The triangle inequality constraint in similarity judgments

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Running text word count: 4752

Abstract

Since Tversky's (1977) seminal investigation, the triangle inequality, along with symmetry and minimality, have had a central role in investigations of the fundamental constraints on human similarity judgments. The meaning of minimality and symmetry in similarity judgments has been straightforward, but this is not the case for the triangle inequality. Expressed in terms of dissimilarities, and assuming a simple, linear function between dissimilarities and distances, the triangle inequality constraint implies that human behaviour should be consistent with Dissimilarity(A,B) + Dissimilarity(B,C) ≥ Dissimilarity(A,C), where A, B, and C are any three stimuli. We show how we can translate this constraint into one for similarities, using Shepard's (1987) generalization law, and so derive the multiplicative triangle inequality for similarities, $Sim(A, C) \geq Sim(A, B) \cdot Sim(B, C)$ where $0 \leq Sim(x, y) \leq 1$. Can humans violate the multiplicative triangle inequality? An empirical demonstration shows that they can.

Keywords

similarity; triangle inequality; Shepard's generalization law; quantum theory

1. Introduction

Tversky's (1977) famous work is widely interpreted as showing that similarity judgments are not consistent with the metric axioms, thus casting a critical eye on the widespread approach to representation and similarity based on psychological spaces (for earlier examination see Attneave, 1950, Rosch, 1975). Specifically, all distances must obey the metric axioms:

Minimality: Distance(A,A) = 0

Symmetry: Distance(A,B) = Distance(B,A)

Triangle Inequality: Distance(A,B) + Distance(B,C) ≥ Distance(A,C)

If we employ distances in psychological spaces to model similarities, should it not be the case that similarities need be consistent with the metric axioms? Then, the common interpretation of Tversky's work is that models of similarity based on distances cannot be adequate.

This interpretation is correct for symmetry and minimality (that the similarity between an item and itself should be maximal and that similarities should be symmetric). However, in fact, Tversky (1977) provided only a much weaker argument regarding the triangle inequality and similarities. He discussed the triangle inequality in relation to a famous example, based on William James. Tversky noted (p.329) "the perceived distance of Jamaica to Russia exceeds the perceived distance of Jamaica to Cuba, plus that of Cuba to Russia – contrary to the triangle inequality." If we equate distances with (some simple function of) dissimilarities, the triangle inequality constraint for these countries can be written as

$$Dissimilarity(Jamaica, Russia) \leq Dissimilarity(Jamaica, Cuba) + Dissmilarity(Cuba, Russia)$$

Jamaica and Russia are highly dissimilar to each other, while Jamaica and Cuba have a very low dissimilarity (because of geographical location) and likewise for Cuba and Russia (because of political affiliation), so violating the triangle inequality – for dissimilarities. In both the original paper and a subsequent one (Tversky & Gati, 1982) Tversky is extremely careful to limit the scope of this conclusion. For example, he said (Tversky, 1977, p.329) "…the triangle inequality implies that if a is quite similar to b, and b is quite similar to c, then a and c cannot be very dissimilar from each other. Thus, it sets a lower limit to the similarity between a and c in terms of the similarities between a and b and between b and c." But, this expression is not a quantitative constraint.

Thus, despite the fact that Tversky's work was nearly 40 years ago, there is currently no precise notion of how the triangle inequality translates into a constraint for *similarities*, as opposed to dissimilarities. Resolving this problem is important both for studies into the foundations of human similarity judgments and, more practically, since the majority of psychological research has focused on similarity, not dissimilarity (e.g., Medin et al., 1990; Minda & Smith, 2001; Nosofsky, 1984; Pothos, 2005).

Why is it not possible to just assume a violation of triangle inequalities and re-express it in terms of similarities? One might be inclined to write a triangle inequality with similarities as

$$Similarity(A, B) + Similarity(B, C) \leq Similarity(A, C)$$

However, such an expression is valid only if we set $Similarity = -Dissimilarity$, which is problematic. Dissimilarities are straightforwardly equated with distances, which have to be positive. But, similarities are also typically considered positive: our intuition of psychological similarity is that of a positive quantity and, operationally, similarity is always measured with positive scales. We can imagine other, 'convenient' functions linking similarity and dissimilarity, but, in the absence of psychological theory, such functions are arbitrary. There is also the complication that in certain cases the two measures may not have a simple inverse relation (Medin, Goldstone & Gentner, 1990), but this possibility is beyond this study.

