Fast bias-constrained optimal FIR filtering for time-invariant state space models

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SUMMARY

This paper combines the finite impulse response filtering with the Kalman structure (predictor/corrector) and proposes a fast iterative bias-constrained optimal finite impulse response filtering algorithm for linear discrete time-invariant models. In order to provide filtering without any requirement of the initial state, the property of unbiasedness is employed. We first derive the optimal finite impulse response filter constrained by unbiasedness in the batch form and then find its fast iterative form for finite-horizon and full-horizon computations. The corresponding mean square error is also given in the batch and iterative forms. Extensive simulations are provided to investigate the trade-off with the Kalman filter. We show that the proposed algorithm has much higher immunity against errors in the noise covariances and better robustness against temporary model uncertainties. The full-horizon filter operates almost as fast as the Kalman filter, and its estimate converges with time to the Kalman estimate.

KEY WORDS: state estimation; finite impulse response filter; Kalman filter; unbiasedness; optimal estimate

1. INTRODUCTION

After more than four decades of developments, the finite impulse response (FIR) filtering is still out of the traditional scope of control and state estimation [1–6]. The computational burden associated with large dimensions of vectors and matrices [7, 8], which cause slow operation, makes the batch FIR estimator highly unattractive for engineering applications, that is, in spite of its inherent bounded input/bounded output stability [9], better robustness [7, 10], and lower sensitivity to noise [11] against the Kalman filter (KF). The tremendous progress in the computational resources did not bring about essential change, and the batch FIR estimators [12–21] still remain mostly on a theoretical level.

Beginning with the work by Kwon, Kwon, and Lee [20], in which recursive forms were shown for FIR filters, there has been some recovery in fast FIR filtering. A receding horizon Kalman FIR filter was designed by Kwon, Kim, and Park in [21]. In [22], Han, Kwon, and Kim have suggested a relevant algorithm for deterministic time-invariant control systems, and Shmaliy derived an iterative algorithm [11] for the p-shift time-invariant unbiased FIR (UFIR) estimator. The latter was further extended in [23] to time-variant models. A distinctive advantage of the iterative UFIR algorithm [24] is that it completely ignores the noise statistics and the initial error statistics, thus, leading to many

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applications in diverse areas [25–27]. However, it does not guarantee optimality in the mean square error (MSE) sense, although its output becomes statistically consistent to the optimal estimate when an averaging horizon of N points occurs to be large, $N \gg 1$.

An in-between solution is the minimum variance unbiased (MVU) FIR filter [7, 16, 28, 29]. It has been shown in [29] that the MVU FIR filter can be obtained by minimizing the variance in the UFIR estimate and that it is equivalent to the optimal FIR (OFIR) filter with the embedded unbiasedness (OFIR-EU). Compared with KF, their filters inherit the aforementioned advantages of the FIR-type methods and do not require initial conditions. All these properties are useful in practical applications, and it is thus highly desirable to have fast and computationally efficient algorithms of the OFIR-EU and MVU FIR methods. This motivated our work presented later.

In this paper, we derive iterative algorithms for the OFIR-EU filter and its MSE and show that the OFIR-EU (or the MVU FIR filter) is full-horizon and Kalman-like. The rest of the paper is organized as follows. In Section 2, we describe the model and formulate the problem. The derivation of the OFIR-EU filter is given in Section 3. The full-horizon form and convergence to the KF are shown in Section 4. Estimation errors are discussed in Section 5. Simulations are provided in Section 6, and conclusions are drawn in Section 7.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a discrete time-invariant linear model represented in state space with

$$x_n = Ax_{n-1} + Bw_n \,, \tag{1}$$

$$y_n = Cx_n + v_n \,, \tag{2}$$

where n is a discrete time index, $x_n \in \mathbb{R}^K$ is the state, $y_n \in \mathbb{R}^M$ is the measurement, and $A \in \mathbb{R}^{K\times K}$, $B \in \mathbb{R}^{K\times P}$, and $C \in \mathbb{R}^{M\times K}$ are some identifiable [30, 31] time-invariant matrices. The process noise $w_n \in \mathbb{R}^P$ and measurement noise $v_n \in \mathbb{R}^M$ are zero mean, $E\{w_n\} = 0$ and $E\{v_n\} = 0$, and white sequences with the covariances $Q = E\{w_n w_n^T\}$ and $R = E\{v_n v_n^T\}$, respectively. The property $E\{w_i v_j^T\} = 0$ holds for all i and j, and (A, C) is assumed to be observable.

The KF estimate referred to (1) and (2) can be given, for our further purposes, in the following form:

$$\begin{aligned}\hat{x}_n = A\hat{x}_{n-1} &+ P_n C^T (R + CP_n C^T)^{-1} \\ &\times (y_n - CA\hat{x}_{n-1}) \,,\end{aligned} \tag{3}$$

$$\begin{aligned}P_{n+1} = AP_n A^T &+ BQB^T - A^T P_n C^T \\ &\times (R + CP_n C^T)^{-1} CP_n A^T \,,\end{aligned} \tag{4}$$

where the initial state x_0 and error P_0 are assumed to be known and $\hat{x}_n \triangleq \hat{x}_{n|n}$ is the estimate obtained via measurements from 0 to n.

To estimate x_n on a horizon of N points from $m = n - N + 1$ to n using FIR filtering, the models (1) and (2) need to be represented on an interval $[m, n]$ as follows [11]:

$$X_{n,m} = A_{n-m} x_m + B_{n-m} W_{n,m} \,, \tag{5}$$

$$Y_{n,m} = C_{n-m} x_m + H_{n-m} W_{n,m} + V_{n,m} \,, \tag{6}$$

where x_m is the initial state at m and $X_{n,m} \in \mathbb{R}^{NK\times 1}$, $Y_{n,m} \in \mathbb{R}^{NM\times 1}$, $W_{n,m} \in \mathbb{R}^{NP\times 1}$, and $V_{n,m} \in \mathbb{R}^{NM\times 1}$ are specified as $X_{n,m} = [x_n^T\, x_{n-1}^T \ldots x_m^T]^T$, $Y_{n,m} = [y_n^T\, y_{n-1}^T \ldots y_m^T]^T$, $W_{n,m} = [w_n^T\, w_{n-1}^T \ldots w_m^T]^T$, and $V_{n,m} = [v_n^T\, v_{n-1}^T \ldots v_m^T]^T$, respectively.

