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Integrability, spin-chains and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence

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Abstract

Building on arXiv:0912.1723 [1], in this paper we investigate the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence using integrability techniques. We present an all-loop Bethe Ansatz (BA) for strings on $AdS_3 \times S^3 \times S^3 \times S^1$, with symmetry $d(2,1;\alpha)^2$, valid for all values of α. This construction relies on a novel, α-dependent generalisation of the Zhukovsky map. We investigate the weakly-coupled limit of this BA and of the all-loop BA for strings on $AdS_3 \times S^3 \times T^4$. We construct integrable short-range spin-chains and Hamiltonians that correspond to these weakly-coupled BAs. The spin-chains are alternating and homogenous, respectively. The alternating spin-chain can be regarded as giving some of the first hints about the unknown CFT_2 dual to string theory on $AdS_3 \times S^3 \times S^3 \times S^1$. We show that, in the $\alpha \to 1$ limit, the integrable structure of the $d(2,1;\alpha)^2$ model is non-singular and keeps track of not just massive but also massless modes. This provides a way of incorporating massless modes into the integrability machinery of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence.

1 Introduction

The gauge/string duality [2–5] offers a fundamentally new approach to understanding strongly-coupled systems. The strongly coupled system is believed to have a gravitational dual; in this gravitational dual one is often able to compute quantities which are physically important but difficult to compute in the original strongly coupled system. While conceptually striking, it is at present not clear how general this approach is. Over the last few years some of the most powerful evidence for the gauge/string correspondence has come from the study of the maximally supersymmetric example: $3+1$-dimensional $\mathcal{N}=4$ Super-Yang-Mills (SYM) superconformal field theory (SCFT) and its dual Type IIB string theory on $AdS_5 \times S^5$. In this example, using integrability techniques, a proposal exists which can be used to calculate anomalous dimensions of generic, unprotected, gauge theory operators and match them with the energies of string states in the dual gravitational spacetime.[1] This proposal has a passed a large number of stringent tests [7–11]. This remarkable progress allows for the calculation of quantities at all values (small, intermediate and large) of the gauge theory coupling (see for example [11–14]), and provides some of the strongest evidence for this particular duality. More importantly, the integrability approach provides a description of how gauge/string duality actually works in practice.

Following the success of the integrability approach in the above mentioned maximally supersymmetric example, other dual pairs have been investigated using these techniques: these included duals involving $3+1$-dimensional gauge theories with less supersymmetry, and $2+1$-dimensional super-Chern-Simons with matter theories. It has been found that in all these examples the anomalous dimensions of operators in the field theory, and the energies of the corresponding string states are encoded in a Bethe Ansatz (BA). At small values of the coupling constant the BA reduces to that of an integrable short-range interaction spin-chain. At large values of the coupling, in the thermodynamic limit, it is best described by finite-gap equations which follow from the classical integrability of the string equations of motion.[2] A recent review of the developments in this field can be found in [15]. For another recent application of the algebraic approach to spacetimes with less supersymmetry see [16].

In this paper we will investigate string theories on backgrounds with an AdS_3 factor which preserve 16 supersymmetries and their CFT_2 duals using the integrability approach.[3] This programme was initiated in [1].[4] One of the main conceptual advantages of this approach is that one can investigate the dual pairs *without* having to perform an S-duality on the gravitational side. Most previous investigations of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence used WZW models for string theory (with NS-NS flux) on AdS_3 which are

[1]The results based on integrability techniques are limited to the planar limit of the gauge theory, in the spirit of 't Hooft's original suggestion about using $1/N^2$ as a small expansion parameter [6].

[2]The string equations of motion in these settings turn out to be equivalent to a flat Lax connection.

[3]The $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence has been extensively studied since the early days of the gauge/string correspondence; see for example [17–23].

[4]For earlier work on integrability in this context see [24,25] and more recently [26]. Integrability has also recently been investigated in the context of the hybrid string formulation on $AdS_3 \times S^3$ in [27–29] and for classical strings in the BTZ black hole background [30]

S-dual to string theory with R-R flux. In the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the strong coupling dual of the $Sym^N(T^4)$ CFT_2 is IIB string theory on $AdS_3 \times S^3 \times T^4$ with R-R flux. As a result, tests of this duality for unprotected quantities should in the first instance be done for string theory in the R-R background; this is the analogue of the planar limit in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality.

There are in fact two types AdS_3 geometries which preserve 16 supersymmetries: $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$.[5] The two backgrounds preserve, respectively, *small* and *large* $(4,4)$ superconformal symmetry; correspondingly, the finite-dimensional sub-algebras of these superconformal algebras are $psu(1,1|2)^2$ and $d(2,1;\alpha)^2$. The superalgebras $d(2,1;\alpha)$ depend on a parameter α which is related to the relative size of the radii of the geometry [31]. Denoting by l the AdS_3 radius and by $R_\pm$ the radii of the two S^3's the background solves the supergravity equations of motion when

$$\frac{1}{R_+^2} + \frac{1}{R_-^2} = \frac{1}{l^2}\,. \tag{1.1}$$

In terms of these geometric quantities, α is defined as

$$\alpha = \frac{l^2}{R_+^2} \equiv \sin^2\phi\,, \qquad \frac{l^2}{R_-^2} \equiv \cos^2\phi\,. \tag{1.2}$$

A candidate CFT_2 dual to string theory on $AdS_3 \times S^3 \times T^4$ is a sigma model on the moduli space of $Q_1 \in \mathbb{N}$ instantons in a $U(Q_5)$ gauge theory on T^4; this is a natural choice given that the $AdS_3 \times S^3 \times T^4$ background arises as the near-horizon limit of Q_1 D1-branes coincident with Q_5 D5-branes. On the other hand, very little is known about the CFT_2 dual of the $AdS_3 \times S^3 \times S^3 \times S^1$ background.[6] Some reasons for this ignorance are discussed in [37]; one of the main obstacles to identifying a suitable CFT_2 is that the supergravity approximation to the full string theory is not as useful in this case as in other examples – for example the BPS states of the finite dimensional sub-algebra of the full superconformal algebra need not be BPS in the full *large* super-Virasoro algebra!

In [1], it was found that string theory on the two AdS_3 backgrounds could be treated on equal footing: equations of motion for Green-Schwarz strings on both backgrounds could be written as flatness conditions for a Lax connection.[7] From this a set of finite-gap integral equations was formulated. In the case of the $AdS_3 \times S^3 \times T^4$ background as well as the $AdS_3 \times S^3 \times S^3 \times S^1$ background with the radii of the two S^3 factors equal it was possible to use the integral equations to postulate an all-loop discrete BA much as was done for the case of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality in [49]. This BA may be viewed as giving some of the first concrete prediction for what the elusive CFT_2 dual to the equal radius

[5]For the purpose of the present paper the $AdS_3 \times S^3 \times K3$ background can be treated as an orbifold of $AdS_3 \times S^3 \times T^4$ and shares many of it's features.

[6]Early work on this backgorund and its dual description can be found in [18, 31–36].

[7]GS actions for strings in curved spacetimes were first constructed in [38, 39]. However, these require the knowledge of the complete super-geometry rather than just the bosonic spacetime solution, making it harder to use them. An algebraic approach to GS actions in flat [40] and curved spacetimes [1, 41–45] leads to much simpler, though equivalent [46], expressions in spacetimes with enough (super-)symmetry. This algebraic approach has proven to be particularly useful since it leads to Lax-integrable equations of motion [47] due to the existence of a $\mathbb{Z}_4$-automorphism on the underlying supercurrents [48].

$AdS_3 \times S^3 \times S^3 \times S^1$ background should be like. There were however two unresolved puzzles. Firstly, how to generalise the BA construction when the radii of the two S^3 factors were not equal. Secondly, it was found that the string worldsheet theory on these backgrounds has a number of massless modes[8] which were not incorporated into the finite-gap (and hence also into the BA) equations. This second problem highlights the fact that at present we do not know how to incorporate massless worldsheet excitations into the integrability approach to the gauge/string correspondence.

In this paper we extend the integrability analysis of the $\text{AdS}_3/\text{CFT}_2$ correspondence begun in [1]. In section 2 we find a discrete all-loop BA, valid for all values of the radii of the two S^3's. We show that at strong coupling in the thermodynamic limit this BA reduces to the finite-gap integral equations that follow from the Lax-pair formuation of the GS string on $AdS_3 \times S^3 \times S^3 \times S^1$. These results resolve the first of the above puzzles.

In order to gain insight into what a CFT_2 dual to these backgrounds should be like, in section 2.2 we obtain the weak-coupling limit of the all-loop BA of section 2. In section 3 we construct an *alternating* spin-chain, together with an integrable Hamiltonian. We show that the BA for this Hamiltonian matches the weak coupling limit of the all-loop BA of section 2.2. The construction of the Hamiltonian is first done in an $sl(2|1)$ subsector of the theory and then lifted to the full $d(2,1;\alpha)^2$ spin-chain. It relies on the universal R-matrix of $sl(2|1)$ first found in [56]. A novel feature of the spin-chain we construct is that it has both left- and right-moving momenta, much as one would expect for a CFT_2. We expect that these results will be useful in the eventual identification of the correct CFT_2. At the equal-radius value $\alpha = 1/2$ the superalgebra is in fact $osp(4|2)^2$. Restricting to just one $osp(4|2)$ factor the spin-chain we construct is closely related to the ABJM spin-chain constructed in [57]. We discuss the relation of our BA and spin-chain to that of the ABJM BA and spin-chain in section 3.

In section 5 we consider the $AdS_3 \times S^3 \times T^4$ dual pair. In particular, in section 5.1, we show how the weakly coupled BA equations for this background [1] can be obtained as the $\alpha \to 1$ limit of the $AdS_3 \times S^3 \times S^3 \times S^1$ BA equations. In section 5.2 we construct a spin-chain whose energies are described by the weakly coupled $AdS_3 \times S^3 \times T^4$ BA equations. The spin-chain in this case is *not* alternating; instead it is homogenous. It would be very interesting to see how such a spin-chain emerges from the recent analysis of the weakly-coupled CFT_2 [58]. When restricted to just the left- or right-movers this spin-chain is closely related to the $psu(1,1|2)$ spin-chain one encounters in $\mathcal{N} = 4$ SYM [59]. Finally, in section 5.3, we make a proposal for how to incorporate massless modes into the integrable description of the gauge/string correspondence. We argue that the missing massless modes puzzle can be resolved by keeping track of the integrable structure of the alternating $d(2,1;\alpha)^2$ spin-chain in the $\alpha \to 1$ limit.

In section 6 we explore some of the features of the weak-coupling BA: we find twist-one solutions in a closed $sl(2)$ subsector, which are important in the identification of the spin-chain as an alternating chain; we find a degeneracy in the spectrum of states that is similar in nature the degeneracy of states in the ABJM model [60]; we investigate the behaviour of certain simple solutions to the BA equations in the $\alpha \to 1$ limit. We conclude in section 7. Some of the technical details are relegated to the appendices.

[8]The appearance of these massless modes is most easily seen from the plane-wave analysis [1,50–55].

2 All-loop Bethe equations for $d(2,1;\alpha)^2$

In this section we propose an all-loop BA for $d(2,1;\alpha)^2$. We show that in the continuum limit this BA reproduces the finite-gap equations obtained from the GS string constructed in [1]. We also obtain the weak-coupling limit of the all-loop BA. In this limit, the all-loop BA reduces to a conventional BA for $d(2,1;\alpha)^2$ in a particular representation.[9] The weights of the representation relevant to our case can then be easily read-off. This will in turn be useful in the construction of a spin-chain for the small-coupling BA which we do in the following section.

In [1] a set of Bethe equations for the coset model on $\mathrm{OSp}(4|2)^2$ were proposed:

$$
\begin{aligned}
\left(\frac{x^+_{1,i}}{x^-_{1,i}}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_1} \frac{x^+_{1,i}-x^-_{1,k}}{x^-_{1,i}-x^+_{1,k}} \frac{1-\frac{1}{x^+_{1,i}x^-_{1,k}}}{1-\frac{1}{x^-_{1,i}x^+_{1,k}}} \sigma^2(x_{1,i},x_{1,k}) \\
&\quad\times \prod_{k=1}^{K_2} \frac{x^-_{1,i}-x_{2,k}}{x^+_{1,i}-x_{2,k}} \prod_{k=1}^{K_{\bar{2}}} \frac{1-\frac{1}{x^-_{1,i}x_{\bar{2},k}}}{1-\frac{1}{x^+_{1,i}x_{\bar{2},k}}} \prod_{k=1}^{K_{\bar{1}}} \sigma^{-2}(x_{1,i},x_{\bar{1},k})\,,
\end{aligned}
\tag{2.1a}
$$

$$
1 = \prod_{k=1}^{K_1} \frac{x_{2,i}-x^+_{1,k}}{x_{2,i}-x^-_{1,k}} \prod_{k=1}^{K_3} \frac{x_{2,i}-x^+_{3,k}}{x_{2,i}-x^-_{3,k}} \prod_{k=1}^{K_{\bar{1}}} \frac{1-\frac{1}{x_{2,i}x^+_{\bar{1},k}}}{1-\frac{1}{x_{2,i}x^-_{\bar{1},k}}} \prod_{k=1}^{K_{\bar{3}}} \frac{1-\frac{1}{x_{2,i}x^+_{\bar{3},k}}}{1-\frac{1}{x_{2,i}x^-_{\bar{3},k}}}\,,
\tag{2.1b}
$$

$$
\begin{aligned}
\left(\frac{x^+_{3,i}}{x^-_{3,i}}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_3} \frac{x^+_{3,i}-x^-_{3,k}}{x^-_{3,i}-x^+_{3,k}} \frac{1-\frac{1}{x^+_{3,i}x^-_{3,k}}}{1-\frac{1}{x^-_{3,i}x^+_{3,k}}} \sigma^2(x_{3,i},x_{3,k}) \\
&\quad\times \prod_{k=1}^{K_2} \frac{x^-_{3,i}-x_{2,k}}{x^+_{3,i}-x_{2,k}} \prod_{k=1}^{K_{\bar{2}}} \frac{1-\frac{1}{x^-_{3,i}x_{\bar{2},k}}}{1-\frac{1}{x^+_{3,i}x^-_{\bar{2},k}}} \prod_{k=1}^{K_{\bar{3}}} \sigma^{-2}(x_{3,i},x_{\bar{3},k})\,,
\end{aligned}
\tag{2.1c}
$$

$$
\begin{aligned}
\left(\frac{x^-_{\bar{1},i}}{x^+_{\bar{1},i}}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_{\bar{1}}} \frac{x^+_{\bar{1},i}-x^-_{\bar{1},k}}{x^-_{\bar{1},i}-x^+_{\bar{1},k}} \frac{1-\frac{1}{x^+_{\bar{1},i}x^-_{\bar{1},k}}}{1-\frac{1}{x^-_{\bar{1},i}x^+_{\bar{1},k}}} \sigma^2(x_{\bar{1},i},x_{\bar{1},k}) \\
&\quad\times \prod_{k=1}^{K_{\bar{2}}} \frac{x^-_{\bar{1},i}-x_{\bar{2},k}}{x^+_{\bar{1},i}-x_{\bar{2},k}} \prod_{k=1}^{K_2} \frac{1-\frac{1}{x^-_{\bar{1},i}x_{2,k}}}{1-\frac{1}{x^+_{\bar{1},i}x^-_{2,k}}} \prod_{k=1}^{K_1} \sigma^{-2}(x_{\bar{1},i},x_{1,k})\,,
\end{aligned}
\tag{2.1d}
$$

$$
1 = \prod_{k=1}^{K_{\bar{1}}} \frac{x_{\bar{2},i}-x^+_{\bar{1},k}}{x_{\bar{2},i}-x^-_{\bar{1},k}} \prod_{k=1}^{K_{\bar{3}}} \frac{x_{\bar{2},i}-x^+_{\bar{3},k}}{x_{\bar{2},i}-x^-_{\bar{3},k}} \prod_{k=1}^{K_1} \frac{1-\frac{1}{x_{\bar{2},i}x^+_{1,k}}}{1-\frac{1}{x_{\bar{2},i}x^-_{1,k}}} \prod_{k=1}^{K_3} \frac{1-\frac{1}{x_{\bar{2},i}x^+_{3,k}}}{1-\frac{1}{x_{\bar{2},i}x^-_{3,k}}}\,,
\tag{2.1e}
$$

$$
\begin{aligned}
\left(\frac{x^-_{\bar{3},i}}{x^+_{\bar{3},i}}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_3} \frac{x^+_{\bar{3},i}-x^-_{\bar{3},k}}{x^-_{\bar{3},i}-x^+_{\bar{3},k}} \frac{1-\frac{1}{x^+_{\bar{3},i}x^-_{\bar{3},k}}}{1-\frac{1}{x^-_{\bar{3},i}x^+_{\bar{3},k}}} \sigma^2(x_{\bar{3},i},x_{\bar{3},k}) \\
&\quad\times \prod_{k=1}^{K_2} \frac{x^-_{\bar{3},i}-x_{\bar{2},k}}{x^+_{\bar{3},i}-x_{\bar{2},k}} \prod_{k=1}^{K_{\bar{2}}} \frac{1-\frac{1}{x^-_{\bar{3},i}x_{2,k}}}{1-\frac{1}{x^+_{\bar{3},i}x^-_{2,k}}} \prod_{k=1}^{K_{\bar{3}}} \sigma^{-2}(x_{\bar{3},i},x_{3,k})\,.
\end{aligned}
\tag{2.1f}
$$

[9] BA equations for any (super)-algebra in certain classes of representations have been proposed in [61].