The most widely adopted function linking distances and similarities is Shepard's (1987) law of generalization, according to which $Similarity = e^{-distance}$. Shepard's law is still very much at the heart of influential cognitive theories, such as Nosofsky's (1984) Generalized Context Model or the Minda-Smith version of prototype theory (Minda & Smith, 2001). Shepard's law assumes that similarity is a ratio scale between 0, 1. While this seems like a strong assumption (e.g., Tversky & Gati, 1982, focused on ordinal relations), note that most empirical similarity measures are based on Likert scales. When using a Likert scale, a common (if not sometimes tacit) assumption is that such scales are linear and so correspond to interval, possibly ratio, scales. For example, naïve observers are able to make fine discriminations of similarity. Moreover, accepting that there are pairs of stimuli that have zero psychological similarity indicates a ratio scale for psychological similarity. It is possible to question this assumption of linearity, which would undermine the present discussion. However, the present authors are not aware of any evidence against linearity and hypothetical arguments to the contrary appear contrived.

We can use Shepard's (1987) law to derive a constraint for similarities, from the triangle inequality:

$$Distance(A, C) \leq Distance(A, B) + Distance(B, C) \Leftrightarrow$$

$$e^{-Distance(A,C)} \geq e^{-\left(Distance(A,B)+Distance(B,C)\right)} \Leftrightarrow$$

$$e^{-Distance(A,C)} \geq e^{-Distance(A,B)} \cdot e^{-Distance(B,C)}$$

which gives:

$$Sim(A, C) \geq Sim(A, B) \cdot Sim(B, C)$$

We call this latter inequality the multiplicative triangle inequality (MTI) and it indicates that, if we consider the similarity of two stimuli (A,C) to a third one (B), then the product of the similarities to the third one provides a lower bound for the similarity of the two initial stimuli. For example, for three objects, table, chair and bed, the lower bound for the similarity between a table and a chair is the product of the similarities between table and bed and chair and bed. As far as we know, the MTI is a unique proposal for how human similarity judgments are constrained, it is the most straightforward way to derive a constraint on *similarities* from the triangle inequality, and it has not been empirically investigated before (we further justify this last comment shortly below)

Note, the literature has also considered similarity functions using a Gaussian, rather than exponential form. However, according to Nosofsky (1992), the Gaussian similarity function applies with "protracted identification training involving asymptotic performance with highly confusable stimuli" (p.29). With a Gaussian similarity function, we have:

$$e^{-Distance(A,C)^2} \geq e^{-\left(Distance(A,B)+Distance(B,C)\right)^2} \Leftrightarrow$$

$$e^{-Distance(A,C)^2}$$

$$\geq e^{-Distance(A,B)^2} e^{-Distance(B,C)^2} e^{-2Distance(A,B)\cdot Distance(B,C)}$$

which gives:

$$Sim(A,C) \geq Sim(A,B) \cdot Sim(B,C) \cdot x$$

where $0 \leq x \leq 1$. Thus, with a Gaussian similarity function, we do not reproduce the MTI, but a weaker form, since stimuli which violate the exponential MTI may be consistent with its Gaussian form. In this work, we employ distinguishable stimuli, which are presented only once, and for which only one response is made. Thus, the (limited) literature only allows to motivate the exponential form of the MTI and we will only consider this henceforth.

The MTI clearly has a distinct form compared to the triangle inequality. Note, that a violation of the MTI implies a violation the triangle inequality and vice versa. This is easily seen by noting that if $Distance(AB) + Distance(BC) < Distance(AC)$ (a violation of the triangle inequality) then $e^{-Distance(AB)-Distance(BC)} > e^{-Distance(AC)} \Leftrightarrow Sim(AB) \cdot Sim(BC) > Sim(AC)$ (a violation of the MTI). Is there empirical evidence that the MTI is violated and is it established how violations of the MTI can arise from similarity models? We suggest that the answer is no for both questions. First, Tversky's (1977) anecdotal example for how dissimilarities violate the triangle inequality perhaps suggests that the MTI would be violated as well. However, this is not a direct empirical demonstration and in fact we are not aware of empirical reports focusing on violations of the MTI (that is, violations of the triangle inequality, as translated for similarities; cf. Tversky & Gati, 1982). There are some reports in the literature which may look like relevant evidence, but this is not the case. For example, Voorspoels et al. (2011), as part of a similarity study, reported on violations of the triangle inequality. But, they derived a similarity matrix based on feature vectors and it is possible that the situation regarding the triangle inequality/ MTI would be different with direct similarity ratings. Also, the highest rate of triangle inequality violations was 0.13%, which indicates, if anything, no violations.

