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Welfare theorems for random assignments with priorities*

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Abstract

We provide priority-constrained versions of the ordinal efficiency welfare theorem for school choice lotteries. Moreover, we show that a priority-constrained version of a cardinal second welfare theorem fails to hold, but can be restored for a relaxed notion of equilibrium with priority-specific prices.
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1 Introduction

The assignment of students to schools (Abdulkadiroglu and Sönmez, 2003) is one of the major applications of matching theory. A school choice mechanism assigns students to schools taking into account the preferences of students and priorities of the students at the different schools. Coarse priorities are a generic feature in school choice. In practice, students are prioritized according to coarse criteria (e.g. based on catchment areas, or having a sibling in the school) such that many students have the same priority for a seat at a school. Thus, one can sometimes not avoid treating students differently ex-post even though they have the same priorities and preferences. However, ex-ante, some form of fairness can be restored by the use of lotteries. This motivates the study of school choice lotteries.

Respecting priorities in school choice mechanisms can lead to efficiency losses (Abdulkadiroglu and Sönmez, 2003). This motivates the question under which conditions

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priority-constrained efficiency can be achieved. Our paper contributes to this discussion by relating different notions of priority-constrained efficiency and providing welfare theorems. Our first main result (Theorem 1) establishes that two different notions of priority-constrained ordinal efficiency are equivalent, and are moreover, equivalent to the existence of a cardinal utility profile consistent with the ordinal preferences such that social-welfare is maximized subject to priority constraints. Thus, we generalize the ordinal efficiency welfare theorem (McLennan, 2002) to the school-choice context. We introduce a family of priority-constrained efficiency notions of different strength under which the ordinal efficiency welfare theorem holds. In particular, we define a natural priority-constrained efficiency notion where efficiency is subject to a set of admission policies at different schools that specify a priority cut-off for each school. This notion of constrained efficiency (taking into account a set of admission constraints) is a refinement of the notion of two-sided efficiency studied by He et al. (2018).

Having related different efficiency concepts, we apply them to the pseudo-markets of He et al. (2018). In their setting, a random assignment is generated by a market for probability shares. Each agent has a budget of tokens and can "buy" probability shares at the different schools. Agents face different prices depending on their priority. In this context, we prove (Theorem 2) a version of the ordinal efficiency welfare theorem for ex-ante stable[1] random assignments, where supporting prices and budgets can be constructed along the utility profile, such that the random assignment under consideration is an equilibrium assignment in the sense of He et al. (2018) for the economy defined by the constructed utility profiles and budgets. This naturally leads to the question whether the result can be strengthened in such a way that for any utility profile and priority-constrained efficient random assignment, there exist corresponding budgets such that the random assignment is an equilibrium assignment in the sense of He et al. (2018). In other words, does a constrained second welfare theorem hold? We show by means of a counterexample that this is not the case. However, with a natural relaxation of the notion of equilibrium with priority-specific prices we obtain (Theorem 3) a second welfare theorem.

1.1 Related Literature

Ordinal efficiency welfare theorems for probabilistic assignments have been studied in the context of object allocation without priorities (McLennan, 2002; Manea, 2008) and in the case of marriage markets (Doğan and Yıldız, 2016). An ordinal efficiency welfare theorem establishes that ordinal efficiency (with welfare evaluated by first-order stochastic dominance) for a random assignment is equivalent to the existence of a cardinal utility profile consistent with the ordinal preferences under which the random assignment is Pareto efficient when lotteries are evaluated according to expected utility. Pareto efficiency can be strengthened to social welfare efficiency. The original ordinal welfare theorem is due to McLennan (2002) answering a question raised by Bogomolnaia and Moulin (2001). A constructive proof was provided by Manea (2008). We generalize

[1]In an ex-ante stable random assignment a student only can obtain a seat at a school with positive probability if there is no higher priority student for that school who obtains a less desirable school with positive probability (Kesten and Ünver, 2015).

the ordinal efficiency welfare theorem to the case with (weak) priorities and quotas.

Kesten and Ünver (2015) initiate the study of ex-ante stable school choice lotteries. For the classical marriage model, the condition was first considered by Roth et al. (1993). Kesten and Ünver (2015) consider mechanisms that implement ex-ante stable lotteries and satisfy constrained ordinal efficiency properties that are generally more stringent than the ones we consider. He et al. (2018) define an appealing class of mechanisms that implement ex-ante stable lotteries. These mechanisms generalize the pseudo-market mechanisms of Hylland and Zeckhauser (1979) by allowing for priority-specific prices (agents with different priorities are offered different prices). Cardinal welfare theorems for random assignments are counterparts to the classical welfare theorems for exchange economies. For cardinal welfare theorems, utility profiles are the primitive of the model. Hylland and Zeckhauser (1979) show that the equilibria of their pseudo markets are Pareto efficient and hence establish a cardinal first welfare theorem. Miralles and Pycia (2017) establish a cardinal second welfare theorem that demonstrates that each Pareto efficient random assignment can be decentralized as a pseudo-market equilibrium by appropriately choosing budgets and prices. We show that their result only generalizes to the pseudo-markets of He et al. (2018), if the notion of equilibrium with priority-specific prices is weakened.

2 Model

There is a set of n agents N and a set of m schools M. A generic agent is denoted by i and a generic school by j. In each school j, there is a finite number of seats $q_j \in \mathbb{N}$. We assume that there are as many school seats as agents, $\sum_{j\in M} q_j = n$. A lottery over schools is a probability distribution over M. We denote the set of all lotteries over schools by $\Delta(M)$.

Agents have preferences over lotteries over schools. Preferences of agents can be modeled in two different ways: In the first version, each agent i has a **preference relation** R_i over different schools. We call $R = (R_i)_{i\in N}$ a **preference profile**. We write $j\, P_i\, j'$ if $j\, R_i\, j'$ but not $j'\, R_i\, j$, and $j\, I_i\, j'$ if $j\, R_i\, j'$ and $j'\, R_i\, j$. The preferences can be extended to a partial preference order over lotteries using the stochastic dominance criterion: A lottery π' **weakly first-order stochastically dominates** lottery π with respect to preferences R_i, if for each $j \in M$ we have

$$\sum_{j'\in M : j'\, R_i\, j} \pi'_{j'} \geq \sum_{j'\in M : j'\, R_i\, j} \pi_{j'}.$$

In this case, we write $\pi'\, R_i^{sd}\, \pi$. We write $\pi\, I_i^{sd}\, \pi'$ if all of the above weak inequalities hold with equality, and $\pi'\, P_i^{sd} \pi$ if at least one of the inequalities is strict. In the latter case we say that π' **strictly first-order stochastically dominates** lottery π.

In the second version, each agent i has a **von-Neumann-Morgenstern (vNM) utility** vector $U_i = (u_{ij})_{j\in M} \in \mathbb{R}_+^M$. We call $U = (U_i)_{i\in N}$ a **utility profile**. Lotteries are evaluated according to expected utility. Thus, agent i prefers lottery π' to lottery π if

$$\sum_{j\in M} u_{ij}\pi'_j > \sum_{j\in M} u_{ij}\pi_j.$$

A utility vector contains more information than a preference relation. In addition to ranking the schools, the vNM-utilities express the rates with which agents substitute probabilities of obtaining seats at the different schools. Utility vector U_i is **consistent** with preferences R_i, if for each pair of schools $j, j' \in M$ we have $jR_ij' \Leftrightarrow u_{ij} \geq u_{ij'}$. Each utility vector U_i is consistent with one preference relation R_i that we call the preference relation **induced** by U_i. It is a standard result (see e.g. Proposition 6.D.1 in Mas-Colell et al., 1995), that if lottery π' strictly first-order stochastically dominates lottery π according to preferences R_i, then lottery π' yields higher expected utility than π according to any vNM-utilities U_i consistent with R_i.

Each school j has a weak (reflexive, complete and transitive) priority order $\succsim_j$ of the agents. We let $i \sim_j i'$ if and only if $i \succsim_j i'$ and $i' \succsim_j i$. We let $i \succ_j i'$ if and only if $i \succsim_j i'$ but not $i' \succsim_j i$. The priorities $\succsim_j$ of a school j partition N in equivalence classes of equal priority agents, i.e. in equivalence classes with respect to $\sim_j$. We call these equivalence classes **priority classes** and denote them by $N_j^1, N_j^2 \ldots, N_j^{\ell(j)}$ with indices decreasing with priority. Thus, for $\ell < \ell'$, $i \in N_j^\ell$ and $i' \in N_j^{\ell'}$ we have $i \succ_j i'$. In that case, we also write $N_j^\ell \succ_j N_j^{\ell'}$. We use the notation $i \succsim_j N_j^\ell$ to indicate that i has higher or equally high priority at j than the agents in the priority class N_j^ℓ.

A **deterministic assignment** is a mapping $\mu : N \to M$ such that for each $j \in M$ we have $|\mu^{-1}(j)| = q_j$. A **random assignment** is a matrix $x = (x_{ij}) \in \mathbb{R}^{N \times M}$ such that

$$0 \leq x_{ij} \leq 1, \quad \sum_{j \in M} x_{ij} = 1, \quad \sum_{i \in N} x_{ij} = q_j,$$

where x_{ij} is the probability that agent i is matched to school j. By the Birkhoff-von Neumann Theorem, each random assignment corresponds to a lottery over deterministic assignments and, vice versa, each such lottery corresponds to a random assignment (see Kojima and Manea (2010) for a proof in the set-up that we consider). For each $i \in N$ we write $x_i = (x_{ij})_{j \in M}$ and for each $j \in M$ we write $x_j = (x_{ij})_{i \in N}$.

