Dynamic Pricing of Substitutable Products in the Presence of Capacity Flexibility

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Firms that offer multiple products are often susceptible to periods of inventory mismatches where one product may face shortages while the other has excess inventories. In this paper, we study a joint implementation of price- and capacity-based substitution mechanisms to alleviate the level of such inventory disparities. We consider a firm producing substitutable products via a capacity portfolio consisting of both product dedicated and flexible resources and characterize the structure of the optimal production and pricing decisions. We then explore how changes in various problem parameters affect the optimal policy structure. We show that the availability of a flexible resource helps maintain stable price differences across products over time even though the price of each product may fluctuate over time. This result has favorable ramifications from a marketing standpoint as it suggests that even when a firm applies a dynamic pricing strategy, it may still establish consistent price positioning among multiple products if it can employ a flexible replenishment resource. We provide numerical examples for the price stabilization effect and discuss extensions of our results to a more general multiple product setting.

1. Introduction

Virtually all manufacturing and service industries are susceptible to periods of supply and demand mismatches. Due to capacity limitations and demand uncertainties, firms producing multiple products can frequently encounter instances where one of their products faces shortages while the other has excess inventories. In order to alleviate the level of such inventory mismatches, firms can utilize several tools to alter supply or demand. Our focus in this paper will be a joint analysis of two of these mechanisms, namely, dynamic pricing and capacity flexibility.

On the demand side, through price discounts and price surcharges that shift demand from one product to another as well as stimulate or reduce the overall demand, dynamic pricing is an effective

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tool to better match demand with supply. As an example, consider a firm offering two substitutable products during a period in which one of the products faces low demand while the other experiences high demand leading to backlogs. A reduction in the price of the product with low demand can induce more customers to prefer that product, thereby reducing its inventory while relieving the firm from excessive backlogs on the other product. Hence dynamic pricing may reduce inventories for one product while simultaneously reducing backlogs for the other. Dynamic pricing has long been used in airline revenue management where prices respond strongly to the availability of various seat classes. More recently, with the advent of e-commerce and the ability to frequently change and advertise prices, dynamic pricing has also been increasingly used in many other industries such as electronics and automobiles. As an example from the automotive industry, Copeland et al. (2005) provide empirical observations on whether vehicle prices are correlated with inventory fluctuations and they conclude that a significant negative relationship exists between inventories and prices.

On the supply side, flexible manufacturing systems can also be utilized to align supply with demand. In the last decade, firms in many industries have invested in flexible manufacturing systems that enable the production of multiple variations of products in the same factory. This enables the firm to easily alter its product mix if demand for one product increases while demand for another decreases. Flexible systems may be considered to have advantages over dynamic pricing as they operate to match demand and supply without sacrificing product revenues (increasing the price to lower backlogs by suppressing sales can lead to a reduction of overall revenue for the firm). Although flexible systems are beneficial if product demands are negatively correlated, the ability to change production from one item to another has limited benefits if both products have a surge in their demand or if both products experience low demand.

It is interesting that many firms use both flexible manufacturing and dynamic pricing simultaneously. For example, in the auto industry, demand for SUVs and sedans fluctuate depending on gas prices and other economic factors. In a recent paper, Moreno and Terwiesch (2011) empirically investigated how different auto manufacturers reacted to shifts in demand. They identified that companies (such as Honda) that invested in more flexible plants were able to decrease production levels of SUVs when demand for SUVs dropped, and increase production of other vehicles made in the same assembly line. On the other hand, companies that had not invested in flexible plants during the same period (such as Ford) reacted to decreases in SUV demand by significantly increasing incentives on their SUVs (i.e., reducing prices, for example Moreno and Terwiesch estimate that deploying flexibility enables manufacturers to reduce the use of incentives typically by between \$200 and \$700 per vehicle). Similarly, during the "great recession" of 2009, demand for larger sized

(42 inches and above) LCD TVs slowed down in the U.S. as consumers trimmed their budgets and preferred smaller sized and lower priced models, according to the market research firm DisplaySearch. Thus an LCD TV manufacturer that produces multiple models of different sizes had the following choices to respond to this change in demand: 1) It can decrease the price of larger sized models to stimulate more demand, 2) it can switch more of their production to smaller sized models (e.g., 32, 37 and 40 inch), or a combination of the two policies.

Moreno and Terwiesch (2011) demonstrate that even firms that deploy flexibility still offer incentives (i.e., adjust prices) when demand fluctuates but do so less than firms with less flexible production capabilities. This raises the following interesting questions which constitute the main objective of our study: how does a firm offering multiple substitutable products decide (*i*) on the price charged for each product, (*ii*) how much of each product it should produce, and (*iii*) how should the flexible resource be allocated among products in a given period. We are also interested in understanding how operational flexibility influences a firm's pricing strategy.

Our first contribution in this paper is to provide a full characterization of joint optimal production and pricing decisions for substitutable products with limited production capacities in the form of product dedicated and flexible resources. We show that the optimal production policy can be characterized by modified base-stock levels. Regarding the optimal pricing policy, we find that the optimal price policy consists of a list price region for each product in addition to regions where price markups and markdowns are given depending on product inventory levels. Interestingly, when both products are understocked and share the flexible capacity, the optimal pricing scheme maintains a constant price difference between products. Hence, our second major finding is that the availability of a flexible resource helps maintain stable price differences across products over time even though the price of each product may fluctuate over time. This result has favorable ramifications from a marketing standpoint as it suggests that even when a firm applies a dynamic pricing strategy, it may still establish consistent price positioning among multiple products if it can employ a flexible replenishment resource. Several studies in marketing and economics show that firms often use price to signal quality differences among products, and that consumers also use price as an indicator of quality or benefits (Monroe, 1973) especially when they do not have sufficient knowledge to judge quality (Rao and Monroe, 1998). Therefore, keeping a consistent price gap between different quality products is very important for retaining product positioning and brand equity. Indeed, Mela, Gupta, and Jedidi (1998), Mela, Gupta, and Lehmann (1997), and Jedidi, Mela, and Gupta (1999) find that deep and frequent price promotions may have long term negative effects on brand equity. Thus, we find that flexible production capacity may have benefits that go beyond operational cost savings, and can help firms retain their product positioning and brand equity by maintaining consistent price differences and resorting to less frequent price promotions. Our finding is consistent with the empirical findings of Moreno and Terwiesch (2011) that flexibility provides price stabilization.

2. Literature Review

There exists a vast literature on dynamic pricing. We will restrict our attention to only those studying joint pricing and replenishment decisions. Extensive reviews on the interplay of pricing and production decisions have been provided by Elmaghraby and Keskinocak (2002), Bitran and Caldentey (2003), Chan et al. (2004), and more recently by Chen and Simchi-Levi (2010). Single product settings have been the focus of much of the earlier work in this area. Whitin (1955) is among the first to consider joint pricing and inventory control for single period problems under both deterministic and stochastic demand models. For a finite horizon, periodic review model, Federgruen and Heching (1999) shows that the optimal policy is of a base-stock, list-price type. When it is optimal to order, the inventory is brought to a base-stock level and a list-price is charged. For inventory levels where no ordering takes place, the optimal policy assigns a discounted price. In a subsequent work, Li and Zheng (2006) extends the setting studied by Federgruen and Heching (1999) to include yield uncertainty for replenishments. Chen and Simchi-Levi (2004) further extend the results of Federgruen and Heching (1999) to include fixed ordering costs and show that a stationary (s,S,p) policy is optimal for both the discounted and average profit models with general demand functions. In such a policy, the period inventory is managed based on the classical (s,S) policy, and price is determined based on the inventory position at the beginning of each period.

Recently, settings consisting of multiple substitutable products have received more attention. Aydin and Porteus (2008) study a single period inventory and pricing problem for an assortment consisting of multiple products. They investigate various demand models and show that a price vector accompanied by corresponding inventory stocking levels constitute the unique solution to the profit maximization problem although the profit function may not necessarily be quasi-concave in product prices. Tang and Yin (2007) examine a retailer's pricing and quantity decisions in a single period setting for two substitutable products with deterministic demand that share a common resource limiting the total order quantity. Zhu and Thonemann (2009) study a periodic review, infinite capacity, joint production and pricing problem with two substitutable products assuming a linear additive demand model. They show that production for each product follows a base-stock policy which is nonincreasing in the inventory level of the other product. They also show that the optimal pricing decisions do not necessarily exhibit monotonicities with respect to inventory levels except for settings where the demand processes for both products are influenced by identical cross price elasticities. They find that a list price is optimal whenever an order is placed for a product, regardless of the inventory level of the other product and a discount is given for any product that is not ordered. Ye (2008) extends their results to an assortment of more than two products and shows that under a similar linear additive demand model and identical cross price elasticities, a base-stock, list-price policy extends to an arbitrary number of products. Song and Xue (2007) also consider dynamic pricing and inventory decisions for a set of substitutable products with price-dependent random demand. They study more general additive and multiplicative demand models for multiple products and provide characterizations for the optimal pricing and inventory policy structure as well as algorithms to compute the optimal policy. Aside from the single period model of Tang and Yin (2007), all these papers assume infinite production capacity. In this paper, we consider a general capacity portfolio consisting of both finite dedicated resources, and more importantly, a shared finite flexible resource. If production capacity is limited, charging list prices for a product whenever an order is placed for that product is no longer optimal. Intuitively, one would expect to charge a higher price when the desired production quantity is restricted by a limited capacity. We show that this expectation is indeed true. Consequently, as opposed to the results for the infinite capacity setting, whenever an order is placed for a product, its price is no longer independent of the inventory level of the other product. In addition, our main results highlight the impact of the flexible resource on the firm's optimal pricing policy.