Second, regarding theoretical accounts, there have been several influential similarity proposals, notably from Krumhansl (1978) and Ashby and Perrin (1988), which all purport to cover Tversky's (1977) key findings, including violations of the triangle inequality. So, exactly how theoretically pertinent is it to still research the triangle inequality (or the MTI)? Is it not the case that, across a research tradition spanning several decades, we now have several satisfactory similarity theories?

Both Krumhansl's (1978) and Ashby and Perrin's (1988) theories, for all their significant overall contributions to our understanding of similarity, actually provide a poor account of violations of the triangle inequality. Krumhansl's (1978) explanation for the triangle inequality is based on the idea that similarity judgments emphasize dimensions and features that objects have in common. As a result, stimuli which are far apart in an overall psychological space may be close to each other in a low dimensionality subspace, corresponding to the common dimensions between the stimuli. For example, Russia and Cuba are similar in the subspace of Communism, which corresponds to their common dimension. Krumhansl (1978, p.12) notes "Subspaces defined by obvious stimulus dimensions would seem to be likelier projections than subspaces not corresponding to such dimensions" and goes on to observe that such a scheme may be able to account for similarity relations inconsistent with the triangle inequality. But, why should similarity be assessed in a subspace for the triangle inequality comparisons and not in other cases? Krumhansl's model does not provide any guidance as to when similarity should be assessed in subspaces or the way to determine the relevant subspaces.

Regarding Ashby and Perrin (1988), they showed how one can manipulate the perceptual effects distributions, so that two stimuli can be both dissimilar to each other and both similar to a third stimulus, hence violating the triangle inequality. Such a situation can be mapped to Tversky's (1977) Russia-Cuba-Jamaica example. However, this argument assumes (see their Figure 4, p.133) asymmetric and inequivalent perceptual effects distributions for the three stimuli. This is an unlikely assumption in the case of, for example, comparisons between Russia, Cuba, and Jamaica. Why would the distributions for such countries have a different shape?

Note, finally, that Nosofsky's (1984) influential Generalized Context Model can produce violations of the triangle inequality, through manipulations of its attentional parameters. But, without an independent way to predict the setting of the attentional weights, this is a post hoc explanation. An analogous argument applies to Tversky's (1977) own contrast model, which relies on parameter setting to accommodate violations of the metric axioms (though note again that, regarding the triangle inequality on similarities, no direct demonstration or model fit was offered by Tversky, 1977). There other less well-known accounts, that are potentially relevant. For example, Jaekel et al.'s (2008) proposal of similarity metrics, based on a Hilbert space (a kind of vector space) and Shepard's (1987) generalization law, can produce violations of the triangle inequality, though the concurrent coverage of violations of symmetry and minimality is unclear (all the other similarity accounts considered here aim for a comprehensive coverage of Tversky's, 1977, key results). Overall, it is a misleading impression that violations of the triangle inequality can be straightforwardly explained by dominant similarity approaches, which makes it unlikely that they can produce violations of the MTI in a satisfactory way too.

Recently, we proposed a similarity model based on the mathematics of quantum theory (QT), specifically so as to account for Tversky's (1977) key findings as naturally as possibly (Pothos et al., 2013). In fact, this similarity approach can naturally cover putative violations of the MTI (and the triangle inequality, if one considers dissimilarities). That this is the case can be explained fairly directly, without detailed modelling. We note that this does not preclude that other similarity approaches may be extended to cover putative violations of the MTI in a natural way, though, the corresponding detailed argument is beyond this paper.

