give consideration to some of the important issues relating to the use of goodness of fit tests in validating extreme hypotheses, especially those concerning highly skewed distributions.

The research reported in this thesis was primarily empirically based and it has sought to advance the understanding of aggregate demand processes and the underlying nature of usage values for replacement spare parts inventory items. An empirical approach to research generally starts with an observation of the state of nature of a system, or of system behaviour, either as a result of direct observation, or indirectly through the collection and examination of data from the system. Initial interest might generally stem from the recognition that a particular system under study exhibits a phenomena that has some, as yet, unexplained regularity in terms of established theory and prior knowledge. Such behaviour might be the first clue to some underlying process at work governing the states or the behaviour of the system. Alternatively, the system may be seen to be behaving chaotically or randomly where regularity might otherwise be expected to exist, indicating the possible existence of some mechanism or process causing departure from regularity. Such observations may be the first clue that some previously held theory does not hold under the circumstances observed.

Using an empirical approach we set out to discover reasons and explanations for the observed system states or its long run behaviour, and eventually try to determine the existence of any underlying processes that explain the observations made. This would generally lead to a generalised statement of the states or processes observed and explained in terms of a lawlike relationship. The methodology should then attempt to try and reach some understanding of the boundary or convergence conditions for which the lawlike relationships hold. In this way the range of conditions can be specified over which any laws, and any derived decision rules, can be expected to apply. This makes for valid transferability of the laws and any derived decision rules to alternative, but similar systems and for predicting possible outcomes of system behaviour.

The establishment of boundary conditions involves determining just how far one can validly move, or extrapolate, from the particular to the more general case with the general law-like relationships still holding true. This whole procedure is generally referred to as the 'empirical - inductive' approach to basic research. To put this into perspective in the context of this research it would mean establishing that the lognormal distribution is the appropriate model to use to graduate period usage values in the DAF inventory system. Then establishing the range of conditions over which the model applies e.g. time, range of items and level of inventory (dealer level, wholesale level and factory level). Finally we would need to verify to what extent the methodology can be applied to other spare parts systems, and eventually to test against non-spare parts inventory systems. Thus the process develops from the particular to the general. The general being expressed as a theory of a law like relationship. A process of retroduction is discussed later whereby a generalised expression or developed theory is then used to explain the observed empirical facts.

We can also see the general empirical approach to research as gradually unfolding over four levels of investigation.

(a) the observational-descriptive level.

By chance, or positive search, the recognition of some, as yet, unexplained phenomena or behaviour followed by a clear description of the nature of the observations and the conditions under which they are observable.

(b) the explanation level.

To deduce by suitable analysis and explanation, using proven methodology, whether the data, system behaviour, or the system characteristics could have arisen by chance, or are most likely to be the result of some underlying process. Then to postulate possible processes or mechanisms or models that could generate the system characteristics under the conditions in which they were observed.

(c) the interpretive level

To formulate appropriate hypotheses to test models against the observed system with a view to selecting and developing a system model that can be used to interpret the behaviour of the system to achieve greater understanding of its characteristics. This procedure includes seeking alternative processes such as stochastic evidence which supports(and indeed suggests) the choice of particular models.

(d) the predictive level

Establishing the boundary conditions within which the phenomena is known to operate and hold true, including the specification of conditions under which the underlying processes converge to stable conditions. Also to clearly establish how far the results and conditions of operation can be extended into alternative systems for predictive purposes.

2.1 The evaluation of extreme hypotheses.

The graduation of empirical data by a particular theoretical model and the acceptance (or rejection) of the model, as that which explains the regularity in the data, is what some authors refer to as the testing of extreme hypotheses. (in particular see - Ijiri and Simon [1977, page 109]). It is argued by Simon and others, eg. Hanson (1961), that acceptance or rejection of a particular model using the classical non parametric methods of statistics, such as the Chi Squared test or the Kolmogorov Smirnov test, is basically unsound. Their argument is that the burden of proof in testing theories of this type rest on disproving the theory on the basis of significant deviations from the law. If the deviations are not significant then the law is not rejected, or disproved, but more importantly neither is it proved. Simon also argues that many situations may exist where a law holds yet aberrant deviations may show significant differences as tested by the classical approaches, especially in cases where large samples are examined due to the size of the deviations. Ijiri and Simon then go on to define the law-finding process as -

(1) *Finding simple generalisations that describe the facts to some degree of approximation.*

(2) *Finding limiting conditions under which the deviations of the facts from generalisations might be expected to decrease.*

(3) *Explaining why the generalisation 'should fit the facts'.*

Taking the process to the third stage and explaining why the generalisations fit the facts is called RETRODUCTION by Ijiri and Simon (op cit), after work by Hanson (1961), and this is clearly regarded by them as an important part of the law establishing process. Retroduction can be regarded as the process of going from facts to generalisations, ie to the law like relationships, and back again. The facts suggest the form of the generalisation (the law), but the generalisation must explain the facts. In the case of testing extreme hypotheses, Ijiri and Simon put forward the following argument and procedure for the law finding and testing process -

> *"We have examined several aspects of the problem of testing theories, and particularly those important theories that take the form of extreme hypotheses. In part our argument has been aimed at a negative goal to show that when we look at realistic examples from natural and social science, statistical theory is not much help in telling us how theories are retroduced or tested. As an alternative to standard probabilistic and statistical accounts of these matters, we have proposed that we take into account a whole sequence of events:*
>
> (1) *The enterprise generally starts with empirical data, rather than with a hypotheses out of the blue.*
>
> (2) *Striking features of the data (e.g. that they are linear on a log scale with slope of minus one) provide for a simple generalisation that summarises them - approximately.*

(3) *We seek for limiting conditions that will improve the approximation by manipulating variables that appear to affect it s goodness.*

(4) *We construct simple mechanisms to explain the simple generalisations- showing that the latter can be deduced from the former.*

(5) *The explanatory theories generally make predictions that go beyond the simple generalisations in a number of respects, and hence suggest new empirical observations and experiments that allow them to be tested further. The very process that generates a theory (and particularly a simple generalisation) goes a long way toward promising it some measure of validity."*

2.2 A branching -converging process

Almost by necessity an empirical -inductive approach is likely to be iterative in the sense that starting with a simple observation, the need to explain and interpret may lead to the search for further empirical evidence to support findings and further the understanding of the system. Furthermore such a procedure is likely to become a branching process, as each step is likely to reveal awareness of other branches of investigation that could, and should be pursued. Inevitably at some stage in the process a framework needs to be established within which a more structured programme can be pursued so that the research process will be refined and developed in a more systematic way.

The actual research process following an inductive approach may well be largely exploratory in the early phases, with no set objectives other than to achieve greater understanding of the observations and empirical facts in fairly broad terms. It may even be prudent to delay setting specific objectives until a later stage of the research when low level inferences and conclusions have been drawn. This was the procedure adopted in this research, and a more systematic approach was only formulated when a better understanding of the use and nature of the lognormal distribution curve as applied to inventories was achieved. In a comparatively new area of investigation such as that presented here it may be difficult to plan the process in detail at an early stage. Only when a comparatively large area of understanding has been mapped out in terms of empirical observation and analysis does it become really possible to establish particular research objectives. It might be argued that this approach could be regarded as unsystematic. However, we argue that it is sound and scientific because the development of specific research hypotheses is delayed until our empirical observational knowledge of particular areas is much better developed. Indeed the development of initial inferences and preliminary working hypotheses do give strong clues and direction to valuable areas of further investigation and hence what additional information to seek and how to examine such evidence once we have obtained it.

Once it became clear that analysis of usage values in the DAF inventory system strongly indicated the likely existence of some underlying stochastic process then a much broader systematic search of the literature was undertaken. There was two main approaches to this search. First the literature was examined to see what had been reported in fields other than inventory theory on the application of lognormal models. Related to this was the search also for stochastic processes that can lead to lognormal, or very similar models. The second main approach was to examine demand processes at a fundamental level and ultimately to search for links between the basic demand for individual items and the aggregated demand for a whole inventory. In view of the fact that the inventory literature provided very little insight into possible stochastic processes relating to aggregate economic variates, the author had to research widely in the fields of economics, econometrics, biometrics, operations research, applied statistics, market research and buyer behaviour literature.

2.3 Judging plausible theories.

The empirical distributions observed in this research for parts prices, period usage values and volumes (both short run and long term) were all seen to be highly skewed. We have seen earlier in this chapter Simon's reasons for not relying on standard statistical tests alone to judge the validity of such distributions. In addition to Simon's views there are other considerations that put the use of standard significance tests in question in the case of highly skewed distribution. Although it depends on the value of the parameters of a particular distribution the highly skewed distributions seen in economics and business data usually have very long tails in the upper region. This is especially true of the majority of lognormal distributions. As a result of this characteristic the frequency of values in the upper tail region will be very small. One of the requirements of the Chi Squared test is that the frequency in any cell must be equal to or greater than five. If such a test is applied to a highly skewed distribution then, to meet the cell frequency criteria, much of the fine structure in the long tail is lost. (This is not a problem with the Kolmogorov Smirnov test however as we discuss in section 2.4 below). Choosing an appropriate test to differentiate between competing candidates to fit highly skewed data is not a straightforward situation, so we discuss this problem in more detail later in this chapter. Suffice to say this issue additionally supports the view that evidence must be sought from other sources to support a particular model and the process of retroduction should be applied to the law testing process.

A further perspective concerning the testing of theories involving fitting empirical data to statistical models has been provided by Aitchison and Brown (1957). Their view is that there are at least two important reasons for seeking a more fundamental basis for explaining the apparent description of a set of empirical data than relying on standard statistical tests of significance. First, by providing such a basis a clearer insight may be obtained into the underlying natural or sociological process, which in turn may suggest wider application of the system. Then second a knowledge of the elementary assumptions from which the law of frequency may be derived will enable modifications to be applied to the law to meet the needs of new circumstances. These same authors then go on to argue that it may be more satisfactory to use a system of frequency curves for which there is a plausible basis for the genesis of a particular distribution than a system which is solely just more successful in graduating the sample observations of empirical data.

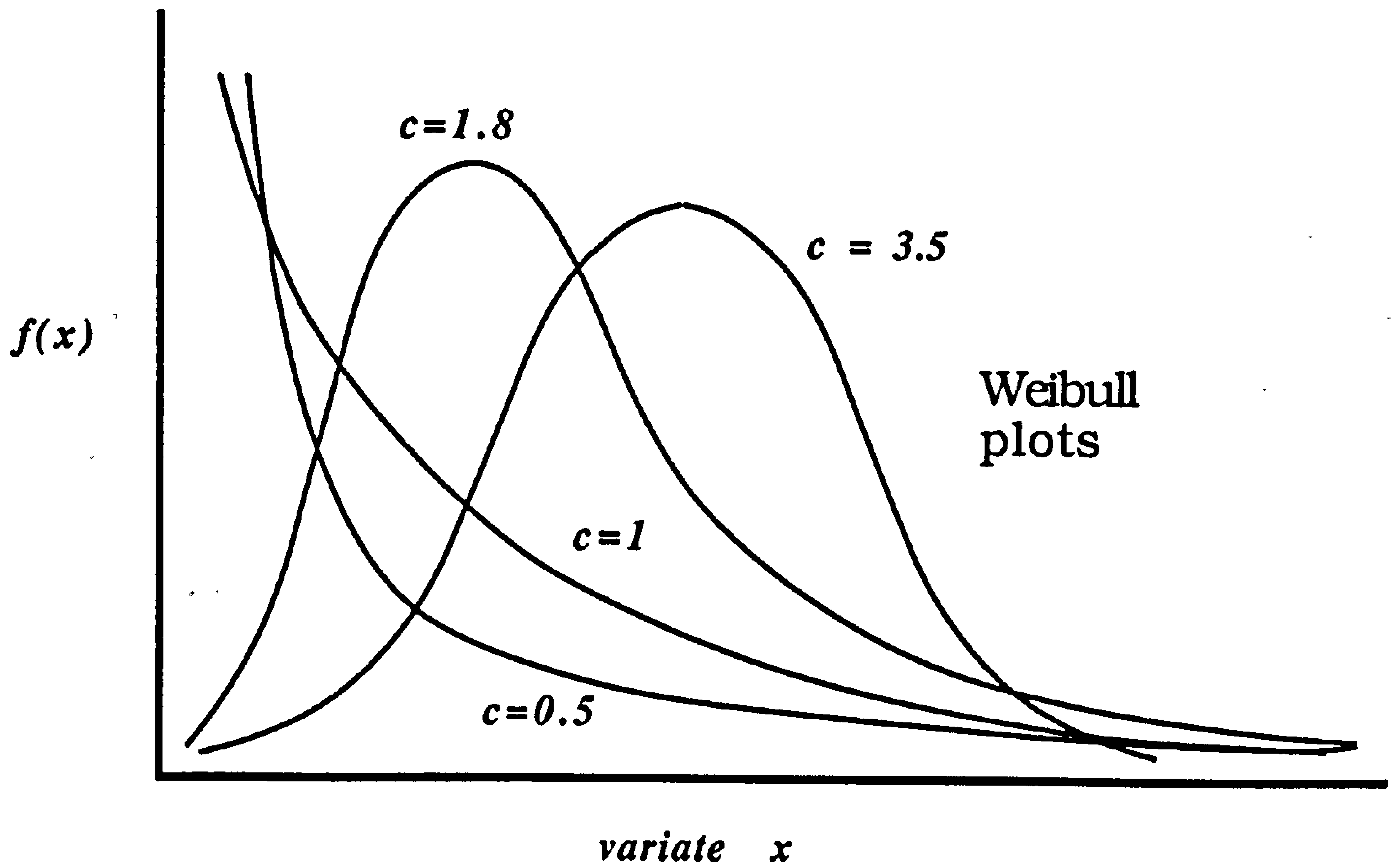
Aitchison and Brown's point above, which effectively says that success in graduation alone is not a sufficient criteria in choosing a particular distribution, can be seen in the right perspective when one looks at the nature of distributions such as the Weibull and the RS. In one sense these distributions can be regarded as synthetic distributions, mathematically contrived to fit a wide range of data. The Weibull distribution, attributed to Walodo Weibull, is a very flexible two parameter distribution whose density function is of the form as shown-

$$f(x)=\left[\frac{cx^{c-1}}{b^c}\right]\exp\left[-\left(\frac{x}{b}\right)^c\right]$$

In this form 'c' is the shape parameter, and 'b' is the scale parameter. A few of the great variety of forms it can take for different values of 'c'

are shown in the diagram below. It can be seen that when 'c' has the value of unity the distribution is exponential, when 'c' is 1.8 it is very close to a gamma distribution and virtually 'normal' at 'c' equal to 3.5. A value of 'c' around 1.5 will give a form very much like a lognormal model.

Figure 2.1
Various Weibull distribution plots



If the distribution does not start at zero then a third parameter 'δ' locates the distribution on the horizontal axis. The RS distribution reported by Dudewicz, Ramberg and Tadikamalla (1974) is a four parameter distribution of even greater flexibility than the Weibull. Almost any regular data set of the types that occur naturally, or as a result of sociological or economic processes, can be graduated by RS distributions by appropriate choice of the parameters; yet the data graduated may well have been generated by a well established stochastic process that leads to a known distribution. For example, the Poisson process of events occurring over time leads to the Poisson distribution of event occasions per unit time provided the time period is short. Any slight aberration in the system may well lead to a poor fit of the data to a Poisson model yet the underlying process may still be Poisson. Under such circumstances the RS distribution may well provide a better fit to the data than the Poisson itself, and yet provide no basis at all for application to the data other than a good statistical fit by traditional methods of judgment.

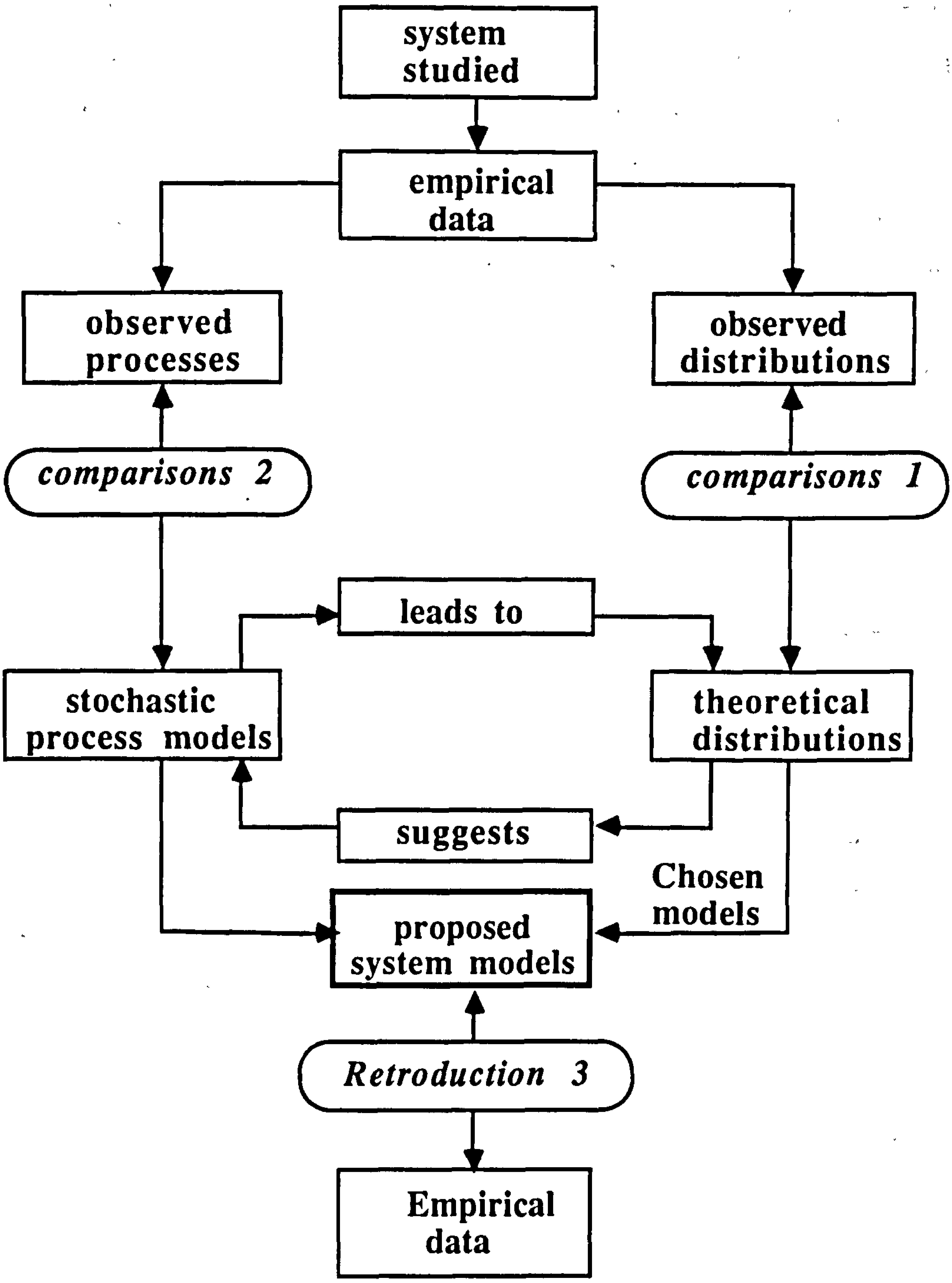
From the preceding discussion we have attempted to show that goodness of fit alone is not sufficient when attempting to find a model which explains the occurrence of regularity in empirical data. Thus, for example, even though DAF period usage values indicated a strong fit to a lognormal model it was considered essential to research at this level to find confirmatory evidence by alternative approaches such as simulation and retroduction.

Whilst this author has taken careful heed of the critical views of Ijiri and Simon, regarding statistical tests, and those of Aitchison and Brown, regarding the need to seek stochastical support , the traditional statistical methods were not ditched completely. It was felt that they still have a part to play, if only at the early stage of exploration and testing providing one is aware of their limitations in use. In later chapters we use both the Chi Squared and the Kolmogorov Smirnov tests quite extensively, but only as a support to the law establishing process. We discuss some of the important issues concerning goodness of fit tests in section 2.4.

The complete procedure we have adopted in this work is formulated and shown in figures 2.2 and 2.3 which follow.

Figure 2.2

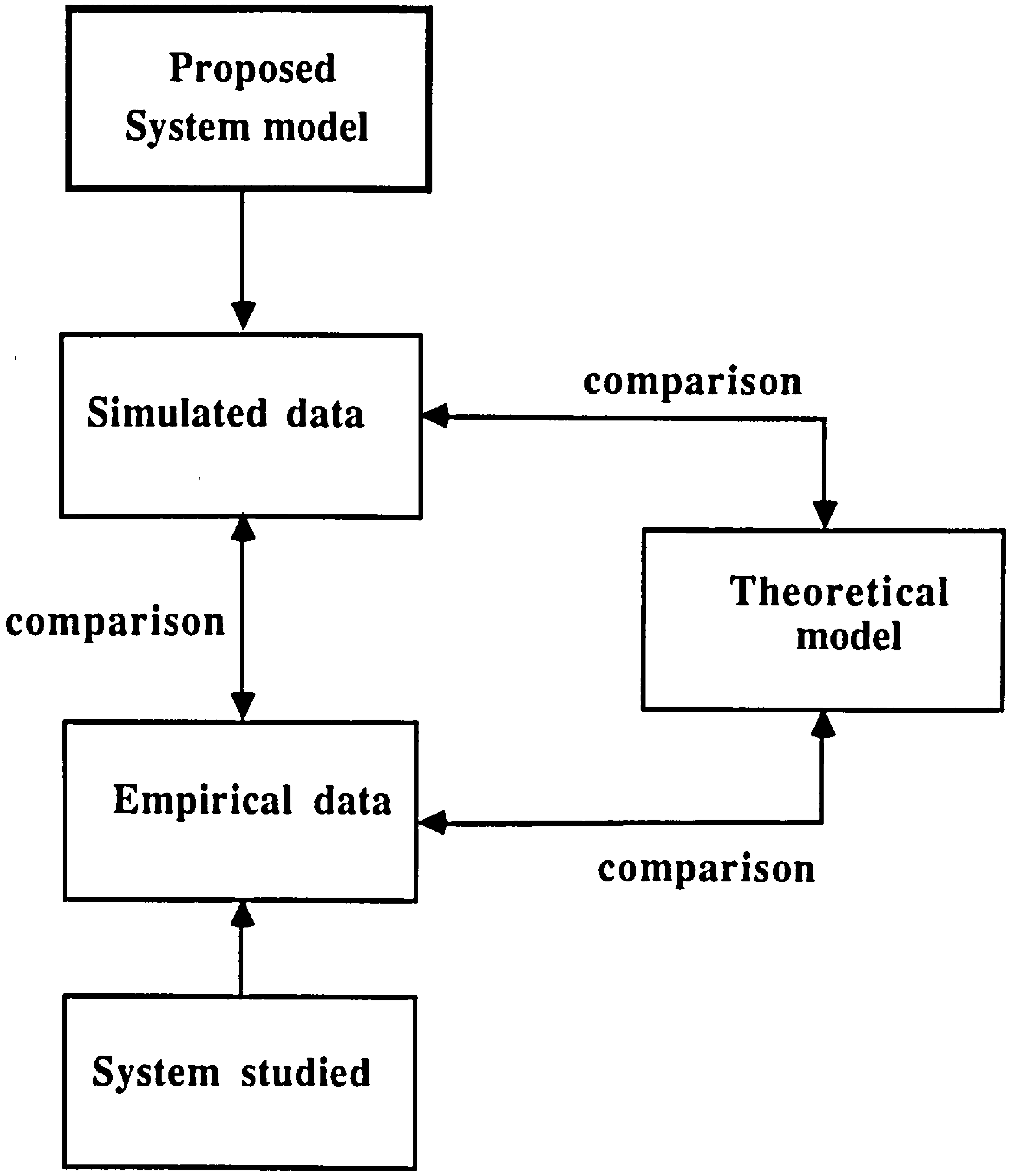
Outline of the empirical data testing process



From figure 2.2 we can see that the observed empirical data and distributions are characterised by comparison with theoretical distribution models. (Comparison point 1 in figure 2.2). These theoretical distribution models may in turn suggest the kind of stochastic process models that should be examined and ultimately tested against the observed empirical processes. (Comparison point 2 in figure 2.2). Support for the theoretical distribution models in turn is found from theoretical stochastic processes that are known to generate the theoretical distribution models. We seek strong independent evidence from the same system of empirical data and observed processes that the favoured stochastic process model is very likely to be operating and the loop of the research process of figure 2.2 is closed by applying supporting/confirmatory tests to the data sets to test for the particular stochastic process (Comparison point 2 again in figure 2.2). Therefore before a particular theoretical distribution can be fully accepted as the correct distribution to use, it must also be shown that a stochastic process exists that can generate such a model. In turn both the theoretical distribution model and the stochastic process model must explain the facts observed. The most promising theoretical distributions and stochastic process models together form the underlying basis of the structure of the '**proposed system models**'. By the process of 'retroduction' our proposed system models are used to explain the observed empirical facts (Comparison point 3 in figure 2.2). The input evidence for plausible stochastic process models and theoretical distribution models is drawn from empirical and theoretical work reported in the appropriate literature sources.

A second major stage we have used in the verification and acceptance of our proposed system models is to use these models to generate simulated data. We then compare this simulated data against the theoretical distributions and the original empirical data. This is shown diagrammatically below in figure 2.3 -

Figure 2.3
Stochastic model validation scheme



As will be discussed in chapter four a stochastic process model that leads to the lognormal distribution is the Law of Proportionate Effect. Hence the strong indication from empirical evidence that usage values are lognormal leads one to examine this process. In turn the conditions under which this stochastic mechanism is known to operate may suggest other statistical distributions (eg Yule), which should be examined as potentially competing candidates to graduate the data. Furthermore independent tests can be applied to the original data sets to support or refute the existence of the Law of Proportionate Effect. By the process of retroduction the lognormal distribution can be used to explain the facts seen.

2.4 Goodness of fit tests.

Because the testing of different statistical models was an important part of the work in research reported in this thesis we consider here some of the considerations to the choice and limitations of the usual goodness of fit tests. The work of two particular authors is considered in some detail because of their extensive consideration of the problems and issues in choosing goodness of fit tests for testing highly skewed statistical distributions.

The classical statistical goodness of fit test is the Chi Squared and it is probably the most widely employed in standard statistical analysis. Yet its use poses a number of methodological problems that we address below. It is a non parametric test and the criterion value of χ^2 (Chi Squared) is calculated from the following formula-

$$\chi^2 = \sum \left| \frac{O - E}{E} \right|^2$$

where O = the observed data frequency
E = the theoretically expected frequency
for a given statistical model.

The value of χ^2 is statistically distributed as a Chi Squared distribution and the test gives an overall measure of departure of the observed values from those expected from particular models. Strictly speaking the criterion value is only asymptotically distributed as a Chi Squared distribution for large samples(Easton 1975), but the departures from this condition do not appear to be critical. One of the major criticisms of the test is its use as a goodness of fit test in the case of highly skewed distributions. In a lengthy appraisal of goodness of fit tests Quandt (1966) has put forward four reasons why the Chi Squared test is unsuited as a goodness of fit test for such situations. -

(a) *"The test criterion χ^2 is sensitive to the particular grouping selected for the observations and in the case of closely similar hypotheses the answer might depend on what grouping was chosen.*

(b) *The test is not as powerful as it might be if its validity rested on the assumption of some specific alternative hypothesis; but it is precisely this kind of situation that is encountered when we wish to choose between, say, the Pareto and the Lognormal distributions.*

(c) *The test can not be validly applied if the expected frequencies are very small. This however, is precisely the case when we are interested in distributions with very long right tails.*

(d) *The value of χ^2 may be small, indicating a good fit, but the fitted cumulative distribution and the sample cumulative may still show very significant systematic divergence from each other."*

Easton (op cit) took up these points up and went on to say -

"The first point may be accommodated by investigating the sensitivity of the test to data groupings. The second point is a general one common to all applications of this particular test. However, point (c) is a key criticism and one which alone would preclude the use of χ^2 tests in the case of highly skewed distributions with very long tails."

Cochran (1954), amongst others, suggests that the problem is easily overcome by combining cells so that the minimum number of frequencies in any given cell is five or more. So in the case of distributions with long tails some of the cells in the critical long region are combined to achieve frequencies equal to or greater than five in the combined cells. However, as already discussed, this process results in a loss in fine structure of the distribution being examined. The Chi Squared test creates a further problem; in the case of very large samples any departure from the theoretical values are likely to cause the rejection of the null hypothesis due to easily discernible variation. We have already discussed at length Ijiri and Simon's (op cit) skeptical view regarding the use of goodness of fit tests to verify extreme hypotheses. According to these same authors empirical data sets that are collected from natural or sociological or econometric systems will almost always contain a certain amount of noise, which the test statistic is likely to measure as significant departures from expected values in large sample analysis.

The Kolmogorov Smirnov test is also a non parametric test frequently used in goodness of fit situations. This too is appraised at length by Quandt, although he is less critical of it compared to the Chi Squared test. Since it involves comparing cumulative distributions it cannot be excluded on the same grounds as the Chi Squared test, and Quandt points out that it has two particular advantages over the Chi Squared test, namely-

(a) *It is not necessary to combine any observations to attain cell greater frequencies so this makes it a more powerful test. It is reactive to all the fine detail.*

(b) *It can be used for very small samples when the Chi Squared maybe quite impractical because of small frequencies in given groups.*

Now given a sample of 'n' observations x_1, x_2, x_3, etc. to x_n and empirical and theoretical distribution functions S(x) and F(x) respectively then the Kolmogorov Smirnov test statistic is given by -

$$D = \max.[S(x_i) - F(x_i)]$$

In this formulation D measures the distance between the empirical and theoretical cumulative distributions. The distribution yielding the smallest D statistic for a given sample would be declared the best fit to the sample data. However, according to Quandt (op. cit.) his points (b) and (d) apply also to this test and in addition he says that it has not proved possible to calculate a table of of critical values for the case where the parameters of the distribution are estimated from the sample. It is clear from Quandt's work that although he believes the Kolmogorov Smirnov test to be more powerful than the Chi Squared there is little reason for believing that either method is appropriate for testing extreme hypotheses involving highly skewed distributions. However, we argue that these issues are not a major problem in the case of the lognormal distribution because it will transform to the symmetrical normal distribution by taking the $\log_e$ values of the variate of interest. The only major concerns are then with the test being very sensitive to fine departure from normality and with the small number of frequencies in the extreme tails.

To overcome some of the perceived problems of the Chi Squared and Kolmogorov Smirnov tests Quandt proposed an alternative method in his 1966a paper which treats the sample points of the cumulative distribution as order statistics. If $F(x_1)$ and $F(x_2)$ are the values of the cumulative distribution then the critical sum 'S' is defined as -

$$s = \sum \left[\frac{\{F(x_i) - F(x_{i-1})\} - 1}{\{n+1\}} \right]^2 \quad \text{for } i = 1 \text{ } to \text{ } n+1$$

The parameters of the fitted distribution are those which minimise S and the critical values for S, in probability terms, are obtained by sampling experiments.

However, Easton (op cit) has criticised this approach on the following grounds-

> " *There are two major problems involved in employing this test. Firstly sampling experiments have to be carried out to construct tables of critical values, for the appropriate range of values, for each distribution fitted. This would represent an enormous amount of computation for the size of the data in the current research* (**author comment -Easton here refers to his own data sets**). *This is in itself not sufficient to reject the test. Secondly, the numerical minimisation technique is also rather suspect. Quandt comments that some of the results obtained in his research on size distributions of firms are inconsistent and he suggests that local, rather than global, minima may have been reached".*

Quandt further draws attention to two important attributes of goodness of fit tests. Namely the 'closeness of fit 'as measured by S at the minimum, and the 'randomness of fit' which can be measured in several ways that Quandt demonstrated. The value of the randomness of fit can be seen by examining the following table.

Table 2.1
Fitting models to Empirical data

size	observed frequency	deviations model 1	deviations model 2
1	80	-5	6
2	28	-3	-3
3	13	-1	1
4	8	-1	0
5	5	0	0
6	3	0	-1
7	2	1	1
8	2	1	0
9	1	0	1
10	5	4	3
	total absolute deviations	16	16

From the above tabulation although both models give the same degree of 'closeness of fit' as measured by the total deviations, the 'randomness of fit', is clearly quite different. The variations shown by model one are more systematic than those of model two. Such systematic variation is usually regarded as an indication of a wrong or poor fitting model. The more random variation shown by model two is a much more desirable attribute that is likely to arise mainly from random noise. Even in cases where the random noise is quite marked the model giving this might be regarded as a better choice than one where the closeness of fit is better, but where undesirable systematic variation is present.

Easton (op. cit.) suggested two goodness of fit measures in an attempt to overcome the short-comings of the standard measures. First he proposed using the mean absolute deviation defined by -

$$\text{MAD} = \sum \frac{|o - e|}{n}$$

where o = the observed frequencies
e = the expected frequencies
n = the number of cells in the tabulation.

In this author's opinion this test will give problems of interpretation as it is not entirely clear what one is comparing, also the test statistic will vary considerably according to the number of cells chosen. Easton also proposed using a relative mean absolute deviation (RMAD) as the test statistic and defined this as -

$$\text{RMAD} = \sum \frac{|o - e|}{n} / \frac{\sum e}{n}$$

Additionally Easton proposed a RMAD measure just for the tail of a distribution by, in general, leaving out the large first cell, which in the case of highly skewed distributions contains by far the greater proportion of frequencies. The justification for this approach is that in attempting to discriminate between competing distributions that are highly skewed it is the differences in their tails that is critical. Easton's RMAD(tail) is based on the measured test statistic below-

$$\text{RMAD(tail)} = \frac{\sum |o - e|}{\sum e} \quad \text{for } n = 2 \text{ to } m \ (m = \text{number of cells})$$

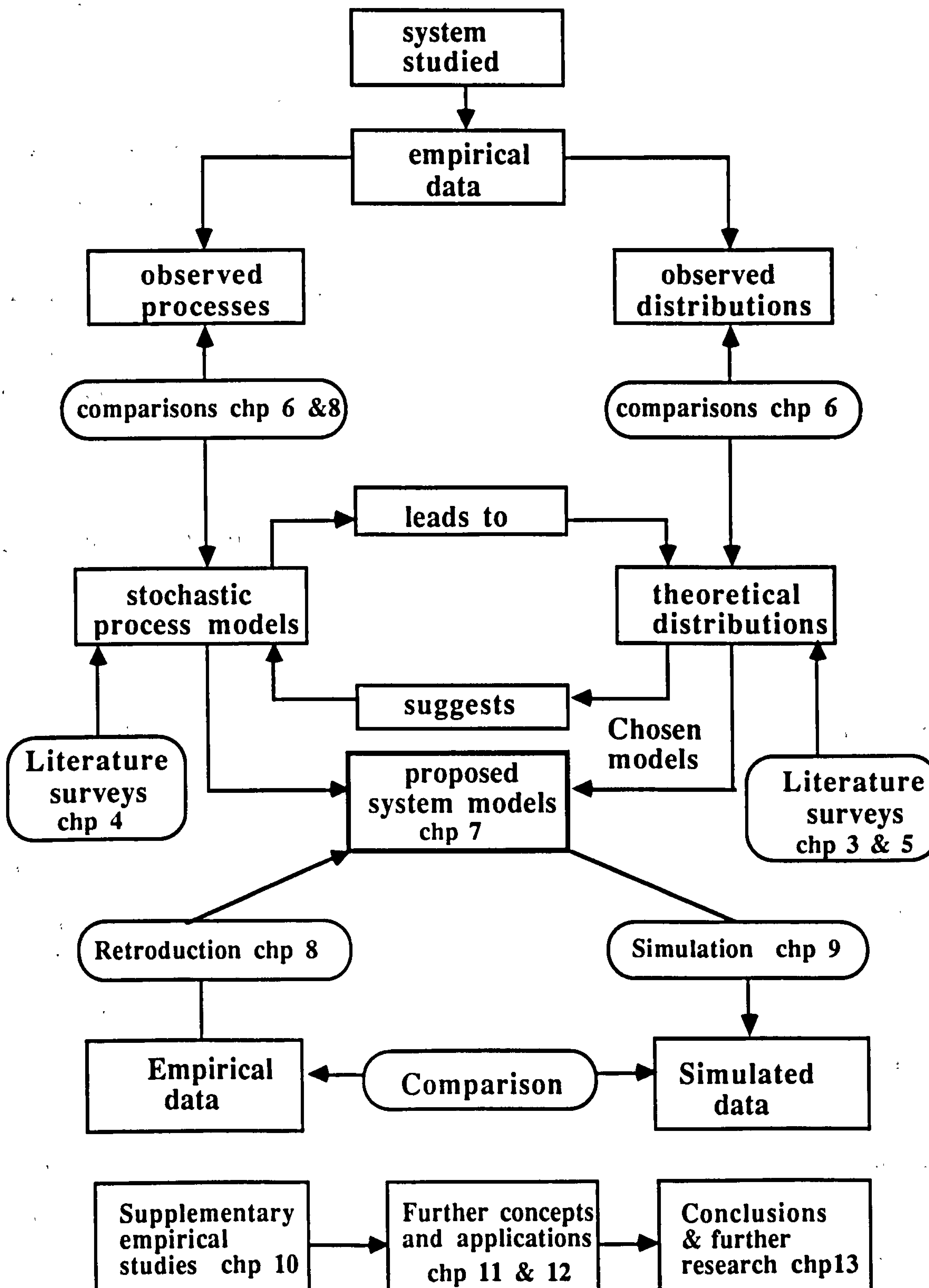
It is claimed by Easton that these measures are an improvement over the classical tests. The concept of a test that focuses on differences solely in the tail region of highly skewed distributions has significant merit, and is an issue we will return to in later chapters particularly when we consider the Negative binomial and Stuttering Poisson distributions.

2.5 Conclusions and Research Organising Framework

We have examined a number of methodological issues in this chapter. Particularly important are the conclusions of both Ijiri and Simon, and Aitchison and Brown, that explaining regularity in empirical data is far more than just testing models by goodness of fit tests. Stochastic evidence must be assembled that goes some considerable way to explaining the facts observed in terms of process models. In turn the process models suggest likely theoretical distribution models that should be tested against the empirical data before they can be accepted as the correct models to use. This is the process called retroduction by Simon.

We have considered some important aspects of the two principal non parametric methods of testing data and considered the criticisms of various authors. In the light of such criticisms these methods have been used in later chapters with a clearer knowledge of their limitations. As a result goodness of fit tests have played only one part in the wider process of developing and testing models against the various empirical data used in this work. We have relied heavily on the processes of retroduction, and to a lesser extent on simulation, to provide evidence that the models developed in this research explain the empirical facts observed. Finally we have examined empirical data from other spares systems and tested such data against the models developed in this thesis. Using figures 2.2 and 2.3 we are now able to present the complete organising framework for the research process adopted in this work as shown in figure 2.4 below. We start the process in the next chapter by presenting a literature review of the essential background knowledge of the lognormal distribution.

Figure 2.4
The Research Organising Framework



The Lognormal Distribution, History, Occurrence and General Properties.

3.0 Introduction.

In this chapter we begin the process of setting the necessary background development and theory of the lognormal distribution. We also examine the range of applications to which the distribution has been put in various business and economics fields of study and enquiry. A review is given of the rather unique properties of the distribution and we show how these can be used to set aggregate inventory standards based on the pioneering work of Robert Goodell Brown.

3.1 Early History

The historical development of the lognormal distribution is confused, but according to Aitchison and Brown (1957) the origin of the development of a definitive theory is probably attributed to McAlister (1879), who apparently based his work on a suggestion by Galton (1879). Galton himself had apparently derived his ideas based on the so called 'Weber Fechner Law' relating responses to stimuli, which asserts that the response is proportional to the logarithm of the stimulus, Weber (1834).

Pearson (1895) evolved his system of curves, which in contemporary statistical theory forms the basis for classifying theoretical statistical distributions, including highly skewed distributions. In 1903 Kapteyn published a text on the use of skewed frequency curves in biology and statistics in which he laid down the foundation for an extensive system of frequency curves. According to Aitchison and Brown (Op. cit.) Kapteyn apparently firmly established the genesis and existence of the lognormal distribution in this work. Kapteyn also developed a machine for generating samples from lognormal populations. There then followed heated exchanges between Pearson and Kapteyn, because according to Aitchison and Brown (Op. cit.), Pearson had a general mistrust of the transformation techniques used by Kapteyn. This exchange is now regarded as a classic debate in the early history of the development of the theory of the lognormal distribution. In 1917 Wickersell independently developed a theory of the genesis of the lognormal similar to that of Kapteyn and he used the method of moments for parameter estimation purposes. Since that time, and up to the milestone publication of Aitchison and Brown (1957), the lognormal function has been used in the analysis and modelling of a wide variety of naturally occurring phenomena, and many empirical distributions have been shown to fit the lognormal very well. Aitchison and Brown (Op. cit.) have compiled a detailed listing of work and discussions of the use of the lognormal distribution in areas such as biology, astronomy, sociology and economics. Since 1957 the lognormal has been found to have utility in a wide variety of technological and economic situations.

3.2 The general properties of the lognormal distribution.

If we consider a positive variate X such that $0 < X > \infty$ and $Y = \log_e X$, then the variate X is lognormally distributed if Y is normally distributed. Aitchison and Brown (1957). [Note: we use capitals for variates concerned with populations eg X, and lower cases for sample variates eg 'x' is consistent with the work of Aitchison and Brown].

The lognormal distribution has a probability density function given by -

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(\log_e x - \mu)^2\right] \quad \text{-------} \ 3.1$$

where σ is the shape parameter of the distribution and μ is its location parameter. σ and μ are also the standard deviation and mean respectively of the transformed distribution, that is $\log_e x$. The distribution has moments of any order. If the jth moment about the origin is denoted by λ_j then -

$$\lambda_j = \int_0^\infty x^j d(x)$$

The mean α of the distribution is given by the first moment about the origin -

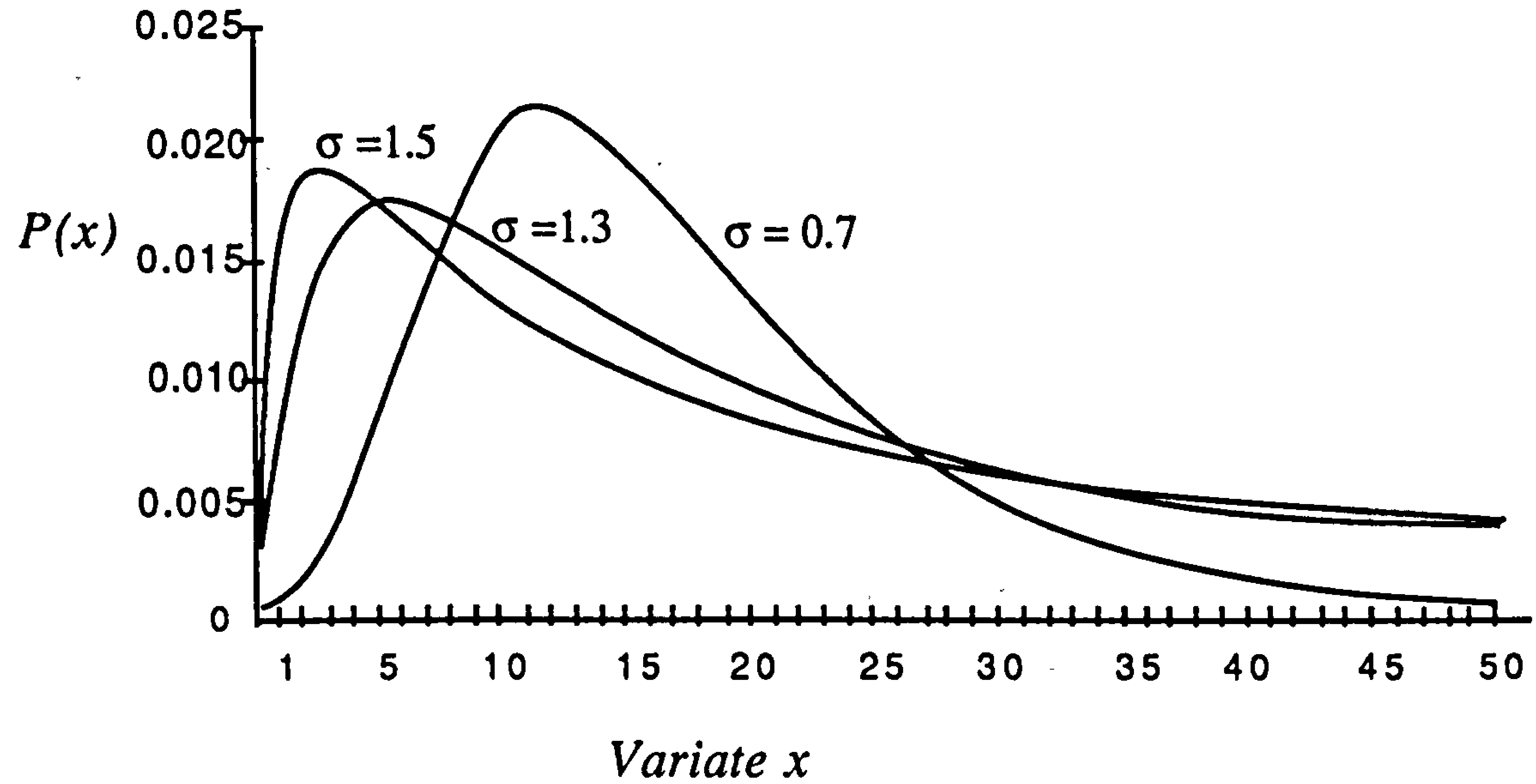
$$\alpha = e^{\mu + 1/2\sigma^2} \qquad \text{-------------------------- } 3.2$$

and from this it follows that the variance β^2 is given by-

$$\beta^2 = e^{\mu + 1/2\sigma^2}(e^{\sigma^2} - 1) \qquad \text{---------------------} 3.3$$

As the shape parameter σ of the distribution approaches 0 the lognormal density function approaches the normal density function, and as the parameter σ increases in value the lognormal function becomes progressively more skewed as shown below -

Figure 3.1
Three different lognormal distributions



The interpretation of the location parameter μ is straightforward as it is the logarithm of the geometric mean of 'x'. The shape parameter σ is of much greater significance and value to this research because of its relationship with the proportion of value accruing to a given proportion of items.

The property of the lognormal distribution that is of special value in inventory management is that if a variable 'x' is lognormally distributed then any power of 'x', say x^k is also lognormally distributed with the same standard deviation, Aitchison and Brown (1957). Furthermore the average of any power of 'x', say again $\overline{x^k}$, is given by the function :

$$\overline{x^k} = \int_0^{\infty} \bar{x}^k f(x) d(x) = \bar{x}^k e^{k(k-1)(\sigma^2/2)} \qquad \text{--------------3.4}$$

In the case of inventories the average value of x could represent inventory usage values or sales in a particular inventory. Thus it can be seen from 3.4 above that the average of the kth power (ie '$\overline{x^k}$') is equal to the average value of 'x' raised to the same power 'k' times the factor -

$$e^{k(k-1)(\sigma^2/2)}$$

that just involves the power 'k ' and the shape parameter σ of the lognormal distribution. It is the formulation 3.4 above that enables valuable aggregate inventory standards to be set and it was R.G. Brown in 1959 who first recognised this and its significant value in inventory management. This is shown in a later section of this chapter.

Because of the importance of function 3.4 above the proof, with minor modifications, as given by R.G.Brown (1959, page 199,) is given here -

First let $x^k = e^{k \ln x}$

Substitute this relationship in the distribution function for the lognormal distribution 3.1 then we can write -

$$f(x)d(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right] dx$$

From which we can write

$$\overline{x^k} = \frac{1}{x\sigma\sqrt{2\pi}} \int_0^{\infty} \exp-\left[\frac{-2\sigma^2 k \ln x + \ln^2 x - 2\mu \ln x - \mu^2}{2\sigma^2}\right] dx$$

By completing the square in the exponent part of the formula the result can be reduced to -

then $$\overline{x^k} = \exp\left[\mu k + \frac{k^2\sigma^2}{2}\right]\int_0^\infty f^{'}(x)dx \qquad \text{------------3.5}$$

where f' (x) is another lognormal distribution with the same standard deviation σ and new mean given by-

$$\mu' = \mu + k\,\sigma^2 \qquad \text{------------------3.6}$$

Now since the sum of the probabilities must equal zero then -

$$\int_0^\infty f'(x)dx \quad = \quad 1$$

and therefore $$\overline{x^k} = e^{\mu k + (k^2\sigma^2/2)} \qquad \text{------------------3.7}$$

Now since the average value of x is -

$$\bar{x} = e^{\mu + \sigma^2/2} \qquad \text{------------------3.8}$$

and $$\bar{x}^k = e^{k\mu + k\sigma^2/2} \qquad \text{------------------3.9}$$

now if we write

$$\overline{x^k} = e^{k\mu + k\sigma^2} e^{k(k-1)(\sigma^2/2)}$$

$$\overline{x^k} = e^{k(k-1)(\sigma^2/2) + k\mu + k\sigma^2}$$

from which it then follows that -

$$\overline{x^k} = \bar{x}^k e^{k(k-1)(\sigma^2/2)}$$

which is function 3.4 as already given above. Also from this relationship we can find the variance of the usage values because we can write-

$$\text{Variance}\ (x) = \bar{x}^2 (e^{\sigma^2} - 1) \qquad \text{--------------3.10}$$

This in turn arises out of functions 3.2 and 3.3 and therefore the standard deviation σ_x of the observed usage values is related to the standard deviation of the logarithms of x, ie σ by the relationship -

$$\sigma_x = \bar{x}\sqrt{(e^{\sigma^2} - 1)} \qquad \text{--------------3.11}$$

Thus if the distribution of say period usage values can be shown to fit a lognormal function then the entire population can be estimated from just a knowledge of the mean of the population and the shape parameter (σ) of the appropriate lognormal function. This is demonstrated in section 3.5.

The parameter σ is also the standard deviation of the logarithms of 'x' and hence it is a measure of the dispersion of the variate $\log_e x$. In inventory applications some authors seem to prefer to measure the dispersion by the so called STANDARD RATIO designated as ρ . [In particular, see Brown (1959), Schary and Howard (1970, page 32), and Bestwick and Lockyer (1982, page 122).] ρ is related directly to (σ) by the relationship -

$$\text{Log}_e\, \rho = \sigma \qquad \text{-------------------- 3.12}$$

The major and perhaps only merit of the standard ratio (ρ) is that it can be readily determined directly from the slope of a graphical plot of data on lognormal graph paper and hence it provides a quick way of determining the value of σ . However, there are more efficient ways of measuring this parameter and for use in inventory calculations a more accurate value of σ is required than that given by graphical measures.

The lognormal distribution has a number of valuable reproductive properties as reviewed by Aitchsion and Brown. We give here just a few that are directly useful to aspects of the work in later chapters.

(i) If a variate 'x' is lognormal with parameter (μ , σ^2) then the variate αx^{β} is also lognormal with parameters ($\alpha + \beta \mu$, $\beta^2 \sigma^2$) providing α and β are constants.

(ii) If a variate 'x 'is lognormal with parameters (μ, σ^2) then $1/x$ is lognormal with parameters ($-\mu$, σ^2).

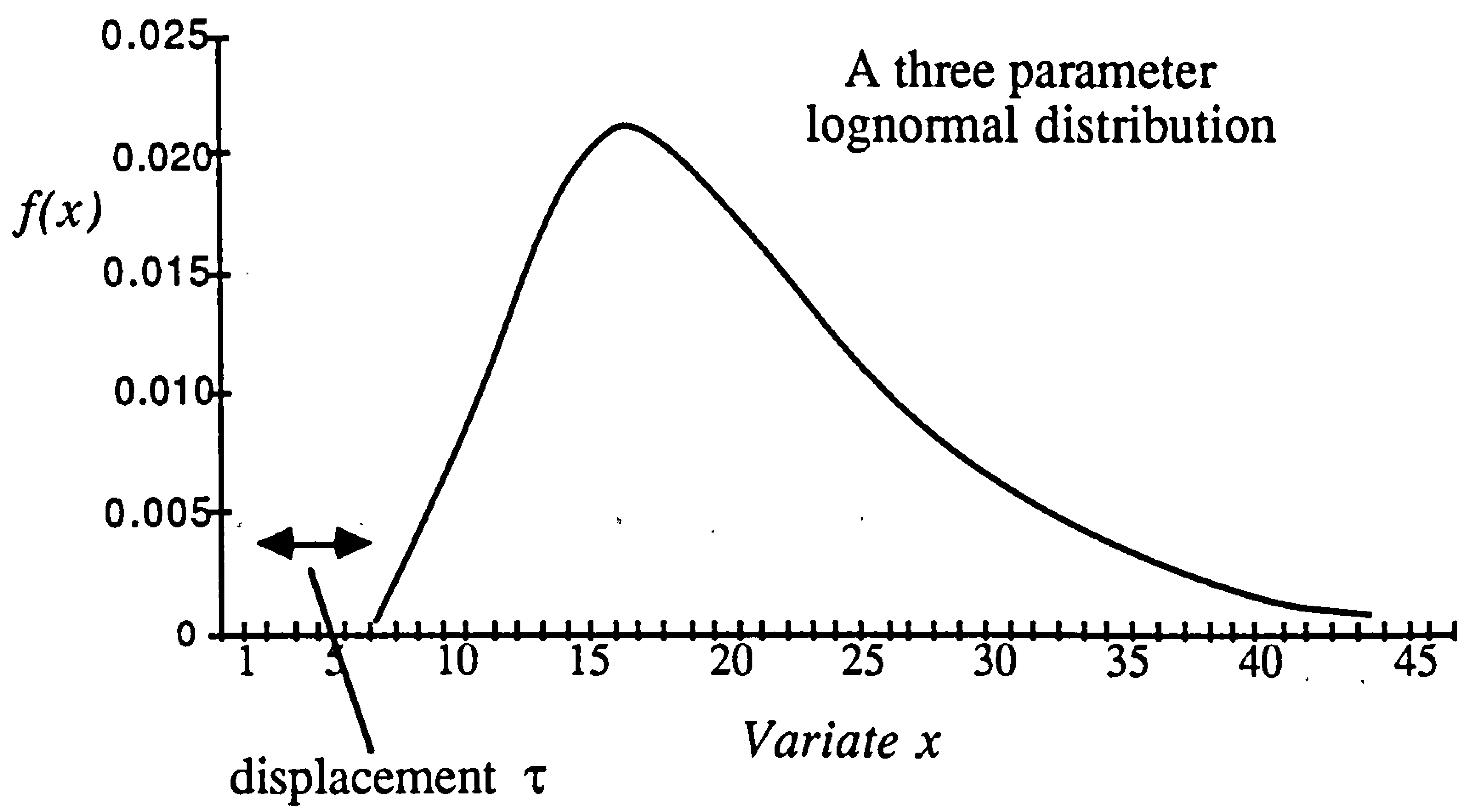
(iii) If x_1 and x_2 are independent lognormal variates with parameters (μ_1, $\sigma^2{}_1$) and ($\mu^2{}_2$, $\sigma^2{}_2$) then the product $x_1 x_2$ is lognormal with parameters ($\mu_1 + \mu_2$, $\sigma^2{}_1 + \sigma^2{}_2$)

(iv) From (iii) above it follows that if the product of two independent variates x_1 and x_2 is lognormal then x_1 and x_2 are each both also lognormal, accept for the special case where one of the variates is a constant, then the other is lognormal .