In the area of flexible capacity allocation, Evans (1967) studies a periodic review problem with two products produced by a single shared resource and characterizes the optimal allocation policy for the flexible resource. DeCroix and Arreola-Risa (1998) study extensions to multiple products. For an infinite horizon problem with homogenous products where all products have identical cost parameters and resource requirements, they derive structural results regarding the optimal allocation of the flexible capacity. Bish et al. (2005) study the impact of flexibility and various capacity allocation policies on supply chain performance with a focus on production swings and variability. Besides these periodic review models, continuous time formulations and corresponding results can also be found in works such as Glasserman (1996) and Ha (1997). These papers on flexible capacity allocation treat the demand process as exogenous whereas our focus is to simultaneously study dynamic pricing decisions that influence the demand for each product.

There have also been other studies that investigate investments in capacity flexibility in the context of substitutable products. Chod and Rudi (2005) study the effects of resource flexibility and price setting in a single period model in which the firm first decides on the capacity investments prior to demand realizations. Following the realization of demand, capacity allocations and product pricing decisions are given. They show that investment in flexible capacity increases in both demand variability and correlation. Lus and Muriel (2009) analyze the impact of product substitution on the optimal mix of dedicated and flexible capacities the firm should invest in. They compare alternative metrics of product substitutability that are commonly used in the economics literature and show that investment in manufacturing flexibility tends to decrease as the products are more substitutable. Bish and Suwandechochai (2010) study the flexibility investment problem by considering the postponement strategies regarding whether the quantity decisions are given before or after prices are set and demand is realized. These works complement our study in the sense that our model takes the capacity investment decisions as given and focuses on the implications of capacity flexibility on a firm's optimal dynamic pricing strategy.

3. Problem Formulation

We consider a firm that produces two substitutable products through a capacity portfolio consisting of product-dedicated and flexible resources. Prices and replenishment quantities for both products are dynamically set at the beginning of each period over a finite planning horizon of length T. At the beginning of period t, the firm reviews the current inventory levels (x_1^t, x_2^t) and decides on (i) the optimal order up to levels (y_1^t, y_2^t) and (ii) the prices, (p_1^t, p_2^t) to charge during the period which will influence the demands (d_1^t, d_2^t) observed within the period. We assume the demands for both products are correlated by the following linear additive price-demand model.

$$\begin{aligned} d_1^t(p_1^t, p_2^t, \epsilon_1^t) &= b_1^t - a_{11}^t p_1^t - a_{12}^t p_2^t + \epsilon_1^t \\ d_2^t(p_1^t, p_2^t, \epsilon_2^t) &= b_2^t - a_{21}^t p_1^t - a_{22}^t p_2^t + \epsilon_2^t \end{aligned} \tag{1}$$

In (1), b_i denotes the demand intercept whereas $a_{ii}^t > 0$ and $a_{ij}^t \leq 0$ for $i, j = \{1, 2\}$, $j \neq i$, refers to the individual and cross price elasticities for product type-i, respectively. The assumption on the signs of elasticity terms reflects the substitutable nature of the products where the demand for a product is decreasing in its own price and increasing with the price of the other product. We assume strict diagonal dominance on price sensitivities, i.e., $a_{11}^t > |a_{12}^t|$ and $a_{22}^t > |a_{21}^t|$. This implies that the income effect is stronger than the substitution effect, i.e., a price change on a product influences its demand more than the demand for the other product. Further, we also assume $a_{ij}^t = a_{ji}^t$. This symmetric relationship implies settings where demands for both products can be influenced by different individual price elasticities but they experience identical cross price elasticities. In other words, the change of the expected demand for a product with respect to the price of the other product is equivalent for both products. This same property is also inherently present in Multinomial Logit (MNL) type demand models that we describe in Section 5. The property is a common assumption in works analyzing multiple product settings such as in Ye (2008) and in the monotonicity results of Zhu and Thonemann (2009). This assumption enables several desired structural properties of the objective function (described further in the next section) which in turn enables us to provide a characterization of the optimal production and pricing policy structure when there is a flexible resource present in the firm's capacity portfolio. Although our results and main insights rely on the assumption of symmetric cross price elasticities, in Section 5, we also numerically explore and show that the insights indeed extend to more general demand functions that violate this assumption. Finally, in (1), we let ϵ_1^t and ϵ_2^t refer to independent random variables having continuous probability distributions with zero mean and nonnegative support on the product demands.

We adapt a general capacity portfolio which allows the firm to utilize any combination of dedicated capacities $K_1, K_2 \geq 0$ for the production of each product exclusively, as well as a limited flexible resource, $K_0 \geq 0$, that can be assigned partially or entirely for the production of either product. We assume that a unit of flexible resource can be used towards producing a unit of each type of product. In each period, the optimal production quantities are bounded by the corresponding available flexible and product-dedicated capacity levels. We let c_i^t denote the unit production cost for product type-i in period t and assume that this unit cost is applicable to both dedicated and flexible production systems when producing the same product. This is especially applicable when the production cost for a product constitutes mostly of the raw materials or when the processing costs differ across products yet remain constant across types of resources. (We consider instances where production on a flexible resource is more expensive compared to production on a dedicated resource in Section 6.) At the end of period t, the firm incurs holding and backorder costs of $h_i^t(x_i^t) = h_i^{t+} x_i^{t+} + h_i^{t-} x_i^{t-}$ where $x_i^{t+} := \max(0, x_i^t)$, $x_i^{t-} := \max(0, -x_i^t)$, and h_i^{t+} and h_i^{t-} refer to holding and backorder cost per unit, respectively. In Section 6, we also consider the setting where the firm does not backorder any demand missed in the current period but uses a more expensive expedited delivery option for any units in shortage. To simplify the notation throughout the subsequent analysis, we suppress the superscript t on demand and cost parameters a_{ij}^t, b_i^t, c_i^t, h_i^{t+}, and h_i^{t-}.

Letting $V^t(x_1^t, x_2^t)$ denote the expected discounted profit-to-go function under the optimal policy starting at state (x_1^t, x_2^t) with t periods remaining until the end of the horizon, the problem can be expressed as a stochastic dynamic program satisfying the following recursive relation:

$$V^t(x_1^t, x_2^t) = \max_{\substack{y_1^t, y_2^t \in \mathcal{F}(x_1^t, x_2^t) \\ p_1^t, p_2^t}} R(p_1^t, p_2^t) - \sum_i c_i(y_i^t - x_i^t) + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}\Big(- \sum_i h_i(y_i^t - \bar{d}_i^t - \epsilon_i^t) + \beta V^{t-1}(y_1^t - \bar{d}_1^t - \epsilon_1^t, y_2^t - \bar{d}_2^t - \epsilon_2^t)\Big) \tag{2}$$

where $\mathcal{F}(x_1^t, x_2^t)$ denotes the set of admissable values for the order-up-to decisions and is defined as $\mathcal{F}(x_1^t, x_2^t) := \Big\{(y_1^t, y_2^t) | x_i^t \le y_i^t \le x_i^t + K_0 + K_i \ \forall i = 1,2 \text{ and } y_1^t + y_2^t \le x_1^t + x_2^t + K_0 + K_1 + K_2\Big\}$. The term $R(p_1^t, p_2^t) := \sum_i p_i^t \bar{d}_i^t(p_1^t, p_2^t)$ represents the expected revenue in period t where the mean demands $\bar{d}_1^t$ and $\bar{d}_2^t$ are given by $\bar{d}_1^t(p_1^t, p_2^t) = b_1 - a_{11}p_1^t - a_{12}p_2^t$ and $\bar{d}_2^t(p_1^t, p_2^t) = b_2 - a_{21}p_1^t - a_{22}p_2^t$. We define $V^0(x_1^t, x_2^t) = 0$ to be the terminal value function and let β denote the discount factor.

We define a new set of variables, (z_1^t, z_2^t), such that $z_i^t := y_i^t - \bar{d}_i^t$. An economic interpretation of z_i^t is that it represents the target safety-stock level for product i after the current inventory position is augmented by the replenishment quantity and depleted by the expected demand for that product. Then, the dynamic programming formulation given in (2) can be rewritten as:

$$V^t(x_1^t, x_2^t) = c_1 x_1^t + c_2 x_2^t + \max_{\substack{z_1^t, z_2^t \in \mathcal{F}'(x_1^t, x_2^t, p_1^t, p_2^t) \\ p_1^t, p_2^t}} J^t(z_1^t, z_2^t, p_1^t, p_2^t) \tag{3}$$

where

$$J^t(z_1^t, z_2^t, p_1^t, p_2^t) = R'(p_1^t, p_2^t) - c_1 z_1^t - c_2 z_2^t + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}\Big(-\sum_i h_i(z_i^t - \epsilon_i^t) + \beta V^{t-1}(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)\Big) \tag{4}$$

In (3), $\mathcal{F}'(x_1^t, x_2^t, p_1^t, p_2^t)$ represents the set of admissable decisions for (z_1^t, z_2^t) with $\mathcal{F}'(x_1^t, x_2^t, p_1^t, p_2^t) := \{(z_1^t, z_2^t) | x_i^t \le z_i^t + b_i - a_{i1}p_1^t - a_{i2}p_2^t \le x_i^t + K_0 + K_i \ \forall i = 1,2 \text{ and } z_1^t + z_2^t + b_1 + b_2 - (a_{11} + a_{21})p_1^t - (a_{12} + a_{22})p_2^t \le x_1^t + x_2^t + K_0 + K_1 + K_2\}$. The term $R'(p_1^t, p_2^t)$ in (4) denotes a modified expected revenue function with $R'(p_1^t, p_2^t) := \sum_i (p_i^t - c_i)\bar{d}_i^t(p_1^t, p_2^t)$. In this reconstructed formulation, it can be observed that the objective function, $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$, is separable in the decision variables (z_1^t, z_2^t) and (p_1^t, p_2^t). In addition, the profit-to-go function, $V^{t-1}(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)$, now only depends on the set of variables (z_1^t, z_2^t), which facilitates the derivation of several structural results on the value function that we require in the analysis of the optimal policy.