QT provides rules for assigning probabilities to events, from quantum mechanics, without the physics. Some researchers have been pursuing QT cognitive models, especially for behaviors at odds with the more established classical probability theory (Aerts & Aerts, 1995; Busemeyer & Bruza, 2011; Pothos & Busemeyer, 2013; Haven & Khrennikov, 2013). Regarding similarity, in the QT model, representations are subspaces in a multidimensional vector space. A subspace can have a higher or lower dimensionality, depending on the extent of knowledge we have for the corresponding stimulus or concept. The mental state is modeled by a state vector, $|\psi\rangle$. Each subspace is associated with a projection operator, which computes the overlap between a vector (e.g., the mental state vector) and a subspace. Following from Tversky's (1977) triangle inequality example, if the projection operator for Russia is P_{Russia}, then the overlap between the Russia subspace and the mental state is the vector $P_{Russia} \cdot |\psi\rangle$, whose length squared, $|P_{Russia} \cdot |\psi\rangle|^2$ is the probability that the mental state is about Russia. Similarity is defined as $Similarity(Russia, Jamaica) = |P_{Jamaica} \cdot P_{Russia} \cdot |\psi\rangle|^2$. Regarding the MTI, we have $Sim(A,C) \geq Sim(A,B) \cdot Sim(B,C) \Leftrightarrow |P_A \cdot P_C \cdot |\psi\rangle|^2 \geq |P_A \cdot P_B \cdot |\psi\rangle|^2 \cdot |P_B \cdot P_C \cdot |\psi\rangle|^2$. But, e.g. with one dimensional subspaces (and appropriate state vectors, cf. Pothos et al., 2013), this implies $cos^2\theta_{AC} \geq cos^2\theta_{AB} \cdot cos^2\theta_{BC}$, where the angles are between the corresponding rays, as indicated. It is clearly possible to violate the MTI with the QT similarity model, e.g., with $\theta_{AB} = \frac{\pi}{4} = \theta_{BC}$ and $\theta_{AC} = \frac{\pi}{2}$, which gives $0 \geq 0.25$.

The more important point is that a prediction of MTI violation for Tversky's (1977) triangle inequality example emerges from the QT similarity model, in the sense that it follows from the corresponding representations and no further assumptions are needed. The representation in Figure 1 was put together on the basis of three, intuitive/ reasonable assumptions: the property of Communism should be as unrelated as possible to the property of being in the Caribbean; Russia should have as much overlap as possible with the Communism property and as little as possible with the in the Caribbean one and vice versa for Jamaica; Cuba should have overlap with both the Communism and the in the Caribbean properties. These assumptions are indeed the ones Tversky (1977) made in arguing for violations of the triangle inequality. Note, this is toy representation, since no psychologically plausible representation would involve simple rays. Nevertheless it is useful for illustration. Given Figure 1, a violation of the MTI is readily predicted, since $\theta_{Jamaica,Russia} = \theta_{Jamaica,Cuba} + \theta_{Russia,Cuba}$, which implies $cos^2\theta_{JR} \leq cos^2\theta_{JC} \cdot cos^2\theta_{RC} \Leftrightarrow Sim(JR) \leq Sim(JC) \cdot Sim(RC)$. violating the MTI (Pothos et al., 2013).

Our argument that violations of the MTI are natural in the QT model has been that, given the Figure 1 (reasonable) representation, then a violation of the MTI just follows. However, it needs be pointed out that perhaps, ideally, empirical similarity data would be used to derive a representation (along the lines of that in Figure 1). and then examine directly (without further fits) whether the MTI is violated or not. Unfortunately, the QT similarity program is not at this point yet. The problem is that we do not know how to determine the optimal dimensionality for the subspace corresponding to each concept (in the above example they are all rays); this is an important objective for future work.