3.3 . The Three-Parameter Distribution.

The lognormal distribution can be extended by the introduction of a third parameter. It may be that a simple displacement of 'x', say $x_i = (x - \tau)$, is lognormally distributed and not the variant itself. The range of x is then seen as ($\tau < x < \infty$) , where the distribution is displaced along the 'x' axis by the value of τ as shown in the diagram below :

Figure 3.2



The two parameter distribution is then the special case for which $\tau = 0$. Since the parameter, τ , defines a lower bound to the range of x it is usually termed the threshold of the distribution.

In certain circumstances the value of τ can be found on 'a priori' grounds, in which case it may not be considered an unknown and therefore does not need to be estimated. Should this be the case $x - \tau$ may be considered in place of x; when a value of x is then given the corresponding value of x_i is immediately known. x_i has all the properties of the two parameter distribution of x and no new theory arises.

If however, τ is an unknown the estimation procedures developed for the two parameter case are not directly applicable to the distribution of $x - \tau$. Since the frequency curve of $(x - \tau)$ is that of x displaced by τ, the location parameters are, therefore, each increased by τ, the mean being at $\alpha + \tau$ where α is defined by equation 3.2. The median is at $x = \tau + e^{\mu}$ and the mode at $x = \tau + e^{\mu - \sigma}$. The moments about τ are:-

$$(x - \tau)^k = \exp[k\mu + 1/2k^2\sigma^2]$$

So that the moments about the mean and hence the standard deviation remain unchanged.

3.4 Truncated and Censored Distributions

It is possible to consider data that is truncated at some particular value such that the remaining data can be treated as lognormal. Aitchison and Brown(op cit) give the appropriate methods for testing such situations and for parameter estimation. Censored data is treated similarly where the occurrence of particular values is not possible or are not available. We will not consider the tools for testing and parameter estimation here as they are not used in this work. However, Aitchison and Brown also discuss a special case of censored data which they refer to as 'The Distribution of Counts'. (Aitchison and Brown op. cit. -page 92). This is of significant value to our work because it can be used to model discrete data by a lognormal function; or what we have referred to in chapter six and other places in this work as an integer lognormal form.

The general form of such censored distributions are as given below :

$$\phi_r = \int_{r}^{r+1} d\Lambda(x{:}\ \mu, \sigma^2)$$

$$\phi_r = \int_{\ln r}^{\ln(r+1)} dN(y{:}\ \mu, \sigma^2)$$

According to Aitchison and Brown such distributions occur in experiments on counts of insects, for example aphids on leaves; and other cases such as spores on culture plates. In chapter six in this work we consider some discrete demand volume data that behaves in all respects as if drawn from lognormal populations, and therefore the validity of accepting discrete data as lognormal is very important. Aitchison and Brown then go on to say that-

> *"The number of zeros in the distribution* [the distribution of counts] *appears as the part of the normal distribution in the range* (-∞, 0). *The treatment suggested by Thompson(1951) is to consider the variate Y where*
>
> $$P(Y < \text{or} = y) = N(y{:}\,\mu\,,\,\sigma^2)$$
>
> *and to estimate the parameters μ and σ from the transformed sample. The transformation in this case is-*
>
> $$y = \log_e(r+1)$$
>
> *and the problem really concerns a normal distribution, censored at the origin for which the censored portion appears as a discrete probability mass at the origin".*

3.5 Some application of the lognormal distribution.

This section briefly reviews some of the applications of the distribution as give in the literature. It is not a comprehensive survey of all known applications, this would be far too widespread and voluminous. The review has been limited to selected areas of the business and management literature to show the range and flexibility of the function and the value it has as a statistical distribution.

3.5 (a) industrial applications

In the work published by R.G.Brown (1959) and (1963), as mentioned in chapter one , it was demonstrated that finished goods inventories (specifically spare parts) when measured by their period usage values can be modelled by lognormal functions. Brown (1959) has also shown, by using the relevant parameters of the fitted lognormal curves and some of the properties of the lognormal distribution, how valuable aggregate inventory standards can be calculated. [This valuable finding will be discussed in more depth later in this chapter]

In a study of service times in a variety of systems Horvath (1959) noted that the service times at a tool crib and in a library could be satisfactorily modelled by the lognormal distribution. This work is of interest because it was time intervals between events that were shown to be lognormal distributed and not the events themselves.

Bovaird and Zagor (1961) have modelled equipment downtimes in complex electronic systems and they found that lognormal models provided a good fit to the empirical data they analysed. Furthermore they used measurements of the parameters of the particular lognormal models to predict the likely downtime distributions to be expected in new equipments.

More recently Nickerson (1973) et al. have discussed the use of lognormal functions in a paper concerned with the design of fishing trawlers, although certain aspects of the work and methodology has been criticised by Coleman and Saipes (1977) as being unsound. Husband and Schofield (1976) have used the lognormal distribution with good success to model pay structures in chemical and engineering companies, although on what grounds they chose the lognormal was not made clear.

3.5 (b) income distributions

The early classical approach to the modelling of income distributions followed the pioneering work of Vilfredo Pareto (1897) using his now classic Pareto curves. Developed somewhat later, but parallelling Pareto's work, several authors have used lognormal functions to model and describe income distributions in a variety of social and economic groups and communities. In particular, Kapteyn (1916), Kapteyn and Van Uven (1916) and Gibrat (1931), and more recently Aitchison and Brown (1954), Lyndal (1959) and Thatcher (1968). According to Easton (1974) the generally accepted position with regard to modelling income distributions is that while the lognormal distribution has been shown to provide a good overall fit to empirical income distribution data the Pareto distribution (of the first kind) provides a better fit in the upper tail of most empirical distributions seen.

3.5 (c) the sizes of firms

There is now an extensive literature on the distribution of sizes of business organisations. A number of authors have claimed that empirical data on firm sizes can be modelled by lognormal functions under specified circumstances. In particular Hart and Prais (1956), Simon and

Bonini (1958), Nelson (1963), Pashignan (1966), Quandt (1966), Silberman (1967) and Ijiri and Simon (1977). The general view appears to be that the lognormal distribution can provide a description of the size of enterprises provided the so called 'Law of Proportionate Effect' governs the growth rate of the firms, and that entries and exits to the industry remain constant. These points are considered in more detail in chapter four which is concerned with the Law of Proportionate Effect and its importance to the genesis of lognormal functions.

Easton (1975) has modelled the industrial purchase size of distributions of industrial organisations for a number of industries and he found that the lognormal distribution provides a good fit to single period purchase decisions. The fit was not good and was deemed failed however, for multiple periods. Easton (op. cit.) made a very careful comparisons in his work between the lognormal and the Yule, Pareto, Gamma and Negative Binomial distributions and he concluded that both on statistical and stochastic grounds that the lognormal was the appropriate one to use. It was in fact after reviewing Easton's work that this author was first made aware of the possible use of the so called Poisson Gamma model in this current work and also lead this author into the literature on consumer purchase theory.

3.5 (d) purchase theory

In a market research study Lawrence (1980) has shown that the lognormal distribution is a satisfactory model to represent purchasing trip frequencies in consumer goods markets. In particular his study was on dentifrice purchasing in US markets. The research reported was also claimed to be part of a larger research which postulates that the time between successive purchasing events is lognormally distributed at the individual purchaser level. The same author also, interestingly, acknowledged that the use of the lognormal in his study conflicts with the wealth of modelling work in consumer repeat purchasing studies using the Poisson Gamma model (to be discussed in Chapter five). However, Lawrence's empirical work is very convincing in terms of the degree of fit between empirical data and the theoretical frequencies of the lognormal distribution although this work is rather light in terms of stochastic support. Lawrence also justified the use of the lognormal, distribution as a continuous distribution to describe discrete events, by using the distribution of counts theory.

3.5 (e) financial theory

Hilliard and Clayton (1982), in an essentially theoretical study of the financial returns from a portfolio of investments, assumed that the lognormal distribution of returns was the appropriate model to use based on empirical work of Breiman (1961). However, a suitable argument using fundamental stochastic grounds was not presented in their work.

3.5 (f) inventory theory

The use of the lognormal distribution in inventory theory has been reported in two main areas, (i) as a model of demand distribution in successive time periods and (ii) as a means of setting aggregate inventory standards based on the original and milestone work of R.G. brown (1959).

In the first case it has been demonstrated by Holt, Modigliani, Muth and Simon (1960) that the sales rates of a number of selected finished product items in successive time periods can be modelled by lognormal distributions. In the mainly consumer product examples quoted by these authors the lognormal was reported to be a marginally better fit than the gamma distribution and much superior to the Poisson distribution. These latter two distributions are often the models of first choice to represent sales rates in many product fields. It is of interest to note that the use of the lognormal distribution in the work quoted by Simon and Muth was primarily justified on goodness of fit grounds, yet is was Simon, who subsequently in later work [eg Ijiri and Simon 1977)] was so critical of relying on goodness of fit tests alone to justify the use of a model to graduate empirical data. Lewis (1975) has briefly discussed the value of the use of the lognormal distribution instead of the normal distribution in modelling inventory demands during replenishment lead times. Tadikamalla (1964) has also discussed and shown in his work that the lognormal can prove to be a satisfactory model of lead time demand.

A perusal of the inventory theory literature will show however, that many different skewed distributions have been proposed and used to model demand in successive time periods. A factor that has to be considered is whether the lead time itself is regarded as fixed or variable. In turn the exponential, Poisson and various modified Poisson models, especially the so called 'Stuttering Poisson', model have been proposed and used for the fixed lead time case by many authors. For variable lead times the gamma, Negative binomial, Weibull, as well as the lognormal, are all examples of distributions that have been suggested and reported by various authors. Lewis(op cit) has discussed the fact that as one generally moves from the retail level through the wholesale level and back through the distribution channel towards the manufacturers, then period inventory demands tend to become progressively more skewed from normal through Poisson to exponential in form. What we can conclude from the foregoing discussion is that given the wide diversity and behaviour of industrial distribution channels and the variety of products that flow through them it is not difficult to see the opportunity to specify many different functions to model item demands. Tadikamalla (op. cit.) has also pointed out that the lognormal, Weibull and gamma distributions for example, can look very much alike when their means and variances are the same. Hence the conclusion is that depending on one's choice a case can be made for one or the other. However, as we have already discussed in chapter two, empirical evidence and goodness of fit alone are not sufficient grounds to make a strong case for a particular model.

More recently Bagchi (1987) has attempted a classification of functions, that include the lognormal, to model lead time demands and he has come up with a novel system of compound distributions for the lumpy demand variable lead time case. In particular Bagchi showed that three variables must be considered, the 'Order Intensity' (the number of orders placed at each demand occasion), the 'Order Rate' (often Poisson in independent demand cases), and the lead time variability. The particular combination of distributions used for each of these variables will determine the overall lead time distribution of demand. For example, the 'Lognormal Poisson Gamma model' (LPG in Bagchis' terminology) would have lognormal order size, Poisson order rate and a gamma lead time. Suffice to say that among the many skewed distributions that have been considered in modelling item sales demands the lognormal is one that has found favour with some authors.

3.6 The use of the lognormal distribution to set aggregate inventory standards.

The value of the lognormal distribution to set aggregate inventory standards rests primarily on equation 3.4 previously given ie :

$$\overline{x^k} = \bar{x}^k e^{k(k-1)(\sigma^2/2)}$$

Although the underlying theory of this was given by Aitchison and Brown (1957) it was, as already discussed, R.G.Brown (1959) who made the intuitive leap to recognise how to make use of this theory in the context of inventory calculations to determine aggregate inventory standards. As previously stated according to the above function the average value of the k th power of a lognormal variate 'x' is equal to the average value of 'x' raised to the same power k times the function 3.4 already given - $\exp[k(k-1)(\sigma^2/2)$. This was an extremely valuable phenomena in inventory analysis applications because many inventory calculations involve the use of power functions. The classical economic order quantity model and its many variations all involve power functions and can, in most cases, be reduced to the simple form-

$$EOQ = c(UV)^{\alpha} \qquad \text{-------------------------} \quad 3.13$$

where c = constant

$\alpha = 0.5$

UV = the usage value of each item.

Thus if the usage values are lognormally distributed across an inventory range, then by just knowing the average value of the usage values and the shape parameter of the fitted lognormal distribution, as will be shown below, the total investment in cycle stock can be calculated for the entire range.

It is not the intention in this thesis to review aspects of inventory control theory such as the many models that have been researched and developed to control inventories. The literature is extremely rich in a variety of models and decision systems relating to many different operating circumstances. Amongst the texts which provide excellent starting points for research into modelling procedures are Hadley and Whitin (1963), Peterson and Silver (1979) and Lewis (1981). We will consider here only elementary models as an illustration of the use of the properties of the lognormal distribution.

3.6 (a) calculation of aggregate cycle stocks

Economic order models are still extensively reported in the literature on statistical inventory theory and they form the basis of reorder calculations in most of todays inventory control computer programmes. Thus in most systems under statistical inventory control the cycle stocks are proportional to the order size as calculated by an EOQ formulation. In the simple case of a reorder level policy with replenishment lot sizes given by the most basic of all EOQ model the average cycle stock will be 1/2 the EOQ.

Thus from -

$$EOQ = \sqrt{(2C_oR / C_hP)} \quad \text{-------------3.14}$$

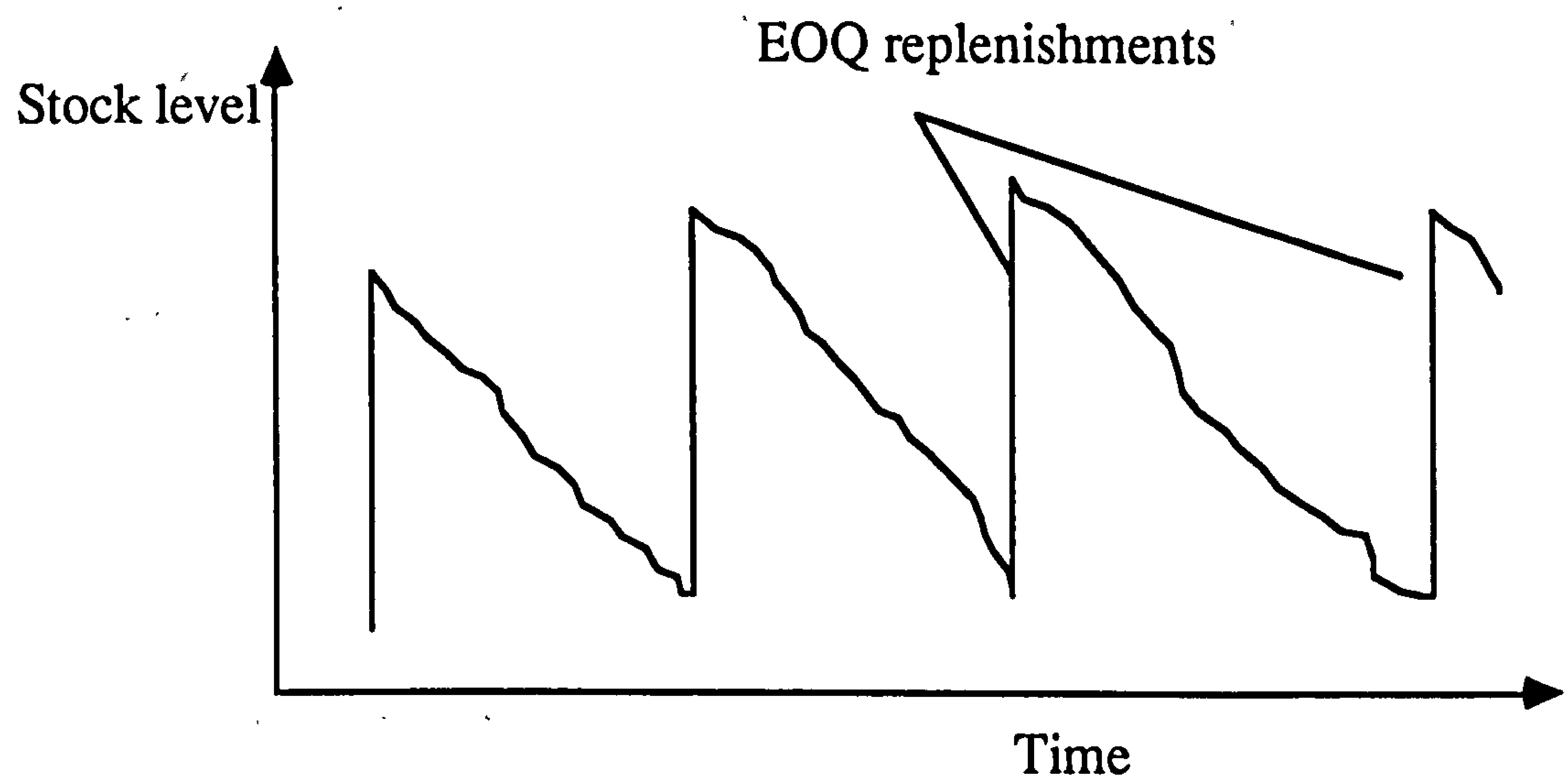
where R = annual item demand
C_o = the cost per order
C_h = the holding cost as a fraction of P
P = the item price

Then the average investment in cycle stock volume ($AICS$) will be given by -

$$EOQ = 1/2\sqrt{(2C_oR / C_hP)}$$

Inventory fluctuations for the majority of items in an independent demand inventory situation will be as shown in the diagram below -

Figure 3.3



Now from equation 3.11 above, the **value** of the EOQ lot size will be-

$$EOQvalue = P\sqrt{(2CoR / C_hP)}$$

or $$EOQvalue = P\sqrt{(2C_oRP / C_hP^2)}$$

and $$EOQvalue = \sqrt{(2C_oS / C_h)}$$

where $RP = S$ the usage value so we can write -

$$EOQvalue = \sqrt{(2C_o / C_h)}\sqrt{S}$$

Now S is the period sales rate or usage value and this formulation is exactly equivalent to equation 3.13 above with the constant 'c' being equal to $\sqrt{(2C_o/C_h)}$.

Assume now that the 'S' values for an entire inventory range of 'n' items are lognormally distributed with shape parameter σ, and that the value of the average sales value is $\bar{S}$. Then from this information the value for the investment in cycle stock for the whole range of 'n' items can be calculated from equation 3.4 ie :-

$$\overline{x^k} = \bar{x}^k e^{k(k-1)(\sigma^2/2)}$$

Then by substitution we have that average investment in cycle stock (*AICS*) over 'n' items is given by-

$$AICS = 1/2\left[n(\sqrt{2C_o/C_h})(\overline{S^{0.5}})\right]$$

By replacing $\overline{S^{0.5}}$ by $\bar{S}^{0.5}$ and using equation 3.4 we can write -

$$AICS = 1/2\left[n(\sqrt{2C_o/C_h})\right]\left[\bar{S}^{0.5}(e^{k(k-1)1/2\sigma^2}\right] \quad \text{----------3.15}$$

Once we have the average value of S, with k equal to 0.5, then all we need specify here is the number of items 'n' to determine the *AICS*, because the values of the constants σ, C_o and C_h, are all known. The *AICS* will then have been calculated without recourse to a complete enumeration of all items in the inventory range. If 'n' runs to tens of thousands of items, which it often does in spare parts systems (sometimes even hundreds of thousands), then a considerable amount of computation time is saved. A similar procedure can be used to calculate investments in safety stock, because as will be discussed below, such calculations can also be based on power functions. Hence total stock investments can then be determined.

3.6 (b) calculation of aggregate safety stocks

It has been shown by many authors that the dispersion of demand (usually measured as the standard deviation) in a time period '*t*' can be related to the actual level of demand by a formulation of the type shown below [see for example Holt et. al. (1960), Brown (1963), Hadley and Whitin (1963), Schary and Howard(1970) and Peterson and Silver (1979).] [We also demonstrate this relationship with DAF data in appendix three].

$$\sigma_{s,t} = \alpha \bar{S}^{\mu}$$

where $\sigma_{s.t}$ = the standard deviation of demand of S in period t
$\bar{S}$ = the average sales rate, or usage values.
α and μ are empirically derived constants.

The empirical constants are normally estimated by regression analysis of many pairs of $\sigma_{s.t}$ and $\bar{S}$ values.

In many simple reorder level systems the safety stock is often set to equal a multiple of the dispersion of the period demand (usually the standard deviation). Hence using the lognormal properties it is possible to calculate the total investment in safety stock for an inventory of '*n*' items based on calculations just for the average item. Thus using equation 3.4 for the calculation of the total investment in safety stock (*TISS*) we have -

$$TISS = n\left[(sf)\alpha(\bar{S})^{\mu}(e^{\mu(\mu-1)1/2\sigma^2})\right] \quad \text{-----------3.16}$$

where *sf* is the safety stock factor (ie the number of standard deviations to give the appropriate service level required from the system).

give us the overall investment in stock based on a system following a simple reorder level / EOQ approach. The same principles and basis formula can be applied to many of the more sophisticated method of controlling independent demand inventory items; it only remains for the analysts and management scientists to make the applications.

3.6 (c) application to sub groups of inventory

The same principles shown above can be applied to sub groups of an inventory range. For example, inventory managers will often prefer to set differential service or investment levels for inventory sub groups related perhaps to an ABC type categorisation. All one need do is determine the number of items in the appropriate sub group of the inventory range, and the average sales rate for that sub group and then apply the lognormal function to each sub group separately.

For example let us consider the situation where the cycle stocks of the 'A' class group in an ABC classification are under control by use of an EOQ model to cover shortage costs eg-

$$EOQ = \sqrt{\left[\frac{2RC_s}{C_1}\right]}\sqrt{\left[\frac{C_1 + C_2}{C_1}\right]}$$

where R = annual demand
C_S = set up cost (= Co)
C_1 = holding cost/item/unit time
C_2 = shortage cost/item/unit time

(see Wild (1971) for the derivation of this formula)

Now the value of this formulation will be-

$$EOQ = P\sqrt{\left[\frac{2RC_s}{C_1}\right]}\sqrt{\left[\frac{C_1 + C_2}{C_1}\right]}$$

Where P = unit price of item i

hence we can write the formula as-

$$EOQ = P\sqrt{\left[\frac{2RPC_s}{P^2 C_i}\right]}\sqrt{\left[\frac{C_1 + C_2}{C_1}\right]}$$

Where C_i = holding cost fraction (ie $C_1 = P * C_i$)

Thus

$$EOQ = \sqrt{\left[\frac{2SC_s}{C_i}\right]}\sqrt{\left[\frac{C_1 + C_2}{C_1}\right]}$$

$$EOQ = (S)^{0.5}\sqrt{\left[\frac{2C_s}{C_i}\right]}\sqrt{\left[\frac{C_1 + C_2}{C_1}\right]}$$

where as before $S = RP$

Now if 'n' is the entire inventory range and $0.2n$ is the top 20% of items (ie A class items based on a usage value ranking), then, if we calculate the average value of S for this fraction we have the information necessary to carry out aggregate calculations for cycle stocks in the same manner as before, but now for the top 20% of items, by use of the formulation-

$$AICS = 1/2(0.2n)\left[\bar{S}^{0.5}(e^{k(k-1)1/2\sigma^2})\right]\sqrt{\left[\frac{2C_s}{C_i}\right]}\sqrt{\left[\frac{C_1+C_2}{C_1}\right]}$$

3.6 (d) accuracy of the lognormal methods

One of the great attractions of using the properties of the lognormal distribution to measure aggregate inventory standards is that it can be extremely accurate, providing the underlying variate which the calculations are based upon is truly lognormally distributed. Wharton (1975) demonstrated in his paper that providing usage values are lognormal, then aggregate inventory calculations to within 2% to 4% of the true values calculated by complete enumerations are quite feasible, even using data where there is some departure from lognormality in the extreme tails of the distribution. When the data is a very close fit throughout the distribution then an accuracy to less than 1% error can be achieved. This makes the use of the lognormal distribution valuable and efficient for a large variety of inventory calculations where answers could otherwise only be obtained by complete enumeration. The latter could be extremely tedious and time consuming in cases where the inventory is very large and it would almost certainly require special programming. Furthermore, Wharton also showed that because of the highly skewed nature of lognormal variates, aggregate calculations based on various sampling methods can yield errors larger than 10% of the true population value. This was a valuable observation and on this basis alone one could justify the use of the lognormal methods for aggregate calculations where the variate of interest can be proven lognormal.

3.7 Literature applications of lognormal functions in inventory control.

The use of the properties of the lognormal distribution are not limited to setting aggregate investment levels, a variety of other valuable relationships and calculations are possible. However, only a comparatively small number of authors have published work on the use of the lognormal distribution of usage values in inventory control applications. Of these only Brown (1967), Heron (1968), (1970), (1974), (1976), (1978) and (in Wild 1981 ed.), and Schary and Howard (1970) and (1971) have really extended the applications beyond the use in setting simple aggregate inventory standards. Even amongst these authors the proposed applications and various inventory norms that have been developed use only the basic theory of the lognormal as applied to usage rates. Very little by way of further basic theory has been advanced. We consider some of the more valuable developments in the following sections.

Heron (1974) has shown how cost factors for reorder costs, cycle stock carrying cost, safety stock carrying cost and stockout costs can be formulated in terms of power functions of the sales rates for the purpose of aggregate calculations. Heron also provides a comparison of the use of such calculations and shows that in practice the results are very close to those obtained by more conventional calculation means. From a retail chain store study Heron (1976) has shown that an excellent lognormal relation existed between annual sales per store and store ranking by sales with a lognormal shape parameter (σ) of 0.98. The annual material handling cost (C_i) per store was estimated from eight stores and expressed as a function of annual dollar sales (S_i). This gave an expression of the form:-

$$C_i = 26{,}800 + 0.094(S_i)^{0.82}$$

from which it was possible, using the lognormal properties, to estimate annual handling costs for all 125 stores in the chain. In the same article Heron demonstrated aggregate trade off analyses of investments against service levels; and he showed how profitability analysis on items in a product line could be performed in aggregate by developing an equation of R_i, the expected return on a product (i) in terms of (S_i) the annual sales rate of item (i).

Schary and Howard (1970) made a very clear presentation of the way various aggregate inventory calculations can be determined and they developed a method of recalculating aggregate investments in inventory after dropping slow moving items from the range. To achieve this they presented a valuable formula (without proof) for recalculating the standard ratio given the proportion (α) of items remaining after the range reduction:-

$$\rho_1 = \rho^{(\log_{10} \alpha - 1)}$$

where ρ is the 'old' standard ratio and $\log_e \rho = \sigma$

These same authors tabulate values of the new standard ratio given (α) the old standard ratio and (β) the proportion (α - 1) of items dropped from an inventory range.

In a follow on paper Howard and Schary (1971) extended the idea of dropping slow moving items (from a retail inventory) and showed the conditions under which it is possible to optimise savings in inventory costs against the reductions in gross margin for eliminated slow moving lines. This was tabulated and presented graphically for various standard ratios and gross product margins. In the same article the authors then went on to discuss characteristics of distribution channels in terms of lognormal characteristics at retail and wholesale levels in a distribution chain. They concluded from their study that the differences that may be observed in the standard ratio of retailers to those for their wholesalers is due to the retailers reluctance to stock or hold on to slow moving items - hence this results in a smaller standard ratio at the retail level. We can deduce from this that the phenomena can be expected to be observed throughout the whole channel from retailer through wholesaler to manufacturers and the authors concluded that :

> *"The change in the value of the standard ratio over the channel therefore demonstrates an important characteristic of current logistics channel systems, the tendency to shift slower-moving items backwards in the distribution channel towards the source of production"*.

This author believes that in their second paper Schary and Howard (1971) only just began to explore what is potentially a very fruitful area of application of the lognormal theory. By building on their work a whole range of problems and issues relating to inventory item and investment decisions in a variety of distribution channel networks become amenable to aggregate analysis instead of relying on sampling estimates or by using the complete enumeration approach, item by item. In chapter 12 we give some new and potentially valuable areas of application of the theory.

Of the remaining authors, other than R.G.Brown, nothing has really been added, in terms of advancing the applications of the lognormal distribution in the inventory field. In a largely theoretical and methodological based paper Wharton (op cit), as we previously discussed, showed that estimating aggregate inventory measures using the properties of the lognormal distribution is extremely efficient statistically. Wharton showed by example that even with a sample of data where there was noticeable departure from lognormality in the tails of the empirical distribution use of the lognormal properties gave a result within 2% of the value obtained by a complete enumeration. This compared very favourably with aggregate values obtained by two frequency grouping methods on the same data which were in error by 11% and 4%.

Brown's paper of 1963, which is largely concerned with a detailed discussions of how to apply lognormal theory to setting aggregate measures for cycle stocks and safety stocks, does show how turnover rates must rise slowly for identical inventory policies for business with different levels of gross sales. To demonstrate this Brown calculates the total investment in inventory for five different levels of gross sales using the same cyclic and safety stock policies in each case and he found the following result:

Table 3.1

Gross sales versus turnover ratio

Gross Sales (millions)	Turnover ratio
6	2.16
7	2.30
8	2.43
9	2.53
10	2.64

Whilst such comparatively small changes in turnover are not likely to be too alarming to practising management it does show that turnover values are clearly related to the absolute level of the gross sales.

Articles and publications by Schary and Howard (op cit), Gargiulo (1969), Wachter (1975), Van Hees and Monhemius (1972), Peterson and

Silver (1979), Christopher (1972) , essentially all deal with the basic theory of computing aggregate inventory standards and with showing how to apply the method in practice. Basically they are all restatements of the earlier works of either R.G. Brown or Heron, but applied to different inventories.

3.8 Conclusions

We have shown in this chapter from various literature sources that the lognormal distribution has had wide application in business and economics. The particular unique mathematical properties of this distribution make it a valuable one to use in various applications. In particular the ability to use it in setting various aggregate calculations make it a valuable tool in many inventory issues without the need to do complete enumerations or develop special programmes.

As will be shown and demonstrated in later parts of this work the distribution and its parameters can give a deeper insight into the fundamental nature of spare parts inventories in terms of the nature of aggregate demands, how demand patterns build up and the effects they cause through distribution channels.

Stochastic Processes and the Genesis of the Lognormal Distribution.

4.0 Introduction

In this chapter the author presents the core theoretical framework upon which plausible mechanisms for the genesis of the lognormal distribution are discussed and evaluated. A review of the literature showed that the only areas where the lognormal distribution has received any significant discussion in terms of its genesis is mainly in a consideration of the growth and distribution of economic variates such as firm sizes and incomes in the economic and applied statistics literature. As was briefly mentioned in the previous chapter, it is the so called 'Law of Proportionate Effect' that seems to be the generally accepted stochastic mechanism that gives rise to this distribution. There are however, some methodological problems to be resolved in the application of this process to spare parts inventory systems. These issues will also be explored in some depth in this chapter.

The Law of Proportionate Effect is examined in some depth here because of its central importance in generating lognormal distributions. Furthermore, as we shall see, variates that are undergoing growth processes, that are governed by this law, do not necessarily always lead to lognormal functions. Depending on a number of assumptions concerning the structure of the systems studied other highly skewed distributions can be obtained as the equilibrium distributions of the variate concerned. Hence it is important to examine these conditions so that we can make the best judgment about the processes and the equilibrium distributions attained in the empirical work in this research.

There is one other documented process in the literature that can lead to lognormality and it is known as the 'Theory of Breakage'. We will examine this process also, to see how far it is applicable to the systems researched by this author.

4.1 The nature of stochastic processes

Before we consider the nature of those stochastic processes of particular relevance to this research we will briefly consider the general meaning of the term stochastic process. According to Bartholomew (1967)

> *" A stochastic process is one which develops in time according to probabilistic laws".*

The word stochastic in fact means random and systems that behave randomly or stochastically lead to uncertain outcomes. However, by knowing something of the nature of the variables (system variables) involved in the system we can make attempts to predict outcomes of system behaviour, or the equilibrium state of nature that it might reach.

The Poisson process (discussed in depth in the next chapter) is an example of a stochastic process of significant importance to many management systems. In a single server queuing system, for example, the system variables are, the arrival rate to the system, its reciprocal- the mean time between arrivals, and the service rate. If the time between arrivals is distributed as a negative exponential distribution with a mean 'μ ', then the process is simple Poisson. In stochastic terms by knowing just this fact enables us to predict under stable conditions (μ remaining constant) the arrival pattern to the system and the likelihood of queues and the extent of queuing in the system. Furthermore if we know the nature of the service rate then we can predict exits from the system and the volume of work in the system at given points in time. This simple model is well documented and classical queuing theory provides explicit formulas to solve most of the problems associated with such systems. Additional information about the behaviour and states of the system can be predicted if we know the extent and direction of change of any of the system variables, e.g. that the system moves to a new equilibrium state, or perhaps begins to become unstable with the passage of time and continued operation of the system.

Most stochastic systems can be seen as operating through time and are time dependent. This is certainly true of most of those of interest to management and we can see that they involve the interaction of a number of random variables, as in the Poisson process, which will lead to stochastic outcomes. It should be noted in passing here that a stochastic process may have an important spatial dimension as well. This is certainly so in some fields of ecology and biometrics as we shall see in the next chapter. Our general problem in stochastic model building and analysis is to define systems and the system variables in such a way as to enable tractable solutions and meaningful predictions to be made from the systems. The literature on stochastic systems is already large and one has only to peruse texts such those by Bartholomew (1967), [Stochastic Models for Social Processes], and Grassman (1981), [Stochastic Systems for Management] to appreciate the diversity and range of concepts and theory that has been developed. No attempt is made here to give a review of the broad theory, only those areas that are directly relevant to this research are considered in this chapter and the one that follows.

4.2 The Law of Proportionate Effect

Aitchison and Brown (1957) have drawn attention to the fact that whenever the Law of Proportionate Effect governs a statistical variable then that variable is very likely to be lognormally distributed. That is a variate defined in terms of the product of a number of elementary variates tends to lognormally distributed. This, according to Aitchison and

Brown, forms the groundwork on which all the existing theories of the genesis of the lognormal distribution are based. It arises out of the original work and theory of Kapteyn (1916), and later work by Gibrat (1931) and Kalecki (1945).

The Law of Proportionate Effect holds whenever the change in a variate at any step in a process is a random proportion of the previous value of the variate and does not depend on any other factor. This, according to Ijiri and Simon (1977) is known as the Gibrat assumption. Ijiri and Simon have in fact challenged the work of Aitchison and Brown as, in effect, being incomplete. They maintain that depending on the assumptions regarding the boundary conditions, in respect of entries and exits to the system, the lognormal, Yule or even the Pareto distribution may be obtained as the equilibrium distribution. This point will be returned to at a later stage in this chapter.

Aitchison and Brown (1957, page 22) define the law more explicitly as follows-

> *" A variate subject to a change process is said to obey the Law of Proportionate Effect if the change in the variate at any step in the process is a random proportion of the previous value of the variate".*

Thus using the methodology of Aitchison and Brown we can formulate the process and say that at the *j*th step : -

$$(x_j) - (x_{j-1}) = \varepsilon_j (x_{j-1})$$

Where x_j is the value attained by the variate at step *j*, and x_{j-1} being the value of the variate at the previous step. The randomising set of elements ε_j are mutually independent, and also independent of the set (x_j). Thus-

$$\frac{(x_j)-(x_{j-1})}{(x_{j-1})}=\varepsilon_j$$

so that

$$\sum\frac{(x_j)-(x_{j-1})}{(x_{j-1})}=\sum\varepsilon_j$$

If we consider the effect at each stage of the process to be very small then we can write the process as -

$$\frac{(x_j)-(x_{j-1})}{(x_{j-1})}=\int_{x_0}^{x_n}\frac{dx}{x}=(\log_e x_n-\log_e x_0)$$

Which then gives-

$$\log_e x_n=\log_e x_0\ +\varepsilon_1+\varepsilon_2+\ \text{------}\varepsilon_n \qquad \text{------------}4.1$$

Then, according to Aitchison and Brown (1957), by the additive form of the central limit theorem $\log_e x_n$ is asymptotically normally distributed and hence x_n is lognormally distributed.

Furthermore following from (x_j)-$(x_{j-1}) = \varepsilon_j(x_{j-1})$ by rearranging and taking logs we obtain -

$$\log_e(x_j)=\log_e(1+\varepsilon_j)+\log_e(x_{j-1})$$

This implies that the law of proportionate effect can be tested by regressing $\log_e x_t$ against $\log_e x_{t-1}$ to give a straight line of slope unity. In fact Easton (1977), Singh and Whittington (1974), and Ijiri and Simon (1977) all refer to the regression of $\log_e (x_j)$ against $\log_e (x_{j-1})$ as a test for the above form of the law. This was the approach used by this author in chapter eight.

4.3 A Random Walk Approach

The Law of Proportionate Effect can be approached in a more heuristic way as shown by Aitchison and Brown from the original work of Kalecki (1945) and Gibrat(1931) -

Given that $$(x_j)-(x_{j-1})=\varepsilon_j(x_{j-1})$$

so $$(x_j)=(1+\varepsilon_j)(x_{j-1})$$

and therefore for 'n' stages in the process

$$x_n = x_o(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)\text{--------}(1+\varepsilon_n)$$

For small time intervals and hence small values of ε_j we can say that-

$$\log_e x_n = \log_e x_o + \varepsilon_1 + \varepsilon_2 + \text{---------} \varepsilon_n \qquad \text{-------4.2}$$

Thus the log of the variate at any time or step 'n' is therefore the outcome of the addition of many small independent random variables acting on the log of the initial size. This is equivalent to a random walk process on a logarithmic scale. If the randomising elements ε_j are identically distributed with mean μ and variance σ^2 then by the central limit theorem the sum of the elements ε_j over 'n' growth stages are normally distributed with mean $n\mu$ and variance $n\sigma^2$. As 'n' tends to ∞ the distribution of $\log_e x_n$ will be normally distributed with mean $n\mu$, and variance $n\sigma^2$, and hence x_n will be lognormally distributed.

According to Easton (1975), both Kapteyn (1916) and Gibrat (1931) regard this log random walk just as fundamental as the simple random walk process. Furthermore, according to Kapteyn (1916) many naturally occurring variates could be envisaged where the proportionate size change would depend only on the immediately previous value.

There is however, a major drawback with the above reasoning, because according to Kalecki (op cit) the process becomes dissipative. As 'n' the number of growth stages approaches ∞ so too does the variance of the process variate. Aitchison and Brown (op cit.) have also drawn attention to the fact that continuous application of the general form of the Law of Proportionate Effect will lead to an equilibrium distribution with increasing variance. This may not prove a major problem in a field of enquiry such as the study of the growth of biological species where growth is terminated at some stage by natural forces. However, it does present problems in economic and social systems. Aitchison and Brown have pointed out that in the case of the distribution of earnings for example it would mean that the concentration of total income would reside with progressively fewer wage earners. This is of course contrary to general experience, because although we know that wealth is disproportionately distributed in most societies there are social and economic limits to its ultimate level or degree of concentration. When applied to the distribution of firm sizes it would mean increasing industrial concentration with the largest firms becoming increasingly larger with the passage of time. Other forces do play their part in restraining growth such as political controls and market dynamics.

4.4 Stabilisation of the Equilibrium Distributions.

To make sense of the observed facts concerning the growth and statistical distribution of certain economic variates in terms of the Law of Proportionate Effect we must consider the possibility of stabilisation conditions in the process. Stabilisation can be considered as a result of constraining economic forces or other factors that bring about convergent conditions.

Hart and Prais (1956) have considered this problem in their work on the growth of firms. They concluded that a simple lognormal model can be used to explain the growth of firm by monetary sizes. Furthermore they conclude that the variance of the transformed distribution changes over time according to the equation-

$$V(x_{t+1}) = \beta^2 V(x_t) + \sigma^2 \quad \text{----------------4.3}$$

where the variances at two dates t and $t+1$ are given by $V(x_t)$ and $V(x_{t+1})$ and the residual variance by σ^2. Thus when firms grow by a random proportion of their monetary size the relative dispersion of the distribution will tend to increase over time only if the coefficient of the above equation is greater than one. If the coefficient is less than or equal to one then the variance will not be dissipative. Whilst this is not, in itself, an explanation of stabilisation it does indicate that Hart and Prais have observed industry conditions where stabilisation has occurred and they have quantified this. For example, Hart (1957) has shown that the growth process can be interpreted in economic terms in the theory of the optimum firm size. From studies of empirical data from the brewing industry he has shown that the stochastic growth process is not always dissipative provided the regression coefficient is below unity.

In the same paper Hart has drawn attention to the question of entries and exits to the system of firm sizes and the effect of this on the variance of the equilibrium distribution. In general his conclusion is that new entries to the system will decrease the variance whilst exits will increase it. This same author has further shown that a change in the total variance of the logarithm of firm sizes, as the number of firms in the system increases, depends on the mean and variance of the births which are related to the mean and variance of the total number of firms as follows-

$$\sigma^2 = w_1\sigma_1^2 + w_2\sigma_2^2 + Tw_1w_2(x_1 + x_2)^2 \qquad \text{----------4.4}$$

In this equation $\sigma^2{}_1$ and x_1 refer to the variance and the mean of the logarithms of the sizes of the surviving firms, whilst w_1 refer to their proportion in the total number of surviving firms plus births (new firms). Similarly $\sigma^2{}_2$, x_2 and w_2 refer to the births. Clearly from Hart's work the rates of entry and exit to the system of the variate under study has a direct effect on the variance of the equilibrium distribution of that variate.

Aitchison and Brown have approached this problem of increasing variance in a somewhat more heuristic way concerning the variance of incomes. Their explanation which follows, counters earlier criticisms to the Law of Proportionate Effect as the process of income growth. Their modification to the process is seen from the quote-

> *"In attempting to explain incomes we may first think of a completely homogeneous group of wage earners each with a claim to an equal share. We then take into account that the*

> *group is not homogeneous, each earner possessing to a different extent attributes and talents which influence the magnitude of his claim. The outcome of these many different effects acting in accordance with the Law of Proportionate Effect is again to produce a lognormal distribution of incomes. At other points in time the distribution of incomes may be thought to arise in a similar way. The reason for the stability of the variance is then sought in the distribution of the attributes and talents in relation to the evaluation of these by contemporary society. Secular changes in this evolution may lead to a drift in the value of the variance".*

There is no indication from this explanation whether any drift that may occur in the value of the variance is an increase or a decrease. One can deduce however, that secular changes which favour the minority already in the high earner group will increase the variance of the process. Likewise such changes which favour a disadvantaged majority will decrease the variance. What is clear however, is that plausible heuristic reasons can be put forward that go some way to explaining the underlying stability seen in practice and yet still allows the system to be interpreted in terms of the Law of Proportionate Effect.

In even earlier work than that of Hart, or Aitchison and Brown, Kalecki (1945) has put forward arguments to show that the variance can be stabilised or even decrease over time. He has shown that the increasing variance assumption of the basic law is unrealistic for economic reasons because no tendency exists for the variance to continually increase, and he then goes on to use income issues as the example. Furthermore, he asserts that to a great extent, the changes in the variance of a lognormally distributed variate generated as a result of stochastic economic pressures, is governed largely by economic forces.

In a more quantitative approach to this issue Kalecki goes on to show that the Gibrat assumption can be modified to admit two important cases both of which can lead to the lognormal distribution as the equilibrium state of the system observed. Firstly a case where the variance is assumed to remain constant, and secondly the more general case where the variance of log x changes through time. According to Kalecki this second case may happen in three ways-

(a) *The second moment increases merely under the influence of random shocks- the dissipative approach.*

(b) *The change is fully determined by economic forces.*

(c) *The influences of economic forces acting upon the second moment is not so rigid as to prevent it from being influenced by random shocks.*

In the case where the variance of $\log_e x_j$ is assumed constant it implies a negative correlation between $\log_e x_j$ and $\log_e(1+ \varepsilon_j)$ where ε_j belongs to the set of randomising elements from-

$$\log_e x_n = \log_e x_0 + \log_e(1+\varepsilon_1) + \log_e(1+\varepsilon_2) + \text{---------} \log_e(1+\varepsilon_n)$$

From which Kalecki provides us with the following formulation of the correlation-

$$y_j = -\alpha Y_j + Z_j$$

where $\qquad y_j = \log_e(1+\varepsilon_j) \quad \text{and} \quad Y_j = \log_e x_j$

This is on the assumption that the regression of $\log_e(1 + \varepsilon_j)$ on y_j is linear and that Z_j is independent of Y_{j-1}. The new stochastic generating equation then becomes -

$$x_j = (x_{j-1})^{1-\alpha_j} e^{z_j}$$

where the regression coefficient α is given by -

$$\alpha = \sum y^2 / 2\sum Y^2$$

For a detailed proof of this development the reader is referred to Kalecki's original paper (1945). What is important for our research, from this original work by Kalecki, is not the particular formulations above and others developed by Kalecki, but the fact that it can be shown that the Law of Proportionate Effect does not necessarily lead to a dissipative process with increasing variance.

From all the foregoing discussions we can conclude that good arguments exist to allow stabilising conditions and the stable equilibrium distributions that we show in chapter six are clearly fully consistent with established theory.

4.5 The form of the Equilibrium Distribution

So far in this thesis we have shown from the wealth of work in the econometrics literature that the Law of Proportionate Effect is the most likely candidate to explain the convergence of a variate to lognormality. Indeed, apart from the Theory of Breakage- itself a special case of the law, there appears, as yet, no other convincing mechanism. However, we must now consider the question of the final form or nature of the equilibrium distribution of a variate undergoing a growth process governed by the Law of Proportionate Effect.

As we have already mentioned in this chapter according to Aitchison and Brown (1957) whenever the Law of proportionate Effect governs a variate then the distribution of that variate is very likely to be lognormal. They have not however, discussed any alternative distributions that may come about as a result of the operation of this law. Their work has been criticised by Simon and Bonini (1958) and Ijiri and Simon (1977, see page 143 in particular). The criticism generally hinges on the boundary conditions of the system considered. Ijiri and Simon assert that by introducing some simple variations into the assumptions of the stochastic model of firm size growth, but retaining the Law of Proportionate Effect as the central feature then various skewed distributions are obtained. They say :

> *" ---we can generate the lognormal, the Pareto, the Yule and the Log Series distributions"*

They also go on to say -

> *"If we assume a random walk of firms already in the system at the beginning of the time interval under consideration, with zero mean change in size, we obtain the lognormal distribution. If we assume a random walk, but with a steady introduction of new firms from below* [author comment - they mean new small businesses] *we obtain the Yule distribution".*

Hence these authors assert two conditions for the achievement of lognormality under the assumption of the operation of the Law of Proportionate Effect-

(a) All items in the system must all start the growth process at the same time .

(b) There must be a constant size of the system, i.e. no entries or exits.

In their extensive treatment of the Yule distribution, Ijiri and Simon (op cit) develop stochastic processes that utilise the Law of proportionate Effect and with the assumption of a steady increase of new firms they obtain the Yule distribution as the long run distribution of the process variate (ie firm sizes). It would seem from this same work (especially chapters 10 and 11) concerning the size, growth and distribution of firm sizes, that with the additional effect of serial correlation in growth rates, and combined with the effects of mergers, acquisitions and dissolutions, that the Pareto distribution is the most likely candidate for the equilibrium distribution of firm sizes. At no point in the same literature source do they give any conditions that can lead to the Log Series distribution as the equilibrium function of a variate undergoing growth based on the assumption of the Law of Proportionate Effect.

4.6 The Theory of Breakage

The other process that can lead to lognormality of a variate is the so called 'theory of breakage' we mentioned in the previous section. An examination of this theory was found to be marginally useful in this research when the form of parts price distributions were considered This is discussed in chapter six. For the moment we confine ourselves to the theoretical foundations of the theory. The development of this is based on the empirical work of Kolmogoroff (1941) whereby items undergoing a physical breakage process are often found to be lognormally distributed. In consequence as pointed out by Aitchison and Brown (1957) the theory has, not surprisingly, found application in research on particle size statistics. We present the theory here in the form given by Aitchison and Brown(1957, page 26) .

"Suppose there is a set of elements each of which has some

positive measure - the dimension of the element. Let the elements be subjected to a sequence of independent breakage operations. If at the jth breakage $G_j(x{:}u)$ describes the distribution of elements arising from elements of dimension 'u' prior to the breakage, then the Law of Proportionate Effect is equivalent to the statement that $G_j(x{:}u)$ depends only on the ratio x/u, thus we may write -

$$G_j(x{:}u) = H_j\left(\frac{x}{u}\right)$$

Then

$$F_j(x) = \int_u H_j(\tfrac{x}{u})dF_{j-1}(u)$$

If X_j and T_j *are the variates associated with the distribution functions* $F_j(x)$ and $H_j(t)$ then -

$$X_j = T_j X_{j-1} \quad \text{so that} \quad X_n = X_0 \prod_{j=1}^{n} T_j$$

From this development Aitchison and Brown (op cit., page 27) then conclude that the final distribution tends to lognormally distributed.

The literature contains very few applications of this theory to business situations. Horvath (1959) is one of the few and he presents a simple case where the theory of breakage is considered to be the underlying explanation of the lognormality of service times in a tool crib application. According to Aitchison and Brown the central idea behind the principle is a theory of classification. They go on to say -

> *"It is a curious fact that when a large number of items is classified on some homogeneity principle the variate defined as the number of items in each class is often approximately lognormal."*

Several examples are quoted where these authors have observed the phenomena none of which have interest here. However, it is possible to see the application of the theory to the distribution of prices of physical items such as spare parts. We consider this issue in the next section.

4.7 Inferences and Conclusions from the Law of Proportionate Effect

We have seen earlier in this chapter that the Law of Proportionate Effect is a stochastic process that can result in a variate becoming lognormally distributed after 'n' growth stages. As we have shown Aitchison and Brown (op cit) have discussed the possibility that the lognormal distribution can, in some cases, be seen as the result of a process which is known as the Theory of Breakage, which itself is really a special case of the Law of Proportionate Effect.

When we consider the Law of Proportionate Effect as the mechanism to explain lognormality of inventory usage values we must take into account the fact that a usage value is the product of item cost(price) and volume demanded in a particular period. These two component parts must be considered separately in order to reach a clearer overall view as to just what is affecting the usage value distribution. Firstly, if we consider demand volumes we can readily see these developing as growth processes in the following way. If we progressively extend the time period from a point time, or epoch, to a large finite period, then the quantity demanded per period will obviously increase also, from something close to zero at the epoch to whatever has accumulated at the time the process is considered terminated. It then remains for us to see what evidence exists to suggest that such growth, when considered in aggregate, conforms to the Law of Proportionate Effect. In the paper by Singh and Whittington (1974), and the text by Ijiri and Simon (1977) methods of testing such growth processes against the Law of Proportionate Effect are given, and we use these in chapter eight where we tested for the law against empirical data. This was an important part of theory validation based on our models developed in chapter seven.

The question of the distribution of prices presents a bigger problem because it is not easy to envisage growth processes associated with this variable. At the factory level the prices are already fixed and in general they are marked up at each distribution level thereafter by proportionate increments until the final retail price is reached. The Theory of Breakage briefly discussed earlier would seem to have some value here to explain lognormality in the prices. We can consider a physical analogy to the breakage processes in rock ores. If the total population of parts in a truck are all laid out before us in their smallest divisible size we will see hundreds of very small components and only a very small number of very large components such as an engine block or crankshaft. If we start with the very largest components and commence grouping by some classification scheme, such as weight or size, and progressively work through the whole range, then, if the theory of breakage applies, we are very likely to find the distribution of the number of items in each class forming a lognormal distribution. Furthermore because component cost is very likely to be strongly correlated with size (and weight), costs are also very likely to be lognormally distributed. Whilst we can appreciate the general logic to this principle we can also see its imperfections. A rough machined bolt costing pence may have the same size and weight as a precision machined needle valve costing several pounds in value! An alternative classification scheme could be based on the proportion of time it takes to manufacture a component and group on the basis of equal increments of production time. Unfortunately whatever scheme we envisage there are no easy ways to test such propositions in analytical terms. The limited literature on the Theory of Breakage provides no direct methods of testing the process in mathematical terms. Hence at this stage we are largely left with the classical tests and the analytical form of the actual distribution found. This problem will be considered again at a later stage.

Returning now to the distribution of volumes we have another methodological problem to resolve. As we have discussed according to Aitchison and Brown (op cit) whenever the Law of Proportionate Effect rules a variable then that variable is most likely to be lognormally distributed. However, as mentioned already , this is in conflict with some of the views expressed by Ijiri and Simon (op cit). In the systems considered by them entries and exits are an integral part of the process and this they have argued leads to the Yule distribution. In the case of the DAF spares system there has always been a steady increase in the number of new parts to the system as well as a steady, but smaller, number of departures through obsolescence and rationalisation activities. Furthermore Ijiri and Simon have made the point strongly that if the Law of Proportionate Effect is to yield the lognormal distribution all the items in the system must all start the growth process at the same time. If we consider again the DAF inventory if new spare parts are continually being added into the system this should violate the conditions for lognormality. But this is contrary to the extensive empirical evidence assembled by this author and presented in chapter six, where lognormal forms are clearly the final and stable long run distributions obtained.

For some while this author was in some difficulty in reconciling the facts observed in the DAF Trucks data in relation to the Law of

Proportionate Effect given the views and theory of Ijiri and Simon. The usage values and volumes were both clearly close fits to lognormal distributions and yet the stochastical literature [particularly that according to Ijiri and Simon (op cit)] indicated it should be the Yule distribution that should be obtained, if indeed the Law of Proportionate Effect was operating. The question of distribution differentiation had to be considered at this stage. In general it can often be difficult to differentiate between highly skewed distributions of the type considered in this work and it is often only in the extreme upper tail of such distributions that significant departures from one another are usually seen. However, in the DAF data, preliminary analysis showed that the Yule and the Pareto distributions were very poor fits to the data observed. This was tested by plotting item size, (either as volume or usage value) on log log graph paper against size rank order, according to the method shown by Ijiri and Simon (op. cit.). If either the Yule or Pareto distributions were valid considerations then an approximate straight line should have been obtained. In all cases attempted marked curves of the type shown in chapter six [section *6.2 (d)*] were obtained. It is only in the extreme upper tail that a fit to a Yule plot is obtained. Given the large size of the samples tested in this way (n>100) it could be confidently reasoned that the Yule and the Pareto were unlikely to provide a satisfactory description of our empirical usage value and usage volume distributions.

