Moonshine and the Meaning of Life

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The elliptic modular function, j, invariant under $PSL(2,\mathbb{Z})$, has Fourier expansion

$$j(q) = \frac{E_4(q)^3}{\Delta(q)} = \sum_{m=-1}^{\infty} c_m q^m = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots \ , \tag{1}$$

as $z \to i\infty$, where $q = e^{2\pi i z}$ is the nome for z. $E_4(z) = 1 + 240 \sum\limits_{n=1}^{\infty} \sigma_3(n) q^n$ is the theta series for the E_8 lattice, $\sigma_3(n) = \sum\limits_{d|n} d^3$ and

$$\Delta(q) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{m=1}^{\infty} \tau_m q^m = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \dots \tag{2}$$

is the modular discriminant [S]. There are two new congruences

OBSERVATIONS: • [JM] $\left(\sum\limits_{m=1}^{24} c_m^2\right) \bmod 70 \equiv 42$; • [YHH] $\left(\sum\limits_{m=1}^{24} \tau_m^2\right) \bmod 70 \equiv 42$
.

The vector $\omega = (0, 1, 2, \dots, 24 : 70)$ lives in the Lorentzian lattice $II_{25,1}$ in 26 dimensions as an isotropic Weyl vector [C], allowing us to construct the Leech lattice as $\omega^{\perp}/\omega$. Watson's [L, W] unique non-trivial solution to $\sum\limits_{i=1}^{n} i^2 = m^2$ is $(n, m) = (24, 70)$.

Indeed, the second author's observation 35 years ago that

$$196884 = 196883 + 1 \tag{3}$$

sparked the field of "Monstrous Moonshine" [B, CN], underlying so much mathematics and physics, relating, inter alia, modular functions, finite groups, lattices, conformal field theory, string theory and gravity (see [G] for a review of some of the vast subjects encompassed) in which the j-invariant and the Leech lattice are central. As we ponder the meaning of life, we should be aware of the prescient remarks of the author [A], Douglas Adams:

> *"The Answer to the Great Question ... is ... Forty-two," said Deep Thought, with infinite majesty and calm.*

References

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