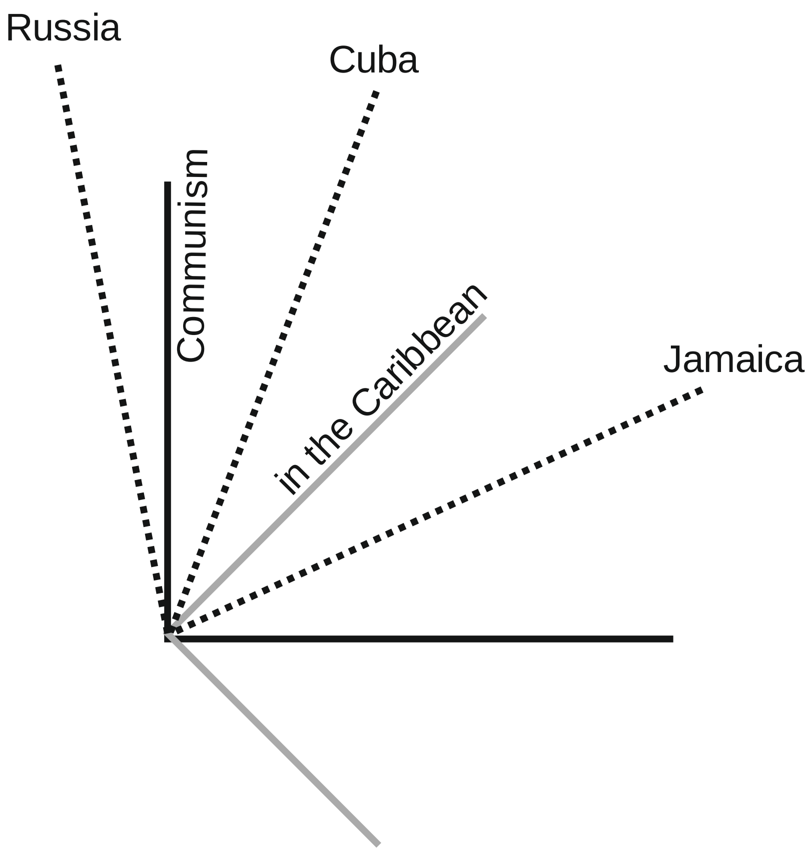


Figure 1. A plausible toy representation for Russian, Cuba, Jamaica, relative to the properties of Communism and in the Caribbean. Given this representation, a violation of the MTI from the QT similarity model readily emerges.

Overall, Tversky's (1977) Jamaica, Cuba, Russia example makes it fairly plausible that violations of the MTI will be observed in similarity judgments. However, it is impossible to establish this without detailed measurements. If similarity judgments are mostly consistent with the MTI, then this would make suspect the QT similarity approach and force us to rethink the motivation for the model. To anticipate our results, this is not the case.

2. Participants, materials, and methods

We tested 431 experimentally naïve participants, recruited through CrowdFlower, for a small payment ($1; due to a computer error, the payment was not administered correctly and we could only manually pay participants who got in touch with us. The payment error manifested itself after the experimental tasks). The sample size was a priori set to 400 participants, but the recruitment process (automated through CrowdFlower) overshot. Participants were randomly divided between two conditions, which employed different stimuli.

We constructed two lists of stimulus triplets, one consisting of 19 country triplets and another consisting of 21 general stimulus triplets (Appendix 1). The triplets were constructed so that two pairs of stimuli were expected to lead to a high similarity while the third pair would have low similarity, e.g., for countries, Mexico, USA, Canada and for general stimuli Razor, Knife, Fork, but no piloting was carried out, since we were not intending detailed modelling. Participants were randomly assigned either to the countries or the general stimuli.

To assess putative MTI violations for each triplet, we required three similarity ratings, so that for the countries stimuli there were overall 57 similarity ratings and for the general stimuli 63; in both cases, participants performed the ratings in a random order. Each trial involved showing the two stimuli concurrently on a screen, with the prompt to rate their similarity on a 1-9 scale. The stimuli remained on the screen until a response was provided.

3. Results

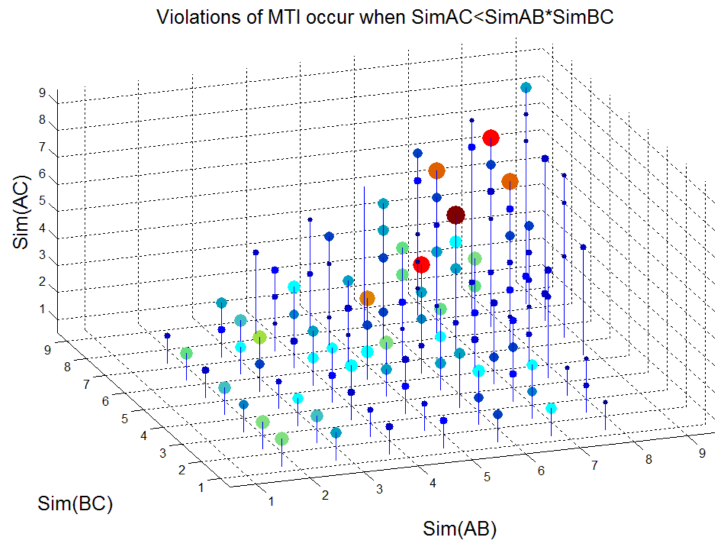
For each participant we computed the variance of all their similarity judgments and removed participants with either very high (e.g., participants using only 1 or 9) or very low (participants not using the full scale) variances. Cutoffs for high, low variance were 10, 0.7 respectively. This procedure retained 191 out of 212 participants for the countries stimuli, and 197 out of 219 for the general stimuli. Similarity ratings for the remaining participants were then converted via a linear rescaling to a 0 to 1 scale, since, recall, similarities in the MTI were derived using Shepard's (1987) generalization law and so bounded by 0,1.