Extended matrices $A_{n-m} \in \mathbb{R}^{NK\times K}$, $B_{n-m} \in \mathbb{R}^{NK\times NP}$, $C_{n-m} \in \mathbb{R}^{NM\times K}$, and $H_{n-m} \in \mathbb{R}^{NM\times NP}$ are represented as respectively:

$$A_i = [(A^i)^T (A^{i-1})^T \cdots A^T I]^T ,$$
$$B_i = \begin{bmatrix} B & AB & \cdots & A^{i-1}B & A^iB \\ 0 & B & \cdots & A^{i-2}B & A^{i-1}B \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & B & AB \\ 0 & 0 & \cdots & 0 & B \end{bmatrix} , \tag{7}$$

$$C_i = \bar{C}_i A_i = [(CA^i)^T \; C_{i-1}^T]^T , \tag{8}$$

$$H_i = \bar{C}_i B_i = \begin{bmatrix} CB & CA\bar{B}_{i-1} \\ 0 & H_{i-1} \end{bmatrix} , \tag{9}$$

with $\bar{C}_i = \text{diag}(\underbrace{C\,C \cdots C}_{i+1})$, where $\bar{B}_{i-1}$ denotes the first row vector of B_{i-1}. Note that x_m in (5) and (6) is assumed to be known and w_m is thus zero valued.

The convolution-based FIR estimate of x_n is given by the following:

$$\hat{x}_n = K_{N-1} Y_{n,m} , \tag{10}$$

where K_{N-1} is the FIR filter gain determined by a given cost function or specific constraints. In the minimum MSE sense, the OFIR was derived in [11] for the purposes of system identification [32, 33] (both w_n and v_n are filtered out) and in [23] as a regular filter (only v_n is filtered out). In turn, the UFIR filter shown in [24] satisfies only the unbiasedness constraint:

$$A^{N-1} = K_{N-1} C_{N-1} . \tag{11}$$

We now formulate the problem. Given the models (1) and (2), we would like to find a fast iterative Kalman-like form for the batch OFIR-EU, which gain K_{N-1} is defined by the minimization problem:

$$\hat{K}_{N-1} = \underset{K_{N-1}}{\arg\min}\, E\{(x_n - \hat{x}_n)\,(x_n - \hat{x}_n)^T\} \tag{12}$$

subject to the constraint (11), where $E\{\cdot\}$ means averaging. We also wish to investigate properties of this filter and compare it to the UFIR filter [24] and the KF under diverse operation conditions.

3. OPTIMAL FINITE IMPULSE RESPONSE FILTER WITH THE EMBEDDED UNBIASEDNESS

Following the derivation procedure given in [34], we substitute x_n in (12) with the first row of (5) and $\hat{x}_n$ as (10), use the trace operator, and embed (11) to get the following:

$$\begin{aligned} \hat{K}_{N-1} = \underset{K_{N-1}}{\arg\min}\; & E\{\text{tr}[((K_{N-1}H_{N-1} - \bar{B}_{N-1})W_{n,m} \\ & + K_{N-1}V_{n,m})(\cdots)^T]\} , \end{aligned} \tag{13}$$

where $(\cdots)$ denotes the term that is the same as the previous term. For uncorrelated noise sources, (13) becomes

$$\begin{aligned} \hat{K}_{N-1} = \underset{K_{N-1}}{\arg\min}\; & \text{tr}[(K_{N-1}H_{N-l} - \bar{B}_{N-1})Q_{N-1}(\cdots)^T \\ & + K_{N-1}R_{N-1}K_{N-1}^T] , \end{aligned}$$

where the matrices with respect to the noises are given by $Q_{N-1} = \mathrm{diag}(\underbrace{Q\,Q\cdots Q}_{N})$ and $R_{N-1} = \mathrm{diag}(\underbrace{R\,R\cdots R}_{N})$, respectively. Following [7], a solution to (13) can be found as follows:

$$\begin{aligned}\hat{K}_{N-1} = A^{N-1}&(C_{N-1}^T Z_{w+v,N-1}^{-1} C_{N-1})^{-1} C_{N-1}^T \\ &\times Z_{w+v,N-1}^{-1} + \bar{B}_{N-1} Q_{N-1} H_{N-1}^T Z_{w+v,N-1}^{-1} \\ &\times (I - C_{N-1}(C_{N-1}^T Z_{w+v,N-1}^{-1} C_{N-1})^{-1} \\ &\times C_{N-1}^T Z_{w+v,N-1}^{-1}),\end{aligned} \tag{14}$$

in which $Z_{w+v,N-1} = Z_{w,N-1} + R_{N-1}$ and $Z_{w,N-1} = H_{N-1} Q_{N-1} H_{N-1}^T$. Provided (14), the batch OFIR-EU is summarized by the following theorem.

Theorem 1
Given models (1) and (2) with zero mean white Gaussian and mutually uncorrelated noise components, the batch OFUR-EU estimate is as follows:

$$\begin{aligned}\hat{x}_n = A^{N-1}&(C_{N-1}^T Z_{w+v,N-1}^{-1} C_{N-1})^{-1} C_{N-1}^T \\ &\times Z_{w+v,N-1}^{-1} Y_{n,m} + \bar{B}_{N-1} Q_{N-1} H_{N-1}^T Z_{w+v,N-1}^{-1} \\ &\times (I - C_{N-1}(C_{N-1}^T Z_{w+v,N-1}^{-1} C_{N-1})^{-1} C_{N-1}^T \\ &\times Z_{w+v,N-1}^{-1}) Y_{n,m}.\end{aligned} \tag{15}$$

As can be deduced, the batch form (15) is complex and computationally inefficient from the engineering perspective. Fast Kalman-like computation is thus required.

3.1. *Iterative form*

In order to avoid matrices of large dimensions, later, we find for (15) an iterative form that involves original matrices of small dimensions.

If to introduce an iterative variable l and define

$$N_l^{-1} = C_l^T Z_{w+v,l}^{-1} C_l, \tag{16}$$

$$F_l = \bar{B}_l Q_l H_l^T, \tag{17}$$

then (15) can equivalently be rewritten at $m+l$ as follows:

$$\hat{x}_{m+l} = \hat{x}_{m+l}^a + \hat{x}_{m+l}^b - \hat{x}_{m+l}^c, \tag{18}$$

where

$$\hat{x}_{m+l}^a = A^l N_l C_l^T Z_{w+v,l}^{-1} Y_{m+l,m}, \tag{19}$$

$$\hat{x}_{m+l}^b = F_l Z_{w+v,l}^{-1} Y_{m+l,m}, \tag{20}$$

$$\hat{x}_{m+l}^c = F_l Z_{w+v,l}^{-1} C_l N_l C_l^T Z_{w+v,l}^{-1} Y_{m+l,m}. \tag{21}$$