4. Structure of the Optimal Production and Pricing Policy

In this section, we first present the structure of the optimal production decisions which also consists of the allocation of the flexible resource. We then present the characterization of the pricing decisions and show the impact of capacity flexibility on optimal pricing. Under the assumptions outlined in the preceding section, our first result establishes several structural properties on the objective function that are preserved throughout the planning horizon.

LEMMA 1. *For all $t = 1, 2, \cdots T$,*

(a) $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$ *is strictly concave,*

(b) $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$ *is submodular in* (z_1^t, z_2^t)*,*

(c) $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$ *possesses the following strict diagonal dominance property* $\forall\ i,j;\quad i \neq j$*:* $\frac{\partial^2 J^t}{\partial z_i^t \partial z_i^t} < \frac{\partial^2 J^t}{\partial z_i^t \partial z_j^t}$

Proof: The proof of Lemma 1 and all subsequent results are provided in the Online Supplement.

Strict concavity of the objective function $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$ implies the uniqueness of an optimal solution and that the production policy is of a modified base-stock type. In addition, the submodularity and diagonal dominance properties allow for the characterization of the optimal allocation of the flexible resource and the monotonicity of the optimal production and pricing decisions with respect to starting inventory levels.

4.1. Optimal Production Policy and Resource Allocation

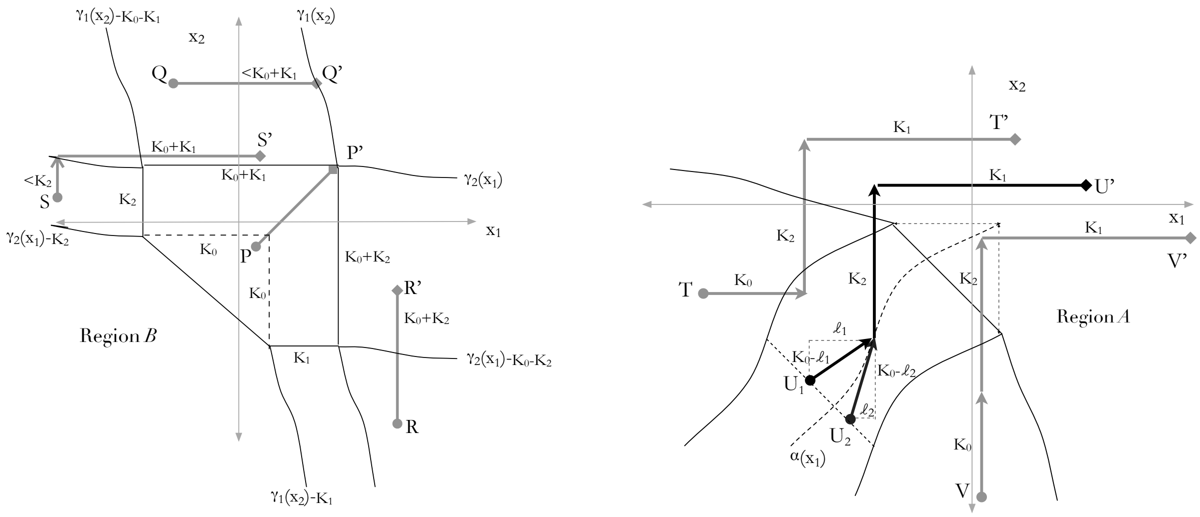
In order to convey the structure of the optimal production policy and how the flexible resource is allocated, we segment the state space into two broad regions based on the initial inventory levels of the products. The first region, denoted by Region A, corresponds to states where there remains some resource (either dedicated or flexible) that is not fully utilized. The second, denoted as Region B, corresponds to initial inventory levels for which all resources are fully utilized. The boundaries of these two regions are described by two monotone functions $\gamma_1^t(x_2^t)$ and $\gamma_2^t(x_1^t)$ that are specified in Theorem 1. Region A is further divided into several subregions with respect to the inventory level of each product according to the following definition.

Definition 1. *Consider initial inventory levels* (x_1^t, x_2^t) *and the functions* $\gamma_1^t(x_2^t)$ *and* $\gamma_2^t(x_1^t)$ *and define* $\bar{x}_1^t$ *and* $\bar{x}_2^t$ *such that* $\bar{x}_1^t = \gamma_1^t(\bar{x}_2^t)$ *and* $\bar{x}_2^t = \gamma_2^t(\bar{x}_1^t)$*. Further, let* $\hat{\gamma}_1^t(x_2^t)$ *(and* $\hat{\gamma}_2^t(x_1^t)$ *in a similar fashion) be given by*

$$\hat{\gamma}_1^t(x_2^t) := \begin{cases} \gamma_1^t(x_2^t) - K_1 & \text{for } \ x_2^t \leq \bar{x}_2^t - K_0 - K_2 \\ \gamma_1^t(x_2^t) + \bar{x}_2^t - x_2^t - K_0 - K_1 - K_2 & \text{for } \ \bar{x}_2^t - K_0 - K_2 < x_2^t \leq \bar{x}_2^t - K_2 \\ \gamma_1^t(x_2^t) - K_0 - K_1 & \text{for } \ \bar{x}_2^t - K_2 < x_2^t \end{cases}$$

Then, product 1 (and product 2) is classified as: (a) "overstocked" if $x_1^t > \gamma_1^t(x_2^t)$*, (b) "moderately understocked" if* $\gamma_1^t(x_2^t) \geq x_1^t > \hat{\gamma}_1^t(x_2^t)$*, and (c) "critically understocked" if* $\hat{\gamma}_1^t(x_2^t) \geq x_1^t$*.*

Defining a product as overstocked means the product requires no further replenishment. A moderately understocked product requires production for which the available capacity is adequate to reach the desired base-stock level whereas a critically understocked product can not be brought to the desired base-stock level due to capacity restrictions. Region A represents all states in which at most one product is critically understocked whereas Region B corresponds to initial inventory levels for which both products are critically understocked. The segmentation of the state space is illustrated in Figure 1 and formally derived within the proof of Theorem 1 which describes the optimal production policy.



(a)Production in region A (b)Production in region B

Figure 1 Optimal production policy

Theorem 1. *The optimal production policy is a state-dependent modified base-stock policy characterized by three monotone functions $\gamma_1^t(x_2^t)$, $\gamma_2^t(x_1^t)$, and $\alpha^t(x_1^t)$ such that*

1. *In states corresponding to initial inventory levels for which at most one product is critically understocked (i.e. in Region A),*

(a) *the optimal production policy for product i $(i=1,2)$ is to produce up to the modified base-stock level* $\min\left(x_i+K_0+K_i, \gamma_i^t(x_{3-i}^t)\right)$.

(b) *the modified base-stock level for product i is nondecreasing with x_i^t and nonincreasing with x_j^t, $j \neq i$.*

2. *In states corresponding to initial inventory levels for which both products are critically understocked (i.e. in Region B),*

(a) *the optimal production policy for product 1 and product 2 is to produce up to the modified base-stock level $x_1^t + K_1 + l^t(x_1^t, x_2^t)$ and $x_2^t + K_2 + K_0 - l^t(x_1^t, x_2^t)$, respectively, where $l^t(x_1^t, x_2^t)$ denotes the amount of flexible capacity allocated to product 1.*

(b) *$l^t(x_1^t, x_2^t) = 0$ if $x_2^t \leq \alpha^t(x_1^t) - K_0$, $l^t(x_1^t, x_2^t) = K_0$ if $x_2^t \geq \alpha^t(x_1^t + K_0)$. Otherwise, $l^t(x_1^t, x_2^t)$ satisfies $l^t(x_1^t, x_2^t) + \alpha^t(x_1^t + l^t(x_1^t, x_2^t)) = x_2^t + K_0$ and the modified base-stock levels for either product is a function of the starting inventory levels through their sum.*

(c) *$l^t(x_1^t, x_2^t)$ is decreasing with x_1^t and increasing with x_2^t.*

(d) *The modified base-stock levels for product i is nondecreasing with either product's inventory level.*

As Theorem 1 suggests, the optimal production policy has a number of properties depending on the inventory state at the beginning of a period. Figure 1(a) illustrates the optimal production policy in region A. When both products are moderately understocked, as shown by the starting inventory level P on Figure 1(a), it is optimal to produce both products up to the point $(\bar{x}_1^t, \bar{x}_2^t)$, denoted by P', and the optimal base-stock levels in this region are independent of initial inventories. Initial inventory levels Q and R in Figure 1(a) are examples of states where one product is overstocked and the other is understocked. Starting at Q, with a base-stock level of $\gamma_1^t(x_2^t)$ for product 1 and no production for product 2, the optimal policy is to move to point Q'. Note that, point Q' refers to a base-stock level for product 1 which is lower than the one suggested by P' (Theorem 1, part 1(b)). The reason is twofold. First, the overstocked product 2 results in a price decrease for that product which in turn increases its demand and decreases the demand for product 1 which further decreases the base-stock for product 1. Second, the overstocked product 2 reduces the potential workload on the flexible resource for that product and increases the availability of the flexible capacity for product 1 in future periods. This allows for fewer units of product 1 to be produced in the current period. An initial inventory state R shows an instance for which no production takes place for overstocked product 1 and all available capacity is used to produce a critically understocked product 2. Point S refers to a state where product 1 is critically understocked and product 2 is moderately understocked. In this case, Theorem 1 states that it is optimal to produce $K_0 + K_1$ units of product 1 and to bring the inventory of product 2 to $\gamma_2^t(x_1^t)$, as shown by point S'. Note that point S' corresponds to a base-stock level higher than the one implied by P' with similar but reverse dynamics as discussed previously.