Here $\sigma(x,y)$ is a dressing phase factor[10], and the variables $x^{\pm}_{l,i}$ satisfy

$$x^{\pm}_{l,i} + \frac{1}{x^{\pm}_{l,i}} = x_{l,i} + \frac{1}{x_{l,i}} \pm \frac{i}{2h}\,, \tag{2.2}$$

where $h = h(\lambda)$ is a function of the worldsheet coupling constant λ. At large coupling it behaves as

$$h(\lambda) \approx \frac{\sqrt{\lambda}}{2\pi}\,, \quad (\lambda \to \infty)\,. \tag{2.3}$$

We will assume that $h(\lambda) \to 0$ as $\lambda \to 0$, so that there is a weak coupling limit of the Bethe equations.

Given a solution of (2.1), the corresponding total energy E and momentum P is given by

$$E = ih \sum_{l=1,3,\bar{1},\bar{3}} \sum_{j}^{K_l} \left(\frac{1}{x^{+}_{l,j}} - \frac{1}{x^{-}_{l,j}} \right)\,, \tag{2.4}$$

$$e^{iP} = \frac{\prod_{l=1,3}}{\prod_{l=\bar{1},\bar{3}}} \prod_{j} \frac{x^{+}_{l,j}}{x^{-}_{l,j}} \equiv 1\,. \tag{2.5}$$

We here propose a generalization of the above equations to the full symmetry group $\mathrm{D}(2,1;\alpha)^2$ for any α. The Bethe equations, energy and momentum take exactly the same form as in (2.1), (2.4) and (2.5) respectively. However, the Zhukovsky map in (2.2) is deformed to

$$x^{\pm}_{l,i} + \frac{1}{x^{\pm}_{l,i}} = x_{l,i} + \frac{1}{x_{l,i}} \pm \frac{iw_l}{2h}\,, \tag{2.6}$$

where

$$w_1 = w_{\bar{1}} = 2\alpha\,, \quad w_3 = w_{\bar{3}} = 2(1-\alpha)\,. \tag{2.7}$$

Since $x^{\pm}_2$ and $x^{\pm}_{\bar{2}}$ do not appear in (2.1), we do not need to specify w_2 and $w_{\bar{2}}$. The elementary magnons now have the dispersion relation

$$\epsilon_l(p) = \sqrt{m_l^2 + 4h^2 \sin^2 \frac{p}{2}}\,, \tag{2.8}$$

where the masses are

$$m_1^2 = \alpha\,, \qquad\qquad m_3^2 = 1-\alpha\,. \tag{2.9}$$

2.1 Classical Bethe equations for $d(2,1;\alpha)^2$

Effectively, the above generalization of the Bethe equations from $osp(4|2)^2$ to $d(2,1;\alpha)^2$ takes the form of a rescaling of the coupling that varies between the different Dynkin

[10] In general the dressing phase can take a different form than the BES/BHL dressing phase [62, 63] that appears in $\mathcal{N}=4$ SYM and ABJM. As we will see below, the classical limit of the BA requires the leading strong coupling behavior of σ to be given by the AFS phase [64].

nodes. The classical limit will be almost identical in the two cases. For large h, $x^{\pm}$ behave as

$$x^{\pm}_{l,i} = x_{l,i} \pm \frac{iw_l}{2h}\frac{x^2_{l,i}}{x^2_{l,i}-1} + \mathcal{O}(1/h^2)\,. \tag{2.10}$$

Expanding the Bethe equations, the only effect of the extra factor w_l will be an overall factor for each factor in the equations.

There are two important points that makes this limit work out. Firstly, there is no direct link between the $x_{2,i}$ and $x_{3,k}$ roots in the Bethe equations. Secondly, the dressing phase only links nodes in the Dynkin diagram that are associated with the same value of w_l (*i.e.*, it either involves two roots $x_{l,i}$ and $x_{l,k}$ at the same node, or one root $x_{l,i}$ from one copy of $d(2,1;\alpha)$ and another root $x_{\bar{l},k}$ from the corresponding node of the second copy of $d(2,1;\alpha)$.)

The only subtlety is in the coupling dependence of the dressing phase. To get a nice form in the strong coupling limit we need to rescale the explicit coupling dependence in the phase coupling $x_{l,i}$ with $x_{l,k}$ or $x_{\bar{l},k}$ by a factor $1/w_l$. For large h, the dressing phase then reduces to the AFS phase [64]

$$\sigma(x_l,y_l) \approx \frac{1-\frac{1}{x_l^- y_l^+}}{1-\frac{1}{x_l^+ y_l^-}}\left[\frac{\left(1-\frac{1}{x_l^+ y_l^-}\right)\left(1-\frac{1}{x_l^- y_l^+}\right)}{\left(1-\frac{1}{x_l^+ y_l^+}\right)\left(1-\frac{1}{x_l^- y_l^-}\right)}\right]^{\frac{ih}{w_l}\left(x_l+\frac{1}{x_l}-y_l-\frac{1}{y_l}\right)} \tag{2.11}$$

With these modifications, the classical Bethe equations can be derived for general α in the exact same way as for $\alpha=1/2$. The resulting equations read

$$\begin{aligned}\pm 4\pi\alpha\frac{\mathcal{E}x}{x^2-1} + 2\pi n^{\pm}_{1,i} = 4\alpha \fint dy\frac{\rho_1^{\pm}}{x-y} - 2\alpha\int dy\frac{\rho_2^{\pm}}{x-y}\\ - 4\alpha\int\frac{dy}{y^2}\frac{\rho_1^{\mp}}{x-\frac{1}{y}} + 2\alpha\int\frac{dy}{y^2}\frac{\rho_2^{\mp}}{x-\frac{1}{y}}\,,\end{aligned} \tag{2.12a}$$

$$\begin{aligned}2\pi n^{\pm}_{2,i} = -2\alpha\int dy\frac{\rho_1^{\pm}}{x-y} - 2(1-\alpha)\int dy\frac{\rho_3^{\pm}}{x-y}\\ + 2\alpha\int\frac{dy}{y^2}\frac{\rho_1^{\mp}}{x-\frac{1}{y}} + 2(1-\alpha)\int\frac{dy}{y^2}\frac{\rho_3^{\mp}}{x-\frac{1}{y}}\,,\end{aligned} \tag{2.12b}$$

$$\begin{aligned}\pm 4\pi(1-\alpha)\frac{\mathcal{E}x}{x^2-1} + 2\pi n^{\pm}_{3,i} = 4(1-\alpha)\fint dy\frac{\rho_3^{\pm}}{x-y} - 2(1-\alpha)\int dy\frac{\rho_1^{\pm}}{x-y}\\ - 4(1-\alpha)\int\frac{dy}{y^2}\frac{\rho_3^{\mp}}{x-\frac{1}{y}} + 4(1-\alpha)\int\frac{dy}{y^2}\frac{\rho_1^{\mp}}{x-\frac{1}{y}}\,.\end{aligned} \tag{2.12c}$$

Setting $\alpha=\sin^2\phi$, these equations exactly agree with the classical Bethe equations of [1].

2.2 Weak-coupling limit

Let us now turn to the weakly-coupled limit of the all-loop BA proposed above. In this limit, $h\to 0$, the variables x_i will behave as

$$x_i \approx \frac{u_i}{h}\,, \tag{2.13}$$

where $u_i \sim \mathcal{O}(1)$ as $h \to 0$. Hence

$$x_i^{\pm} \approx \frac{u_i \pm i w_i/2}{h}. \tag{2.14}$$

Moreover, we will assume that $\sigma(x,y) \to 1$ as $h \to 0$. This means that in the weak coupling limit, the Bethe equations in (2.1) decouple into two sets of equations involving only variables with either un-bared or bared indices. Inserting the above weak coupling expressions for x_i and $x_i^{\pm}$ into the first three equations in (2.1) we get

$$\left(\frac{u_{1,i}+i\alpha}{u_{1,i}-i\alpha}\right)^L = \prod_{\substack{k=1\\k\neq i}}^{K_1} \frac{u_{1,i}-u_{1,k}+2i\alpha}{u_{1,i}-u_{1,k}-2i\alpha} \prod_{k=1}^{K_2} \frac{u_{1,i}-u_{2,k}-i\alpha}{u_{1,i}-u_{2,k}+i\alpha}\,, \tag{2.15a}$$

$$1 = \prod_{k=1}^{K_1} \frac{u_{2,i}-u_{1,k}-i\alpha}{u_{2,i}-u_{1,k}+i\alpha} \prod_{k=1}^{K_3} \frac{u_{2,i}-u_{3,k}-i(1-\alpha)}{u_{2,i}-u_{3,k}+i(1-\alpha)}\,, \tag{2.15b}$$

$$\left(\frac{u_{3,i}+i(1-\alpha)}{u_{3,i}-i(1-\alpha)}\right)^L = \prod_{\substack{k=1\\k\neq i}}^{K_3} \frac{u_{3,i}-u_{3,k}+2i(1-\alpha)}{u_{3,i}-u_{3,k}-2i(1-\alpha)} \prod_{k=1}^{K_2} \frac{u_{3,i}-u_{2,k}-i(1-\alpha)}{u_{3,i}-u_{2,k}+i(1-\alpha)}\,. \tag{2.15c}$$

We now extract the weights of the $d(2,1;\alpha)$ representation to which the above BA applies. Recall, that for a general (super)-Lie algebra with simple roots $\vec{\alpha}_q$ we can write a set of Bethe equations in the representation give by weights $\vec{w}$ as [61, 65]

$$\left(\frac{u_{l,i}+\frac{i}{2}\vec{\alpha}_l\cdot\vec{w}}{u_{l,i}-\frac{i}{2}\vec{\alpha}_l\cdot\vec{w}}\right)^L = \prod_{\substack{k=1\\k\neq i}}^{K_l} \frac{u_{l,i}-u_{l,k}+\frac{i}{2}\vec{\alpha}_l\cdot\vec{\alpha}_l}{u_{l,i}-u_{l,k}-\frac{i}{2}\vec{\alpha}_l\cdot\vec{\alpha}_l} \prod_{l'\neq l} \prod_{k=1}^{K_{l'}} \frac{u_{l,i}-u_{l',k}+\frac{i}{2}\vec{\alpha}_l\cdot\vec{\alpha}_{l'}}{u_{l,i}-u_{l,k}-\frac{i}{2}\vec{\alpha}_l\cdot\vec{\alpha}_{l'}} \tag{2.16}$$

As we will argue shortly the spin-chain we are interested in will be alternating. In the next section we will compare equations (2.15) and (2.16) to find that the spin-chain should have sites which alternate between the representations $\left(-\frac{\alpha}{2};\frac{1}{2};0\right)$ and $\left(-\frac{1-\alpha}{2};0;\frac{1}{2}\right)$.

3 An integrable $d(2,1;\alpha)^2$ spin-chain

In the previous section we have made a proposal for an all-loop Bethe Ansatz for the energies of massive string states in the $AdS_3\times S^3\times S^3\times S^1$ background. In equation (2.15) above, we extracted the weak-coupling limit of this BA. In this section, we will construct an alternating spin-chain with integrable Hamiltonian whose energies are described precisely by (2.15). This spin-chain Hamiltonian should provide vital clues in identifying the elusive CFT_2 dual of string theory on this background.

To see what kind of spin-chain we should consider note the following observations. In section 6 below, we show that the weak-coupling BA equations 2.15 have non-trivial solutions for twist-one operators. More specifically, we identify an $sl(2)$ subsector of the BA (2.15) and show it has non-trivial solutions of length $L=1$. This indicates that the spin-chain is not of the conventional homogenous type, where, by definition, no such solutions exist.[11] Further, in section 4.1 we show that at $\alpha = 1/2$ the weak-coupling BA equations match precisely the $osp(4|2)$ subsector of the weak-coupling spin-chain [57] of the ABJM theory. Both these observations lead us to conclude that the spin-chain related to the BA (2.15) should be alternating.

3.1 The alternating $d(2,1;\alpha)$ spin-chain

Before constructing the Hamiltonian, we will collect here some facts about $d(2,1;\alpha)$ representations. In this discussion we follow closely [66]; a very nice review of Lie superalgebras can be found in [67]. If we pick a suitable real form, the algebra $d(2,1;\alpha)$ has a bosonic subalgebra $sl(2,\mathbb{R}) \oplus su(2) \oplus su(2)$. The corresponding generators are denoted as S_μ ($\mu = 0,1,2$), L_m ($n = 3,4,5$) and $R_{\dot m}$ ($\dot m = 6,7,8$). The vector indices of the bosonic generators are raised and lowered using $\eta_{\mu\nu} = \mathrm{diag}(-++)$, δ_{mn} and $\delta_{\dot m\dot n}$, respectively. There are eight fermionic generators transforming as a tri-spinor under the bosonic subgroup and denoted by $Q_{a\alpha\dot\alpha}$, where each index takes values $+$ or $-$. The (anti)-commutation relations of $d(2,1;\alpha)$ are[12]

$$
\begin{aligned}
&[S_\mu, S_\nu] = i\,\epsilon_{\mu\nu\rho}\,S^\rho\,, && [S_\mu, Q_{a\alpha\dot\alpha}] = \tfrac{1}{2} Q_{b\alpha\dot\alpha}\gamma^b_{\mu\,a}\,,\\
&[L_m, L_n] = i\,\epsilon_{mnp}\,L^p\,, && [L_m, Q_{a\alpha\dot\alpha}] = \tfrac{1}{2} Q_{a\beta\dot\alpha}\gamma^\beta_{m\,\alpha}\,,\\
&[R_{\dot m}, R_{\dot n}] = i\,\epsilon_{\dot m\dot n\dot p}\,R^{\dot p}\,, && [R_{\dot m}, Q_{a\alpha\dot\alpha}] = \tfrac{1}{2} Q_{a\alpha\dot\beta}\gamma^{\dot\beta}_{\dot m\,\dot\alpha}\,,
\end{aligned}
\tag{3.1}
$$
$$
\{Q_{a\alpha\dot\alpha}, Q_{b\beta\dot\beta}\} = -\Big(S_\mu\,\gamma^\mu_{ab}\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta} + \alpha\,L_m\,\epsilon_{ab}\,\gamma^m_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta} + (1-\alpha)\,R_{\dot m}\,\epsilon_{ab}\,\epsilon_{\alpha\beta}\,\gamma^{\dot m}_{\dot\alpha\dot\beta}\Big)\,.
$$

The simple roots for $d(2,1;\alpha)$ are given by

$$
\begin{aligned}
\vec\alpha_1 &= \left(\sqrt{2\alpha}; 0; -\sqrt{2\alpha}\right)\,,\\
\vec\alpha_2 &= \left(-\frac{\sqrt{\alpha}+\sqrt{1-\alpha}}{\sqrt{2}}; 1; \frac{\sqrt{\alpha}-\sqrt{1-\alpha}}{\sqrt{2}}\right)\,,\\
\vec\alpha_3 &= \left(\sqrt{2(1-\alpha)}; 0; \sqrt{2(1-\alpha)}\right)\,,
\end{aligned}
\tag{3.2}
$$

where the signature is $(+\;-\;+)$. The Cartan matrix $A_{ij} = \vec\alpha_i \cdot \vec\alpha_j$ is

$$
A = \begin{pmatrix} 4\alpha & -2\alpha & 0 \\ -2\alpha & 0 & -2(1-\alpha) \\ 0 & -2(1-\alpha) & 4(1-\alpha) \end{pmatrix}\,.
\tag{3.3}
$$

[11] The fact that the $d(2,1;\alpha)$ spin-chain is alternating can also be anticipated from the fact that it contains two momentum-carrying roots. We would like to thanks Kostya Zarembo for a discussion of this point.