Employing the decomposition of H_l specified by (9) and taking into account that

$$\bar{B}_l Q_l \bar{B}_l^T = BQB^T + A\bar{B}_{l-1} Q_{l-1} \bar{B}_{l-1}^T A^T, \tag{22}$$

we further provide

$$Z_{w,l} = \begin{bmatrix} C\bar{B}_l Q_l \bar{B}_l^T C^T & CA\bar{B}_{l-1} Q_{l-1} H_{l-1}^T \\ H_{l-1} Q_{l-1} \bar{B}_{l-1}^T A^T C^T & H_{l-1} Q_{l-1} H_{l-1}^T \end{bmatrix},$$
$$R_l = \begin{bmatrix} R & 0 \\ 0 & R_{l-1} \end{bmatrix}.$$

Now, $Z_{w+v,l}$ can be decomposed as $Z_{w+v,l} = \Delta_l + \Theta_l$ to have the components

$$\Delta_l = \begin{bmatrix} R & 0 \\ 0 & Z_{w+v,l-1} \end{bmatrix}, \ \Theta_l = \begin{bmatrix} CU_l C^T & CAF_{l-1} \\ F_{l-1}^T A^T C^T & 0 \end{bmatrix},$$

where $U_l = \bar{B}_l Q_l \bar{B}_l^T$. By the matrix inversion lemma [35]

$$(X + Y)^{-1} = X^{-1} - X^{-1}\left(I + YX^{-1}\right)^{-1} YX^{-1}, \tag{23}$$

we represent the inverse of $Z_{w+v,l}$ as follows:

$$Z_{w+v,l}^{-1} = \Delta_l^{-1}(I + \Theta_l \Delta_l^{-1})^{-1}. \tag{24}$$

Later, we derive iterative algorithms for all of the functions involved in (18) and come up with the iterative form for (15).

3.1.1. Iterations for (16). Using (24), referring to (8), and doing some arrangements, we first transform (16) to

$$\begin{aligned} N_l^{-1} &= \left[{A^l}^T C^T R^{-1} \ \ C_{l-1}^T Z_{w+v,l-1}^{-1} \right] \left(I + \Theta_l \Delta_l^{-1}\right)^{-1} C_l \\ &= \left[{A^l}^T C^T R^{-1} \ \ C_{l-1}^T Z_{w+v,l-1}^{-1} \right] S_l^{-1} C_l, \end{aligned} \tag{25}$$

where $S_l = I + \Theta_l \Delta_l^{-1} = \begin{bmatrix} S_{l11} & S_{l12} \\ S_{l21} & S_{l22} \end{bmatrix}$ has components $S_{l11} = I + CU_l C^T R^{-1}$, $S_{l12} = CAF_{l-1} Z_{w+v,l-1}^{-1}$, $S_{l21} = F_{l-1}^T A^T C^T R^{-1}$, and $S_{l22} = I$. With the Schur complement of S_{l11} [36] described by the following,

$$\bar{S}_{l11} = I + C\,\Xi_l C^T R^{-1}, \tag{26}$$

$$\Xi_l = U_l - AF_{l-1} Z_{w+v,l-1}^{-1} F_{l-1}^T A^T, \tag{27}$$

the inverse matrix S_l^{-1} can be computed using

$$\begin{aligned} S_l^{-1} &= \begin{bmatrix} \bar{S}_{l11}^{-1} & -\bar{S}_{l11}^{-1} S_{l12} S_{l22}^{-1} \\ -S_{l22}^{-1} S_{l21} \bar{S}_{l11}^{-1} & S_{l22}^{-1}(I + S_{l21}\bar{S}_{l11}^{-1} S_{l12} S_{l22}^{-1}) \end{bmatrix} \\ &= \begin{bmatrix} \bar{S}_{l11}^{-1} & -\bar{S}_{l11}^{-1} S_{l12} \\ -S_{l21}\bar{S}_{l11}^{-1} & I + S_{l21}\bar{S}_{l11}^{-1} S_{l12} \end{bmatrix}. \end{aligned}$$

At this point, (25) reduces to

$$\begin{aligned} N_l^{-1} &= [L_l \ \ C_{l-1}^T Z_{w+v,l-1}^{-1} - L_l S_{l12}] C_l \\ &= C_{l-1}^T Z_{w+v,l-1}^{-1} C_{l-1} + L_l (CA^l - S_{l12} C_{l-1}) \\ &= N_{l-1}^{-1} + L_l \bar{X}_l, \end{aligned} \tag{28}$$

where

$$L_l = X_l^T C^T R^{-1} \bar{S}_{l11}^{-1}, \tag{29}$$

$$X_l = A^l - AF_{l-1}Z_{w+v,l-1}^{-1}C_{l-1}\,, \tag{30}$$

$$\bar{X}_l = CX_l\,. \tag{31}$$

Using (23), we provide

$$N_l = N_{l-1} - N_{l-1}(I + L_l\bar{X}_lN_{l-1})^{-1}L_l\bar{X}_lN_{l-1}\,. \tag{32}$$

3.1.2. Iterations for $\hat{x}^a_{m+l}$. Reasoning similarly, (19) can be decomposed as follows:

$$\hat{x}^a_{m+l} = A^lN_lC_{l-1}^TZ_{w+v,l-1}^{-1}Y_{m+l-1,m} + A^lN_lL_l\bar{y}_{m+l} \tag{33}$$

with

$$\bar{y}_{m+l} = y_{m+l} - CAF_{l-1}Z_{w+v,l-1}^{-1}Y_{m+l-1,m}\,. \tag{34}$$

Next, substituting the first N_l on the right-hand side of (33) with (32) yields

$$\begin{aligned}\hat{x}^a_{m+l} &= A^lN_{l-1}C_{l-1}^TZ_{w+v,l-1}^{-1}Y_{m+l-1,m}\\ &\quad - A^lN_{l-1}(I + L_l\bar{X}_lN_{l-1})^{-1}L_l\bar{X}_lN_{l-1}\\ &\quad \times C_{l-1}^TZ_{w+v,l-1}^{-1}Y_{m+l-1,m} + A^lN_lL_l\bar{y}_{m+l}\\ &= A\hat{x}^a_{m+l-1} - A^lN_lL_l\hat{x}^{a-c}_{m+l-1} + A^lN_lL_l\bar{y}_{m+l}\,,\end{aligned} \tag{35}$$

in which

$$\hat{x}^{a-c}_{m+l-1} = CA\left(\hat{x}^a_{m+l-1} - \hat{x}^c_{m+l-1}\right)\,. \tag{36}$$