Part 2 of Theorem 1 corresponds to the states in region B where both products are critically understocked. As illustrated in Figure 1(b), Theorem 1 part 2 states that when the initial inventory levels for both products fall within a 'band' defined by $\{(x_1^t, x_2^t)$, s.t. $(x_1^t, x_2^t) \in$ Region B, and $\alpha(x_1^t + K_0) > x_2^t > \alpha(x_1^t) - K_0\}$, the optimal policy allocates $l^t(x_1^t, x_2^t) > 0$ units of the flexible resource to product 1 and the remaining $K_0 - l^t(x_1^t, x_2^t) > 0$ units to product 2. Moreover, for any two inventory states corresponding to the same total inventory (points U_1 and U_2 in Figure 1(b)), the intermediate inventory levels after the flexible resource is utilized are identical. From this point on, additional units of each product are produced to the full extent of their dedicated resources. For initial inventory levels that fall outside this band, the flexible resource is fully assigned to the product which experiences the most severe shortage. For example, in Figure 1(b), points T and V refer to instances where all flexible capacity is used towards product 1 and product 2, respectively. Part 2 (c) of Theorem 1 states that the share of flexible resource a product receives is decreasing with its own inventory and increasing with the other product's inventory. Referring to Figure 1(b), since the initial inventory level of product 1 corresponding to point U_1 is less than that of corresponding to U_2, the amount of flexible capacity allocated to product 1 when starting at U_1 is larger than the one starting at U_2. As we will discuss next, we find that the optimal prices charged for each product have a specific relationship within this band.

4.2. Optimal Pricing Policy

When making pricing decisions, it is often helpful to think in terms of markdowns and markups where a markdown (markup) corresponds to a price discount (surcharge) relative to a current period list price. Earlier results in the literature on pricing of substitutable products focused on infinite capacity settings, hence the optimal pricing policy was characterized by a list price and markdown policy. In the presence of capacity limitations however, the characterization of the optimal pricing policy relies on a third component, namely price markups. We let $m_i^t(x_1^t, x_2^t)$ denote the price markup/markdown for product i in period t with $m_i^t < 0$ corresponding to markdowns and $m_i^t > 0$ corresponding to markups in reference to a current period list price p_{iL}^t. Thus, in period t we have,

$$p_i^t(x_1^t, x_2^t) = p_{iL}^t + m_i^t(x_1^t, x_2^t) \tag{5}$$

The following theorem defines the optimal pricing policy.

THEOREM 2. *For all $i = 1, 2$, in period t, we have the following:*

a) In Region A, if a product i is moderately understocked, then $m_i^t(x_1^t, x_2^t) = 0$ and it is optimal to charge a list price, p_{iL}^t, for that product where $p_{i,L}^t = \frac{a_{3-i,3-i}b_i - a_{12}b_{3-i}}{2(a_{11}a_{22} - a_{12}{}^2)} + \frac{c_i}{2}$. If a product i is overstocked, then $m_i^t(x_1^t, x_2^t) < 0$, i.e., it is optimal to give a price discount to that product. If on the other hand, product i is critically understocked, then $m_i^t(x_1^t, x_2^t) > 0$, indicating that it is optimal to give a price markup to that product.

b) In Region B, $m_i^t(x_1^t, x_2^t) > 0$, hence the optimal policy marks up the price of both products. Furthermore, if (x_1^t, x_2^t) is such that $0 < l^t(x_1^t, x_2^t) < K_0$, then $m_1^t(x_1^t, x_2^t) = m_2^t(x_1^t, x_2^t)$ resulting in

$$p_2^t(x_1^t, x_2^t) = p_1^t(x_1^t, x_2^t) + C^t \quad \text{where } C^t = p_{2L}^t - p_{1L}^t.$$

c) The optimal price $p_i^t(x_1^t, x_2^t)$, $i = 1, 2$ is decreasing with respect to x_1^t and x_2^t.

Figure 2 illustrates the optimal pricing policy in terms of markups and markdowns for each product. It is optimal to give discounts on a product if it is overstocked, apply the list price on the product if it is moderately understocked and to markup the price of the product if it is critically understocked. Part (b) of Theorem 2 suggests an interesting fact about the pricing policy when the

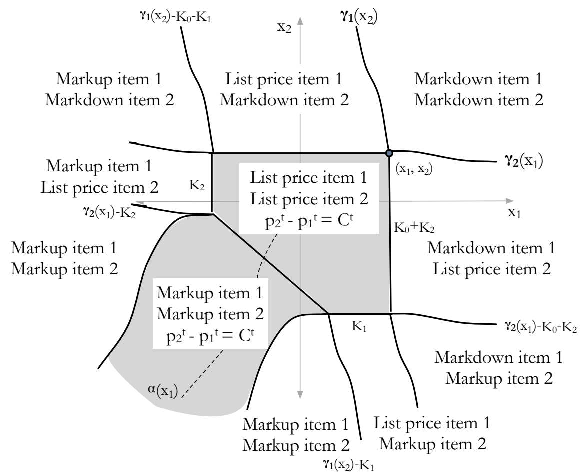


Figure 2 **Optimal pricing policy with the shaded area indicating constant price difference between products**

inventory level falls within the band in region B where both products use a positive share of the flexible capacity. In states corresponding to this region, both products are marked up by exactly the same amount. This results in the price difference between products to remain identical to the difference between their list prices.

This special structure of the optimal price policy has favorable ramifications from a marketing standpoint. Capacity flexibility may be viewed as a significantly beneficial tool when firms use dynamic pricing and are sensitive to maintaining consistent price differences among products in order to preserve price positioning across products. In Section 5, we will demonstrate through a numerical study that this insight can be extended to other demand models such as the Multinomial Logit Model (i.e., flexibility still enables consistent price differences among products when we consider different demand distributions).

4.3. Sensitivity of the Optimal Policy

Having characterized the optimal production and pricing policy, we are also able to explore the sensitivity of the optimal policy structure with respect to changes in various problem parameters. Specifically, we analytically explore the sensitivity of the optimal policy to (i) cost parameters including the production, holding and backorder costs, (ii) capacity parameters, and (iii) demand parameters including demand intercepts and individual price elasticities. The sensitivity results provided in Table 1 have been obtained analytically by studying the effects of an increase in the current-period value of the corresponding parameter to optimal pricing and production decisions. (For brevity, we omit the proofs which are available from the authors.) Where applicable, we report the effects of changes in the parameters corresponding to product 1 only as the results for the

	Price of prod. 1	Price of prod. 2	Base-stock for prod. 1	Base-stock for prod. 2
Production cost (c_1)	$\uparrow$	$\downarrow^{\ddagger}\ \uparrow^{\ddagger'}$	$\downarrow$	$\uparrow$
Holding cost (h_1^+)	$\downarrow$	$\downarrow\ \uparrow$	$\downarrow$	$\downarrow\ \uparrow$
Backorder cost (h_1^-)	$\uparrow$	$\downarrow\ \uparrow$	$\uparrow$	$\downarrow\ \uparrow$
Dedicated capacity (K_1)	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow^{\ddagger'}\ \uparrow^{\ddagger}$
Flexible capacity (K_0)	$\downarrow$	$\downarrow$	$\downarrow^{\dagger'}\ \uparrow^{\dagger}$	$\downarrow\ \uparrow$
Demand intercept (b_1)	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow^{\ddagger}\ \uparrow^{\ddagger'}$
Price Sensitivity (a_{11})	$\downarrow$	$\downarrow\ \uparrow$	$\downarrow$	$\downarrow\ \uparrow$

Table 1 Sensitivity of the optimal policy to various problem parameters

parameters for product 2 are symmetric. In Table 1, the symbols $\downarrow$ and $\uparrow$ denote nonincreasing and nondecreasing, respectively. In addition, † denotes the states for which product 1 is critically understocked while †′ refers to all remaining states, and ‡ refers to states where both products are critically understocked and the flexible resource is shared between the products whereas ‡' denotes the remaining states.

As a particular case, we would like to highlight the results corresponding to an increase in the dedicated or flexible capacity levels. When capacity increases, one would expect that the price for both products would decrease. As shown in Table 1, we note that this expectation is true. A capacity increase in either the dedicated resource or the flexible resource helps reduce instances where products are critically understocked which limits price markups and hence reduces prices. Regarding the modified base-stock levels, an increase in the current period dedicated capacity for product 1 leads to an increase in the modified base-stock level for product 1. When both products share the flexible resource and are critically understocked, an increase in the current period dedicated capacity for product 1 allows more flexible capacity to be allocated to product 2 increasing product 2's modified base-stock level. In all other instances, the modified base-stock level for product 2 decreases. The logic for the results corresponding to an increase in the flexible capacity is similar.

