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Generalized quark-antiquark potential in AdS/CFT

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In this talk we present a family of Wilson loop operators which continuously interpolates between the 1/2 BPS line and the antiparallel lines, and can be thought of as calculating a generalization of the quark–antiquark potential for the gauge theory on $S^3 \times \mathbf{R}$. We evaluate the first two orders of these loops perturbatively both in the gauge and string theory. We obtain analytical expressions in a systematic expansion around the 1/2 BPS configuration, and comment on possible all-loop patterns for these Wilson loops.

1 Overview

One of the most fundamental observables in a quantum field theory is the potential between charged particles, which in a gauge theory is captured by a long rectangular Wilson loop, or a pair of antiparallel lines representing the trajectories of infinitely heavy quarks. Such quark-antiquark potential can be also considered in the maximally supersymmetric $\mathcal{N}=4$ SYM theory, where "quarks" are modeled by infinitely massive W-bosons arising from a Higgs mechanism [1].

The expectation value of this observable was calculated very early after the introduction of the AdS/CFT correspondence by the effective action of a string ending along the curve on the four-dimensional AdS boundary, and is in fact a seminal example of the duality itself. In this context of a conformal field theory the potential is fixed to be Coulomb-like and the whole dynamical content is in the corresponding coefficient, for which the weak and strong coupling ('t Hooft coupling λ) previously obtained results read

$$V_{q\bar{q}}(\lambda, L) = -\frac{1}{L}\, c(\lambda)\,, \qquad c(\lambda) = \begin{cases} \frac{\lambda}{4\,\pi}\Big[1-\frac{\lambda}{2\,\pi^2}\Big(\ln\frac{2\pi}{\lambda}-\gamma_E+1\Big)+\mathcal{O}(\lambda^2)\Big], & \lambda \ll 1 \\ \frac{\sqrt{\lambda}\,\pi}{4\,\mathbb{K}(\frac{1}{2})^2}\,\Big[1+\frac{a_1}{\sqrt{\lambda}}+\mathcal{O}\Big(\frac{1}{(\sqrt{\lambda})^2}\Big)\Big], & \lambda \gg 1\,. \end{cases} \tag{1}$$

Above, L is the distance between the lines, $\mathbb{K}$ is the complete elliptic integral of the first kind and the weak-coupling expansion is the field-theoretical calculation of [2, 3, 4]. On the string theory (strong coupling) side, the question of evaluating the first quantum string correction a_1 to the classical result of [1] [1] is a hard mathematical problem. The absence of parameters in the problem (the only one, L, being fixed by conformal invariance) precludes considering special scaling limits in which nice results in σ-model perturbation theory have been obtained for some relevant string solutions (see, for example, [6, 7] and reference therein). The coefficient a_1 was presented formally in [8, 9], evaluated numerically in [10] to be $a_1 = 1.33459$ and simplified further in [11] to an analytic one–dimensional integral representation.

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[1] This is actually the $AdS_5 \times S^5$ counterpart of the so-called "Lüscher term", which in flat space is a coulombic term proportional to the number of transverse dimensions [5].

It is hard to guess how to connect the two regimes of (1). It is tempting to think about the chance of exploiting the integrability of the underlying AdS/CFT system and describe correctly the interpolation of $c(\lambda)$ between the two regimes of (1), as in the by now most famous example of smooth interpolation for a non-protected quantity - the cusp anomaly of $\mathcal{N}=4$ SYM [12].