However, when it was realised in most cases of the analysis undertaken on usage values and usage volumes by this author that the data was effectively being treated as a closed system, then the observations could be reconciled with the theory. For example, in parts of the analysis (treated in detail in chapter six) the monthly demand volumes of 200 randomly selected parts were cumulated over successive months to show a convergence to lognormality. By preselecting 200 at the outset and keeping this number constant the system was closed to entries of new parts, although not effectively to exits of very slow moving members of the system. As the exit of parts in general from the parts range in total was very small, for all practical purposes the analysis was on a closed system. Hence once this was realised it was not too surprising to find the lognormal as the equilibrium distribution. In other cases of analysis on whole parts range data (from ABC printouts comprising anything from around 7000 items in the 1970's data to around 11,000 in the '80s data), the rate of new entries and exits, although positive were very small in relation to the total size of the system. Also in time terms the periods were comparatively short, just one calendar year. Hence, again the system, for practical purposes, could be regarded as a near closed systems. It can be concluded that although the lognormal distribution does indeed seem to be the final stable distribution it may not however, be the final form if we examined the data over several years. In this case we may find the Yule distribution emerging. From an inventory management viewpoint however, this would have no value at all and would not lead to any practical applications. We are confident that the lognormal distribution is a satisfactory representation of inventory usage values over time periods of value to practical inventory and logistics issues.

We also need to give careful consideration to the fact that the systems studied by Ijiri and Simon would have been very limited in size in terms of the individual members of the systems, ie firms in an industry, and entries and exits may well have been large in comparison to the number of members of the system. In the case of our spare parts systems the number of individuals was very large and entries and exists (on a per annum basis) were very small in comparison. Hence we must be very careful how far we translate findings from one system to another with such a variation in size and structure.

Stochastic Process Models of Independent Demands

5.0 Introduction

The stochastic modelling approach of chapter four was based on what this author chose to regard as the top down approach. That is start with the assumption that usage values in aggregate are lognormal. In this way the research rested on finding and verifying process models that are known to yield lognormal distributions and testing them against the data and circumstances that give rise to lognormal distributions in the systems studied in this research. As discussed in the previous chapter, the Law of Proportionate Effect is a very good candidate to explain a stochastic growth process that leads to lognormality. However, the law does not provide an explanation of the the origin of the process, nor does it offer any insight into the observed regularity in aggregate demand for very short time periods.

This chapter explores and reviews stochastic process models for independent demand items with particular reference to the Poisson process, spare parts systems and cases where the value of demand variance exceeds the mean. We make no attempt at a complete review of all the literature concerned with recurrent event processes and stochastic models of compound demand processes. This literature is already extraordinarily large, and a great deal of the contemporary work in the inventory area owes much to the pioneering works of Arrow, Karlin and Scarf (1958), and Hadley and Whitin (1963), in particular. Any reader interested in following some of the early stochastic developments would find it of value to start with these sources.

Our concern here is to summarise only those models and concepts from single item demand theory that have proved useful as a step towards our objective of developing models for aggregate inventory demands. We show here the particular importance of the Poisson process to the development of a scheme that ultimately leads to a lognormal distribution of aggregate demands for spare parts: this was the bottom up approach. In chapter seven we then draw together the work of this chapter, that of chapter four and the empirical work of chapter six to show how we can move from a consideration of individual items to many items in aggregate in one overall stochastic scheme. The first clue that this might be a viable line of research came from the work of Easton (1975 and 1980), who attempted to model aggregate industrial buyer behaviour using the lognormal distribution..

Although only partially successful in his endeavour Easton nevertheless showed that it might be possible to start with elementary assumptions about demand processes, in his case at the level of the individual industrial buyer, and build up to a more complex aggregate case of many industrial buyers over an entire industry. Easton also considered in some depth the possible use of the so called Poisson Gamma model, which has been used extensively to model 'consumer' buyer behaviour. With this model the demand at the level of the individual consumer for frequently purchased items consumer items is assumed to be Poisson. When buyer behaviour is aggregated over many consumers using a gamma distribution of purchasing rates the Negative Binomial distribution is obtained -the so called Poisson Gamma model. The pioneering work with this model was carried out by Andrew Ehrenberg (1959) and since then it has been used extensively to model consumer purchases for individual consumer items. In Easton's investigation its use was not successful in modelling the purchase behaviour across individual industrial firms, but the insights he gave to more complex systems have proved helpful to this work.

We postulated that starting from similar elementary assumptions to Easton a fruitful line could be developed that might yield a mechanism to explain the regularity seen for parts aggregate demand. This approach was encouraged by the wealth of information in the literature of both an empirical and theoretical nature to support the view that demands for spare parts at the level of individual items are almost always Poisson in nature. We also note and demonstrate that the validity for using a Poisson process in this research has a much stronger basis than the modelling work of Ehrenberg et. al. in the consumer purchase field.

We start by first considering the simple Poisson process and then see what important modifications have been proposed and used in various fields of inquiry. Finally consideration is given to theoretical and empirical evidence that supports the use of certain modified Poisson process models in spare parts systems.

An important note to the reader must be given at this stage. We consider here only Poisson process functions that model demand in fixed intervals of time. Our concern is not with independent demands in variable lead times, that is necessary if one is considering lead time demand for the purpose of setting safety stocks and reorder levels. Hence a lack of a detailed consideration of variable lead time models must not be considered an omission in this work. Our concern is ultimately with aggregate demand behaviour in fixed time intervals and not inventory control models for individual items. The variability in a re-supply lead time is a factor introduced by the supply channel and it must of course be considered in inventory control system design. Many models have been developed and presented in the literature to take into account the three principal variables that need to be considered in such a problem, namely the order rate per unit time, the order intensity (ie the actual amount ordered) and the lead time distribution itself. The reader interested in this area is referred to the schema for classifying such modelling approaches by Bagchi (1987) and the text by Silver et al. (1979). We give no further direct consideration to this issue.

5.1 The Simple Poisson Process

A Poisson process with rate λ is a process in which the state variable x is Poisson distributed with parameter λ as -

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

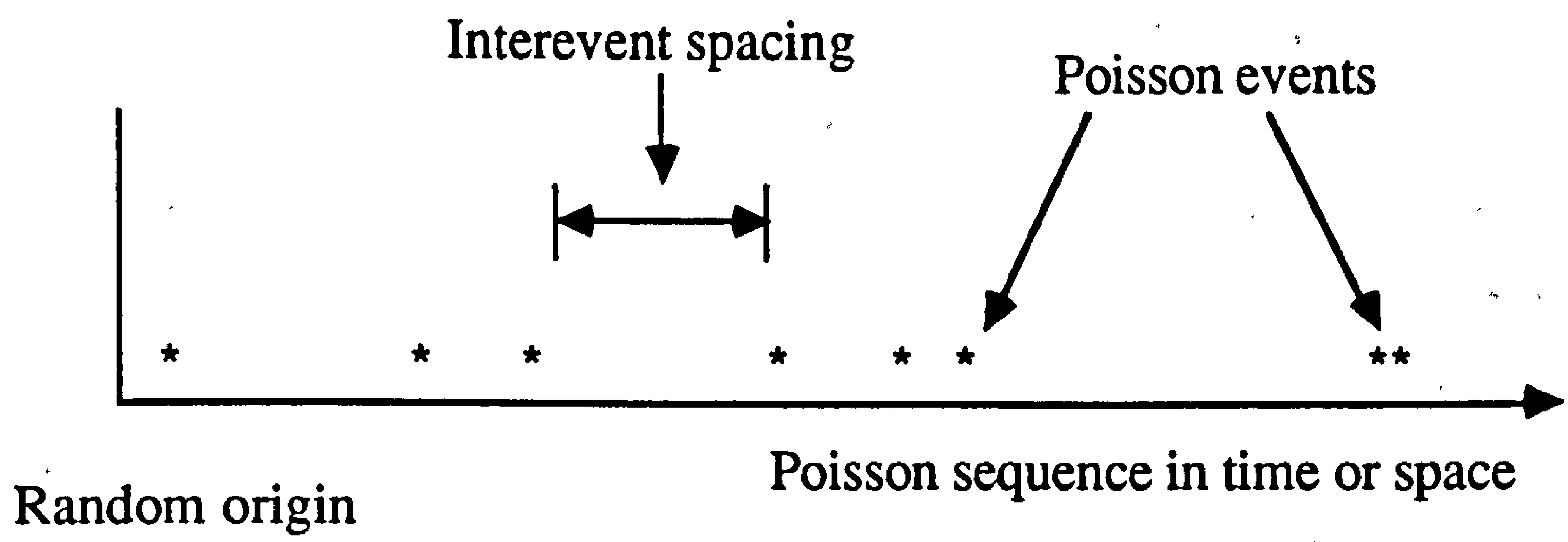
The rate λ is usually obtained by estimating the expected number of Poisson events per unit of time period. For example, if the number of arrivals to a tollgate is on the average 5 per hour then the Poisson rate is 5 and the probability of observing any other value i in a time period of one hour is -

$$P(x = i) = \frac{(5)^i e^{-5}}{i!}$$

In a Poisson process with rate λ the time between two Poisson events is exponentially distributed with expectation $1/\lambda$ (where $\mu = 1/\lambda$). Conversely, if we have a string of independent events, such as arrivals, and if the times between consecutive events are exponentially distributed with expectation μ, then the process is Poisson with rate $1/\mu$ or λ. A further characterisation of the Poisson distribution that is of concern with aspects of methodology in this work is that the mean and variance of the Poisson distribution are always equal numerically.

According to Jewell (1960), Haight (1967), and others, a <u>necessary</u> and <u>sufficient characterisation</u> of the Poisson process is that the probability distribution of the distance between successive events (the interevent distribution) is negative exponential. The Poisson process can be regarded as a maximum disorder process and events that occur by such a process are as random as possible, Jewell (1960), Haight (1967) and Grassman (1981). A Poisson sequence can be envisaged as follows in figure 5.1:

Figure 5.1
Simple Poisson Process



The occurrence of a sequence of Poisson events may be in terms of time or space, or both, and Poisson events are independent outcomes. In most economic and management systems a simple Poisson can be regarded as a stochastic process in which the Poisson sequence occurs in time. For example, the arrival of customers to a supermarket checkout when the system is lightly loaded is mostly likely to be a simple Poisson process. Each arrival is random and completely independent of any other arrivals and we would expect to find the distribution of inter-arrival times to be distributed as a negative exponential function. However, when such a system becomes heavily loaded the build up of queues at check outs can influence the behaviour of other shoppers, who may delay their arrival at the check out points. Under these circumstances the simple Poisson process would begin to break down and more complex stochastic processes would take place. An excellent study of a variety of complex stochastically derived operational systems is presented in the classic text by Morse (1957) together with methods of analysis and solution.

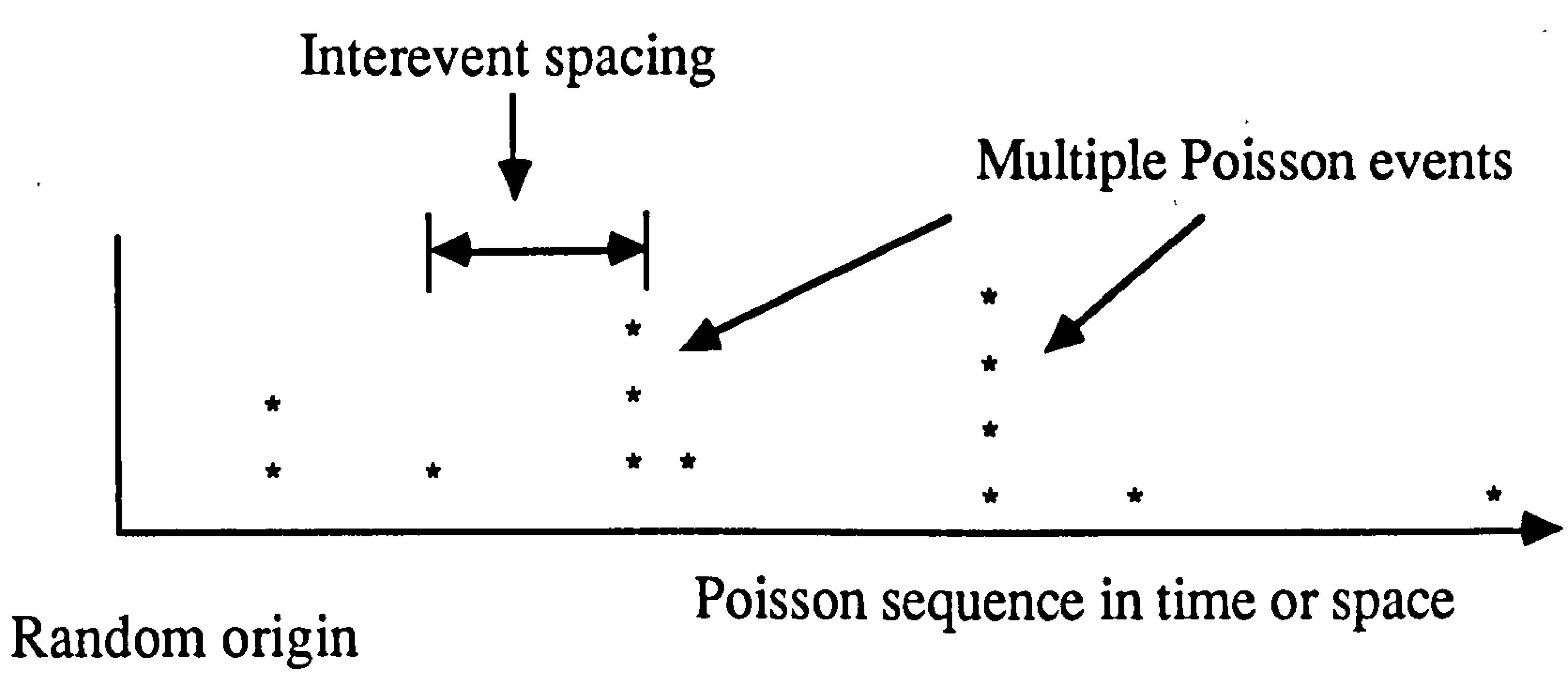
In terms of inventory theory the demands at the retail and wholesale levels of an inventory echelon are usually independent and if the interevent spacing between successive demands is exponential then the process is Poisson and the counting distribution of the number of demands expected in a unit interval of time will follow a Poisson distribution. However, as noted by Morse (1957), Jewell (1960), and Sherbrooke (1968), amongst others, many operational situations occur where the inputs are Poisson in character yet the variance of the process is much greater than that predicted by the simple Poisson process. In inventory terms for example this means we can often observe that the variance of demand in any period is greater than the value of the demand mean for the same period. This characteristic generally results in a highly skewed distribution of the number of inputs per unit time to the operational system. As a result of the wide spread occurrence of this phenomena the literature on inventory theory is rich in the variety of skewed distributions that have been put forward from time to time to model various demand processes of individual items in such circumstances. The issues and the various approaches to solve the associated demand estimation problems for skewed demand situations in both fixed and variable periods has been addressed by many authors including Morse (op. cit.), Galliher, Morse and Simmond (1959), Jewell (1960), Adelson (1966), Feeney and Sherbrook (1966), Gallagher (1969), Bott (1977), Ward (1978), Nahmias and Demmy (1982), Mitchell et. al. (1983), and Bagchi(1983 and 1987), amongst others. A common view of these authors is that demand can be represented by so called compound Poisson models, that we consider in the next section. The obvious generalisation of the simple Poisson Process is a compounding process in which the interevent spacing between some of the Poisson events is reduced to zero and multiple Poisson events can and do occur simultaneously.

Gallagher D.J. (1969) refers to the simultaneous occurrence of multiple Poisson events as the 'Stuttering of events' and to the process as the Stuttering Poisson process. Strictly speaking the term Stuttering Poisson distribution is confined to a particular member of a general class of compound Poisson distributions as will be discussed shortly in this chapter. According to Morse (op. cit.) and Jewell (1960) compounding in a Poisson process always increases the variance of the process and results in a compound Poisson distribution as the counting distribution of events in fixed time intervals.

5.2 Modified Poisson Processes

We can now formally consider the development of two important extensions to the simple Poisson process. In the first, as already stated, the spacing between Poisson events is allowed to become zero so that two or more Poisson events can occur together. In the second the Poisson rate parameter (λ) itself is regarded as a variable and distributed according to a statistical distribution; a process regarded and termed Poisson mixing by some authors. The first situation can be depicted as shown in figure 5.2 -

Figure 5.2
Compound Poisson Process



According to Haight (op. cit.) there is some confusion over the precise meaning of the terms Poisson compounding and Poisson mixing in the literature on statistical theory. However, we have chosen here to use the term Poisson compounding when a discrete magnitude is applied to the number of events occurring on each Poisson occasion, and Poisson mixing when a continuous magnitude (or distribution) is applied to the Poisson rate parameter (λ). This is consistent with the terminology of Haight (Op. cit.) in his definitive study of the Poisson distribution and with the majority of authors, who discuss such distributions in the context of inventory theory.

5.2 (a) Poisson compounding.

When a Poisson process is compounded this gives rise to a counting distribution of the number of events occurring in a unit time with a variance and coefficient of variation much greater than that predicted by the simple Poisson process. The compounding and the increase in the variance are due to the simultaneous occurrence of two or more events and furthermore the number of events occurring on each occasion are themselves statistically distributed. It should be noted however, that if the concern is solely with the number of occasions when a Poisson event took place, rather than with the actual value of the events themselves, then the process defined in this way reduces to a simple Poisson process, even though compounding may be taking place. Hence this is a Poisson process of demand occasions and not demand quantity in the same time unit. This distinction between Poisson demand occasion (or demand incidence) and Poisson demand quantity is important and is an issue we will return to in later sections and chapters.

In theory a great many discrete statistical distributions could be used to model the distribution of the number of events occurring on each

Poisson occasion in a compound Poisson process. However, it would seem from Haight's work (op. cit.) that only a few distributions are considered important in applied statistics. We consider five here that would cover the majority of operational situations likely to be encountered in practice. Following the notation used by Bott(1977), and Johnson and Kotz (1969) we first consider three possibilities. If the number of compounding events follow a Binomial distribution, the resulting overall compound distribution is the Poisson Binomial; if the events follow a Negative Binomial we have a Poisson Pascal compound distribution; and if the number of events follow a Poisson distribution then we have a Poisson Poisson (or Neyman Type A) distribution.The choice between these three in any particular operational situation would depend on the variance to mean ratio of the distribution of the compounding events themselves. It the ratio is less than one then the Binomial model is the preferred choice with the Poisson Binomial being the model to use as the overall compounded demand model. If the ratio is equal to one then the Poisson is the choice, and if the ratio is greater than one then the Negative Binomial is the choice, and the overall compounding model would be the Poisson Pascal. Bott gives the probability density functions for each of these three models, which we reproduce below.

Poisson Binomial -

$$P(d=k,r,p)=\binom{r}{k}p^k q^{r-k}$$

for $r = 1, 2, 3,$ etc
$0 \leq p \leq 1$
$k = 0, 1, 2,$ etc.

Poisson Poisson-

$$P(d=k,\rho)=\frac{e^{-\rho}\rho^k}{k!}$$

for $k = 0, 1, 2,$ etc.

Poisson Pascal -

$$P(d=k,r,p)=\binom{r+k-1}{k}p^r q^k$$

$$\begin{aligned}&\text{for } r=1,2,3,\text{etc}\\&0\leq p\leq 1\\&k=0,1,2,\text{etc.}\end{aligned}$$

Mitchell Rappold and Faulkner (1983) discuss the use of the constant Poisson model in the context of aircraft supplies to several air force bases. An underlying assumption behind this model is that the compounding is fixed at the mean rate of the compounding events. We see no particular merits of this model, other than simplification, and Mitchell et al (op cit) themselves then go on to show in the same paper that the Stuttering Poisson distribution (that we discuss below) provided better overall results.

In those situations where the variance of the quantity of compounding events is potentially large, then the Log Series distribution and the Geometric distribution seem to offer significant utility as the compounding model of demand events. We now consider the two particular overall compound distributions that arise from these situations, because of their value in the theoretical development in this research and particularly where demand is lumpy. If the number of events occurring at each Poisson occurrence are distributed as Fisher's Log Series distribution then the unconditional distribution of events occurring in a unit time period is the Negative Binomial distribution (NBD) and the process has become known as the Afwedson process in the literature after its originator, Afwedson (1955). This process has received attention from many authors amongst the most prominent being Philipson (1957), Thyrion (1960) and Haight (op. cit.). The compounded version of the

NBD gives rise to the distribution in an integer form as shown :

$$P(x) = \frac{(k+x-1)!\ p^x}{(k-1)!x!\ q^{k+x}} \qquad \text{------------------------} \quad 5.1$$

where $p = q$-1 and both p , $k > 0$ for $x = 0, 1, 2, 3,$ etc.

When k is not integral the factorials in the above formula become gamma functions to give a 'gamma' version of the distribution as will be seen in a later section when we consider a 'mixing' approach to the development of the NBD model.

If the number of events at each Poisson occasion are distributed as a Geometric distribution then the unconditional distribution of the number of events occurring in a unit time period is the Stuttering Poisson distribution. This statistical distribution has been used by a number of authors in the context of inventory theory and it seems particularly relevant in cases of very lumpy or spasmodic demand processes. Ward (1978) characterises the Stuttering Poisson distribution in the following form :

$$R_n = \frac{(1-\rho)\lambda t}{n} \sum_{j=1}^{j=m} j\rho^{j-1} R_{n-j} \qquad \text{--------------------} 5.2$$

where R_n is the probability that 'n' units will be demanded in a time interval of length t with Poisson rate parameter λ and distribution parameter ρ.

Both the NBD and the Stuttering Poisson distributions have found application in inventory theory and in particular Bagchi's recent paper (1987) discusses both distributions in the context of lead time demand models for slow moving independent demand inventory items.

5:2 (b) *Poisson compounding and lumpy demand processes*

Ward (op. cit.) discussed the common observation in spares and supplies type inventories that individual item demand patterns are often 'lumpy' and very erratic. From discussions in Ward's paper, and from this author's own observations of spare parts inventories, periods of high demand are often followed by periods of low or zero demand, and sometimes periods of low demand are often punctuated by random demand spikes - thus giving lumpy demand behaviour. This can be especially true of slower moving items, although for very slow moving parts compounding is not likely to be a phenomena observed to any marked degree. Demand variation that is skewed need not necessarily be very 'lumpy', but lumpiness in demand always leads to a highly skewed distribution. It would also seem that compounding of demands by highly skewed compounding models, such as the LSD, will impart 'lumpiness' to the process.

The inventory control problems and the associated analytical solutions for very lumpy demand patterns under a variety of circumstances have been discussed by a number of authors; in particular Galliher, Morse and Simmond (1959), Mitchell (1962), Feeney and Sherbrook (1966), Sherbrook (1968), Adelson (1966), Gallagher (1969), Silver (1970 and 71), Silver, Ho and Deemer(1971), Ward (1978) and more recently Mitchell et al. (1983) and Bagchi (1983 and 1987). A general view shared by most of these authors is that whilst in many independent demand inventory situations the demands can be regarded as derived from a Poisson process, in the case of very lumpy demands the simple Poisson process, and the Poisson distribution do not provide an adequate description of the processes observed. These authors are in general agreement that the variation of demand and the coefficient of variation are often much greater than that predicted from a simple Poisson process. In these cases it is appropriate compound Poisson distributions that are put forward as the most promising models. In particular the Stuttering Poisson distribution is a favoured model to fit and explain the demand character frequently observed. Van Hees and Monhemius (1972) discuss a relationship between spare parts demand and the variance of demand. From limited data they show for very low demand volumes the simple Poisson model is adequate, but for progressively higher demand volumes the variance increases markedly away from the mean, with increasing values of the variance to mean ratio.The valid assumption of simple Poisson demand for slow moving spare parts is also behind the work of Mitchell (1962) at the National Coal Board. Feeney and Sherbrook (op. cit.) describe how a Poisson compound process can be applied to aircraft spare parts inventory systems and say :

> *" One of the reasons we are interested in generalising the assumptions of Poisson demand to distributions with a larger variance is that our demand data usually produces variances that exceed the mean. Furthermore the physical model of customers who can order several units appears to be a reasonable description of many supply processes".*

In the simple Poisson process the mean demand is equal to the variance of demand, whereas in most lumpy demand situations the variance is much greater than the mean even though the demand process is Poisson in character in other respects. Feeney and Sherbrooke (op. cit.) go on to give four possible explanations for the high variance of demand observed in the demand for aircraft spares:

(1) *Sympathetic replacement or undetected malfunctions - a maintenance man discovers a defective item on one aircraft and, as a result, inspects that item on other aircraft, replacing incipient failures.*

(2) *Initial wear-out - some components like vacuum tubes may have a high probability of failure shortly after installation.*

(3) *Damage during installation - maintenance personnel may damage some parts like windshields during installation.*

(4) *Flying programmes are usually correlated between aircraft.*

Therefore although the demand for spare parts can arise by a process that is Poisson in character the random failure of one part can cause the sympathetic replacement (and hence demand) of other parts of the same type in the same type or sometimes other but similar equipment, or cause the replacement of all parts of the same type in the same equipment. The result is to effectively compound the Poisson demand process. This author has noted, from commercial vehicle spares demand processes, the occurrence of a 'cluster demand' effect with certain parts and this gives rise to compounding of demands. For example, an engine is stripped down to replace a burnt out exhaust valve and whilst the equipment is stripped the opportunity is taken to replace all the exhaust, and inlet valves, which are showing signs of wear although technically speaking not failed parts at the time. This phenomena has also been referred to by Van Hees and Monhemius (1972) page 287. It is also very likely to add to the observed 'lumpiness' of spares parts demand seen in many systems.

5.2 (c) the Stuttering Poisson distribution.

As discussed earlier when the distribution of Poisson events at each Poisson occasion is geometric the counting distribution of item demands in a unit time period can be described by the Stuttering Poisson distribution. It is a two parameter distribution with an order arrival rate of λ orders per unit time. The average numbers of orders arriving in the time interval Δt is $\lambda \Delta t$. Each order is for one or more units, the actual amount being given by the geometric distribution with the second parameter p.

Ward (1978) gives the following properties of a stuttering Poisson distribution of demand over an interval of time of length Δt.

(1) mean $\mu = \lambda \Delta t / (1 - p)$

(2) variance $(v) = \lambda \Delta t (1 + p) / (1 - p)^2$

(3) coefficient of variation $(c) = [(1 - p) / \lambda \Delta]^{1/2}$

(4) ratio of variance to mean $c^2/m = (1 + p)/(1 - p)$

Where $\lambda > 0$ and $0 < p < 1$

The average order size is $1/(1-p)$ units and the probability R_n that n units will be demanded in the time interval Δt is given by equation 5.2 as previously shown -

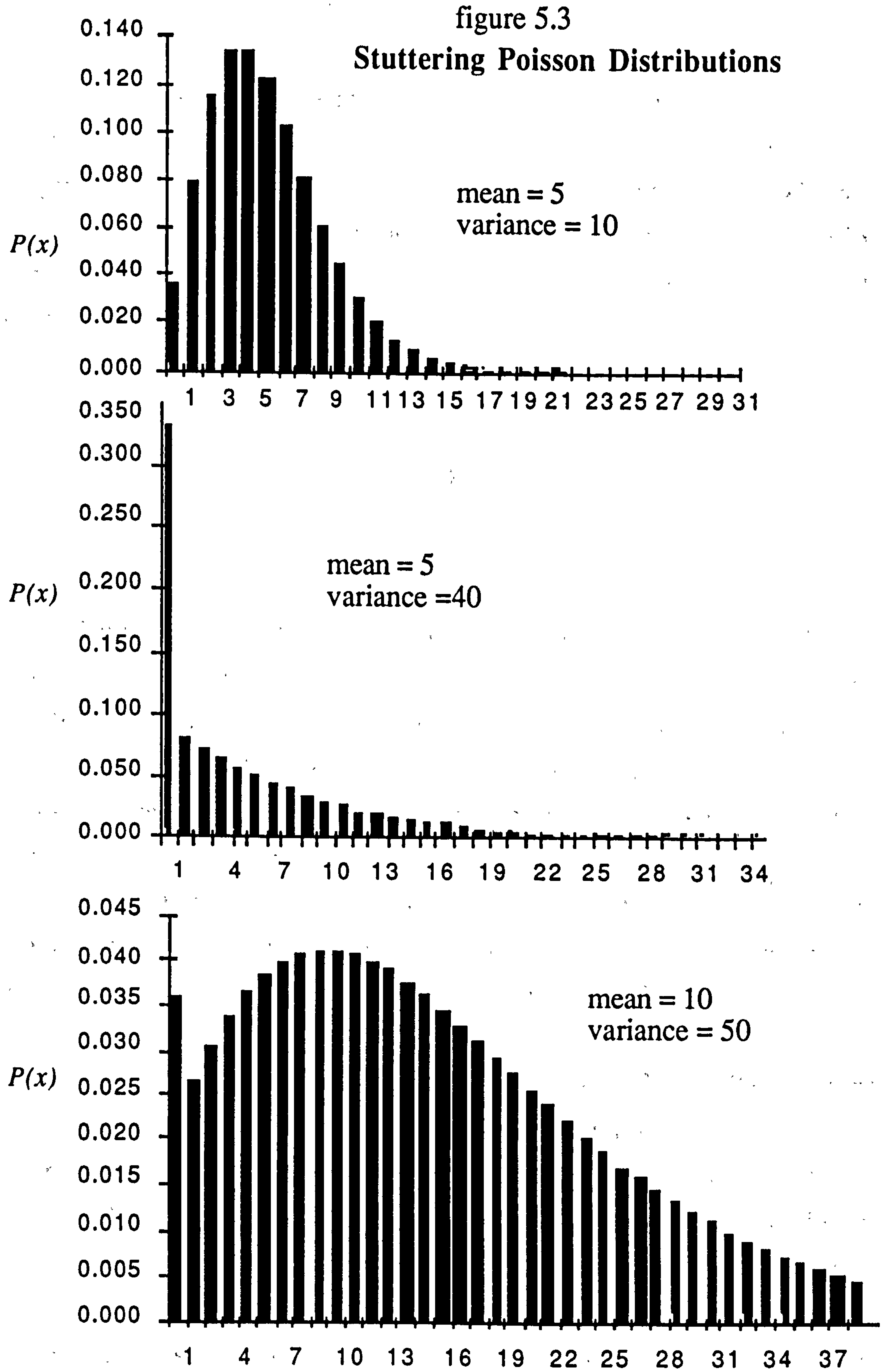
$$R_n = \frac{(1-\rho)\lambda t}{n} \sum_{j=1}^{j=m} j\rho^{j-1} R_{n-j}$$

This is a recurrence formula for generating such distributions and is given by Ward (op cit) after the original work of Adelson (1966). Given a time series of demand history accumulated over a sequence of equal time intervals of length T the parameters of the Stuttering Poisson distribution can be estimated from the mean (μ) and variance (v) of the observations as demonstrated by Ward (op cit). In the same paper Ward showed how reorder points can be determined in the lumpy demand case using the Stuttering Poisson distribution and he demonstrated improved results using this distribution over the normal distribution with empirical data.

Clearly both the Negative Binomial distribution, via the Afwedson process, and the Stuttering Poisson distribution are examples of compound distribution that can be used to model the purchase quantities of 'lumpy' demand items, although the Stuttering Poisson appears to be the more favoured distribution in operational situations. The difference between the two models is only through the form of the compounding model used ie Geometric and Log series. Given that the LSD has a greater degree of variance than a corresponding Geometric then it could be argued that the NBD might be a better model in cases of extreme lumpiness in demand. The degree to which one or the other fits empirical data is however, only one of the criteria that should be considered, additional theoretical support should be found to indicate the underlying process at work. In a detailed comparison of these two models Sherbrooke (1968) attempted to find conditions where the two models produce identical results. He concluded that it depends on the application under consideration, but by using a variance to mean ratio 'q' he found that in one application when 'q' was < 3 the two models give almost identical results, but gradually diverge when 'q' is > 3. In chapter ten we show some operational data, from a car spares system, where the NBD and the Stuttering Poisson give very close results; and in appendix one we show results which indicate the closeness of the Stuttering Poisson and the NBD to model DAF period demands for selected spare parts. The Stuttering Poisson does have one unusual property discussed below that could mark it out as a favoured compounding distribution is particular circumstances.

The figure 5.3 below shows three example Stuttering Poisson distributions produced from this author's own computer developed model for generating probability density functions from such distributions. What can be clearly seen are the variety of shapes the distribution is capable of assuming including the somewhat unusual property (for theoretical probability models) of being bimodel. This phenomena was also show by Sherbrooke (1968) in his limited tabulations. Although it was only a side issue in our work we did in fact cover a greater range of values sP variance to mean values than Sherbrooke.

figure 5.3
Stuttering Poisson Distributions



5.2 (d) Poisson mixing

Whereas a discrete magnitude is associated with each point in Poisson compounding, in a Poisson mixing process a continuous magnitude is associated with the Poisson rate parameter (λ) which is statistically distributed over the range $0 \rightarrow \infty$ with a distribution function $u(x)$ [after Haight's terminology (1967).] The resulting mixed Poisson distribution π_x from Haight has the general form :-

$$\pi_x = \int_0^\infty p_x(\lambda)dU(\lambda) \qquad x = 0,\ 1,\ 2,\ \text{etc} \qquad \text{--------5.3}$$

Haight refers to the distribution $U(\lambda)$ as the mixing distribution [i.e. it is the distribution of the Poisson rate parameter (λ) and the resulting distribution π_x is called the mixed distribution]. Haight (1967) reviews a number of examples of Poisson mixing, but only the so called 'Polya Process' is considered here because of its extensive application reported in the literature on consumer buying theory and practice. The other examples of mixed distributions given by Haight seem to be of more theoretical interest as far as can be judged from their lack of reported use in the operational inventory and consumer purchase literatures.

In the Polya process the mixing distribution $U(\lambda)$ is a Pearson type III (or Gamma) distribution and the resulting unconditional mixed distribution π_x is the Negative Binomial distribution (NBD) as previously given, but obtained (in theory at least) by a different mechanism. Because of the way the NBD is obtained by this particular process i.e. gamma mixing of a Poisson process, it is extensively referred to, in the consumer behaviour literature in particular, as the 'Poisson Gamma model' (see Ehrenberg 1959 and 1972, in particular). In its continuous ('gamma' function) form the NBD has the following probability density function -

$$P(r) = (1+\alpha)^{-k} \bullet \frac{\Gamma(k+r)}{\Gamma(k+r)\Gamma(k)} \bullet \frac{\alpha^r}{(1+\alpha)} \qquad \text{----------5.4}$$

For the probability of 'r ' occurrences in a unit time period with exponent 'k ' and mean αk. $\Gamma(x)$ is the gamma function of x.

The NBD is a positively skewed distribution with variance is given by m (1+m /k), where m /k is sometimes replaced by α. The best estimate of the mean 'm' is simply the observed mean from sample data: this is the unbiased maximum likelihood estimator. The exponent 'k ' can be determined by equating the observed variance to m (1+m /k), although according to Ehrenberg (1972, p.58) this can give large errors.

5.2 (e) concurrent Poisson compounding and mixing

Conceptually it can be reasoned that there could be many demand situations where different consumers for a particular item will exhibit different demand rates (i.e. variable rate parameter) and also compound their purchase quantities giving rise to a Poisson process comprising both compounded and mixed Poisson events. However, no references to such a process has been observed in any of the literature examined. As far as the management literature is concerned the general position appears to be that those authors concerned with consumer purchase theory discuss and use Poisson mixing models particularly the Poisson Gamma, whereas those authors coming more from an operational inventory position tend to favour the use of compounding models. There are a few exceptions to this however, as will be discussed later in this chapter. The only models that begins to approach concurrent compounding and mixing are those reported by Bagchi(1987) and Nahmias and Demmy (1982). In both cases the authors consider what is effectively compounding and mixing in a sequential fashion to develop novel models of lead time demand with, in each case, the lead time variable providing the mixing equation.

From an analytical point of view Poisson mixing and compounding could be expected to coexist in many inventory systems, particularly those of a spare parts variety. There is the question however of the form of the distribution so obtained. If we consider an underlying Poisson process with gamma mixing we obtain an NBD (in its gamma form). What then if we consider the possibility of compounding occurring at each NBD occasion. Do we still have an overall NBD, but with increased variance, or is a new distribution obtained ? Conversely if we obtain the NBD by compounding and then admit the possibility of the gamma mixing of the Poisson rate parameter do we still retain the NBD as the overall distribution ? This is an issue we return to at a later stage in this work.

5.2 (f) non Poisson compounding

It can be envisaged that in some systems the interevent distribution of the time between demands might be more regular than those described by the exponential distribution. In particular the Erlang distribution has been suggested by Jewell (1960) and Galliher, Morse and Simmond (1959) for use in situations where the interevent distribution is more regular than the negative exponential. Such a distribution might be appropriate to situations where there is a degree of regularity in ordering, as for example between retailer and distributor, or distributor and a factory level, due to a regular inventory restocking effect.

The Erlang distribution of order 'r ' is of the form-

$$f(x) = \frac{x^{r-1} e^{-x/\mu}}{\mu^r \Gamma(r)} \qquad \text{------------------5.5}$$

for a process with rate μ where r is a positive integer and specifies the order of the distribution. When $r = 1$ the distribution reduces to the exponential distribution.

If the interevent distribution follows such a distribution with r greater than unity, say 2, 3, etc., then, without compounding, the process would yield a demand stream with a variance less than the mean. If however, compounding is present, then the variance will increase and could give rise to a situation where the variance exceeds the mean. Hence it is possible to envisage a process where demand ordering is more regular than that described by an negative exponential interevent distribution, but where the variance is still greater than the mean as a result of significant compounding of demand. Indeed Jewell discusses such a possibility and makes reference to a 'Stuttering Erlang Process', but he provides no analytical derivation. Such a process would however, not be a Poisson in form and although theoretically interesting has not been applied to our work here.

5.3 The Poisson Gamma Model

We now turn our attention to a more detailed consideration of this important stochastic model. The underlying assumption for the Poisson Gamma model is that events occur as random draws from a Poisson process with mean rate μ. In the simple process where μ is constant the occurrence of events is given by the simple Poisson distribution. If the model is extended, as will be shown, to assume that this Poisson rate parameter μ is itself a variable and takes the form of a Gamma distribution then, through a Poisson mixing process, the unconditional distribution of Poisson events that is obtained is the Negative Binomial distribution in its gamma form as shown previously by equation 5.3 :

$$P(r) = (1+\alpha)^{-k} \bullet \frac{\Gamma(k+r)}{\Gamma(k+r)\Gamma(k)} \bullet \frac{\alpha^r}{(1+\alpha)}$$

As we indicated earlier this Poisson Gamma model has been used extensively in the field of buyer behaviour research and has been discussed at length by Massey, Montgomery and Morrison (1970), Ehrenberg (1972) and to a lesser extent by Graham (1974) and Easton (1980). Goodhardt, et al (1984) provided an extensive list of references which report the use of the NBD model, and variants mainly the Dirchlet model, in the consumer purchase research. The references cited by both Massey and Ehrenberg are extensive and demonstrate the wealth of modelling work that has been reported following the pioneering work in this field by Ehrenberg (1959). According to Massey (op.cit.) Ehrenberg is credited as being the first author to develop an explicit model of heterogeneous buyer behaviour, which has since become known as the Poisson-Gamma model. It is strictly speaking just a negative binomial distribution, but referring to it as the Poisson Gamma model gives us the process by which is has been derived. It should be distinguished of course from the Afwedson process whereby the Negative Binomial distribution is obtained by a compounding process of Poisson events using the Log Series distribution.

The attainment of the NBD model by a mixing process is best developed and most easily appreciated by reference to consumer purchase theory based on the work of Ehrenberg. The starting assumptions were:-

(1) *Purchases of a given consumer at successive points in time can be regarded as independent drawings from a Poisson distribution with mean rate λ , i.e. -*

$$P(r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad \text{-------------------------5.6}$$

for $r = 0, 1, 2,$ et

(2) *The average rate of purchasing of different consumers in long run differ, their distribution being gamma with exponent k and mean m and the consumer has a mean purchasing rate λ in the long run, hence:*

$$u(\lambda) = \frac{e^{-\lambda/\alpha}\lambda^{k-1}}{\alpha^k \Gamma(k)} \qquad \text{-------------------5.7}$$

where $\alpha = m/_k$ and $\Gamma(k)$ is the gamma function of k

The function u(λ) represents the distribution of λ over many consumers. Thus applying equation 5.7 as the mixing distribution on the process equation 5.6 we can derive the unconditional mixed Poisson distribution π_x of equation 5.2, or $P(r)$ for the probability of r purchases-

$$\pi_x = \int_0^\infty \frac{e^{-\lambda/k}\lambda^{k-1}}{\alpha^k \Gamma(k)} \bullet \frac{e^{-\lambda}\lambda^r}{r!} d\lambda \qquad \text{--------------5.8}$$

This can be integrated and then simplified to give the unconditional Negative Binomial distribution previously shown by equation 5.3.

$$P(r) = (1+\alpha)^{-k} \bullet \frac{\Gamma(k+r)}{\Gamma(k+r)\Gamma(k)} \bullet \frac{\alpha^r}{(1+\alpha)}$$

Zacks (1969, p.153) in a largely theoretical paper on a Bayesian approach to setting stock levels gives a detailed theoretical proof that this mixing process leads to a Negative Binomial Distribution.

The NBD model has also been developed and depicted in a useful graphical way by Easton (1975) and is shown in tabular form as follows-

Figure 5.4

Poisson Gamma Model

Consumer	Periods 1	2	3	4	5	n	Long run average	Distribution horizontally
A	x_1^A	x_2^A	x_3^A	---	-----	x_n^A	λ_A	Poisson
B	x_1^B	x_2^B		----	-----	x_n^B	λ_B	Poisson
C	x_1^C	----	x_3^C	-	-----	x_n^C	λ_C	Poisson
D	x_1^D		---	----	----	----	⋮	
E	⋮					⋮	⋮	
F	⋮					⋮	⋮	
Mean	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$			$\bar{\bar{x}}_n$	
vertical distrib'n	NBD	NBD	NBD	NBD	NBD		Gamma	

We use this form of the Poisson Gamma model in chapter seven to assist with our approach to aggregate model building and we return to it at that stage.

5.4 Application of the Poisson Gamma Model

The fact that Poisson events with Gamma mixing lead to an unconditional Negative Binomial Distribution is now well known in several fields. In accident statistics several authors have used it to model accident proneness or have referred to its use in that context, Arbous and Kerrick (1951), Bates and Neyman (1952), Edwards and Gurland (1961) and Bhattacharya and Holla (1965). Also it has had reported use in ecology by Quenouille (1949) and in O.R. (See Ehrenberg (1972) and Grahn (1969, p.73).

However, the most widely used and reported use of this model and derivations from it is in the field of consumer buyer behaviour. In particular, Andrew Ehrenberg and Gerald Goodhardt between 1959 and 1972 published nearly fifty papers in the field of consumer purchasing many of which refer to the methodology and results of applying the Poisson-Gamma model, and derivations, to various consumer purchases. Ehrenberg (1972) gives a comprehensive listing of these references and many more by other authors. The text by Massey et al, (1970) also contains an extensive reference listing to publications in the consumer purchase field. The Poisson Gamma model as applied in consumer purchase research can clearly be regard as a major milestone in both the theoretical and applied developments in this field of enquiry. It has however, also found application in related fields which are relevant to the research by this author.

As previously mentioned Easton (op cit) has discussed the model in his work on patterns of industrial buying. Haber and Sitgreaves (1970) have developed the same model using a Bayesian approach and then used it to estimate usage rates for submarine spares in single periods. They assumed that the variation in Poisson demand rates for given parts from different sources could be regarded as Gamma distributed. Hollier (1980) has also indicated the validity of the Poisson Gamma model in the context of spare parts in a paper devoted to the problem of estimating the all time requirement of vehicle spare parts. In both the Haber and Sitgreaves, and Hollier papers the development and use of the model was in terms of the expected frequency of purchase incidence or purchase occasion for a single item type in a single period. In both cases although the concern was operational inventory issues the model was formulated in terms of mixing with the Poisson rate parameter considered to be gamma distributed. It must also be noted that in the application of the model to consumer buying theory, the model is developed explicitly for and interpreted in terms of single product types over a range of consumers during single periods. Hence the Poisson Gamma model can be classified as a single product single period model. Furthermore, as has been discussed by Ehrenberg (1959 & 1972) amongst others, it is a model of stationery purchase behaviour. That is to say no sales trend should exist for the period being analysed. Hence the analysis period must be kept short, or alternatively, successive analysis periods should be treated independently and the assumption of stationarity should be applied to each separately. Massey (1970) points out that there is a further methodological problem in consumer purchase theory relating to the analysis period. If the period is too short then purchase event feedback is an important consideration. That is purchases made in one time period influence the decision whether to buy or not in the next period. However, this is not a problem in the context of the work here because when a spare part fails in service it must, in general, be replaced irrespective of whether the failure occurred after three hours or three months. This assumes of course an operational policy that requires the equipment concerned be kept in a fully operational condition.

Taylor (1961) has shown how the NBD distribution can be applied to the problem of setting stock levels for individual aircraft spares at British Airways. He developed the model by first assuming and subsequently verifying for his data that demand for spares was simple Poisson and that the resupply lead time could be considered to be distributed as a Gamma distribution. He then went on to formulate a NBD model for the resupply time demand with variable lead times by a mixing process, although he does not discuss mixing as such. This is an interesting and valuable development because this approach clearly leads to a model of purchase (or demand) quantity , whereas Ehrenberg's development of the NBD model was in terms of purchase occasions independent of quantity. However, in the case of Ehrenberg's consumer purchase studies the work was in the context of such consumer products as toothpaste, detergents, and breakfast cereals and similar fast moving consumer products and fixed intervals of time. The problem of considering a unit quantity of purchase in such merchandise areas is clearly evident with the multiplicity of pack sizes available. In the case of spare parts however, this problem does not exist to anything like the same degree. A spare part as defined by most, if not all, original equipment manufacturers (OEM's) is a discrete item of unit quantity and coded and catalogued as such. Some consumable spare items may well be carded in units of 10 or a dozen for example, but they are generally very small in number compared to the size of most spare parts ranges. Other spare parts such as engines, axles etc. will be aggregates of simpler spare units but again they will be coded and catalogued accordingly. It can be argued that for many spare part demands at the retail level, purchase occasion is very likely to equate with a purchase quantity of unity. At the distributor level restocking decisions from the dealers will create multiple order quantities for many items and hence significantly compound demand for those items as a result.

Taylor's formulation of the NBD may then be considered in a way which is consistent with Ehrenberg's approach. By considering the variable resupply lead time to be Gamma distributed, the Poisson mean rate of demand will also be a variable quantity proportional to the lead time variation and it too will then be Gamma distributed. This is effectively equivalent to the Poisson mixing and we can regard Taylor's model as a Poisson gamma model of quantity. Interestingly Taylor makes no reference to demand models in fixed time intervals nor does he discuss the possibility of compounding, which, from evidence discussed earlier in this work, one would expect it to occur in the systems studied by Taylor.

5.5 Modifications to the NBD Model

A number of modifications to the basic Poisson Gamma model have been proposed by various authors. In particular, Herniter (1971), Chatfield and Goodhardt (1973), Jeuland et al(1980) and Morrison and Schmittlein (1981) have all considered situations where purchases are more regular than that assumed by a negative exponential interpurchase distribution. In these cases, they considered the Erlang family of distributions of order two or more to model interpurchase times and this resulted in what has become known as a 'Condensed Poisson Gamma' model. However, according to Chatfield and Goodhardt (op cit) the results of such formulations and use on consumer purchase data only gave marginal improvements at best over the basic NBD model, and Dunn (1983) has concluded that the NBD must be considered as a very robust model of purchase incidence. Lawrence (1980) has suggested the use of the lognormal distribution as an alternative mixing distribution to the gamma distribution of consumer purchase rates and claims success on his chosen data sets (toothpaste purchases) using this approach. More recently Sichell (1982) has developed a much more flexible version of the Poisson Gamma model which he named the Inverse Gaussion Poisson model (IPG). In this approach demand at the level of the single buyer is assumed to be Poisson, but Sichell uses the Inverse Gaussian distribution as the mixing distribution of λ, the Poisson rate parameter, on the Poisson process. This results in a fearsome looking distribution of the form :

$$f(x) = \left[\frac{2(1-\theta)^{1/2} / \alpha\theta}{2K_\gamma\left[\alpha(1-\theta)^{1/2}\right]} \right] \lambda^{\gamma-1} e^{-[(1/\theta-1)-1]\lambda-(\alpha^2\theta/4\gamma)}$$

Where $-\infty < \gamma < \infty$
and $0 \le \theta \le 1$
and $\alpha \ge \theta$

$K\gamma\,(Z)$ is a modified Bessel function of the second kind of order γ and argument Z. The flexibility of this distribution is seen by the variety of forms it can take. If $\gamma > 0$ and if we then let $\alpha \to \infty$ then it becomes the Gamma (Pearson type III) distribution. If $\gamma < 0$ and we let $\alpha \to 0$ then it becomes a Pearson type V distribution.

In the same paper Sichell then demonstrates an improved fit of this model, to several consumer products groups, over the Negative Binomial model, and according to Sichell's work it seems to offer substantial advantages due to the greater flexibility of the Inverse Gaussian distribution over the gamma distribution. However, the model has not, so far, been reported widely in the literature.

Other modifications to the basic Poisson Gamma model have been proposed by various authors, such as Goodhardt's Beta Binomial model to account for repeat purchase in multiple periods, (see Ehrenberg,1972). More recently the generalised Dirichlet model has found favour with consumer purchase researchers to take into account broader purchasing conditions such as consumer brand choice and store choice. In particular Goodhardt, Ehrenberg and Chatfield (1984), showed that the NBD is just a special case of the Dirichlet model. And Morrison, Schmittlein and Colombo (1987) have extended the basic NBD model via the Dirichlet to non stationary conditions. However, neither the Beta Binomial or

Dirichlet models have any direct application to the work in this thesis.

5.6 Support for Poisson and Gamma Processes

In this section we will now consider what evidence exists, both theoretical and empirical, which supports and justifies the use of Poisson process in the development of both compound distributions and Poisson Gamma type models. We also examine the Gamma assumption behind the Poisson mixing processes.

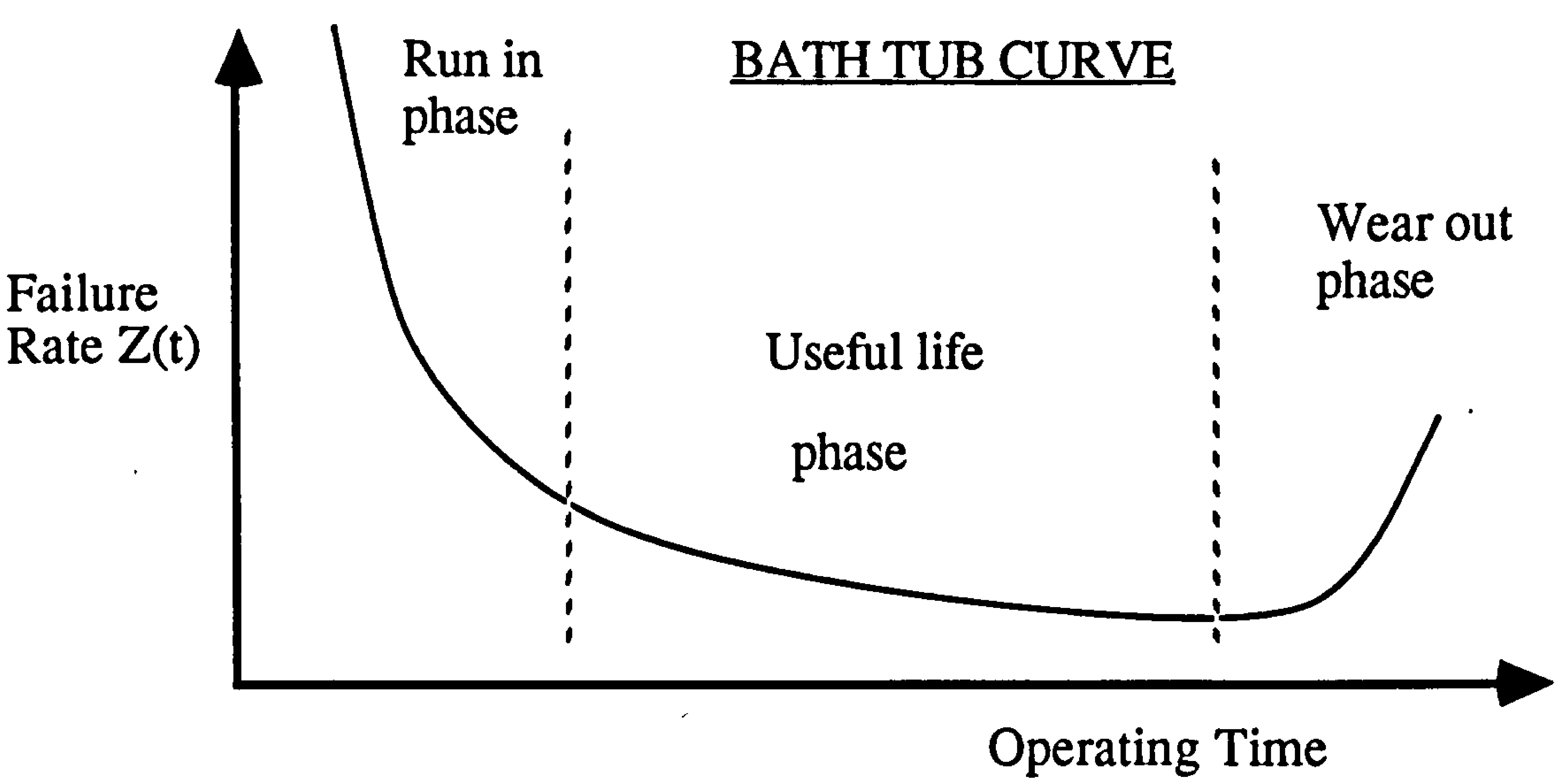
The Poisson Gamma model, and all Poisson compound models, start with the basic assumption that demands at the consumer level are Poisson generated. In the extensive literature on the application of Poisson - Gamma models, especially in the consumer purchase field, the argument for a Poisson process is on the basis that consumer demands are independent and that the customer base is generally large. This is providing, as pointed out by Ehrenberg (1972), that the analysis period is not so short that purchase event feedback becomes a problem. The assumption of independence seems somewhat shallow in the context of consumer purchasing, but the success with which the Poisson - Gamma model has been applied to consumer purchase field would seem to justify it. In the context of this research the underlying assumption that demands generated for spare parts items are Poisson can be justified and supported by a considerable amount of evidence.

5.6 (a) from failure models of complex systems

The literature on reliability theory, failure processes and renewal theory is extensive and a review here would not be appropriate. However, some consideration to basic theory is justified together with a consideration of some of the papers that refer to spares inventory issues.