5. Implications of Capacity Flexibility on Optimal Pricing

We have previously shown in Theorem 2 that the existence of flexible capacity in a firm's portfolio results in an extended region where the price differences between products remain constant. We now numerically further explore how capacity flexibility influences a firm's optimal pricing strategy. Specifically, we compare the optimal prices charged over a planning horizon for several problem instances (e.g., different demand models and parameters, different processing times) where the share of the flexible resource in the capacity portfolio is gradually increased.

The numerical results presented below were obtained in two steps. As a first step, we solved the finite horizon stochastic dynamic program by using a series of fine discretization and value

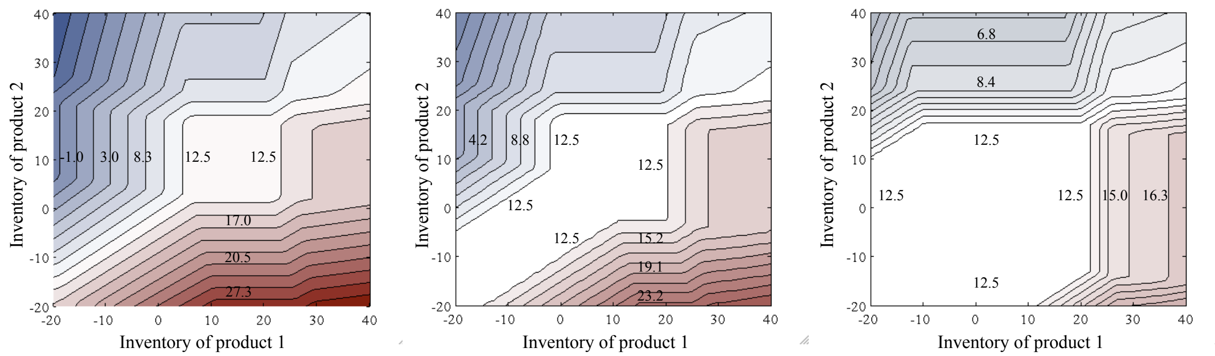


Figure 3 **Price gap between products 1 and 2 ($p_2 - p_1$) reflected by the optimal pricing policy for each product in period 10 for a system with (a) only dedicated resources, (b) hybrid portfolio of dedicated and flexible resources (c) a fully flexible resource**

function approximations for all initial inventory states at each period. We then recorded the optimal production and pricing policy over the entire planning horizon. For the second step, we initialized the starting inventory levels at state $(0,0)$ and ran 500 randomly generated sample paths that result from the optimal policies for the corresponding state at each period. Our first numerical study considers the following demand model.

$$d_1^t(p_1^t, p_2^t, \epsilon_1^t) = 35 - 0.75p_1^t + 0.25p_2^t + \epsilon_1^t, \;\; d_2^t(p_1^t, p_2^t, \epsilon_2^t) = 30 + 0.25p_1^t - 0.5p_2^t + \epsilon_2^t \tag{6}$$

The remaining problem parameters are set as $c_1 = 15, c_2 = 20, h_1^+ = 3, h_2^+ = 4, h_1^- = 20, h_2^- = 25$, and $\beta = 0.8$. We let ϵ_1^t and ϵ_2^t be randomly drawn from a uniform distribution over the interval [-10,10] with a positive support on the realized demand. We selected the parameters corresponding to demand intercepts, cross price elasticities and production costs to construct a setting where the two products have a reasonable list price difference. Additionally, as demand for lower-priced products generally exhibits higher sensitivity to price, we let the product with the lower list price (product 1) be more sensitive to changes in its own price and the product with the higher list price (product 2) be less sensitive to changes in its own price. The model yields list prices of $p_{1,L}^t = 47.5$ and $p_{2,L}^t = 60.0$. In the first setting, we consider a firm with dedicated production capacities, $K_1 = K_2 = 15$ and no flexible capacity, $K_0 = 0$. In the second setting, the firm employs a 'hybrid' portfolio of dedicated and flexible resources where $K_0 = K_1 = K_2 = 10$. Finally, in the third setting, the firm utilizes full flexibility with $K_0 = 30$ and $K_1 = K_2 = 0$. These parameters correspond to utilizations of approximately 95% and 80% for product 1 and product 2, respectively, in the dedicated capacity only setting and approximately 90% overall utilization in the flexible capacity only setting.

Figure 3 displays contours of the price difference between products 1 and 2 (i.e., $p_2 - p_1$) resulting from the optimal pricing policy for each product in period 10 for the three capacity settings.

The white areas in each figure indicate the regions where the price difference between products is identical to the difference between their list prices. The figures collectively show how the constant price region gets larger as the share of the flexible resource in the capacity portfolio increases.

We next consider price evolutions over multiple periods. Table 2 reports the average and standard deviation (with their 95% confidence intervals) of the prices and price differences observed along the planning horizon of 15 periods for the 500 sample paths. Average price 1 reports the mean (over all the sample paths) of the average price for that product along the 15-period horizon. Similarly, the standard deviation of price 1 reports the mean (over all the sample paths) of the standard deviation of the price for that product along the 15-period horizon. Lastly, the standard deviation of the price difference is the mean of the standard deviations for the price difference between products along the planning horizon.

The most interesting aspect of the results in Table 2 is that when flexible systems are used, the difference between the prices charged for products 1 and 2 remain very stable across periods. (In Table 2, compare the standard deviations 2.47, 0.64, and 0.37 for price differences between the two products respectively, for the dedicated only, hybrid, and fully flexible capacity settings.) We actually proved in Theorem 2 that the price difference between the two products will be constant when both products are either moderately understocked or critically understocked and share the flexible resource. When following the optimal policy, we expect the inventory positions for products through the sample path to generally fall within or close to this combined constant price region (white areas in Figures 3 (b) and (c)), thus yielding the results observed in Table 2. In Table 2, we also report the standard deviations for percentage price differences. Most economics papers assume that absolute price differences affect consumer choice between products (Azar 2011). However, behavioral decision theory has shown that in some situations customers are influenced more by percentage price differences rather than actual price differences (Kahneman and Tversky 1984, Darke and Freedman 1993). The recent empirical work by Azar (2011) considers relative price

	Only Dedicated	Hybrid Portfolio	Only Flexible
Average Price 1	49.40 ± 0.003	48.64 ± 0.002	48.53 ± 0.002
Average Price 2	61.27 ± 0.003	61.00 ± 0.002	60.99 ± 0.002
Std. Dev. of Price 1	2.13 ± 0.002	1.60 ± 0.002	1.57 ± 0.002
Std. Dev. of Price 2	1.99 ± 0.002	1.66 ± 0.002	1.64 ± 0.002
Std. Dev. of Price Difference	2.47 ± 0.003	0.64 ± 0.001	0.37 ± 0.001
Std. Dev. of % Price Difference	0.037 ± 0.00003	0.010 ± 0.00002	0.007 ± 0.00001

Table 2 **Price statistics for systems with (i) only dedicated resources ($K_0 = 0, K_1 = K_2 = 15$), (ii) a hybrid portfolio of dedicated and flexible resources ($K_0 = K_1 = K_2 = 10$) and (iii) a fully flexible resource ($K_0 = 30, K_1 = K_2 = 0$)**

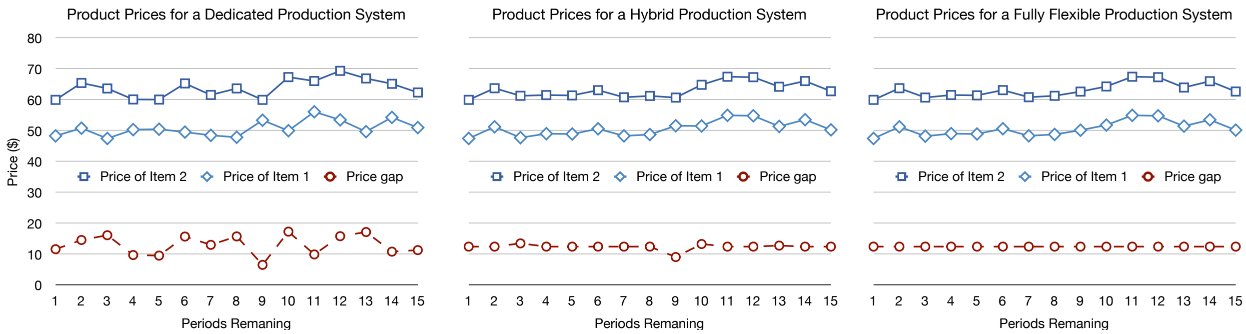


Figure 4 **Prices for, and price difference between products 1 and 2 for a particular sample path for systems with (a) only dedicated resources ($K_0 = 0, K_1 = K_2 = 15$), (b) hybrid portfolio of dedicated and flexible resources ($K_0 = K_1 = K_2 = 10$) (c) a fully flexible resource ($K_0 = 30, K_1 = K_2 = 0$)**

differences and shows that absolute price differences are important where there is a perceived and quantifiable quality gap between products and where the consumers can directly attribute a value for the increased quality. But Azar also identifies certain other situations where percentage price differences can also be important. As depicted in Table 2, flexibility also significantly reduces the standard deviation of percentage price difference between the products.