Our proposal [13] for addressing the problem relies on the introduction of *extra parameters* in the initial setup. They do not make the perturbative or supergravity calculation any harder and allow, in fact, to interpolate between protected, much simpler, operators and the desired observable. The first deformation parameter (indicated below with θ) allows for the two lines to couple to two different scalar fields, and was already introduced in [1]. In the general expression of the Maldacena-Wilson loop

$$W=\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\left[\oint(iA_\mu\dot{x}^\mu+\Phi_I\Theta^I|\dot{x}|)ds\right], \tag{2}$$

we allow two different values of $\vec{\Theta}$ of relative angle θ on the two long edges of the rectangle. For $\theta=0$ the two lines couple to the same scalar field, say Φ_1. When $\theta=\pi/2$ the two lines couple to $\Phi_1\pm\Phi_2$, which are orthogonal to each-other. Then for $\theta=\pi$ they couple to the field Φ_2, but with opposite signs, which means that the lines are effectively parallel, rather than antiparallel. In that case the two lines share eight supercharges and the correlator is trivial. The other deformation parameter (indicated below with ϕ) is geometric, and a way to illustrate it is to replace the theory on $\mathbb{R}^4$ with the theory on $\mathbb{S}^3\times\mathbb{R}$ (related by the exponential map). We consider a pair of antiparallel lines separated by an angle $\pi-\phi$ on $\mathbb{S}^3$. For $\phi=0$ the two lines are antipodal and mutually BPS, while for $\phi\to\pi$ the lines get very close together. "Zooming in" to the vicinity of the lines by a conformal transformation we get a situation very similar to the original antiparallel lines in flat space. An equivalent picture is that of a cusp in the plane in $\mathbb{R}^4$. For $\phi=0$ the cusp disappears and the system is that of a single infinite straight line.

In the $\mathbb{S}^3\times\mathbb{R}$ picture the expectation value of the Wilson loop calculates the effective potential $V(\phi,\theta,\lambda)$ between a generalized quark-antiquark pair. In the case of a cusp in $\mathbb{R}^4$ the loop suffers from logarithmic divergences [14]. The expectation values of the loop in the two pictures are respectively

$$\langle W\rangle\approx\exp\Big[-T\,V(\phi,\theta,\lambda)\Big]\,,\qquad\qquad \langle W_{\text{cusp}}\rangle\approx\exp\Big[-\log(R/\epsilon)\,V(\phi,\theta,\lambda)\Big]\,. \tag{3}$$

The logarithmic divergence is exactly the same as the linear time divergence, and the cutoffs of the two calculations are related by $\log(R/\epsilon)\sim T$.

The effective potential $V(\phi,\theta,\lambda)$ depends on the 't Hooft coupling $\lambda=g^2N$ (we do not consider non-planar corrections) and it can be expanded at weak coupling and at strong coupling in the two relevant asymptotic expansions

$$V(\phi,\theta,\lambda)=\begin{cases}\sum_{n=1}^\infty\left(\frac{\lambda}{16\pi^2}\right)^n V^{(n)}(\phi,\theta), & \lambda\ll 1\\ \frac{\sqrt{\lambda}}{4\pi}\sum_{l=0}^\infty\left(\frac{4\pi}{\sqrt{\lambda}}\right)^l V^{(l)}_{AdS}(\phi,\theta), & \lambda\gg 1\,.\end{cases} \tag{4}$$

Below, we will present the evaluation of the first two terms of both regimes, adopting the picture of a cusp in $\mathbb{R}^4$ at weak coupling and the $\mathbb{S}^3\times\mathbb{R}$ picture at strong coupling. In particular, at strong coupling the coefficients in the perturbative expansions are complicated functions of the angles ϕ and θ which are given only implicitly (at the classical level) or in integral form (one–loop). We consider therefore the expansion of these functions around $\phi=\theta=0$. This is an expansion around the $1/2$ BPS line (related to the circle via conformal transformation), one of the most simple observables in the theory. As a consequence, we obtain here *analytic results* at both weak and strong coupling.

Focussing on the first coefficients of this expansion, we argue below how they should receive contributions only from a subset of graphs in perturbation theory – the most connected graphs. At variance with the case of the circular Wilson loop, where in the Feynman gauge only ladder diagrams contribute and all interacting graphs combine to vanish [3, 15, 16], we find here an observable which gets contributions only from the most interacting graphs. To our surprise, from the explicit calculation of the 2–loop graphs, we find that the result of these internally–connected graphs is simpler than the internally–disconnected one and does not involve polylogarithms. Since summing up ladder graphs is rather easy [2], it would be very interesting to explore the 3–loop graphs and see whether a similar pattern persists and perhaps learn how to calculate the most connected graphs to all orders.