A random assignment x is **ex-ante blocked** by agent i and school j if there is some agent $i' \neq i$ with $x_{i'j} > 0$ and $i \succ_j i'$ and some school $j' \neq j$ with $x_{ij'} > 0$ and $j \, P_i \, j'$. In this case, we say that i has **justified envy** at school j. A random assignment is **ex-ante stable** if it is not blocked by any agent-school pair. The definition extends to the case where agents have vNM-utilities, by considering the preference profile induced by the utility profile.

2.1 Constrained Efficiency

Next, we introduce priority-constrained efficiency notions. In some situations, priorities of schools contain welfare relevant information that we want to respect. Schools' welfare can be evaluated by a first-order stochastic dominance ranking derived from priorities. For a school $j \in M$ and random assignments x and y we let $y_j \succsim_j^{sd} x_j$ if for each $1 \leq \ell \leq \ell(j)$ we have

$$\sum_{i \succsim_j N_j^\ell} y_{ij} \geq \sum_{i \succsim_j N_j^\ell} x_{ij}.$$

We write $y_j \succ_j^{sd} x_j$ if at least one of the inequalities is strict. A welfare improvement for agents is uncontroversial if it leaves schools' welfare unchanged or improves it. This leads us to the notion of priority-constrained efficiency. We introduce ordinal and cardinal versions of the notion.

Random assignment y **first-order stochastically dominates** (sd-dominates) random assignment x if for each $i \in N$ we have $y_i \, R_i^{sd} \, x_i$ and for at least one $i \in N$ we have $y_i \, P_i^{sd} \, x_i$. Random assignment x is **priority-constrained sd-efficient** if for each random assignment y that sd-dominates x there is a school j such that $y_j \not\succsim_j^{sd} x_j$.

For the case without priorities, sd-efficiency is equivalent to the absence of cycles of mutually beneficial bilateral trades (see Lemma 4 in Bogomolnaia and Moulin, 2001). One generalization of this notion to the case with priorities is that there should not be cycles of mutually beneficial bilateral trades where priority is weakly increasing through the cycle. Formally a **strong stable improvement cycle** for an assignment x is a sequence of agents $i_0, i_1, \ldots, i_K$ and schools $j_0, j_1, \ldots, j_K$ such that the following holds:

1. $x_{i_k, j_k} > 0$ for each $0 \le k \le K$,
2. $j_K \, R_{i_{K-1}} \, j_{K-1} \, R_{i_{K-2}} \, j_{K-2} \ldots j_1 \, R_{i_0} \, j_0 \, P_{i_K} \, j_K$,
3. $i_0 \succsim_{j_1} i_1 \succsim_{j_2} \ldots i_K \succsim_{j_0} i_0$.[2]

We will show in Theorem 1 that the absence of strong stable improvement cycles is equivalent to priority-constrained sd-efficiency.

The definition of priority-constrained efficiency extends to cardinal utility: Random assignment y **Pareto dominates** random assignment x with respect to vNM-utility profile U if for each $i \in N$ we have

$$\sum_{j \in M} u_{ij} y_{ij} \ge \sum_{j \in M} u_{ij} x_{ij},$$

and the inequality is strict for at least one agent. Random assignment x is **priority-constrained efficient** if for each random assignment y that Pareto dominates x there is a school j such that $y_j \not\succsim_j^{sd} x_j$. A random assignment y **dominates** random assignment x **in social welfare terms** with respect to vNM-utility profile U if

$$\sum_{i,j} u_{ij} y_{ij} > \sum_{i,j} u_{ij} x_{ij}.$$

Random assignment x **maximizes social welfare subject to priority constraints** if for each assignment y that dominates x in social welfare terms with respect to U there is a school j such that $y_j \not\succsim_j^{sd} x_j$.

[2]We use the term, "strong stable improvement cycle", because an ex-ante stable random assignment x that permits such a cycle can be sd-improved upon without violating ex-ante stability, by reallocating probability shares along the cycle. Erdil and Ergin, 2008 consider a weaker notion of a stable improvement cycle for deterministic stable matchings (and strict ordinal preferences), which was generalized to the probabilistic set-up by Kesten and Ünver (2015). In comparison to a strong stable improvement cycle, in the definition of a (weak) stable improvement cycle the third item in the definition is replaced by the weaker condition that for $0 \le k \le K$, i_k is one of the highest priority agents at school j_{k+1} that prefers j_{k+1} to some of the schools that he is matched to with positive probability. Here we take indices modulo $K + 1$.

He et al. (2018) consider a slightly stronger notion than priority-constrained efficiency, called two-sided efficiency, in which also priority-improvements that leave agents' welfare unchanged are ruled out. Formally, a random assignment y makes a **welfare-indifferent priority improvement** on random assignment x if for each $i \in N$ we have $\sum_{j\in M} u_{ij}y_{ij} = \sum_{j\in M} u_{ij}x_{ij}$, for each $j \in M$ we have $y_j \succsim_j^{sd} x_j$ and for at least one $j \in M$ we have $y_j \succ_j^{sd} x_j$. A random assignment x is **two-sided efficient** if it is priority-constrained efficient and does not admit a welfare-indifferent priority improvement.

2.1.1 Stronger Constrained Efficiency Notions

The concept of priority-constrained efficiency can be naturally strengthened: For priority profile $\succsim$, let $\succsim \subseteq \succsim'$ be a profile of weaker priorities. A random assignment x is **partially priority-constrained efficient** for $\succsim'$ if it is priority-constrained efficient under $\succsim'$. Partial priority-constrained efficiency thus only takes into account a subset of the priority comparisons as welfare relevant for schools, respectively as constraints when maximizing welfare for agents.

We are particularly interested in the case of lower bounds (or cut-offs) on priorities. If priorities are for example derived from exam scores or grade point averages,[3] then cut-offs are specified by minimum scores that grant admission at the different schools, where randomization can be used to ration seats among applicants who achieve exactly the minimum score. The concept is particularly natural for priority-respecting assignments: For strict priorities and preferences, priority cut-offs uniquely determine stable matchings (Azevedo and Leshno, 2016) and hence the welfare for the students. For probabilistic assignments, an ex-ante stable random assignment is determined by priority cut-offs and by a rationing rule for admission at the cut-off in case that there is over-demand at the priority cut-off. See Section 3.2.1. Thus, efficiency subject to priority lower bounds precludes inefficient rationing for given priority cut-offs.

Formally, we define the **cut-off** $C_j(x)$ for a school j under random assignment x to be the lowest priority class containing an agent that has a positive probability of obtaining a seat in school j in x. Formally

$$C_j(x) := N_j^{\max\{\ell : \exists i \in N_j^\ell, x_{ij} > 0\}}.$$

A school j uses a **more lenient admission policy** under random assignment y than under random assignment x, if either the school has a strictly lower cut-off in y than in x or it has the same cut-off, but admits a bigger fraction of the students in the cut-off class, i.e.

$$C_j(x) \succ_j C_j(y) \text{ or } [C_j(x) = C_j(y) \text{ and } \sum_{i \in C_j(x) = C_j(y)} y_{ij} > \sum_{i \in C_j(x) = C_j(y)} x_{ij}].$$

[3]These kinds of priorities occur for example frequently in centralized college admission (e.g. in China, and several European countries, see `http://www.matching-in-practice.eu/higher-education/`). In this context "schools" are universities or colleges. In practice, different methods are used to ration seats among equal score students, such as lotteries, the use of additional tie-breaking criteria, or (in the case of Hungary) leaving seats unassigned.

A random assignment x is **(priority)-cut-off-constrained sd-efficient** if for each random assignment y that sd-dominates it there exists a school that uses a more lenient admission policy under y than under x. Similarly, a random assignment x is **(priority)-cut-off-constrained efficient** if for each random assignment y that Pareto dominates it there exists a school that uses a more lenient admission policy under y than under x. A random assignment x **maximizes social welfare subject to priority cut-offs** if for each random assignment y that dominates x in social welfare terms with respect to U there is a school j that uses a more lenient admission policy under y.[4] One immediately sees that cut-off-constrained efficient random assignments are partially priority-constrained efficient with an ordering $\succsim'$ under which agents are partitioned into three priority classes at each school: Those that are ranked above the cut-off priority class, those that are in the cut-off class, and those ranked below the cut-off class. Formally for $j \in M$ and $i, i' \in N$

$$i \succsim'_j i' \Leftrightarrow [i \succ_j C_j(x) \text{ or } (i \sim_j C_j(x) \succsim_j i') \text{ or } (C_j(x) \succ_j i \text{ and } C_j(x) \succ_j i')].$$

3 Results

3.1 The Constrained Ordinal Efficiency Welfare Theorem

We now relate priority-constrained ordinal efficiency and priority-constrained cardinal efficiency by providing an ordinal efficiency welfare theorem for random assignments with priorities.

Theorem 1 (Constrained Ordinal Efficiency Welfare Theorem)**.** *For a random assignment x, preferences R and priorities $\succsim$ the following statements are equivalent:*

(1) x has no strong stable improvement cycle,

(2) x is priority-constrained sd-efficient for R and $\succsim$,

(3) there exists a vNM-utility profile U consistent with R, such that x maximizes social welfare subject to priority constraints for U and $\succsim$.