3.1.3. Iterations for $\hat{x}^b_{m+l}$. In order to find a similar form for $\hat{x}^b_{m+l}$, we first define F_l recursively by the following:

$$F_l = \left[U_lC^T \;\; AF_{l-1}\right]\,. \tag{37}$$

Accordingly, by referring to (25), $\hat{x}^b_{m+l}$ becomes

$$\begin{aligned}\hat{x}^b_{m+l} &= \left[\Lambda_l \;\; AF_{l-1}Z_{w+v,l-1}^{-1} - \Lambda_lS_{l12}\right]Y_{m+l,m}\\ &= A\hat{x}^b_{m+l-1} + \Lambda_l\bar{y}_{m+l}\,,\end{aligned} \tag{38}$$

where

$$\Lambda_l = \Xi_lC^TR^{-1}\bar{S}_{l11}^{-1}\,. \tag{39}$$

3.1.4. Iterations for $\hat{x}^c_{m+l}$. By combining (21) and (34), $\hat{x}^c_{m+l}$ can be rewritten as follows:

$$\begin{aligned}\hat{x}^c_{m+l} &= (AF_{l-1}Z_{w+v,l-1}^{-1}C_{l-1} + \Lambda_l\bar{X}_l)N_l\\ &\quad \times (C_{l-1}^TZ_{w+v,l-1}^{-1}Y_{m+l-1,m} + L_l\bar{y}_{m+l})\end{aligned}$$

and further transformed to

$$\begin{aligned}\hat{x}^c_{m+l} &= AF_{l-1}Z_{w+v,l-1}^{-1}C_{l-1}N_lC_{l-1}^TZ_{w+v,l-1}^{-1}Y_{m+l-1,m}\\ &\quad + \Lambda_l\bar{X}_lN_lC_{l-1}^TZ_{w+v,l-1}^{-1}Y_{m+l-1,m}\\ &\quad + AF_{l-1}Z_{w+v,l-1}^{-1}C_{l-1}N_lL_l\bar{y}_{m+l}\\ &\quad + \Lambda_l\bar{X}_lN_lL_l\bar{y}_{m+l}\,.\end{aligned} \tag{40}$$

Finally, substituting N_l with (32), taking into account (27) and (30), and doing some rearrangements, we have

$$
\begin{aligned}
\hat{x}^c_{m+l} = {} & A\hat{x}^c_{m+l-1} - AF_{l-1}Z^{-1}_{w+v,l-1}C_{l-1}N_l L_l \hat{x}^{a-c}_{m+l-1} \\
& + AF_{l-1}Z^{-1}_{w+v,l-1}C_{l-1}N_l L_l \bar{y}_{m+l} \\
& + \Lambda_l \bar{X}_l N_l L_l \bar{y}_{m+l} + \Lambda_l \hat{x}^{a-c}_{m+l-1} \\
& - \Lambda_l \bar{X}_l N_l L_l \hat{x}^{a-c}_{m+l-1} ,
\end{aligned} \tag{41}
$$

where $\hat{x}^{a-c}_{m+l-1}$ is specified by (36).

3.1.5. An iterative form for (15). Recursions (33), (38), and (41) can now be combined in (18) along with $\bar{y}_{m+l} - \hat{x}^{a-c}_{m+l-1} = y_{m+l} - CA\hat{x}_{m+l-1}$, in order to compute (18) iteratively as follows:

$$
\begin{aligned}
\hat{x}_{m+l} &= A\hat{x}_{m+l-1} + \Lambda_l(y_{m+l} - CA\hat{x}_{m+l-1}) \\
&\quad + (A^l - AF_{l-1}Z^{-1}_{w+v,l-1}C_{l-1} - \Lambda_l \bar{X}_l)N_l L_l \\
&\quad \times (\bar{y}_{m+l} - \hat{x}^{a-c}_{m+l-1}) \\
&= A\hat{x}_{m+l-1} + \Lambda_l(y_{m+l} - CA\hat{x}_{m+l-1}) \\
&\quad + (I - \Lambda_l C)X_l N_l L_l (y_{m+l} - CA\hat{x}_{m+l-1}) .
\end{aligned} \tag{42}
$$

By (29) and (39), the estimate (42) attains the Kalman form of

$$
\hat{x}_{m+l} = \quad A\hat{x}_{m+l-1} + \Psi_l \left(y_l - CA\hat{x}_{m+l-1} \right) , \tag{43}
$$

in which

$$
\begin{aligned}
\Psi_l = {} & [\Xi_l + X_l N_l X_l^T - \Xi_l C^T (R + C\Xi_l C^T)^{-1} \\
& \times CX_l N_l X_l^T] C^T (R + C\Xi_l C^T)^{-1} .
\end{aligned} \tag{44}
$$

Note that Ψ_l depends on Ξ_l, X_l and N_l and is still given in the batch form. To find an iterative form for (44), later, we first represent N_l given by (32) as follows:

$$
\begin{aligned}
N_l = {} & N_{l-1} - N_{l-1}[I + X_l^T C^T (R + C\Xi_l C^T)^{-1} \\
& \times CX_l N_{l-1}]^{-1} X_l^T C^T (R + C\Xi_l C^T)^{-1} CX_l N_{l-1}
\end{aligned} \tag{45}
$$

and then derive similar relations for Ξ_l and X_l.

Transforming (27) with respect to $l + 1$ by opening the aforementioned defined functions leads to the iterative form of the following:

$$
\begin{aligned}
\Xi_{l+1} &= AU_l A^T + BQB^T \\
&\quad - A\left[\Lambda_l \;\; AF_{l-1}Z^{-1}_{w+v,l-1} - \Lambda_l S_{l12}\right] \\
&\quad \times \begin{bmatrix} CU_l^T \\ F_{l-1}^T A^T \end{bmatrix} A^T \\
&= A\Xi_l A^T + BQB^T - A\Lambda_l C\Xi_l A^T ,
\end{aligned} \tag{46}
$$

which, using (39), can further be represented at l as follows:

$$
\begin{aligned}
\Xi_l = {} & A\Xi_{l-1}A^T + BQB^T - A\Xi_{l-1}C^T \\
& \times (R + C\Xi_{l-1}C^T)^{-1} C\Xi_{l-1}A^T .
\end{aligned} \tag{47}
$$

In a similar manner, we represent X_{l+1} as follows:

$$X_{l+1} = A^{l+1} - AF_l Z_{w+v,l}^{-1} C_l \\ = (A - A\Lambda_l C)\, X_l\,,$$

and transform it to

$$X_l = [A - A\Xi_{l-1} C^T (R + C\Xi_{l-1} C^T)^{-1} C] X_{l-1}\,.$$

Finally, by introducing an auxiliary variable Υ_l,

$$\Upsilon_l = C^T (R + C\Xi_l C^T)^{-1}\,, \tag{48}$$

the iterative OFIR-EU is stated by the following theorem.