[12] The anti-symmetric symbols $\epsilon_{\mu\nu\rho}$, ϵ_{mnp} and $\epsilon_{\dot m\dot n\dot p}$ are normalized so that $\epsilon_{012} = \epsilon_{345} = \epsilon_{678} = 1$. The gamma-matrices are given by

$$
(\gamma^\mu)^a{}_b = (-\sigma_3, i\sigma_2, -i\sigma_1)\,,\quad (\gamma^m)^\alpha{}_\beta = (\sigma_1,\sigma_2,\sigma_3)\,,\quad (\gamma^{\dot m})^{\dot\alpha}{}_{\dot\beta} = (\sigma_1,\sigma_2,\sigma_3)\,.
$$

There are three special values for α where the algebra simplifies. For $\alpha = 1/2$ it is equivalen to $d(2,1) = osp(4|2)$. When $\alpha = 0$ or $\alpha = 1$ it reduces to $psu(1,1|2)$.

A highest-weight module of $d(2,1;\alpha)$ consists of a number of heighest-weight modules of the bosonic sub-algebra $sl(2) \times su(2) \times su(2)$. The fermionic generators of the algebra map these into each other while preserving algebra relations; this is the conventional induced representation way of constructing a superalgebra module [68, 69]. Heighest weight representations of $d(2,1;\alpha)$ are specified by highest weights of the bosonic subalgebra [66, 68]. To denote such a representation we will use the notation

$$(p\,;\,q\,;\,r)$$

where $p \in \mathbb{C}$ is the spin of the non-compact $sl(2)$ and $q\,,\,r \in \frac{1}{2}\mathbb{Z}$ are the compact spins. The corresponding Dynkin labels are

$$[2q; -2p - 2\alpha q - 2(1-\alpha)r; 2r]\ .$$

A general representation $(p;q;r)$ decomposes under the bosonic subalgebra as

$$\begin{aligned}(p;q;r) \to \Big\{&(p,q,r), (p-\tfrac{1}{2}, q \pm \tfrac{1}{2}, r \pm \tfrac{1}{2}), (p-\tfrac{1}{2}, q \pm \tfrac{1}{2}, r \mp \tfrac{1}{2}),\\ &(p-1, q \pm 1, r), (p-1, q, r \pm 1), (p-1, q, r)^{\oplus 2},\\ &(p-\tfrac{3}{2}, q \pm \tfrac{1}{2}, r \pm \tfrac{1}{2}), (p-\tfrac{3}{2}, q \pm \tfrac{1}{2}, r \mp \tfrac{1}{2}), (p-2, q, r)\Big\}\,.\end{aligned} \tag{3.4}$$

This decomposition is only valid for a generic module and sometimes has to be modified in two important ways [66]. Firstly, for a heighest-weight module $(p;q;r)$, if $q < 1$ or $r < 1$ the right-hand-side of equation (3.4) will contain at most terms (p', q', r') for which $q'\,,\,r' \geq 0$.[13] Secondly, as in all superalgebras, there are heighest weight $d(2,1;\alpha)$ modules for which the heighest weight state is anihilated not only by the raising operators, but also by a subset of the fermionic lowering operators; such modules are called atypical, short or BPS. Atypical modules occur only when p, q and r satisfy so-called shortening conditions. In the present case these are [66, 68]

$$0 = p + \alpha q + (1-\alpha) r\,, \tag{3.5}$$
$$0 = p - \alpha(q+1) + (1-\alpha) r\,, \tag{3.6}$$
$$0 = p + \alpha q - (1-\alpha)(r+1)\,, \tag{3.7}$$
$$0 = p - \alpha(q+1) - (1-\alpha)(r+1)\,. \tag{3.8}$$

Let us now consider the $d(2,1;\alpha)$ modules that will feature in our spin-chain. The discussion at the start of the present section strongly suggests that the spin-chain we are after is alternating. Comparing the weak-coupling BA (2.15) with the general BA for any

[13] For special values of q and r it may also not contain some (p', q', r') for which $q'\,,\,r' \geq 0$. This may happen if a particular (p', q', r') for which $q'\,,\,r' \geq 0$ is a descendant of a sub-module (p'', q'', r'') for which $q''\,,\,r'' < 0$.

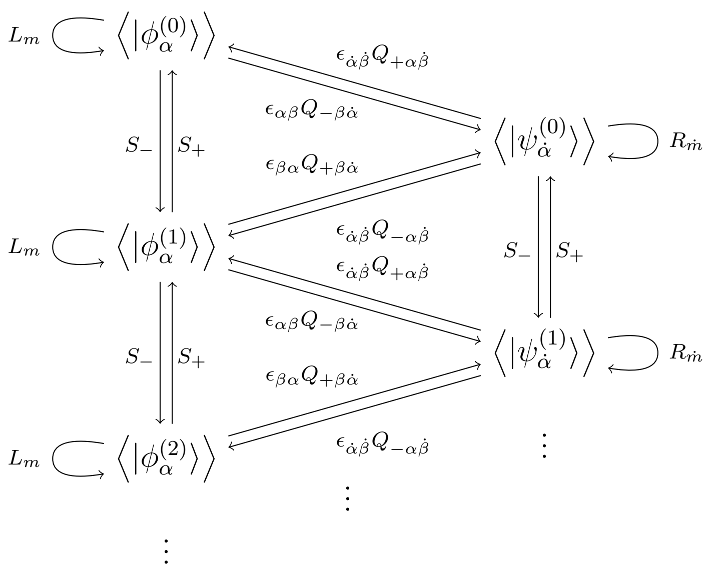


Figure 1: An illustration of the short $\left(-\frac{\alpha}{2};\frac{1}{2},0\right)$ $d(2,1;\alpha)$ module. For clarity, in the figure we have neglected to include the exact coefficients of the linear maps for the representation. These can be read-off from equations (A.4)-(A.6) in Appendix A. To emphasise this we use the notation $\left\langle\dots\right\rangle$ to denote the span of $\dots$. The states in the left-hand side column have $(L_5,R_8)=(\pm\frac{1}{2},0)$ and those in the right-hand side column have $(L_5,R_8)=(0,\pm\frac{1}{2})$. Starting at the top, the rows in the diagram have S_0 eigenvalues equal to $-\frac{\alpha}{2}$, $-\frac{\alpha+1}{2}$; $-\frac{\alpha+2}{2}$, ...

weights given in (2.16), we may then read-off the weights of the $d(2,1;\alpha)$ representations at the even and odd sites to be

$$\left(-\tfrac{\alpha}{2};\tfrac{1}{2};0\right) \qquad\qquad \text{and} \qquad\qquad \left(-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}\right)\,. \tag{3.9}$$

These representations satisfy shortening conditions in equation 3.5 and are particularly simple since q and r are also very small for them. In terms of representations of the bosonic part of the algebra they decompose as

$$\left(-\tfrac{\alpha}{2};\tfrac{1}{2};0\right)=\left\{(-\tfrac{\alpha}{2},\tfrac{1}{2},0),(-\tfrac{\alpha+1}{2},0,\tfrac{1}{2})\right\}\,, \tag{3.10}$$
$$\left(-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}\right)=\left\{(-\tfrac{1-\alpha}{2},0,\tfrac{1}{2})\,,\,(-\tfrac{2-\alpha}{2},0,\tfrac{1}{2})\right\}\,. \tag{3.11}$$

A representation for these representations is given in appendix A. The heighest-weight state of the modules $(-\frac{\alpha}{2},\frac{1}{2},0)$ and $(-\frac{\alpha+1}{2},0,\frac{1}{2})$ of the bosonic sub-algebra will be denoted by $|\phi^{(0)}_{\alpha=+}\rangle$ and $|\psi^{(0)}_{\dot\alpha=+}\rangle$, respectively. The subscripts $_\alpha$ and $_{\dot\alpha}$ indicate that these states are part of a doublet representation under L_m, and $R_{\dot m}$, respectively. When acted on by S_- both these states generate a discrete infinite dimensional representation of $sl(2)$, with descendants denoted by the super-script $^{(n)}$ for $n\in\mathbb{N}$. In figure 1 we denote pictorially this representation.

The $\left(-\frac{1-\alpha}{2};0;\frac{1}{2}\right)$ module can be easily obtained from the $\left(-\frac{\alpha}{2};\frac{1}{2};0\right)$ module by exchanging the two sets of $su(2)$ generators and replacing α with $1-\alpha$.

3.2 The integrable $d(2,1;\alpha)$ hamiltonian

In the previous sub-section we presented the free alternating spin-chain relevant to our problem. In this section we will construct an integrable Hamiltonian for this spin-chain. We do this using the R-matrix approach. In section 3.2.1 we first construct the Hamiltonian in an $sl(2|1)$ subsector of $d(2,1;\alpha)$ using the universal R-matrix found in [56]. In section 3.3, we show that there is a unique lift of this R-matrix to the full $d(2,1;\alpha)$; this lift is similar in spirit to those in [70] and [60]. As an example, in section 3.4 we write down explicitly the Hamiltonian for the $su(2)^2$ subsector of the alternating $d(2,1;\alpha)$ spin-chain. Finally, in section 3.5 we construct the spin-chain and Hamiltonian of the full $d(2,1;\alpha)^2$ symmetry relevant to the AdS/CFT duality.

3.2.1 The $sl(2|1)$ subsector

We begin this sub-section by reviewing some facts about the $sl(2|1)$ algebra, its embedding into $d(2,1;\alpha)$ and its representations. Recall that $sl(2|1)$ is a maximal regular sub-algebra of $d(2,1;\alpha)$ [67]; an explicit embedding is given by

$$\begin{aligned} J_0 = -S_0\,, \quad J_\pm = \mp S_1 + i\,S_2\,, \quad B = -\alpha\,L_5 + (1-\alpha)\,R_8\,, \\ Q^+ = Q_{--+}\,, \quad Q^- = Q_{-+-}\,, \quad S^+ = Q_{+-+}\,, \quad S^- = Q_{++-}\,. \end{aligned} \tag{3.12}$$

It is easy to check that these generators satisfy the $sl(2|1)$ algebra

$$\begin{gathered} [J_0, J_\pm] = \pm J_\pm\,, \quad [J_+, J_-] = 2J_0\,, \\ [B, Q^\pm] = \pm\tfrac{1}{2}Q^\pm\,, \quad [J_0, Q^\pm] = +\tfrac{1}{2}Q^\pm\,, \quad [J_-, Q^\pm] = S^\pm\,, \\ [B, S^\pm] = \pm\tfrac{1}{2}S^\pm\,, \quad [J_0, S^\pm] = -\tfrac{1}{2}S^\pm\,, \quad [J_+, S^\pm] = Q^\pm\,, \\ \{Q^+, Q^-\} = +J_+\,, \quad \{Q^+, S^-\} = -J_0 + B\,, \\ \{S^+, S^-\} = -J_-\,, \quad \{S^+, Q^-\} = -J_0 - B\,. \end{gathered} \tag{3.13}$$

The irreducible representations of $sl(2|1)$ are labeled by the highest weights (j,b) under J_0 and B. Atypical representations have $b = \mp j$, and are called chiral and anti-chiral. The chiral representation $(\frac{\alpha}{2}, -\frac{\alpha}{2})$ and the anti-chiral representation $(\frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ will be useful in the construction of the integrable Hamiltonian below. The chiral representation $(\frac{\alpha}{2}, -\frac{\alpha}{2})$ can be obtained from the $d(2,1;\alpha)$ representation $\left(-\frac{\alpha}{2}; \frac{1}{2}; 0\right)$ constructed in Appendix A. In particular, the highest-weight state, $|\phi^{(0)}_{\alpha=+}\rangle$, of the $\left(-\frac{\alpha}{2}; \frac{1}{2}; 0\right)$ module is taken as the highest-weight state of the $(\frac{\alpha}{2}, -\frac{\alpha}{2})$ module. The only fermionic generator that does not annihilate this state is Q^- which acts as

$$Q^-\,|\phi^{(0)}_{\alpha=+}\rangle = -\sqrt{\alpha}\,|\psi^{(0)}_{\dot\alpha=+}\rangle\ . \tag{3.14}$$

Acting with the bosonic lowering operator J_+ we obtain the (countably infinite) descendents $|\phi^{(n)}_{\alpha=+}\rangle$ and $|\psi^{(n)}_{\dot\alpha=+}\rangle$, where $n \geq 0$, which form the basis for the $(\frac{\alpha}{2}, -\frac{\alpha}{2})$ module. In summary then, starting from the $d(2,1;\alpha)$ module $\left(-\frac{\alpha}{2}; \frac{1}{2}; 0\right)$ whose basis are the states $|\phi^{(n)}_{\alpha}\rangle$ and $|\psi^{(n)}_{\dot\alpha}\rangle$ we can obtain the $sl(2|1)$ module $(\frac{\alpha}{2}, -\frac{\alpha}{2})$ by restricting to ${}_{\alpha=+}$ and ${}_{\dot\alpha=+}$ states. In figure 2 we provide a pictorial representation of this module. Anti-chiral

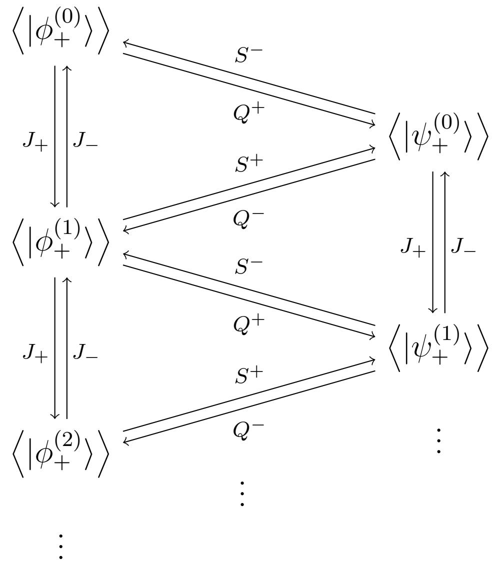


Figure 2: An illustration of the short $(\frac{\alpha}{2}, -\frac{\alpha}{2})$ $sl(2|1)$ module. For clarity, in the figure we have neglected to include the exact coefficients of the linear maps for the representation. These can be read-off from equations (A.4)-(A.6) in Appendix A using the embedding (3.12). To emphasise this we use the notation $\left\langle \dots \right\rangle$ to denote the span of $\dots$. The states in the left-hand side column have $B = -\frac{\alpha}{2}$ and those in the right-hand side column have $B = \frac{1-\alpha}{2}$. Starting at the top, the rows in the diagram have J_0 eigenvalues equal to $\frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2}, \dots$

modules $(\frac{1-\alpha}{2}, \frac{1-\alpha}{2})$ can be obtained in an analogous fashion from the $d(2,1;\alpha)$ modules $\left(-\frac{1-\alpha}{2}; 0; \frac{1}{2}\right)$.