In the rest of the talk we present a summary of our results at weak and strong coupling (Section 2), the explicit analytic expressions of the expansion around the BPS configuration and a short discussion on how the relevant coefficients can be evaluated via insertions of local operators into the loop (Section 3). The results obtained are suggestive of the framework in which an efficient description of the weak-to-strong coupling interpolation for these Wilson loops might take place. Certainly, they represent a set of analytic data to be of reference if an all-loop calculation will ever emerge.

2 Results at weak and at strong coupling

At *weak coupling*, we work with the cusp in $\mathbb{R}^4$ [18] and allow for an extra angle θ in $\mathcal{N}=4$ SYM. For the potential $V(\phi,\theta)$ up to two-loops we found [3]

$$
\begin{aligned}
V^{(1)}(\phi,\theta) &= -2\,\frac{\cos\theta-\cos\phi}{\sin\phi}\,\phi\,;\\
V^{(2)}(\phi,\theta) &= V^{(2)}_{\text{lad}}(\phi,\theta)+V^{(2)}_{\text{int}}(\phi,\theta)\,,\\
V^{(2)}_{\text{lad}}(\phi,\theta) &= -4\,\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[\mathrm{Li}_3\left(e^{2i\phi}\right)-\zeta(3)-i\phi\left(\mathrm{Li}_2\left(e^{2i\phi}\right)+\frac{\pi^2}{6}\right)+\frac{i}{3}\phi^3\right],\\
V^{(2)}_{\text{int}}(\phi,\theta) &= \frac{4}{3}\,\frac{\cos\theta-\cos\phi}{\sin\phi}\,(\pi-\phi)(\pi+\phi)\phi\,,
\end{aligned}
\tag{5}
$$

where $V^{(2)}$ is written as a sum of the contribution of ladder [4] and interacting graphs.

The analytic expressions (5) undergo various checks. In the BPS case [21], where $\phi=\pm\theta$, then $V^{(1)}=V^{(2)}=0$ as expected. At large imaginary angle, the prefactor of the linear term matches indeed a quarter of the perturbative expansion of the cusp anomalous dimension [22]. Formulas (5) also reproduce (and generalize) the antiparallel lines result of [2]. Taking the $\phi\to\pi$ limit and specializing to the case $\theta=0$, the resulting expression matches the one in [2] with the replacement $L\to\pi-\phi$. It is interesting to notice that the complicated interacting graphs result in a contribution much simpler than the one due to the 2–loop ladder graph and without polylogarithmic functions [5]. Indeed it is proportional to the 1–loop result with a ratio which is just is a polynomial in ϕ.

At *strong coupling*, Wilson loops are described by macroscopic strings [1, 23]. The classical solutions are found in global Lorentzian AdS_5 [6] starting from a time-independent ansatz, the boundary conditions being lines separated by $\pi-\phi$ on the boundary of AdS and θ on S^5. The relevant solutions (written down in the case of $\theta=0$ in [19] and for $\theta\neq 0$ in Appendix C.2 of [24]) can be found for arbitrary values of ϕ and θ as the solutions of transcendental equations. The result for the generalized potential is then found in

[2] In [17], an integral equation was written whose solution gives the contribution of ladder graphs to all orders in perturbation theory.

[3] The calculation of $V^{(1)}$ at one–loop order was done in [19]. The $\theta=0$ case is in [17] (see also [20]), where expressions were written in integral form. Here we have extended the expressions to $\theta\neq 0$ and computed the integrals in closed form.

[4] After subtracting the exponentiation of the $O(\lambda)$ term.