Theorem 2
Given the batch OFIR-EU estimate (15), then its iterative algorithm is the following:

$$\hat{x}_{m+l} = A\hat{x}_{m+l-1} + [\Xi_l + (I - \Xi_l \Upsilon_l C) X_l N_l X_l^T] \Upsilon_l \\ \times (y_{m+l} - CA\hat{x}_{m+l-1})\,, \tag{49}$$

where Υ_l is given by (48), and

$$\Xi_l = A\Xi_{l-1} A^T + BQB^T - A\Xi_{l-1} \Upsilon_{l-1} C\Xi_{l-1} A^T\,, \tag{50}$$

$$X_l = A(I - \Xi_{l-1} \Upsilon_{l-1} C) X_{l-1}\,, \tag{51}$$

$$N_l = N_{l-1} - N_{l-1} (I + X_l^T \Upsilon_l C X_l N_{l-1})^{-1} \\ \times X_l^T \Upsilon_l C X_l N_{l-1}\,, \tag{52}$$

with initial states

$$\Xi_{\alpha-1} = \bar{B}_{\alpha-1} Q_{\alpha-1} \bar{B}_{\alpha-1}^T \\ - AF_{\alpha-2} Z_{w+v,\alpha-2}^{-1} F_{\alpha-2}^T A^T\,, \tag{53}$$

$$X_{\alpha-1} = A^{\alpha-1} - AF_{\alpha-2} Z_{w+v,\alpha-2}^{-1} C_{\alpha-2}\,, \tag{54}$$

$$N_{\alpha-1} = \left(C_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1} C_{\alpha-1}\right)^{-1}\,, \tag{55}$$

$$\hat{x}_{m+\alpha-1} = \left(A^{\alpha-1} N_{\alpha-1} C_{\alpha-1}^T + \bar{B}_{\alpha-1} Q_{\alpha-1} H_{\alpha-1}^T \right. \\ - \bar{B}_{\alpha-1} Q_{\alpha-1} H_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1} C_{\alpha-1} \\ \left. \times N_{\alpha-1} C_{\alpha-1}^T\right) Z_{w+v,\alpha-1}^{-1} Y_{m+\alpha-1,m}\,, \tag{56}$$

where F_l is specified by (17), $\alpha = max\{K, 2\}$ (K is the number of the states) guarantees the invertibility of the matrix $N_{\alpha-1}$, and l ranges from α to $N-1$.

Proof
Proof is provided by (16)–(48).

As a result, instead of the slow and computationally inefficient batch form (15), we now have a fast iterative one (49)–(56) stated by Theorem 2. The question then arises whether further algorithmic progress is possible in OFIR-EU filtering or not, which is the subject of the next section. □

4. FULL-HORIZON FORM AND CONVERGENCE TO KALMAN FILTER

As the OFIR-EU minimizes the MSE and the variance of the white Gaussian noise is reduced by averaging as a reciprocal of N, one may suppose that the optimal horizon N_{opt} for the OFIR-EU lies at infinity. If that is the case, the OFIR-EU filter is full-horizon. This section analyzes this, and we show that the OFIR-EU estimate becomes exactly the Kalman one when N reaches infinity.

To state that the OFIR-EU is full-horizon, one needs to show that its estimate converges to the KF estimate by putting N to infinity. We prove it with a theorem.

Theorem 3
The iterative OFIR-EU given by (49)–(56) is full-horizon; that is, its N_{opt} lies at infinity.

Proof
The full-horizon iterative OFIR-EU algorithm appears from (49) to (56) by substituting $N = n+1$ and $l = n$ [24]:

$$\begin{aligned}\hat{x}_n = A\hat{x}_{n-1} &+ \left[\Xi_n + (I - \Xi_n \Upsilon_n C)X_n N_n X_n^T\right]\Upsilon_n \\ &\times (y_n - CA\hat{x}_{n-1}),\end{aligned} \tag{57}$$

where Υ_n is given by (48), and

$$\Xi_n = A\Xi_{n-1}A^T + BQB^T - A\Xi_{n-1}\Upsilon_{n-1}C\Xi_{n-1}A^T, \tag{58}$$

$$X_n = A(I - \Xi_{n-1}\Upsilon_{n-1}C)X_{n-1}, \tag{59}$$

$$\begin{aligned}N_n = N_{n-1} &- N_{n-1}(I + X_n^T \Upsilon_n C X_n N_{n-1})^{-1} \\ &\times X_n^T \Upsilon_n C X_n N_{n-1}.\end{aligned} \tag{60}$$

The initial conditions are specified with (53)–(56).

By introducing $G_{n-1} = \Xi_{n-1}\Upsilon_{n-1}C$, (59) becomes

$$X_n = A(I - G_{n-1})X_{n-1}. \tag{61}$$

Considering the fact that the spectral radius $\rho(G_{n-1})$ of G_{n-1} does not exceed unity in stable filtering, $\rho(G_{n-1}) < 1$, and using the Lyapunov property [37], we have $\lim_{n\to\infty} X_n = 0$, which transforms the full-horizon OFIR-EU estimate (57) at $n = \infty$ to the Kalman estimate given by (3),

$$\begin{aligned}\hat{x}_n &= A\hat{x}_{n-1} + \Xi_n \Upsilon_n (y_n - CA\hat{x}_{n-1}) \\ &= A\hat{x}_{n-1} + \Xi_n C^T (R + C\Xi_n C^T)^{-1} \\ &\quad \times (y_n - CA\hat{x}_{n-1}).\end{aligned} \tag{62}$$

The proof is complete.

The convergence of OFIR-EU estimate to KF estimate is also supported by the fact that the KF has infinite impulse response (IIR) and the full-horizon OFIR-EU with $n = \infty$ turns to the optimal IIR filter with EU filter. On the other hand, a complete convergence of the OFIR-EU with $N = \infty$ to KF means that the unbiasedness no longer affects the estimate. Thus, the full-horizon OFIR-EU with $n \gg 1$ is essentially the OFIR filter. It can also be shown that this filter combines the properties of the UFIR filter with $n < N_{\text{opt}}$ and of the OFIR filter with $n > N_{\text{opt}}$.

To demonstrate the full-horizon form more clearly, a code of the full-horizon OFIR-EU filtering algorithm is given in Table I.

Table I. Full-horizon OFIR-EU algorithm code

Input: y_n, Q, R, α=max$\{K$,2$\}$

Initialization:

$\quad F_{\alpha-2} = \bar{B}_{\alpha-2} Q_{\alpha-2} H_{\alpha-2}^T$

$\quad \Xi_{\alpha-1} = \bar{B}_{\alpha-1} Q_{\alpha-1} \bar{B}_{\alpha-1}^T - A F_{\alpha-2} Z_{w+v,\alpha-2}^{-1} F_{\alpha-2}^T A^T$

$\quad X_{\alpha-1} = A^{\alpha-1} - A F_{\alpha-2} Z_{w+v,\alpha-2}^{-1} C_{\alpha-2}$

$\quad N_{\alpha-1} = (C_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1} C_{\alpha-1})^{-1}$