In constructing R-matrices it is often useful to know the tensor product decompositions of the constituent representations. The tensor products of two atypical representations are given by

$$(j_1, \pm j_1) \otimes (j_2, \pm j_2) = (j, \pm j) \oplus \bigoplus_{n=0}^{\infty} \left(j + \tfrac{1}{2} + n, \pm(j - \tfrac{1}{2})\right), \tag{3.15}$$

$$(j_1, \pm j_1) \otimes (j_2, \mp j_2) = \bigoplus_{n=0}^{\infty} (j + n, \pm \bar{j}), \tag{3.16}$$

where $j = j_1 + j_2$ and $\bar{j} = j_1 - j_2$. For the representations we are interested in this implies,

$$\left(\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}\right) \otimes \left(\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}\right) = (\alpha, -\alpha) \oplus \bigoplus_{n=1}^{\infty} \left(\alpha - \tfrac{1}{2} + n, \tfrac{1}{2} - \alpha\right), \tag{3.17}$$

$$\left(\tfrac{1-\alpha}{2}, \tfrac{1-\alpha}{2}\right) \otimes \left(\tfrac{1-\alpha}{2}, \tfrac{1-\alpha}{2}\right) = (1 - \alpha, 1 - \alpha) \oplus \bigoplus_{n=1}^{\infty} \left(\tfrac{1}{2} - \alpha + n, \tfrac{1}{2} - \alpha\right), \tag{3.18}$$

$$\left(\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}\right) \otimes \left(\tfrac{1-\alpha}{2}, \tfrac{1-\alpha}{2}\right) = \bigoplus_{n=0}^{\infty} \left(\tfrac{1}{2} + n, \tfrac{1}{2} - \alpha\right). \tag{3.19}$$

The R-matrix $R_{ab}(u)$ acts on the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$ and can be decomposed as

$$R_{ab}(u) = \sum_c R^c_{ab}(u)\mathcal{P}_c\,, \tag{3.20}$$

where the sum is over all irreducible representations in the decomposition of the product, and $\mathcal{P}_c$ is a projector. To simplify the notation we will use the labels $a, b = \pm$ for the matrix acting on chiral respectively anti-chiral representations, and write the projectors as

$$\mathcal{P}_{(\frac{1}{2}-\alpha+n,\frac{1}{2}-\alpha)} \equiv \mathcal{P}_{n^+}\,, \qquad \mathcal{P}_{(\alpha-\frac{1}{2}+n,\frac{1}{2}-\alpha)} \equiv \mathcal{P}_{n^-}\,, \qquad \mathcal{P}_{(\frac{1}{2}+n,\frac{1}{2}-\alpha)} \equiv \mathcal{P}_{n}\,, \tag{3.21}$$

with corresponding notation for the label c of the R-matrix.

In [56] the R-matrix acting on an arbitrary representation of $sl(2|1)$ was derived. For two chiral or anti-chiral states we get $(n \geq 1)$

$$R^{n^-}_{--}(u) = (-1)^n N_{--}(\tilde{u})\frac{\Gamma(+\tilde{u}+\alpha+n)}{\Gamma(-\tilde{u}+\alpha+n)}\,, \tag{3.22}$$

$$R^{n^+}_{++}(u) = (-1)^n N_{++}(\tilde{u})\frac{\Gamma(+\tilde{u}+1-\alpha+n)}{\Gamma(-\tilde{u}+1-\alpha+n)}\,, \tag{3.23}$$

while a chiral with an anti-chiral state gives $(n \geq 0)$

$$R^{n}_{+-}(u) = R^{n}_{-+}(u) = (-1)^n N_{-+}(\tilde{u})\frac{\Gamma(+\tilde{u}+1+n)}{\Gamma(-\tilde{u}+1+n)}\,. \tag{3.24}$$

Here $\tilde{u} = u/c$ and c and $N_{\pm\pm}(\tilde{u})$ are arbitrary and correspond to trivial symmetries of the Yang-Baxter equation. We find it convenient to choose

$$N_{--}(\tilde{u}) = \frac{\Gamma(-\tilde{u}+\alpha)}{\Gamma(+\tilde{u}+\alpha)}\,, \quad N_{++}(\tilde{u}) = \frac{\Gamma(-\tilde{u}+1-\alpha)}{\Gamma(+\tilde{u}+1-\alpha)}\,, \quad N_{\mp\pm}(\tilde{u}) = \frac{\Gamma(-\tilde{u}+1)}{\Gamma(+\tilde{u}+1)}\,, \tag{3.25}$$

and set $c = 2$. Then the above expressions can be written as

$$R^{n^-}_{--}(u) = \prod_{k=0}^{n-1}\frac{u+2\alpha+2k}{u-2\alpha-2k}\,, \tag{3.26}$$

$$R^{n^+}_{++}(u) = \prod_{k=0}^{n-1}\frac{u+2(1-\alpha)+2k}{u-2(1-\alpha)-2k}\,, \tag{3.27}$$

$$R^{n}_{\mp\pm}(u) = \prod_{k=0}^{n}\frac{u+2k}{u-2k}\,. \tag{3.28}$$

With the above normalization we also have

$$R^{(\alpha,-\alpha)}_{--} = 1\,, \qquad\qquad R^{(1-\alpha,1-\alpha)}_{++} = 1\,. \tag{3.29}$$

We will now use these R-matrices to derive a Hamiltonian for the alternating spin-chain. Our notation closely follows the notation of [71]. The spin-chain has chiral representations sitting at odd sites and anti-chiral representations at even sites, which we will label by a_i and $\bar{a}_i$, respectively. For a chain with L sites (of each kind) we can write down two transfer matrices, with the auxiliary space in either the chiral representation (with an a index), or the anti-chiral representation (indicate by an $\bar{a}$ index),

$$T_a(u) = R_{aa_1}(u)\, R_{a\bar{a}_1}(u)\, R_{aa_2}(u) R_{a\bar{a}_2}(u) \cdots R_{aa_L}(u) R_{a\bar{a}_L}(u)\,, \tag{3.30}$$

$$T_{\bar{a}}(u) = R_{\bar{a}a_1}(u)\, R_{\bar{a}\bar{a}_1}(u)\, R_{\bar{a}a_2}(u) R_{\bar{a}\bar{a}_2}(u) \cdots R_{\bar{a}a_L}(u) R_{\bar{a}\bar{a}_L}(u)\,. \tag{3.31}$$

Taking the traces over the auxiliary spaces we define

$$\tau(u) = \mathrm{tr}_a\, T_a(u)\,, \qquad \bar{\tau}(u) = \mathrm{tr}_{\bar{a}}\, T_{\bar{a}}(u)\,. \tag{3.32}$$

The Yang-Baxter equation now ensures the commutation relations

$$[\tau(u), \tau(v)] = [\tau(u), \bar{\tau}(v)] = [\bar{\tau}(u), \bar{\tau}(v)] = 0. \tag{3.33}$$

The Hamiltonian is given by

$$H = C(\tau(0)\bar{\tau}(0))^{-1}\, \frac{d}{du}(\tau(u)\bar{\tau}(u))\bigg|_{u=0}\,, \tag{3.34}$$

where C is a normalization constant. Since $\tau(u)$ and $\bar{\tau}(v)$ commute, we can write this as

$$H = C(\tau(0)\bar{\tau}(0))^{-1}\, [\tau'(0)\bar{\tau}(0) + \tau(0)\bar{\tau}'(0)] \tag{3.35}$$

$$= C(\tau(0)\bar{\tau}(0))^{-1}\, [\tau'(0)\bar{\tau}(0) + \bar{\tau}'(0)\tau(0)]\,. \tag{3.36}$$

We note that

$$R_{ab}(0) = \mathcal{P}_{(\alpha,-\alpha)} + \sum_{n=1}^{\infty}(-1)^n \mathcal{P}_{n^-}\,, \tag{3.37}$$

$$R_{\bar{a}\bar{b}}(0) = \mathcal{P}_{(1-\alpha,1-\alpha)} + \sum_{n=1}^{\infty}(-1)^n \mathcal{P}_{n^+}\,. \tag{3.38}$$

The representations appearing in the sums are symmetric for even n and anti-symmetric for odd n. Hence the above operators act as two-site exchange operators

$$R_{ab}(0) = P_{ab}\,, \qquad R_{\bar{a}\bar{b}}(0) = P_{\bar{a}\bar{b}}\,. \tag{3.39}$$

The R-matrix acting on a chiral and an anti-chiral representation is, at $u = 0$,

$$R_{a\bar{b}}(0) = \sum_{n=0}^{\infty}(-1)^n \mathcal{P}_n\,. \tag{3.40}$$

Introducing explicit indices I_a and $\bar{I}_a$ for the states of the chiral and anti-chiral representations, we can write the above operators as

$$R_{ab}(0) = \delta_{I_a}^{J_b}\delta_{I_b}^{J_a}\,, \qquad R_{\bar{a}\bar{b}}(0) = \delta_{\bar{I}_a}^{\bar{J}_b}\delta_{\bar{I}_b}^{\bar{J}_a}\,, \qquad R_{a\bar{b}}(0) = \mathcal{M}_{I_a\bar{I}_b}^{J_a\bar{J}_b}\,, \tag{3.41}$$

where the last equality defines $\mathcal{M}$. We also note that

$$\mathcal{M}^{K_a\bar{K}_b}_{I_a\bar{I}_b}\mathcal{M}^{J_a\bar{J}_b}_{K_a\bar{K}_b} = \delta^{J_a}_{I_a}\delta^{\bar{J}_b}_{\bar{I}_b}\,. \tag{3.42}$$

It then follows that

$$(\tau(0)\bar{\tau}(0))^{J_1\bar{J}_1J_2\bar{J}_2\cdots J_L\bar{J}_L}_{I_1\bar{I}_1I_2\bar{I}_2\cdots I_L\bar{I}_L} = \delta^{J_2}_{I_1}\delta^{\bar{J}_2}_{\bar{I}_1}\delta^{J_3}_{I_2}\delta^{\bar{J}_3}_{\bar{I}_2}\cdots\delta^{J_1}_{I_L}\delta^{\bar{J}_1}_{\bar{I}_L}\,, \tag{3.43}$$

which acts as a two site shift operator.

We also need the derivatives of the R-matrix at $u=0$,

$$\mathcal{A} \equiv R'_{ab}(0) = \sum_{n=1}^{\infty}(-1)^n\left[\psi(\alpha+n)-\psi(\alpha)\right]\mathcal{P}_{n^-}\,, \tag{3.44}$$

$$\bar{\mathcal{A}} \equiv R'_{\bar{a}\bar{b}}(0) = \sum_{n=1}^{\infty}(-1)^n\left[\psi(1-\alpha+n)-\psi(1-\alpha)\right]\mathcal{P}_{n^+}\,, \tag{3.45}$$

$$\mathcal{B} \equiv R'_{a\bar{b}}(0) = \sum_{j=1}^{\infty}(-1)^n\left[\psi(n+1)+\gamma_{\mathrm{E}}\right]\mathcal{P}_{n}\,, \tag{3.46}$$

where $\psi(z)$ is the digamma function and γ_{E} is the Euler-Mascheroni constant.[14] Putting everything together, the Hamiltonian can now be written as

$$\begin{aligned} H &= C\sum_{l=1}^{L}\left(\mathcal{M}^{K\bar{K}}_{I_l\bar{I}_l}\mathcal{A}^{J_{l+1}L}_{KI_{l+1}}\mathcal{M}^{J_l\bar{J}_l}_{L\bar{K}} + \mathcal{B}^{K\bar{K}}_{I_l\bar{I}_l}\mathcal{M}^{J_l\bar{J}_l}_{K\bar{K}}\right) \\ &\quad + C\sum_{l=1}^{L}\left(\mathcal{M}^{\bar{K}K}_{\bar{I}_lI_{l+1}}\bar{\mathcal{A}}^{\bar{J}_{l+1}\bar{L}}_{\bar{K}\bar{I}_{l+1}}\mathcal{M}^{\bar{J}_lJ_{l+1}}_{\bar{L}K} + \mathcal{B}^{\bar{K}K}_{\bar{I}_lI_{l+1}}\mathcal{M}^{\bar{J}_lJ_{l+1}}_{\bar{K}K}\right)\,. \end{aligned} \tag{3.47}$$

We observe in particular that for $\alpha=1/2$,

$$\mathcal{A} = \bar{\mathcal{A}} = \sum_{n=1}^{\infty}(-1)^n\left(2h(2n-1)-h(n-1)\right)\mathcal{P}_{(n,0)}\,, \tag{3.48}$$

$$\mathcal{B} = \sum_{n=1}^{\infty}(-1)^n h(n)\mathcal{P}_{(n+\frac{1}{2},0)}\,. \tag{3.49}$$

For this value of α the Hamiltonian exactly coincides with the Hamiltonian for the $sl(2|1)$ sector of ABJM [71].

3.3 The lift to $d(2,1;\alpha)$

The lift of the $sl(2|1)$ R-matrix in (3.26)–(3.29) to the full $d(2,1;\alpha)$ is now straightforward. Comparing the tensor product decompositions (3.17)–(3.19) with the corresponding products in the larger group, (A.1)–(A.3), it is easy to see that there is a direct map between the individual states. The $d(2,1;\alpha)$ R-matrix is then given by replacing the projectors in the R-matrix of the previous sector by the projector of the full group. It would be interesting to verify the validity of this uplifting procedure by a direct check of the YBE equation for the $d(2,1;\alpha)$ R-matrix.[15]

[14] We can also express these coefficients in terms of analytically continued harmonic numbers using the relation $h(z)=\psi(z+1)+\gamma_{\mathrm{E}}$.

[15] We would like to thanks Kostya Zarembo for a discussion of this point.

3.4 The $su(2) \times su(2)$ sector

The largest compact subalgebra of $d(2,1;\alpha)$ is $su(2) \times su(2)$. In this sector, the tensor product between odd and even sites is truncated to

$$(-\tfrac{\alpha}{2};\tfrac{1}{2};0) \otimes (-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}) \to (-\tfrac{1}{2};\tfrac{1}{2};\tfrac{1}{2})\,, \tag{3.50}$$

$$(-\tfrac{\alpha}{2};\tfrac{1}{2};0) \otimes (-\tfrac{\alpha}{2};\tfrac{1}{2};0) \to (-\alpha;1;0) \oplus (-\alpha;0;0)\,, \tag{3.51}$$

$$(-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}) \otimes (-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}) \to (-(1-\alpha);0;1) \oplus (-(1-\alpha);0;0)\,. \tag{3.52}$$

The corresponding products in $sl(2|1)$ read

$$\left(\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}\right) \otimes \left(\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}\right) \to (\alpha, -\alpha) \oplus \left(\alpha + \tfrac{1}{2}, \tfrac{1}{2} - \alpha\right)\,, \tag{3.53}$$

$$\left(\tfrac{1-\alpha}{2}, \tfrac{1-\alpha}{2}\right) \otimes \left(\tfrac{1-\alpha}{2}, \tfrac{1-\alpha}{2}\right) \to (1-\alpha, 1-\alpha) \oplus \left(\tfrac{3}{2} - \alpha, \tfrac{1}{2} - \alpha\right)\,, \tag{3.54}$$

$$\left(\tfrac{\alpha}{2}, -\tfrac{\alpha}{2}\right) \otimes \left(\tfrac{1-\alpha}{2}, \tfrac{1-\alpha}{2}\right) \to \left(\tfrac{1}{2}, \tfrac{1}{2} - \alpha\right)\,. \tag{3.55}$$

As we previously noted, the representations $\left(\alpha + \tfrac{1}{2}, \tfrac{1}{2} - \alpha\right)$ and $\left(\tfrac{3}{2} - \alpha, \tfrac{1}{2} - \alpha\right)$ are anti-symmetric. Hence

$$\mathcal{P}_{(\frac{1}{2}+\alpha,\frac{1}{2}-\alpha)} = \tfrac{1}{2}(1 - P_{I_l I_{l+1}})\,, \qquad \mathcal{P}_{(\frac{3}{2}-\alpha,\frac{1}{2}-\alpha)} = \tfrac{1}{2}(1 - P_{\bar{I}_l \bar{I}_{l+1}})\,, \tag{3.56}$$

This gives the Hamiltonian

$$H = \frac{C}{2} \sum_l \left[\frac{1}{\alpha}(1 - P_{I_l I_{l+1}}) + \frac{1}{1-\alpha}(1 - P_{\bar{I}_l \bar{I}_{l+1}}) \right]\,. \tag{3.57}$$

As expected this is the sum of two Heisenberg spin-chain Hamiltonians.