It is generally accepted by a number of authors who have published in the field of reliability and maintenance, and also by practising Operations Managers, that the classical 'bath tub' curve of figure 5.5 represents a reasonable model of the failure rate observed in many complex equipment situations, for example Trusscott (1970), Caplen (1972), Bestwick and Lockyer (1982), and Nahmias (1989) amongst others.

Figure 5.5



This diagram shows the now well known three phases of operating life for complex systems. The 'run in phase' is characterised by a high, but rapidly reducing failure rate due to run-in and early random failures. The 'useful life' phase is characterised by a low and near constant random failure rate; and the 'wear out' phase is characterised by an increasing failure rate due to wear out and age deterioration and random failures. For complex equipment, which may be assembled from hundreds, or even thousands, of parts and components, the bath tub curve represents the aggregated failure pattern of all components and parts. Individual component types may exhibit very complex failure patterns, but when considered 'in situ' with other component parts the aggregated picture is often not complex due to levelling or smoothing effects.

The useful life phase of the bath tub curve is characterised by a constant failure rate and this condition is applicable to complex systems over quite long periods of operating life, (provided they are repaired on failure and are regularly maintained), Bazovsky (1961). Under these circumstances the exponential failure rate equation can be shown to apply, and the failure density is given by the function :

$$f(t) = \lambda e^{-\lambda t}$$

where λ = the constant failure rate, and t = the operating period.

From this equation the reliability function R(t) is readily derived as given by Barlow (1967), Truscott (1970).

$$R(t) = e^{-\lambda t}$$

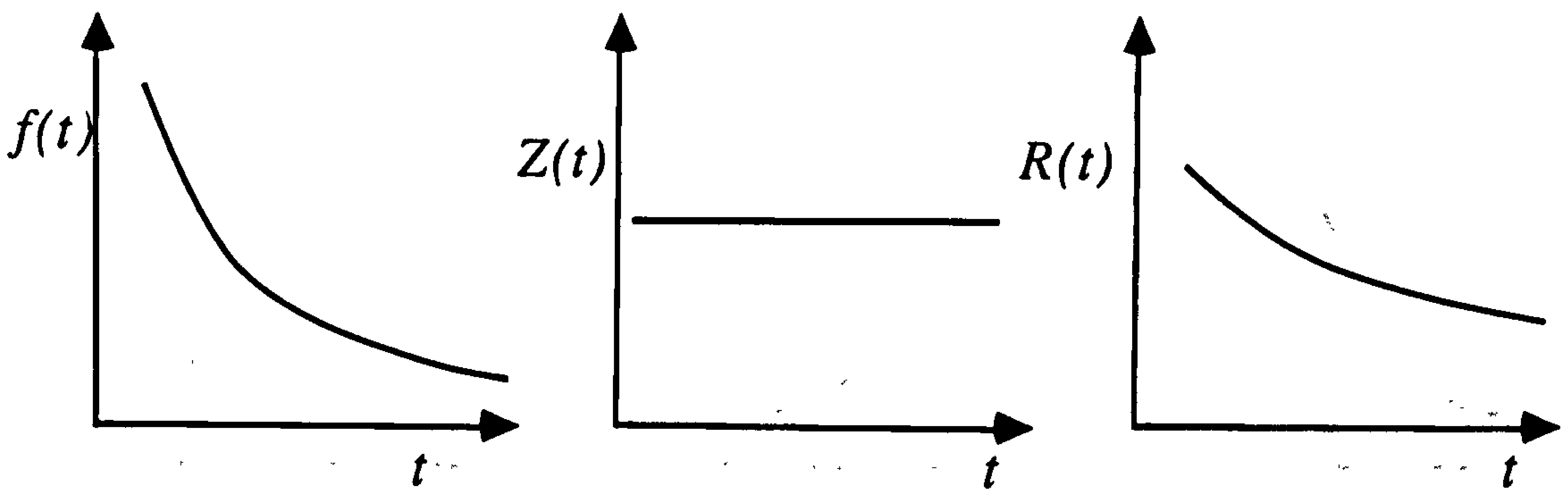
and the associated failure rate $Z(t)$ given by -

$$Z(t) = f(t)/R(t)$$

$$Z(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}}$$

The form of each of these expressions for the useful life phase are as shown:-

figure 5.6
Equipment Failure Modes



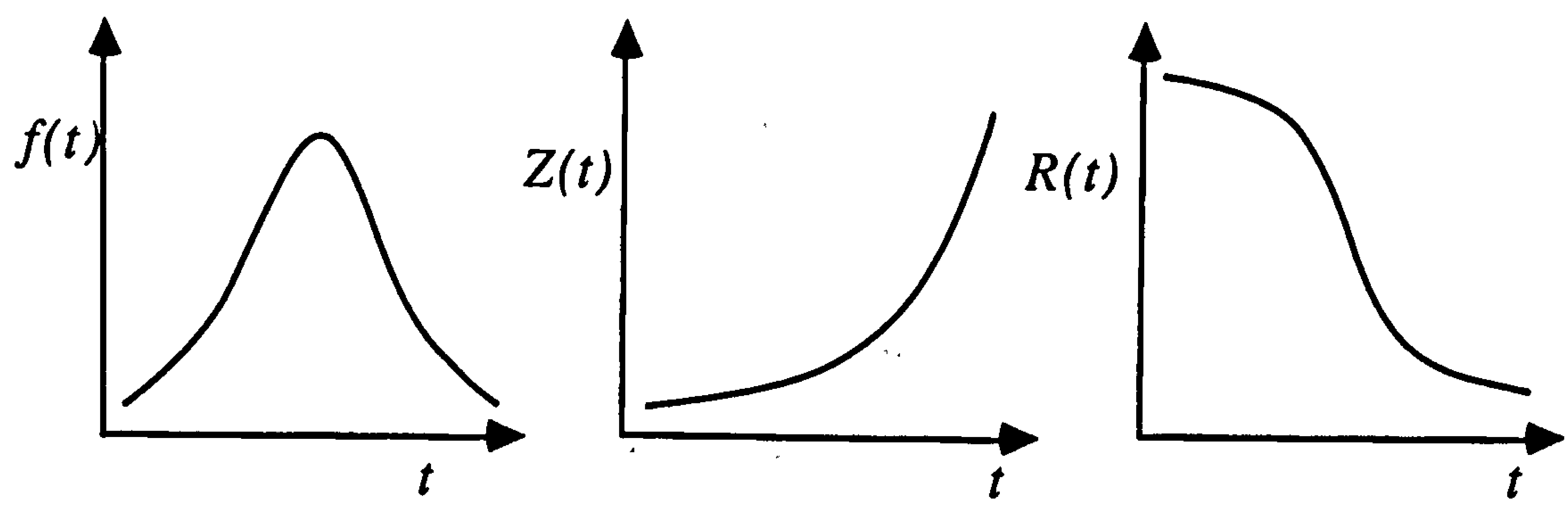
Hence, during the 'normal operating life' of equipment the 'negative exponential failure rate mode' is widely considered to be fully justified to model the time between failures and the prime condition of the simple Poisson process is satisfied. We can therefore expect replacement demands occasions in unit periods to be Poisson distributed. The actual amount demanded could be simple or compound Poisson, this will depend on the nature of the parts being replaced.

5.6 (b) wear out failure modes

Whilst the simple exponential failure equation adequately describes the useful life phase it is not an adequate description of the failure pattern in short intervals when complex equipment moves into its 'wear out' phase. In such regions of a 'life curve' the normal and sometimes lognormal distributions are used to model the failure density. The associated plots of $f(t)$, $Z(t)$ and $R(t)$ for the wear out phase are given in figure 5.7 (from Truscott 1970).

figure 5.7

Wear out failure modes

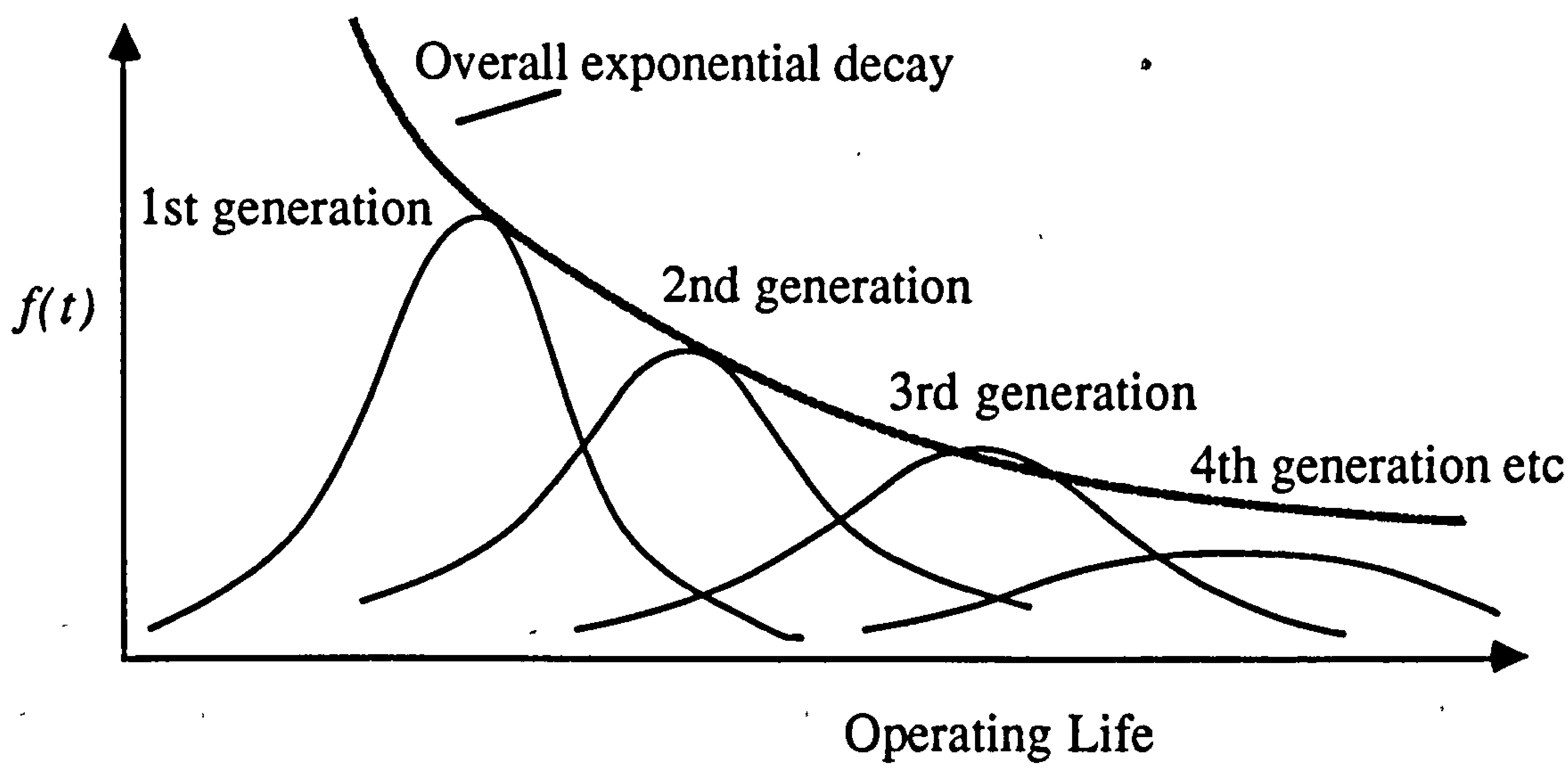


If however, we consider long intervals of time then, even in a wear out region, if parts are replaced on failure, in complex systems, there is a strong tendency for the frequency of failure to converge to an exponential form, Truscott (1970) and Barlow (1967). Barlow (page 19) in particular quotes:

> "*consider a system consisting of many components, each subject to an individual pattern of malfunction and replacement, and all parts making up the failure pattern of the equipment as a whole. Under reasonably general conditions the distribution of the time between equipment failures tends to be exponential as the complexity and the time of operation increases*".

Barlow then goes on to develop this analytically from first principles and shows the convergence to exponential failure frequency over a given period for a large number of components in the system. Truscott (1970) reaches the same result by a simple practical illustration shown as in figure 5.8 that follows:

figure 5.8
Exponential Failure Decay



It can be seen from the above diagram that because of the phased introduction of second, third etc., component generations the variance about the mean life of each generation increases markedly with each successive generation. Under these circumstances the overall exponential convergence can be readily seen.

Menzler (1953) writes

> *"A study of the statistics of bus equipment confirms that in normal circumstances the failures are to all intents and purposes independent occurrences".*

That is they satisfy one of the basic conditions of a Poisson process. Davies (see Kendrick 1960) presented a considerable amount of failure data including that from manufactured products such as vacuum tubes, bus engines and his data was compared with normal and exponential theories of failure and in his conclusions Davies writes:-

"The exponential theory of failure appears to describe most of the systems examined here. Those systems that exhibit reasonable agreement with this failure theory are characterised by a predominance of human errors as the cause, or a careful and well developed operating technique for minimising failures. Systems which are subject to a wide range of environmental conditions also appear to follow the exponential failure law ".

The general conclusion from the foregoing is that provided we are concerned with complex equipment comprising a large number of parts and components subject to replacement on failure then the exponential distribution is a satisfactory model of the failure frequency of all three phases of the life curve of the equipment. This also includes the 'run-in' phase because according to Truscott (1970), Caplen (1972) and Barlow (1967), although the failure frequency is generally regarded as hyper-exponential in this region it will quickly converge to the exponential form mainly because the period of 'run in' is often very short compared to the whole life of typical equipment. Hence the overall interevent distribution between one failure and the next can, in most failure modes for complex equipment, be regarded as exponential and the primary requirement for a Poisson process is satisfied throughout the major part of the complete operating life.

5.6 (c) support from inventory theory

The literature on inventory theory contains a large number of publications which refer to the demand processes of spare parts as being adequately modelled by a Poisson - exponential process. Pitt (1946) is accredited as being the first author to give an exact solution of a steady state probability of reorder levels for spare parts and demand was assumed to be a captive Poisson. Karush (1957), Garrett (1958), Takacs (1956) and Karlin and Scarf (1958) have published early papers and all have either, stated, or implied, that spare parts demand processes are Poisson in character.

Several authors already quoted earlier in this chapter have published inventory papers related to spare parts. Galliher, Morse and Simmond (1959) in a study concerned, in part, with military equipment spares, concluded that the demand generation process was Poisson, but the demand itself in unit time periods consisted of bunches or bursts of demand. They further concluded that whilst the incidence (or occasion) of bursts could be regarded as Poisson the quantity demanded per unit time was compounded as a Stuttering Poisson distribution. In a similar paper by Feeney and Sherbrooke (1966) on establishing reorder levels for (s-1, S) inventory policies of aircraft spare parts they found demand arrivals to be Poisson distributed, but the quantity demanded per period was described by a compound Poisson distribution.

In Taylor's (1961) paper, devoted to establishing the lead time demand for aircraft spare parts, he concluded that the demand processes he studied satisfied the requirements of a Poisson process. Additional papers by Sherbrooke (1966) and (1968), and Bartakke (1981) concluded that spare parts demand can be assumed Poisson in character. Ward (1978) in a study of an inventory of 30,000 spare parts (of unspecified type) and Haber & Sitgreaves (1970) in a study of 25,000 submarine spares both concluded that spares demand processes were Poisson in nature. In Hollier's (1980) paper concerning the all time requirement of vehicle spares he proposed that the Poisson process was the appropriate theory to use. More recently Mitchell et al. (1983), in a study which was also based on aircraft spares, it was found that from over 6,000 order arrivals (demand incidence) into air force resupply bases that the order incidence process was simple Poisson. Bagchi, et al, (1983) in a study of slow moving independent demand items concluded that Poisson unit demands are appropriate given that the interval between demands is exponential; and Federgruen (1984) et. al. used compound Poisson distributions as the appropriate models to represent the behaviour of independent demand incidence in multi-echelon inventory channels. In a more theoretical paper Bagchi (1987) concluded that the Poisson distribution is the appropriate model to represent demand occasion when demands are independent and completely random. Bagchi then used various models to represent the customer order distribution and the lead time. For example, his GPG model is Poisson order occasion (the order intensity), Gamma lead time and Geometric order size (the order rate).

This author confidently concludes from the weight of reported evidence in the literature that failure rates and the follow on demand rates for spare parts are Poisson in character for wide range of spares types and the unit time demands are distributed as either simple, compound Poisson or mixed Poisson, or even composites of each.

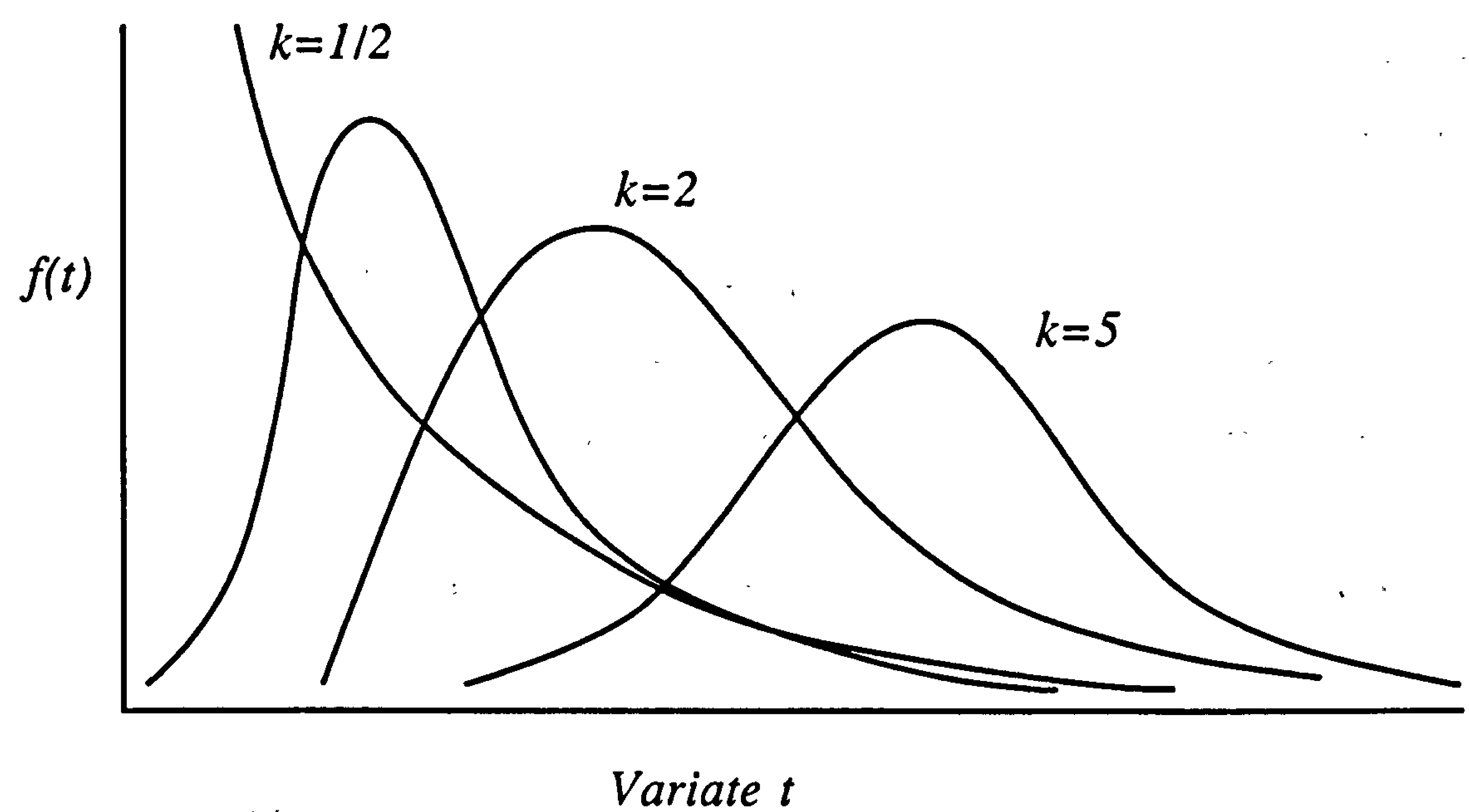
5.6 (d) support for the gamma assumption

The second major assumption of the NBD model is that the consumers long run average rate of purchasing is gamma distributed. Intuitively it can be reasoned that consumers of the same product will have different purchase propensities and it would not be surprising to find that the variation can be summarised by a statistical distribution as flexible as the gamma. Ehrenberg 1959 & 1972 amongst other publications has shown that the result predicted by the NBD model are very close to those actually found for a very wide range of consumer products. Despite attempts by Sichell (1982), Graham (1969) and Chatfield and Goodhardt (1973) to improve the general NBD model, the robustness of this distribution and its underlying assumptions hold up extremely well.

Burgin (1975) in a detailed discussion of the use of the gamma distribution in inventory control shows the great flexibility of this model for different values of the parameter k.

Figure 5.9

Various Gamma Distributions



Ehrenberg (1959) drew attention to the fact that it is the appropriate distribution to use having the correct general form and always positively skewed. Ehrenberg and Goodhardt (1979) discuss the form of the appropriate distribution and justify the use of the gamma distribution in purchase theory on the following grounds:

(1) *"different consumers buy the brand independently of buying each of the other brands in the market and if:-*

(2) *a consumers buying of the product is independent of how much of that consumers total product class purchases the brand accounts for then it can be proved*

> *mathematically that the difference consumer purchase probabilities are gamma distributed."*

A detailed development of the proof behind the above reasoning is given by Goodhardt and Chatfield (1973).

In the context of spare parts it can be reasoned from an intuitive position that users of equipment, such as commercial vehicles for which spare parts replacements are required from time to time, will demand identical spares at different rates. This will arise because of the different environments in which the parts are used. In turn, this occurs due to the many operational variables that have impact on the wear rate and ultimate failure of a part. For example, different rates of vehicle usage, different loads carried, different driver behaviour, service level policies ranging from strict maintenance scheduling to drive it to failure attitudes etc. These are all aspects that can significantly alter the life and hence demand rate of many identical parts from vehicle to vehicle and operator to operator.

In papers previously quoted, Hollier (1980) and Haber and Sitgreaves (1970) have supported the view that the gamma distribution is appropriate to use in the context of spare parts as a model of variable demand rates over a range of consumers. Zacks (1969) using a more theoretical approach to inventory - problems supported the use of the gamma as the appropriate distribution to use in the development of the NBD model.

From the foregoing discussion, and the wealth of work in the consumer purchase and accident statistics fields, the underlying assumptions of a Poisson process with gamma mixing are well founded and supported by a considerable body of evidence.

5.7 Conclusion

A review of the relevant literature has shown that for inventory items whose demand is derived independently from a large consumer population then the demand process is very likely Poisson. In the case of spare parts items evidence from failure processes and reliability theory give further support to an underlying Poisson process, but the demand variance is often greater than the mean implicating compound Poisson functions as demand models. It has been shown from the literature that both Poisson compounding, and mixing, increase the variance of the demand in unit periods of time, whilst the underlying Poisson process is still preserved. Several compound models have been considered that can be used in operational situations, but in the case of very lumpy spare parts demand systems the NBD and the Stuttering Poisson distributions have been proposed as models of the demand quantity per unit time. The NBD can also be derived by a mixing process, but as such it is strictly speaking a distribution model of purchase occasions or incidence. In those cases where no compounding of purchase size occurs then the NBD, derived by mixing, can then be regarded as a model of purchase quantity.

The development of the NBD by mixing processes is somewhat confusing. We have seen from various authors that it can be developed from a mixing process by two approaches depending on the intentions and assumptions made by the respective authors. In one case the Poisson rate parameter is regarded as a Gamma variate and we are given an NBD of purchases in fixed intervals of time. This has been essentially the consumer purchase approach. In the other case the Poisson rate parameter is assumed constant and Gamma mixing is introduced by considering a Gamma resupply variable. This has essentially been the inventory theory approach using the NBD to model lead time demands.

Whilst NBD models have been discussed in the context of single item types there is some empirical and theoretical evidence we consider in some detail in chapter seven that supports the view that similar processes may be applicable to certain classes of heterogeneous (or mixed) item populations. It is this finding that provides a possible link between single item demand and aggregate item demand. Based on the considerations of this chapter and those of chapter four we can now move to a more formalised statement of working hypotheses -

(i) Demand occurrences of individual spare part inventory items are Poisson distributed. The overall demand quantities, in fixed intervals of time, are compounded to varying degrees from spare part to spare part. This compounding leads to fixed interval demand models that are most likely NBD or Stuttering Poisson.

(ii) Where no compounding or mixing occurs then the simple Poisson model is sufficient to model fixed interval demand.

(iii) Concurrent mixing and compounding can be expected in the demand for some items and this will in all probability lead to an overall demand distribution with a variance value greater than that expected from either compounding or mixing alone.

(iv) In the long run aggregate inventory item usage rates are lognormally distributed as the stable long run equilibrium distribution. The convergence of usage rates to lognormality is governed by the Law of Proportionate Effect.

(v) Furthermore, as usage values are the product of item prices and item demand volumes, then these two factors are also lognormally distributed providing that the period over which the demand is measured is sufficiently long for the process to have converged.

Empirical Data Analysis

6.0 Introduction

This chapter contains most of the empirical data analysis relating to the DAF Trucks spare parts data, and therefore a brief outline of the company and the DAF spares environment is given as an introduction.

DAF Trucks (GB) limited, was until mid 1987, the wholly owned subsidiary of the parent Dutch company DAF BV Eindhoven. The UK company commenced sales and marketing operations in 1971 to sell DAF built trucks of gross vehicle weight 14 tons and over in the UK market. The truck range consisted of some 15 basic models, but considerable variation was possible around modular units, eg engine type, wheel configuration and rigid and tractor bodies. 46 trucks were sold in 1972 and by 1979 the annual sales of trucks had reached 2,110. After a deep industry recession in 1980 and 1981 DAF sales reached 2,680 truck units in 1985. To support the sales of trucks and to supply all necessary regional technical, marketing and sales support DAF developed a network of sole franchised regional dealers. These numbered some 22 in 1979 and by 1985 they had increased to 30. In some areas the regional dealer also routed spare parts to local service dealers, who maintained small stocks of fast moving consumable items.

The number of spare parts lines stocked in the UK rose steadily over the period 1975 to 1986, even though the Dutch parent company had a vigorous programme of variety reduction and standardisation. Inevitably in a changing technical market truck modifications and new models increased the number of catalogued spare parts. In 1975 the active parts range in the UK was around 7,000; by 1986 this had risen to some 12,000 items with over 25,000 part numbers listed. In Holland the parent company had a catalogue of over 60,000 part numbers. This large range of spares supported a number models not exported to the UK; these were principally light commercial vehicles.

6.1 Inventory Analysis

In this section we report the detailed analysis undertaken at DAF Trucks UK headquarters operation at Marlow, Buckinghamshire. As previously discussed in chapter one the initial analysis was focused on the aggregate properties of the entire parts inventory, and to see how far the lognormal distribution could be applied to this environment. We were anxious to validate, or refute, the claims of the mainly American studies which reported the validity of the lognormal distribution to model usage values of inventory items. Furthermore it was the intention to achieve a deeper understanding of the nature of the distribution in relation to spares inventory items, the meaning of the parameter values and ultimately to explain, if possible, by appropriate and testable theory the genesis of the lognormal distribution in this environment. The justification for this was to reach an understanding of the validity of the theory, to appreciate the range of applicability and to see what further insights into aggregate inventory properties could be achieved. As will be seen in the following sections the applicability of the lognormal distribution was proven valid and we show a deeper understanding of the nature of demand volumes and usage values. Also a better understanding of the stability of the processes at work is also seen. The analysis in this chapter, together with the literature work from the previous chapters, has provided a sound basis upon which the theory development of chapter seven has been based.

The more detailed analysis and results here are generally based on 1979, 1983 and 1985 as example years. The corresponding results for all years over the period 1975 to 1985 are presented in a summarised form. In this way it is hoped that a more effective and economic presentation is achieved. Even so the chapter is large due to the extensive analysis that was undertaken. 1979 was chosen because it was the first year that this author analysed in depth and it is one of the middle years of the entire span of DAF data analysed. 1985 was chosen because it was the last set of full year data to be analysed in depth. 1983 was a completely arbitrary choice. In the case of demand volumes for our three example years we examined the forms of the distribution of demand volumes from one period (four weeks) upwards to show how such distributions change shape and converge to an integer form of the lognormal distribution as the stable long run model. We also present an analysis of usage values for a sample of DAF regional dealers based on 1980 data. We were anxious to achieve a view of the processes at work at the dealer level and to examine their value and relationship to the distributor level in the system.

6.2 Analysis of Usage Values

In chapter one of this thesis we gave some preliminary results on the analysis of usage values by presenting the results for the whole year of 1979 from the DAF inventory at Marlow. We now reconsider this data set below in some detail to verify the lognormality of usage values. In section 6.5 we then consider the data for 1983 and 1985 for comparison. For the years 1975 to 1981 the analysis was based on computer generated ABC listings of annual parts sales by usage value. This enabled whole population data to be used for usage values. In 1979 when this author commenced the initial data collection and analysis by fortuitous circumstances the ABC listings for years 1975 to 1978 were available in the Parts department office, (gathering dust in an old filing cupboard). This was extremely fortunate otherwise the data for earlier years would not have been available. For an examination of parts prices and parts sales volume other company documents had to be used and data was extracted by sampling methods. In the majority of these cases sample sizes of 200 items were always selected. This was felt to be a reasonable compromise between the need to keep sampling errors down to the smallest values possible and the time needed to work through very large computer generated documents (some as large as 500 pages). The important sampling issues that had to be considered are discussed in appendix two together with criteria for determining sample sizes.

6.2 (a) Graphical tests of lognormality

By using the ABC print out of usage values for the whole year 1979 the following histogram and data tabulation were constructed in classes of equal logarithmic value bands. The symmetry in the usage value data is evident from both the tabulated data and the histogram giving a characteristic 'normal' curve form.

figure 6.1

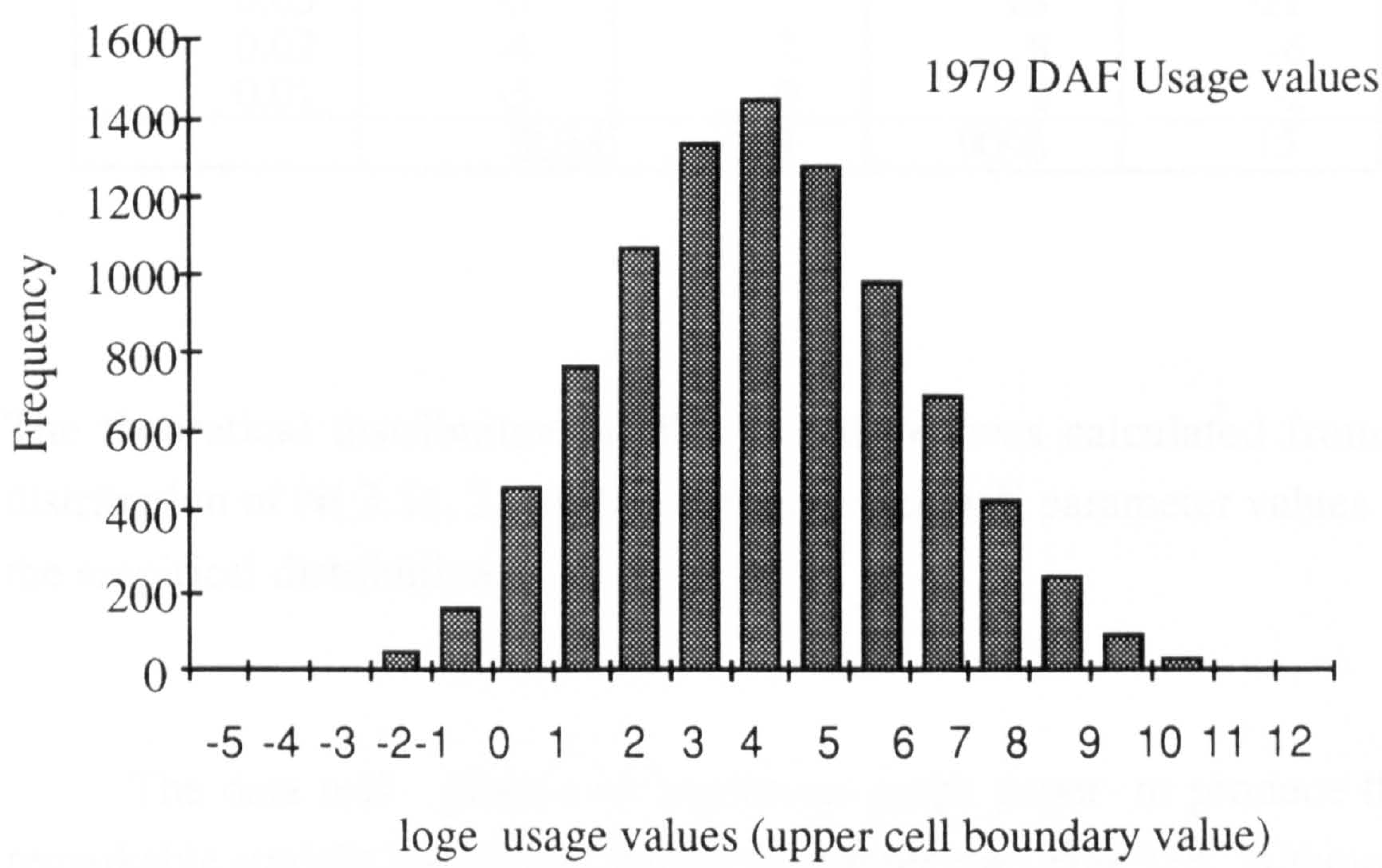


table 6.1

1979 Usage value data

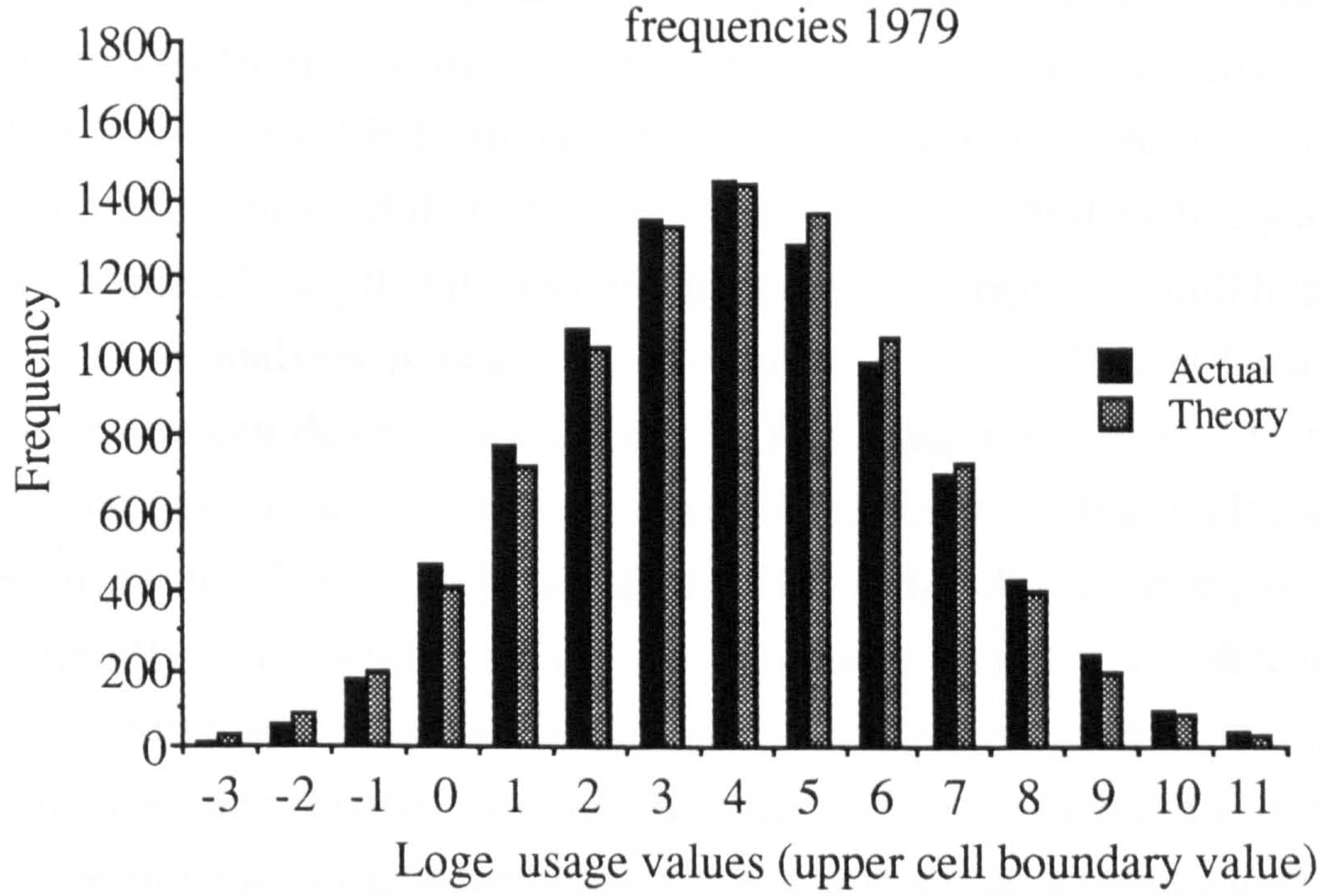
Usage value upper bound	loge Usage value	frequency of items	theoretical frequency	difference
442413.00	13	1	2	-1
162754.00	12	0	8	-8
59874.00	11	35	28	7
22026.00	10	94	83	11
8103.00	9	241	196	45
2980.90	8	427	401	26
1096.60	7	697	724	-27
403.43	6	987	1052	-65
148.41	5	1285	1371	-86
54.60	4	1452	1439	13
20.08	3	1344	1331	13
7.39	2	1072	1022	50
2.72	1	774	723	51
1.00	0	461	408	53
0.37	-1	171	192	-21
0.13	-2	50	80	-30
0.05	-3	7	28	-21
0.02	-4	2	8	-6
0.01	-5	0	2	-2
	SUM	9100	9098	15

The theoretical distribution in table 6.1 above was calculated from a distribution of N(3.51, 2.48), i.e. the same 'normal' parameter values as the empirical distribution.

The data was plotted on lognormal graph paper to produce the remarkable straight line as was shown in chapter one, page 35a. Although the lognormal graphical presentation in chapter one, together with the histograms presented here, are entirely visual they do provide us with strong support to the validity of claiming the data to be lognormally distributed. It can be clearly seen from the graph in chapter one that it is only in the extreme tails of the empirical distribution that any significant deviations occur from a theoretical normal curve. It is argued that data fitting a straight line as good as this over the whole percentage range, and well into the one percentage point of each tail on lognormal graph paper, must be regarded as lognormally distributed. Professor Gerald Goodhardt[1] has pointed out (private communication, 1984) that one should not be too surprised to find such a very high degree of regularity with social science data of this type, given the large effective size of the data set (9,100 in the case of 1979), provided the model is the appropriate one to use. Even so the degree of fit is so close that a critical reader might conclude that the data had been corrupted in some way. Figure 6.2 gives a direct visual comparison between the theoretical and actual frequencies.

1 Professor Goodhardt has published widely and extensively in the field of consumer behaviour and has vast experience of modelling work with highly skewed distributions in social science data.

figure 6.2
Comparison of actual and theoretical
frequencies 1979



To enable any follow on researchers to validate the data or conduct any additional analysis the critical first ten pages from the relevant DAF computer ABC print out is given in appendix seven. (A copy of the entire print out may be obtained at nominal charge by contacting the author at City University, London.)

6.2 (b) Goodness of fit tests

A Chi Squared test was determined comparing the 1979 data with the corresponding theoretical normal distribution (ie same parameter values), the results were as follows-.

Chi Squared statistic $= \Sigma (O - E)^2/E$

Observed Chi Squared = 82.65

The degrees of freedom 'v' =19 -1- 2 = 16

Criterion test value of Chi Square = 26.3 at the 5% level

Criterion test value of Chi Square = 32.0 at the 1% level

It can be seen from the above results that based on this classical goodness of fit test alone one would reject the null hypothesis, namely that the data is from a normal distribution, and accept the alternative that the data is not normal in form. However, as discussed in chapter two there are a number of methodological problems associated with the Chi squared test one of them being that the test will almost always reject the null hypothesis when the analysis is based on very large samples. This is because the procedure can detect (or react to) the fine departure, or noise, from the theoretical curve even in very good fit situations. The Kolmogorov - Smirnov test based, as it is, on cumulative distribution comparisons, is generally considered a fairer test in these situations. As discussed in chapter two according to Kendall (1979), this test is generally much more sensitive and accurate than the Chi Squared as a goodness of fit test. If Fo represents the cumulative observed distribution (as a proportion) and Fe the expected cumulative distribution (also as a proportion) then the Kolmogorov test statistic Dn for sample size '*n*' is computed as follows-

$$Dn = \text{Max}\ |Fo - Fe|$$

If the maximum observed value of Dn exceeds the tabulated critical value of Dn at the chosen level of significance, then the null hypothesis is rejected and the observed distribution is assumed to be non-normal. Conversely if he observed Dn value is less than the critical value then the null hypothesis is accepted. Although of course it is not proven as correct either; only that we can not disprove it on the given evidence.

The critical values of Dn are given by Kendall (1979) and for sample sizes greater than 35 the 1% and 5% levels of significance can be calculated as follows-

At the 5% level, $Dn = 1.36/\sqrt{n}$
and at the 1% level $Dn = 1.63/\sqrt{n}$

Therefore for a population of size 9,100 the significant values are 0.0142 and 0.017 respectively. The appropriate tabulation for the 1979 distribution is shown below -

table 6.2

Chi Squared test on 1979 Usage values

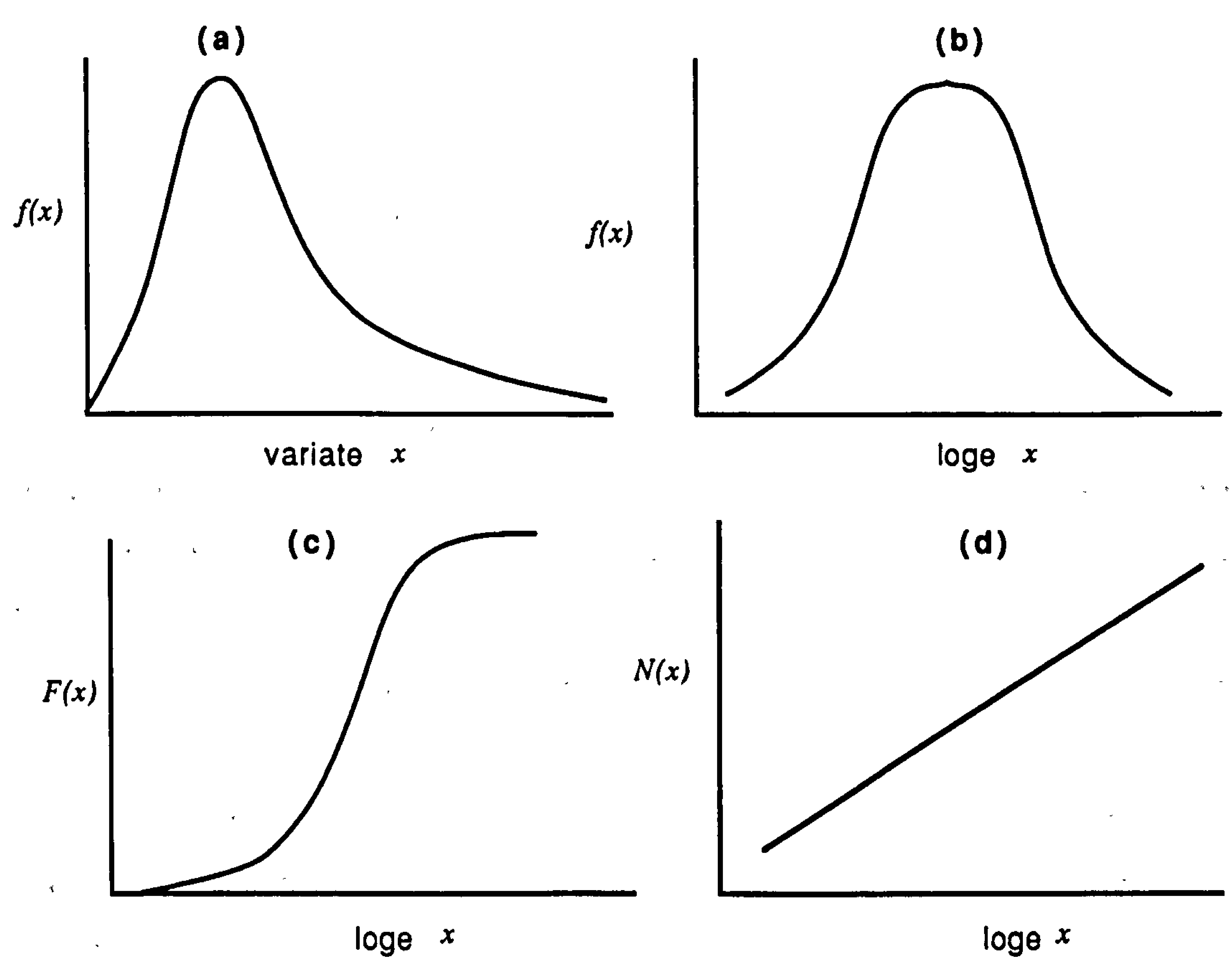
log usage value	Cum.frequency observed (Fo)	Cum frequency theoretical (Fe)	difference ABS(Fo-Fe)
13	0.0001	0.0020	
12	0.0001	0.0011	
11	0.0039	0.0042	
10	0.0143	0.0133	
9	0.0408	0.0349	
8	0.0877	0.0790	0.0087
7	0.1646	0.1585	
6	0.2727	0.2742	
5	0.4141	0.4248	
4	0.5735	0.5829	
3	0.7212	0.7292	
2	0.8390	0.8415	
1	0.9240	0.9209	
0	0.9747	0.9658	
-1	0.9935	0.9870	
-2	0.9990	0.9958	
-3	0.9997	0.9988	
-4	0.9999	0.9990	

The maximum difference between the observed and expected cumulative frequency distributions from the table above was 0.0108, which is less than both critical values of Dn at 0.0142 and 0.017 respectively. Hence the observed distribution satisfies the Kolmogorov - Smirnov test for normality and therefore the conclusion based on this test is that there is a very high probability that usage values are lognormal.

6.2 (c) Regression test for lognormality

It is possible to perform a regression test on the cumulative period usage data as explained and shown in detail in appendix four. In essence this means regressing $N(x)$ against $\log_e x$ after the data has been subjected to the following transformation shown in figure 6.3 below. (It is a very similar transformation to the transformation shown in figure 1.4 on page 36 in chapter one; the only difference being the inverse distribution function $R(x)$ was used in figure 1.4 instead of the distribution function)

figure 6.3
Lognormal transformations



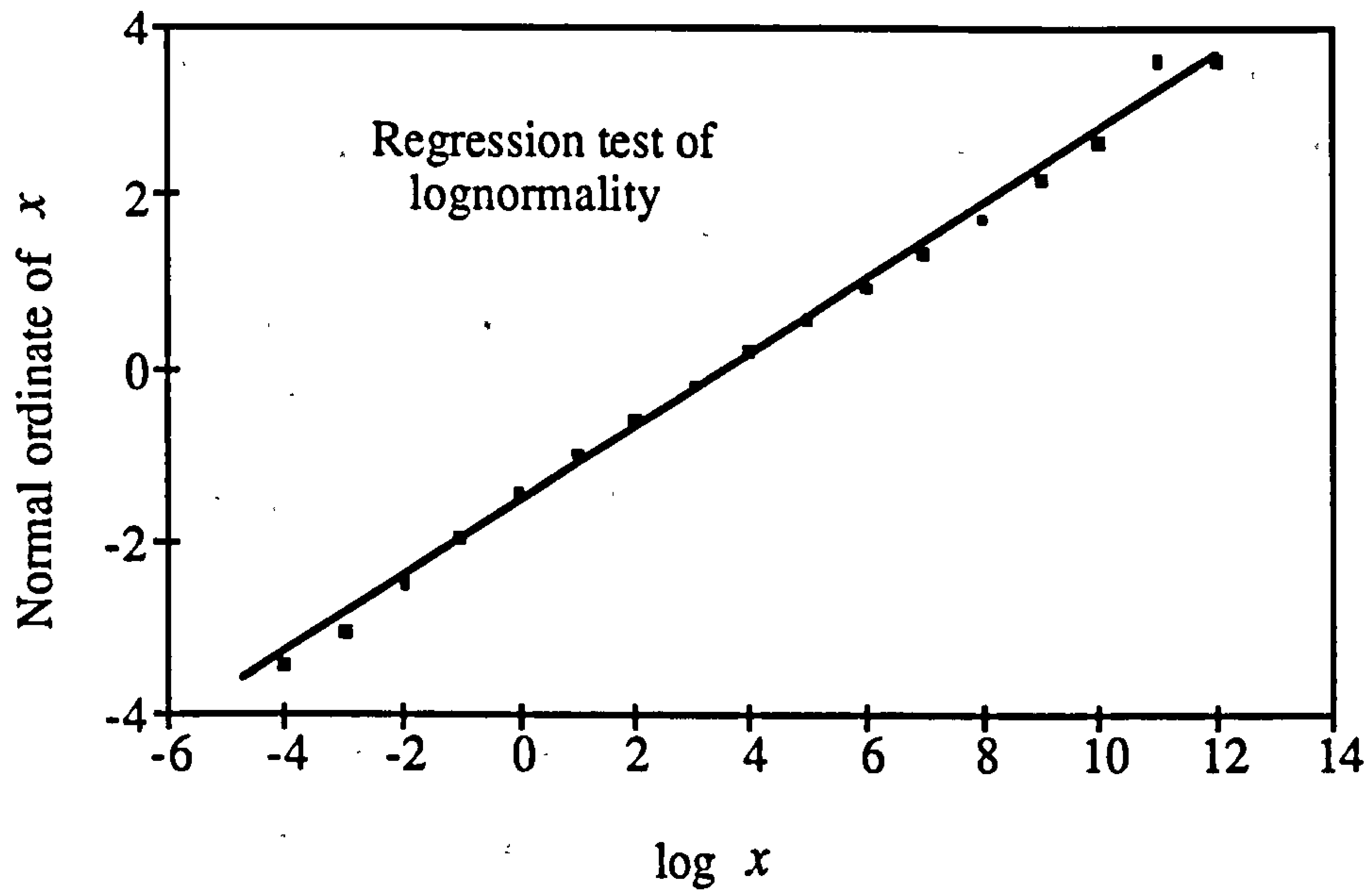
In the above diagrams (a) to (d) $N(x)$ is the normal probability ordinate of the transformed values of x and $\log_e x$ is simply the natural logarithm of the period usage values. For a theoretical lognormal distribution $N(x)$ will linearly regress perfectly with $\log_e x$. In our case there will be some noise in the data, that will be shown by the goodness of the fit of the regression line as measured by the coefficient of determination and the scatter of the regression residuals. Also the Durbin Watson test will give us the opportunity to measure any autocorrelation in the residuals. All considered the regression test for normality in the transformed data is probably the best and most sophisticated of the all the tests we have available to determine lognormality.

The result of regressing $N(x)$ against $\log_e x$ for 1979 gave the following values :

Correlation coefficient = 0.998
Coefficient of Determination = 99.6%
Standard error = 0.151
Durbin Watson statistic = 1.475

Statistically the correlation coefficient and the coefficient of determination are highly significant values and the coefficient of determination shows that greater than 99% of the variation in $\log_e x$ can be explained by the variation in the normal ordinate. The Durbin Watson statistic at 1.475 falls in the inconclusive regions for a sample of the size taken. As a general rule, if the Durbin Watson statistic is between 1.5 and 2.5 then no autocorrelation exists (although it is sample size dependent). If the statistic is between 1.0 and 1.5, or between 2.5 and 3.0 the result is inconclusive. If the statistic is above 3 or below 1 then autocorrelation is present. A graphical plot of $N(x)$ against $\log_e x$ for 1979 is shown below in figure 6.4.

figure 6.4



We have a very strong case at this stage to regard the usage value data as being drawn from a lognormal population. The graphical and regression tests used are very powerful in determining normal forms and this strongly supports the assertion that period usage values based on annual data are lognormally distributed. The failure of the Chi Squared test was only to be expected, in fact the closeness of the actual Chi Squared statistic given the size of the sample is in itself confirmatory evidence that we have a distribution very close to a normal curve. With such large samples a Chi Squared test would require an almost perfect fit between the empirical data and the theoretical distribution to provide a positive result, at the usual levels of significance for such statistical tests. Given the prior evidence from this work on the Law of Proportionate Effect and the wealth of supporting work quoted in the literature on the occurrence of lognormality in spares environments for most purposes the usage values of the DAF data can be regarded as drawn from lognormal populations. Following the general approach of Herbert Simon to this kind of data fitting problem (see Ijiri and Simon 1977 in particular) we should in fact be surprised to find that the data fits as remarkable close, as it does, given all the economic and technical factors that could potentially disturb the underlying processes at work. From Simon's view if the process of retroduction is applied and a stochastic process can support the existence of the model which in turn can explain the stochastic process then we could accept a degree of data fit far more inferior to what has been shown here. Hence, even without strong stochastic evidence to support the existence of lognormality, the data looks very convincing as a sample drawn from an underlying lognormal population.

6.2 (d) Comparison with the Yule and Pareto distributions.

In view of Simon's work on the Yule distribution and the possible genesis of this distribution by a modified version of the Law of Proportionate Effect (as discussed in chapter four) it was considered necessary to test the DAF usage value data against both the Yule and the Pareto distributions. The Yule distribution is somewhat difficult to handle analytically and there are difficulties in estimating its parameters. Furthermore the only source in which it is given any substantial modern treatment is Ijiri and Simon (1977). In fact according to Easton (1974), it is only Simon who has given this distribution any major consideration in contemporary modelling work on economic variates. [Note: in making this comment Easton was referring to Simon's earlier work of (1955) on the Yule distribution, which then appeared in his definitive text of 1977, joint authored with Uji Ijiri]. Hence the source books for applied work using this distribution are very limited. Fortunately Ijiri and Simon give a graphical test that can be applied to empirical data to see it can be summarised by the Yule distribution. A similar graphical procedure can be applied to the Pareto distribution. For the Yule distribution the procedure involves ordering the data set from the largest to the smallest data element (usage values here) then plotting the size of each element against its rank order on log - log graph paper. The largest element is given rank 1 the next rank 2 etc. If the data is Yule distributed this process will give a straight line on log log graph paper.