[5] Note the uniform transcendentality three (when $e^{2i\phi}$ is considered rational) of both interacting and ladder graphs at this order.

[6] This is the appropriate strong coupling dual of the gauge theory on $\mathbb{S}^3\times\mathbb{R}$.

terms of elliptic integrals $\mathbb{K}$ and $\mathbb{E}$ [7]

$$V^{(0)}_{AdS}(\phi,\theta)=\frac{\sqrt{\lambda}}{2\pi}\frac{2\sqrt{b^4+p^2}}{b\,p}\left[\frac{(b^2+1)p^2}{b^4+p^2}\mathbb{K}(k^2)-\mathbb{E}(k^2)\right], \tag{6}$$

where the elliptic modulus k and the parameter b are functions of p, q, which are in turn related to ϕ, θ via transcendental equations.

Quadratic fluctuations around the classical solution can be considered, based on the Nambu-Goto type action in the static gauge. The mass matrix in the resulting quadratic fluctuation Lagrangian, depending in general on the two parameters of the problem, becomes diagonal in the two limiting cases $\theta = 0$ (equivalently $q = 0$) and $\phi = 0$ (the limit $p \propto q \to \infty$). In particular, for these values all the quadratic fluctuation operators, which have a trivial time dependence, can be written in the form of one-dimensional single-gap *Lamé* differential operators [8]. The latter point is crucial. It makes it possible to trade the explicit evaluation of the eigenvalue spectrum for the relevant operators with the resolution of the associated differential equation (an approach known as Gelfand-Yaglom method, see also the analysis in [13]). Relying on the knowledge of the solutions to the Lamé spectral problem, all fluctuations determinants can be then computed analytically. The resulting (regularized) effective action Γ_{reg}, which is the ratio of determinants including the contribution of the trivial time direction $\mathcal{T}{=}\int d\tau$, is then expressed as a single integral [9] and defines the one-loop correction to the generalized quark-antiquark potential as follows (e.g. in the $\theta = 0$ case)

$$V^{(1)}_{AdS}(\phi,\theta)=\frac{\Gamma_{\text{reg}}}{T}=-\frac{\mathcal{T}}{2\,T}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\frac{\epsilon^2\omega^2\det^8\mathcal{O}^\epsilon_F}{\det^5\mathcal{O}^\epsilon_0\det^2\mathcal{O}^\epsilon_1\det\mathcal{O}^\epsilon_2}\,. \tag{7}$$

The explicit expressions for the 1d determinants can be found in [13], here we report as representative the bosonic contribution

$$\det\mathcal{O}^\epsilon_2\cong-\frac{\sinh(2\mathbb{K}(k_2^2)\,Z(\alpha_2))}{\epsilon^2\,\omega\sqrt{\omega^4+(2-4k^2)\omega^2+1}}\,,\qquad \text{sn}(\alpha_2|k_2^2)=\frac{\sqrt{1+k_2^2+\omega_2^2}}{k_2}\,, \tag{8}$$

where Z is the Jacobi Zeta function, sn is the Jacobi elliptic sine, k_2 is a rational function of k and ω_2 a rational function of k and ω. Above, ϵ is the standard infrared regulator curing the linear divergence expected at the boundary, the determinant is taken at leading order in a $\epsilon \simeq 0$ expansion and an explicit subtraction of the remaining divergences (a regularization artifact) is made.

It is possible to see that both the classical and the one-loop strong coupling results, (6) and (7)-(8), reproduce the known expressions for the antiparallel lines, in [1, 23] and [10, 11] respectively, in the $\phi \to \pi,\ \theta = 0$ limit [10]. This happens, as in the weak coupling case, once the replacement of the pole $\pi - \phi \to L$ is performed.

It is straightforward to evaluate the integral (7) numerically for arbitrary values of ϕ, as well as in the analog case of $\phi = 0$ and arbitrary θ, while, in general, we do not know how to calculate it analytically [11]. To gain more analytic control over the form of $V^{(1)}_{AdS}$ we will proceed in a systematic expansion around $\theta = 0$ and $\phi = 0$, to which the next section is devoted.