3.5 The full $d(2,1;\alpha)^2$ spin-chain

The symmetry of the weak coupling Bethe equations constructed above is $\mathrm{D}(2,1;\alpha)$. However, the symmetry group of superstrings on $AdS_3 \times S^3 \times S^3$ is $\mathrm{D}(2,1;\alpha)^2$. The two factors of the full group act independently on the left- respectively right-moving sectors of the theory. As seen in the full Bethe equations in section 2, the two sets of equations are coupled via the dressing phase and via fermionic inversion symmetry links. To leading order at weak coupling these interactions are trivial, and we can treat the left- and right-movers separately.[16] Hence, the situation is similar to the $su(2) \times su(2)$ sector of ABJM, which also consists of two independent subsectors which at weak coupling only couple through the momentum constraint [57].

Hence, the weak coupling Bethe equations in this section only describe the left-moving sector of the theory. To describe the full spectrum, we need an additional set of equations describing the right-movers. As argued above, at weak coupling the left- and right-mover equations are independent. However, there is still a coupling between

[16] We assume that the dressing phase is trivial at weak coupling, as is the case in $AdS_5 \times S^5$ and $AdS_4 \times CP^3$.

them from the momentum constraint which requires that the total momentum for any physical operator vanishes,[17]

$$e^{iP_{\rm tot}} = e^{i(P_{\rm L}-P_{\rm R})} = \frac{\prod_{k=1}^{K_1} \frac{u_{1,k}+i\alpha}{u_{1,k}-i\alpha} \prod_{k=1}^{K_3} \frac{u_{3,k}+i(1-\alpha)}{u_{3,k}-i(1-\alpha)}}{\prod_{k=1}^{\bar{K}_1} \frac{u_{\bar{1},k}+i\alpha}{u_{\bar{1},k}-i\alpha} \prod_{k=1}^{\bar{K}_3} \frac{u_{\bar{3},k}+i(1-\alpha)}{u_{\bar{3},k}-i(1-\alpha)}} = 1\,, \tag{3.58}$$

where the momentum of excitations in the left- and right-moving sectors are counted with different signs.

Since we have two sets of Bethe equations coupled only via the momentum constraint, there are many more physical operators than there would have been if we required each sector to have vanishing momentum by itself. In fact, *any* solution to the Bethe equations for the left-movers can be turned into a physical solution with vanishing momentum, provided we can find a solution to the equations for the right-movers with equal momentum. This is easily accomplished by just setting $u_{\bar{l},i} = u_{l,i}$.

The string theory background also contains an S^1 factor, which couples to the rest of the geometry only via the Virasoro constraints. At weak coupling this means that also solutions to the Bethe equations which do not have zero total momentum should be considered, since any additional momentum can be attributed to one or more massless excitation on this circle.

Even though a physical operator only needs to have zero total momentum, it is interesting to consider solutions to the left-moving Bethe equations that satisfy the momentum constraint by themselves. As we will see below, such solutions for example display extra degeneracies not directly explained by the manifest symmetries of the model. In addition the similarities of the Bethe equations proposed here and those describing $\mathcal{N} = 4$ SYM and the ABJM model are made more apparent when we concentrate on such a chiral sector.

4 Fermionic duality

The Dynkin diagram of a superalgebra is not unique, since there are multiple inequivalent choices of the simple roots. For example, in figure 3 we show two examples of Dynkin diagrams for $d(2,1;\alpha)$. The structure of the Bethe equations is intimately related to a chosen Dynkin diagram and one can transform the Bethe equations between different such choices using a fermionic duality. The Bethe equations we have been dealing with so far are related to the Dynkin diagram in figure 3a. In this section we will perform a fermionic duality on our Bethe equations in order to obtain a form of them related to the Dynkin diagram in figure 3b. The new form of Bethe equations will allow us, in the next section, to take the $\alpha \to 1$ limit more easily.

The procedure for fermionic duality on Bethe equations used in this section closely follows the general prescription discussed in [72–75]. The case at hand is very similar to

[17] On the string theory side of the AdS_3/CFT_2 duality the momentum constraint arises from the level matching condition fo closed strings. Since the CFT_2 is unknown it is not clear what this corresponds to at weak coupling.

the $osp(6|4)$ spin-chain studied in [49, 57]. The original Bethe equations read

$$\begin{aligned}
\left(\frac{u_{1,i}+i\alpha}{u_{1,i}-i\alpha}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_1}\frac{u_{1,i}-u_{1,k}+2i\alpha}{u_{1,i}-u_{1,k}-2i\alpha}\prod_{k=1}^{K_2}\frac{u_{1,i}-u_{2,k}-i\alpha}{u_{1,i}-u_{2,k}+i\alpha}\,,\\
1 &= \prod_{k=1}^{K_1}\frac{u_{2,i}-u_{1,k}-i\alpha}{u_{2,i}-u_{1,k}+i\alpha}\prod_{k=1}^{K_3}\frac{u_{2,i}-u_{3,k}-i(1-\alpha)}{u_{2,i}-u_{3,k}+i(1-\alpha)}\,,\\
\left(\frac{u_{3,i}+i(1-\alpha)}{u_{3,i}-i(1-\alpha)}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_3}\frac{u_{3,i}-u_{3,k}+2i(1-\alpha)}{u_{3,i}-u_{3,k}-2i(1-\alpha)}\prod_{k=1}^{K_2}\frac{u_{3,i}-u_{2,k}-i(1-\alpha)}{u_{3,i}-u_{2,k}+i(1-\alpha)}\,.
\end{aligned}\tag{4.1}$$

The middle equation can be expressed as

$$P(u_{2,i})=0\,,\tag{4.2}$$

where $P(u)$ is the polynomial

$$\begin{aligned}
P(u) = &\prod_{k=1}^{K_1}(u-u_{1,k}+i\alpha)\prod_{k=1}^{K_3}(u-u_{3,k}+i(1-\alpha))\\
&-\prod_{k=1}^{K_1}(u-u_{1,k}-i\alpha)\prod_{k=1}^{K_3}(u-u_{3,k}-i(1-\alpha))\,.
\end{aligned}\tag{4.3}$$

$P(u)$ has in total K_1+K_3-1 zeros. Of these, K_2 are the original roots $u_{2,k}$. Hence we can write $P(u)$ as

$$P(u)=2i(\alpha K_1+(1-\alpha)K_3)\prod_{k=1}^{K_2}(u-u_{2,k})\prod_{k=1}^{\tilde{K}_2}(u-\tilde{u}_{2,k})\,,\tag{4.4}$$

where we have introduced $\tilde{K}_2 = K_1+K_3-K_2-1$ dual roots $\tilde{u}_{2,k}$. An important observation is that these dual roots satisfy the same Bethe equation as the original roots. By evaluating $P(u)$ in $u_{1,i}\pm i\alpha$ and $u_{3,i}\pm i(1-\alpha)$ we can now transform our Bethe equations to the dual form

$$\begin{aligned}
\left(\frac{u_{1,i}+i\alpha}{u_{1,i}-i\alpha}\right)^L &= \prod_{k=1}^{K_3}\frac{u_{1,i}-u_{3,k}-i}{u_{1,i}-u_{3,k}+i}\prod_{k=1}^{\tilde{K}_2}\frac{u_{1,i}-\tilde{u}_{2,k}+i\alpha}{u_{1,i}-\tilde{u}_{2,k}-i\alpha}\,,\\
1 &= \prod_{k=1}^{K_1}\frac{\tilde{u}_{2,i}-u_{1,k}+i\alpha}{\tilde{u}_{2,i}-u_{1,k}-i\alpha}\prod_{k=1}^{K_3}\frac{\tilde{u}_{2,i}-u_{3,k}+i(1-\alpha)}{\tilde{u}_{2,i}-u_{3,k}-i(1-\alpha)}\,,\\
\left(\frac{u_{3,i}+i(1-\alpha)}{u_{3,i}-i(1-\alpha)}\right)^L &= \prod_{k=1}^{K_1}\frac{u_{3,i}-u_{1,k}-i}{u_{3,i}-u_{1,k}+i}\prod_{k=1}^{\tilde{K}_2}\frac{u_{3,i}-\tilde{u}_{2,k}+i(1-\alpha)}{u_{3,i}-\tilde{u}_{2,k}-i(1-\alpha)}\,.
\end{aligned}\tag{4.5}$$

Compared to the equations we started with we note a few changes:

- There are no self-interactions in any equation. This means that this corresponds to a Dynkin diagram where all nodes are fermionic.
- The interaction of the $u_{2,k}$ roots with $u_{1,k}$ and $u_{3,k}$ has switched sign.
- There is a new interaction between the $u_{1,k}$ and $u_{3,k}$ roots.

(a) (b)

Figure 3: Two of the Dynkin diagrams for $d(2,1;\alpha)$. The crossed notes are fermionic and the labels indicate the momentum carrying roots in the Bethe equations. The original equations (4.1) corresponds to the diagram (a), while the dualized equations (4.5) corresponds to (b).

4.1 Dualization of the full Bethe equations?

The dualization of the full Bethe equations is more tricky, due to the links between, *e.g.*, the $x_{1,k}$ and $x_{\bar{2},k}$ nodes. Here we will work in the sector where one of the $d(2,1;\alpha)$ factors carry no excitations, *i.e.*, $K_{\bar{1}} = K_{\bar{2}} = K_{\bar{3}} = 0$. The procedure is then very similar to the one-loop case and we get the dual equations

$$\begin{aligned}
\left(\frac{x^+_{1,i}}{x^-_{1,i}}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_1} \frac{1-\frac{1}{x^+_{1,i}x^-_{1,k}}}{1-\frac{1}{x^-_{1,i}x^+_{1,k}}}\sigma_1^2(x_{1,i},x_{1,k}) \prod_{k=1}^{\tilde{K}_2} \frac{x^+_{1,i}-\tilde{x}_{2,k}}{x^-_{1,i}-\tilde{x}_{2,k}} \prod_{k=1}^{K_3} \frac{x^-_{1,i}-x^+_{3,k}}{x^+_{1,i}-x^-_{3,k}}\,, \\
1 &= \prod_{k=1}^{K_1} \frac{\tilde{x}_{2,i}-x^+_{1,k}}{\tilde{x}_{2,i}-x^-_{1,k}} \prod_{k=1}^{K_3} \frac{\tilde{x}_{2,i}-x^+_{3,k}}{\tilde{x}_{2,i}-x^-_{3,k}}\,, \\
\left(\frac{x^+_{3,i}}{x^-_{3,i}}\right)^L &= \prod_{\substack{k=1\\k\neq i}}^{K_3} \frac{1-\frac{1}{x^+_{3,i}x^-_{3,k}}}{1-\frac{1}{x^-_{3,i}x^+_{3,k}}}\sigma_3^2(x_{3,i},x_{3,k}) \prod_{k=1}^{\tilde{K}_2} \frac{x^+_{3,i}-\tilde{x}_{2,k}}{x^-_{3,i}-\tilde{x}_{2,k}} \prod_{k=1}^{K_1} \frac{x^-_{3,i}-x^+_{1,k}}{x^+_{3,i}-x^-_{1,k}}\,.
\end{aligned} \tag{4.6}$$

Note that these equations are valid for any value of α.

Let us now consider the $\alpha = 1/2$ case, and put $K_1 = K_3 = K$, $\tilde{K}_2 = 0$ and $x^\pm_{1,k} = x^\pm_{3,k} = x^\pm_k$. The above equations then reduce to

$$\left(\frac{x^+_i}{x^-_i}\right)^L = -\prod_{\substack{k=1\\k\neq i}}^{K} \frac{x^-_i - x^+_k}{x^+_i - x^-_k} \frac{1-\frac{1}{x^+_i x^-_k}}{1-\frac{1}{x^-_i x^+_k}} \sigma^2(x_i,x_k)\,. \tag{4.7}$$

An equation of exactly this form appears in ABJM [49], and, apart from the minus sign on the right hand side, it is the starting point for deriving the Eden-Staudacher [76], Beisert-Eden-Staudacher [62] and Freyhult-Rej-Staudacher [77] equations. However, these equations heavily rely on the exact form of the BES/BHL dressing phase [62, 63], and there is no particular reason that the dressing phase in (4.7) should take the same form as the corresponding phases in $\mathcal{N} = 4$ SYM and ABJM.[18] Hence, the solutions of (4.7) can in general be very different from the previously known cases.

[18] In order to reproduce the classical Bethe equations in (2.12) the leading strong coupling behavior of σ should take the AFS form [64].

5 The $\alpha \to 1$ limit and $AdS_3 \times S^3 \times T^4$

In the $\alpha \to 1$ limit, the $d(2,1;\alpha)$ algebra turns into $psu(1,1|2)$.[19] This allowed for a unified treatment of the Green-Schwarz superstring action on both the $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$ backgrounds [1]. In this section we will investigate the $\alpha \to 1$ limit for the weakly coupled Bethe Ansatz and corresponding spin-chain described in sections 2.2 and 3 above. Firstly, in sub-section 5.1, we will show that taking the weak-coupling limit of the all-loop Bethe equations for $AdS_3 \times S^3 \times T^4$ [1] gives the same equations as one obtains by taking the $\alpha \to 1$ limit of the weak-coupling $\alpha \neq 1$ Bethe equations discussed in section 4. Then, in sub-section 5.2 we construct a homogenous spin-chain which leads to the $\alpha = 1$ weak-coupling Bethe equations of the previous sub-section. In sub-section 5.3 we discuss how this homogenous spin-chain arrises from the $\alpha \to 1$ limit of the alternating chain constructed in section 3. We show that the integrable structure underlying the alternating spin-chain constructed in section 3 remains finite in the $\alpha \to 1$ limit, and argue that the "missing massless states" puzzle mentioned in the introduction can be resolved by a careful analysis of this limit.

5.1 The weakly-coupled $\alpha = 1$ Bethe equations

To find the Bethe equations for $\alpha = 1$, we start with the dualized form of the equations in (4.5). To get the conventional notation for the Dynkin labels of $psu(1,1|2)$ we exchange the labels 1 and 2. Setting $\alpha = 1$ and rescaling the Bethe roots by a factor 2, the equations read

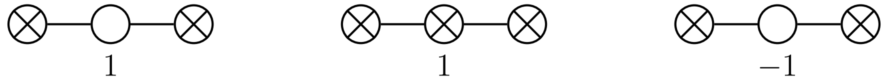
$$1 = \prod_{k=1}^{K_2} \frac{\tilde{u}_{1,i} - u_{2,k} + \frac{i}{2}}{\tilde{u}_{1,i} - u_{2,k} - \frac{i}{2}} \,, \tag{5.1a}$$

$$\left(\frac{u_{2,i} + \frac{i}{2}}{u_{2,i} - \frac{i}{2}}\right)^L = \prod_{k=1}^{\tilde{K}_1} \frac{u_{2,i} - \tilde{u}_{1,k} + \frac{i}{2}}{u_{2,i} - \tilde{u}_{1,k} - \frac{i}{2}} \prod_{k=1}^{K_3} \frac{u_{2,i} - u_{3,k} - \frac{i}{2}}{u_{2,i} - u_{2,k} + \frac{i}{2}} \,, \tag{5.1b}$$

$$1 = \prod_{k=1}^{K_2} \frac{u_{3,i} - u_{2,k} - \frac{i}{2}}{u_{3,i} - u_{2,k} + \frac{i}{2}} \,. \tag{5.1c}$$

This is the standard Bethe equations for a nearest-neighbor spin-chain in the $psu(1,1|2)$ representation with Dynkin labels $[0;1;0]$, and the Dynkin diagram in figure 4b. By performing a fermionic duality transformation on either of the outer nodes of the diagram we can make either the compact $su(2)$ sector or the non-compact $sl(2)$ sector manifest.

[19]The $\alpha \to 0$ limit also results in such a reduction; the only difference is which $su(2)$ sub-algebra one looses. Since the $d(2,1;\alpha)$ is isomorphic to the $d(2,1;1-\alpha)$ algebra the two limits are related and correspond to chosing which of the two S^3 decompactifies.



(a) (b) (c)

Figure 4: Three Dynkin diagrams for $psu(1,1|2)$.