The Yule distribution has the density function -

$$f(i) = AB(i, \rho + 1)$$

where A' is a normalising constant
and ρ is the characteristic parameter
for element i

In this form B is the complete Beta function with parameter τ given by -

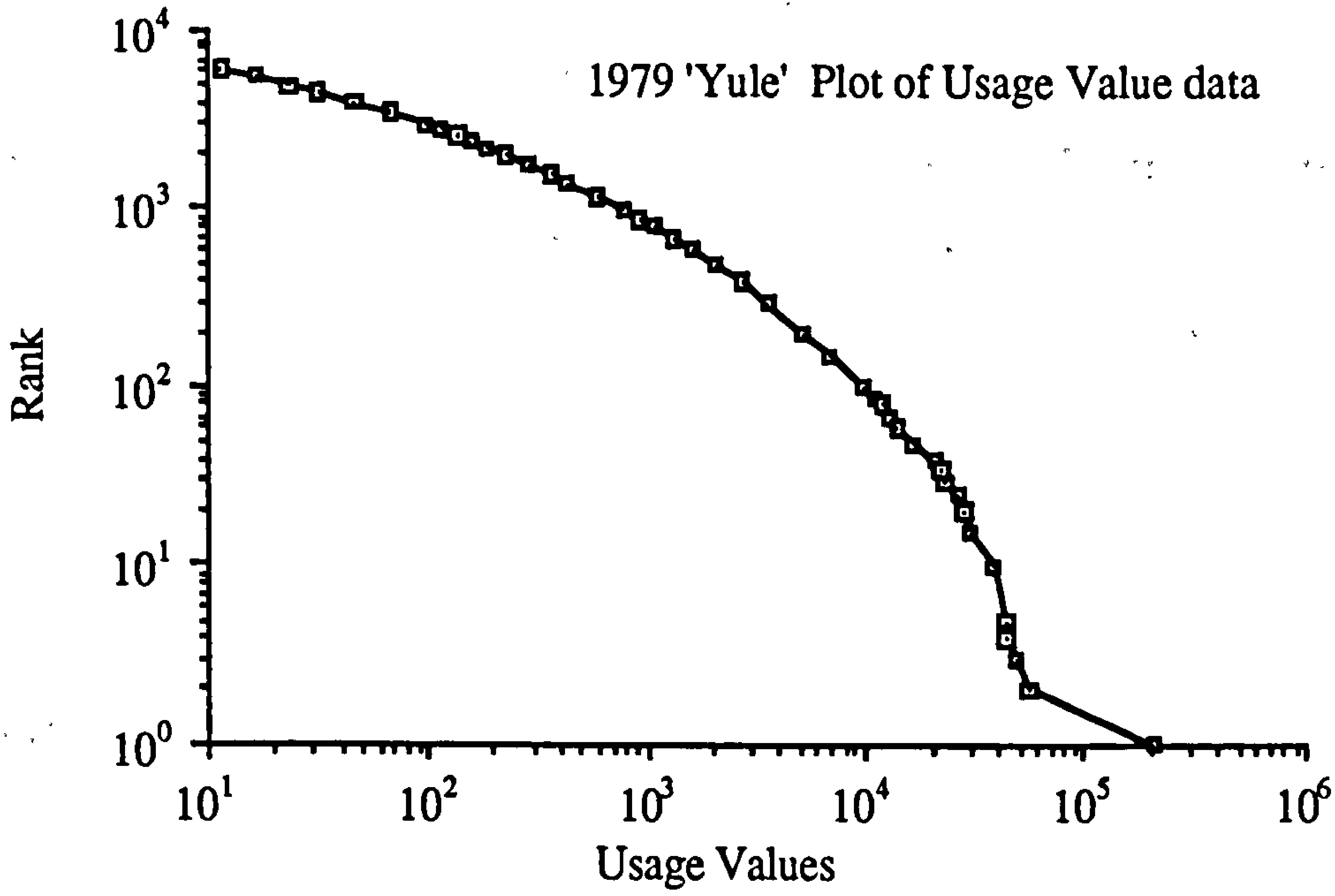
$$B(i, \rho + 1) = \int \tau^{i-1}(1-\tau)^{\rho} d\tau$$

The slope of the straight line from a Yule plot is given by β which is related to the distribution characteristic parameter ρ by the following formula -

$$\rho = \left[(1 - \frac{1}{i})^{\beta} - 1 \right] i$$

The results of an attempted Yule plot is given on the following graph. It can be seen that the data does not fall anywhere near a straight line, although there is a high degree of regularity in the plot. The convexity in Yule plots is referred to by Ijiri and Simon (1977) in the context of data that does not provide good Yule fits, but was data where a Yule fit might otherwise have been expected to exist. However, the convexity in the Yule plots we obtained in this work was far more marked than those shown by Simon.

Figure 6.5



In our analysis we also have to give consideration to the fact that the Yule distribution is a discrete distribution and hence its use to model continuous data would present some difficulty with interpretation. Furthermore it has no modal value after unity and it is always reverse 'J' shaped from the left. Our Usage value data is always unimodel, which is of course a characteristic of all lognormal distributions. We concluded from these foregoing facts that the evidence here for a Yule distribution is not at all strong, certainly so for periods up to one year.

A Pareto distribution can be checked against the same data by a plot on log log graph paper. In this case it is F(t) which is plotted against 't' by effective transformation of the following cumulative form of the Pareto distribution-

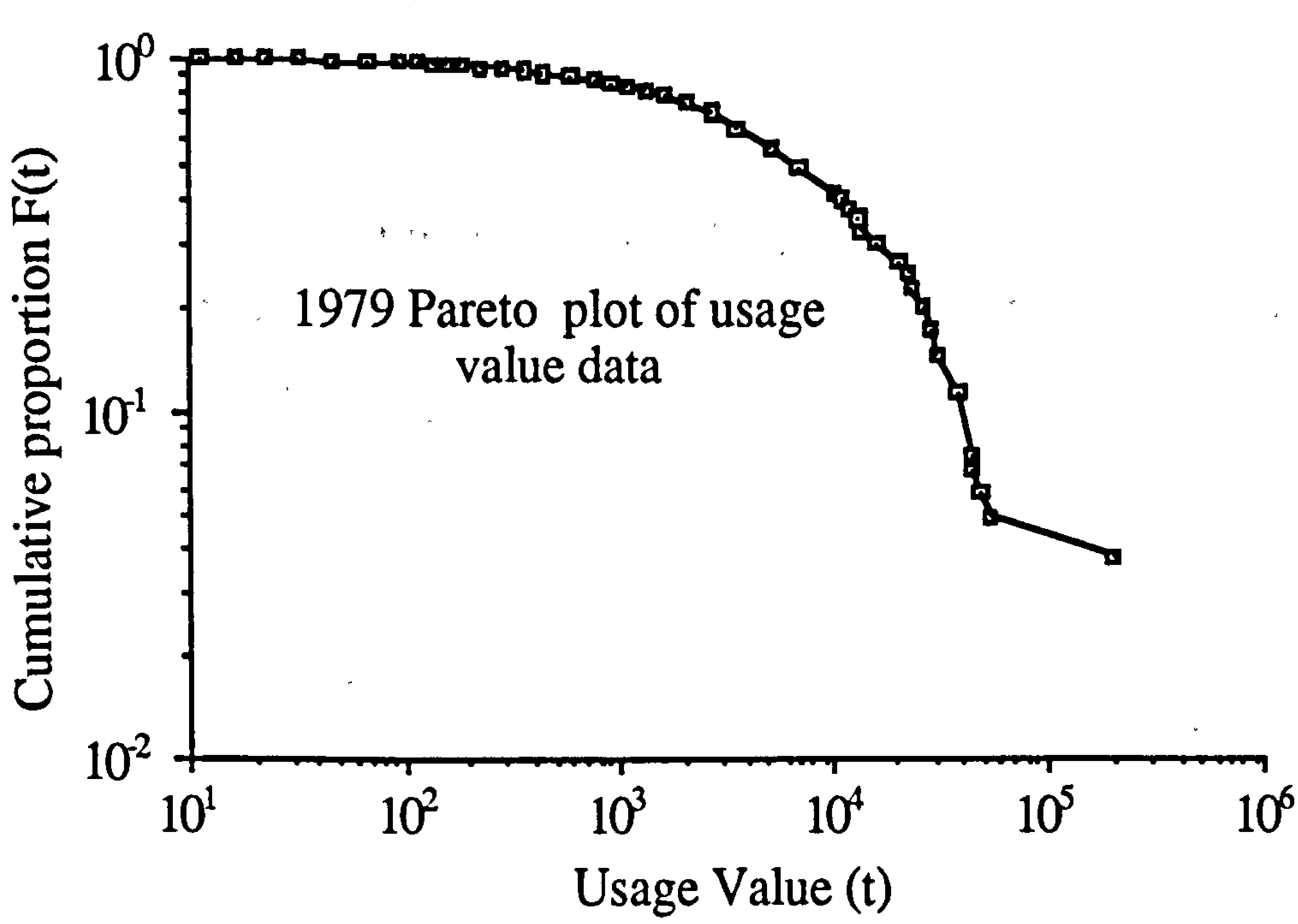
$$F(t) = Kt^{-\alpha}$$

then $$\log_e F(t) = (\log_e K) - \alpha(\log_e t)$$

The result of an attempted Pareto plot for the DAF usage value data a markedly convex plot that could not be regarded as Pareto in form. Furthermore like the Yule this integer form of the Pareto distribution is reverse J shaped from the left and therefore it ignores the unimodal nature of our usage value data. Both the Yule and the Pareto graphical plots should be contrasted with the high degree of fit obtained using the same data set on lognormal graph paper, shown in chapter one, and the lognormal regression line given in figure 6.4 . One might be tempted to say 'no contest'.

The attempted Pareto plot is shown below in figure 6.6, where as with our Yule plot, the graph produced a markedly convex line not at all very convincing as a Pareto plot.

Figure 6.6



At this stage we must therefore conclude that whilst the data under consideration here has in all probability come from a process driven by the Law of Proportionate Effect, which can lead to the Lognormal, Yule or Pareto distributions, little evidence exists to support either of the latter two as far as usage values are concerned. All the years of DAF usage value data examined produced similar markedly curved Yule and Pareto plots. Given the wealth of evidence so far for the lognormal distribution no further considerations were given to the either the Yule or the Pareto in the context of usage values.

6.2 (e) short period usage values

In view of the preceding results we also tested the proposition that usage values might be also lognormal in a short time durations such as one period (4 weeks in DAF terms). This could only be achieved by resorting to sampling, so single period demands for 200 randomly selected parts were obtained, from a demand history print out, and each part was multiplied by the relevant price of that part to give a sample of 200 usage values. The prices were obtained from the master price file. The tabulation of the sample is as follows-

table 6.3

Short Period Usage Values

log Usage Value	frequency of items	theoretical frequency	difference
10	0	0	0
9	2	1	1
8	0	3	-3
7	7	6	1
6	20	14	6
5	27	25	2
4	28	33	-5
3	30	36	-6
2	36	33	3
1	30	24	6
0	10	14	-4
-1	8	6	2
-2	1	3	-2
-3	1	1	0
-4	0	0	0

The fit seen in table 6.4 is not good by the standards of most of the goodness of fit tests we have used in this thesis. In terms of 'closeness' of fit the match between the theoretical and empirical distribution is poor, whilst the 'randomness' of fit is quite good, ie there are no runs of positive or negative differences. The actual Chi Squared value is very large at 381 compared with the theoretical $\chi_{0.05}$ and $\chi_{0.01}$ values of 18.307 and 23.209 respectively. Interestingly the difference between the two distributions did not fail the Kolmorgorov Smirnov test. However, there is also stochastic evidence, to be discussed later in this work, that the distribution of the short term usage values should not be lognormal because the volume distribution for such a short period is not lognormal. The real test of the use of the lognormal distribution for such short period usage values is how accurate any aggregate estimates will be that are based on such a distribution.

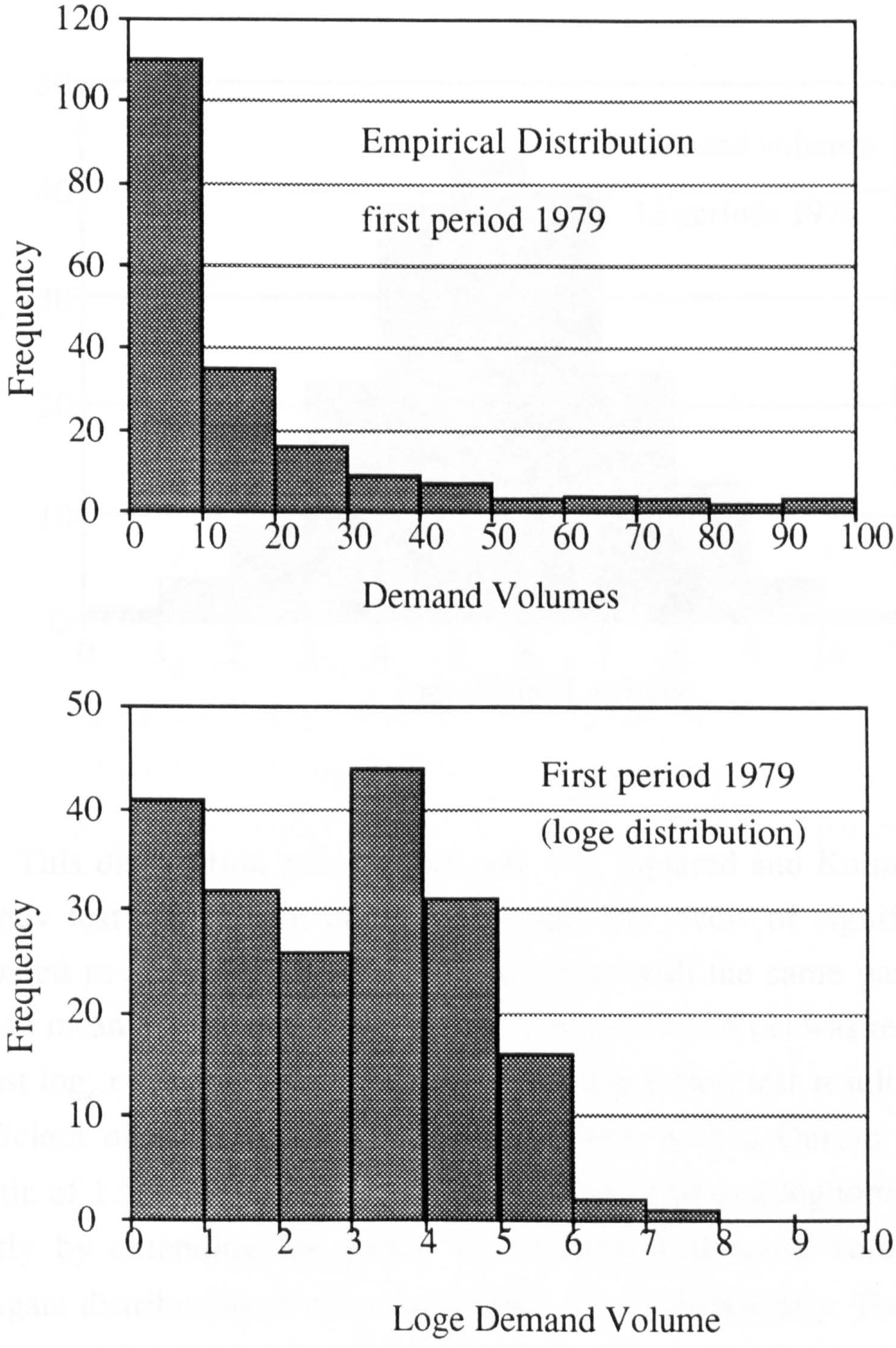
6.3 Analysis of Period Volumes

In this section we show in some detail the phenomena of the convergence to a lognormal form when demand volumes are cumulated over progressively longer time periods.

6.3 (a) convergence to lognormality

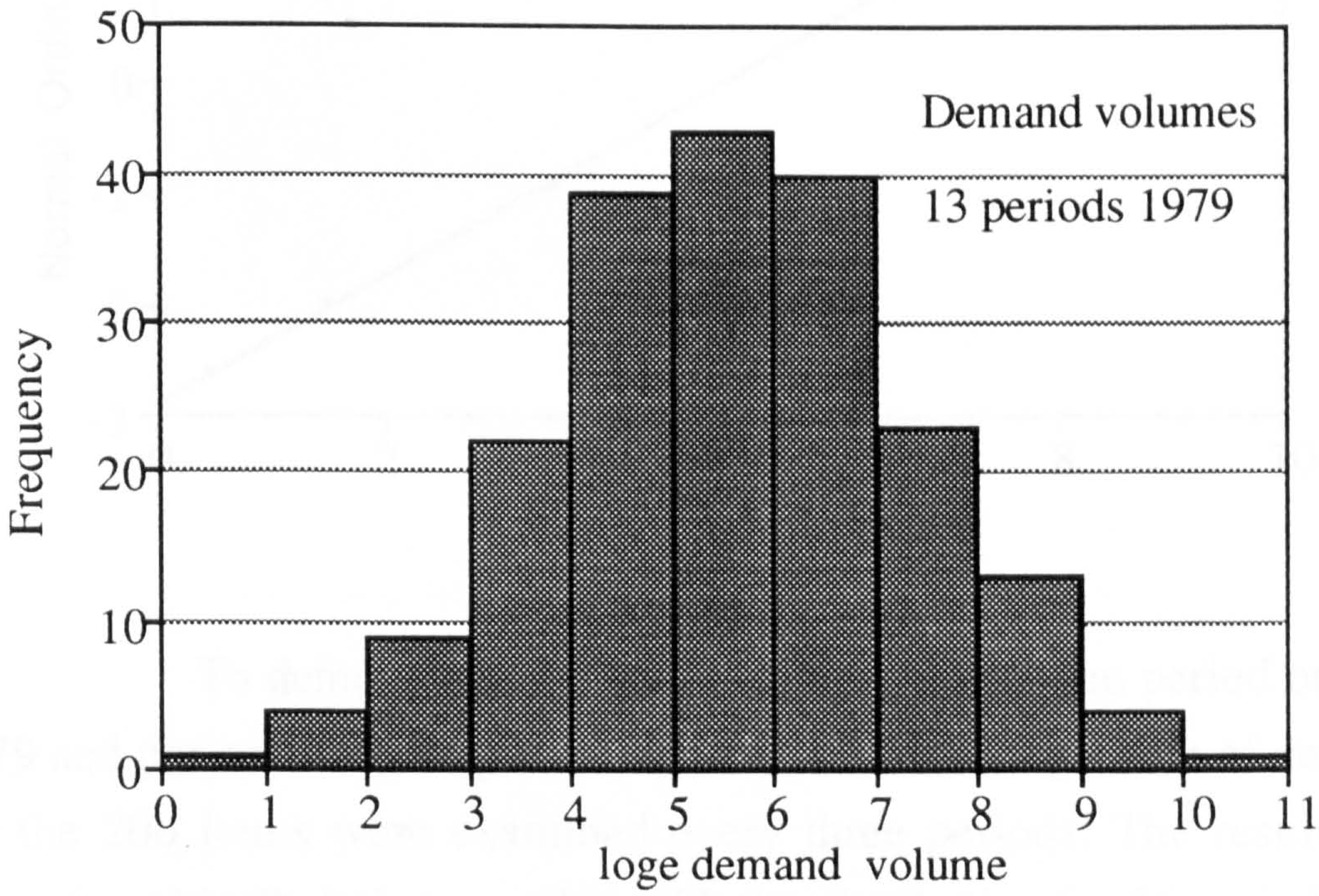
It was mentioned in the first chapter that according to the basic theory of lognormal distributions that if two independent variates 'A' and 'B' are both lognormally distributed then the product 'AB' is also lognormally distributed. Furthermore according to Aitchison and Brown (1957) if 'A' is lognormal as $\Lambda(\mu_1,\sigma^2_1)$ and 'B' is $\Lambda(\mu_2,\sigma^2_2)$ then the distribution of 'AB' is $\Lambda(\mu_1+\mu_2, \sigma^2_1+\sigma^2_2)$. The converse of this is also true, that is if 'A' and 'B' are both lognormally distributed then the product 'AB' is lognormal. The consequence of this is that if period usage values are lognormal then so to should be the separate components of price and volume. Early exploratory analysis revealed this to be true for prices, but it was only true for volumes provided the period considered was sufficiently long. The preliminary considerations of chapter one indicated that aggregate demand volumes for a sample of 200 randomly selected items from the first period 1979 (a four week duration) gave an empirical distribution that was far from lognormal in form. This distribution is shown below together with its logarithmic form. It can be seen that the logarithmic distribution does not give the characteristic normal shape it would if the original data were lognormal. Furthermore the early analysis strongly indicated that this empirical distribution could be fitted very well to a Log Series distribution. (The detailed analysis and testing of short period aggregate demand volumes are presented chapter eight).

Figure 6.7



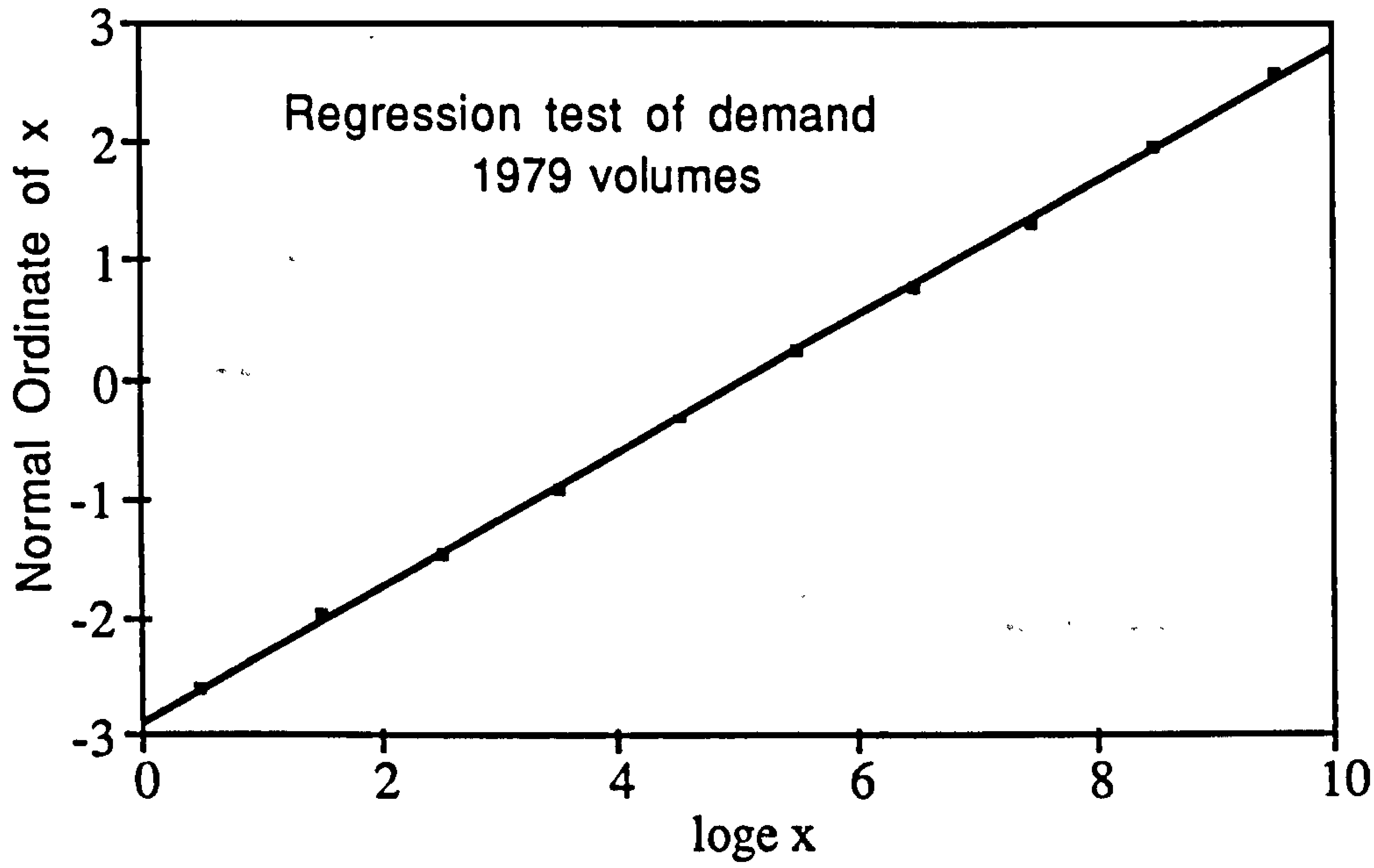
In marked contrast to the above first period distribution, when the demand volumes for the same 200 items were examined for one calendar year (13 periods) a completely different empirical distribution was seen. This is shown below in its logarithmic form where it can be seen that the characteristic normal distribution shape is obtained.

Figure 6.8



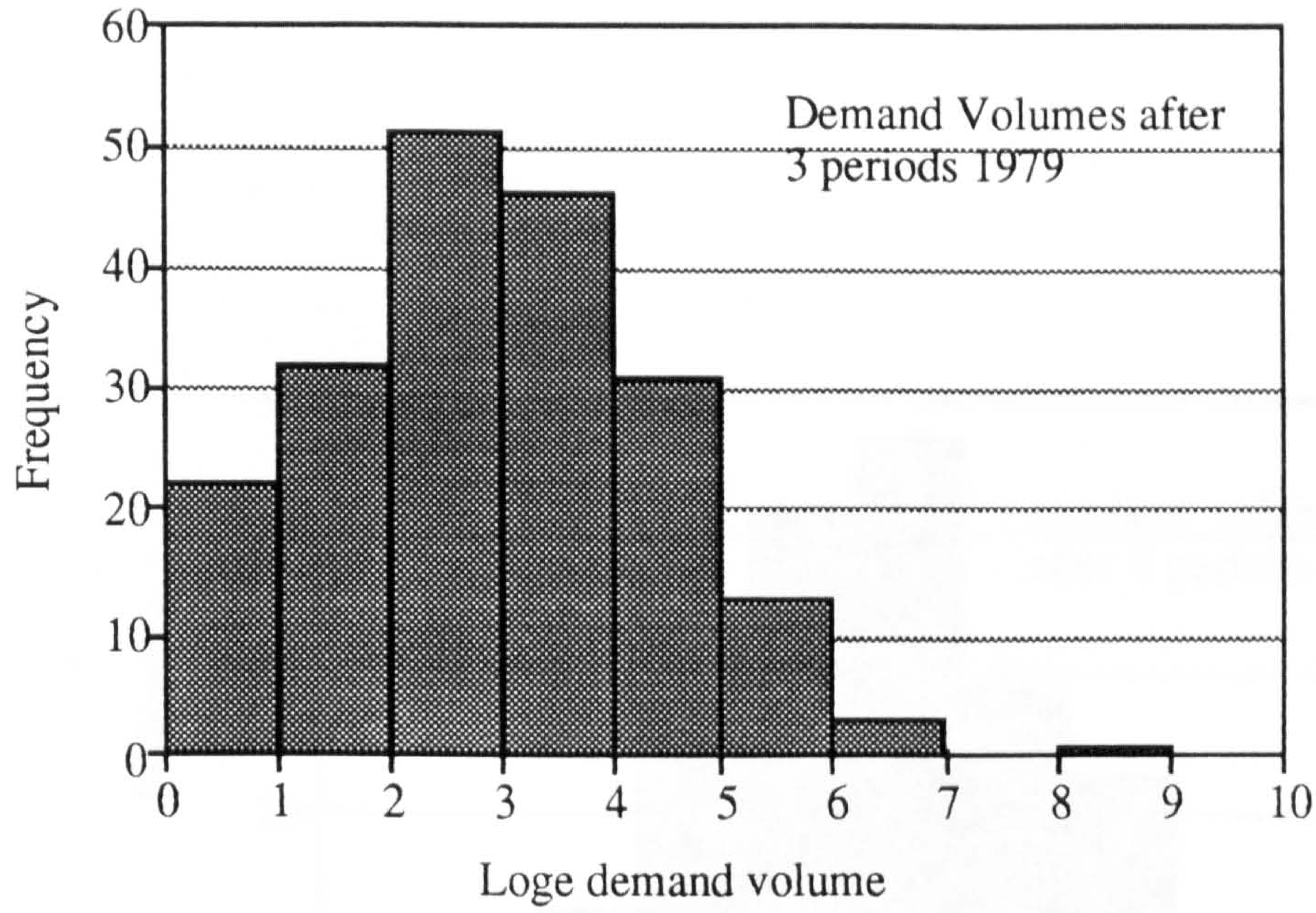
This distribution gave significant Chi Squared and Kolmogorov Smirnov test statistics at both the 1% and 5% levels of significance, compared to a theoretical normal distribution with the same parameter values (mean 5.575, standard deviation 1.789). When $N(x)$ was regressed against $\log_e x$ for this data we obtained a regression test result with a coefficient of determination greater than 99% with a Durbin Watson statistic of 1.550 indicating a very high degree of fit to a lognormal line. Clearly by extending the period for measuring demand volume the aggregate distribution of volumes changes form substantially. The graph of the regression test is given below in figure 6.9

Figure 6.9



To demonstrate what was happening between period one of 1979 and period 13 the empirical distributions of the logarithm of demand for the 200 items were examined every three periods. The results are shown graphically below together with the distribution for 26 periods (i.e. the whole of 1979 and 1980). The 26 period distribution was included to see how stable this convergence process was over a long time period.

Figure 6.10


Demand Volumes after
3 periods 1979
Frequency
0
10
20
30
40
50
60
0
1
2
3
4
5
6
7
8
9
10
Loge demand volume

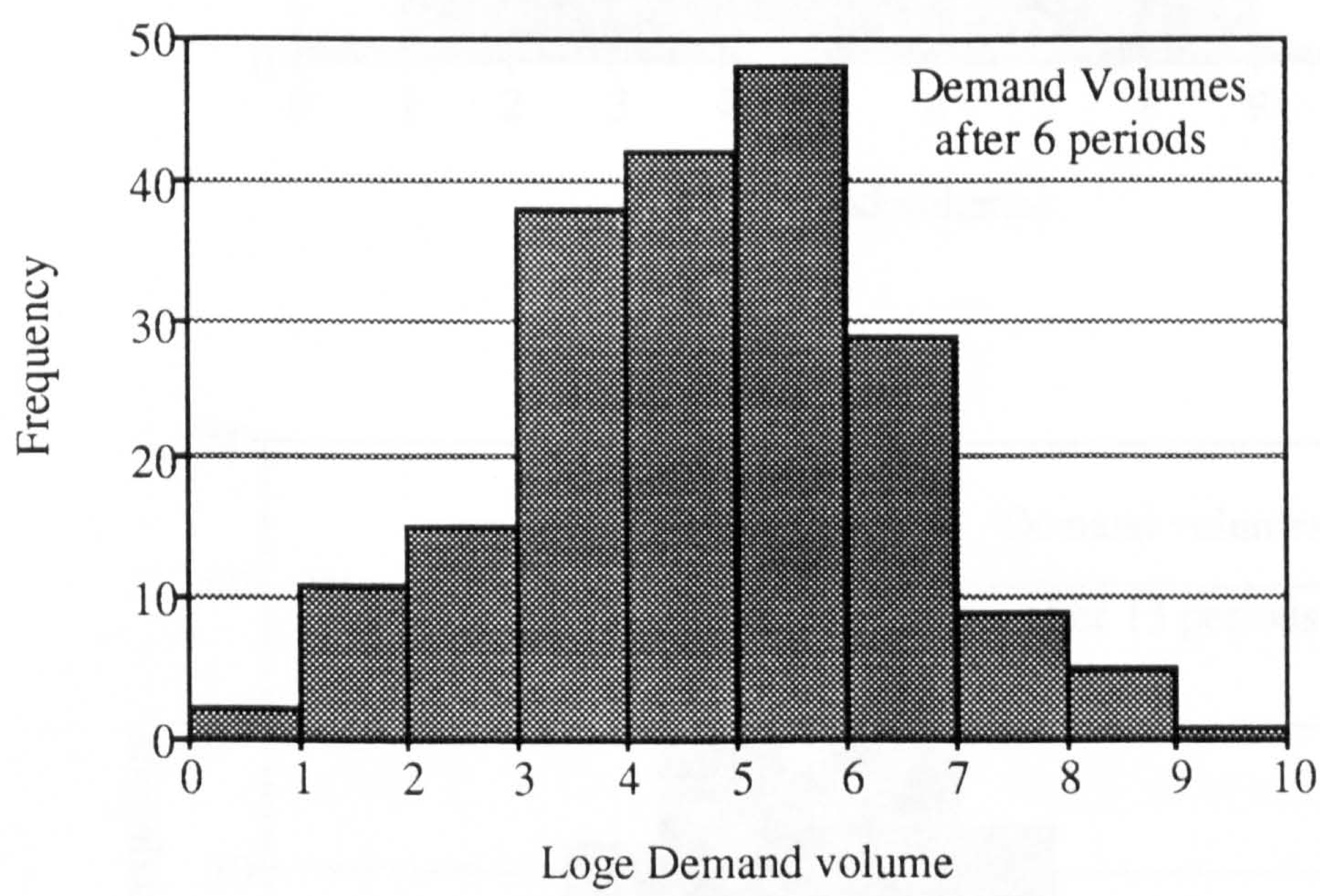
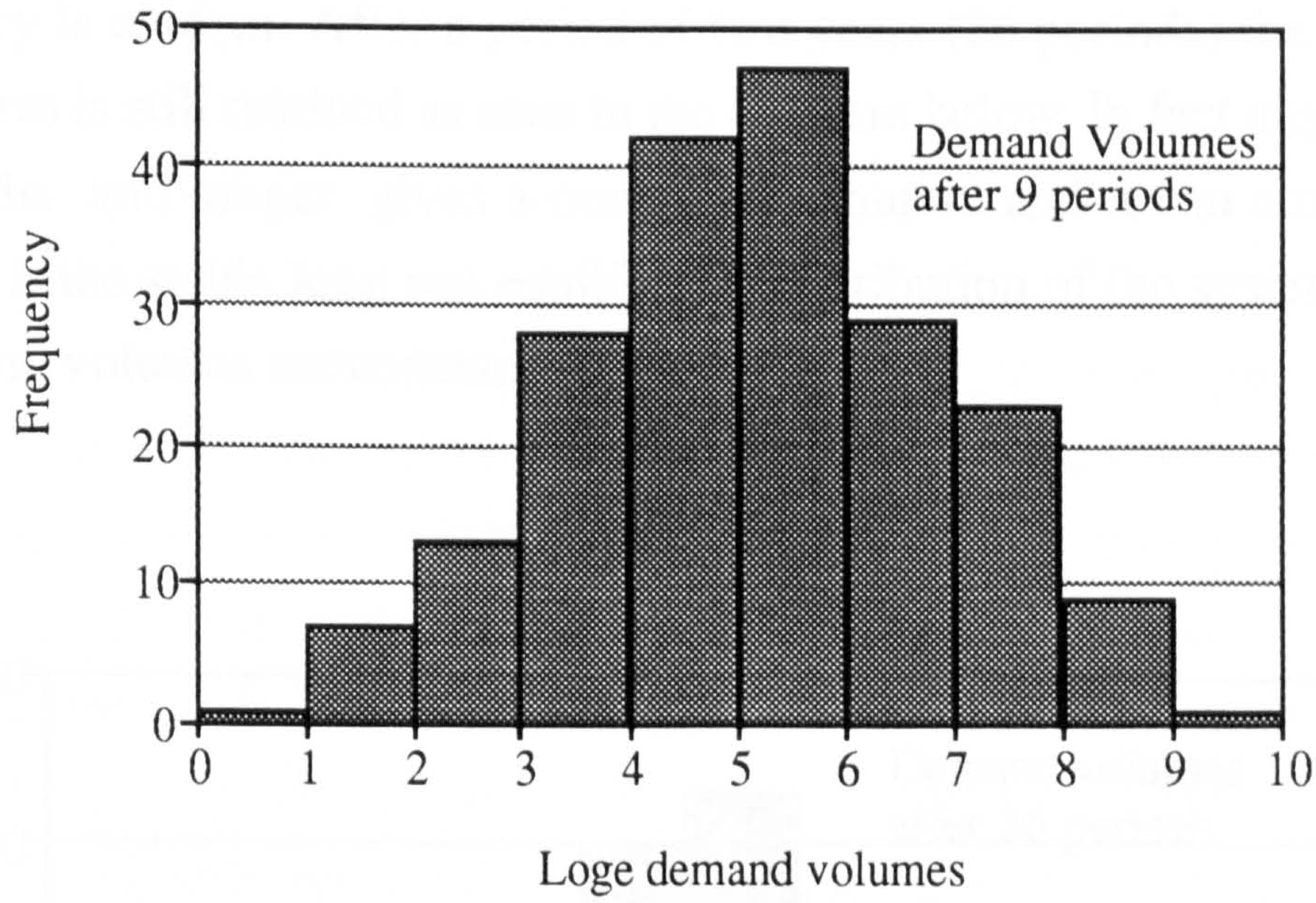
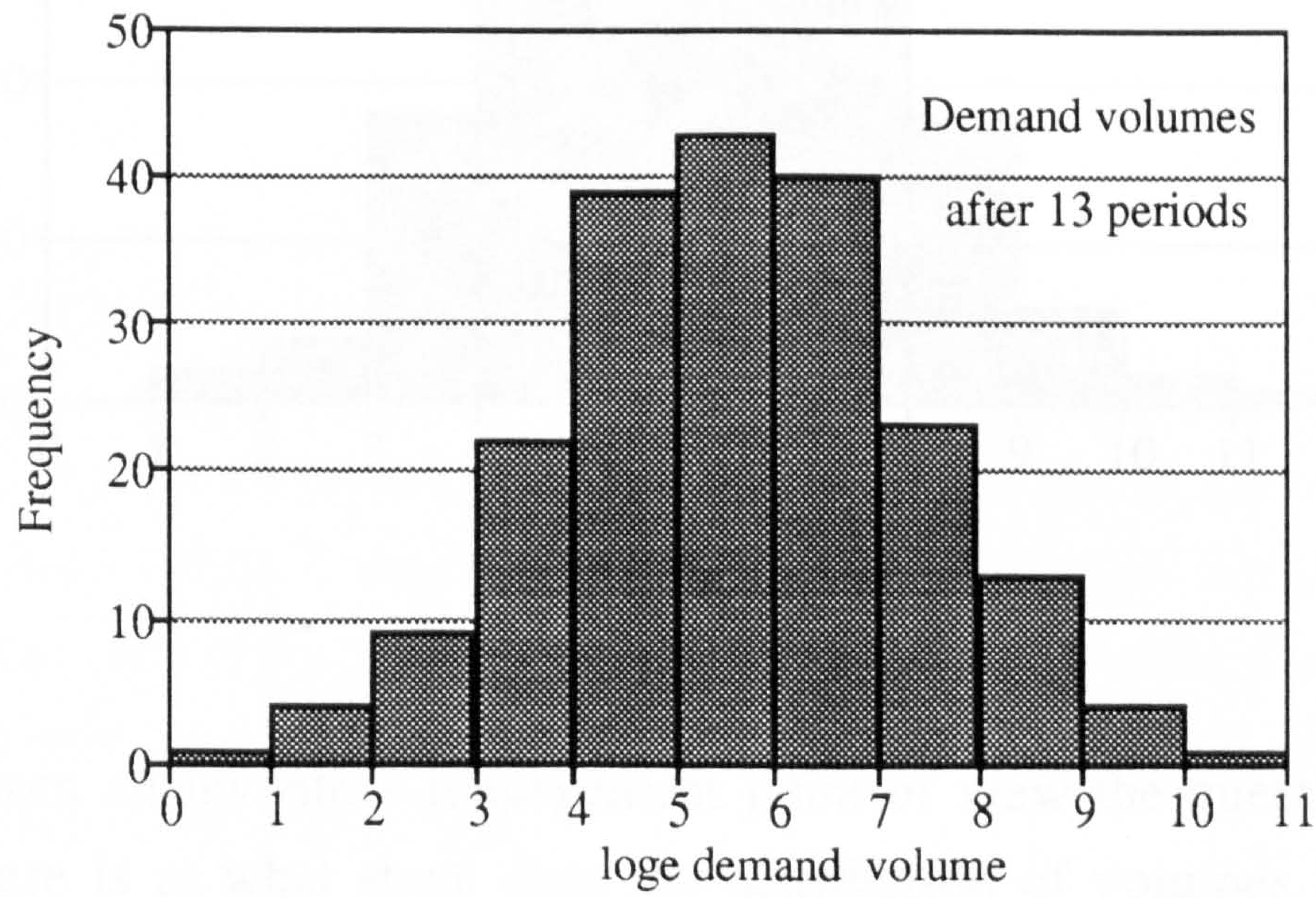

Demand Volumes
after 6 periods
Frequency
0
10
20
30
40
50
0
1
2
3
4
5
6
7
8
9
10
Loge Demand volume

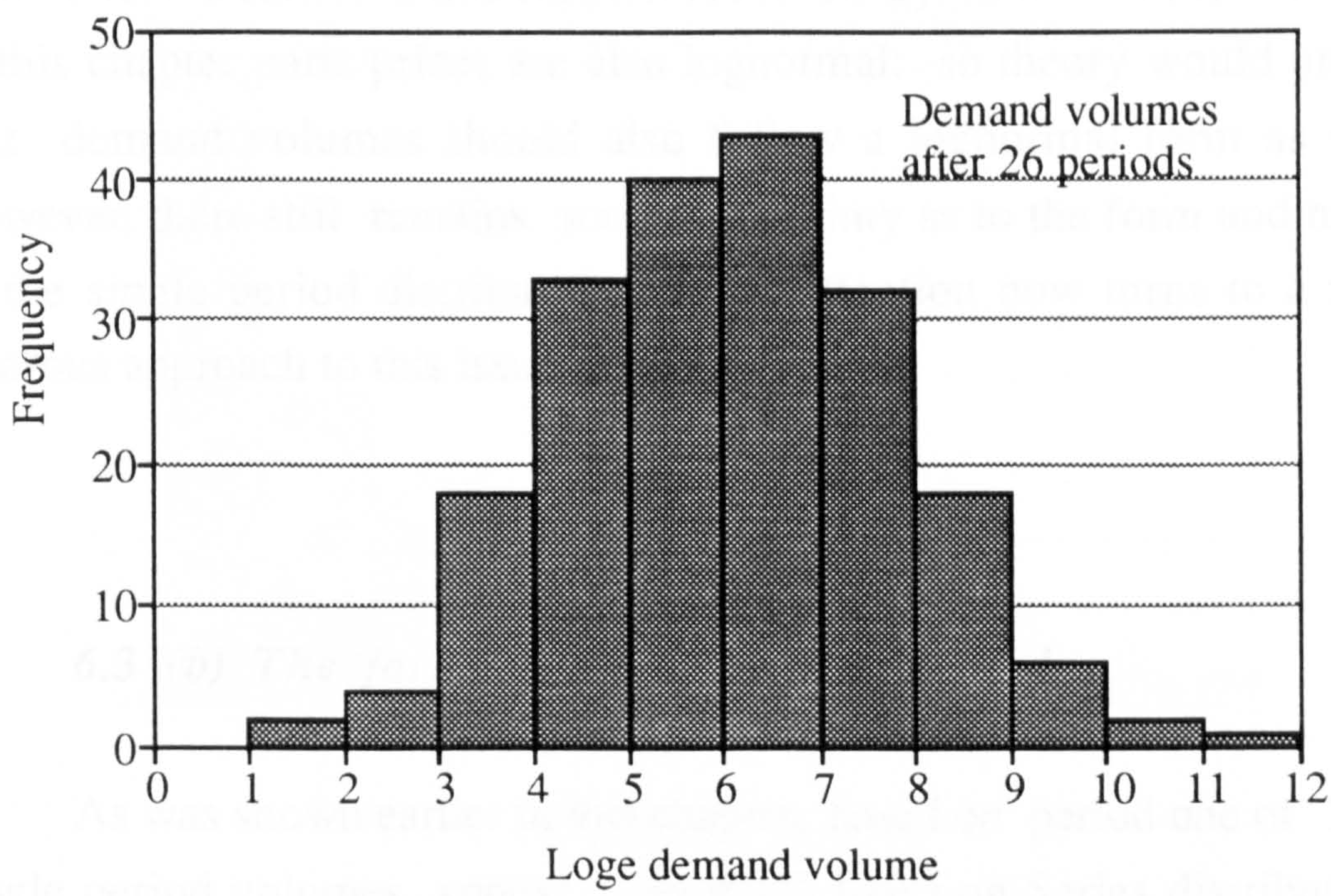
Figure 6.10 (continued)





It can be clearly seen from the above graphs that in a period as short as four weeks the logarithm of demands is not 'normal' in form. As we consider progressively longer time periods, as shown above, it can be observed that the distribution does begin to take on a characteristic normal form. After nine periods the distribution begins to take on a symmetrical form. After one year (13 periods) the distribution passes all the standard tests for a normal distribution and the high degree of symmetry is evident. After a period of two years (26 periods) the normal curve form is still retained as seen in the diagram below. In fact any period 13 months and longer gives a normal distribution and it can concluded that this is the stable long run equilibrium distribution of the system as far as demand volumes are concerned.

Figure 6.11



From an inventory management point of view the question that arises here is at what stage does the distribution of volumes become normal. It can be seen that up to three periods (three months) it is certainly not normal, whereas from nine periods onwards the distribution becomes very symmetrical and normal in form. Around periods six to nine we have a grey area where the demand volumes first begin to approach a normal curve form. Clearly this issue is related to the accuracy required of the typical aggregate inventory calculations and it will be discussed in a later section. Our concern here has been to show this unique finding that the distribution of demand volumes change in the way shown and that the long run stable distribution is a integer form of the lognormal distribution.

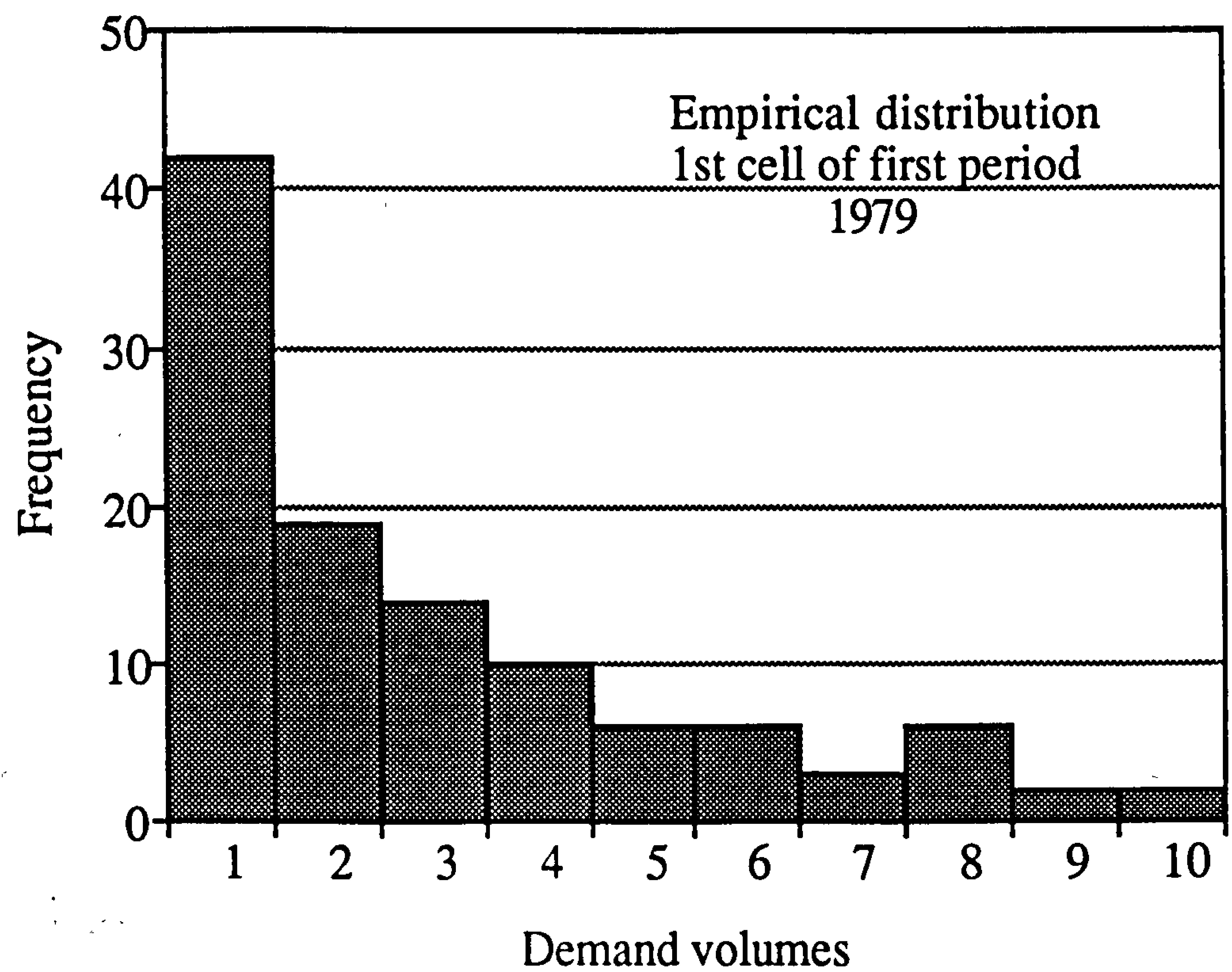
Clearly an interesting phenomena exists in the data examined, namely a distribution of demand volumes that is reverse 'J' shaped in very short time periods and which gradually converges to an integer version of a lognormal distribution as the time period is extended. The validity of the lognormal distribution as the stable long run form for usage values is well validated earlier in this chapter. Additionally, as will be demonstrated in this chapter parts prices are also lognormal; so theory would predict that demand volumes should also follow a lognormal form as well. However, there still remains some uncertainty as to the form and nature of the single period distribution; so our attention now turns to a more rigorous approach to this issue.

6.3 (b) The form of single period demands

As was shown earlier in this chapter, based on period one of 1979, single period volumes appear very much like Log Series distributions. This is now tested more formally against the corresponding theoretical LSD and other potentially competing distributions using period one as the example empirical distribution.

As can be seen from figure 6.3 the distribution is highly skewed and reverse 'J' shaped. It was easy to show that this distribution was not lognormal because a histogram of the logarithms of the demands did not produce the characteristic shape of the normal distribution (again see figure 6.3). Also it was possible to show that it was a true reverse 'J' shaped distribution by examination of the data in the first cell (that is the values 1 to 10) This is shown in figure 6.7 below where it can be seen that the distribution has no modal value after the value one and the frequencies rapidly decline with each value thereafter. We can confidently characterise the distribution as highly skewed and true reverse 'J' shaped for all positive integer values

figure 6.12



The next question that needed to be addressed was could a theoretical distribution fully explain the shape and, more importantly, the occurrence of such a distribution. Certainly prior exploratory analysis seemed to strongly favour the LSD, but other possibilities had to be examined. Any theoretical distribution to be a candidate must be a discrete distribution, reverse 'J' shape for all positive values, or be capable of taking this form with appropriate values of its parameters. The prime candidates from the various common theoretical distributions, were the lognormal, binomial, Poisson, geometric, negative binomial and the log series distributions. A number of other distributions are capable of taking an extreme reverse 'J' shape form such as the exponential, hyper-exponential, and the gamma distribution. However, these are continuous distributions and whilst it may prove possible to graduate the shape of the empirical distribution they would present problems of interpretation of the origin of discrete data.

The binomial was quickly eliminated because such distributions have a variance less than the mean. The Poisson was eliminated for a similar reason because for Poisson distributions the mean equals the variance, also both distributions are not true reverse 'J' shaped. The mean and variance of the empirical distribution of figure 6.3 were 16.813 and 398.27 respectively. The lognormal was eliminated because it has a modal value greater than unity. Also, as shown previously, when the distribution of the logarithms were examined no normal curve form was obtained. The LSD, geometric and the NBD distributions were all attractive because of their connection with Poisson compounding and Poisson mixing processes. The NBD however, can be regarded as a special case of the LSD and also it includes zero values and it may have a modal value greater than unity. The consideration of this distribution as a candidate to explain the regularity observed is considered in more detail at a later stage. Thus of the common discrete theoretical distributions we are left with the LSD and the Geometric. The hypergeometric was ultimately rejected because no stochastic evidence could be found to support its use.

6.3 (c) Fitting LSD and geometric distributions.

The Log Series distribution is a discrete distribution, highly skewed and true reverse 'J' shaped with the following probability density function-

$$P(x) = \frac{-q^x}{x\ln(1-q)} \qquad \text{-------------- 6.1}$$

Where 'q' is the parameter of the distribution ranging between 0 and 1, and x, the variate, is a positive integer for all values > 1.