Proof. To show that (3) $\Rightarrow$ (2), note that if a random assignment y sd-dominates x under the preferences R induced by U, then y Pareto dominates x with respect to U and in particular y yields higher social welfare than x with respect to U. To show that (2) $\Rightarrow$ (1) note that if there is a strong stable improvement cycle, we can construct an assignment y that sd-dominates x by reallocating probabilities across the cycle: Let $i_0, \dots, i_K$ and $j_0, \dots, j_K$ be the agents and schools in the cycle. Choose $\epsilon > 0$ such that $\epsilon < \min\{x_{i_k,j_k}, 1 - x_{i_k,j_{k+1}}\}$ for each $0 \le k \le K$ (taking indices modulo $K+1$), and define $y_{i_k j_k} = x_{i_k,j_k} - \epsilon$ and $y_{i_k j_{k+1}} = x_{i_k,j_{k+1}} + \epsilon$ for each $0 \le k \le K$ (taking indices modulo $K+1$). Leave the assignment otherwise un-changed. By construction $y_j \succsim_j^{sd} x_j$ for each $j \in M$, $y_i \, R_i^{sd} \, x_i$ for each $i \in N$ and $y_i \, P_i^{sd} \, x_i$ for at least one $i \in N$. Thus, (2) $\Rightarrow$ (1). To show (1) $\Rightarrow$ (3), we use the following result due to Manea (2008):

[4] A stronger efficiency notion would define leniency only in terms of the cut-off and would allow for a smaller fraction of admitted students within the cut-off. We will later show (see Example 2) that this stronger efficiency notion is too stringent for our purposes.

Lemma 1 (Proposition 2 in Manea, 2008)**.** *Let $\rhd$ and $\bowtie$ two disjoint binary relations over a finite set $\mathcal{O}$. Let $\unrhd := \rhd \cup \bowtie$. If $\unrhd$ is acyclic, i.e. if there exists no sequence $o_1, o_2, \ldots, o_k$ with $o_1 \unrhd o_2 \unrhd \ldots \unrhd o_k \rhd o_1$, then there exist a mapping $u : \mathcal{O} \to \mathbb{R}_+$ such that for all $o, o' \in \mathcal{O}$ we have*

$$o \rhd o' \Rightarrow u(o) > u(o'),$$
$$o \bowtie o' \Rightarrow u(o) \geq u(o').$$

We choose $\mathcal{O}$ to be the set of all priority classes, i.e. $\mathcal{O} := \{N_j^\ell : j \in M, 1 \leq \ell \leq \ell(j)\}$ and define $\unrhd$ such that it reflects a common component of the agents' preferences: if an agent is contained in two priority classes (at different schools), consumes from the first one with positive probability, but (weakly) prefers consuming from the second one, then that second priority class is (weakly) preferred to the first one according to $\unrhd$. Formally, for $j \neq j'$, $1 \leq \ell \leq \ell(j)$ and $1 \leq \ell' \leq \ell(j')$ we let

$$N_j^\ell \rhd N_{j'}^{\ell'} \Leftrightarrow (\exists i \in N_j^\ell \cap N_{j'}^{\ell'}, x_{ij} > 0, j' \, P_i \, j),$$
$$N_j^\ell \bowtie N_{j'}^{\ell'} \Leftrightarrow (N_j^\ell \not\rhd N_{j'}^{\ell'} \text{ and } \exists i \in N_j^\ell \cap N_{j'}^{\ell'}, x_{ij} > 0, j' \, I_i \, j).$$

We rank priority classes at the same school such that a lower priority is weakly preferred to a higher priority, i.e. for $j \in M$ and $1 \leq \ell, \ell' \leq \ell(j)$ we let

$$N_j^{\ell'} \bowtie N_j^\ell \Leftrightarrow \ell < \ell'.$$

The absence of strong stable improvement cycles implies that $\unrhd$ is acyclic. Indeed suppose there is a cycle

$$N_{j_K}^{\ell_{r_K}} \bowtie \ldots \bowtie N_{j_K}^{\ell_{r_{K-1}+1}} \unrhd N_{j_{K-1}}^{\ell_{r_{K-1}}} \bowtie \ldots \bowtie N_{j_{K-1}}^{\ell_{r_{K-2}+1}} \unrhd \ldots \unrhd N_{j_0}^{\ell_{r_0}} \bowtie \ldots \bowtie N_{j_0}^{\ell_0} \rhd N_{j_K}^{\ell_{r_K}},$$

for indices $0 \leq r_0 \leq r_1 \leq \ldots \leq r_K$ and schools $j_0 \neq j_1 \neq \ldots \neq j_K \neq j_0$. For $0 \leq k \leq K-1$, choose $i_k \in N_{j_k}^{\ell_{r_k}} \cap N_{j_{k+1}}^{\ell_{r_k+1}}$ with $x_{ij_k} > 0$ such that $j_{k+1} \, R_{i_k} \, j_k$, and choose $i_K \in N_{j_K}^{\ell_{r_K}} \cap N_{j_0}^{\ell_0}$ with $x_{ij_K} > 0$ such that $j_0 \, P_{i_K} \, j_K$. Observe that as $N_{j_1}^{\ell_{r_1}} \bowtie N_{j_1}^{\ell_{r_0+1}}$, we have $\ell_{r_0+1} < \ell_{r_1}$ and therefore $i_0 \succsim_{j_1} i_1$, etc. We obtain $i_0, \ldots, i_K$ with $x_{i_k, j_k} > 0$, $j_{k+1} \, R_{i_k} \, j_k$ and $i_k \succsim_{j_{k+1}} i_{k+1}$ for $0 \leq k \leq K-1$, and $j_0 \, P_{i_K} \, j_K$ and $i_K \succsim_{j_0} i_0$. We have found a strong stable improvement cycle.

We use the lemma to define the utility profile U. Consider a mapping $u : \mathcal{O} \to \mathbb{R}_+$ as in Lemma 1. We define utilities such that for each priority class $N_j^\ell \in \mathcal{O}$ all agents in N_j^ℓ that are matched to j with positive probability have the same utility $u(N_j^\ell)$ for attending j. Moreover, this utility will be the maximal utility that any agent in N_j^ℓ has for attending j. Thus, we require for each $i \in N$ and $N_j^\ell \in \mathcal{O}$ that

$$x_{ij} > 0, i \in N_j^\ell \Rightarrow u_{ij} = u(N_j^\ell), \tag{1}$$
$$x_{ij} = 0, i \in N_j^\ell \Rightarrow u_{ij} \leq u(N_j^\ell). \tag{2}$$

We show that we can construct U consistent with R such that Conditions (1) and (2) hold. For $i \in N$ let $M_i := \{j \in M : x_{ij} > 0\}$ and $\mathcal{O}_i := \{N_j^\ell \in \mathcal{O} : i \in$

$N_j^\ell\}$. Order the elements of M_i consistently with R_i, i.e., let $M_i = \{j_0, \ldots, j_K\}$ with $j_K \, R_i \, j_{K-1} \, R_i \, \ldots \, R_i \, j_0$. For $0 \leq k \leq K$, let $1 \leq \ell_k \leq \ell(j_k)$ be the index such that $i \in N_{j_k}^{\ell_k}$ and let for each $j \in M$:

$$j \, I_i \, j_k \Rightarrow u_{ij} = u(N_{j_k}^{\ell_k}).$$

Define U_i otherwise consistent with R_i such that (2) holds, by requiring for $0 \leq k \leq K-1$ that

$$j_{k+1} \, P_i \, j \, P_i \, j_k \Rightarrow u(N_{j_k}^{\ell_k}) < u_{ij} < \min_{N_{j'}^\ell \in \mathcal{O}_i : j_{k+1} \, R_i \, j' \, P_i \, j_k} u(N_{j'}^\ell),$$

and that

$$j \, P_i \, j_K \Rightarrow u(N_{j_K}^{\ell_K}) < u_{ij} < \min_{N_{j'}^\ell \in \mathcal{O}_i : j' \, P_i \, j_K} u(N_{j'}^\ell),$$

$$j_0 \, P_i \, j \Rightarrow 0 \leq u_{ij} < \min_{N_{j'}^\ell \in \mathcal{O}_i : j_0 \, R_i \, j'} u(N_{j'}^\ell).$$

By the construction of our ordering this is possible and yields Conditions (1) and (2), as for each $0 \leq k \leq K$, $j \in M$ with $j \, P_i \, j_k$, and $N_j^\ell \in \mathcal{O}_i$, we have $N_j^\ell \rhd N_{j_k}^{\ell_k}$ and therefore $u(N_j^\ell) > u(N_{j_k}^{\ell_k})$, and for each $0 \leq k \leq K$, $j \in M$ with $j \, I_i \, j_k$ and $N_j^\ell \in \mathcal{O}_i$ we have $N_j^\ell \bowtie N_{j_k}^{\ell_k}$ and therefore $u(N_j^\ell) \geq u(N_{j_k}^{\ell_k}) = u_{ij}$.