There are two subtle issues which affect the analysis of results. First, a violation of the MTI occurs when $Sim(A, C) \leq Sim(A, B) \cdot Sim(B, C)$. Recall, the empirical procedure involved triplets of items, A, B, C, for which we collected empirical data for all pairwise similarities. For each triplet, we could identify the lowest similarity and so seek putative MTI violations. However, this procedure is inappropriate and would simply increase Type I errors, since for each triplet the stimuli A, B, C were specifically selected so that one similarity would be low and the other two similarities higher. So, we tested for violations of the MTI just in terms of which pairwise similarity was expected to be lower than the other two (note, the relationships between the A, B, C stimuli are obvious; see Appendix 1). Second, the MTI as a constraint on similarity judgments makes most sense when a consistent/ fixed order is employed throughout all relevant pairwise comparisons and this is the approach we adopted (with future work we will explore the implications from possible violations of symmetry).

We first considered the reliability of the data, using a measure of MTI violation (since the MTI states that $Sim(A,C) \geq Sim(A,B) \cdot Sim(B,C)$ we computed $Sim(A,C) - Sim(A,B) \cdot Sim(B,C)$, which will be negative if the MTI is violated). This reliability analysis indicates whether participants consistently produce a greater level of violations of the MTI for some triplets of stimuli than for others. For the countries and general stimuli Cronbach's alpha was .70 (N=191) and .84 (N=197), respectively. The materials, therefore, differed in the degree to which they consistently showed possible violations of the MTI (see also Figure 2).

The null hypothesis is that the MTI is a psychological constraint, so that similarity judgments will always be consistent with the MTI, excluding the possibility of random variation in ratings. Thus, for each triplet, there is a possibility that a violation of the MTI will be observed by chance. To compute this chance probability, we considered all possible combinations of 1-9 similarity ratings, for each triplet (converted to a 0,1 scale), and counted the percentage of triplets in which an MTI violation was observed: this was 25%. This is a conservative estimate of random error, since, given a null hypothesis that similarity judgments are always consistent with the MTI, we still assume a rate of by chance MTI violation for any triplet of ¼. Note also that in considering a triplet, $Sim(A,C), Sim(A,B),\ \ Sim(B,C)$, the three similarities are not completely independent. However, when considering correlations between triplet similarities, the random error rate is reduced. We can see this by noting that the MTI is not violated when the three similarities are equal or when $Sim(A,C)$ is equal to one of the other two similarities. Instead, MTI violations occur when there is a mismatch between the similarities, with $Sim(A,C)$ small but the other two large. Thus, if anything, the effect of taking into account correlations would be to make it easier to reject the null hypothesis.

We conducted an item-based analysis, testing that the proportion of MTI violations for a given triplet was higher than the 25% error rate expected by chance (participants were treated as a random effect). Note, there are no expectations as to whether the MTI is consistently violated in a set of items. Instead, the null hypothesis is that the MTI is a psychological constraint and rejecting the null hypothesis involves existence proof that there are some items for which the MTI is violated. The MTI violation count for each triplet was based on when $Sim(A, C) - Sim(A, B) \cdot Sim(B, C) < 0$, where A,C was the pair of stimuli assumed a priori to be most similar. Using a dependent variable based on binary counts (for each triplet, for each participant, checking whether the MTI was violated or not) is justified because the distribution of $Sim(A, C) - Sim(A, B) \cdot Sim(B, C)$ values was not normal (this is because when it is in principle possible to violate the MTI; most violations are observed for small positive values of this quantity).



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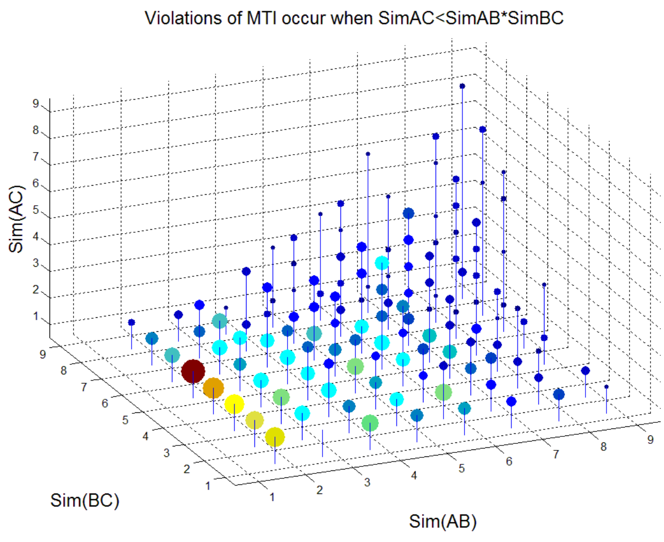

B

Figure 2. Diagrams (A: countries stimuli; B general stimuli) indicating the distribution of violations of the MTI, across participant responses. Each data point means that the corresponding combination of Sim(A,B), Sim(B,C), Sim(A,C), for which an MTI violation occurs, was observed at least twice (we suppressed points indicating a single violation, to unclutter the diagrams). Larger circles and color from blue to red both indicate greater frequencies.