$\quad \Upsilon_{\alpha-1} = C^T (R + C \Xi_{\alpha-1} C^T)^{-1}$

$\quad \hat{x}_{\alpha-1} = (A^{\alpha-1} N_{\alpha-1} C_{\alpha-1}^T + \bar{B}_{\alpha-1} Q_{\alpha-1} H_{\alpha-1}^T$
$\quad\quad - \bar{B}_{\alpha-1} Q_{\alpha-1} H_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1} C_{\alpha-1} N_{\alpha-1} C_{\alpha-1}^T)$
$\quad\quad \times Z_{w+v,\alpha-1}^{-1} Y_{\alpha-1,0}$

for n=α: ∞ **do**

$\quad \Xi_n = A \Xi_{n-1} A^T + B Q B^T - A \Xi_{n-1} \Upsilon_{n-1} C \Xi_{n-1} A^T$

$\quad X_n = A(I - \Xi_{n-1} \Upsilon_{n-1} C) X_{n-1}$

$\quad N_n = (N_{n-1}^{-1} + X_n^T \Upsilon_n C X_n)^{-1}$

$\quad \Upsilon_n = C^T (R + C \Xi_n C^T)^{-1}$

$\quad \Psi_n = \Xi_n \Upsilon_n + (I - \Xi_n \Upsilon_n C) X_n N_n X_n^T \Upsilon_n$

$\quad \hat{x}_n = A \hat{x}_{n-1} + \Psi_n (y_n - C A \hat{x}_{n-1})$

and for

Output: $\hat{x}_n$

5. ESTIMATION ERRORS

We finish our investigations with an analysis of the MSEs in the OFIR-EU. Most generally, the instantaneous MSE in the OFIR-EU estimate can be defined at time index n by the following:

$$J_n = E\{e_n e_n^T\}, \tag{63}$$

where $e_n = x_n - \hat{x}_n$ is the estimation error, $\hat{x}_n$ is given by (10), and x_n can be expressed with the first vector row of (5) as follows:

$$x_n = A^{N-1} x_m + \bar{B}_{N-1} W_{n,m} . \tag{64}$$

With (10) and (64), invoke the orthogonality condition, and provide the averaging, then (63) can be written as follows:

$$J_n = J_{n,x} + J_{n,w} + J_{n,v} , \tag{65}$$

where $J_{n,x}$, $J_{n,w}$, and $J_{n,v}$ are given by $J_{n,x} = (K_{N-1} C_{N-1} - A^{N-1}) \Phi_m (\cdots)^T$, $J_{n,w} = (K_{N-1} H_{N-1} - \bar{B}_{N-1}) Q_{N-1} (\cdots)^T$ and $J_{n,v} = K_{N-1} R_{N-1} K_{N-1}^T$, respectively, and $\Phi_m = E\{x_m x_m^T\}$.

5.1. Batch form

Due to (11), prior knowledge of the initial state is not required by the OFIR-EU. This means that the OFIR-EU estimate does not depend on the mean square initial state Φ_m and we thus can let $J_{n,x} = 0$. By virtue of this, the MSE can be found in the batch form using (65) with $\hat{K}_{N-1}$ (14) as follows:

$$J_n = (\hat{K}_{N-1}H_{N-1} - \bar{B}_{N-1})Q_{N-1}(\cdots)^T + \hat{K}_{N-1}R_{N-1}\hat{K}_{N-1}^T\,. \tag{66}$$

5.2. *Iterative form*

Similarly to the batch estimate, the batch MSE can also be represented with an iterative form. Towards this end, by changing a variable to $m + l$ and substituting (43) into (63), we get

$$J_{m+l} = E\left\{[Ae_{m+l-1} + Bw_{m+l} - \Psi_l\,(y_{m+l} - CA\hat{x}_{m+l-1})]\,[\cdots]^T\right\}\,. \tag{67}$$

Next, express y_{m+l} in terms of x_{m+l-1} as follows:

$$y_{m+l} = CAx_{m+l-1} + CBw_{m+l} + v_{m+l}\,, \tag{68}$$

combine (67) with (68), and arrive at

$$J_{m+l} = E\left\{[(A - \Psi_l CA)\,e_{m+l-1} + (B - \Psi_l CB)w_{m+l} - \Psi_l v_{m+l}]\,[\cdots]^T\right\}\,. \tag{69}$$

Assuming white Gaussian components, (69) can further be transformed to the iterative form of the following:

$$\begin{aligned} J_{m+l} &= (A - \Psi_l CA)\,J_{m+l-1}(\cdots)^T + (B - \Psi_l CB) \times Q(\cdots)^T + \Psi_l R\Psi_l^T \\ &= (I - \Psi_l C)(AJ_{m+l-1}A^T + BQB^T)(\cdots)^T + \Psi_l R\Psi_l^T\,, \end{aligned} \tag{70}$$

where l ranges from α to $N-1$, Ψ_l is the l-variant filter gain (44), and the MSE at n corresponds to $l = N-1$. Using (66), the initial MSE $J_{m+\alpha-1}$ can be found as follows:

$$J_{m+\alpha-1} = \left(\hat{K}_{\alpha-1}H_{\alpha-1} - \bar{B}_{\alpha-1}\right)Q_{\alpha-1}\,(\cdots)^T + \hat{K}_{\alpha-1}R_{\alpha-1}\hat{K}_{\alpha-1}^T\,,$$

where $\hat{K}_{\alpha-1}$ is the batch filter gain at $m + \alpha - 1$ specified by

$$\begin{aligned} \hat{K}_{\alpha-1} = {} & A^{\alpha-1}N_{\alpha-1}C_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1} + \bar{B}_{\alpha-1}Q_{\alpha-1}H_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1} \\ & \times (I - C_{\alpha-1}N_{\alpha-1}C_{\alpha-1}^T Z_{w+v,\alpha-1}^{-1})\,. \end{aligned} \tag{71}$$

5.3. *Full-horizon form*

Because the OFIR-EU is full-horizon, its MSE can also be represented in a fast full-horizon form. To get the relevant algorithm, we first specify the initial MSE $J_{\alpha-1}$ at time $\alpha - 1$ using Table I as follows:

$$J_{\alpha-1} = (\hat{K}_{\alpha-1}H_{\alpha-1} - \bar{B}_{\alpha-1})Q_{\alpha-1}(\cdots)^T + \hat{K}_{\alpha-1}R_{\alpha-1}\hat{K}_{\alpha-1}^T\,,$$

where $\hat{K}_{\alpha-1}$ is given by (71). The MSE for the full-horizon OFIR-EU can then be found by transforming (70) to the following:

$$J_n = (I - \Psi_n C)(AJ_{n-1}A^T + BQB^T)(\cdots)^T + \Psi_n R\Psi_n^T\,, \tag{72}$$

where n ranges starting with α and Ψ_n is given by (44).