To visualize the effect of flexible capacity on the optimal pricing policy demonstrated in Table 2, we next illustrate a particular sample path over the 15-period horizon. Figure 4 depicts the optimal prices at each period for the three settings for the same sample path and highlights the advantages of flexible resources. Our main observation from Figure 4 is that the availability of flexible capacity enables price difference between the products to be fairly stable across periods. (We have observed similarly that percentage price differences also are much more stable with flexible capacity.) As the characteristics of the structure of the optimal policy does not depend on an individual parameter set, we obtain similar policy results as displayed in 3 with varying parameters in equation (6). For brevity we omit the results from additional numerical tests in which we varied the demand intercepts and individual and cross-sensitivities in equation (6). The main insight that flexibility significantly reduces the variability of price differences holds in all tested instances of differing demand parameters.

5.1. Asymmetric Cross Price Elasticities

Although our theoretical results on the policy structure and the above numerical study assume identical cross price elasticities, we are also interested to see whether a similar price behavior extends to settings with asymmetric cross price elasticities. To that end, we use the parameters discussed in the previous setting as a base case and gradually increase the cross price elasticity differential, δ, between the products. We run computations with the elasticity values $a_{12} = -0.25 + \delta$

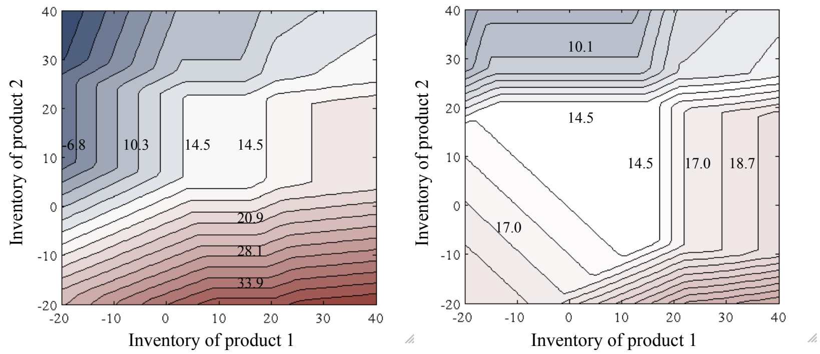


Figure 5 **Price difference between products 1 and 2 reflected by the optimal pricing policy in period 10 corresponding to asymmetric cross price elasticities ($\delta = 0.10$) and for a system with (a) only dedicated resources, (b) a fully flexible resource**

and $a_{21} = -0.25 - \delta$ where we increase δ from 0 to 0.20 with increments of 0.02. As the changes in cross price elasticities affect the list price demand levels, we set the available capacities in each problem instance so as to maintain identical utilization levels with the base setting. Figure 5 displays the price difference between the optimal prices for product 1 and product 2 in period 10 in an instance with asymmetric cross price elasticities ($\delta = 0.10$) and for a system with only dedicated resources and a fully flexible resource. In Figure 5(b), we observe that with asymmetric cross price elasticities, the price differences in the critically understocked region are no longer constant. However we see that flexibility nevertheless continues to provide a less variable price difference compared to the price gap exhibited in the pure dedicated capacity case shown in Figure 5(a).

To consider the price evolution over multiple periods, we again run simulations varying the cross price differential δ from 0 to 0.20. For brevity, in Table 3, we only report the price statistics for $\delta = 0.10$, i.e., where $a_{12} = -0.15$ and $a_{21} = -0.35$. We find that the average standard deviation of the price difference between products 1 and 2 over the 15-period horizon is 2.95 ± 0.003 and 0.50 ± 0.001 for the dedicated only and fully flexible capacity settings, respectively. Thus, flexible capacity continues to reduce the variation in price differences.

	Only dedicated	Only flexible
Average Price 1	47.46 ± 0.003	47.56 ± 0.002
Average Price 2	63.67 ± 0.004	62.39 ± 0.003
Std. Dev. of Price 1	1.85 ± 0.002	1.69 ± 0.002
Std. Dev. of Price 2	2.82 ± 0.003	2.05 ± 0.003
Std. Dev. of Price Difference	2.95 ± 0.003	0.50 ± 0.001

Table 3 **Asymmetric cross price elasticity: Price statistics for the instance where cross price elasticity differential, $\delta = 0.10$ for systems with (a) only dedicated resources and (b) a fully flexible resource**

5.2. Asymmetric Processing Times

Next, we explore the effects of different processing times for the two products on the flexible resource. We let κ denote the processing time differential such that a unit of product 1 requires $1/\kappa$ units of the flexible resource, whereas a unit of product 2 requires κ units of the flexible resource. Hence, as κ increases, the same amount of flexible capacity can produce more of product 1 compared to product 2. We vary κ from 1 to 2 by increments of 0.2 where $\kappa = 1$ corresponds to the base case studied earlier. We adjust the capacity levels such that the utilization of the flexible resource in each setting is identical to that of the base setting.

We observe that different processing times on the flexible resource do not result in an extended constant price difference region and yield to higher markups for the item with the longer processing time. To see why, consider a deviation from identical processing times that reduces the processing time of product 1 on the flexible resource. When both products are understocked, offering a relatively higher markup for product 2 enables the firm to suppress product 2 demand as opposed to the demand for product 1 with the shorter processing time. This in turn shifts more of the backlog to product 1 that can be quickly produced, helping the firm better recover from understocked inventory levels. Considering the price evolution over multiple periods, we again run 500 sample paths using the demand and cost parameters of the base setting and for each instance of κ. We find that the average standard deviation of price difference for the entire range of κ is 1.52. Compared to the standard deviation of 2.47 for the pure dedicated capacity setting, this suggests that capacity flexibility continues to provide a smoothing effect for the price difference between products when there is processing time discrepancies for the products that share the flexible resource. We find that a higher κ value yields a higher standard deviation of the price difference between products. One would expect to see this since increasing the deviation of the processing time results in even higher relative markups for the product with the slower processing time, thereby increasing the price difference variability.

5.3. MNL Demand Models

Finally, we also explore whether the main insight that flexibility provides stability in the price difference between products extends to other demand models. For this purpose, we consider the Multinomial Logit (MNL) demand model. For a detailed discussion of MNL demand models in this context, we refer the reader to Aydin and Porteus (2008). Following Aydin and Porteus (2008), we let $u_i^t - p_i^t + \zeta_i$ denote the surplus utility of a customer who purchases product i where ζ_i is a Gumbel error term with shape parameter μ . The demand for product i is then given by $\Theta\left(\exp(\frac{u_i^t - p_i^t}{\mu})\right) / \left(1 + \sum_j \exp(\frac{u_j^t - p_j^t}{\mu})\right) + \epsilon_i^t$ where Θ denotes the market size and ϵ_i^t is an additional

	$K_0 = 0, K_1 = K_2 = 15$	$K_0 = 30, K_1 = K_2 = 0$
Average Price 1	7.84 ± 0.0054	7.81 ± 0.0050
Average Price 2	9.84 ± 0.0056	9.81 ± 0.0050
Std. Dev. of Price 1	0.40 ± 0.0003	0.35 ± 0.0003
Std. Dev. of Price 2	0.45 ± 0.0003	0.35 ± 0.0003
Std. Dev. of Price Difference	0.48 ± 0.0004	0.01 ± 0.0001

Table 4 **MNL demand model: Price statistics for systems with (i) only dedicated resources ($K_0 = 0, K_1 = K_2 = 15$) and (ii) a fully flexible resource ($K_0 = 30, K_1 = K_2 = 0$)**

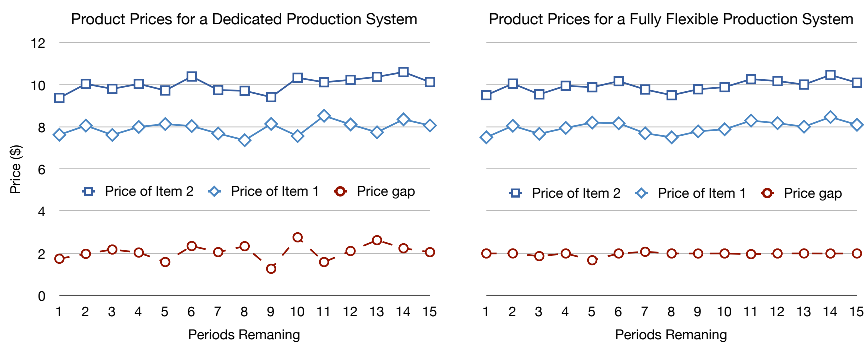


Figure 6 **Multinomial Logit (MNL) demand model: Prices for, and price gap between products 1 and 2 for systems with (a) only dedicated resources, (b) a fully flexible resource**

additive demand uncertainty term. Table 4 similarly displays the results for 500 sample paths for a setting where $u_1 = 8$, $u_2 = 10$, $\mu = 1$, $\Theta = 30$ and with $c_1 = 3, c_2 = 5, h_1 = 1.5, h_2 = 2.5, \pi_1 = 6, \pi_2 = 10$, and $\beta = 0.8$. We find that the insights we have gained by the linear demand model regarding the price difference stabilizing effects of flexible capacity continue to strongly hold under the MNL demand model. (For an illustration of a particular sample path, see Figure 6.)