Now that we have defined U, we can show that x maximizes social welfare subject to priority constraints with respect to U and $\succsim$. Suppose for random assignment y we have

$$\sum_{j \in M} \sum_{i \in N} u_{ij} y_{ij} > \sum_{j \in M} \sum_{i \in N} u_{ij} x_{ij}.$$

By Condition (1), we have

$$\sum_{j \in M} \sum_{i \in N} u_{ij} x_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} \sum_{i \in N_j^\ell} u_{ij} x_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} u(N_j^\ell) (\sum_{i \in N_j^\ell} x_{ij}).$$

By Conditions (1) and (2), we have

$$\sum_{j \in M} \sum_{i \in N} u_{ij} y_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} \sum_{i \in N_j^\ell} u_{ij} y_{ij} \leq \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} u(N_j^\ell) (\sum_{i \in N_j^\ell} y_{ij}).$$

Thus,

$$\sum_{j \in M} \sum_{\ell=1}^{\ell(j)} u(N_j^\ell) (\sum_{i \in N_j^\ell} y_{ij}) > \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} u(N_j^\ell) (\sum_{i \in N_j^\ell} x_{ij}).$$

Rearranging the terms and noting that $\sum_{i \succsim_j N_j^{\ell(j)}} y_{ij} = \sum_{i \in N} y_{ij} = q_j = \sum_{i \in N} x_{ij} = \sum_{i \succsim_j N_j^{\ell(j)}} x_{ij}$ we have

$$0 < \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} u(N_j^\ell)(\sum_{i \in N_j^\ell} (y_{ij} - x_{ij})) = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)-1} (u(N_j^\ell) - u(N_j^{\ell+1}))(\sum_{i \succsim_j N_j^\ell} (y_{ij} - x_{ij})).$$

By construction, $u(N_j^\ell) \leq u(N_j^{\ell'})$ for $\ell < \ell'$ and therefore $u(N_j^\ell) - u(N_j^{\ell+1}) \leq 0$ for $\ell = 1, \ldots, \ell(j) - 1$ for each $j \in M$. Thus, there is a $j \in M$ and $1 \leq \ell \leq \ell(j)$ with $\sum_{i \succsim_j N_j^\ell} (y_{ij} - x_{ij}) > 0$ and therefore $y_j \not\precsim_j^{sd} x_j$. □

Remark 1. *An analogous ordinal efficiency theorem can be obtained for two-sided efficiency. In this case, the notion of a strong stable improvement cycle has to be weakened as follows: For the sequences of agents $i_0, i_1, \ldots, i_K$ and schools $j_0, j_1, \ldots, j_K$ the following holds:*

1. $x_{i_k, j_k} > 0$ *for each* $0 \leq k \leq K$,
2. $j_K \, R_{i_{K-1}} \, j_{K-1} \, R_{i_{K-2}} \, j_{K-2} \ldots j_1 \, R_{i_0} \, j_0 \, R_{j_K} \, j_K$,
3. $i_0 \succsim_{j_1} i_1 \succsim_{j_2} \ldots i_K \succsim_{j_0} i_0$,
4. $j_0 \, P_{i_K} \, j_K$ *or* $i_K \succ_{j_0} i_0$.

Moreover, instead of priority-constrained sd-efficiency, we have to consider two-sided sd-efficiency, which can be defined as the combination of priority-constrained sd-efficiency and the absence of ordinal welfare-indifferent priority improvements, i.e. there should not be a random assignment y such that for each $i \in N$ we have $y_i \, I_i^{sd} \, x_i$, for each $j \in M$ we have $y_j \succsim_j^{sd} x_j$ and for at least one $j \in M$ we have $y_j \succ_j^{sd} x_j$. The absence of improvement cycles as above is equivalent to two-sided sd-efficiency and to the existence of a vNM-utility profile under which the assignment under consideration is two-sided efficient. We sketch the argument in the appendix.

An immediate consequence of the theorem and the discussion in Section 2.1.1 is the following version of an ordinal efficiency welfare theorem for cut-off-constrained efficiency. For ex-ante stable random assignments, the result can be further strengthened, as we show in Section 3.2.1.

Corollary 1 (Ordinal Efficiency Welfare Theorem for cut-off-constrained efficiency)**.** *For a random assignment x, preferences R and priorities $\succsim$ the following statements are equivalent:*

(1) x is cut-off-constrained sd-efficient for R and $\succsim$,

(2) there exists a vNM-utility profile U consistent with R, such that x maximizes social welfare subject to priority cut-offs for U and $\succsim$.

3.2 Equilibria with Priority-Specific prices

Next, we consider the pseudo-markets with priority-specific prices of He et al. (2018). A random assignment is generated by a pseudo-market of probability shares. Each agent has a budget of tokens and can "buy" probability shares at the different schools. Agents face different prices depending on their priority.

A **pseudo-market** is a triple $(U, b, \succsim)$ consisting of a vNM-utility profile U, a vector of budgets $b \in \mathbb{R}^M_+$, and priorities $\succsim$. We consider two equilibrium notions for pseudo-markets. The first relaxes the equilibrium notion of He et al. (2018): An **equilibrium with priority-specific prices** for the pseudo market $(U, b, \succsim)$ is a pair (x, p) consisting of a random assignment x, and prices $p = (p_{j,\ell})_{j \in M, 1 \leq \ell \leq \ell(j)}$ for each priority class that are (weakly) decreasing with priority, i.e. for each $j \in M$ we have $0 \leq p_{j,1} \leq p_{j,2} \leq \ldots \leq p_{j,\ell(j)} \leq \infty$, such that for each $i \in N$ the lottery x_i is an optimum for the problem

$$\max_{\pi \in \Delta(M)} \quad \sum_{j \in M} u_{ij} \pi_j$$
$$\text{subject to} \sum_{j,\ell : i \in N_j^\ell} p_{j,\ell} \pi_j \leq b_i.$$

We denote the set of equilibria with priority-specific prices for $(U, b, \succsim)$ by $\mathcal{E}(U, b, \succsim)$.

A stronger notion of constrained equilibrium was considered in He et al. (2018). A **cut-off-constrained equilibrium** is a pair $(x, \bar{p})$ of a random assignment x and prices $\bar{p} \in \mathbb{R}^M_+$ such that $(x, p) \in \mathcal{E}(U, b, \succsim)$ for

$$p_{j,\ell} := \begin{cases} 0 \text{ for } N_j^\ell \succ_j C_j(x), \\ \bar{p}_j \text{ for } N_j^\ell = C_j(x), \\ \infty \text{ for } C_j(x) \succ_j N_j^\ell, \end{cases}$$

We denote the set of cut-off-constrained equilibria for $(U, b, \succsim)$ by $\bar{\mathcal{E}}(U, b, \succsim)$. Requiring zero prices above the cut-offs guarantees that cut-off-constrained equilibrium assignments are ex-ante stable by construction: If an agent i and school j ex-ante block the random assignment x, then $i \succ_j C_j(x)$ and i can obtain j for free. Thus, i could afford a better lottery where he substitutes probability shares at a worse school for probability shares at school j. He et al. (2018) show that cut-off-constrained equilibrium allocations are priority-constrained efficient[5] if the following tie-breaking assumption is made: Whenever multiple lotteries are optimal for an agent, he chooses a cheapest one. Their argument immediately generalizes to equilibrium random assignments under priority-specific prices. For completeness, the appendix contains a proof.

Proposition 1 (Constrained First Welfare Theorem)**.** *For each pseudo-market $(U, b, \succsim)$, each equilibrium random assignment for priority-specific prices such that each*

[5] The result can be strengthened to two-sided efficiency, if prices in the cut-offs are positive. However, two-sided efficiency can fail to hold if some prices in the cut-off are zero: Consider two agents i_1, i_2, two schools j_1, j_2 each with one seat, arbitrary budgets b_1, b_2, and priorities $i_1 \succ_{j_1} i_2$ and $i_2 \succ_{j_2} i_1$. Suppose $u_{11} = u_{12} > 0$ and $u_{21} = u_{22} > 0$. Consider the random assignment x with $x_{ij} = \frac{1}{2}$ for all i and j. Since both agents are indifferent between both schools, but agent i_1 is ranked higher at j_1 and agent i_2 is ranked higher at j_2, the assignment that assigns i_1 to j_1 for sure and i_2 to j_2 for sure yields a welfare-indifferent priority improvement over x. Thus, x is not two-sided efficient. However, it can be decentralized with prices $p_{j_1} = p_{j_2} = 0$ and the second priority as cut-off.

agent chooses a cheapest lottery whenever multiple lotteries are optimal, is priority-constrained efficient with respect to U and $\succsim$.

An immediate consequence of the proposition and our definition of cut-off-constrained efficiency is that cut-off-constrained equilibrium assignments are cut-off-constrained efficient.

Corollary 2. *For each pseudo market $(U, b, \succsim)$, each cut-off-constrained equilibrium allocation such that each agent chooses a cheapest lottery whenever multiple lotteries are optimal is cut-off-constrained efficient.*

Proof. Consider the auxiliary priority profile $\succsim'$ constructed in Section 2.1.1. By definition $\bar{\mathcal{E}}(U, b, \succsim') = \bar{\mathcal{E}}(U, b, \succsim)$. Thus, by the proposition, each cut-off-constrained equilibrium allocation x in $(U, b, \succsim)$ is priority-constrained efficient in $(U, b, \succsim')$ and hence cut-off-constrained efficient in $(U, b, \succsim)$. □

The result in the corollary is strictly stronger than the first welfare theorem under priority-constrained efficiency even for ex-ante stable random assignments, as cut-off-constrained efficiency is strictly stronger than priority-constrained efficiency.