The mean (w) of the distribution is given from-

$$w = \frac{-q}{[(1-q)\ln(1-q]}$$

This expression cannot be solved directly for q, however, Ehrenberg (1972, page 159) has developed an approximation for 'q' given by-

$$q = \frac{(w-1.4)}{(w-1.15)}$$

This gives 'q' to an accuracy of 2% for the range $2 < w < 20$. Hence to calculate theoretical frequencies for an LSD we simple use the mean of the empirical distribution to calculate 'q' and then use the probability density function to determine the probability of any value 'x'. With a mean of 16.813 for the empirical distribution we obtained a value for 'q' of 0.985. For the geometric distribution we use the following probability density function-

$$f(x) = P(1-P)^{x-1}$$

where $P = 1/\text{mean}$

Hence to determine P we equate the mean of the empirical distribution to P in the above equation. Thus with a mean of 16.813 we obtained a geometric parameter of 0.059. Using these procedures we obtained the appropriate theoretical frequencies of the LSD and geometric distributions as shown in the following table -

Table 6.4
Testing Short Period Demand Volumes

Value range cell mid pt.	Actual distribution	LSD	Difference	Geometric	Difference
5	110	112	2	87	-27
15	35	32	-3	47	-12
25	16	17	1	26	-10
35	10	10	0	14	-4
45	7	7	0	8	-1
55	3	5	2	4	-1
65	4	4	0	2	2
75	3	3	0	1	2
85	2	2	0	1	1
95	3	1	-2	1	2
	Σ193	Σ193	Σ 0		Σ-48

It can be seen from the above table that the LSD provides a very close fit to the distribution throughout the entire range, whereas the geometric distribution provides a very poor fit in the lower value range. Quite clearly just on visual inspection alone the LSD meets the criteria of goodness of fit to the empirical data. The geometric distribution can be rejected because the fit is so poor, that even with good stochastical support, its use in the context here would be very dubious. We can conclude at this stage that the aggregate distribution of demand for a period as short as one month is very likely to be a LSD. However, as further support we use the Kolmogorov Smirnov test to test the empirical distribution against the LSD. The tabulation for this is shown below-

Table 6.5

Kolmogorov Smirnov Test of Short period Demand Volumes

Value Range cell mid pt.	Actual Distribution	Proportion	LSD	Proportion	Difference
5	110	0.570	112	0.580	-0.010
15	35	0.751	32	0.746	0.005
25	16	0.834	17	0.834	0.000
35	10	0.886	10	0.886	0.000
45	7	0.992	7	0.922	0.000
55	3	0.938	5	0.948	-0.010
65	4	0.959	4	0.969	-0.010
75	3	0.974	3	0.984	-0.010
85	2	0.984	2	0.995	-0.010
	3	1.000	1	1.000	0.000
	Sum 193		Sum 193		

The largest Kolmogorov Smirnov test statistic value (Dn) given from the above tabulation was 0.010 which compares with the theoretical Dn values of 0.0978 and 0.0120 at the 1% and 5% levels of significance respectively. As shown in section 6.2(b) the theoretical Dn values were calculated from-

$$Dn\ 5\% = 1.36/\sqrt{193} \quad \text{and} \quad Dn\ 1\% = 1.63/\sqrt{193}$$

Hence we can regard the result as significant at both significance levels and we can not prove that the empirical distribution is not LSD. To add further weight to the conclusion that the empirical distribution is most likely LSD we have examined the first cell, value by value, and fitted these to a LSD distribution using the same procedures shown above for calculating theoretical frequencies. The tabulation is shown in table 6.6 :-

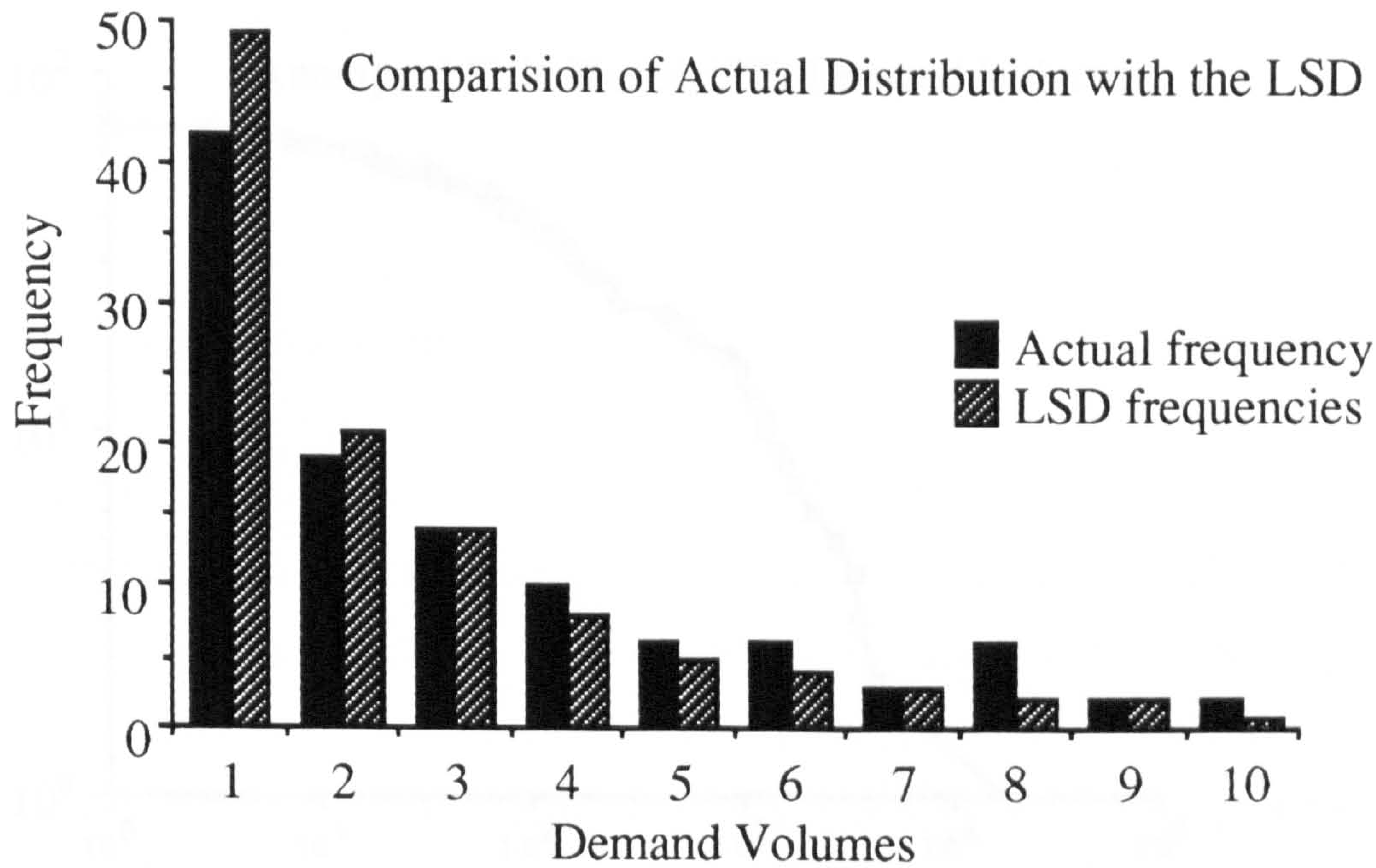
Table 6.6

First Cell, First Period Volume 1979

Value	Actual Frequency	Theoretical Frequency	Difference
1	42	49	-7
2	19	21	-2
3	14	14	0
4	10	8	2
5	6	5	1
6	6	4	2
7	3	3	0
8	6	2	4
9	2	2	0
10	2	1	1
			Σ =1

The very close correspondence between the two distributions can be readily seen from the above tabulation and figure 6.13 below :-.

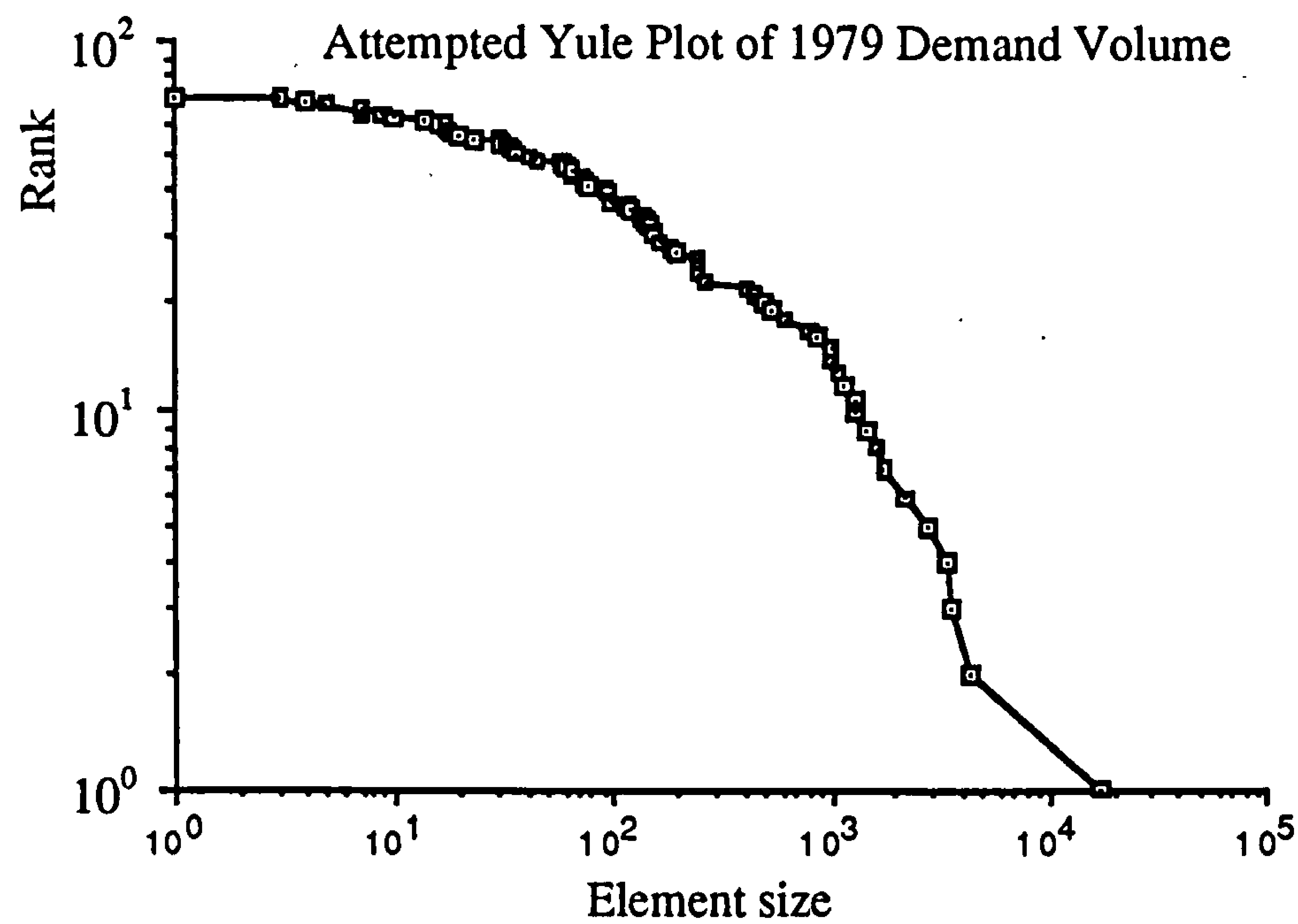
Figure 6.13



6.3 (d) Testing volumes against the Yule distribution

Our last stage of analysis on 1979 demand volumes was to examine the possibility that volumes might be Yule distributed. We selected a random sample of part numbers and ranked the annual demand volume for each part then plotted the rank against the size of each element. The result was as shown in the following graph-

Figure 6.14



In view of the success in graduating demand volumes with lognormal curves and the very poor fit to a straight line in the above graph it was concluded that the evidence for a Yule distribution was very weak. Hence we were not encouraged to search any further for a fit of Yule distributions to our empirical data.

6.4 Analysis of Price Distributions

The other factor of concern in considering usage values of spare parts is the price of each item and the form of distribution of prices for the inventory range. Given that usage values have been shown to be lognormally distributed and that demand volumes are of a lognormal form (provided we consider a time period of at least nine months), then we could take it that prices too must be lognormal, which the theory says they must be. However, to verify the theory this section considers the analysis of prices and their distribution. We are also concerned here with the parameters of the price distribution so that when considered jointly with the parameters values of demand volumes we can see their respective contribution to the parameters of the parent usage values over the period 1975 to 1985.

6.4 (a) Data capture considerations

As with demand volumes the only feasible way to examine the distribution of prices was on a sampling basis. From the DAF computer system various programmes were available that generated reports containing parts price data e.g. - ABC reports by volume and by value, Stock detail reports (stock status - in stock, on order etc), stock valuation reports, and master parts price lists. The prices appearing on all such reports were taken from the master price file, hence it did not matter which document was used to extract price information provided the samples were always randomly selected. Two protocols were possible in selecting item prices. Either the same part numbers as used for parts demand volume analysis could be used for consistency, or parts numbers could be selected at random independently of the volume data. In fact as it turned out it did not matter either way, the results were the same within sampling error. However, it was considered a fairer test if the two sets of data for volumes and prices were independently sampled. In this way any sampling bias inadvertently introduced in any one set would not be carried over in a second set of data. The price data used was also based on cost price to the company i.e. that paid by DAF (GB) to the parent company in Holland (called landed cost price by DAF GB). In this way any related aggregate inventory calculations using lognormal theory, such as investment in cycle and safety stocks, would be based on cost price to the company at Marlow. There was no evidence from within the company that prices were arranged in Holland to reflect any profit leakage policies back to the parent organisation (i.e. paying excessive amounts for certain parts, although one could not be absolutely sure). As far as was made known to this author the prices were consistent in that a fixed percentage was added to all part numbers by the parent company. Hence it was assumed that landed cost price for any DAF spare part reflects a fair and consistent value for that part.

6.4 (b) Price distributions

A simple random sample of 200 part numbers from the stock detail report of 1979 gave the following $\log_e$ distribution of prices-

Table 6.7

1979 Price Distribution

loge price cell mid pt.	Frequency	Theoretical Frequency *	Difference
-5.5	0	1	-1
-4.5	2	2	0
-3.5	7	6	1
-2.5	17	14	3
-1.5	21	24	-3
-0.5	31	34	-3
0.5	45	38	7
1.5	31	34	-3
2.5	19	24	-5
3.5	19	14	5
4.5	6	6	0
5.5	1	2	-1
6.5	1	1	0
	n=200	n=200	Σ=0

*The theoretical frequencies were calculated for a normal distribution with the same mean and standard deviation as the empirical distribution.(ie mean $\log_e x = 0.495$, and standard deviation $\log_e x = 2.085$).

A Chi Squared test on this data gave a highly significant result with an observed Chi Squared value of 7.33 against theoretical values at the 1% and 5% levels of significance of 19.67 and 24.725 respectively at 11 degrees of freedom. A Kolmogorov-Smirnov test on this same set of values gave an observed maximum Dn value at 0.02 compared with 1% and 5% critical values of Dn at 0.1152 and 0.0962 respectively. The actual and theoretical frequencies are shown in the following histogram where the close correspondence between them can be seen.

A regression test of $N(x)$ (the normal ordinate) against $\log_e x$ gave a highly significant result with a correlation coefficient of 0.999 with coefficient of determination at 0.997. The Durbin Watson test gave a DW statistic of 2.45 which is just on the acceptable limit for no autocorrelation between regression residuals. However, the actual size of the residuals is extremely small, hence the DW test in this cases should not be given too much weight. Given that this price distribution test was carried out on a random sample of 200 price elements the very high degree of correlation obtained was really quite staggering.

Figure 6.15

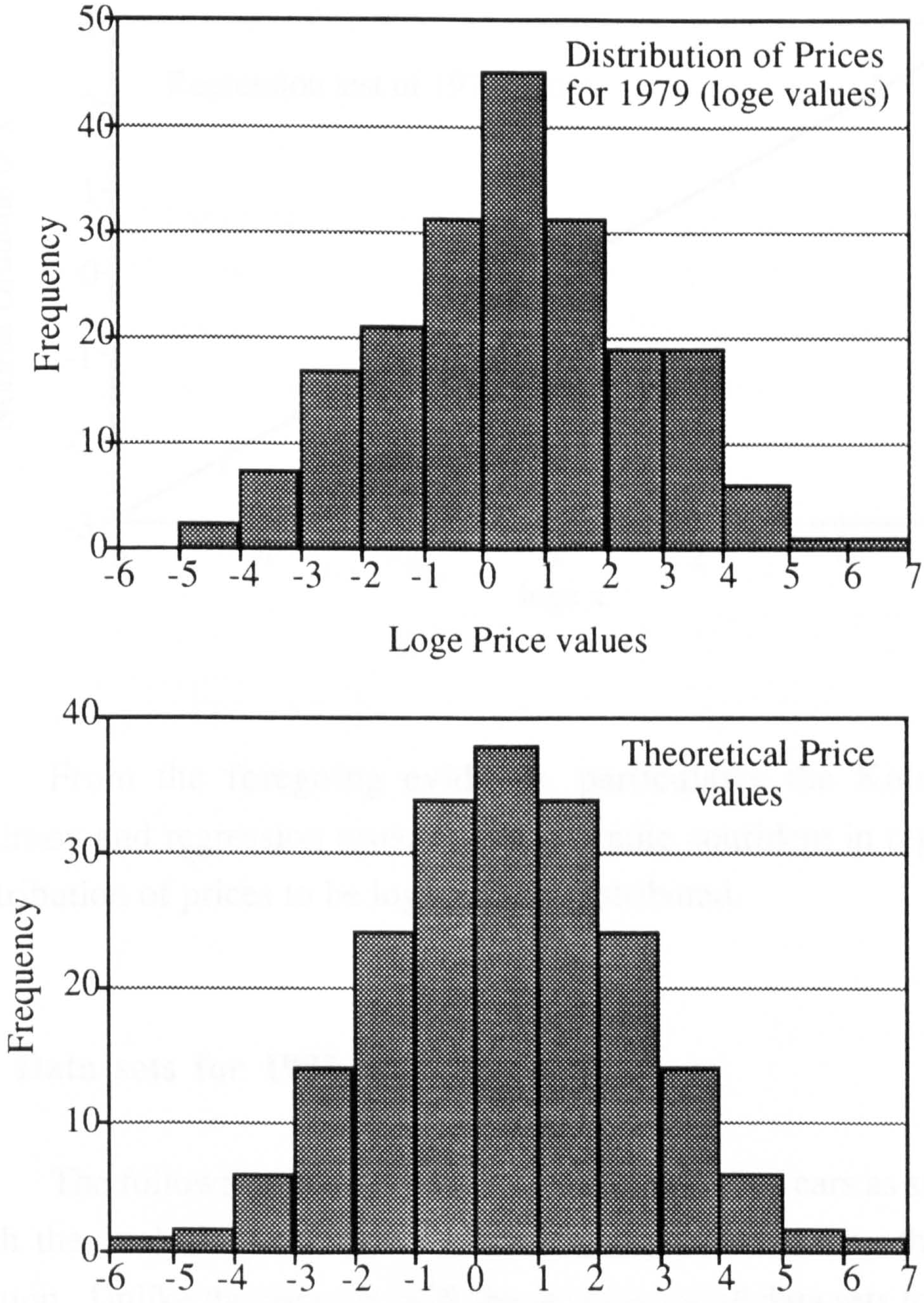
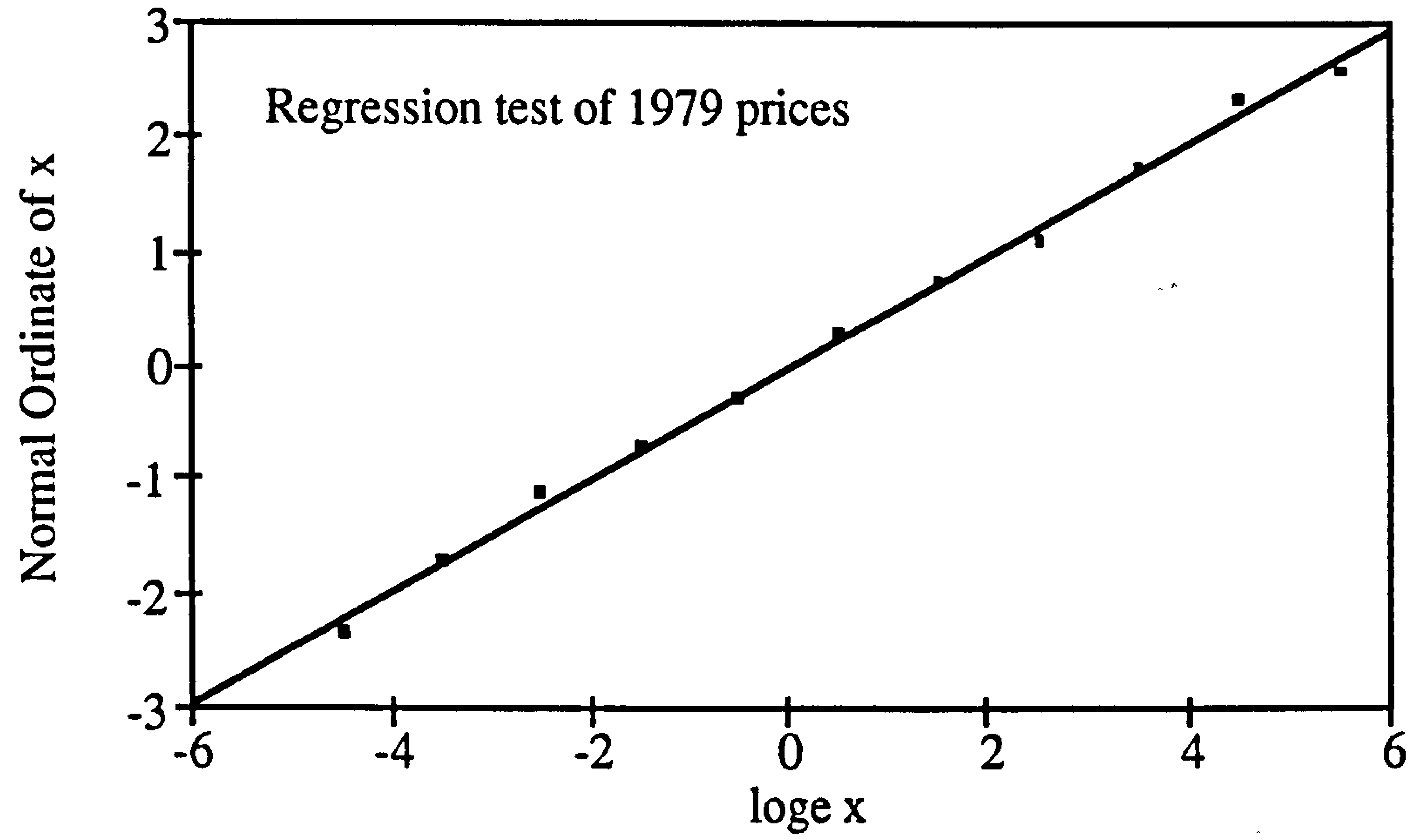

Distribution of Prices for 1979 (loge values)
Frequency
0
10
20
30
40
50
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
Loge Price values
Theoretical Price values
Frequency
0
10
20
30
40
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
Theoretical Prices (loge values)

Figure 6.16



From the foregoing evidence, particularly the Kolmogorov - Smirnov and regression tests, we can be quite confident in regarding the distribution of prices to be lognormally distributed.

6.5 Data sets for 1983 and 1985

The following data sets and analysis are for the years as stated above with the analysis presented in a more compact form than the previous section. Unlike the data for 1979 the analysis for all data sets for 1983 and 1985 were based on sample data because the convenient ABC listings were not available to the author for these periods. The company had during the period 1982-1983 switched from an IBM system 34 to an IBM system 38 and some standard routines were not converted to run on the larger mini system. The ABC listing routine was one of them because it was not in regular use by management. However, this author had free access to the demand history files, master price files and stock detail reports (the current stock status reports). In general data analysis was based on sample sizes of 200 selected by a simple random number process from the appropriate reports and listings.

6.5 (a) Period usage values

Period usage values were obtained for approximately 200 parts for each of the two years by sampling from the parts demand history file then applying the appropriate part price from either the master price file or stock detail report. The usage value distribution for 1983 follows -

Table 6.8

1983 Usage Value Distribution

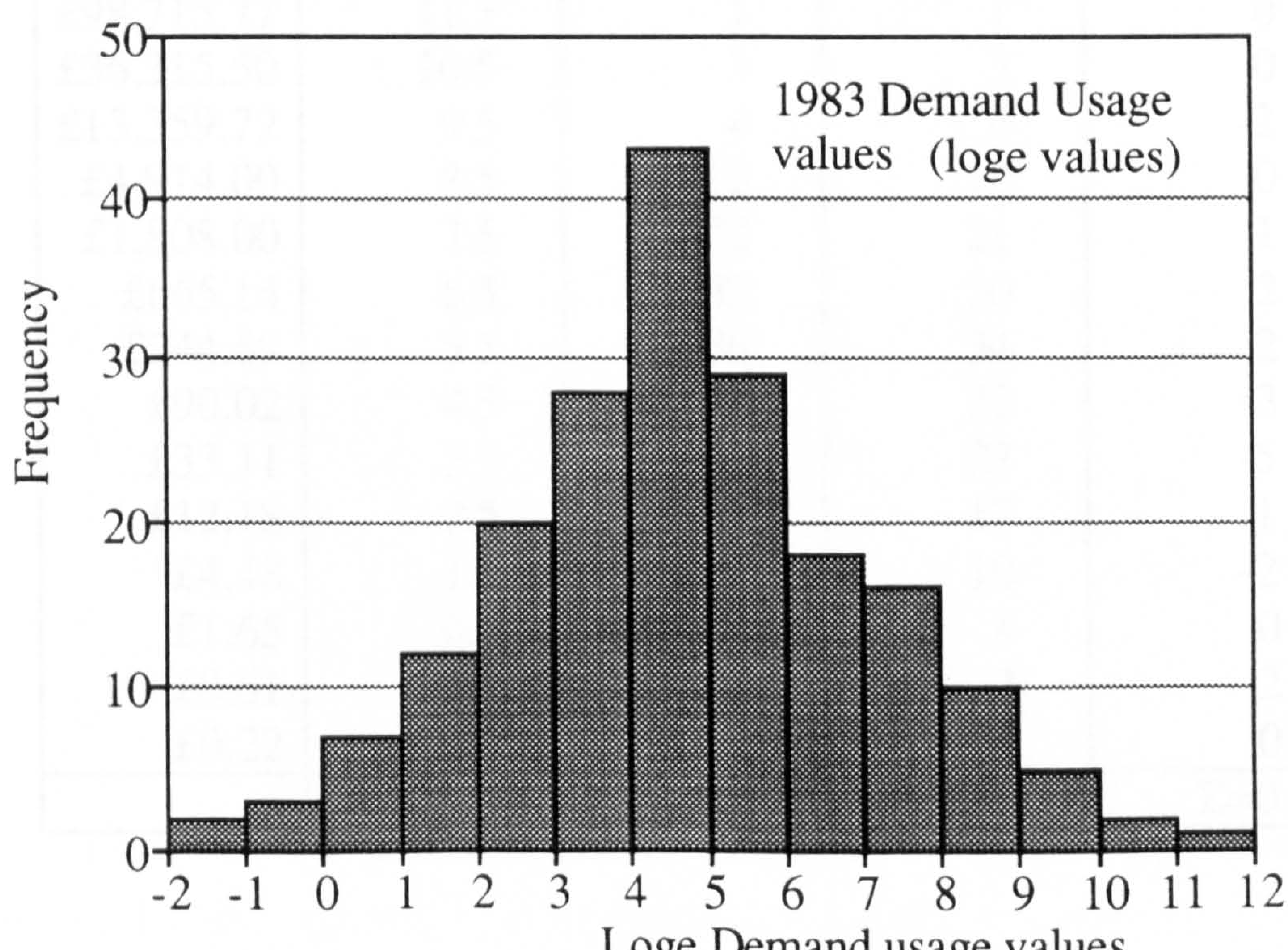
Usage value Cell mid pt.	loge Usage value	Frequency of items	Theoretical frequency	Difference
£98,715.77	11.5	1	1	0
£36,315.50	10.5	2	2	0
£13,359.72	9.5	5	3	2
£4,914.00	8.5	10	6	4
£1,808.00	7.5	16	13	3
£665.14	6.5	18	20	-2
£244.69	5.5	29	28	1
£90.02	4.5	43	31	12
£33.11	3.5	28	31	-3
£12.18	2.5	20	26	-6
£4.48	1.5	12	16	-4
£1.65	0.5	7	11	-4
£0.61	-0.5	3	5	-2
£0.22	-1.5	2	2	0
		n= 196	n=195	Σ=1

The mean and standard deviation of the empirical distribution were used to calculate the corresponding theoretical frequencies in the above table.

The differences in the above table between the actual and theoretical frequencies gave an actual Chi Squared value of 15.71 compared to theoretical Chi Squared values of 16.91 and 23.309 at the 1% and 5% levels of significance respectively (for 12-1-2 degrees of freedom). The distribution also passed the Kolmogorov Smirnov test at both 1% and 5% levels of significance respectively. The typical 'normal curve' symmetry of the data is clearly seen by the following histogram. (Although the distribution is somewhat more peaked in the model cell than a corresponding theoretical distribution it is nevertheless well within sampling error).

It is of interest to note here that in this case the sample passed the Chi Squared test, whereas the 1979 usage value distribution based on 9,100 items did not, despite the fact that is was far more symmetrical and normal in form than the above distribution for 1983. This comparison demonstrates how over-sensitive the Chi Squared test can be to small deviations in very large samples.

Figure 6.17



A regression test of $N(x)$ against $\log_e x$ for this same price distribution gave the following results-

r = 0.999

R^2=0.998

Standard error = 0.067

Durbin Watson test result = 1.254

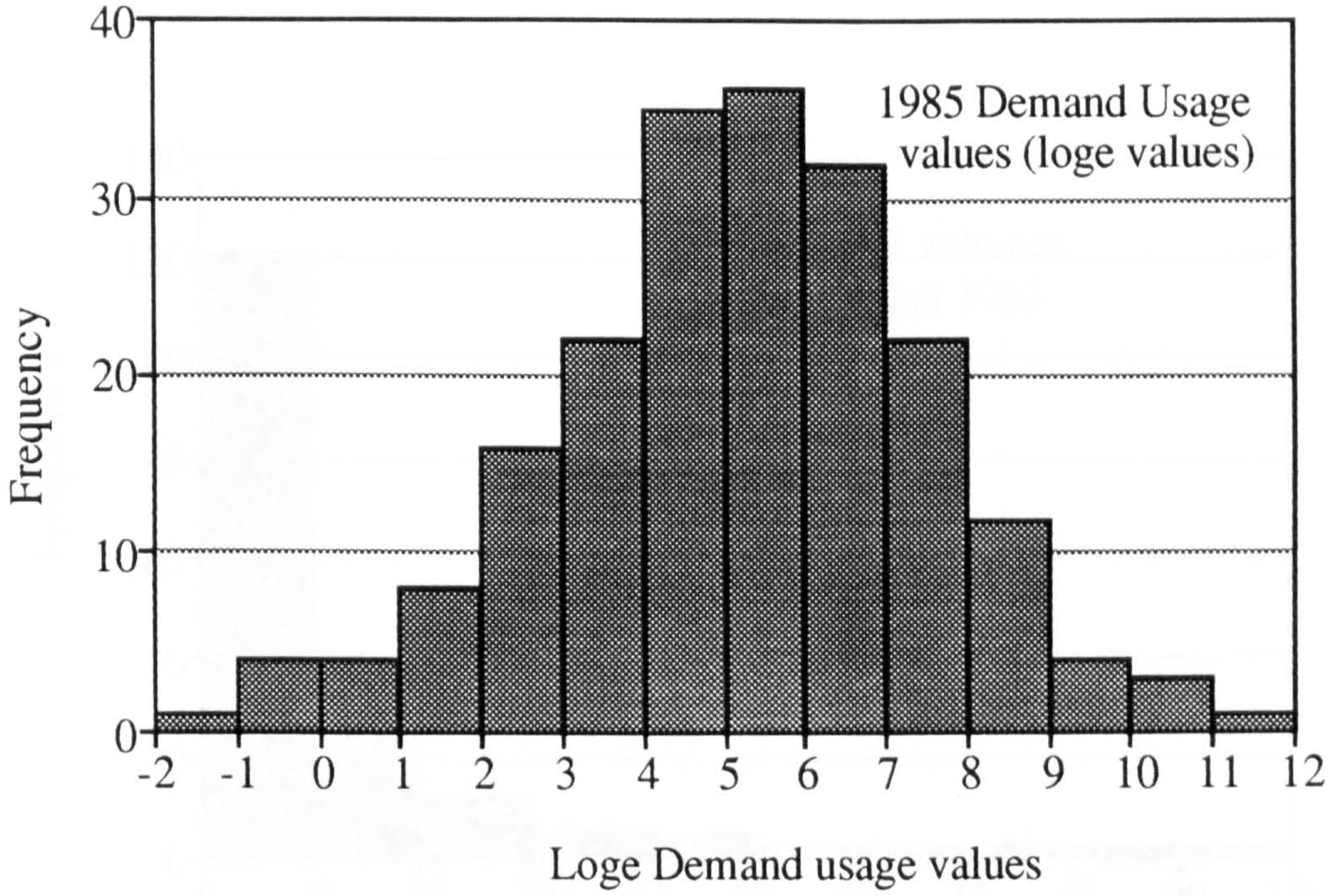
The corresponding usage value analysis for 1985 gave the following tabulation and histogram-

Table 6.9

1985 Usage Values

Usage value Cell mid pt.	loge Usage value	Frequency of items	Theoretical frequency	Difference
£98,715.77	11.5	1	1	0
£36,315.50	10.5	3	3	0
£13,359.72	9.5	4	6	-2
£4,914.00	8.5	12	12	0
£1,808.00	7.5	22	21	1
£665.14	6.5	32	30	2
£244.69	5.5	36	34	2
£90.02	4.5	35	32	3
£33.11	3.5	22	27	-5
£12.18	2.5	16	17	-1
£4.48	1.5	8	10	-2
£1.65	0.5	4	4	0
£0.61	-0.5	4	2	2
£0.22	-1.5	1	1	0
		n=200	n=200	Σ=0

Figure 6.18



In this case we obtained an actual Chi Squared of 3.697 compared with theoretical values at 1% and 5% levels of significance respectively at 16.91 and 23.309. A regression test gave -

r = 0.999

R^2=0.997

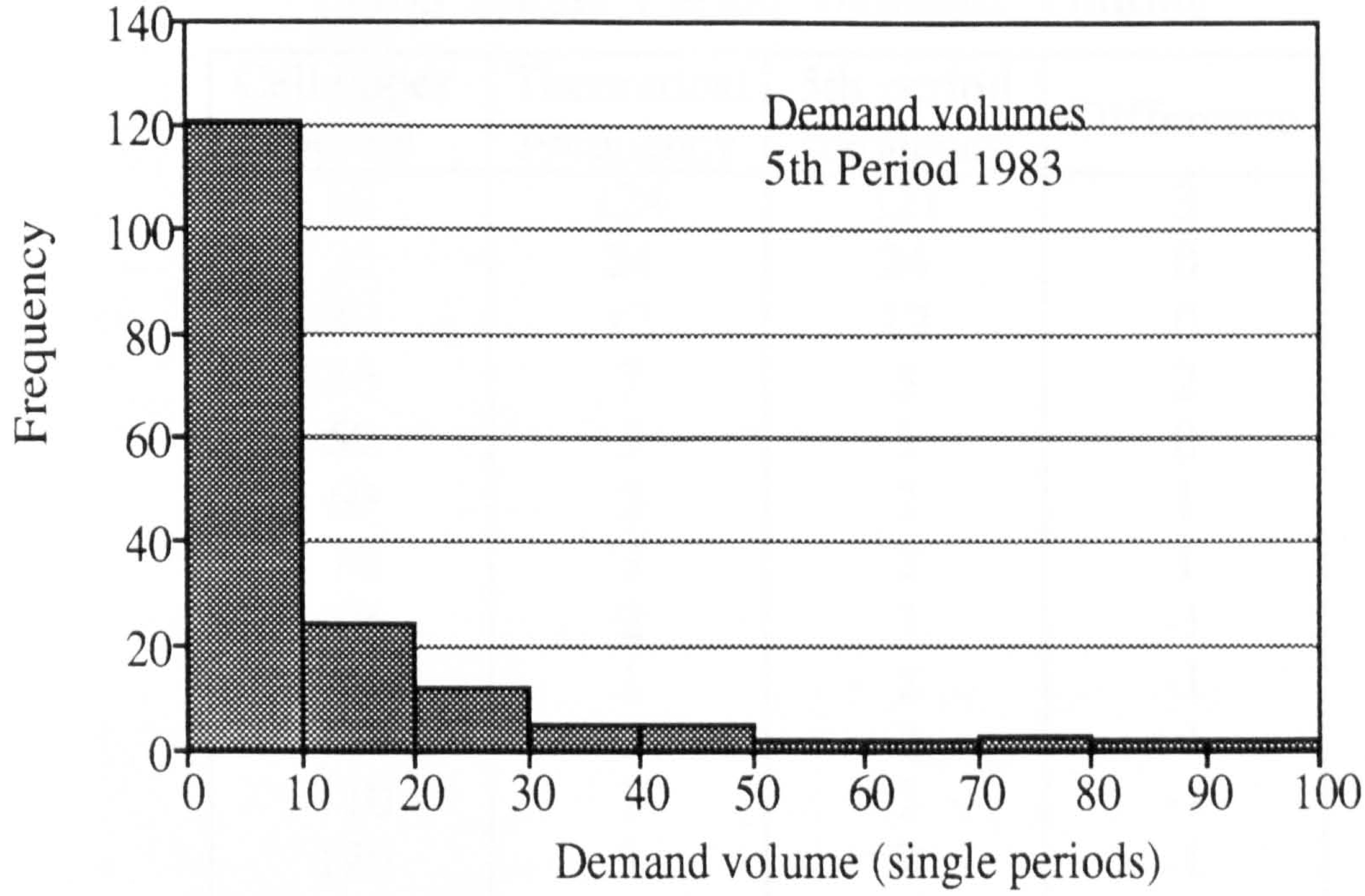
Standard error =0.091

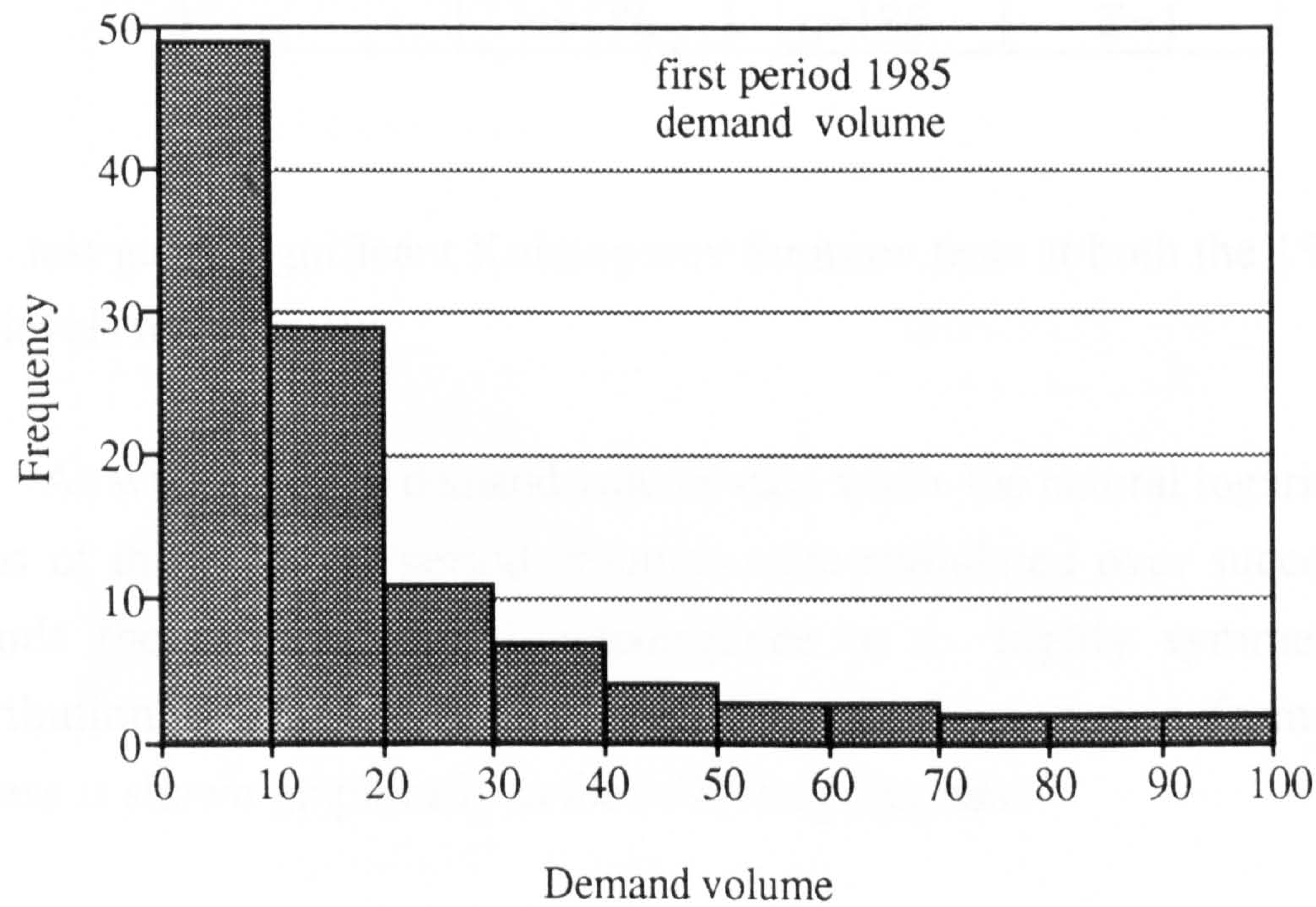
DW test = 1.148

6.5 (b) Period volumes

The single period aggregate volumes for any selected period during 1983 and 1985 show the the same characteristic reverse 'J' shape distribution as for the 1979 data. A single period selected at random from both 1983 and 1985 showed the following histogram forms-

figure 6.19





The 5th period of 1983 was compared with a theoretical log series distribution with the same parameter value (q = 0.987) as shown below-

Table 6.10

1983 Single Period Demand Volume

Cell upper bound	Theoretical Frequency	5th period frequency	Difference
10	124	121	3
20	24	24	0
30	12	12	0
40	7	5	2
50	5	5	0
60	3	2	1
70	3	2	1
80	2	3	-1
90	1	2	-1
100	1	2	-1
110	1	2	-1
120	1	2	-1
130	1	1	0
140	1	1	0
150	0	1	-1
	n=186	n=185	Σ=1

This test gave significant Kolmogorov Smirnov tests at both the 1% and 5% levels respectively.

As with the 1979 demand volume data when the natural logarithmic forms of these single period volumes were cumulated over successive periods the same gradual convergence to a highly symmetrical distribution was obtained, that proved to be lognormal in form. The process is shown graphically in the following diagrams-

figure 6.20

Cumulation of 1983 Demand Volumes

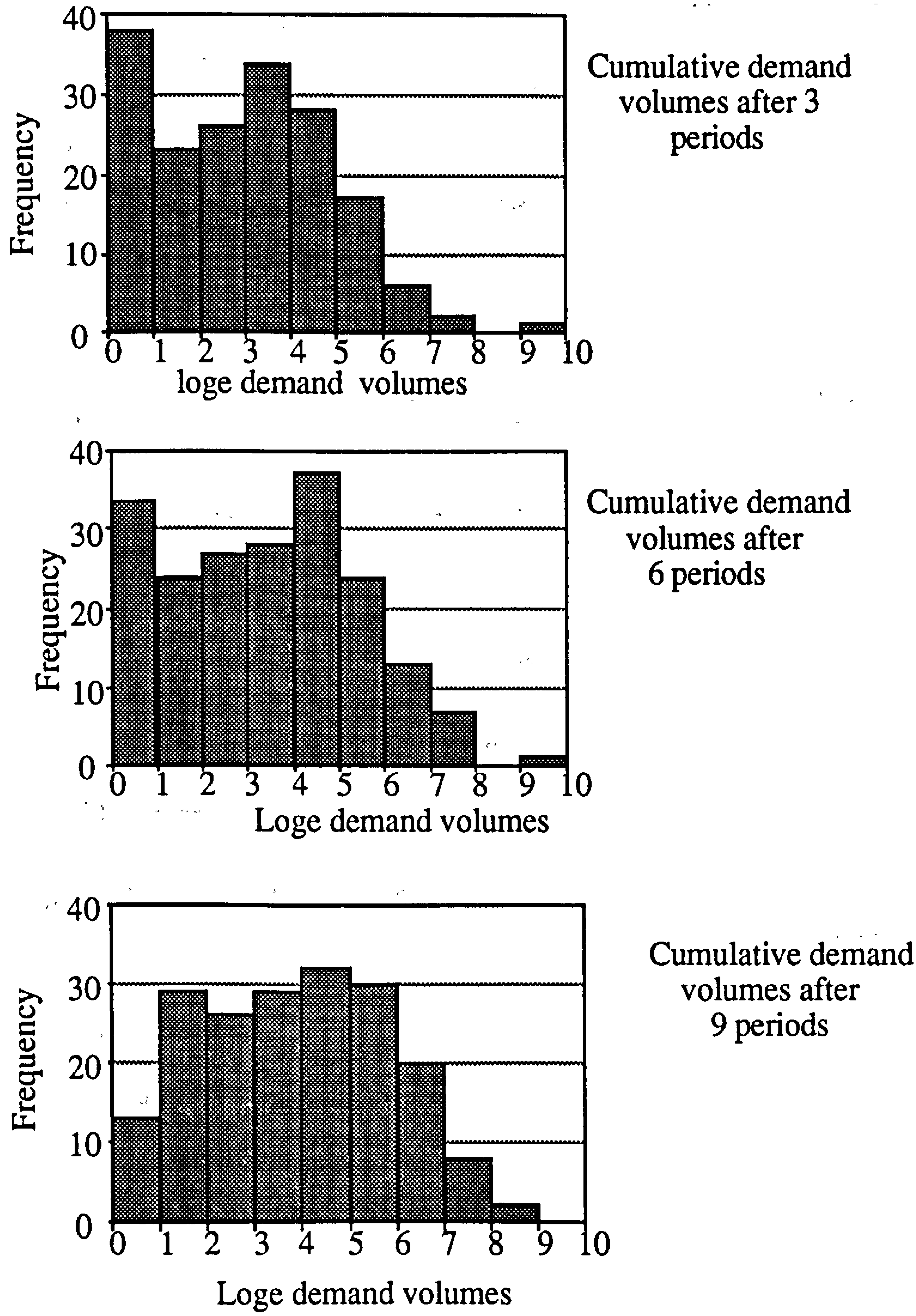
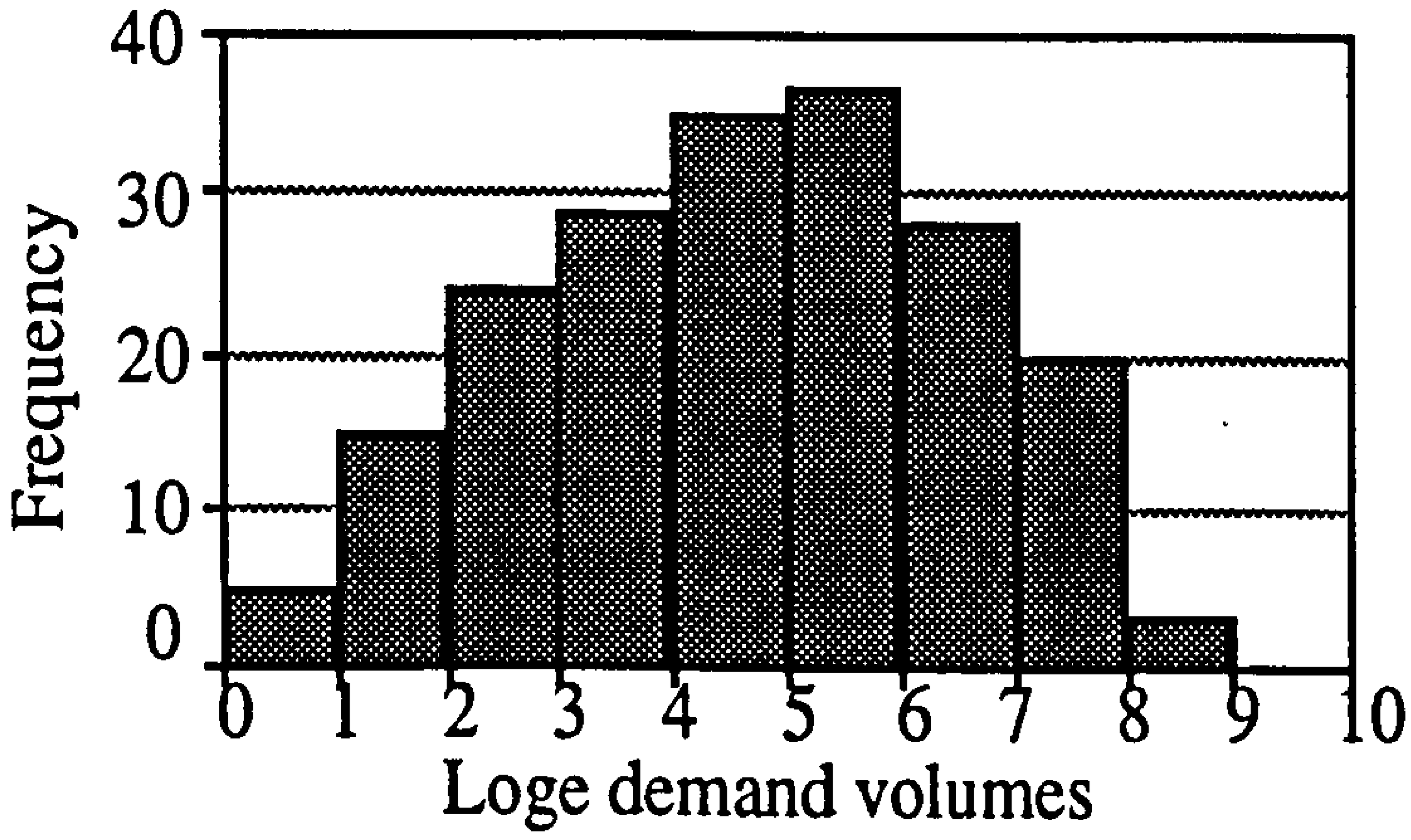
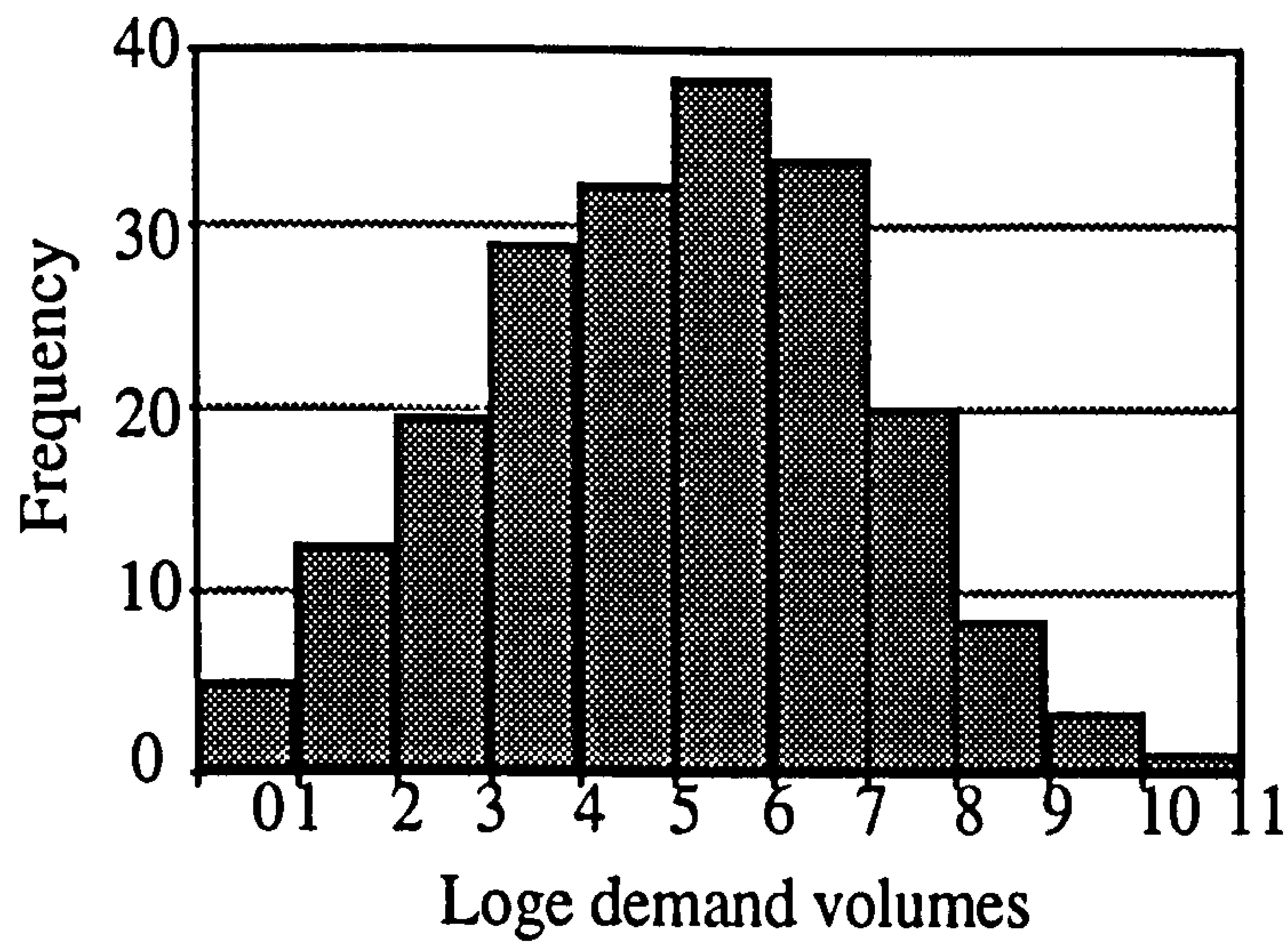


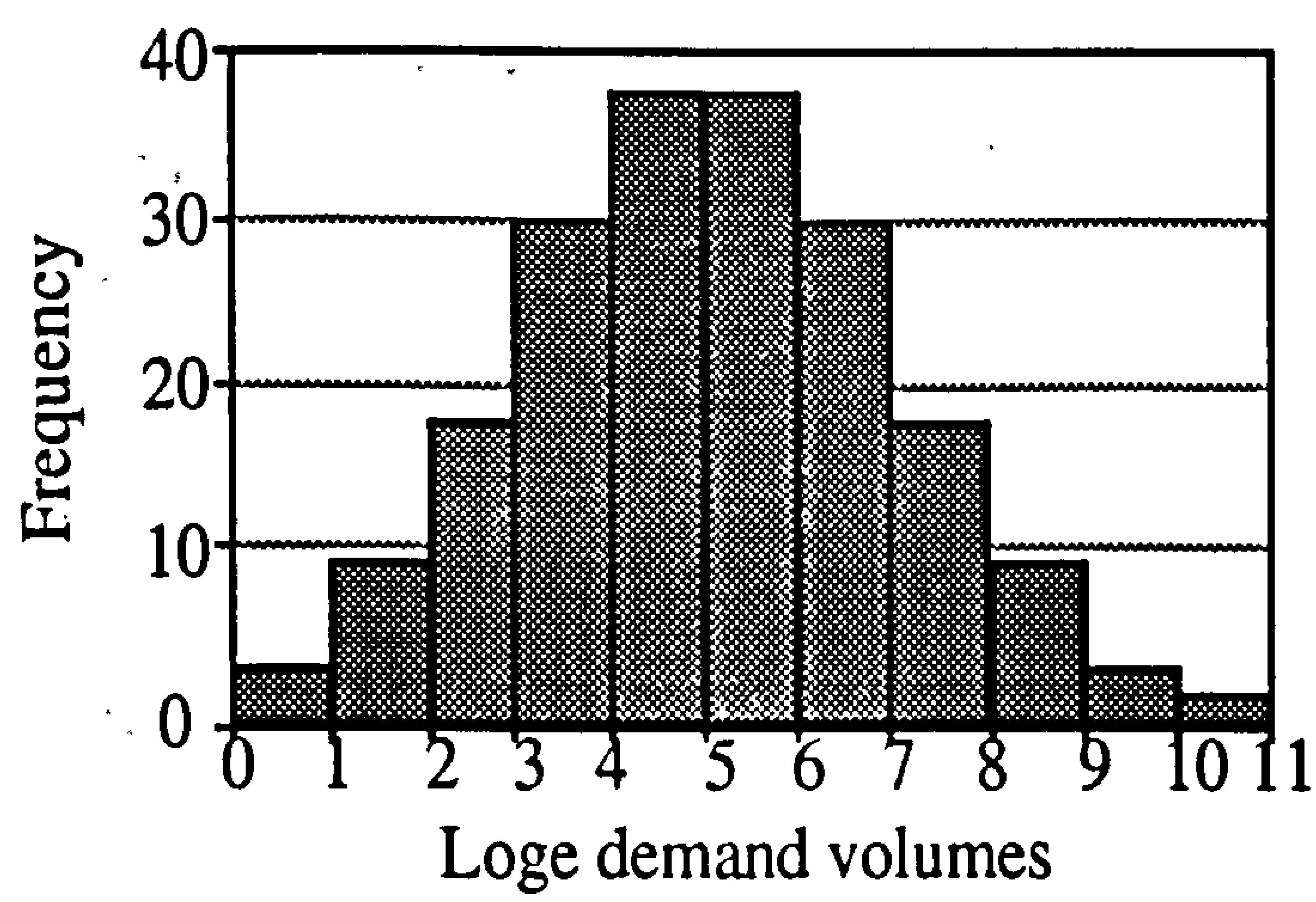
figure 6.20 continued



Cumulative demand volumes after 12 periods



Cumulative demand volumes after 15 periods



Theoretical distribution for 15 periods

The comparison of the corresponding theoretical normal distribution with the empirical distribution after 15 periods is shown below-

Table 6.11

1983 Whole Year Demand Volumes

loge value range	Actual distribution	Theoretical distribution	Difference
0	0	1	-1
1	5	3	2
2	12	9	3
3	19	18	1
4	29	30	-1
5	31	38	-7
6	38	38	0
7	34	30	4
8	20	18	2
9	8	9	-1
10	3	3	0
11	1	1	0
	n=200	n=198	Σ=2

This analysis gave an actual Chi Squared value of 6.26 which is statistically significant at both the 1% and 5% levels. Hence based on this test we have no reason to reject the null hypothesis that the empirical distribution is from a normal population.