In the first case, corresponding to the diagram figure 4a the Bethe equations read

$$1 = \prod_{k=1}^{K_2} \frac{u_{1,i} - u_{2,k} - \frac{i}{2}}{u_{1,i} - u_{2,k} + \frac{i}{2}}\,, \tag{5.2a}$$

$$\left(\frac{u_{2,i} + \frac{i}{2}}{u_{2,i} - \frac{i}{2}}\right)^L = \prod_{\substack{k=1\\k\neq i}}^{K_2} \frac{u_{2,i} - u_{2,k} + i}{u_{2,i} - u_{2,k} - i} \prod_{k=1}^{K_1} \frac{u_{2,i} - u_{1,k} - \frac{i}{2}}{u_{2,i} - u_{1,k} + \frac{i}{2}} \prod_{k=1}^{K_3} \frac{u_{2,i} - u_{3,k} - \frac{i}{2}}{u_{2,i} - u_{3,k} + \frac{i}{2}}\,, \tag{5.2b}$$

$$1 = \prod_{k=1}^{K_2} \frac{u_{3,i} - u_{2,k} - \frac{i}{2}}{u_{3,i} - u_{2,k} + \frac{i}{2}}\,, \tag{5.2c}$$

while the equations corresponding to figure 4c are

$$1 = \prod_{k=1}^{K_2} \frac{\tilde{u}_{1i} - u_{2,k} + \frac{i}{2}}{\tilde{u}_{1i} - u_{2,k} - \frac{i}{2}}\,, \tag{5.3a}$$

$$\left(\frac{u_{2,i} + \frac{i}{2}}{u_{2,i} - \frac{i}{2}}\right)^L = \prod_{\substack{k=1\\k\neq i}}^{K_2} \frac{u_{2,i} - u_{2,k} - i}{u_{2,i} - u_{2,k} + i} \prod_{k=1}^{\tilde{K}_1} \frac{u_{2,i} - \tilde{u}_{1k} + \frac{i}{2}}{u_{2,i} - \tilde{u}_{1k} - \frac{i}{2}} \prod_{k=1}^{\tilde{K}_3} \frac{u_{2,i} - \tilde{u}_{3k} + \frac{i}{2}}{u_{2,i} - \tilde{u}_{3k} - \frac{i}{2}}\,, \tag{5.3b}$$

$$1 = \prod_{k=1}^{K_2} \frac{\tilde{u}_{3i} - u_{2,k} + \frac{i}{2}}{\tilde{u}_{3i} - u_{2,k} - \frac{i}{2}}\,. \tag{5.3c}$$

In [1] a set of all-loop Bethe equations for $AdS_3 \times S^3 \times T^4$ were proposed. From these we readily obtain the equations presented above by taking a weak coupling limit of the left-moving sector of the full equations in a very similar way to the procedure discussed in section 2.2. As discussed in section 3.5, the complete model has another set of identical equations describing the right-moving sector, with the two sectors being coupled only through the momentum constraint given in equation (3.58). From this analysis, we are lead to conclude that taking the $\alpha \to 1$ limit commutes with taking the weak-coupling limit in the full Bethe ansatz. This was not guaranteed *a priori*, and suggests that investigating the $\alpha \to 1$ limit at weak-coupling is a physically meaningful procedure.

The Bethe equations derived above have appeared in [78], in the context of the $psu(1,1|2)$ sector of $\mathcal{N} = 4$ SYM. We are then lead to the natural result that the integrable spin-chain which gives rise to the Bethe equations (5.3) or (5.2) is of the homogenous type. A confirmation of this observation can be found by noting that, unlike the Bethe equations for $\alpha \neq 1$ (2.15), these Bethe equations do not have any

$L = 1$ solutions. In the next sub-section, we will discuss the spin-chain which leads to the Bethe equations (5.3) and (5.2).

5.2 The integrable $psu(1,1|2)$ spin-chain

In this sub-section we construct the R-matrix and hamiltonian for a spin-chain which can be solved by the $psu(1,1|2)$ BA equations (5.3) and (5.2). As discussed at the end of the previous sub-section these equations correspond to a *non-alternating*, homogenous, spin-chain with $psu(1,1|2)$ symmetry. From the Bethe ansatz equations of the previous sub-section we find that the weights of the $psu(1,1|2)$ representations at each site are $(-\frac{1}{2};\frac{1}{2})$.[20] This is a short representation, and we refer the reader to appendix B, where we present an explicit realisation for it.

To construct the R-matrix for this spin-chain we make two observations. Firstly, note the decomposition into irreducible representations of the tensor product of two $(-\frac{1}{2};\frac{1}{2})$ $psu(1,1|2)$ modules

$$(-\tfrac{1}{2};\tfrac{1}{2})\otimes(-\tfrac{1}{2};\tfrac{1}{2}) = (-1;1)\oplus\bigoplus_{j\geq 0}(-1-j;0)\,. \tag{5.4}$$

Secondly, in the construction we will also use the embedding of the maximal sub-algebra $sl(2|1)$ into $psu(1,1|2)$. The embedding in terms of generators is obtained by setting $\alpha = 1$ in equation (3.12). Under this embedding the $sl(2|1)$ module $(\frac{1}{2},-\frac{1}{2})$ is a submodule of the $psu(1,1|2)$ module $(-\frac{1}{2};\frac{1}{2})$. The decomposition of the tensor product of these sub-modules can be obtained by setting $\alpha = 1$ in equation (3.17) to get

$$\left(\tfrac{1}{2},-\tfrac{1}{2}\right)\otimes\left(\tfrac{1}{2},-\tfrac{1}{2}\right) = (1,-1)\oplus\bigoplus_{n=1}^{\infty}\left(\tfrac{1}{2}+n,-\tfrac{1}{2}\right)\,. \tag{5.5}$$

The R-matrix can be constructed using the universal R-matrix [56]. It has the general form given in equation (3.20). This R-matrix is just the $R_{--}(u)$ R-matrix of section 3.2.1 evaluated with $\alpha = 1$: the coefficients in front of the projection operators are given in equation (3.26) evaluated at $\alpha = 1$. The transfer matrix is then the conventional one for a homogenous spin-chain

$$T_a(u) = R_{aa_1}(u)\,R_{aa_2}(u)\cdots R_{aa_L}(u)\,. \tag{5.6}$$

Taking the traces over the auxiliary space, which is also taken in the $(\frac{1}{2}\,,\,-\frac{1}{2})$ representation of $sl(2|1)$, we define

$$\tau(u) = \mathrm{tr}_a\,T_a(u)\,. \tag{5.7}$$

The Hamiltonian is given by

$$H = C(\tau(0))^{-1}\,\frac{d}{du}\tau(u)\bigg|_{u=0}\,, \tag{5.8}$$

[20] We use the notation $(p;q)$ to label a $psu(1,1|2)$ module, with p and q being the highest weights of the bosonic sub-algebras $su(1,1)$ and $su(2)$.

where C is a normalization constant. Comparing the tensor product decompositions of the $sl(2|1)$ modules given in equation (5.5) with those of the $psu(1,1|2)$ modules given in equation (5.4) we see that there is an isomorphism between the irreducible representations of the two decompositions. As a result, just as in section 3.3, we can uniquely lift the $sl(2|1)$ R-matrix and Hamiltonian to a $psu(1,1|2)$ R-matrix and Hamiltonian. As expected, the resulting $psu(1,1|2)$ Hamiltonian is precisely the one studied in [59].

5.3 The $psu(1,1|2)$ spin-chain from the $d(2,1;\alpha)$ spin-chain

In this subsection we collect some observations about the $\alpha \to 1$ limit.[21] In this limit the algebra changes from $d(2,1;\alpha)$ to $psu(1,1|2)$. One key feature of this limit is that the number of massless BMN states changes [1]. Recall that for generic α the BMN states consist of $2+2+2=6$ massive bosons and two massless bosons as well as their fermionic superpartners. The massive bosons have masses squared proportional to 1, α and $1-\alpha$. As $\alpha \to 1$, two bosons and two fermions become massless. We will argue below that understanding the $\alpha \to 1$ limit is intimately related to understanding the way the massless modes should enter the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence.

5.3.1 The $\alpha \to 1$ limit in the BA equations

Consider first taking the $\alpha \to 1$ limit in the BA equations. As we saw in section 5.1 above, it is possible to write down BA equations for which this limit can be taken smoothly. From this we can conclude that any solution of the $\alpha = 1$ BA equations (5.1) can be uplifted to a solution of the $\alpha \neq 1$ BA equations (2.15). However, this does not mean that all solutions of the $\alpha \neq 1$ BA equations (2.15) map smoothly to solutions of the $\alpha = 1$ BA equations (5.1). In section 6.1 below, we illustrate this by considering some simple solutions which lie in a compact $su(2)$ subsector of the $d(2,1;\alpha)$ BA (2.15) and investigate their $\alpha \to 1$ limit. We observe that the energies of some solutions in this sector diverge, while other solutions' energies remain finite. The presence of solutions to the BA (4.5) whose energies diverge in this limit shows that the $\alpha = 1$ BA (5.1) "loses" some of the states from the $\alpha \neq 1$ BA.

This divergent behaviour can be expected on general grounds. To see this we note that the magnon dispersion relation of the $u_{3,i}$ Bethe roots is given by

$$\epsilon_3(p) = \sqrt{1-\alpha+4h^2\sin^2\frac{p}{2}}\,, \tag{5.9}$$

In the weak coupling limit, we expand this as

$$\epsilon_3(p) = \sqrt{1-\alpha} + \frac{2h^2}{\sqrt{1-\alpha}}\sin^2\frac{p}{2} + \mathcal{O}(h^4)\,. \tag{5.10}$$

When $\alpha \to 1$, the $u_{3,i}$ excitations become massless, and this expansion is not valid anymore. Instead

$$\epsilon_3(p) \to 2h\left|\sin\frac{p}{2}\right|\,. \tag{5.11}$$

[21] Similar comments apply to the equivalent $\alpha \to 0$ limit.

However, the weak coupling Bethe equations, and the corresponding spin-chain Hamiltonian derived in section 3 gives the leading correction to the energy for generic α, and is therefore proportional to h^2. Hence, the divergences appearing in the energies in the $\alpha \to 1$ limit is an indication of the presence of extra massless modes.

What is more, as we see in some examples in section 6.1 below, the BA solutions with divergent terms will generically have, in the $\alpha \to 1$ limit, coincident roots. Such behaviour is not allowed in the conventional BA [79, 80]. We take it as a strong hint that one should be able to construct, at $\alpha = 1$, a "generalised" BA which allows for such coincident roots, and in this way captures the "lost" states described above. We expect this generalised ansatz to contain useful information about the massless states which constitute some of the "lost" states mentioned above.

Naively, one might think that the massless modes in the $AdS_3 \times S^3 \times T^4$ theory would be decoupled from the other degrees of freedom, interacting perhaps at most via the level-matching condition. While it is true that the bosons on T^4 are free, in the GS formulation the fermions are 10d spinors. As such, reduced on $AdS_3 \times S^3 \times T^4$ they will transform as tri-spinors of the three components of this spacetime. As a result, once we fix kappa gauge, there will not be any fermions that decouple from the $AdS_3 \times S^3$ directions. World-sheet superconformal invariance then implies that the bosons of T^4 will also have to have a non-trivial coupling to the $AdS_3 \times S^3$ directions.

5.3.2 The $\alpha \to 1$ limit in the integrable spin-chain

From the above discussion it should be clear that the missing description of massless states is related to the subtleties of the $\alpha \to 1$ limit. As we have argued above, in general the spin-chain Hamiltonian will diverge in this limit. The simplest way to see this is to consider the $su(2) \times su(2)$ subsector discussed in section 3.4 – the Hamiltonian is clearly divergent. One may expect that this divergence comes from an order of limits problem between taking the weak-coupling limit and the $\alpha \to 1$ limit. In general, this would prevent us from extracting useful information about massless modes in the weak coupling limit. However, the system we are considering is far from generic – it is in fact integrable. In this sub-section we point out that the underlying integrable structure of the spin-chain remains *finite* in the $\alpha \to 1$ limit. This indicates that a more detailed analysis of the spin-chain in this limit should yield exact information about the missing massless states of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence.

Let us begin by relating the alternating $\alpha \neq 1$ spin-chain constructed in section 3 to the homogenous $\alpha = 1$ spin-chain constructed in section 5.2. In the alternating $d(2,1;\alpha)$ spin-chain, the odd and even sites transform in the $(-\frac{\alpha}{2}; \frac{1}{2}; 0)$ and $(-\frac{1-\alpha}{2}; 0; \frac{1}{2})$ representations, respectively. In appendix B, we study the $\alpha \to 1$ limit of these representations. The first representation, describing the odd sites of the spin-chain, turns into the $(-\frac{1}{2}; \frac{1}{2})$ spin representation of $psu(1,1|2)$, while the second representation, which sits at even site, becomes *reducible*, decomposing into a pair of singlet states and a $(-\frac{1}{2}; \frac{1}{2})$ module whose heighest weight state is fermionic in the original grading. The $psu(1,1|2)$ Bethe equations and corresponding spin-chain only describe operators in which all even sites contain the singlet state $\bar{\phi}_+$. Indeed one can always replace a homogenous spin-chain by an alternating spin-chain where the extra sites are just singlets of the underlying global symmetry. The energies of the two spin-chains and the BA equations will be indistinguishable.

Having identified the homogenous spin-chain constructed in section 5.2 as a subsector of the $\alpha \to 1$ limit of the alternating spin-chain constructed in section 3, we now turn to the integrable structure's behaviour in the $\alpha \to 1$ limit. The full $d(2,1;\alpha)$ spin-chain actually contains more information about the $\alpha \to 1$ spectrum than the equations (5.1). To see this, let us consider the $\alpha \to 1$ limit of the $sl(2|1)$ R-matrix:

$$\tilde{R}^{n^-}_{--}(u) = \prod_{k=0}^{n-1} \frac{u+2(k+1)}{u-2(k+1)}\,, \quad \tilde{R}^{n^+}_{++}(u) = \prod_{k=0}^{n-1} \frac{u+2k}{u-2k}\,, \quad \tilde{R}^{n}_{\mp\pm}(u) = \prod_{k=0}^{n} \frac{u+2k}{u-2k}\,. \tag{5.12}$$

We use the same notation for the states appearing in the tensor product as in section 3.2.1, though some of the projectors now project onto reducible representations.

The key observation is that while the Hamiltonian of the alternating spin-chain in the $\alpha \to 1$ limit is divergent, the R-matrix remains *finite* and *non-trivial*. This indicates that in order to construct the complete Hamiltonian for the $\alpha \to 1$ limit of the alternating spin-chain, one needs to start with the above R-matrix, which is well defined in this limit, and construct the transfer matrix and Hamiltonian from it. This Hamiltonian will "know" about the massless states appearing in the $\alpha \to 1$ limit. From the finite R-matrix above, one should also be able to construct a set of "generalised" BA equations for the $\alpha = 1$ theory which also "know" about the massless modes. We are pursuing these directions presently and hope to report more fully on these developments in the near future [81].[22]

6 Solutions to the Bethe equations

In this section we collect some solutions to the Bethe equations presented in the paper.

6.1 The $\alpha \to 1$ limit

We will now consider some solutions in the $\alpha \to 1$ limit. The simplest case is to consider the SU(2) that remains compact. Setting $L = 4$, $K_1 = 2$ and $K_2 = K_3 = 0$, and applying the momentum constraint, we have a single solution

$$u_{1,1} = \frac{\alpha}{\sqrt{3}}\,, \qquad u_{1,2} = -\frac{\alpha}{\sqrt{3}}\,. \tag{6.1}$$

This state is in the $(-2;0;2)$ representation of $d(2,1;\alpha)$ and the energy is $E = \frac{3}{\alpha}h^2$.

To find more interesting solutions we set $L = 4$, $K_1 = K_3 = 1$ and $K_2 = 0$, corresponding to the $(-2;1;1)$ representation. Then there are three solutions with zero total momentum, an unpaired solution

$$u^0_{1,1} = 0\,, \qquad u^0_{3,1} = 0\,, \tag{6.2}$$

[22] The above procedure will tell us how to incorporate two massless bosonic modes (and their superpartners) into the integrability machinery. The $AdS_3 \times S^3 \times T^4$ model has four identical massless bosonic modes (and their superpartners). We strongly suspect that once we know how to incorporate two massless modes into the integrability machinery, we will also be able to incorporate the other two.

and the parity pair

$$u_{1,1}^{\pm} = \pm\alpha\,, \qquad\qquad u_{3,1}^{\pm} = \mp(1-\alpha)\,. \tag{6.3}$$

The energies are

$$E^0 = \frac{2}{\alpha}h^2 + \frac{2}{1-\alpha}h^2\,, \qquad\qquad E^{\pm} = \frac{1}{\alpha}h^2 + \frac{1}{1-\alpha}h^2\,. \tag{6.4}$$

As $\alpha \to 1$ these energies diverge as expected from the discussion in section 5.