Example 1. Consider three agents, three schools, each with a single seat ($q_j = 1$ for each j), the following utilities and priorities,

U_{i_1}	U_{i_2}	U_{i_3}	$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$
2	3	5	i_1	$\underline{i_1, i_2, i_3}$	$\underline{i_1, i_2, i_3}$
3	4	4	i_2		
4	9	3	$\underline{i_3}$		

The underlined entries in the priorities are priority cut-offs for the following assignment (here and in the following rows correspond to agents and columns to schools)

$$x = \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0.2 & 0.6 \\ 0.2 & 0.8 & 0 \end{pmatrix}.$$

Assignment x is ex-ante stable, since the only potential blocking pairs are agent i_1 with school j_1 and agent i_2 with school j_1. However, for both agents j_1 is the worst school. We show in the appendix that x is two-sided efficient. However, x is not cut-off-constrained efficient, since the following assignment without more lenient schools dominates it:

$$y = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0.8 & 0 \end{pmatrix}$$

□

It is a natural question whether the constrained efficiency notion in Corollary 2 can be further strengthened. In particular, it seems to be natural to require that the cut-off-constrained equilibrium assignment is un-dominated by any random assignment that uses the same cut-offs but does not necessarily assign the same probability mass to each cut-off. The following example shows that cut-off-constrained equilibrium assignments, in general, fail to satisfy the stronger efficiency notion:

Example 2. Consider four agents, four schools, each with a single seat ($q_j = 1$ for each j), the following utilities and priorities

U_{i_1}	U_{i_2}	U_{i_3}	U_{i_4}	$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$	$\succsim_{j_4}$
4	2	2	4	i_2, i_3	i_4	$\underline{i_2, i_3}$	i_1
3	3	3	1	$\underline{i_1}$	$\underline{i_2, i_3}$	i_1, i_4	$\underline{i_4}$
2	4	4	2	i_4	i_1		i_2, i_3
1	1	1	3				

and budgets

$$b_1 = 1, \quad b_2 = 7, \quad , b_3 = 7, \quad b_4 = 1.$$

The underlined entries in the priorities are cut-offs for the following random assignment

$$x = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

which together with prices

$$p_1 = 3, \quad p_2 = 6, \quad p_3 = 12, \quad p_4 = 3,$$

is a cut-off-constrained equilibrium assignment in $(U, b, \succsim)$. The random assignment is Pareto dominated by the random assignment

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we have $C_j(y) = C_j(x)$ for each $j \in M$.

3.2.1 An Ordinal Efficiency Theorem for Ex-ante Stable Assignments

Next, we provide another version of the ordinal efficiency welfare theorem for the case that the random assignment in question is ex-ante stable. In this case, we can construct prices and budgets along the utility profile such that the random assignment can be decentralized as a cut-off-constrained equilibrium. Moreover, a stronger notion of a strong stable improvement cycle, and hence a weaker ordinal efficiency notion is sufficient to obtain the result. Formally, a strong stable improvement cycle is an **equal-priority stable improvement cycle** if in Item 3 of the definition we have $i_0 \sim_{j_1} i_1 \sim_{j_2} i_2 \ldots i_K \sim_{j_0} i_0$.

The result relies on the following observation: Ex-ante stable random assignments can be interpreted as lotteries that ration seats in the cut-offs: Assigning probabilities for seats in the cut-offs fully determines the welfare obtained by each agent in each ex-ante stable random assignment for these cut-off probabilities. For any agent the remaining probability after probabilities for the cut-offs have been allocated, is allocated to his most preferred school(s) for which he is above the cut-off, since otherwise he could ex-ante block. These school(s) can be interpreted as the agent's "safe school(s)" where he gets a seat if he does not get a more-preferred seat in one of the schools where he is in the cut-off. In a cut-off-constrained equilibrium, an agent can consume these safe schools for free and the seats in the cut-off are rationed by a competitive market with different budgets for the different agents.

We have to make one additional assumption on random assignments for the result. A random assignment x satisfies **no indifferences between assigned cut-off schools and safe schools** if for $i \in N$ $j, j' \in M$ with $i \in C_j(x)$ and $i \succ_{j'} C_{j'}(x)$, we have $j \, I_i \, j' \Rightarrow x_{ij} = 0$. The condition is, for example, redundant if preferences are strict.

Theorem 2 (Ex-ante Stable Ordinal Efficiency Welfare Theorem for Pseudo-Markets)**.** *For a random assignment x that is ex-ante stable and satisfies no indifferences between assigned cut-off schools and safe schools with respect to preferences R and priorities $\succsim$ the following statements are equivalent:*

(1) x has no equal-priority stable improvement cycle according to R and $\succsim$.

(2) x is cut-off-constrained sd-efficient.

(3) There exists a vNM-utility profile U consistent with R, a budget vector b, and prices $\bar{p} = (\bar{p}_j)_{j \in M}$ such that $(x, \bar{p})$ is a cut-off-constrained equilibrium under R and $\succsim$ in which each agent chooses a cheapest lottery whenever multiple lotteries are optimal.

Proof. To show that (3) $\Rightarrow$ (2), note that, by Corollary 2, x is cut-off-constrained efficient under U and $\succsim$ which implies that x is cut-off-constrained sd-efficient for the induced preferences R and priorities $\succsim$. To show that (2) $\Rightarrow$ (1) note that cut-off-constrained sd-efficiency implies priority-constrained sd-efficiency which implies by Theorem 1 the absence of strong stable improvement cycles, thus, in particular, the absence of equal-priority stable improvement cycles.

To show that (1) $\Rightarrow$ (2), define $C_j := C_j(x)$ for each $j \in M$ and reconsider the ordering $\trianglerighteq$ defined in the proof of Theorem 1. Observe that by the absence of equal-priority stable improvement cycles, $\trianglerighteq$ is acyclic on the set $\mathcal{O}' \subseteq \mathcal{O}$ of all cut-off classes $\mathcal{O}' := \{C_j : j \in M\}$. Thus, by Lemma 1, we find a mapping $u : \mathcal{O}' \to \mathbb{R}_+$ such that $C_j \rhd C_{j'} \Rightarrow u(C_j) > u(C_{j'})$ and $C_j \bowtie C_{j'} \Rightarrow u(C_j) \geq u(C_{j'})$. For each $j \in M$ we define $p_j := u(C_j)$. For each $i \in N$ we let $b_i := \sum_{j: i \in C_j} p_j x_{ij}$. Moreover, we choose a number $0 \leq \bar{u} < \min_{j \in M} u(C_j)$ and define U as follows: For each $i \in N$ we choose U_i consistent with R_i such that

$$x_{ij} > 0, i \succ_j C_j \Rightarrow u_{ij} = \bar{u}, \tag{3}$$

$$x_{ij} = 0, i \succ_j C_j \Rightarrow u_{ij} \leq \bar{u}, \tag{4}$$

$$x_{ij} > 0, i \in C_j \Rightarrow u_{ij} = p_j + \bar{u}, \tag{5}$$

$$x_{ij} = 0, i \in C_j \Rightarrow u_{ij} \leq p_j + \bar{u}. \tag{6}$$

By construction of $\trianglerighteq$ and ex-ante stability this is possible: By ex-ante stability, for each $i \in N$, if $i \succ_j C_j$, then $j' \, R_i \, j$ for each $j' \in M$ with $x_{ij'} > 0$ and by the assumption of no indifference between cut-off schools and safe schools, we have $j' \, P_i \, j$ if additionally

$i \in C_{j'}$. Thus, (3) and (4) can be satisfied. For $j, j' \in M$ with $i \in C_j \cap C_{j'}$ and $x_{ij} > 0$ we have that $j' \, R_i \, j$ implies $p_{j'} \geq p_j$ and that $j' \, P_i \, j$ implies $p_{j'} > p_j$. Thus, (5) and (6) can be satisfied.

We show that for each $i \in N$ lottery x_i is optimal given prices and his budget. It suffices to show that x_i is an optimum for the problem:

$$\begin{aligned} \max_{\pi} \quad & \sum_{j: i \succsim_j C_j} u_{ij} \pi_j \\ \text{subject to} \quad & \sum_{j \in M: i \in C_j} p_j \pi_j \leq b_i, \\ & \sum_{j \in M} \pi_j \leq 1, \\ & \pi_j \geq 0, \quad \forall j \in M. \end{aligned}$$

The dual problem is

$$\begin{aligned} \min_{\lambda, \mu} \quad & \lambda b_i + \mu \\ \text{subject to} \quad & p_j \lambda + \mu \geq u_{ij}, \quad \forall j : i \in C_j, \\ & \mu \geq u_{ij}, \quad \forall j : i \succ_j C_j, \\ & \lambda, \mu \geq 0. \end{aligned}$$

The choice of $\lambda = 1$ and $\mu = \bar{u}$ is feasible for the dual, as, by Conditions (5) and (6), for each j with $i \in C_j$ we have

$$p_j \lambda + \mu = p_j + \bar{u} \geq u_{ij},$$

and by Conditions (3) and (4), for each j with $i \succ_j C_j$ we have

$$\mu = \bar{u} \geq u_{ij}.$$

By Conditions (3) and (5) we have

$$\sum_{j \in M} u_{ij} x_{ij} = \sum_{j: i \in C_j} (p_j + \bar{u}) x_{ij} + \sum_{j: i \succ_j C_j} \bar{u} x_{ij} = \sum_{j: i \in C_j} p_j x_{ij} + \bar{u} = b_i + \bar{u} = \lambda b_i + \mu.$$

By linear programming duality, this shows that x_i is an optimal solution to the agent's maximization problem (and $(\lambda = 1, \mu = \bar{u})$ is optimal for the dual). Next, we show that there is no cheaper bundle that maximizes utility: If the budget constraint does not bind, then in the corresponding dual solution $\lambda = 0$, and therefore $\mu \geq \max_{j \in M: i \succsim_j C_j} u_{ij}$. But then by linear programming duality $\sum_{j \in M} u_{ij} x_{ij} = \mu \geq \max_{j \in M: i \succsim_j C_j} u_{ij}$. Thus, $u_{ij} = \mu = \max_{j \in M: i \succsim_j C_j} u_{ij}$ for each $j \in M$ with $x_{ij} > 0$. If there is a $j \in M$ with $i \succ_j C_j$ such that $x_{ij} > 0$, then by the assumption of no indifferences between assigned cut-off schools and safe schools, we have $x_{ij'} = 0$ for each $j' \in M$ with $i \in C_{j'}$, and therefore $b_i = 0$. Otherwise note that for each $j, j' \in M$ with $x_{ij} > 0$, $i \in C_j \cap C_{j'}$ and $u_{ij'} = \max_{j \in M: i \succsim_j C_j} u_{ij} = u_{ij}$ we have $j' \bowtie j$ and therefore $p_{j'} \geq p_j$. In either case, x_i is a cheapest utility maximizing lottery. □

The following example demonstrates that the assumption "of no indifferences between assigned cut-off schools and safe schools" is necessary for the theorem.