Using chi-squared tests with alpha set at .05, we observed 4/19 significant violations for the countries and 10/21 for the general stimuli (highest p-value amongst the significant violations .035, see Appendix 1). The significance level for rejecting the null hypothesis for each triplet was set to .05, so by chance we still expect 1/20 MTI violations for each stimulus category. The proportion 1/20 was significantly different from 4/19 (p=.013), and even more so from 10/21 (p<.0005; in both cases using Fisher's Exact Probability Test on one degree of freedom).

One possible issue with the analysis above is that some ratings of the similarity between stimuli make an MTI violation impossible. For example, if a participant rates the similarity between A and B as 1 out of 9, this implies $Sim(A, B) = 0$, and a violation of the MTI is now impossible whatever the other two similarities. To investigate this we reanalyzed the data, ignoring, for a given triplet of stimuli, responses of 1 for the A,B or B,C similarities (which would lead to a converted similarity of 0, making an MTI violation impossible). This approach changes the frequency of observed MTI violations, reduces the sample size for each triplet of stimuli (because of the eliminated responses; the range of responses for each triplet is now 64-179), and increases the expected rate of obtaining MTI violations by chance from 25% to 33%, for a given triplet. For the general stimuli, the rate of violations (14/21) was still significantly higher than the chance 1/20 rate ($p<.0005$) but for the countries stimuli (3/19) it was now not significantly different from the chance rate ($p=.067$). This gives us confidence that an overall conclusion of MTI violations, in some cases, is independent of the precise way we analyze the data.

4. Discussion and conclusions

Tversky's (1977) seminal influence was that he started a research programme into the algebraic foundations of similarity judgments and, indeed, most major subsequent similarity proposals are often tested against his key empirical conclusions regarding violations of the metric axioms. However, we showed that implications for similarity from the triangle inequality have not been worked out and require a commitment to a function linking distance and similarity. Another seminal influence in psychology, Shepard's (1987) generalization law, was used for this purpose. We thus derived the MTI and, in one experiment, provided an existence proof that the MTI is sometimes violated in similarity judgments. Note, our results offer no guidance as to what might be the proportion of MTI violations, if one were to select a triplet of items randomly, that is, we currently cannot provide guidance into the manipulations which may make violations of the MTI more or less likely (contrast with e.g. Aguilar and Medin, 1999, in relation to symmetry) All the stimuli were selected with an expectation that violations of the MTI may be 'likely' and so, if one were to cast a critical eye on our results, one could say that the evidence for the preponderance of MTI violations in human similarity judgments is not strong. However, as Tversky (1977) intended in his original discussion, our results do provide clear existence proof that the MTI can be violated sometimes.

A researcher insisting on conceptualizing similarity as a function of distance may explore alternative functions linking similarity and distance, such as a Gaussian function. However, violations of the exponential MTI version, for stimuli as the ones employed in this study will still need to be explained or, alternatively, a Gaussian similarity function will need to be motivated more strongly for all kinds of stimuli. This latter possibility is inconsistent with the available evidence (Nosofsky, 1992), though note the issue of exponential vs. Gaussian similarity functions has not attracted much attention recently (indeed some researchers use a free parameter corresponding to the exact form of the similarity function). An alternative approach might be to adopt Nosofsky's (1984, 1992) formalization, which offers parametric flexibility to accommodate both MTI violations and violations of the other metric axioms (such as symmetry, using a directionality parameter). While there is no doubt that his theory is one of the most influential categorization theories, it is arguable as to whether similarity researchers will be satisfied with this approach, unless parametric changes can be motivated independently; currently this is not possible.

