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# Moonshine and the Meaning of Life

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The elliptic modular function,  $j$ , invariant under  $PSL(2, \mathbb{Z})$ , has Fourier expansion

$$j(q) = \frac{E_4(q)^3}{\Delta(q)} = \sum_{m=-1}^{\infty} c_m q^m = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots, \quad (1)$$

as  $z \rightarrow i\infty$ , where  $q = e^{2\pi iz}$  is the nome for  $z$ .  $E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n$  is the theta series for the  $E_8$  lattice,  $\sigma_3(n) = \sum_{d|n} d^3$  and

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{m=1}^{\infty} \tau_m q^m = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \dots \quad (2)$$

is the modular discriminant [S]. There are two new congruences

**OBSERVATIONS:** • [JM]  $\left( \sum_{m=1}^{24} c_m^2 \right) \bmod 70 \equiv 42$  ; • [YHH]  $\left( \sum_{m=1}^{24} \tau_m^2 \right) \bmod 70 \equiv 42$

The vector  $\omega = (0, 1, 2, \dots, 24 : 70)$  lives in the Lorentzian lattice  $II_{25,1}$  in 26 dimensions as an isotropic Weyl vector [C], allowing us to construct the Leech lattice as  $\omega^\perp/\omega$ . Watson's [L, W] unique non-trivial solution to  $\sum_{i=1}^n i^2 = m^2$  is  $(n, m) = (24, 70)$ .

Indeed, the second author's observation 35 years ago that

$$196884 = 196883 + 1 \quad (3)$$

sparked the field of "Monstrous Moonshine" [B, CN], underlying so much mathematics and physics, relating, inter alia, modular functions, finite groups, lattices, conformal field theory, string theory and gravity (see [G] for a review of some of the vast subjects encompassed) in which the  $j$ -invariant and the Leech lattice are central. As we ponder the meaning of life, we should be aware of the prescient remarks of the author [A], Douglas Adams:

*"The Answer to the Great Question ... is ... Forty-two," said Deep Thought, with infinite majesty and calm.*

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