6.2 Twist-one operators

We will now consider the $sl(2)$ sector and set $L = 1$. We will work with the dualized Bethe equations (4.5). For the state $(-K;0;0)$ the excitation numbers are $K_1 = K_3 = K$, $\tilde{K}_2 = 0$. The equations we want to solve are

$$\frac{u_{1,i} + i\alpha}{u_{1,i} - i\alpha} = \prod_{j\neq i}^{K} \frac{u_{1,i} - u_{3,j} - i}{u_{1,i} - u_{3,j} + i}\,, \tag{6.5a}$$

$$\frac{u_{3,i} + i(1-\alpha)}{u_{3,i} - i(1-\alpha)} = \prod_{j\neq i}^{K} \frac{u_{3,i} - u_{1,j} - i}{u_{3,i} - u_{1,j} + i}\,. \tag{6.5b}$$

To find solutions to these equations we introduce the Baxter polynomials [60, 76, 82]

$$Q_{1K}(u) = c_{1K} \prod_{k=1} (u - u_{1,k})\,, \qquad\qquad Q_{3K}(u) = c_{3K} \prod_{k=1} (u - u_{3,k})\,, \tag{6.6}$$

where c_{1K} and c_{3K} are irrelevant normalization constants. The Bethe equations (6.5) can then be rewritten as two coupled difference equations for the polynomials Q_{1K} and Q_{3K},

$$T_{1K} Q_{1K}(u) = (u + i\alpha)\, Q_{3K}(u+i) - (u - i\alpha)\, Q_{3K}(u-i)\,, \tag{6.7a}$$

$$T_{3K} Q_{3K}(u) = (u + i(1-\alpha))\, Q_{1K}(u+i) - (u - i(1-\alpha))\, Q_{1K}(u-i)\,. \tag{6.7b}$$

In general T_{1K} and T_{3K} would be functions of u, but comparing powers of u on the two sides of the equations we find that here T_{1K} and T_{3K} are independent of u.

The energy of a state can be calculated from the polynomials Q_{1K} and Q_{3K} as

$$E_K = 2ih^2 \frac{d}{du} \log Q_{1K}(u)\bigg|_{u=i\alpha} + 2ih^2 \frac{d}{du} \log Q_{3K}(u)\bigg|_{u=i(1-\alpha)}\,. \tag{6.8}$$

For $\alpha = 1/2$ we can set $Q_{1K}(u) = Q_{3K}(u) = Q_K(u)$. The resulting equation has a solution in terms of the Meixner polynomials [60]

$$Q_K(u) = {}_2F_1(-K, iu + \tfrac{1}{2}; 1; 2)\,, \qquad\qquad T_K = (2K+1)i\,, \tag{6.9}$$

and gives the energy

$$E_K^{(\alpha=1/2)} = 8h^2(S_1(K) - S_{-1}(K))\,, \tag{6.10}$$

where $S_a(M)$ is the harmonic sum

$$S_a(M) = \sum_{j=1}^{M} \frac{(\operatorname{sgn} a)^j}{j^{|a|}} . \tag{6.11}$$

For general α we need to solve for both polynomials Q_{1K} and Q_{2K}. For a solution to exist, the constants T_{lK} have to satisfy

$$T_{1K} = 2i(K+\alpha) , \qquad T_{3K} = 2i(K+1-\alpha) . \tag{6.12}$$

Using (6.7a) to eliminate the Q_{1K} dependence in (6.7b) and introducing

$$v = -\frac{iu}{2} , \qquad R_K(v) = Q_{3K}(2iv) . \tag{6.13}$$

we get the equation

$$\begin{aligned} \left(\tfrac{\alpha^2-1}{4} - v^2\right)\Big(R_K(v+1) - 2R_K(v) + R_K(v-1)\Big) & \\ -v\Big(R_K(v+1) - R_K(v-1)\Big) + K(K+1)R_K(v) = 0 . & \end{aligned} \tag{6.14}$$

This is a discrete form of the Legendre differential equation

$$\left(\tfrac{\alpha^2-1}{4} - v^2\right) R_n''(v) - 2vR_n'(v) + n(n+1)R_n(v) = 0 . \tag{6.15}$$

From this equation we can derive a recurrence relation for R_K [83]

$$R_{K+1}(v) - \frac{K^2}{16}\frac{K^2-\alpha^2}{K^2-\frac{1}{4}} R_{K-1}(v) - vR_K(v) = 0 . \tag{6.16}$$

We have not been able to find a closed form solution of this equation, but it is straight forward to find solutions for any K. The first few solutions for Q_{3K} are

$$Q_{31}(u) = -\frac{i}{2}u , \tag{6.17}$$

$$Q_{32}(u) = -\frac{1}{4}u^2 + \frac{1-\alpha}{12} , \tag{6.18}$$

$$Q_{33}(u) = \frac{i}{8}u^3 + \frac{i(3\alpha^2-7)}{40}u , \tag{6.19}$$

$$Q_{34}(u) = \frac{1}{16}u^4 + \frac{3\alpha^2-13}{56}u^2 + \frac{3(1-\alpha^2)(9-\alpha^2)}{560} . \tag{6.20}$$

The Q_{1K} polynomials are obtained from Q_{3K} by substituting $\alpha \to 1-\alpha$. The corresponding energies are

$$E_1 = 2\left(\frac{1}{\alpha} + \frac{1}{1-\alpha}\right) , \tag{6.21}$$

$$E_2 = 6\left(\frac{1}{1+\alpha} + \frac{1}{2-\alpha}\right) , \tag{6.22}$$

$$E_3 = 2\left(\frac{1}{\alpha} + \frac{1}{1-\alpha}\right) - 10\left(\frac{1}{1+\alpha} + \frac{1}{2-\alpha}\right) + 20\left(\frac{1}{2+\alpha} + \frac{1}{3-\alpha}\right) , \tag{6.23}$$

$$E_4 = 20\left(\frac{1}{1+\alpha} + \frac{1}{2-\alpha}\right) - 70\left(\frac{1}{2+\alpha} + \frac{1}{3-\alpha}\right) + 70\left(\frac{1}{3+\alpha} + \frac{1}{4-\alpha}\right) . \tag{6.24}$$

We note that these energies are of the form

$$E_K = \sum_{m=1}^{K} c_m \left(\frac{1}{(m-1)+\alpha} + \frac{1}{m-\alpha} \right), \qquad c_m \in \mathbb{Z}\,, \tag{6.25}$$

with $c_1 = 0$ for even K. Hence the energies of the odd states diverge as α approaches 0 or 1.

6.3 Degeneracies in the spectrum

Let us a ssume that we have a solution to the Bethe equations (2.15) where one of the $u_{2,k}$ roots sits at zero:

$$u_{2,K_2} = 0\,, \tag{6.26}$$

and with total momentum

$$\prod_{i=1}^{K_1} \frac{u_{2,i} + i\alpha}{u_{2,i} - i\alpha} \prod_{i=1}^{K_3} \frac{u_{2,i} + i(1-\alpha)}{u_{2,i} - i(1-\alpha)} = 1\,. \tag{6.27}$$

The equations for the roots at the first and third nodes then read

$$\left(\frac{u_{1,i} + i\alpha}{u_{1,i} - i\alpha} \right)^{L+1} = \prod_{\substack{k=1\\k\neq i}}^{K_1} \frac{u_{1,i} - u_{1,k} + 2i\alpha}{u_{1,i} - u_{1,k} - 2i\alpha} \prod_{k=1}^{K_2-1} \frac{u_{1,i} - u_{2,k} - i\alpha}{u_{1,i} - u_{2,k} + i\alpha}\,, \tag{6.28a}$$

$$\left(\frac{u_{3,i} + i(1-\alpha)}{u_{3,i} - i(1-\alpha)} \right)^{L+1} = \prod_{\substack{k=1\\k\neq i}}^{K_3} \frac{u_{3,i} - u_{3,k} + 2i(1-\alpha)}{u_{3,i} - u_{3,k} - 2i(1-\alpha)} \prod_{k=1}^{K_2-1} \frac{u_{3,i} - u_{2,k} - i(1-\alpha)}{u_{3,i} - u_{2,k} + i(1-\alpha)}\,. \tag{6.28b}$$

and the equation for u_{2,K_2} becomes

$$1 = \prod_{k=1}^{K_1} \frac{u_{2,k} + i\alpha}{u_{2,k} - i\alpha} \prod_{k=1}^{K_3} \frac{u_{2,k} + i(1-\alpha)}{u_{2,k} - i(1-\alpha)}\,. \tag{6.28c}$$

The last equation is automatically fulfilled since we assumed the solution to satisfy the momentum condition. Hence the resulting equations are the same as what we would get for a state with the same $d(2,1;\alpha)$ charges but with the length increased by one and the excitation number of the middle node lowered by one.

The same degeneracy is seen in the $osp(4|2)$ sector of ABJM [60]. However, in that case the two degenerate states are actually part of the same representation of the full $osp(6|4)$ algebra. It would be interesting to undertand the origin of the degeneracy in the spectrum of the $d(2,1;\alpha)$ spin-chain.

7 Conclusions

In this paper we have investagated the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondences with 16 supersymmetries using integrable techniques. There are two classes of such dual pairs: those with

$psu(1,1|2)^2$ symmetry and those with $d(2,1;\alpha)^2$ symmetry, corresponding to $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ spacetimes, respectively. We have presented a set of all-loop BA equations valid for all values of the parameter α, generalising the $\alpha = 1/2$ BA given in [1], resolving one of the puzzles left open in that paper.

In the remainder of the paper we have considered the weakly coupled limit of the $psu(1,1|2)^2$ and $d(2,1;\alpha)^2$ BA equations. This is analogous to the perturbative gauge theory limit in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. We have constructed integrable spin-chains with local interactions whose energies reproduce both sets of weakly coupled BA equations. In section 5 we addressed the second unresolved puzzle mentioned in the introduction: incorporating massless modes into the integrability description of the gauge/string correspondence. Recall that the $AdS_3 \times S^3 \times T^4$ BA is missing four massless bosonic modes (and four fermionic superpartners). On the other hand, the $AdS_3 \times S^3 \times S^3 \times S^1$ BA is missing only two massless bosons (and their superpartners). In the $\alpha \to 1$ limit the Hamiltonian of the alternating spin-chain and certain solutions of the weakly-coupled $d(2,1;\alpha)^2$ BA diverge. We argue that these divergences signal the appearance of massless modes in the $\alpha \to 1$ limit. These massless states are precisely the two *extra* massless bosons (and their superpartners) that the $AdS_3 \times S^3 \times T^4$ BA is missing compared to the $AdS_3 \times S^3 \times S^3 \times S^1$ BA.

In general, it might be very difficult to retain control over such divergences coming from extra massless states appearing as one varies a parameter in the theory. If that were the case, there would be little hope for understanding the massless modes by investigating the $\alpha \to 1$ limit. However, the theory we are considering is integrable; and while the Hamiltonian and certain solutions of the BA diverge in this limit, we show that the integrable structure underlying the alternating $d(2,1;\alpha)^2$ spin-chain remains *non-singular* in the $\alpha \to 1$ limit. This leads us to posit that the $\alpha \to 1$ limit of the alternating $d(2,1;\alpha)^2$ spin-chain R-matrix that we have constructed in this paper, describes not just the massive modes coming from the homogenous spin-chain coming from the weakly coupled $AdS_3 \times S^3 \times T^4$ BA, but also two of the four missing zero modes (and their superpartners). We will return to a more detailed study of this later [81].

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A Representations of the $d(2,1;\alpha)$ superalgebra

We are mostly interested in the short representations $(-\frac{\alpha}{2};\frac{1}{2};0)$ and $(-\frac{1-\alpha}{2};0;\frac{1}{2})$. The tensor product between these state decomposes as

$$(-\tfrac{\alpha}{2};\tfrac{1}{2};0)\otimes(-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}) = (-\tfrac{1}{2};\tfrac{1}{2};\tfrac{1}{2}) \bigoplus_{j\geq 0}(-1-j;0;0)\,. \tag{A.1}$$

We also need the decompositions

$$(-\tfrac{\alpha}{2};\tfrac{1}{2};0)\otimes(-\tfrac{\alpha}{2};\tfrac{1}{2};0) = (-\alpha;1;0) \bigoplus_{j\geq 0}(-\alpha-j;0;0)\,, \tag{A.2}$$

$$(-\tfrac{1-\alpha}{2};0;\tfrac{1}{2})\otimes(-\tfrac{1-\alpha}{2};0;\tfrac{1}{2}) = (-(1-\alpha);0;1) \bigoplus_{j\geq 0}(-(1-\alpha)-j;0;0)\,. \tag{A.3}$$

A.1 Representations

The $(-\frac{\alpha}{2};\frac{1}{2};0)$ representation consists of the bosonic states $|\phi^{(n)}_{\alpha}\rangle$, which transform as a doublet under L_m, and the fermions $|\psi^{(n)}_{\dot\alpha}\rangle$, transforming as a doublet under $R_{\dot m}$. The non-vanishing action of the $d(2,1;\alpha)$ generators on these states are given by

$$\begin{aligned}
&L_5\,|\phi^{(n)}_{\pm}\rangle = \pm\frac{1}{2}\,|\phi^{(n)}_{\pm}\rangle\ , \quad L_+\,|\phi^{(n)}_{-}\rangle = |\phi^{(n)}_{+}\rangle\ , \quad L_-\,|\phi^{(n)}_{+}\rangle = |\phi^{(n)}_{-}\rangle\ ,\\
&R_8\,|\psi^{(n)}_{\pm}\rangle = \pm\frac{1}{2}\,|\psi^{(n)}_{\pm}\rangle\ , \quad R_+\,|\psi^{(n)}_{-}\rangle = |\psi^{(n)}_{+}\rangle\ , \quad R_-\,|\psi^{(n)}_{+}\rangle = |\psi^{(n)}_{-}\rangle\ ,
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
S_0\,|\phi^{(n)}_{\alpha}\rangle &= -\left(\frac{\alpha}{2}+n\right)|\phi^{(n)}_{\alpha}\rangle\ ,\\
S_-\,|\phi^{(n)}_{\alpha}\rangle &= -\sqrt{(n+\alpha)(n+1)}\,|\phi^{(n+1)}_{\alpha}\rangle\ ,\\
S_+\,|\phi^{(n)}_{\alpha}\rangle &= \sqrt{(n-1+\alpha)n}\,|\phi^{(n-1)}_{\alpha}\rangle\ ,\\
S_0\,|\psi^{(n)}_{\dot\alpha}\rangle &= -\left(\frac{\alpha+1}{2}+n\right)|\psi^{(n)}_{\dot\alpha}\rangle\ ,\\
S_-\,|\psi^{(n)}_{\dot\alpha}\rangle &= -\sqrt{(n+1)(n+1+\alpha)}\,|\psi^{(n+1)}_{\dot\alpha}\rangle\ ,\\
S_+\,|\psi^{(n)}_{\dot\alpha}\rangle &= \sqrt{n(n+\alpha)}\,|\psi^{(n-1)}_{\dot\alpha}\rangle\ ,
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&Q_{-\pm\dot\alpha}\,|\phi^{(n)}_{\mp}\rangle = \pm\sqrt{n+\alpha}\,|\psi^{(n)}_{\dot\alpha}\rangle\ , \qquad Q_{+\pm\dot\alpha}\,|\phi^{(n)}_{\mp}\rangle = \pm\sqrt{n}\,|\psi^{(n-1)}_{\dot\alpha}\rangle\ ,\\
&Q_{-\alpha\pm}\,|\psi^{(n)}_{\mp}\rangle = \mp\sqrt{n+1}\,|\phi^{(n+1)}_{\alpha}\rangle\ , \quad Q_{+\alpha\pm}\,|\psi^{(n)}_{\mp}\rangle = \mp\sqrt{n+\alpha}\,|\phi^{(n)}_{\alpha}\rangle\ .
\end{aligned} \tag{A.6}$$

It is straightforward to check that the above expressions satisfy the $d(2,1;\alpha)$ algebra.