Let us finally show that, if $n \to \infty$, the MSE (72) converts to the *a posteriori* estimate covariance $\bar{J}_n$ of the KF, which can be written as follows:

$$\bar{J}_n = P_n - P_n C^T (R + C P_n C^T)^{-1} C P_n \,. \tag{73}$$

With $n \to \infty$, we have $\lim_{n\to\infty} \Psi_n = \Xi_n \Upsilon_n$, and (72) transforms to $J_n = \Xi_n - \Xi_n \Upsilon_n C \Xi_n - \Xi_n C^T \Upsilon_n^T \Xi_n^T + \Xi_n \Upsilon_n (R + C \Xi_n C^T) \Upsilon_n^T \Xi_n^T$. By (48), the last two terms become identically 0, and we get

$$J_n = \Xi_n - \Xi_n C^T (R + C \Xi_n C^T)^{-1} C \Xi_n \,, \tag{74}$$

which is the *a posteriori* estimate covariance (73) of KF.

6. SIMULATIONS

In this section, we test the batch and iterative OFIR-EU algorithms by a two-state polynomial model in different environments. The UFIR filter [24] and KF are chosen as benchmarks. Towards this end, (1) and (2) are specified with $B = I$, $C = [1\ 0]$, and

$$A = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix},$$

where τ is a constant in unit of time. Note that this model is used in moving target tracking [38] and some related results can be found in [7, 21, 24].

6.1. Estimation accuracy

To learn a trade-off in the estimation accuracy, we let $\tau = 0.1$ s, $\sigma_{w1}^2 = 0.1$, $\sigma_{w1}^2 = 0.1/\text{s}^2$, and $\sigma_v^2 = 10$. The initial values are set as $x_{10} = 1$ and $x_{20} = 0.01/\text{s}$. The model and noise statistics are assumed to be known exactly. The process was simulated at 400 points, and the optimal horizon for the UFIR filter found to be $N_{\text{opt}} = 60$. Typical instantaneous estimation errors are given in Figure 1. What can be concluded from this figure is that the OFIR-EU and KF estimates are very consistent and almost indistinguishable. The UFIR filter also produces good estimates, but with a bit larger MSE.

We further learn effect of N on the FIR estimates for $\sigma_v^2 = 10^2$. The root square of $\text{tr}\,(J_n)$ for three filters is given in Figure 2 as a function of N. We first notice that the KF has IIR and is thus N-invariant, whereas the UFIR filter minimizes MSE at $N_{\text{opt}} = 35$. As has been shown before, the OFIR-EU is essentially the UFIR filter with $N < N_{\text{opt}}$, and it approaches the KF estimate with $N > N_{\text{opt}}$. By virtue of this, we have another proof that the OFIR-EU is full-horizon.

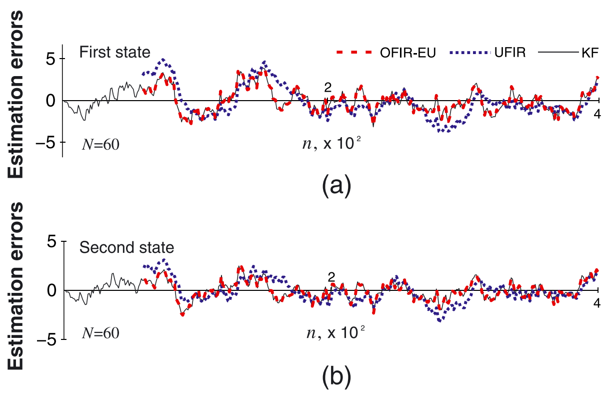

Figure 1. Typical estimation errors in the unbiased finite impulse response (UFIR), optimal finite impulse response filter with the embedded unbiasedness (OFIR-EU), and Kalman filter (KF) estimates: (a) first state and (b) second state.

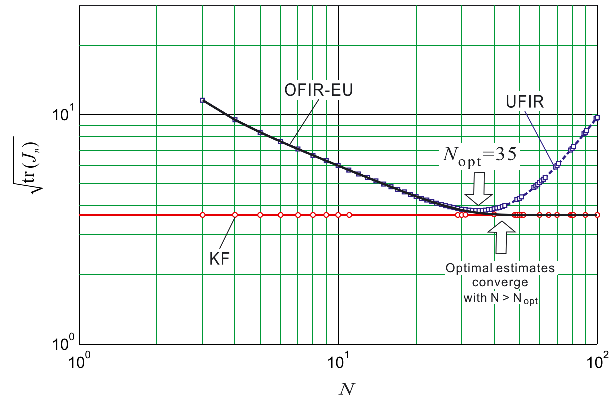

Figure 2. Root mean square errors computed by $\sqrt{\mathrm{tr}\,(J_n)}$ as functions of N. OFIR-EU, optimal finite impulse response filter with the embedded unbiasedness; UFIR, unbiased finite impulse response; KF, Kalman filter.

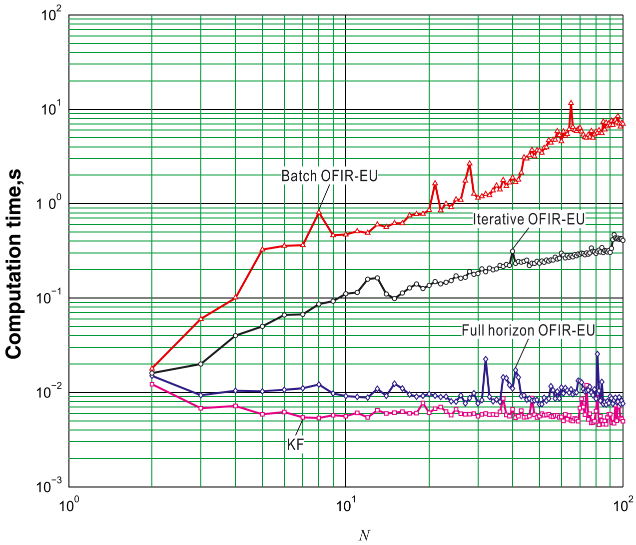

Figure 3. Typical computational time consumed by different optimal finite impulse response filter with the embedded unbiasedness (OFIR-EU) algorithms applied to the two-state polynomial model.

6.2. Computation time

Any batch FIR algorithm consumes more computation time than KF because of larger order and matrix complexity. The iterative algorithm operates faster and the full-horizon one much faster. Later, we give an evidence to this preliminary analysis by measuring the computation time in the FIR and KF algorithms. The process was simulated at 200 points. Figure 3 sketches the computation time consumed by the batch OFIR-EU algorithm, iterative OFIR-EU algorithm, full-horizon OFIR-EU algorithm, and KF. As expected, the batch algorithm demonstrates low efficiency, because its complexity grows with N. Iterations allow the OFIR-EU filter to operate much faster, although its computation time still grows with N. A dramatic progress is achieved in the full-horizon algorithm (Table I), which estimate becomes N-invariant and the computation time almost as large as in the KF.