Example 3. Consider three agents, three schools, each with a single seat ($q_j = 1$ for each j), and the following preferences and priorities.

R_{i_1}	R_{i_2}	R_{i_3}	$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$
j_1, j_2	j_1	j_2	i_1	i_2	$\underline{i_3}$
j_3	j_2	j_3	$\underline{i_2}$	$\underline{i_1, i_3}$	i_1, i_2
	j_3	j_1	i_3		

The underlined entries in the priorities are the cut-offs for the following assignment

$$x = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There exists no blocking pair. Moreover, one can show that the random assignment is cut-off-constrained sd-efficient.

Now suppose there exist a utility profile U consistent with R and decentralizing cut-off prices p and budgets b. Observe that i_1 is indifferent between j_1 and j_2 and is strictly above the cut-off at j_1. Thus, it must be the case that $p_2 = 0$, since otherwise i_1 could obtain probability shares at school j_1 instead to obtain a cheaper bundle. However, it also needs to be the case that $p_2 > p_3$ since otherwise agent i_3 could substitute shares at j_3 by shares at j_2 which he prefers. But then $0 = p_2 > p_3$, a contradiction. □

3.2.2 Second Welfare Theorems

In our ordinal efficiency welfare theorem we started with ordinal preferences and constructed vNM-utilities, budgets, cut-offs and prices to decentralize an ex-ante stable random assignment that is cut-off-constrained sd-efficient as a cut-off-constrained equilibrium. It is a natural question, whether the result can be strengthened in the following way: Start with a profile of vNM-utilities and show that each ex-ante stable random assignment that is cut-off-constrained efficient can be decentralized as a cut-off-constrained equilibrium. We demonstrate by means of a counterexample that this is not possible and a cardinal second welfare theorem does not hold. In the example, each agent obtains different utility from the different schools. Thus, the induced ordinal preferences are strict (in particular there is no indifference between assigned cut-off schools and safe schools) and the example does not rely on indifferences interfering with the construction of decentralizing prices.

Example 4. Consider three agents, three schools, each with a single seat ($q_j = 1$ for each j), and the following utilities and priorities

U_{i_1}	U_{i_2}	U_{i_3}	$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$
1	3	4	i_1	i_2	i_3
2	1	2	$\underline{i_2, i_3}$	$\underline{i_1, i_3}$	$\underline{i_1, i_2}$
5	6	1			

The underlined entries in the priorities are the cut-offs for the following assignment

$$x = \begin{pmatrix} 0.1 & 0.6 & 0.3 \\ 0.8 & 0.2 & 0 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.$$

There exist no blocking pairs in this example, since for all agents their worst school is the only school where they are ranked strictly above the cut-off. Therefore, x is ex-ante stable. We show in the appendix that x is two-sided efficient (which, as there are only two priority classes at each school and utilities are strict, is equivalent to cut-off-constrained efficiency).

Next, we show that x cannot be decentralized as a cut-off-constrained equilibrium. Suppose there are budgets $b \in \mathbb{R}^N_+$ and prices $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) \in \mathbb{R}^M_+$ such that $(x, \bar{p})$ is a cut-off-constrained equilibrium.

Agent i_1's optimization problem is

$$\begin{aligned} \max_{\pi \in \Delta(M)} \quad & \pi_1 + 2\pi_2 + 5\pi_3 \\ \text{subject to } & \bar{p}_2\pi_2 + \bar{p}_3\pi_3 \leq b_1. \end{aligned}$$

Substituting $\pi_1 = 1 - \pi_2 - \pi_3$ and ignoring the constant in the objective function, we obtain the following equivalent optimization problem

$$\begin{aligned} \max_{\pi_1, \pi_2} \quad & \pi_2 + 4\pi_3 \\ \text{subject to } & \pi_2 + \pi_3 \leq 1 \\ & \bar{p}_2\pi_2 + \bar{p}_3\pi_3 \leq b_1 \\ & \pi_2, \pi_3 \geq 0. \end{aligned}$$

If $(x, \bar{p})$ is a cut-off-constrained equilibrium, then $(\pi_1, \pi_2, \pi_3) = (0.1, 0.6, 0.3)$ is a solution to the original problem and hence $(\pi_2, \pi_3) = (0.6, 0.3)$ is a solution to the equivalent problem. Since in this optimum point neither the non-negativity constraints $\pi_2, \pi_3 \geq 0$ nor the constraint $\pi_2 + \pi_3 \leq 1$ bind, the optimal point is not a corner of the LP polytope and the line $\bar{p}_2\pi_2 + \bar{p}_3\pi_3$ is parallel to the objective function line $\pi_2 + 4\pi_3$. That is $\frac{\bar{p}_3}{\bar{p}_2} = 4$.

Agent i_3's optimization problem is

$$\begin{aligned} \max_{\pi \in \Delta(M)} \quad & 4\pi_1 + 2\pi_2 + \pi_3 \\ \text{subject to } & \bar{p}_1\pi_1 + \bar{p}_2\pi_2 \leq b_3 \end{aligned}$$

Similarly to the previous case, we can substitute $\pi_3 = 1 - \pi_1 - \pi_2$ and ignore the constant in the objective function, to obtain an equivalent problem

$$\begin{aligned} \max_{\pi_1, \pi_2} \quad & 3\pi_1 + \pi_2 \\ \text{subject to } & \pi_1 + \pi_2 \leq 1 \\ & \bar{p}_1\pi_1 + \bar{p}_2\pi_2 \leq b_3 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

If $(x, \bar{p})$ is a cut-off-constrained equilibrium, then $(\pi_1, \pi_2, \pi_3) = (0.1, 0.2, 0.8)$ is a solution to the original problem and hence $(\pi_1, \pi_2) = (0.1, 0.2)$ is a solution to the equivalent problem. Since in this optimum point neither the non-negativity constraints $\pi_1, \pi_2 \geq 0$ nor the constraint $\pi_1 + \pi_2 \leq 1$ bind, the optimal point is not a corner of the LP polytope and the line $\bar{p}_1\pi_1 + \bar{p}_2\pi_2$ is parallel to the objective function line $3\pi_1 + \pi_2$. That is $\frac{\bar{p}_1}{\bar{p}_2} = 3$.

Without loss of generality, we can assume that $\bar{p}_2 = 1$. Then $\bar{p}_1 = 3$ and $\bar{p}_3 = 4$. Then, agent i_2's optimization problem becomes:

$$\begin{aligned} \max_{\pi \in \Delta(M)} \quad & 3\pi_1 + \pi_2 + 6\pi_3 \\ \text{subject to } & 3\pi_1 + 4\pi_3 \leq b_2 \end{aligned}$$

Analogously to the previous cases, by substituting $\pi_2 = 1 - \pi_1 - \pi_3$ and ignoring the constant in the objective function, we obtain an equivalent problem

$$\begin{aligned} \max_{\pi_1, \pi_3} \quad & 2\pi_1 + 5\pi_3 \\ \text{subject to } & \pi_1 + \pi_3 \leq 1 \\ & 3\pi_1 + 4\pi_3 \leq b_2 \\ & \pi_1, \pi_3 \geq 0 \end{aligned}$$

If $(x, \bar{p})$ is a cut-off-constrained equilibrium, then $(\pi_1, \pi_2, \pi_3) = (0.8, 0.2, 0)$ is a solution to the original problem and hence $(\pi_1, \pi_3) = (0.8, 0)$ is a solution to the equivalent problem. Since the first constraint is not binding in this solution, the second constraint binds. That is, $b_2 = 3 \times 0.8$. But in this case, the lottery $(\pi_1, \pi_3) = (0, 0.6)$ is feasible, gives a value $5 \times 0.6 = 3$, instead of $2 \times 0.8 = 1.6$. This contradicts the assumption that $(x, \bar{p})$ is a cut-off-constrained equilibrium. □

The second welfare theorem can, however, be restored, if we allow for non-zero prices above the cut-off. More generally, we can obtain a second welfare theorem for equilibria with priority-specific prices by using the second welfare theorem of Miralles and Pycia (2017) and treating priority classes as separate objects that have to be priced. The second welfare theorem for cut-off-constrained equilibria follows immediately as a corollary.

Theorem 3. *For each priority-constrained efficient random allocation x under U and R, there exist priority-specific prices p and budgets b such that $(x, p) \in \mathcal{E}(U, b, \succsim)$.*

Proof. We rely on Theorem 3 of Miralles and Pycia (2017). Importantly, a careful inspection of their proof shows that their Theorem 3 does not hinge on the assumption that the supply of each object is an integer number. Miralles and Pycia prove a more general result for multi-unit demand, however, for our set-up, the following version of their theorem is sufficient.