To get the $(-\frac{1-\alpha}{2};0;\frac{1}{2})$ representation, we from the above expressions, exchange the L_m and $R_{\dot m}$ generators and replace $\alpha\to 1-\alpha$. The corresponding state are $|\bar\phi^{(n)}_{\dot\alpha}\rangle$ and $|\bar\psi^{(n)}_{\alpha}\rangle$, and the actions of the generators are

$$\begin{aligned}
&L_5\,|\bar\psi^{(n)}_{\pm}\rangle = \pm\frac{1}{2}\,|\bar\psi^{(n)}_{\pm}\rangle\ , \quad L_+\,|\bar\psi^{(n)}_{-}\rangle = |\bar\psi^{(n)}_{+}\rangle\ , \quad L_-\,|\bar\psi^{(n)}_{+}\rangle = |\bar\psi^{(n)}_{-}\rangle\ ,\\
&R_8\,|\bar\phi^{(n)}_{\pm}\rangle = \pm\frac{1}{2}\,|\bar\phi^{(n)}_{\pm}\rangle\ , \quad R_+\,|\bar\phi^{(n)}_{-}\rangle = |\bar\phi^{(n)}_{+}\rangle\ , \quad R_-\,|\bar\phi^{(n)}_{+}\rangle = |\bar\phi^{(n)}_{-}\rangle\ ,
\end{aligned} \tag{A.7}$$

$$
\begin{aligned}
S_0\,|\bar\phi^{(n)}_{\dot\alpha}\rangle &= -\left(\frac{1-\alpha}{2}+n\right)|\bar\phi^{(n)}_{\dot\alpha}\rangle\ ,\\
S_-\,|\bar\phi^{(n)}_{\dot\alpha}\rangle &= -\sqrt{(n+1-\alpha)(n+1)}\,|\bar\phi^{(n+1)}_{\dot\alpha}\rangle\ ,\\
S_+\,|\bar\phi^{(n)}_{\dot\alpha}\rangle &= \sqrt{(n-\alpha)n}\,|\bar\phi^{(n-1)}_{\dot\alpha}\rangle\ ,\\
S_0\,|\bar\psi^{(n)}_{\alpha}\rangle &= -\left(\frac{2-\alpha}{2}+n\right)|\bar\psi^{(n)}_{\alpha}\rangle\ ,\\
S_-\,|\bar\psi^{(n)}_{\alpha}\rangle &= -\sqrt{(n+1)(n+2-\alpha)}\,|\bar\psi^{(n+1)}_{\alpha}\rangle\ ,\\
S_+\,|\bar\psi^{(n)}_{\alpha}\rangle &= \sqrt{n(n+1-\alpha)}\,|\bar\psi^{(n-1)}_{\alpha}\rangle\ ,
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
Q_{-\alpha\pm}\,|\bar\phi^{(n)}_{\mp}\rangle &= \pm\sqrt{n+1-\alpha}\,|\bar\psi^{(n)}_{\alpha}\rangle\ , \quad & Q_{+\alpha\pm}\,|\bar\phi^{(n)}_{\mp}\rangle &= \pm\sqrt{n}\,|\bar\psi^{(n-1)}_{\alpha}\rangle\ ,\\
Q_{-\pm\dot\alpha}\,|\bar\psi^{(n)}_{\mp}\rangle &= \mp\sqrt{n+1}\,|\bar\phi^{(n+1)}_{\dot\alpha}\rangle\ , & Q_{+\pm\dot\alpha}\,|\bar\psi^{(n)}_{\mp}\rangle &= \mp\sqrt{n+1-\alpha}\,|\bar\phi^{(n)}_{\dot\alpha}\rangle\ .
\end{aligned}
\tag{A.9}
$$

B The $psu(1,1|2)$ superalgebra

We obtain the $psu(1,1|2)$ algebra by taking the $\alpha\to 1$ (or, equivalently, $\alpha\to 0$) limit of the $d(2,1;\alpha)$ algebra. However, depending on how the limit is approach, slightly different versions of the resulting algebra can be obtained. By comparing with the corresponding coset sigma model in [1] we see that to get the relevant limit we need to rescale the generators $R_{\dot m}\to R_{\dot m}/\sqrt{1-\alpha}$ before letting $\alpha\to 1$. The resulting algebra then reads[23]

$$
\begin{aligned}
[S_\mu,S_\nu] &= i\,\epsilon_{\mu\nu\rho}\,S^\rho\,, & [S_\mu,Q_{a\alpha\dot\alpha}] &= \frac{1}{2}Q_{b\alpha\dot\alpha}\gamma^{b}_{\mu\,a}\,,\\
[L_m,L_n] &= i\,\epsilon_{mnp}\,L^p\,, & [L_m,Q_{a\alpha\dot\alpha}] &= \frac{1}{2}Q_{a\beta\dot\alpha}\gamma^{\beta}_{m\,\alpha}\,,\\
\{Q_{a\alpha\dot\alpha},Q_{b\beta\dot\beta}\} &= -\left(S_\mu\,(\epsilon\gamma^\mu)_{ab}\,\epsilon_{\alpha\beta}+L_m\,\epsilon_{ab}\,(\epsilon\gamma^m)_{\alpha\beta}\right)\epsilon_{\dot\alpha\dot\beta}\,.
\end{aligned}
\tag{B.1}
$$

The $psu(1,1|2)$ algebra is equipped with an outer $su(2)$ automorphism, which commutes with the bosonic generators and under which the supercharges transform as a doublet, as indicated by the third, dotted, index. Hence the action of these $su(2)$ generators on the algebra is the same as the action of the $R_{\dot m}$ generators of the $d(2,1;\alpha)$ algebra in (3.1). Note, however, that these additional generators are *not* identical with $R_{\dot m}$, since the latter, in the limit $\alpha\to 1$, commute both among themselves and with $psu(1,1|2)$.

Let us now consider what happens to the representations in the last section as $\alpha\to 1$. With the above scaling, the $R_{\dot m}$ generators annihilate all states. The rest of the states in the $(-\frac{\alpha}{2};\frac{1}{2};0)$ representation in (A.4)–(A.6) transform as

$$
L_5\,|\phi^{(n)}_{\pm}\rangle = \pm\frac{1}{2}\,|\phi^{(n)}_{\pm}\rangle\ ,\quad L_+\,|\phi^{(n)}_{-}\rangle = |\phi^{(n)}_{+}\rangle\ ,\quad L_-\,|\phi^{(n)}_{+}\rangle = |\phi^{(n)}_{-}\rangle\ , \tag{B.2}
$$

[23]If we take the limit without rescaling the $R_{\dot m}$ generators, these generators would appear as an $su(2)$ outer automorphism of the resulting algebra. Another alternative would be to instead rescale $R_{\dot m}\to R_{\dot m}/(1-\alpha)$. Then $R_{\dot m}$ would appear as three commuting central charges in the anti-commutators of the supercharges (see [84, 85]).

$$
\begin{aligned}
&S_0\,|\phi_\alpha^{(n)}\rangle = -\left(\frac{1}{2}+n\right)|\phi_\alpha^{(n)}\rangle\ , \quad S_0\,|\psi_{\dot\alpha}^{(n)}\rangle = -\,(1+n)\,|\psi_{\dot\alpha}^{(n)}\rangle\ ,\\
&S_-\,|\phi_\alpha^{(n)}\rangle = -(n+1)\,|\phi_\alpha^{(n+1)}\rangle\ , \quad S_-\,|\psi_{\dot\alpha}^{(n)}\rangle = -\sqrt{(n+1)(n+2)}\,|\psi_{\dot\alpha}^{(n+1)}\rangle\ ,\\
&S_+\,|\phi_\alpha^{(n)}\rangle = n\,|\phi_\alpha^{(n-1)}\rangle\ , \qquad\qquad S_+\,|\psi_{\dot\alpha}^{(n)}\rangle = \sqrt{n(n+1)}\,|\psi_{\dot\alpha}^{(n-1)}\rangle\ ,
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
&Q_{+\pm\dot\alpha}\,|\phi_\mp^{(n)}\rangle = \pm\sqrt{n}\,|\psi_{\dot\alpha}^{(n-1)}\rangle\ , \qquad Q_{-\pm\dot\alpha}\,|\phi_\mp^{(n)}\rangle = \pm\sqrt{n+1}\,|\psi_{\dot\alpha}^{(n)}\rangle\ ,\\
&Q_{+\alpha\pm}\,|\psi_\mp^{(n)}\rangle = \mp\sqrt{n+1}\,|\phi_\alpha^{(n)}\rangle\ , \quad Q_{-\alpha\pm}\,|\psi_\mp^{(n)}\rangle = \mp\sqrt{n+1}\,|\phi_\alpha^{(n+1)}\rangle\ .
\end{aligned}
\tag{B.4}
$$

This is the same spin representation that appears in the $psu(1,1|2)$ sector of $\mathcal{N}=4$ SYM [59].

The limit of the $(-\frac{1-\alpha}{2};0;\frac{1}{2})$ representation is more interesting. Setting $\alpha=1$ in (A.7)–(A.9) we obtain

$$
L_5\,|\bar\psi_\pm^{(n)}\rangle = \pm\frac{1}{2}\,|\bar\psi_\pm^{(n)}\rangle\ , \quad L_+\,|\bar\psi_-^{(n)}\rangle = |\bar\psi_+^{(n)}\rangle\ , \quad L_-\,|\bar\psi_+^{(n)}\rangle = |\bar\psi_-^{(n)}\rangle\ , \tag{B.5}
$$

$$
\begin{aligned}
&S_0\,|\bar\phi_{\dot\alpha}^{(n)}\rangle = -n\,|\bar\phi_{\dot\alpha}^{(n)}\rangle\ , \qquad\qquad S_0\,|\bar\psi_\alpha^{(n)}\rangle = -\left(\frac{1}{2}+n\right)|\bar\psi_\alpha^{(n)}\rangle\ ,\\
&S_-\,|\bar\phi_{\dot\alpha}^{(n)}\rangle = -\sqrt{n(n+1)}\,|\bar\phi_{\dot\alpha}^{(n+1)}\rangle\ , \quad S_-\,|\bar\psi_\alpha^{(n)}\rangle = -(n+1)\,|\bar\psi_\alpha^{(n+1)}\rangle\ ,\\
&S_+\,|\bar\phi_{\dot\alpha}^{(n)}\rangle = \sqrt{n(n-1)}\,|\bar\phi_{\dot\alpha}^{(n-1)}\rangle\ , \qquad S_+\,|\bar\psi_\alpha^{(n)}\rangle = n\,|\bar\psi_\alpha^{(n-1)}\rangle\ ,
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
&Q_{+\pm\dot\alpha}\,|\bar\phi_\mp^{(n)}\rangle = \pm\sqrt{n}\,|\bar\psi_\alpha^{(n-1)}\rangle\ , \quad Q_{-\alpha\pm}\,|\bar\phi_\mp^{(n)}\rangle = \pm\sqrt{n}\,|\bar\psi_\alpha^{(n)}\rangle\ ,\\
&Q_{+\pm\dot\alpha}\,|\bar\psi_\mp^{(n)}\rangle = \mp\sqrt{n}\,|\bar\phi_{\dot\alpha}^{(n)}\rangle\ , \qquad Q_{-\pm\dot\alpha}\,|\bar\psi_\mp^{(n)}\rangle = \mp\sqrt{n+1}\,|\bar\phi_{\dot\alpha}^{(n+1)}\rangle\ .
\end{aligned}
\tag{B.7}
$$

Note, that the states $|\bar\phi_\pm^{(0)}\rangle$ are annihilated by all generators. Hence, this representation is *reducible*. The highest weight state of the general α representation becomes a pair of singlets. To identify the rest of the representation we define a new set of bosonic states $|\bar\varphi_\pm^{(n)}\rangle = -\,|\bar\phi_\pm^{(n+1)}\rangle$. The algebra then reads

$$
L_5\,|\bar\psi_\pm^{(n)}\rangle = \pm\frac{1}{2}\,|\bar\psi_\pm^{(n)}\rangle\ , \quad L_+\,|\bar\psi_-^{(n)}\rangle = |\bar\psi_+^{(n)}\rangle\ , \quad L_-\,|\bar\psi_+^{(n)}\rangle = |\bar\psi_-^{(n)}\rangle\ , \tag{B.8}
$$

$$
\begin{aligned}
&S_0\,|\bar\psi_\alpha^{(n)}\rangle = -\left(\frac{1}{2}+n\right)|\bar\psi_\alpha^{(n)}\rangle\ , \quad S_0\,|\bar\varphi_{\dot\alpha}^{(n)}\rangle = -(n+1)\,|\bar\varphi_{\dot\alpha}^{(n)}\rangle\ ,\\
&S_-\,|\bar\psi_\alpha^{(n)}\rangle = -(n+1)\,|\bar\psi_\alpha^{(n+1)}\rangle\ , \quad S_-\,|\bar\varphi_{\dot\alpha}^{(n)}\rangle = -\sqrt{(n+1)(n+2)}\,|\bar\varphi_{\dot\alpha}^{(n+1)}\rangle\ ,\\
&S_+\,|\bar\psi_\alpha^{(n)}\rangle = n\,|\bar\psi_\alpha^{(n-1)}\rangle\ , \qquad\qquad S_+\,|\bar\varphi_{\dot\alpha}^{(n)}\rangle = \sqrt{n(n+1)}\,|\bar\varphi_{\dot\alpha}^{(n-1)}\rangle\ ,
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
&Q_{+\pm\dot\alpha}\,|\bar\psi_\mp^{(n)}\rangle = \pm\sqrt{n}\,|\bar\varphi_{\dot\alpha}^{(n-1)}\rangle\ , \qquad Q_{-\pm\dot\alpha}\,|\bar\psi_\mp^{(n)}\rangle = \pm\sqrt{n+1}\,|\bar\varphi_{\dot\alpha}^{(n)}\rangle\ ,\\
&Q_{+\pm\dot\alpha}\,|\bar\varphi_\mp^{(n)}\rangle = \mp\sqrt{n+1}\,|\bar\psi_\alpha^{(n)}\rangle\ , \quad Q_{-\alpha\pm}\,|\bar\varphi_\mp^{(n)}\rangle = \mp\sqrt{n+1}\,|\bar\psi_\alpha^{(n+1)}\rangle\ .
\end{aligned}
\tag{B.10}
$$

Comparing these expressions to (B.2)–(B.4), we see that this is almost the same representation except that the roles of the bosonic and fermionic states have been interchanged.

C Global charges

The global charges corresponding to a set of Bethe roots can be read off by adding a Bethe root close to $u = \infty$ [78]. From the general equations in (2.16) we get[24]

$$\begin{aligned} 1 &= \left(\frac{u_{l,i} - \frac{i}{2}\vec{\alpha}_l \cdot \vec{w}}{u_{l,i} + \frac{i}{2}\vec{\alpha}_l \cdot \vec{w}} \right)^L \prod_{l'} \prod_{\substack{j \\ (j,l') \neq (i,l)}}^{K_{l'}} \frac{u_{l,i} - u_{l',j} + \frac{i}{2}\vec{\alpha}_l \cdot \vec{\alpha}_{l'}}{u_{l,i} - u_{l,j} - \frac{i}{2}\vec{\alpha}_l \cdot \vec{\alpha}_{l'}} \\ &\approx 1 - \frac{i}{u_{l,i}} \vec{\alpha}_l \cdot \left(L\vec{w} - \sum_{l'} K_{l'} \vec{\alpha}_{l'} \right) + \cdots . \end{aligned} \tag{C.1}$$

For $d(2,1;\alpha)$ the charges for a state with excitation numbers K_1, K_2, K_3 are

$$(p; q; r) = \left(-\tfrac{L+K_2}{2}; \tfrac{L+K_2}{2} - K_1; \tfrac{L+K_2}{2} - K_3 \right) . \tag{C.2}$$

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