6.3. Sensitivity to errors in noise covariances

An important issue in optimal estimation is a typically insufficient knowledge about the noise statistics. Referring to the worst case of errors in the noise covariances, we introduce a correction coefficient p as p^2Q and R/p^2. The root MSEs computed by $\mathrm{tr}(J_n)$ are sketched in Figure 4 for $0.1 \leqslant p \leqslant 10$. Inherently, the UFIR filter ignoring the noise statistics is p-invariant, although it produces a bit larger errors with $p = 1$ than in optimal filters. The KF is most sensitive to p, and the OFIR-EU filter occupies an intermediate position: It is almost insensitive to p with $p < 1$ and is as sensitive to p as the KF when $p > 1$. We consider it as an important advantage of the OFIR-EU filter.

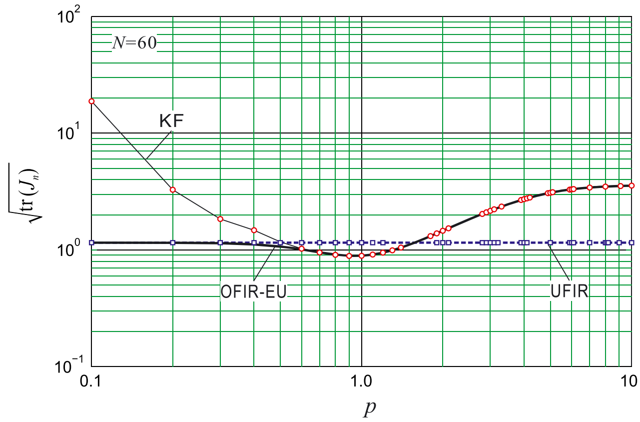

Figure 4. Typical mean square errors in the unbiased finite impulse response (UFIR), optimal finite impulse response filter with the embedded unbiasedness (OFIR-EU), and Kalman filter (KF) estimates caused by the correction coefficient p.

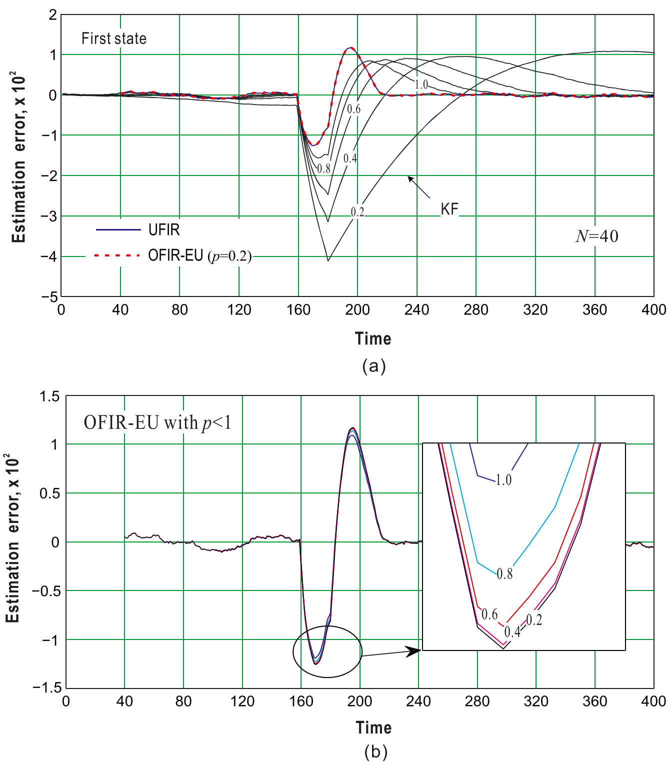

Figure 5. Robustness against temporary model uncertainties in a gap of $160 \leqslant n \leqslant 180$: (a) first state and (b) errors in optimal finite impulse response filter with the embedded unbiasedness (OFIR-EU) with $p \leqslant 1$. UFIR, unbiased finite impulse response.

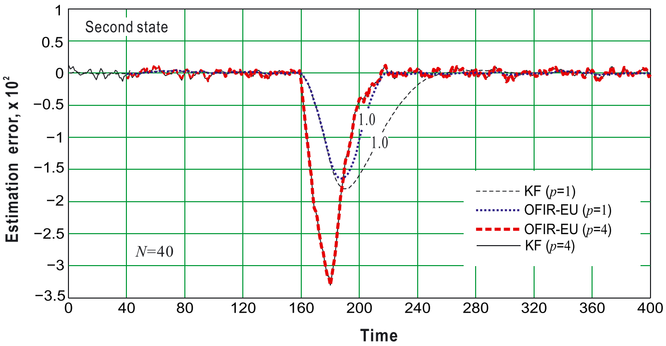

Figure 6. Basic operation diagrams of the Kalman-like unbiased finite impulse response (UFIR) filter and Kalman filter (KF). OFIR-EU, optimal finite impulse response filter with the embedded unbiasedness.

6.4. Robustness against model uncertainness

Robustness against temporary model uncertainties is often required from optimal estimators. We simulate an uncertainty by setting $\tau = 5\,\mathrm{s}$ from $160 \leqslant n \leqslant 180$ and $\tau = 0.1\,\mathrm{s}$ otherwise for $\sigma_v^2 = 10^2$ and $N = 40$. The process is generated at 400 points. Typical responses in the estimates of the first state with $p \leqslant 1$ are shown in Figure 5a. It is seen that the OFIR-EU and UFIR estimates converge with $p = 0.2$ that is in agreement with our early inference. In contrast, KF demonstrates worst robustness for any $p < 1$. Figure 5b gives a more precise picture of what goes on with the OFIR-EU estimates when $p < 1$. One infers here that errors in the noise covariances do not affect the OFIR-EU estimates essentially. The estimation errors in the second state are sketched in Figure 6 for $p > 1$. As can be seen, the OFIR-EU is a bit more successful in accuracy than KF when $p = 1$. However, this advantage vanishes with an increase in p that is also in agreement with the early results shown in Figure 4.

7. CONCLUSIONS

The iterative OFIR-EU algorithm developed in this paper has several useful engineering properties. For practical use, it offers two options. The basic algorithm relying on the horizon length $N > N_{\mathrm{opt}}$, where N_{opt} refers to the optimal horizon of the UFIR filter, can be used whenever the bounded input/bounded output stability, robustness against uncertainties, and low sensitivity to errors in the noise covariances are required. It performs almost as the UFIR filter when $p < 1$; however, its computation time increases with N. The full-horizon algorithm has the KF structure and consumes almost as much computation time as the KF. But, unlike the KF, it ignores the initial conditions. An overall conclusion that can be made is that the iterative computation of OFIR-EU estimates is the next breakthrough solution in FIR filtering. We now work on its practical applications and hope to present the results in the near future.

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