6. Extensions

6.1. Expedited Delivery Option

First we consider a setting where the firm does not backorder any demand missed in the current period. When revenue is only collected from the current period satisfied demand with any unmet demand considered as lost sales, even a single product case with a linear demand function leads to an objective function that is not concave (see for example Petruzzi and Dada (1999), Chen et al. (2006), Federgruen and Heching (1999)). Hence, we instead consider a different case where the firm has an expedited delivery option for any units in shortage and still collects revenue from all incoming demand during the period. The expedited delivery may correspond to a more expensive outsourcing option or the possibility of producing products during overtime. In such a setting, the problem can then be reformulated with a slight modification as:

$$V^t(x_1^t, x_2^t) = c_1 x_1^t + c_2 x_2^t + \max_{\substack{z_1^t, z_2^t \in \mathcal{F}'(x_1^t, x_2^t, p_1^t, p_2^t) \\ p_1^t, p_2^t}} J^t(z_1^t, z_2^t, p_1^t, p_2^t)$$

$$J^t(z_1^t, z_2^t, p_1^t, p_2^t) = R'(p_1^t, p_2^t) - c_1 z_1^t - c_2 z_2^t + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}\Big(-\sum_i h_i'(z_i^t - \epsilon_i^t) + \beta V^{t-1}((z_1^t - \epsilon_1^t)^+, (z_2^t - \epsilon_2^t)^+)\Big),$$

and the term $h_i'(z_i^t - \epsilon_i^t)$ is defined as $h_i'(z_i^t - \epsilon_i^t) := h_i^+(z_i^t - \epsilon_i^t)^+ + s_i(z_i^t - \epsilon_i^t)^-$ where $s_i > c_i$ denotes the unit expedited production cost. Under this formulation, it can be shown that the optimal policy structure outlined in Theorems 1 and 2 are preserved.

6.2. Costly Flexible Production

We next study a problem which takes into account production cost differences between a flexible resource and a dedicated resource, where production cost for the former may be more expensive than the latter. Such a setting may arise in a labor intensive production environment where operating a flexible resource requires additional skills and training. Let δc_1, $\delta c_2 \geq 0$ be the incremental cost of producing product 1 and product 2 on the flexible resource, i.e., $c_i^F = c_i^D + \delta c_i$ where c_i^D and c_i^F denote the cost to produce product i in its corresponding dedicated and flexible resource, respectively. Introducing two new variables w_1^t and w_2^t to represent the amount of product 1 and product 2 produced in the flexible resource, we can rewrite the problem formulation as follows:

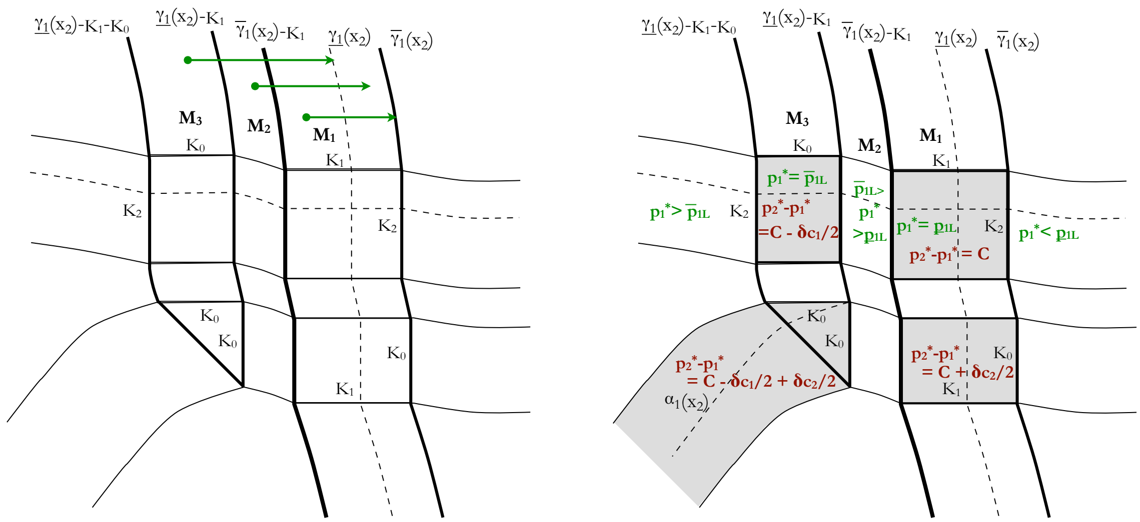
$$V^t(x_1^t, x_2^t) = \max_{\substack{z_1^t, z_2^t, w_1^t, w_2^t \in \mathcal{F}'(x_1^t, x_2^t, p_1^t, p_2^t) \\ p_1^t, p_2^t}} c_1^D x_1^t + c_2^D x_2^t + J^t(z_1^t, z_2^t, w_1^t, w_2^t, p_1^t, p_2^t)$$

$$J^t(z_1^t, z_2^t, w_1^t, w_2^t, p_1^t, p_2^t) = R'(p_1^t, p_2^t) - \sum_i (c_i^D z_i^t + \delta c_i w_i^t) + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}\Big(-\sum_i h_i(z_i^t - \epsilon_i^t) + \beta V^{t-1}(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)\Big)$$

and the term $\mathcal{F}'(x_1^t, x_2^t, p_1^t, p_2^t)$ corresponding to the feasible region is given by $\{(z_1^t, z_2^t, w_1^t, w_2^t) | x_i^t \leq z_i^t + b_i - a_{i1}p_1^t - a_{i2}p_2^t - w_i^t \leq x_i^t + K_i \; \forall i = 1,2, \; w_1^t + w_2^t \leq K_0$, and $w_i^t \geq 0 \; \forall i = 1,2\}$. The optimal policy for the case where production via flexible resources is more costly exhibits identical characteristics of the optimal policy structure given in Theorem 1 when both products are critically understocked. However, the production cost differential between the flexible and the dedicated resources leads to a 'two-tier' modified base-stock level denoted by $\bar{\gamma}_i(x_{3-i})$ and $\underline{\gamma}_i(x_{3-i})$ which segment the moderately understocked region into three subregions for each product (depicted by M_1, M_2, and M_3 in Figure 7). The following theorem summarizes the changes in the optimal policy structure.

Theorem 3. *(a) When at most one product is critically understocked, the optimal production policy for product i is a state-dependent modified base-stock policy consisting of two tiers. If $\underline{\gamma}_i(x_{3-i}^t) - K_i < x_i^t < \bar{\gamma}_i(x_{3-i}^t)$, it is optimal to bring the inventory of product i to $\min(x_i^t + K_i^t, \bar{\gamma}_i(x_{3-i}^t))$. If $x_i^t < \underline{\gamma}_i(x_{3-i}^t) - K_i$, then it is optimal to produce up to $\min(x_i^t + K_i^t + K_0^t, \underline{\gamma}_i(x_{3-i}^t))$. Moreover, $\underline{\gamma}_i(x_{3-i}^t)$ and $\bar{\gamma}_i(x_{3-i}^t)$ is nonincreasing with the inventory level of product j, $j \neq i$.*

(b) The optimal pricing policy for product i is defined by dual list prices $\bar{p}_{i,L}$ and $\underline{p}_{i,L}$ (where $\bar{p}_{i,L} = \underline{p}_{i,L} + \delta c_i/2)$ as well as markup and markdowns. When $\bar{\gamma}_i(x_{3-i}^t) - K_i < x_i^t < \bar{\gamma}_i(x_{3-i}^t)$, it is



(a)Production decisions for product 1 (b)Pricing decisions for product 1

Figure 7 Changes in optimal policy when it is more costly to produce via flexible resources

optimal to charge the lower list price, $\underline{p}_{i,L}$*. If* $\underline{\gamma}_i(x^t_{3-i}) - K_i < x^t_i < \bar{\gamma}_i(x^t_{3-i}) - K_i$*, it is optimal to mark up the price of product* i *such that* $\bar{p}_{i,L} > p^t_i > \underline{p}_{i,L}$*. For* $\underline{\gamma}_i(x^t_{3-i}) - K_i - K_0 < x^t_i < \underline{\gamma}_i(x^t_{3-i}) - K_i$*, it is optimal to charge the higher list price,* $\bar{p}_{i,L}$*. When both products are critically understocked and share the flexible capacity, the optimal policy marks up the price of both products such that the price difference is equivalent to the difference between the high list prices. The price of each product is decreasing with the inventory level of either product.*

For expositional clarity, in Figure 7, we only present the differences in the optimal policy from our earlier main results and only for product 1. As can be observed in Figure 7(a), the production policy consists of a two-tier base-stock level arising due to the production cost difference between the dedicated and the flexible resources. If the initial inventory level of product 1 falls in region M_1, only the dedicated resource for this product is used to bring the inventory level to the upper base-stock level. When the initial inventory level is lower such that the dedicated capacity is not adequate to bring the inventory level to the desired upper base-stock level, the use of the flexible resource is not immediately justified due to its higher cost. Starting in a state within region M_2, it is optimal to fully use the dedicated resource and none of the flexible resource. Region M_3 corresponds to the states where the use of the flexible resource is required. Starting in this region, the inventory is brought up to the lower base-stock level for the product.