3 Near straight-line expansion

In the $\phi \to 0$ limit the cusp disappears and we are left with an infinite straight line in $\mathbb{R}^4$, or a pair of antipodal lines on $\mathbb{S}^3 \times \mathbb{R}$. In this case the analysis indeed simplifies, and allows for explicit analytic expressions at weak and at strong coupling.

[7] The standard linear divergence for two lines along the boundary, canceled as usual by a boundary term, is here removed.

[8] See also [25].

[9] The integration variable ω in (7) is the Fourier-transformed τ variable $\partial\tau = -i\,\omega$.

[10] This limit translate in the conditions $p \to 0\,,\ \frac{q^2}{p} = \text{fixed},\ k^2 = 1/2$ on the parameters relevant at strong coupling.

[11] See however the results of [11] in the limit of antiparallel lines.

At *weak coupling*, the first few orders in the expansion of (5) around $\phi = \theta = 0$ read

$$
\begin{aligned}
V^{(1)}(\phi,\theta) &= \theta^2 - \phi^2 - \frac{1}{12}(\theta^2-\phi^2)^2 + O((\phi,\theta)^6)\,,\\
V^{(2)}(\phi,\theta) &= -\frac{2\pi^2}{3}(\theta^2-\phi^2) + \frac{1}{18}(\pi^2(\theta^2-\phi^2)^2 + 6(\theta^2-\phi^2)(3\theta^2-\phi^2)) + O((\phi,\theta)^6)\,.
\end{aligned}
\tag{9}
$$

All the terms are proportional to $\theta^2 - \phi^2$, and indeed we expect $V(\phi,\theta,\lambda)$ to vanish for $\theta = \pm\phi$, which are BPS configurations [24].

At *strong coupling*, an expansion of the leading semiclassical result leads to

$$
V^{(0)}_{AdS}(\phi,\theta) = \frac{1}{\pi}(\theta^2-\phi^2) - \frac{1}{8\pi^3}(\theta^2-\phi^2)\left(\theta^2-5\phi^2\right) + O((\phi,\theta)^6)\,. \tag{10}
$$

At one–loop order in σ-model perturbation theory, the expansion translates in a small k expansion of all the elliptic functions in the integrand of (7), and results in a power series of regular hyperbolic functions. An integration over the logarithm of this series can then always be performed, and gives

$$
\begin{aligned}
V^{(1)}_{AdS}(\phi,0) &= \frac{3}{2}\frac{\phi^2}{4\pi^2} + \left(\frac{53}{8} - 3\,\zeta(3)\right)\frac{\phi^4}{16\pi^4} + \left(\frac{223}{8} - \frac{15}{2}\zeta(3) - \frac{15}{2}\zeta(5)\right)\frac{\phi^6}{64\pi^6} + O(\phi^8)\,.\\
V^{(1)}_{AdS}(0,\theta) &= -\frac{3}{2}\frac{\theta^2}{4\pi^2} + \left(\frac{5}{8} - 3\,\zeta(3)\right)\frac{\theta^4}{16\pi^4} + \left(\frac{1}{8} + \frac{3}{2}\zeta(3) - \frac{15}{2}\zeta(5)\right)\frac{\theta^6}{64\pi^6} + O(\theta^8)\,.
\end{aligned}
\tag{11}
$$

Focus now on the expansion coefficients around $\phi = \theta = 0$, for example the first (quadratic) one

$$
\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0} = -\frac{1}{2}\frac{\partial^2}{\partial\phi^2}V(\phi,\theta,\lambda)\Big|_{\phi=\theta=0} = \begin{cases} \dfrac{\lambda}{16\pi^2} - \dfrac{\lambda^2}{384\pi^2} + \cdots & \lambda \ll 1\,,\\[2ex] \dfrac{\sqrt{\lambda}}{4\pi^2} - \dfrac{3}{8\pi^2} + \cdots & \lambda \gg 1\,.\end{cases} \tag{12}
$$