A similar pattern of convergence was also seen for the 1985 demand volumes as shown in the following histograms

Figure 6.21

Cumulation of 1985 Demand Volumes

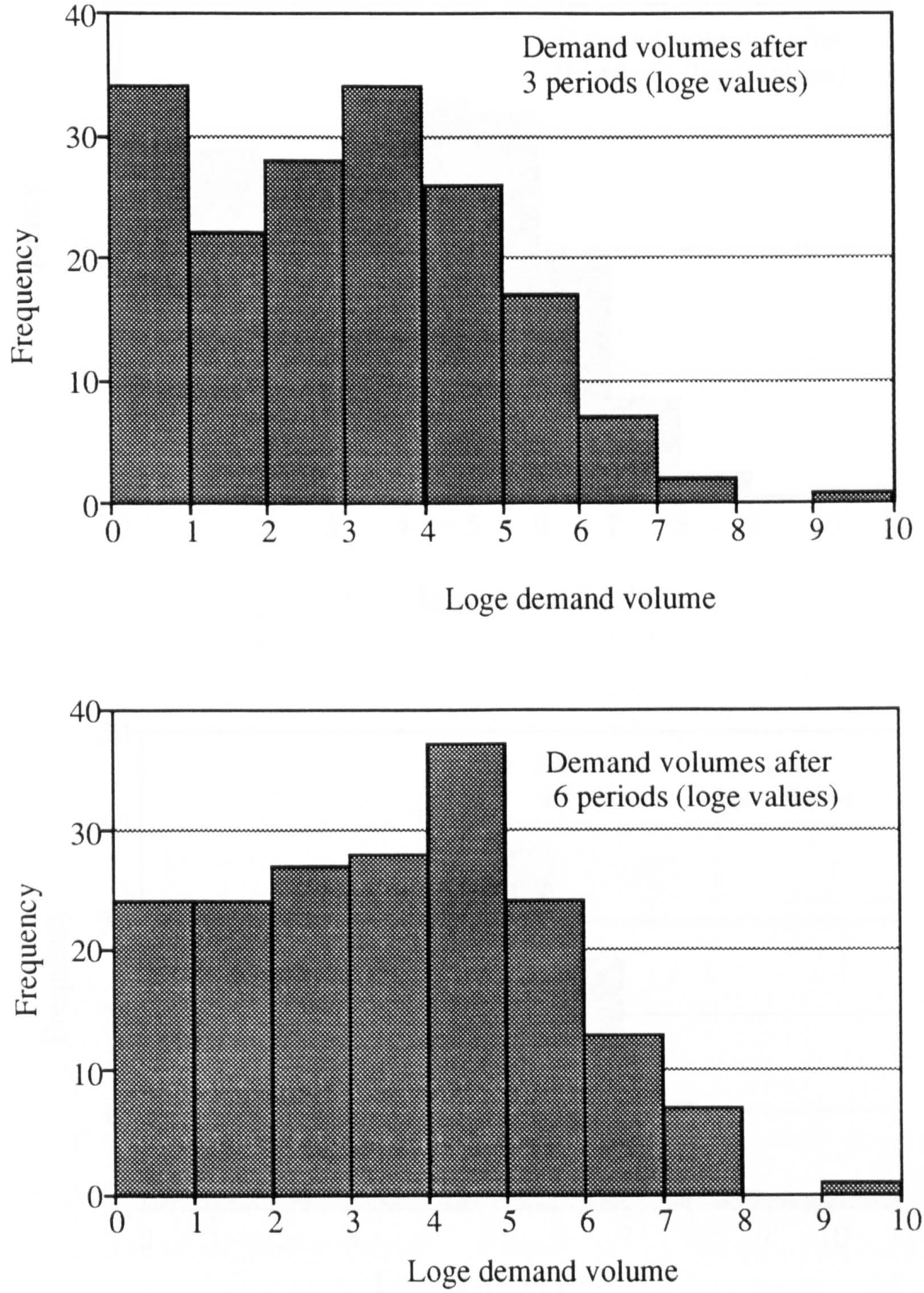
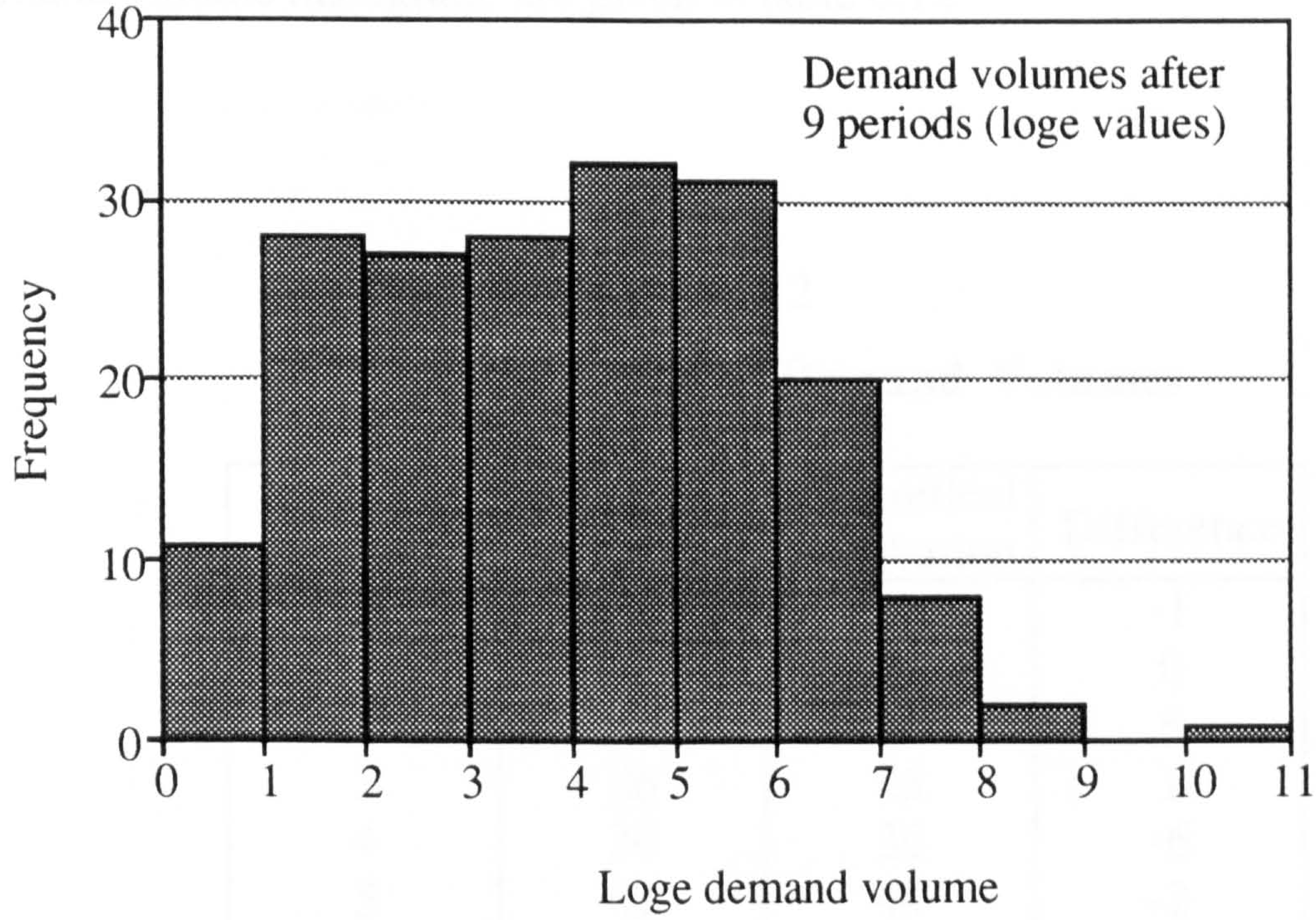
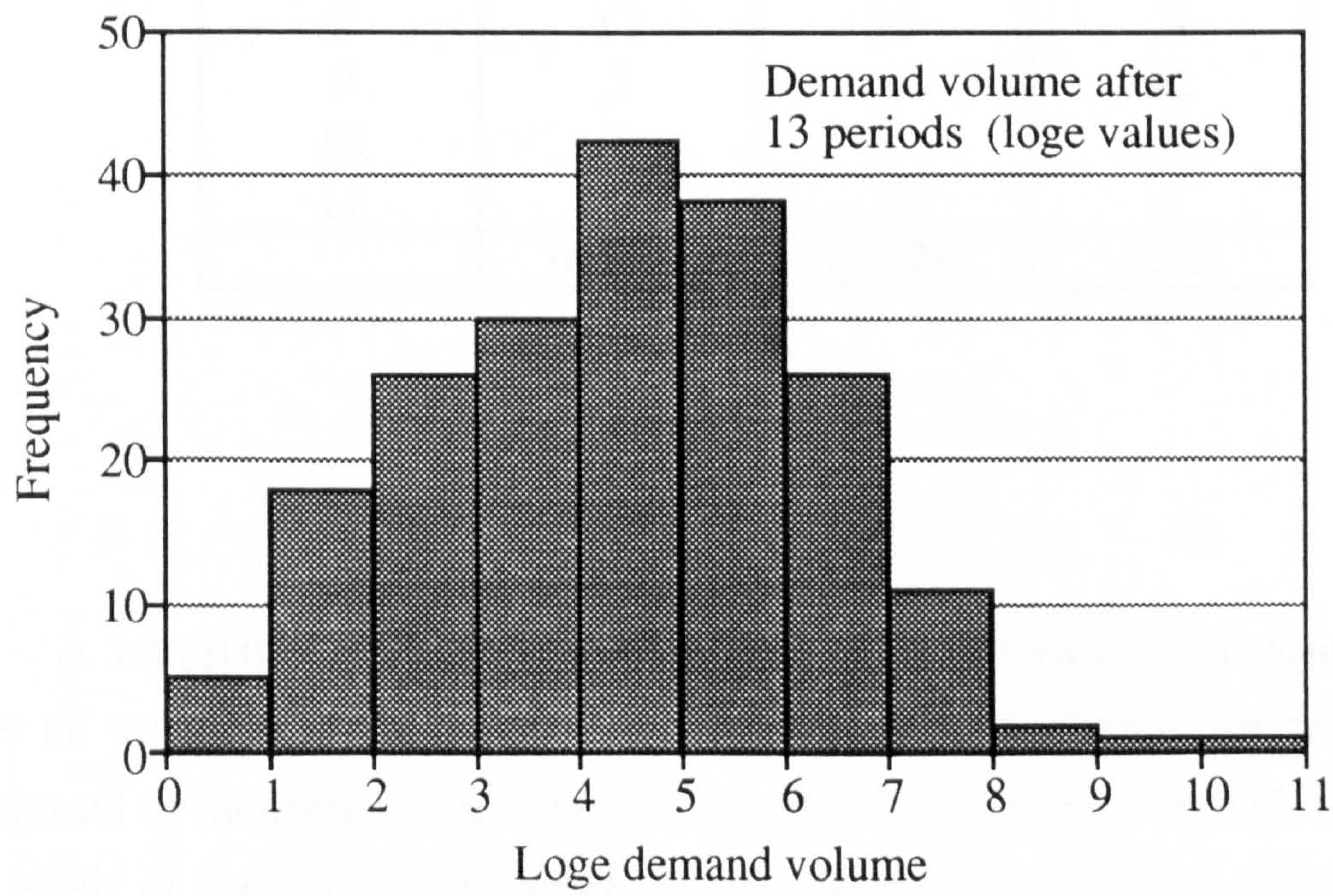


figure 6.21 (continued)





The actual and corresponding theoretical frequencies for the 13 period demand volume histogram are given in table 6.12

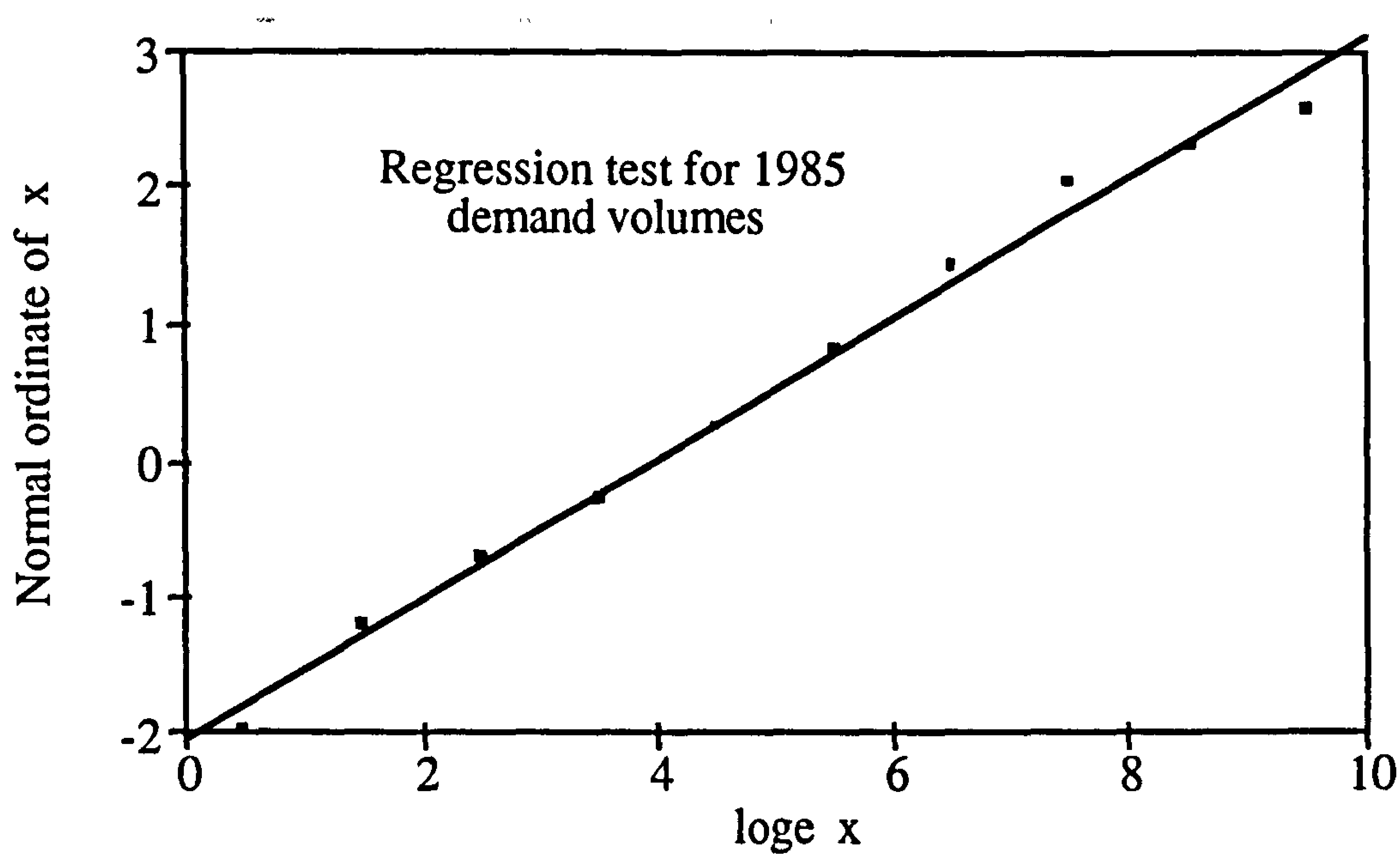
Table 6.12

1985 Whole Year Demand Volumes

loge value range	Actual distribution	Theoretical distribution	Difference
0	0	1	-1
1	5	5	0
2	18	12	6
3	26	25	1
4	30	36	-6
5	42	44	-2
6	38	35	3
7	26	24	2
8	11	11	0
9	2	4	-2
10	1	1	0
11	1	0	1
	n=200	n=198	Σ=2

A comparison of the above theoretical and empirical frequencies gave an actual Chi Squared value of 5.58 at 9-1-2 degrees of freedom compared to theoretical Chi Squared values of 12.592 and 16.812 1% and 5% levels of significance respectively. A regression test on this same 1985 demand volume data gave the following results-

Figure 6.22



r $= 0.996$

$R^2 = 0.991$

Durbin Watson test = 1.433

6.5 (c) Distribution of Prices for 1983 and 1985

The corresponding prices for a 200 item sample taken from the 1983 and 1985 stock detail reports are tabulated below with the corresponding theoretical frequencies. The data is also shown in histogram form.

table 6.13
1983 Price Distribution

loge value range	Actual distribution	Theoretical distribution	Difference
-5	0	1	-1
-4	2	3	-1
-3	8	7	1
-2	18	15	3
-1	21	24	-3
0	28	34	-6
1	44	38	6
2	36	34	2
3	25	24	1
4	17	15	2
5	7	7	0
6	0	3	-3
7	0	1	1
	n=206	n=206	Σ=2

The actual value of Chi Squared value from the above tabulation was 11.22 compared with the theoretical values Chi Squared values at 0.01 and 0.05 levels of significance of 16.91 and 21.67 respectively. The regression of $N(x)$ against $\log_e x$ for the same data gave -

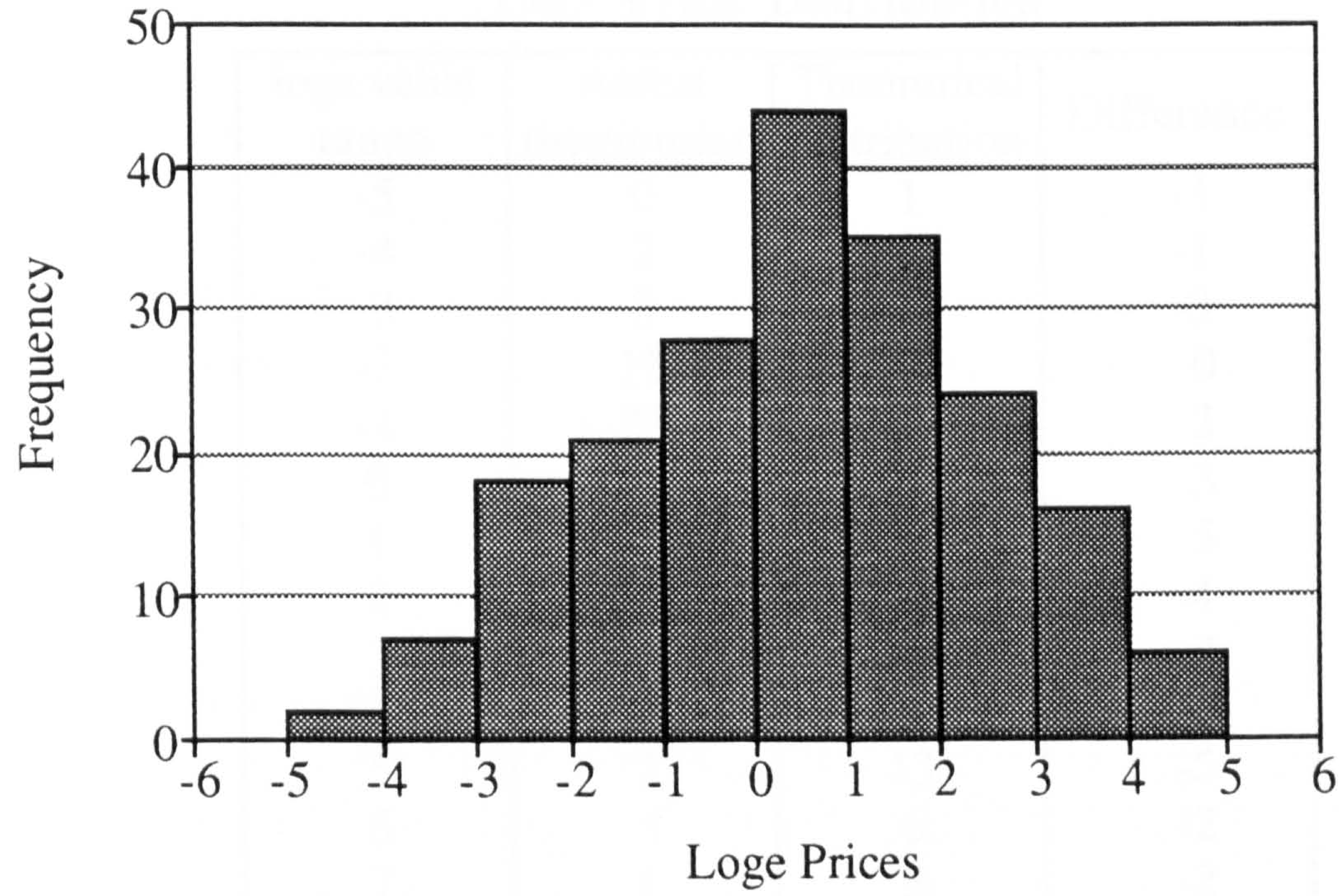
r = 0.998

R^2=99.6%

SE =0.090

DW test = 1.934

Figure 6.23
1983 Log Prices distribution



The corresponding tests for 1985 gave the following results :-

Figure 6.24
1985 Log Prices Distribution

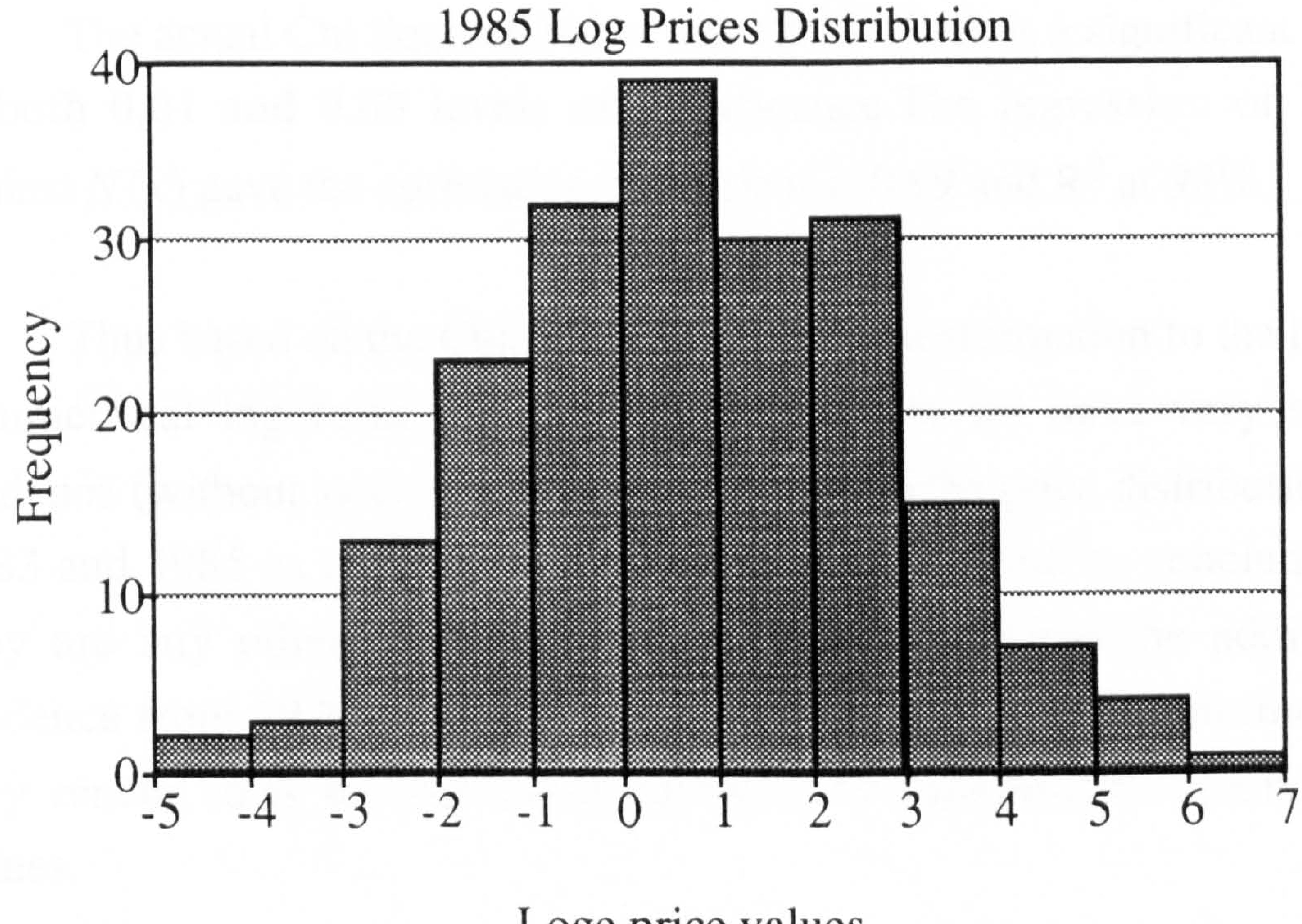


Table 6.14
1985 Price Distribution

loge value range	Actual distribution	Theoretical distribution	Difference
-5	0	1	-1
-4	2	3	-1
-3	3	6	-3
-2	13	13	0
-1	23	20	3
0	32	29	3
1	39	34	5
2	30	34	-4
3	31	29	2
4	26	20	6
5	7	13	-5
6	4	6	-2
7	1	3	-2
8	0	1	-1
	n=211	n=212	Σ=0

The actual Chi Squared value was 12.06 which is a significant value at both 0.01 and 0.05 levels of significance.The regression of $\log_e x$ against $N(x)$ gave the correlation coefficient at 0.99 and R^2 at 98%.

Thus based on the Chi squared test, the transformation to the highly symmetrical log form and the regression tests we have very strong evidence (without stochastical support) to regard the price distributions of 1983 and 1985 as lognormal. It would be very difficult to conclude that they are any other form given the evidence here and the additional evidence from 1979. It would appear then that prices are lognormal and they remain so as the stable long run model of this variable of parts usage values.

6.6 Eleven year summary of usage values

In table 6.15 below we give a summary of the parameters of the lognormal distribution of usage values for the period 1975 to 1985, together with the growth in the number of parts in the active parts range in each year. ie n was the total number of parts with a demand of one or more in the year shown. The parameter ρ is related directly to σ by the relationship $\rho = e^{\sigma}$.

Table 6.15

Eleven Year Usage Value Distribution Parameters

Year	1975	1976	1977	1978	1979	1980
σ	2.491	2.485	2.215	2.338	2.480	2.501
μ	2.564	2.904	3.105	3.310	3.510	3.897
ρ	12.070	11.680	9.160	10.360	11.940	12.190
n	7,111	7,695	7,836	8,547	9,100	7,853

Year	1981	1982	1983	1984	1985
σ	*	*	2.395	2.277	2.313
μ	*	*	4.690	4.825	5.203
ρ			10.968	9.747	10.105
n	9,250	9,870	10,627	11,250	11,625

* in the table above indicates data not collected and measured.

For the period 1975 until 1980 the parameters above were calculated from whole population data using ABC listings of the complete years parts sales. From 1983 onwards the parameters were determined from sample data from various company sources. Stock detail reports and price files for price data , and parts demand history files for sales volume data. (samples sizes were always 200 in each case). Hence these estimates are subect to sampling error which can be quantified at a given confidence level by use of the standard error of the sampling distribution of σ for large samples as shown in the next section.

6.7 Dealer Level Usage Values.

Naturally, given the fact that usage values (likewise volumes and prices) are clearly lognormal at the wholesale (distributor) level in the DAF distribution system it was considered important to see the form of the usage value distributions at the dealer level. In 1980 DAF had 24 dealers covering the entire UK sales and marketing operation including Northern Eireland. Five dealers were chosen as a representative sample and from each was gathered a random sample of of 200 parts usage values. The dealers and their geographical area were -

Moreys	Southern Counties.
Northern Commercials	Humberside and North Yorkshire.
Harris Commercials	Essex region and East London
Midland DAF Trucks	Birmingham and Midland region
North West Trucks	North Lancs. and Cumbria

Each was a fairly typical exclusive DAF dealerships wholly concerned with the sales and marketing of the entire DAF Trucks UK range of commercial vehicles for a designated geographical area. Each was also a fully equipped service dealer trained and competent to provide the complete range of after sales service with spares stockholding and parts marketing. About 60% of the full active parts range was normally carried in stock in each dealership (based largely on sales volume terms). All five dealers had computerised stock control procedures in house running a

DAF developed stock control system called 'Autoparts' - a basic review period replenishment system. About 60% of all stock replenishments were made routinely on a phased two week order cycle with a different set of dealers placing orders at Marlow each week. The remainder of orders were telephoned in to the Parts department as 'dealer out of stock' or 'non stock' item order requests. A portion of these orders were further characterised as VOR orders (which literally meant that a customers vehicle was off the road pending repairs). VOR orders were executed immediately from the DAF HQ with the object of next day delivery to the dealer.

The results of the analysis of the five samples of dealer usage values gave lognormal distributions of usage rates with the parameter values as shown in table 6.16 :-

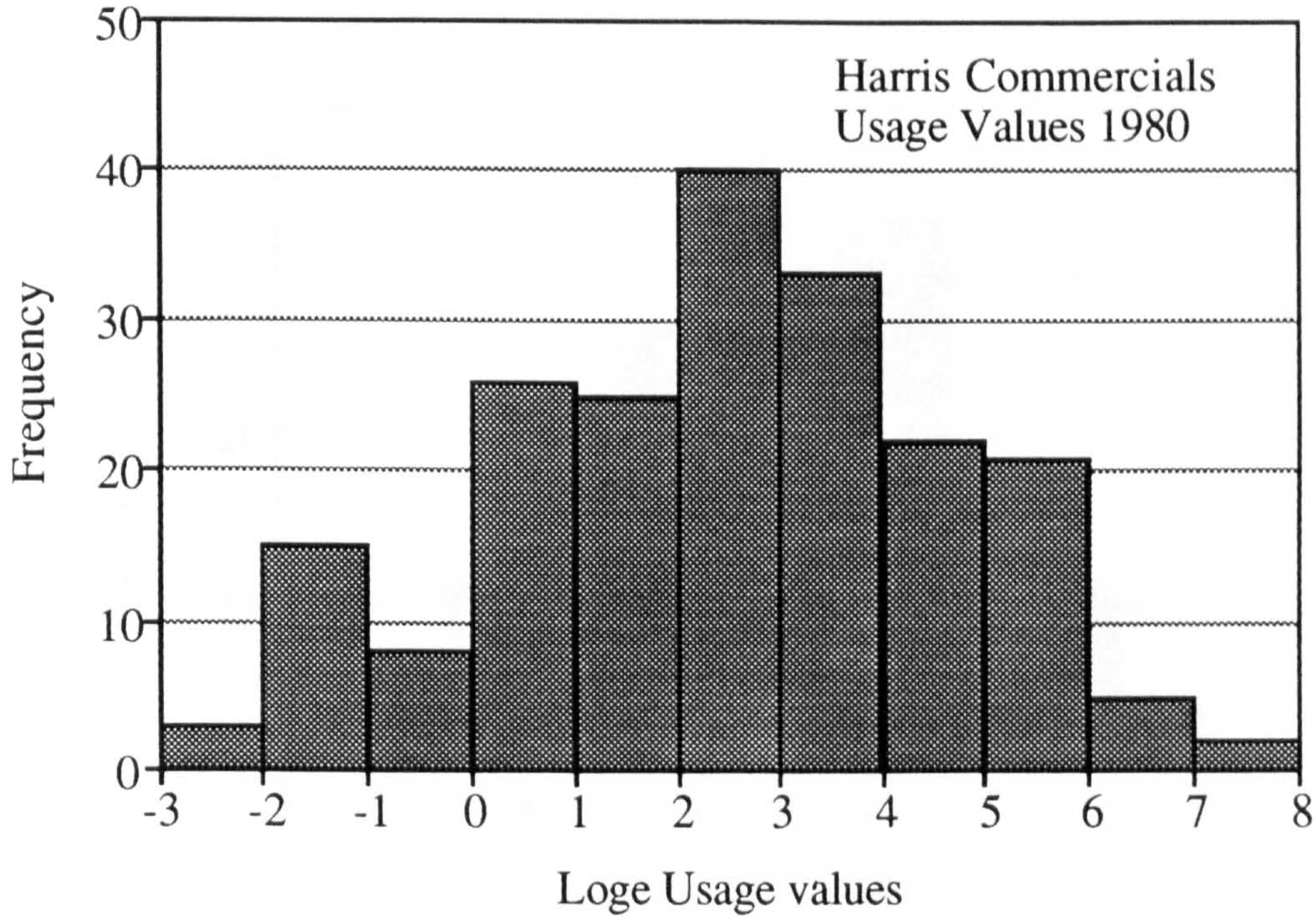
Table 6.16

$\log_e$ usage values distributions

Dealer	mu μ	sigma σ
Harris	2.458	2.166
Northern Commercials	2.928	2.316
Moreys	2.899	2.210
Midland	2.765	2.304
North West Trucks	2.278	2.288

The most striking feature of this data is the very close similarity between all the standard deviations of the normal distribution of $\log_e$ parts usage values. Furthermore each sample produced the characteristic normal curve of the $\log_e$ values. The histograms for Moreys (the best in terms of symmetry), Northern Commercials, and Harris (the worst in terms of shape) are shown below. All three are very significant as normal distributions based on Chi Squared and Kolmogorov Smirnov test criteria.

Figure 6.25


Harris Commercials
Usage Values 1980
Frequency
50
40
30
20
10
0
-3
-2
-1
0
1
2
3
4
5
6
7
8
Loge Usage values

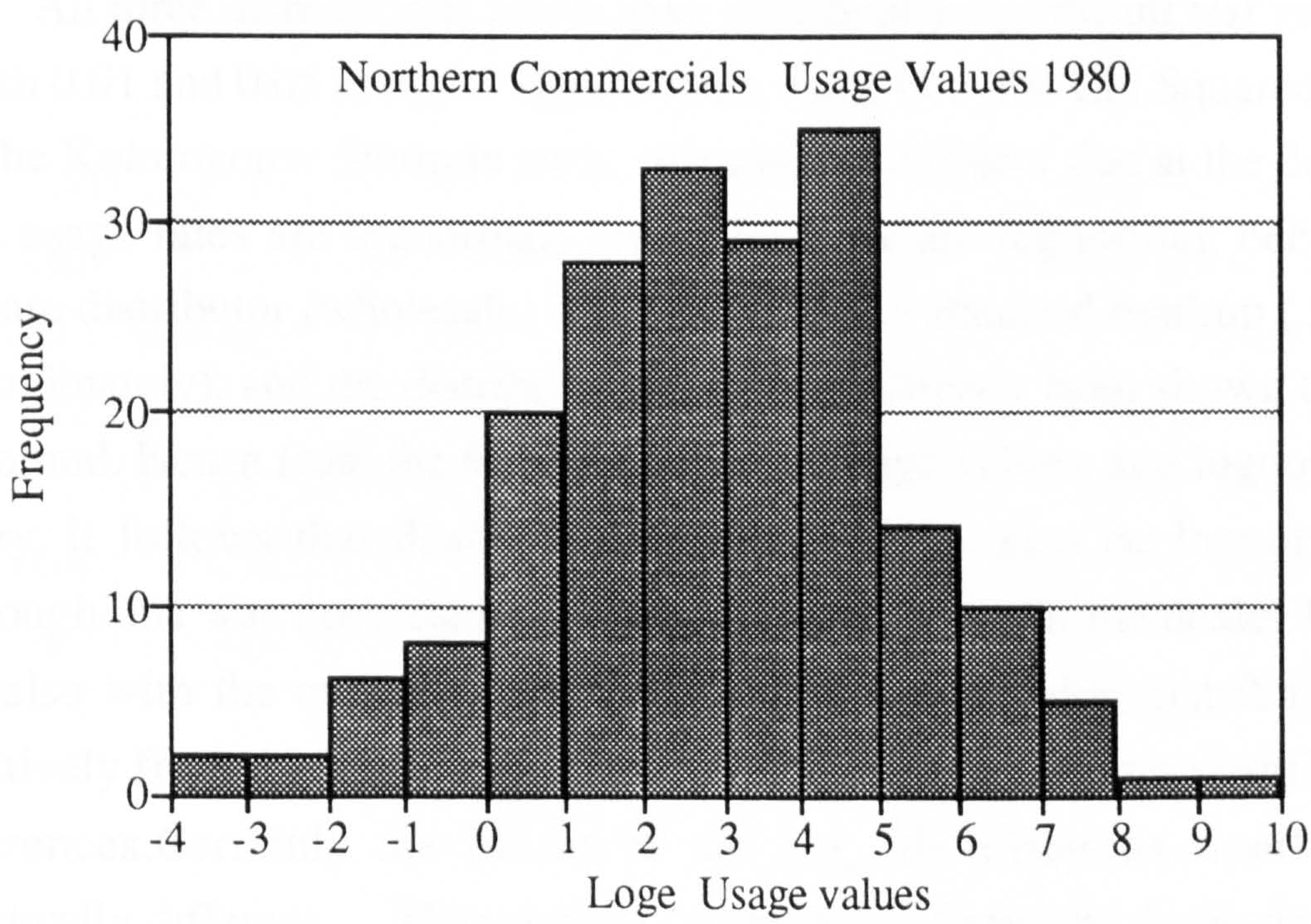
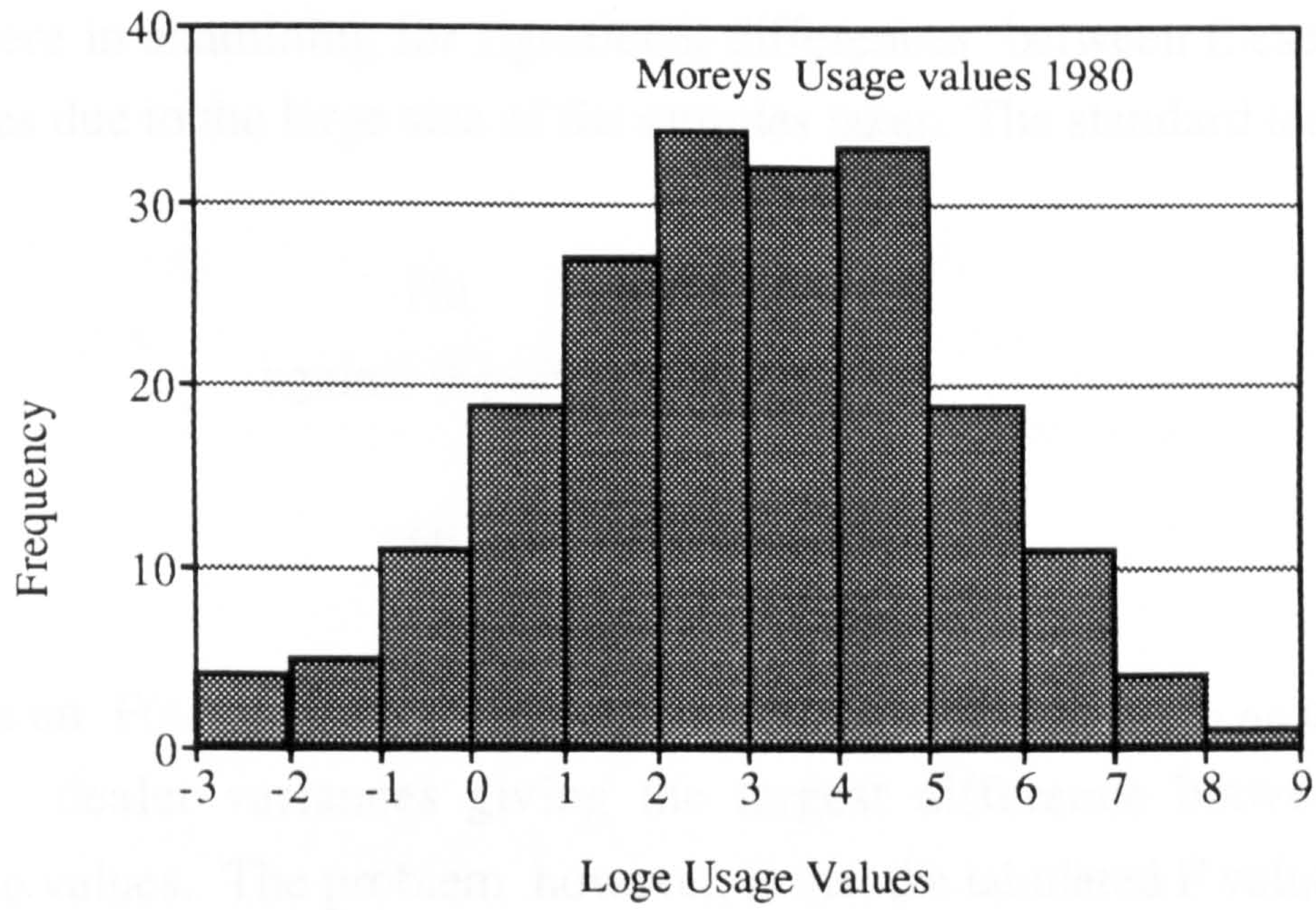

Northern Commercials Usage Values 1980
Frequency
40
30
20
10
0
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
Loge Usage values

Figure 6.25 (continued)



All three distributions above gave statistically significant test values at both 0.01 and 0.05 levels of significance, using both the Chi Squared test and the Kolmogorov Smirnov tests. We can be confident that at the dealer level usage rates are lognormal. Prices certainly are lognormal; because they are distributor (wholesale) level prices with a standard markup (50% approximately), and the distributor prices have already been shown to be lognormal. Hence from the foregoing tests on usage values, and lognormal theory, it follows that dealer level volumes must also be lognormal, (although this was not tested directly). Our concern with the dealer level was also with the question, were these local usage value distributions effectively from the same population, or did they display some significant differences.Certainly the means of the $\log_e$ distributions should be statistically different. They depend on the level of sales by the particular dealer. Variances, however, as will be discussed later, should reflect some characteristics of the local market as they measure the balance or proportion of demand volumes across the range of items.

To compare the variances we turn to the standard statistical tests. The classical test for comparing sample variances relies on the use of the 'F' variance ratio distribution. However, this distribution does not help us much here in examining for significant differences between these dealer variances due to the large size of the samples taken. The standard test ie -

$$\text{Ho} \quad \sigma^2{}_1 = \sigma^2{}_2$$

against the alternative

$$\text{H1} \quad \sigma^2{}_1 \neq \sigma^2{}_2$$

gives us an F(actual) of 1.143 against a tabulated value of $F_{0.05}$ of unity for the dealer variances giving the largest difference between two variance values. The problem however, is that the tabulated F values at all levels of significance for a sample as large as 200 give F at unity. Hence strictly speaking the null hypothesis should be rejected with an F actual at 1.143. However, we have taken it here, given the large size of the sample, that F(actual) is sufficiently close to one for the test to be inconclusive. As a way round this problem we can make use of the confidence intervals of the sample standard deviations 's' as estimates of the underlying parent population standard deviation σ. From statistical theory it is known that provided the parent population is approximately normally distributed, then the 'sampling distribution' of 's' for large samples is normally distributed with mean s and standard deviation $s/\sqrt{2n}$. Hence we can assert with a probability of $(1-\alpha)$ that -

$$s - z_{\alpha/2}\frac{\sigma}{\sqrt{2n}} < \sigma < s + z_{\alpha/2}\frac{\sigma}{\sqrt{2n}}$$

which leads to -

$$\frac{s}{1+\frac{z_{\alpha/2}}{\sqrt{2n}}} < \sigma < \frac{s}{1-\frac{z_{\alpha/2}}{\sqrt{2n}}}$$

Where $z_{\alpha/2}$ is the standard normal deviate at significance level α.

Thus for a sample standard deviation of 2.166 (the smallest of the five dealers) and a sample size of 200 we have the following 95% confidence intervals around the underlying parent population standard deviation-

$$1.97 < \sigma < 2.401$$

For a sample standard deviation of 2.316 (the largest value of the five dealers) we obtain the following confidence intervals -

$$2.109 < \sigma < 2.567$$

Hence we can see at once there is substantial overlap between these confidence intervals. If we call the largest standard deviation s_L then there is only a probability of 0.025 that the true value of s_L will be smaller than 2.109 and the same probability that it will be larger than 2.567. The other three dealer standard deviations fall between the two extremes of 's' at 2.116 and 2.316 - all well within both confidence bands above. Hence we can conclude that there is a very high probability that all five dealer standard deviations are from the same underlying parent population, or certainly from local spare parts markets exhibiting very similar characteristics.

6.8 Conclusions

This long chapter has presented much of the empirical analysis undertaken at DAF Trucks UK Headquarters. In effect the work was spread over the period 1979 until 1985 and into the early part of 1986. It necessitated many visits to the company and a great deal of time consuming data collection and analysis was carried out. Much of the work was conducted on an iterative basis as the search for and acquisition of appropriate theory often suggested the lines of attack on the analysis of empirical data. Because of the extended time involved inevitably some repetition occurred in the analysis and few blind alleys were pursued before a consistent and reliable path of investigation emerged. From this work we are now able to summarise some strong conclusions that go a long way to proving the initial hypotheses of this research. In the next chapter we use this empirical analysis and the theory of chapters four and five to develop the new theory of chapter seven. Subsequent chapters concentrate on validating the theory.

(a) The usage values of DAF spare parts do indeed fit very closely to a lognormal distribution provided the time period for the measurement is at least nine months. After one year usage values fit a lognormal distribution to a very high degree as measured and tested by a variety of measures. Furthermore annual usage values at the company HQ were consistently lognormal over the period 1975 until 1986.

(b) We have also shown that prices are lognormal, and so are demand volumes provided the period is around one year or longer. Limited investigations strongly indicate that an equivalent picture exists at the dealer level, ie. demand volumes, usage values and item prices consistently fit a lognormal distribution.

From these two sets of conclusions we can confidently accept the working hypotheses four (in part) and five of chapter five page 160 as most likely correct.

(c) For short time periods the aggregate distribution of demand volumes is not lognormal and strong evidence exists to show that it is the Log Series distribution that models these short period aggregate demand volumes extremely well.

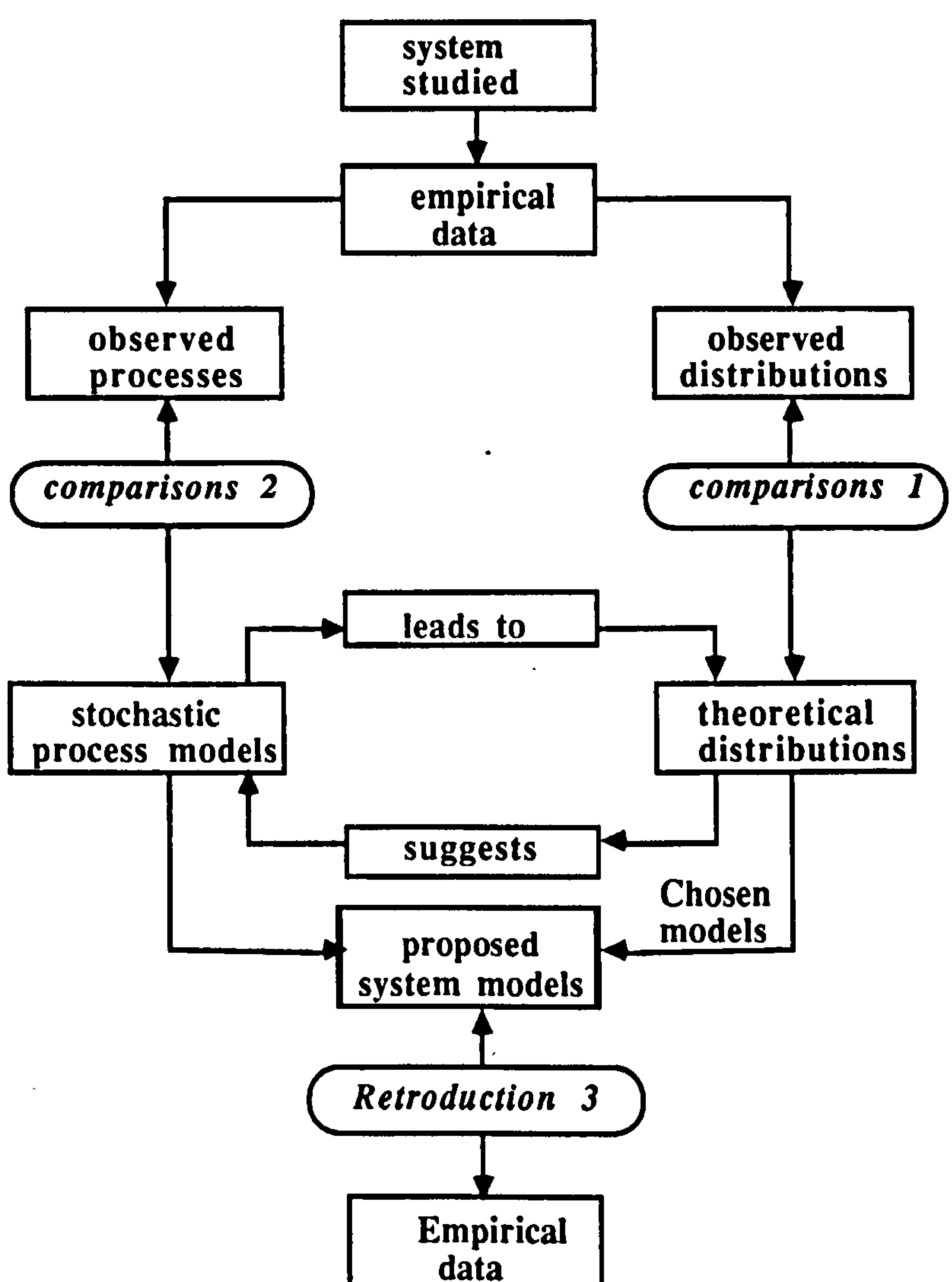
(d) Work from chapter five strongly suggests that at the level of individual items the demand is Poisson in character, although demand variances are, in most cases, much greater than the mean level of demand supporting the view that Poisson compounding, and possibly Poisson mixing, is taking place.

(e) We have shown empirically that single period aggregate demands (possible LSD distributed) when subjected to period by period summation do converge to an integer form of the lognormal distribution, and this remains the stable long run model of aggregate item demand volumes. The work of chapter four suggests that this process is driven by the so called Law of Proportionate Effect.

(f) Over the period of analysis (1975 to 1985) the shape parameters of the fitted usage value lognormal distributions seem very stable, varying by only small amounts from year to year. This suggests that very stable processes are at work controlling the form of the process models despite the great complexity in the inventory range at DAF. The location parameter μ shows a very steady drift from year to year, because this reflects the gradual increase in prices over the period.

If we reconsider our research scheme model of chapter two we can now assert that the demand volumes, prices and usage values have all been proved lognormal and our comparison point 1 in the model (reproduced again below) has been fully evoked. Also the very existence of the lognormal distribution for prices, volumes and usage values strongly supports the view that the underlying process that governs the convergence to this distribution is the Law of Proportionate Effect. We however leave the empirical testing of this law (comparison point 2) until chapter eight. In the next chapter we consider additional theory that together with our empirical work so far then forms the basis for the development of several new models.

Outline of the empirical data testing process



The Development of Theory; Towards an Aggregate Model of Period Usage Values

7.0 Introduction

In chapters four and five we approached the problem of explaining the lognormality of period usage values from two standpoints. First by starting with the assumption that such variates, in aggregate, are lognormal it was shown that lognormality can be achieved by a growth process over time that is explained by the Law of Proportionate Effect. This was the top down approach. However, as was discussed in chapter four, whilst the Law of Proportionate Effect provides evidence for growth it falls short in a number of respects to explain the starting point of such growth processes. It gives us a plausible mechanism for how the process operates, but provides no insight into its origin. The second standpoint has been to examine the nature of Poisson demand processes for individual items and to explore similar models that provide a plausible explanation of the underlying aggregate demand processes at work in short time periods. This was the bottom up approach.

This chapter now examines the possible mechanisms at work that could provide links between demands of individual inventory items to the aggregate demands of a whole range of inventory items; ie we connect the theoretical considerations of chapter four with the groundwork in chapter five. In particular we develop a theoretical scheme which shows that the Afwedson process can be applied to aggregate demand volumes in short time periods to explain the patterns observed in the empirical data. A necessary requirement of this scheme appears to be that the underlying demand process must be Poisson in character and that by application of the Law of Proportionate Effect through successive time periods the process will yield the Lognormal distribution as the long run stable distribution of the system. We start the model development process in this chapter by a detailed examination of the Log Series distribution. This particular statistical model has proved to be of central importance and value in this research. We first consider its properties and use in modelling individual consumer products, then examine its particular relationship with the NBD and the use of the combined LSD/NBD model in fields where it has been applied to heterogeneous populations. This is equivalent to our search for aggregate models of parts demand, and we are able to show that the Poisson Gamma model and the Afwedson process model of the NBD can be applied to certain kinds of heterogeneous populations of item types.

7.1 Properties and Development of the LSD

We saw from the empirical work of the previous chapter that the aggregate distribution of demand volumes for the DAF data in very short time periods was certainly not lognormal. It was true reverse 'J' shaped for all values positive integers and for any single demand period examined. Furthermore empirical investigations showed that it was the Log Series distribution that seemed to be the most likely candidate to explain the form of this single period aggregate data. It met the general requirement of goodness of fit criteria exceedingly well. However, before it can be accepted as a strong candidate to model the data the methodology of this research requires that there should be strong stochastical evidence to support its use. We will now examine this distribution in more detail and consider two stochastic processes that can explain its occurrence.