Lemma 2 ((Adapted) Theorem 3 in Miralles and Pycia, 2017)**.** *Let N be a finite set of agents and let $\mathcal{O}$ be a finite set of objects where object $o \in \mathcal{O}$ is supplied in $\tilde{q}_o \in \mathbb{R}_+$ units such that $\sum_{o \in \mathcal{O}} \tilde{q}_o = |N|$. Suppose each agent $i \in N$ has a set of feasible objects,*

$B_i \subseteq \mathcal{O}$ and a utility function $\tilde{u}_i : B_i \to \mathbb{R}_+$. Let $\tilde{x} = ((\tilde{x}_{io})_{o\in B_i})_{i\in N} \in \times_{i\in N}\Delta(B_i)$ such that for each $o \in \mathcal{O}$ we have $\sum_{i\in N} \tilde{x}_{io} \leq \tilde{q}_o$ and there is no $\tilde{y} \in \times_{i\in N}\Delta(B_i)$ such that for each $o \in \mathcal{O}$ we have $\sum_{i\in N} \tilde{y}_{io} \leq \tilde{q}_o$ and $\tilde{y}$ Pareto dominates $\tilde{x}$ in the sense that for each $i \in N$ we have

$$\sum_{o\in B_i} \tilde{u}_{io}\tilde{y}_{io} \geq \sum_{o\in B_i} \tilde{u}_{io}\tilde{x}_{io}$$

where the inequality is strict for at least one agent $i \in N$. Then there exist prices $p = (p_o)_{o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{O}}_+$ and budgets $b = (b_i)_{i\in N} \in \mathbb{R}^N_+$ such that for each $i \in N$ we have $\sum_{o\in B_i} p_o\tilde{x}_{io} \leq b_i$ and for each $\pi \in \Delta(B_i)$ we have

$$\sum_{o\in B_i} \tilde{u}_{io}\pi_o > \sum_{o\in B_i} \tilde{u}_{io}\tilde{x}_{io} \Rightarrow \sum_{o\in B_i} p_o\pi_o > b_i.$$

We define an auxiliary market as in Lemma 2. We treat priority classes as objects which are supplied with the probability mass allotted to the priority class in the random assignment x. Agents are allowed to buy from a priority class, as long as they are ranked in or above that priority class at the school. Formally, we let $\mathcal{O} := \{N^\ell_j : j \in M, \ell = 1, \dots, \ell(j)\}$, let $\tilde{q}_{j,\ell} := \sum_{i\in N^\ell_j} x_{ij}$, for each $i \in N$ let $B_i = \{N^\ell_j \in \mathcal{O} : i \succsim_j N^\ell_j\}$ and let $\tilde{u}_{i,j,\ell} := u_{ij}$ for $i \succsim_j N^\ell_j$. For each random assignment $\tilde{y} \in \times_{i\in N}\Delta(B_i)$, we can define a corresponding random assignment $y \in \times_{i\in N}\Delta(M)$ by

$$y_{ij} := \sum_{\ell=1}^{\ell(j)} \tilde{y}_{i,j,\ell}. \tag{7}$$

Note that by construction, for each $i \in N$ we have

$$\sum_{j\in M} u_{ij}y_{ij} = \sum_{j\in M} \sum_{\ell:i\succsim_j N^\ell_j} \tilde{u}_{i,j,\ell}\tilde{y}_{i,j,\ell}.$$

Moreover, $\tilde{y}_i \in \Delta(B_i)$ and therefore $\sum_{j\in M} \sum_{\ell:i\succsim_j N^\ell_j} \tilde{y}_{i,j,\ell} = 1$. As $\sum_{N^\ell_j\in\mathcal{O}} \tilde{q}_{j,\ell} = |N|$, this implies that $\sum_{i\in N} \tilde{y}_{i,j,\ell} = \tilde{q}_{j,\ell}$ for each $N^\ell_j \in \mathcal{O}$. Thus, for each $j \in M$ and $1 \leq \ell' \leq \ell(j)$ we have

$$\sum_{i\succsim_j N^{\ell'}_j} y_{ij} = \sum_{\ell=1}^{\ell'} \sum_{i\in N} \tilde{y}_{i,j,\ell} = \sum_{\ell=1}^{\ell'} \tilde{q}_{j,\ell} = \sum_{\ell=1}^{\ell'} \sum_{i\in N^\ell_j} x_{ij} = \sum_{i\succsim_j N^{\ell'}_j} x_{ij},$$

and therefore $y_j \succsim^{sd}_j x_j$. Similarly, we can derive a random assignment $\tilde{x} \in \times_{i\in N}\Delta(B_i)$ from $x \in \mathbb{R}^{N\times M}$ by

$$\tilde{x}_{i,j,\ell} = \begin{cases} x_{ij}, & \text{for } i \in N^\ell_j, \\ 0, & \text{else.} \end{cases}$$

Note that $\tilde{x}$ is Pareto efficient under $\tilde{U}$ among (in expectation) feasible random allocations for $\tilde{q}$, since otherwise if $\tilde{y} \in \times_{i\in N}\Delta(B_i)$ is feasible (in expectation) under $\tilde{q}$ and Pareto dominates $\tilde{x}$ according to $\tilde{U}$, the corresponding $y \in \times_{i\in N}\Delta(M)$, defined by

Equation (7), Pareto dominates x according to U. As $y_j \succsim_j^{sd} x_j$ for each $j \in M$, this contradicts the priority-constrained efficiency of x. Since $\tilde{x}$ is Pareto efficient under $\tilde{U}$ among (in expectation) feasible random allocations for $\tilde{q}$, there exists, by Lemma 2, prices $\tilde{p} \in \mathbb{R}_+^{\mathcal{O}}$ and budgets $b \in \mathbb{R}_+^N$ such that $(\tilde{x}, \tilde{p})$ is an equilibrium for $(\tilde{U}, b)$. For each $j \in M$ and $1 \leq \ell \leq \ell(j)$ define

$$p_{j,\ell} := \min_{\ell \leq \ell' \leq \ell(j), \tilde{q}_{j,\ell'} > 0} \tilde{p}_{j,\ell'},$$

with the usual convention that the minimum over an empty set is ∞. Note that by construction, for each $j \in M$ and $1 \leq \ell \leq \ell' \leq \ell(j)$ we have $p_{j,\ell} \leq p_{j,\ell'}$. We show that also $\tilde{p}$ decentralize $\tilde{x}$ in $(\tilde{U}, b)$ and x in $(U, b, \succsim)$. Note that for each $i \in N$ and $\pi \in \Delta(B_i)$ with $\sum_{j \in M} \sum_{\ell : i \succsim_j N_j^\ell} p_{j,\ell} \pi_{j,\ell} \leq b_i$, the lottery $\tilde{\pi} \in \Delta(B_i)$ defined by

$$\tilde{\pi}_{j,\ell'} := \begin{cases} \sum_{\ell=1}^{\ell(j)} \pi_{j,\ell} & \text{if } \ell' = \min\{1 \leq \ell \leq \ell(j) : i \succsim_j N_j^\ell, \tilde{q}_{j,\ell} > 0\}, \\ 0 & \text{else.} \end{cases}$$

is affordable under $\tilde{p}$, as

$$\sum_{j \in M} \sum_{\ell : i \succsim_j N_j^\ell} \tilde{p}_{j,\ell} \tilde{\pi}_{j,\ell} \leq \sum_{j \in M} \sum_{\ell : i \succsim_j N_j^\ell} p_{j,\ell} \pi_{j,\ell} \leq b_i,$$

and yields the same expected utility for i as π, since

$$\sum_{j \in M} \sum_{\ell : i \succsim_j N_j^\ell} \tilde{u}_{i,j,\ell} \tilde{\pi}_{j,\ell} = \sum_{j \in M} u_{i,j} \sum_{\ell : i \succsim_j N_j^\ell} \tilde{\pi}_{j,\ell} = \sum_{j \in M} u_{i,j} \sum_{\ell : i \succsim_j N_j^\ell} \pi_{j,\ell} = \sum_{j \in M} \sum_{\ell : i \succsim_j N_j^\ell} \tilde{u}_{j,\ell} \pi_{j,\ell}.$$

Moreover, $\tilde{x}_i$ costs the same under p and $\tilde{p}$ and thus is feasible for i under p and b_i. Thus, $\tilde{x}_i$ is still an optimal bundle for $i \in N$ under prices p and budget b_i.

□

Corollary 3. *For each cut-off-constrained efficient random allocation x under U and R, there exist priority-specific prices p and budgets b such that $(x, p) \in \mathcal{E}(U, b, \succsim)$, where for each $j \in M$ there are two prices $\underline{p}_j \leq \bar{p}_j$ such that*

$$p_{j,\ell} = \begin{cases} \underline{p}_j \text{ for } N_j^\ell \succ_j C_j(x), \\ \bar{p}_j \text{ for } N_j^\ell = C_j(x), \\ \infty \text{ for } C_j(x) \succ_j N_j^\ell. \end{cases}$$

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A Proof Sketch for Remark 1

A stable improvement cycle as defined in Remark 1, can be used (in a similar way as in the proof of Theorem 1) to generate a random assignment y such that $y_i \, R_i^{sd} x_i$ for each $i \in M$, $y_j \succsim_j^{sd} x_j$ for each $j \in M$, and $y_i \, P_i^{sd} x_i$ for some $i \in N$ or $y_j \succsim_j^{sd} x_j$ for some j. Thus, if there is such a stable improvement cycle, the random assignment is not two-sided efficient. Moreover, each two-sided sd-efficient random assignment is two-sided efficient for any vNM utilities consistent with the ordinal preferences.