The expansion around the 1/2 BPS straight line can be viewed as a *deformation* of the straight line itself, and as such it can be written in terms of insertions of local operators into the Wilson loop. One can write the latter as a straight ($\phi = 0$) line in the x^1 direction with arbitrary θ

$$
W = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\left[\exp\Big(\int_{-\infty}^{0}(iA_1+\Phi_1)ds\Big)\exp\Big(\int_0^\infty(iA_1+\Phi_1\cos\theta+\Phi_2\sin\theta)ds\Big)\right]\,, \tag{13}
$$

such that it couples to the scalar Φ_1 for all $s < 0$ and to the linear combination $\Phi_1\cos\theta + \Phi_2\sin\theta$ for $s > 0$ [12]. Using that [13]

$$
\frac{\partial^2}{\partial\theta^2}V(0,0) = -\frac{1}{\ln(R/\epsilon)}\frac{\partial^2}{\partial\theta^2}\log\langle W\rangle \approx -\frac{1}{\ln(R/\epsilon)}\frac{\partial^2}{\partial\theta^2}\langle W\rangle, \tag{14}
$$

one finds for the coefficient in (12) [14]

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial\theta^2}V = &-\frac{1}{\ln(R/\epsilon)}\frac{1}{2N}\int_0^\infty ds_1\int_0^\infty ds_2\left\langle \mathrm{Tr}\,\mathcal{P}\left[\Phi_2(s_1)\Phi_2(s_2)\,e^{\int_{-\infty}^{\infty}(iA_1+\Phi_1)ds}\right]\right\rangle\\
&+\frac{1}{\ln(R/\epsilon)}\frac{1}{2N}\int_0^\infty ds_1\left\langle \mathrm{Tr}\,\mathcal{P}\left[\Phi_1(s_1)\,e^{\int_{-\infty}^{\infty}(iA_1+\Phi_1)ds}\right]\right\rangle.
\end{aligned}
\tag{15}
$$

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[12] We fixed the parameterization such that $|\dot{x}| = 1$, so we can ignore the difference between $x^\mu(s_i)$ and s_i.

[13] The first identity is the definition of V. The second follows from $\frac{\partial}{\partial\theta}\langle W\rangle = 0$ and from $\langle W|_{\phi=\theta=0}\rangle = 1$.

[14] The variation with respect to θ is somewhat simpler than the the variation with respect to ϕ, since the latter modifies the path of the loop and is captured by insertions of the field strength $F_{\mu\nu}$ as well as its derivatives into the loop, while the former only introduces local scalar field insertions.

Examining the right-hand side is suggestive of a pattern expected to hold for all values of the coupling. One notices that graphs which involve propagators between the Wilson loop and itself, and not the insertions, will vanish due to the BPS nature of the straight line. At one and two-loop order, only graphs with at most one internally connected component contribute, as the explicit expansion of $V_{\rm int}^{(2)}$ and $V_{\rm lad}^{(2)}$ in (5) easily confirms. The interesting observation is that this argument should apply also to higher order graphs. *Only graphs with one set of connected internal lines attached to the Wilson loop contribute to this term*[15]. Regarding further expansion coefficients, the one of θ^4 will involve for example graphs with at most two disconnected internal components, and so on. Since by explicit calculation we found that the connected (interacting) graphs at 2–loop order had a simpler (without polylogarithms) functional form than the disconnected (ladder) ones, it would be certainly interesting to see if this structure persists at higher orders in perturbation theory and whether it is possible to guess the answer for the most connected graphs at all loop order, and reproduce the strong coupling results in (12).

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[15] This statement is true assuming the cancellation for the straight line does not require integration. Otherwise, there will be boundary terms in the disconnected graphs, which can be regarded as connected ones.