7.1 (a) basic properties and use of the LSD

The Log Series distribution is a discrete distribution, highly skewed and reverse 'J' shaped with the following probability density function-

$$P(x) = \frac{-q^x}{x\ln(1-q)} \qquad \text{------------- } 7.1$$

Where parameter q is between 0 and 1,
x is a positive integer for all values = to & > 1.

The mean (w) of the distribution is given from-

$$w = \frac{-q}{[(1-q)(\ln(1-q]}$$

This expression cannot be solved directly for q, but Ehrenberg (1972, page 159) has developed an approximation for q given by :

$$q = \frac{(w-1.4)}{(w-1.15)}$$

which gives q to an accuracy of 2% for the range $2 < w < 20$.

Nahmias and Demmy (1982, page 668) give a recursive method for determining q given the sample mean and they show this to very accurate, but it is rather laborious to use in practice. In the work here we use the more straightforward approach suggested by Ehrenberg as our values of w were generally within the range 10 to 20.

The variance of the LSD is given by :

$$\sigma^2 = \frac{-q\{(1+q)/\ln(1-q)\}}{(1-q)2\ln(1-q)}$$

This formula can simplified according to a method suggested by Ehrenberg (op. cit. page 161) by the substitution :

$$\alpha = q/(1\text{-}q)$$

$$\text{then} \quad \sigma^2 = w(1+\alpha - w).$$

The origin of the LSD appears from the literature to be attributed to the work of R.A. Fisher (1943) from statistical work on animal ecology. Some important results and implications from Fisher's work will be considered in a later section.

The current author's attention was first drawn to the LSD after reviewing Ehrenberg's (1972) text that we have quoted several times in earlier chapters. In this work he discussed at some length the use of the LSD as a distribution useful in modelling consumer purchases in particular circumstances. Ehrenberg also discussed the close relationship between the NBD and the LSD in the case of so called 'lightly purchased' items (see page 60 in particular). A necessary part of the formation of Ehrenberg's Poisson Gamma approach to the NBD model, is that the consumer group of purchasers of a product should include, for the period considered, a group of non-buyers (who may however be buyers in a subsequent period). It was shown by Ehrenberg that in those cases where the proportion (β) of these non-buyers in a period, is large compared to the buyers, then, if the non buyers are excluded from the group, the distribution of buyers can be modelled very closely by the LSD. For this reason the LSD model has been referred to as a model for lightly purchased items in terms of consumer purchases. Generally the LSD has been shown by Ehrenberg to accurately model consumer purchase behaviour when the proportion of non-buyers is less than or equal to 0.2. In cases where 'β ' is greater than 0.2 then the fit of the LSD to empirical data becomes progressively less efficient and the NBD model is then used in preference. Thus when considered in the light of consumer purchase theory the LSD, like the NBD, is a distribution of purchase occasions for single item types and it can be regarded as a special case of the NBD model.

A close relationship between the LSD and the NBD can also seen from the Afwedson process discussed in chapter five in connection with Poisson compound processes. It was seen that if the number of demands at each Poisson event is distributed as a Log Series distribution then the unconditional outcome is a Negative Binomial distribution of demands in a given period. However, formed in this way, by a compounding process, a model of purchase quantity is obtained for single items. An NBD model derived this way by a compounding process has not received much attention in the literature, especially that concerned with inventory problems. The closely related Galliher process however, where geometrically distributed Poisson events lead to a Stuttering Poisson distribution of demand, has, in contrast, received much more attention in the inventory literature.

From the foregoing discussion and our discussion of chapter five we can see the close interrelationship between the LSD and NBD models in consumer demands and the underlying Poisson process. This can be either through Gamma mixing of lightly purchased item demands, as purchase occasions, or; aggregated across all consumers as an overall NBD model of demand quantity with individual demands being LSD distributed. One process (the former) is a mixing process, and the other is by compounding.

We next make the assumption that we can consider similar processes taking place across may items in a family of products or items. And we reason that there is a connection between the Poisson generated demand for individual items and the aggregated demand for many items in a population in the same time period through Poisson compounding and mixing processes; and the important linkage appears to be the Log Series distribution of R.A. Fisher (1943). It was the chance finding of the early work of Fisher, [quoted by Ehrenberg 1972 and others] and that of Quenouille (1949), that lead this author to the view that similar processes might work in aggregate for many items in a parts range, given that the LSD appeared to be a very good model for short period aggregate demand volumes in DAF data. Additionally when we examined DAF demand volumes, including the zero demands, then the NDB was also seen to provide a very good fit.

It was not clear however, whether the linkage, if it exists, is by a mixing process or by a compounding process or indeed both. Support for this tentative hypothesis was sought from a variety of sources.

7.1 (b) stochastic models of the LSD

A stochastic process very similar to that used to develop the Poisson Gamma model can be formulated for the Log Series distribution as shown by Chatfield (1969). The primary difference between the formulation of the Poisson Gamma model and the process below is in the consideration of the consumer purchase rate distribution, which is assumed Gamma distributed in the NBD model. In the process leading to the LSD it is postulated that there is a group of never buyers of the brand as distinct from those people who have a positive long run rate of buying $\mu > \delta$, where δ is the truncation value. Therefore in formulating the model a truncated Gamma distribution is used such that the consumers long run average rate of purchasing is Gamma distributed in the range $\delta < \mu > \infty$. Hence the frequency of any particular value of μ is given by the truncated Gamma distribution -

$$f(\mu) = \frac{(ce^{-\mu/\alpha})}{\mu} \qquad \text{------------------ 7.2.}$$

where $\mu > \delta$, and δ is very small , α is a parameter of the distribution and c is a constant chosen so that -

$$f(\mu) = \int_{\delta}^{\infty} \frac{(ce^{-\mu/\alpha})}{\mu} d\mu$$

Applying equation 7.2 as the mixing distribution on the Poisson process equation previously given we can develop the LSD as shown by Ehrenberg (1972, page 170)

$$P(r) = c\int_{\delta}^{\infty} \frac{(e^{-\mu}\mu^{r})}{r!} \bullet (e^{\mu/\alpha}) d\mu$$

$$P(r) = \left\{c / \left[r!(1+1/\alpha)^{r}\right]\right\} \int_{\delta}^{\infty} e^{(1+1/\alpha)^{\mu}} \{(1+1/\alpha)\mu\}^{r-1} d\{(1+1/\alpha)\mu\}$$

$$P(r) \cong \left\{c / \left[r!(1+1/\alpha)^{r}\right]\right\} \Gamma(r) \qquad \text{for } r \geq 1 \quad \text{since } \delta \text{ is very small}$$

hence

$$P(r) = \frac{c}{\left\{(1+1/\alpha)^{r} r\right\}}$$

$$P(r) = \frac{cq^{r}}{r}$$

$$P(r) = \frac{qp_{r-1}(r-1)}{r} \qquad \text{with } q = \frac{\alpha}{(1+\alpha)}$$

if $\sum p_r \equiv 1$ for $r \geq 1$ we must have -

$$p_1 = \frac{-q}{\ln(1-q)}$$

and hence the probability *P(r)* of *r* purchases being made in a given time period is

$$P(r) = \frac{-q^r}{r\ln(1-q)} \qquad \text{for } r \geq 1$$

which is the the form of the Log Series distribution.

In the application of this model and the NBD model to consumer purchases by Ehrenberg, Goodhardt, Chatfield et. al. the period of application was single time periods of between one week and one month. To justify this and to account for longer durations Ehrenberg (1959 and 1972) made the following points-

> *"It is of course not necessary to assume that consumers purchasing behaviour actually follows this stochastic model in the long run. As is also the case for the NBD model, it is only necessary to suppose that in any time period, or periods being analysed, the purchases behave as if they were a random sample from the values generated by such a model."*

The significance of this point is that in many product fields where the NBD and LSD theory applies in short time periods the long run equilibrium condition of the system may well be described by models other than the NBD or LSD and this does not negate the validity of the theory in short time periods. Furthermore it does remove the burden of the necessity to model the distribution of individual item demands in successive time periods, which in the case of this research was a complex issue due to mixed nature of the demand process of individual inventory items. (We do however, present some evidence in appendix one which suggests the form of such distributions). Hence we have a stochastic mechanism that leads to the Log Series distribution by a truncated version of the Polya 'Poisson mixing' process. To relate this to our empirical observations of the previous chapter it means in effect that

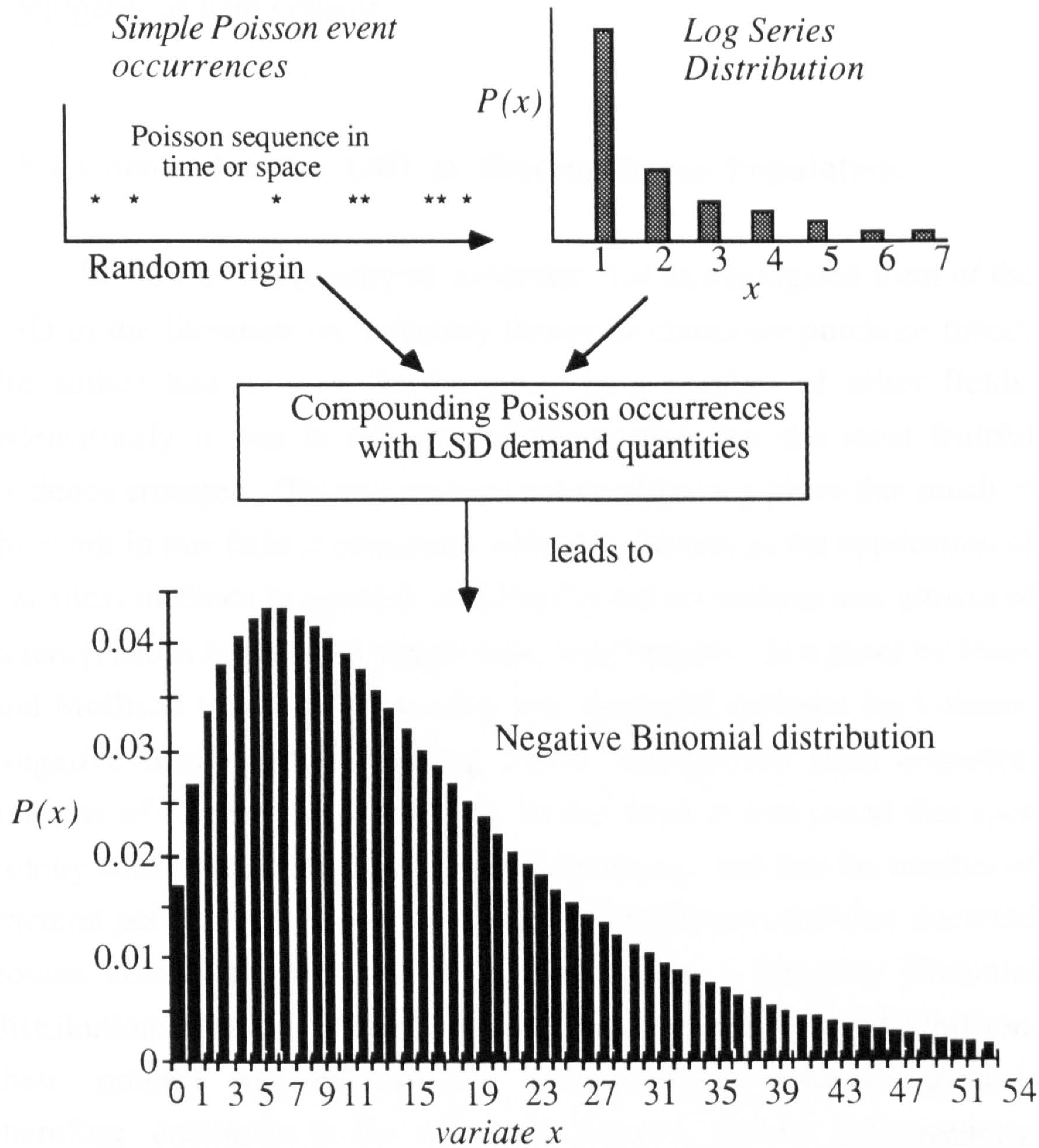
(1) The demands of individual items must be Poisson in nature

(2) The individual Poisson demand rates - $\lambda_1 \quad \lambda_2 \quad \lambda_3$ ---- λ_n etc must be Gamma distributed within the range $\delta < \mu > \infty$.

such that the overall distribution of aggregate demands will be LSD by the stochastic mechanism given above. Our problem so far is that the above mechanism has proven foundation based on the work from consumer purchase theory, but that was in terms of single item types. In our work we need to apply the same principles and the LSD to an aggregate population of heterogeneous items.

The alternative stochastic model involving the LSD is through Poisson compounding by the Afwedson process that we considered in chapter five. This is shown schematically overleaf in figure 7.1:-

Figure 7.1
Schematic illustration of the Afwedson process



If the order rate is Poisson with mean λ and the order intensity is distributed as a Log Series distribution then the overall distribution of demands in fixed intervals of time is the Negative Binomial. Conversely if the overall distribution across many items is NBD and the underlying stochastic nature of the process occurrence is Poisson, then individual demands are LSD. Hence we now have two processes both of which lead to the LSD (and the NBD), one by mixing and the other by compounding, but the important question is can such processes be applied to heterogeneous item systems.

7.2 Evidence for the LSD in Heterogeneous Population.

In view of the paucity of evidence for an aggregated form of the LSD in the literature on inventory theory or consumer purchase theory the author had to scan the literature in a number of other fields. Interestingly it was in the area of Biometrics that the most fruitful evidence emerged. This was perhaps not so surprising given that much of the work in this field is concerned with such matters as the application of statistical methods to quantify and classify the occurrence and growth of heterogeneous species and populations, ie aggregates. In a paper by Jones and Mollison (1948) a relationship was discussed between the Poisson, Negative Binomial and the Log Series distributions from empirical analysis of bacterial colony counts. In this work it was found that such colony counts were Poisson distributed (in space) and that the number of bacteria per colony followed a Log Series distribution, and that bacterial counts across all colonies were distributed as a Negative Binomial distribution. Whilst it was not discussed as such by Jones and Mollison, their process was effectively a compounding mechanism and it is therefore analogous to the Afwedson process, but for <u>heterogeneous populations</u>. In fact it was seven years after the work of Jones and Mollison that Afwedson published his work on collective risk theory that was to lead to a process that eventually bore his name (Afwedson 1955). Hence Jones and Mollison could not, obviously, interpret the results of their empirical in terms of the compounding process named after Afwedson. It is possible to postulate a close analogy of the the Jones and Mollison findings with the spare parts system under study here. If the occurrence of demand for small clusters of spares is Poisson, (the equivalent of the colonies), and if the number of demands across each item in a cluster is distributed as a LSD, then the distribution of demands across all items and all clusters should then be NBD if the analogy is valid.

No proof of the relationship between the three distributions was given by Jones and Mollison, but in a paper by Quenouille (1949) in which he discussed their work he provided a mathematical proof of the relationship between the Poisson, LSD and NBD distributions. The proof as presented by Quenouille is given here-

" Suppose the number of groups observed on any one occasion is distributed Poisson so that the probability of observing 'n' groups is-

$$P(\text{n groups}) = \frac{e^{-m} m^{n}}{n!}$$

then the probability of observing 'S' individuals in any sample is -

$$P(s\text{ individuals}) \quad = \sum_{n=0}^{\infty} P(\text{n groups}) \bullet (S\text{ individuals in '}n\text{' groups})$$

Now the probability of observing 'S' individuals in any one group is-

$$\frac{\alpha x^{s}}{s}$$

or the coefficient of t^{S} *in -*

$$-\alpha \log_{e}(1 - xt)$$

likewise the probability of observing 'S' individuals in 'n' groups *is the coefficient of* t^S *in -*

$$\left\{-\alpha \log_e(1-xt)\right\}$$

thus we have -

$$P(s\text{ individuals}) = \text{coef. of } t^s \text{ in } \sum_0^\infty \frac{e^{-m}m^n}{n!}[-\alpha\ln(1-xt)]^n$$

$$P(s\text{ individuals}) = \text{coef. of } t^s \exp[-m - am\ln(1-xt)]$$

$$P(s\text{ individuals}) = \text{coef. of } t^s \text{ in } (1-xt)^{-\alpha m}e^{-m}$$

$$P(s\text{ individuals}) = (1-\alpha)^{\alpha m} \bullet \frac{(\alpha m+s-1)!}{(\alpha m-1)!\,S!}x^S \quad \text{since } (1-x)=e$$

> *This is the same as the (s+1)th term in a Negative Binomial Distribution with parameter 'x'. Consequently the probability of the number of individuals in a random sample is NBD. Conversely the assumption of any two of the distributions holding leads to the third provided the parameters of the LSD and the NBD are equal."*

This author had some difficulty in following the obscure mathematical logic of this proof, however since the paper containing it was refereed and accepted for publication in the prestigious journal 'Biometrics' and was not subsequently challenged at a later stage, its mathematical validity was assumed by this author to be correct; and meaningful to Biometricians. Also Quenouille was a respected author in statistical circles. His work on the Poisson/LSD/NBD relationship has been cited by other researchers, for example Nahmias and Demmy (1982), in the context of single item compound demands. The important point for this research is that an accepted proof does exist that relates the three distributions in this way. Furthermore, according to Quenouille (op.cit.), for the proof to apply to a particular situation the parameters of the LSD and NBD must be the same. This means that one must be a special case of the other ie removing the zero demands from a Negative Binomial distribution will leave a distribution of the positive integers for all values of unity and greater, and this must be the Log Series distribution. We should note here that the LSD is not always a special case of the NBD. It is only so when the proportion (β) of zero demands is very large in a given period.

Further evidence of the relationship between the Poisson, NBD and the LSD in heterogeneous item systems came from Fisher's original work (1943) in an investigation of the frequency distribution of the number of different species of animals obtained in random samples. Fisher first derived the LSD by considering the distribution of species to be a Poisson process with mean 'm', and assumed that 'm' was distributed across the heterogeneous animal groups as a Gamma distribution of the type shown-

$$P(m) = \frac{\rho^{-k} m^{k-1} e^{-m/\rho}}{(k-1)!}$$

With this assumption Fisher then regarded the process as the superposition of a set of Poisson distributions which resulted in one overall distribution of animal species- the Negative Binomial as previously given. Developed in this way the process is effectively a Poisson-Gamma model with gamma mixing of the Poisson rate parameter to give the NBD. Hence it is exactly analogous to Ehrenberg's derivation of the model of purchase occasions by gamma mixing of consumer purchase rates, but in this case applied to a mixed specie population. Fisher also showed that as the NBD parameter 'k' tended to zero the process gives rise to a distribution whose first term tended to become infinite. However, upon excluding this term as being in general unobservable in empirical data Fisher then obtained the Log Series distribution. This is also analogous to the work of Ehrenberg (1959 and 1972) in which for lightly purchased items, ignoring the first term of his NBD model (i.e. the large group of non buyers in the period) he obtained a LSD.

The interesting features of the work of Fisher, and also that Jones and Mollison, is that they both end up with essentially the same overall model the LSD/NBD as applied to aggregate or heterogeneous specie situations, but by different routes. A process that is closely related to the Afwedson compounding process in the case of the Jones and Mollison work, and a process which is clearly Poisson mixing in Fisher's work.

In a paper by Anscombe (1950) essentially concerned with statistical theory, and published in Biometrika, he showed that as 'n' approaches infinity (n being the number of observations in a sample) and as 'k' approaches zero (k being the exponent parameter in an NBD) the associated LSD can be easily decomposed into the product of Poisson frequency functions. More explicitly Anscombe argued that the LSD as developed by R.A. Fisher was obtained by a limiting process from the Negative Binomial distribution by considering a sample of 'n' readings and letting 'n' tend to infinity and 'k' tend to zero and then neglecting the zero readings. Anscombe regarded the LSD as a multivariate distribution consisting of a set of independent Poisson distributions with mean values -

$$\alpha x^1, \quad \frac{\alpha x^2}{2}, \quad \frac{\alpha x^3}{3}, \quad \text{-------etc to} \quad \frac{\alpha x^n}{n}$$

$$\text{where} \quad \alpha = -\frac{1}{\ln(1-x)} \quad \text{and } x \quad \text{is a constant} < 1$$

And according to Anscombe a sample comprised one reading from each Poisson distribution. Anscombe's work effectively started with the assumption of the interrelation of the three distributions and then provided additional support for the mechanism by which one distribution is formed from the other two.

It is clear from the foregoing discussions that in a number of fields of investigation that the Poisson, LSD and NBD distributions are closely related to each other through stochastic mechanisms. Indeed it now seems certain that a necessary condition for the occurrence and existence of the NBD is that the underlying process being considered must be Poisson; although this can be in time or space. Ehrenberg's work is now classic in the field of consumer purchase theory, but it has only been applied to single item types or brands and to purchase occasions. Early work in the field of Biometrics has however, shown that similar models can be derived both empirically and theoretically and can be applied to multiple specie situations and for item quantities. Furthermore, in all the work so far considered there is general agreement, either because of explicit statements, or by inference, that the LSD can be regarded as a special case of the NBD and in those cases in particular where the first term of a NBD distribution is large (i.e. the number of zero readings) then the LSD is a good model of the remaining data. (ie all. values of x equal to and greater than one).

7.3 Inferences Regarding Heterogeneous Item Demands

In terms of the implications for the research here we can now draw some strong conclusions and inferences. The work and discussions of previous chapters has shown the large body of evidence that indicates that the underlying demand process for spare parts is Poisson. Early empirical work by this author shows that the empirical distribution of demands across all spare part item types is very likely LSD in short time periods. Then, drawing from the work of various authors in the field of Biometrics theoretical evidence is available to support the validity of the use of the LSD in heterogeneous populations such as in spare parts environments. This being the case, then the overall distribution of short period demands for all spare types in aggregate should be distributed as a Negative Binomial distribution, when the zero demands for the period are also taken into account. We are not clear however, whether this is by mixing or by compounding, or indeed both. Hence if the LSD can be shown to converge to a lognormal distribution then, it can be argued, so to will the associated NBD also converge to lognormality. Before we now proceed to a more formal statement of a hypothesis some of the evidence in the literature relating to constraining conditions to the processes will now be considered.

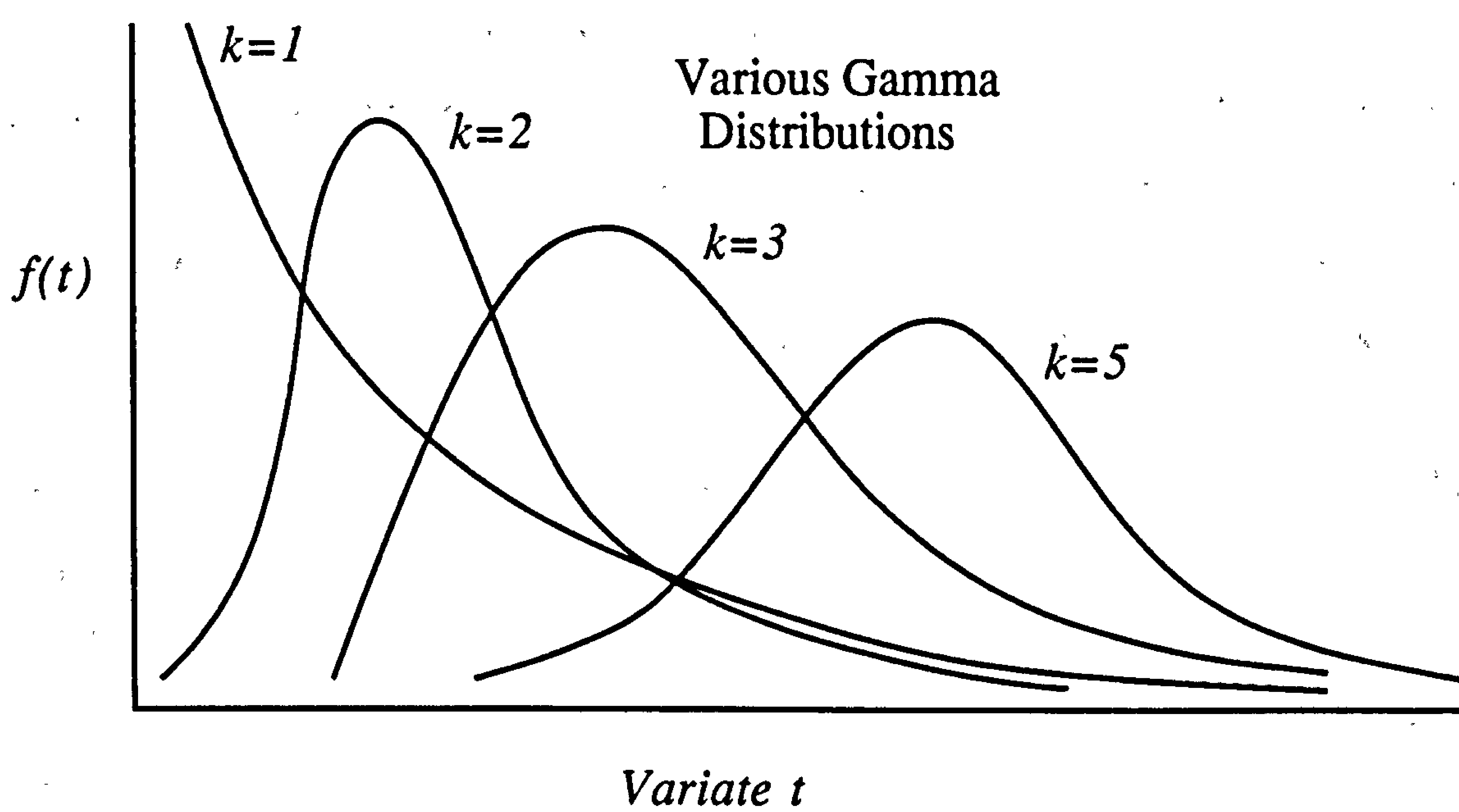
Firstly returning to Anscombe's work we can draw some valuable interpretations. He sees the development of the LSD from a Poisson model as a limiting process of the mechanism that gives rise to the NBD and which depends on two general conditions-

(a) The parameter 'k' (being the exponent parameter in the NBD model) must be very small, i.e.'k' must approach zero.

(b) 'n' the number of observations must be very large. That is 'n' must begin to approach infinity.

If we consider the aggregate NBD as derived by the Polya process then the parameter '*k*' in the NBD model is derived from the Gamma mixing distribution on the Poisson rate parameter. In fact '*k*' is the shape parameter of the Gamma distribution and as '*k*' becomes smaller the Gamma shape becomes more and more skewed as shown by the following diagram-

Figure 7.2



When '*k*' equals one the distribution becomes effectively a negative exponential distribution. When '*k*' becomes 1/2 the form becomes more hyper-exponential in form. In the context of consumer purchase theory the condition of '*k*' being very small has been shown by Ehrenberg (1972) to be valid in those cases where consumer purchases can be modelled by the LSD in place of the NBD model. That is for the so called 'lightly purchased' items. Thus it appears that the mixing distribution which models demand variation across the Poisson occasions must have a reverse 'J' shaped profile of a negative exponential or hyper-exponential form. By way of comparison, in the Afwedson compounding process the variability in demand volume at each Poisson occasion is described by the LSD which, although discrete and 'J' shaped, is of an exponential or hyper-exponential form. Thus the two processes may be closer in nature and form than has been previously recognised.

The condition of 'n' approaching infinity seems somewhat more doubtful in the context of consumer purchase research, because although most of Ehrenberg's data sets were large, 'n' was nevertheless measured in hundreds and the requirement for 'n' to be very large does not seem to have been discussed directly in such research. However, in view of the richness and proof of the efficiency of the NBD model in consumer purchasing one can only assume that data sets measured in hundreds are valid. In terms of the research reported here the requirement for 'n' to be large does not present any problems, as all the data sets examined were very large and could, if required, be measured in thousands of items. The condition that 'k' should be small and approach zero is more complex. If, as evidence now begins to suggest, a Poisson Gamma model can be applied in short time periods to multi specie situations, then the need for 'k' to be small implies that the demand variation across the range of customers for each item should be a Gamma distribution in an exponential or hyper-exponential form. Hence a large proportion of the demands should be very small and only a small proportion of the demands should be relatively much larger. When considered in the aggregate situation this will result in a highly skewed range of demands across all products. It is like taking clusters of random numbers from an exponential distribution and computing the mean value of each cluster. Most of these means will be of very small value, only a few will be larger and very few will be very large. This is what was generally found across the wide variety of parts in the DAF system.

7.4 Convergence Processes

In chapter four we discussed the growth of variates over time and considered the possible mechanisms that can lead to lognormality as the long run stable distribution of a system. Having now established a plausible theoretical base for the likely mechanism and distribution of the aggregate distribution of demands in short time periods we will now reconsider the growth processes that might lead to lognormality. Whilst the literature provided no direct evidence that a summation of independent Log Series distributions could result in a Lognormal distribution the author sought indirect evidence to indicate the plausibility of such a process. The following discussion considers some empirical evidence that supports a linkage between the two main theoretical strands proposed so far, namely that both the Poisson Gamma and Afwedson processes can lead to a Log Series/NBD distribution of aggregate demand for short time periods. And as the demand period is extended the stable long run equilibrium distribution of aggregate demands is an integer lognormal distribution and this achieved by a stochastic growth process governed by the Law of Proportionate Effect.

From the field of military logistics Hadden et al (1959) in a study of military equipment down times, showed that the sum of exponential distributions gradually converge to a lognormal distribution. This was demonstrated graphically by a decomposition of the lognormal distribution obtained into sums of exponential distributions. Apparently this is contrary to statistical theory because according to Cramer (1946) in general one would expect the sum of exponentials to give a Pearson type three distribution (a Gamma distribution). However, it must be borne in mind that with appropriate values of the parameters it is possible for the Gamma and Lognormal distributions to become very similar in form. Indeed as indicated in section 7.3 as 'k' the Gamma distribution parameter approaches unity the distribution approaches the exponential distribution. When 'k' = unity the two distributions become identical. Furthermore whilst empirical evidence seems to contradict statistical theory no rigorous proof of Cramer's statement was given. Hadden's work is particularly relevant to our developments because he showed that a continuous 'J' shaped distribution - the exponential, converged to a continuous lognormal distribution. Hence it can be reasoned that a discrete 'J' shaped distribution of an exponential form, such as the LSD, could be expected to converge to the discrete distribution of counts form of the lognormal distribution. This is exactly what our empirical analysis on DAF demand volume data has shown. The short period empirical data converged to a discrete lognormal form.

Bovaird and Zagor (1961) have also discussed the problem of equipment down times and state that-

> *"There is a growing body of evidence that indicates that in general downtime distributions within various categories are either lognormal or exponential and the total downtime distribution is lognormal provided the equipment is sufficiently complex -otherwise it is exponential."*

By sufficiently complex it is taken that these authors refer to the number and diversity of parts used in the equipment. This same phenomena has also been referred to by Barlow et. al. (1967). The empirical work of Hadden (1959) and Bovaird and Zagor (1961) give support to a proposition that lognormality is reached along two dimensions, a summation through time of a highly skewed distribution, and a space complexity dimension in terms of the diversity of items. It is interesting to note that the work of these authors is concerned with equipment failure processes which can confidently be assumed Poisson in terms of failure rates, and a resulting highly skewed distribution that can be summed to a lognormal distribution.

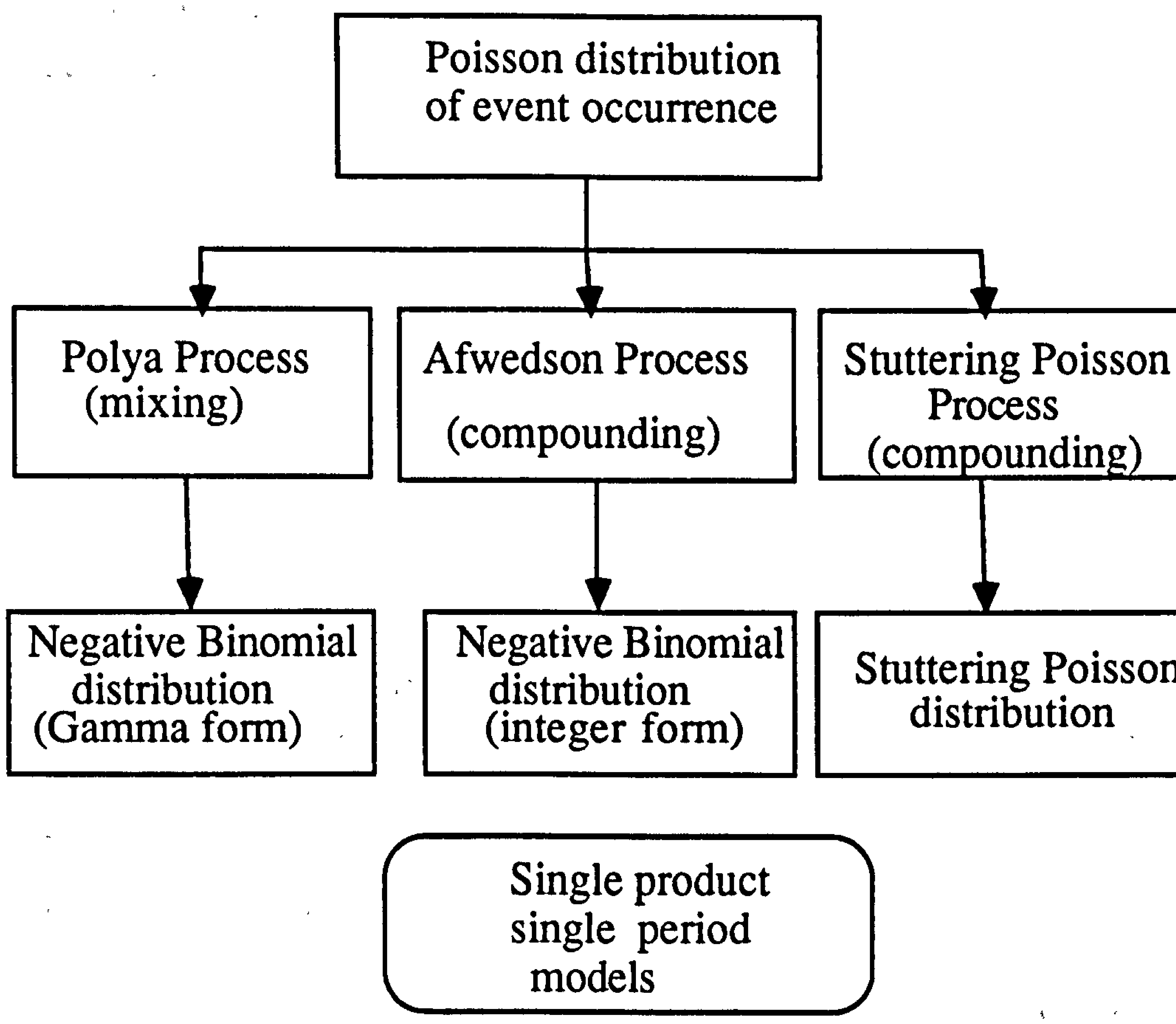
7.5 Model Development

As we have seen the favoured candidate for a statistical model of demand rates for individual items under both mixing (the Polya Process) and compounding (the Afwedson Process) processes is the Negative Binomial distribution. In the case of compounding the Stuttering Poisson is also a favoured candidate for period demand volumes in inventory applications. Whether the same models can be applied satisfactorily to those situations where both compounding and mixing occur simultaneously is not known. In all probability it has already been applied, possibly unknowingly, to such situations as for example in the work of Haber and Sitgreaves on submarine spares, and in Taylor's work on aircraft spares. These authors have not attempted to differentiate between mixing and compounding. However, in both research works demand for spare parts was formulated in a manner equivalent to a mixing process with the Poisson rate parameter λ assumed distributed as a gamma variate over the lead time. Furthermore in both cases one would expect a degree of demand compounding to be taking place with some of the spare parts in use.

It is now quite easy to envisage that in a spare parts environment, such as we have in the DAF Trucks situation, that at the level of individual items both compounding and mixing of demands is very likely to be taking place. Many of the demands, received at the distributor level, especially those for consumable and cheaper spare parts, are to replenish dealer retail stocks; hence these are orders for multiple items that give a compounded demand situation. For many other parts, that are ordered on a unit basis, the effect of different order rates from the various retailers will give a degree of mixing to the demand character. This mixing character will however will be much more marked at the retail level through the different demand rates of the very large ultimate end user truck population. For very slow moving parts the process is very likely to be close to a simple Poisson process with no compounding and very little mixing evident in the demand process. For some spare parts, most likely the faster moving moderately priced items, we would expect to see a substantial degree of both compounding through inventory replenishment effects, and mixing through variable end user rates. Hence we would expect that an examination of the variance of demand of a range of different spare parts types will almost certainly reveal a large variation in that parameter. In general it was found across the DAF parts range that demand variances were indeed almost always much greater than the mean, except for very slow moving items in which cases the variance values were very close to the mean values. Indeed from limited analytical evidence on DAF data we were able to show that both the NBD and Stuttering Poisson distributions provide very good fits to fixed period demand for certain representative parts.(See appendix one).

As a first step in developing a model of aggregate demand behaviour we consider the start of the process as beginning with a recurrent event process where single period demand occasions for individual items are simple Poisson. Then, through varying degrees of both mixing and compounding of individual purchase behaviour, the process yields single period demand quantities that are distributed as either NBD, or the Stuttering Poisson distribution as shown in figure 7.3 below. When no compounding exists and the level of mixing is small the the outcome will be a simple Poisson demand quantity.

Figure 7.3
Recurrent Event Processes



If we then consider this process occurring for a large population of '*n*' products we know that for short periods of time this leads to an aggregate distribution of demand quantities that are highly skewed and 'J' shaped. We know from both theoretical evidence and empirical analysis discussed in this chapter that a strong candidate to model the aggregate outcome in short time periods is the Log Series distribution. Hence given a Poisson demand occurrence outcome for all parts in aggregate combined with an LSD distribution of demand quantity in aggregate then we have strong evidence to suggest the operation of an aggregate form of the Afwedson process. Under these conditions the overall distribution of aggregate demand quantities should then also be the NBD. Now previous empirical and literature evidence so far strongly supports the view that both the Poisson Gamma model and the Afwedson model can be applied to the heterogeneous populations, ie the multiproduct or aggregate item demand case. So we now postulate that in single demand periods (defined as four weeks in the DAF case) the Afwedson model is an adequate description of the aggregate demand quantities.

As a next step we formulate our model of the aggregate process in a manner similar to Easton's(1975) diagrammatic representation of the NBD demand occasions model we first showed in chapter five. A modified version of this model reformulated to suit our aggregate parts process is shown in figure 7.4

Figure 7.4

Poisson Gamma Model

Spare Part	Periods						Long run average	Distribution horizontally
	1	2	3	4	5	n		
A	x_1^A	x_2^A	x_3^A	----	----	x_n^A	λ_A	Modified Poisson
B	x_1^B	x_2^B		-----	--------	--x_n^B	λ_B	"
C	x_1^C	--	--x_3^C		-------	--x_n^C	λ_C	"
D	x_1^D		-----	-----	-----	.	.	
E	.					.	.	
F	.					.	.	
etc						.	.	
Mean	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	-	------	$\bar{\bar{x}}_n$	
vertical distrib'n	LSD/ NBD	LSD/ NBD	LSD/ NBD	LSD/ NBD	----	-	Lognormal	

* The horizontal distributions in figure 7.4 are likely to be predominantly NBD or Stuttering Poisson because each period expectation 'x' can be regarded as a demand generated from a Poisson occurrence model with either geometric or logarithmic compounding for each individual item.

This overall model is intended to be a model of aggregate demand quantity for all items and for multiple time periods. However, to be a valid and comprehensive model, it should be capable of interpretation in terms of both demand quantity and demand occasion. Firstly we attempt to interpret the model in terms of the Afwedson process. This model requires that for the period under consideration the aggregate occurrence of demand in fixed time intervals for all items considered together, irrespective of source or amount of the demand, should be Poisson. Secondly the quantities demanded from Poisson occurrence point to Poisson occurence point should be distributed as a Log Series distribution. Then when the zero demand occurrence points are taken into account this leads to an overall aggregate NBD distribution of demand quantity for the period. This interpretation of the model only requires that the occurrence of demand for an individual item amongst all items be Poisson and this frees up any restriction on the particular nature of demand distribution through successive time periods for individual items. Hence these could be NBD, Stuttering Poisson, simple Poisson, or any other close cousin, but they must be Poisson in nature. This will allow the possibility of true Poisson mixtures for individual item demand streams, ie cases where the period demand for a particular individual item might well be effectively comprised of converged demands of different behaviours. Some of which may be of a compounded nature, because of multiple customer requirements, whilst others may be simple Poisson or Poisson mixing in nature. This model formulation is not in conflict with a Poisson Gamma formulation of demand for individual items. The operation of one is not necessarily a condition for the other, neither does one preclude the operation of the other; because the essential link is the Poisson occurrence of demand at the level of the individual item, which is likely to give a predominance of the Stuttering Poisson demand quantity, and NBD of both demand occasion, through mixing, and demand quantity by compounding.

To interpret the model in terms of a Polya process, ie by Gamma mixing, it requires that for each item in a short time period demand occasion should also be Poisson; and that the variation in the Poisson rate parameter across all items should follow a Gamma distribution. This can only be tested however, by examination of the variation of the mean value of the Poisson rate parameter over several successive periods. To satisfy and clarify the requirements of both aggregate Poisson compounding and aggregate Poisson mixing we need to reformulate the model in two parts, one as an '**aggregate demand occasion model**', the other as an '**aggregate demand quantity model**' as shown in figures 7.5 and 7.6 on pages 254 and 255.

In figure 7.5 the individual demand streams are assumed to be simple Poisson or NBD by mixing of all demands for that individual item. The short period aggregate demand occasion across all parts is assumed simple Poisson, whilst the long run average aggregate demand occasion should be Gamma distributed. To test this model we need to examine the short run and long run aggregate demand occasions to see if we obtain the Poisson and Gamma models respectively.

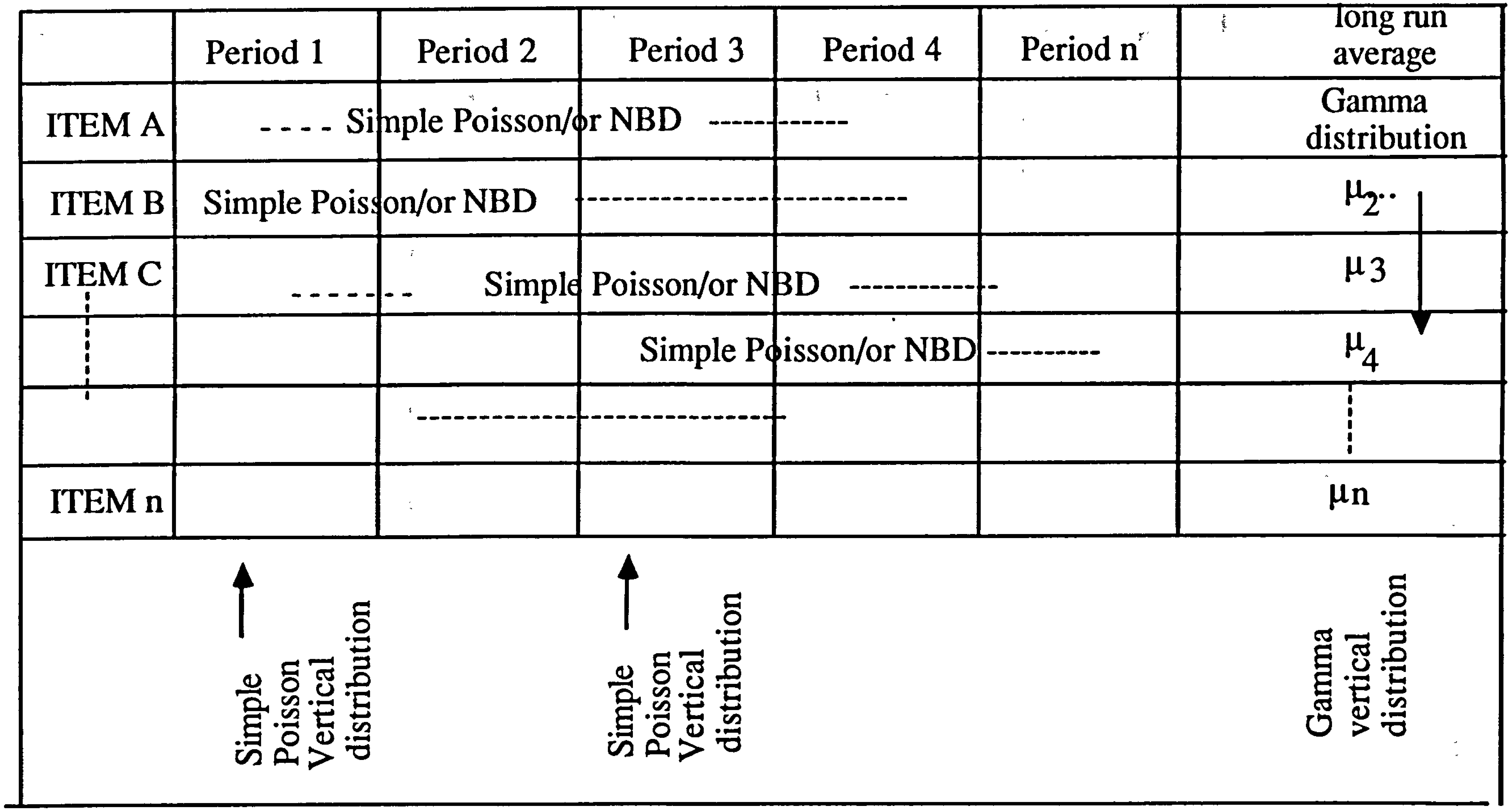
In figure 7.6 the individual demand stream quantities are assumed to be most likely Poisson mixtures. By that we mean that demand for some parts might be simple Poisson others may well be compounded as either NBD or stuttering Poisson or indeed other unidentified Poisson models including a proper mixture of our various Poisson models. However these forms are not critical to our model development. It is the form of the aggregate vertical distribution that is important. We assume this to be the Afwedson LSD/NBD distribution. In the long run the cumulation of these short term distributions must converge to the lognormal if our model is to hold good and explain the lognormal form of long run demand quantities.

7.6 Conclusions and hypotheses

The evidence for a Poisson occurrence of demand for individual spare parts in systems of the type studied in this research is very strong, both from theoretical and empirical evidence reported in the literature. However, the values of the demand variances, for the majority of parts studied individually in the DAF system, always exceeded the corresponding mean values, sometimes substantially. Thus whilst the underlying process for period demand quantity is almost certainly Poisson in nature the simple Poisson process does not provide an adequate description of the demand process in operation for majority of the spare parts examined. In all probability both Poisson mixing and Poisson compounding, to varying degrees, is taking place along individual demand streams from one time period to the next. As we have seen from the work of previous authors such as Ehrenberg, Jewell and Haight, Poisson mixing and compounding always increases the variance of demand. In our work here there is every reason to confidently assume that any mixing is by a Polya Process to yield a Negative Binomial distribution of demand occurrences for individual items in short time periods. In the case of faster moving items there is good evidence from the literature on spare parts demands to indicate that Poisson compounding is also operating. Such compounding of demands is very likely to be from either the Afwedson or the Galliher processes which lead respectively to the NBD and Stuttering Poisson distributions. We do not see these processes as being mutually exclusive.

Figure 7.5

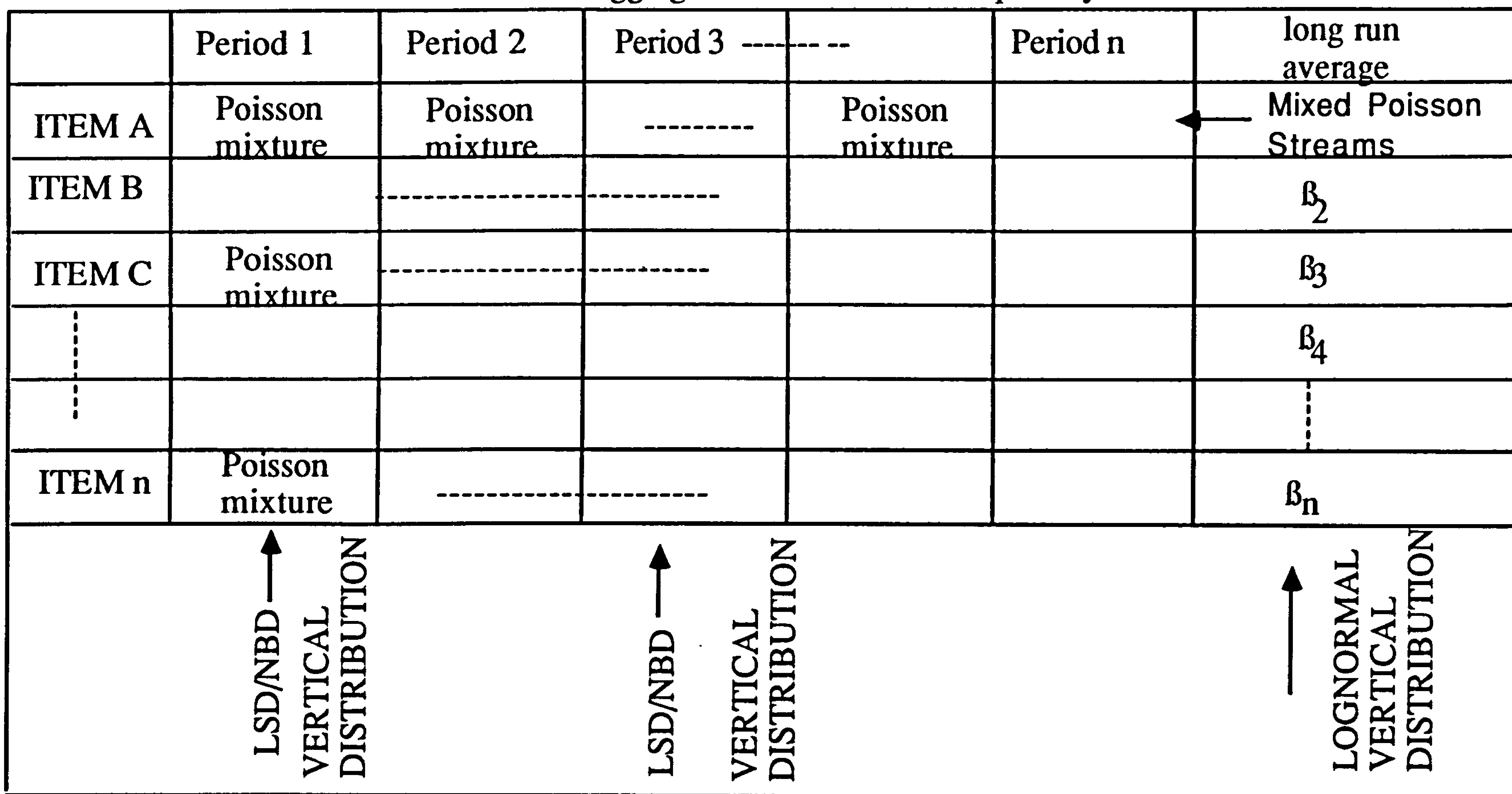
The aggregate model of demand occasions



DEMAND OCCASIONS MODEL

Figure 7.6

The aggregate model of demand quantity



DEMAND QUANTITY MODEL

Empirical and theoretical evidence suggests that the observed aggregate distribution of demand quantities across all spare parts items in short time periods can be modelled by the Log Series distribution. Theoretical evidence also strongly supports the proposition that an aggregate form of the Afwedson process, as shown in figure 7.1 on page 235, is in operation across all parts in short time periods. When aggregate demand quantities are cumulated over successive time periods the observed aggregate distribution is seen to gradually converge to an integer form of the lognormal distribution as the stable long run distribution of aggregate demand quantity. As the LSD is a special case of the Negative Binomial distribution then the NBD itself may also be a distribution to model aggregate short term demand, and in turn be capable of summation to the same lognormal form. The summation or growth process through successive time periods would seem, from the work in chapter four, to be governed by the Law of Proportionate Effect.

Our development so far assumes, indeed requires, that the underlying demand process in aggregate is Poisson in form. Hence demands from the dealers to the distributor should be Poisson and the interevent distribution between demand occasions should therefore be negative exponential. However, at DAF Trucks the demand for some 60% of the parts range occur in response to dealer stock replenishment decisions and under such circumstances this might suggest a demand occurrence interval more regular than the negative exponential distribution for such items. However, the variance of demand in the DAF case for the vast majority of parts was always greater than the mean; substantially so in most cases. Ordinarily an interevent distribution more regular than the exponential would result in a variance to mean ratio less than unity. The factor that could increase the ratio however under more regular ordering conditions, would be significant compounding. For example the possibility of a 'Stuttering Erlang' effect as we discussed in chapter five. We can however take some support from the quotation of

Ehrenberg given on page 233 in this chapter relating to his development of the LSD model of consumer purchases in very time periods.

> *"It is of course not necessary to assume that consumers purchasing behaviour actually follows this stochastic model in the long run. As is also the case for the NBD model, it is only necessary to suppose that in any time period, or periods being analysed, the purchases behave as if they were a random sample from the values generated by such a model."*

Hence what really is important is to show that in any short period the aggregate demands do indeed behave as if they were drawn from an aggregate Afwedson model. We reconsider this issue again in chapter eight when our models are tested retroductively.

Given the empirical evidence of chapter six, and the deductive reasoning of this chapter we can now reconsider our research hypotheses given at the end of chapter five in a much stronger form together with two additional hypotheses concerning the actual form of short period demand volumes. We know from the work of chapters five and six that annual usage values are composed of the product of item price and the quantity used or demanded in a particular period. Furthermore, we can say from the work of this chapter that item prices and item demand volumes are both lognormally distributed subject to certain constraints. Hence we now hypothesise that :

(a) In the case of prices the inventory range must be large and complex in the sense that it must comprise many small value items in addition to very high value items as typically found in spare parts inventories for complex capital equipment such as commercial vehicles, aircraft, tractors etc. In the case of demand volumes the period must be comparatively long and it is a discrete form of the lognormal distribution that is attained as the stable long run distribution by the summation of short period demand volumes.

(b) In comparatively short time periods the aggregate distribution of demand quantity its fully described and modelled by the Log Series distribution of R.A.Fisher.This distribution is itself a special case of the Negative Binomial distribution when the proportion of very low demands in the population is high. In the same time period the aggregate distribution of demand occasions is described by the simple Poisson process.

(c) The underlying stochastic process that explains the occurrence of the Log Series distribution of aggregate inventory item demand quantity is the 'Afwedson Compound Poisson Process' as previously developed and discussed.

(d) The Log Series distribution of aggregate demand will, if cumulated over successive time periods, gradually converge to a distribution, which is discrete and has all the characteristics of the form of the lognormal distribution known as the distribution of counts.

(e) The stochastic process that governs the convergence of demand volumes, and hence also of usage values, to the lognormal distribution of counts is the the Law of Proportionate Effect.

In the next chapter we turn to the process of testing the models we have developed in this chapter.