To show that the absence of stable improvement cycles as defined in Remark 1 implies the existence of a utility profile U consistent with R such that x is two-sided efficient with respect to U and $\succsim$, we have to adapt the proof of Theorem 1 as follows: We modify the ordering $\trianglerighteq$: For $j \neq j'$, $1 \leq \ell \leq \ell(j)$ and $1 \leq \ell' \leq \ell(j')$ we leave $\trianglerighteq$ unchanged. For $j \in M$ and $1 \leq \ell < \ell' \leq \ell(j)$ we now let $N_j^{\ell'} \triangleright N_j^{\ell}$. With the modified notion of a stable improvement cycle in Remark 1, we can show that the modified $\trianglerighteq$ is acyclic. We derive U as before from $\trianglerighteq$. The same argument as before shows that x maximizes social welfare subject to priority constraints. To show that there is no welfare-indifferent priority improvement, observe that for y with

$$\sum_{j \in M} \sum_{i \in N} u_{ij} y_{ij} = \sum_{j \in M} \sum_{i \in N} u_{ij} x_{ij},$$

we can rearrange the terms as before to obtain

$$0 = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)-1} (u(N_j^{\ell}) - u(N_j^{\ell+1}))(\sum_{i \succsim_j N_j^{\ell}} (y_{ij} - x_{ij})).$$

Now by the modified construction of $\trianglerighteq$, we have $u(N_j^{\ell}) < u(N_j^{\ell'})$ for $\ell < \ell'$ and therefore $u(N_j^{\ell}) - u(N_j^{\ell+1}) < 0$ for $\ell = 1, \ldots, \ell(j) - 1$ for each $j \in M$. Thus, for each $j \in M$ and $1 \leq \ell \leq \ell(j)$ we have $\sum_{i \succsim_j N_j^{\ell}} y_{ij} = \sum_{i \succsim_j N_j^{\ell}} x_{ij}$.

B Two-Sided Efficiency of x in Example 1

Let y be a random assignment such that

$$\begin{aligned}
&u_{11} y_{11} + u_{12} y_{12} + u_{13} y_{13} \geq 2 \times 0.6 + 3 \times 0 + 5 \times 0.4 = 3.2, \\
&u_{21} y_{21} + u_{22} y_{22} + u_{23} y_{23} \geq 3 \times 0.2 + 4 \times 0.2 + 9 \times 0.6 = 6.8, \\
&u_{31} y_{31} + u_{32} y_{32} + u_{33} y_{33} \geq 5 \times 0.2 + 4 \times 0.8 + 3 \times 0 = 4.2, \\
&y_{11} \geq 0.6, \quad y_{11} + y_{21} \geq 0.8.
\end{aligned}$$

We show that $y = x$. First we show that agent i_1 obtains the same lottery, i.e. $y_{i_1} = x_{i_1} = (0.6, 0, 0.4)$. As $2y_{11} + 3y_{12} + 5y_{13} \geq 3.2$ and $y_{11} + y_{12} + y_{13} = 1$ we have $y_{12} + 3y_{13} \geq 3.2 - 2 = 1.2$. Since $y_{11} \geq 0.6$ we have $y_{12} + y_{13} \leq 0.4$ and therefore $1.2 \leq y_{12} + 3y_{13} \leq 0.4 + 2y_{13}$. As $y_{12} + y_{13} \leq 0.4$, this implies $y_{13} = 0.4$ and $y_{12} = 0$. Finally, since $y_{11} + y_{12} + y_{13} = 1$, we also have $y_{11} = 0.6$.

Next, we show that agent i_3 obtains the same lottery, i.e. $y_{i_3} = x_{i_3} = (0.2, 0.8, 0)$. As $5y_{31} + 4y_{32} + 3y_{33} \geq 4.2$ and $y_{31} + y_{32} + y_{33} = 1$ we have $2y_{31} + y_{32} \geq 1.2$. As $y_{11} + y_{21} \geq 0.8$ we have $y_{31} \leq 0.2$ and therefore, $0.4 + y_{32} \geq 1.2$ with the last inequality strict only if $y_{31} < 0.2$. Thus, $y_{32} = 0.8$ and $y_{31} = 0.2$. Finally, since $y_{31} + y_{32} + y_{33} = 1$, we also have $y_{33} = 0$.

As agents i_1 and i_3 obtain the same lottery also agent i_2 obtains the same lottery, i.e. $y_{i_2} = x_{i_2} = (0.2, 0.2, 0.6)$.

C Two-Sided Efficiency of x in Example 4

Let y be a random assignment such that

$$
\begin{aligned}
&u_{11}y_{11} + u_{12}y_{12} + u_{13}y_{13} \geq 1 \times 0.1 + 2 \times 0.6 + 5 \times 0.3 = 2.8, \\
&u_{21}y_{21} + u_{22}y_{22} + u_{23}y_{23} \geq 3 \times 0.8 + 1 \times 0.2 + 6 \times 0 = 2.6, \\
&u_{31}y_{31} + u_{32}y_{32} + u_{33}y_{33} \geq 4 \times 0.1 + 2 \times 0.2 + 1 \times 0.7 = 1.5, \\
&y_{11} \geq x_{11} = 0.1, \quad y_{22} \geq x_{22} = 0.2, \quad y_{33} \geq x_{33} = 0.7.
\end{aligned}
$$

We show that $x = y$. First we show that agent i_1 obtains the same lottery, i.e. $y_{i_1} = x_{i_1} = (0.1, 0.6, 0.3)$. By $y_{11} + y_{12} + y_{13} = 1$ and $y_{11} + 2y_{12} + 5y_{13} \geq 2.8$ we have $y_{12} + 4y_{13} \geq 1.8$. As $y_{11} \geq 0.1$, we have $y_{12} + y_{13} \leq 0.9$ and therefore $1.8 \leq y_{12} + 4y_{13} \leq 0.9 + 3y_{13}$ with strict last inequality only if $y_{11} > 0.1$. Therefore $y_{13} \geq 0.3$ and, as $y_{33} \geq 0.7$, we have $y_{13} = 0.3$. Thus, the last inequality from before holds with equality and therefore $y_{11} = 0.1$. Since $y_{11} + y_{12} + y_{13} = 1$ this implies moreover $y_{12} = 0.6$.

Next we show that agent i_2 obtains the same lottery, i.e. $y_{i_2} = x_{i_2} = (0.8, 0.2, 0)$. As $y_{13} = 0.3$ and $y_{33} \geq 0.7$, we have $y_{33} = 0.7$ and $y_{23} = 0$. Thus, $2.6 \leq 3y_{21} + y_{22} + 6y_{23} = 3y_{21} + y_{22}$. As $y_{22} \geq 0.2$ we have $y_{21} \leq 0.8$ and, as $y_{21} + y_{22} \leq 1$ the previous inequality can only hold for $y_{22} = 0.2$ and $y_{21} = 0.8$.

As agents i_1 and i_2 obtain the same lottery also agent i_3 obtains the same lottery, i.e. $y_{i_3} = x_{i_3} = (0.1, 0.2, 0.7)$.

D Proof of Proposition 1

Proof. Let $(x, p) \in \mathcal{E}(U, b, \succsim)$ such that each agent chooses a cheapest lottery if multiple lotteries are optimal. Suppose random assignment y Pareto dominates x. Then for each $i \in N$,

$$\sum_{j \in M} u_{ij} y_{ij} \geq \sum_{j \in M} u_{ij} x_{ij},$$

where the inequality is strict for at least one agent. For an agent i, for which the inequality is strict, we have by revealed preferences

$$\sum_{j \in M} p_{ij} y_{ij} > b_i \geq \sum_{j \in M} p_{ij} x_{ij}.$$

For an agent i, for which equality holds, we have

$$\sum_{j \in M} p_{ij} y_{ij} \geq \sum_{j \in M} p_{ij} x_{ij},$$

since otherwise the tie-breaking rule that a cheapest lottery is chosen in case of multiple optimal lotteries would be violated. Summing the inequalities over all agents, we obtain

$$\sum_{i \in N} \sum_{j \in M} p_{ij} y_{ij} > \sum_{i \in N} \sum_{j \in M} p_{ij} x_{ij}.$$

We can rearrange the right-hand side of the inequality,

$$\sum_{i \in N} \sum_{j \in M} p_{ij} x_{ij} = \sum_{j \in M} \sum_{i \in N} p_{ij} x_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} \sum_{i \in N_j^\ell} x_{ij}.$$

Similarly, we can rearrange the left-hand side of the inequality,

$$\sum_{i \in N} \sum_{j \in M} p_{ij} y_{ij} = \sum_{j \in M} \sum_{i \in N} p_{ij} y_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} \sum_{i \in N_j^\ell} y_{ij}.$$

Thus,

$$\sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} \sum_{i \in N_j^\ell} y_{ij} > \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} \sum_{i \in N_j^\ell} x_{ij}$$

Rearranging the terms:

$$0 < \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} (\sum_{i \in N_j^\ell} (y_{ij} - x_{ij})) = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)-1} (p_{j,\ell+1} - p_{j,\ell})(\sum_{i \succsim_j N_j^{\ell+1}} (y_{ij} - x_{ij}))$$

As $p_{j,\ell} - p_{j,\ell+1} \leq 0$ for each $1 \leq \ell \leq \ell(j) - 1$ and $j \in M$, we thus have $\sum_{i \succsim_j N_j^{\ell+1}} (y_{ij} - x_{ij}) < 0$ for at least one $1 \leq \ell \leq \ell(j) - 1$ and $j \in M$. Thus, there is a $j \in M$ with $y_j \not\succsim_j^{sd} x_j$. $\square$