Our motivation for pursuing this research was exactly because of its potential to provide results which are particularly easy to accommodate within the QT similarity model (Pothos et al., 2013). We interpret violations of the MTI inequality as additional support for the QT similarity model, while of course acknowledging that this is a vast research topic that cannot be settled by any single study. In brief, the psychological explanation for how violations of the MTI arise from the QT similarity model relates generally to the contextual way in which probabilities are assessed in QT. Different regions of psychological space imply different properties or contexts for assessing similarity. For example, regarding the Figure 1 representation, the Communism region is the part of psychological space where countries consistent with this property cluster. So, two countries which are close to each other in the Communism region of psychological space can be said to be similar to each because they are both consistent with the property of Communism, and analogously with the property in the Caribbean etc. Regarding the problem at hand, because of the geometry of how such different regions are arranged, one can easily construct patterns that violate the MTI inequality. It is exactly this contextuality that is characteristic of the QT similarity model (and QT models in general) that provides a natural interpretation of this and related similarity findings (such as the diagnosticity effect, which Tversky, 1977, also reported). Note, as the number of possible stimuli increases, it is likely that a pattern of similarity relations of a certain complexity will constraint the minimum dimensionality of the corresponding QT space; this is an interesting topic for future work.

In closing, understanding the formal properties of similarity judgments is a key objective not only in cognitive science (since similarity is often the building block of cognitive models; Goldstone & Son, 2005; Pothos, 2005; Sloman & Rips, 1998; see also Gärdenfors, 2000), but beyond too. For example, in information retrieval, most models are based on vector spaces (e.g., Salton et al., 1975), and the corresponding ranking algorithms are either obviously consistent with the metric axioms or a detailed assessment is not made (e.g., Manning et al., 2009; Robertson & Spärck Jones, 1976). Similar considerations apply to e.g. latent semantic analysis (e.g., Dumais, 2004). In presenting these results, we hope to provide an important technical modification in our understanding of violations of the triangle inequality and, in addition, a further source of evidence concerning the QT similarity model.

Acknowledgements

EMP was supported by Leverhulme Trust grant RPG-2015-311 and H2020-MSCA-IF-2015 grant 696331.

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Appendix 1. The materials used in the two conditions of the experiment. After each triplet, we show the number of participants violating the MTI (in the countries case 191 participants overall and in the general stimuli case 197 participants). We also give the p value computed from a χ^2test on one degree of freedom, assuming a chance violation rate per participant of 25%.

Country triplets:

Country A	Country B	Country C	% of Ps violating MTI	p Value (one-tailed)
Mexico	USA	Canada	39.3	<.001
Jamaica	Cuba	Russia	41.4	<.001
India	Pakistan	Iran	49.7	<.001
Albania	Greece	Italy	28.3	.29
The Netherlands	Germany	Poland	22.5	-
Norway	Latvia	Slovenia	12.6	-
Portugal	Brazil	Uruguay	27.2	.48
Australia	UK	Zimbabwe	7.9	-
Spain	France	Switzerland	27.2	.48
North Korea	China	Japan	32.5	.017
Saudi Arabia	Nigeria	Ghana	20.4	-
Turkey	Cyprus	Malta	24.1	-
Luxemburg	Cayman Islands	Dominican Republic	15.7	-
Malaysia	Singapore	Gibraltar	16.8	-

Austria	Hungary	Romania	17.3	-
Ireland	Madagascar	Mozambique	16.8	-
Nepal	Mongolia	Greenland	10.5	-
Dubai	Panama	Colombia	16.2	-
Croatia	Serbia	Bulgaria	13.6	-

Triplets of general stimuli:

Item A	Item B	Item C	% of Ps violating MTI	p Value (one-tailed)
Butcher	Surgeon	GP	31.5	.035
Razor	Knife	Fork	59.4	<.001
Cheetah	Bullet	Dart	24.9	-
Skyscraper	Giraffe	Zebra	44.2	<.001
Fossil	Skeleton	Muscle	48.7	<.001
Sheet	Plain	Mountain	18.8	-
Fox	Lawyer	Teacher	17.8	-
Snail	Tortoise	Hamster	16.8	-
Mouse	Cockroach	Locust	19.8	-
Feather	Fur	Bear	54.3	<.001
Oven	Tropics	Ocean	46.2	<.001
Pig	Dirt	Stain	33.5	.006
Porcupine	Cactus	Palm-Tree	45.7	<.001
Book	Magazine	TV-show	17.8	-
Ice	Alaska	Hawaii	46.7	<.001
Mule	Negotiator	Counsellor	7.6	-
Butterfly	Blue-Bird	Crow	24.9	-

Ant	Poppy-seed	Mustard	21.3	-
Doughnut	Wedding-ring	Necklace	32.5	.015
Skunk	Pig-sty	Chicken-shed	28.9	.21
Zebra	Wasp	Fly	13.7	-