Consequently, the production cost difference also yields a two-tier list price for each product, where a lower list price is applied when the initial inventory level is within region M_1, and a higher list price is applied when the starting inventory level is within region M_3. When starting in region

M_2, the optimal price decreases with the initial inventory level and is between the higher and the lower list prices. These results are analogous to our findings in the original model in the sense that list prices are charged when there is adequate capacity to bring the inventory up to a desired level and prices decrease in starting inventory level. In this modified case, the main difference is that we have two sets of desired base-stock levels for the two types of capacity being utilized and hence we have two list prices corresponding to the regions where capacity is adequate to reach the desired base-stock level. When also considering the pricing policy of product 2, the superposition of the two-tier list price policies for both products collectively yield four separate regions where the price difference between the products remain constant individually in each of these regions (shown in Figure 7(b)). Finally, we note that $\delta c_1 = 0$ implies that there is no cost surcharge to use the flexible resource and hence the problem reduces to the original setting. Specifically, the region depicted by M_2 collapses and regions M_1 and M_3 merge to construct the moderately understocked region of the original problem.

6.3. Higher Number of Products

We also consider a general N-product setting. Extending the previously studied demand model in Section 3, we can represent the demand for product i by $d_i^t(p_1^t,...,p_N^t,\epsilon_i^t) = b_i - \sum_{n=1:N} a_{in}^t p_n^t + \epsilon_i^t$. We let the square matrix $\mathbf{A}^t$ with elements a_{ij}^t for $i,j \in \mathcal{N} = \{1,...,N\}$ denote the price-elasticity matrix and $\mathbf{p}^t = (p_1^t,...,p_N^t)$ denote the vector of product prices. Hence, we can write the expected demand vector as a function of the product prices as $\bar{\mathbf{d}}^t = \mathbf{b}^t - \mathbf{A}^t\mathbf{p}^t$. We again assume that $\mathbf{A}^t$ has positive diagonal elements and non-positive off-diagonal elements, that is $a_{ii}^t > 0$ and $a_{ij}^t \le 0$ for $i \neq j$ to reflect the substitutability of the products. We further assume that $\mathbf{A}^t$ possesses strict diagonally dominance property, i.e., $a_{ii}^t > \sum_{j\neq i}|a_{ij}^t|$ and that $\mathbf{A}^t$ is symmetric. Following analogous assumptions as given in Section 3, we can present the problem formulation as:

$$V^t(\mathbf{x^t}) = \mathbf{cx^t} + \max_{\mathbf{z^t}\in\mathcal{F}'(\mathbf{x^t},\mathbf{p^t}),\mathbf{p^t}} J^t(\mathbf{z^t},\mathbf{p^t}) \tag{7}$$

$$J^t(\mathbf{z^t},\mathbf{p^t}) = R'(\mathbf{p^t}) - \mathbf{cz^t} + \mathrm{E}_{\boldsymbol{\epsilon}^t}\Big(-\mathbf{h}(\mathbf{z^t}-\boldsymbol{\epsilon}^t) + \beta V^{t-1}(\mathbf{z^t}-\boldsymbol{\epsilon}^t)\Big) \tag{8}$$

In (7), $V^t(\mathbf{x^t})$ denotes the expected discounted profit starting at state $\mathbf{x^t}$ with t periods remaining until the end of the planning horizon. In (8), the term $R'(\mathbf{p^t})$ stands for the modified expected revenue function with $R'(\mathbf{p^t}) := \mathbf{p^t}'\mathbf{b} + \mathbf{cAp^t} - \mathbf{p^t}'\mathbf{Ap^t} - \mathbf{cb}$. Due to the combinatorial nature of the product-capacity assignments, we first consider a special case where the firm uses dedicated capacities for each product. We will then partially extend the results to a setting consisting of a particular portfolio of dedicated and flexible resources. We let the subset of products requiring production in the current period by $\mathcal{P}$ and the products that are overstocked by $\mathcal{N}\backslash\mathcal{P}$.

In the dedicated capacity case where each product is replenished by its own limited resource K_i, the term corresponding to the feasible region $\mathcal{F}'(\mathbf{x^t}, \mathbf{p^t})$ in (7) is given by $\mathcal{F}'(\mathbf{x^t}, \mathbf{p^t}) := \{\mathbf{z^t} | z_i^t + b_i - \sum_{n=1:N} a_{in} p_n \leq x_i^t + K_i\}$.

THEOREM 4. *(a) The optimal production policy for product $i \in \mathcal{P}$, consists of a modified base-stock level where it is optimal to bring the inventory level of product i up to this level as much as capacity K_i permits. The modified base-stock level for product i is nonincreasing with x_j^t, $j \neq i$.*

(b) For product $i \in \mathcal{P}$, it is optimal to charge a list price $(\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$ if its available capacity K_i is adequate to bring the inventory to its base-stock level. Otherwise, it is optimal to mark up the price of product i. For product $j \in \mathcal{N} \backslash \mathcal{P}$, it is optimal to give a price discount. The optimal price for each product is nonincreasing with the starting inventory level of other products.

The results in Theorem 4 are the extensions of the results given in Theorems 1 and 2 for a portfolio consisting solely of dedicated resources.

Finally, we consider the effects of flexibility on the optimal pricing decisions for a capacity portfolio where the products are indexed such that the first k products are produced by dedicated resources and the remaining $N - k$ products are replenished by a shared flexible resource K_0. For this setting, the term corresponding to the feasible region $\mathcal{F}'(\mathbf{x^t}, \mathbf{p^t})$ in (7) is now given by $\mathcal{F}'(\mathbf{x^t}, \mathbf{p^t}) := \{\mathbf{z^t} | z_i^t + b_i - \sum_{n=1:N} a_{in} p_n \leq x_i^t + K_i, \sum_j \left(z_j^t + b_j - \sum_{n=1:N} a_{in} p_n\right) \leq \sum_j x_j^t + K_0\}$ where $i \leq k$, $j > k$, and $i, j \in N$. The results below summarize the impact of flexibility on optimal prices.

THEOREM 5. *(a) For product $i \leq k$, $i \in \mathcal{P}$, it is optimal to charge a list price $(\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$ if its available capacity K_i is adequate to bring the inventory to its base-stock level, and markup the price otherwise. For product $i' \leq k$, $i' \in \mathcal{N} \backslash \mathcal{P}$, it is optimal to give a price discount.*

(b) If there remains some flexible capacity that is not fully utilized, then it is optimal to set the price of each product $j > k$, $j \in \mathcal{P}$ at its list-price given by $(\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_j/2$ and give a discount to a product $j' > k$, $j' \in \mathcal{N} \backslash \mathcal{P}$. If the flexible capacity is fully utilized, then it is optimal to mark up the price of each product $j > k$, $j \in \mathcal{P}$ and the optimal markup amount is identical for each product.

Theorem 5 outlines the instances when applying list prices, charging markups or offering discounts are optimal for each of the two product groups based on whether the product is produced by a dedicated resource or shares a flexible resource with other products. (For the case of a product portfolio consisting of three products where two products share a flexible resource and one has a dedicated resource or all three products share a flexible resource, it can be also shown that the optimal prices are decreasing with respect to the inventory levels of all products.) Among the products that share the flexible resource, Theorem 5 part (b) indicates that the constant price difference region exhibited by the availability of a shared flexible resource extends beyond two products to an arbitrary number of products for a setting with comparable demand models. The issue of choosing k optimally ultimately requires an analysis that considers several dimensions such as the investment cost for each type of resource and an evaluation of process requirements for the products. From a practical standpoint, capacity flexibility possesses a significant benefit in reducing the complexity of optimal price selection for a product portfolio consisting of a large number of products. Our result indicates that in instances where the firm has to apply price markups for any subset of products that share the flexible resource, it only needs to identify one markup level that will be applied across all products that are understocked.

7. Conclusions

In this paper, we studied a joint mechanism of dynamic pricing and capacity flexibility to mitigate demand and supply mismatches. We considered a firm producing two products with correlated demands utilizing limited product-dedicated and flexible resources and characterized the structure and sensitivity of the optimal production and pricing decisions. We found that the presence of a flexible resource may significantly reduce the fluctuations of price differences across products over time. Thus, the existence of a flexible resource in the firm's capacity portfolio helps maintain stable price differences across products over time. This enables the firm to establish consistent price positioning among multiple products even if it uses a dynamic pricing strategy. Finally, we have extended our results to a more general setting with multiple products and showed that the availability of a flexible resource continues to induce constant price differences among multiple products sharing a single flexible resource.

References

Aydin, G., E. Porteus. 2008. Joint Inventory and Pricing Decisions for an Assortment. *Operations Research*, **56:5**, 1247-1255.

Azar, O. H. 2011. Relative Thinking in Consumer Choice Between Differentiated Goods and Services and its Implications for Business Strategy. *Judgement and Decision Making*, **6:1**, 176-185.

Bish, E. K., A. Muriel, S. Biller. 2005. Managing Flexible Capacity in a Make-to-Order Environment. *Management Science*, **51:2**, 167-180.

Bish, E. K., R. Suwandechochai. 2010. Optimal capacity for substitutable products under operational postponement. *European Journal of Operational Research*, **207**, 775-783.

Bitran, R.G., R. Caldentey. 2003. Pricing Models for Revenue Management. *M&SOM*, **5**, 203-229.

Chan, L.M.A., Z.J.M. Shen, D. Simchi-Levi, J.L. Swann. 2004. Coordination of Pricing and Inventory Decisions: A Survey and Classification. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era*, Kluwer Academic Publisher, Boston.

Chen, X., D. Simchi-Levi. 2004. Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Finite Horizon Case. *Mathematics of Operations Research* **29:3**, 698-723.

Chen, X., D. Simchi-Levi. 2010. Coordinated Pricing and Inventory Management. To appear in *Handbook of Pricing Management*, eds. O. Ozer and R. Phillips.

Chod, J., N. Rudi. 2005. Resource Flexibility with Responsive Pricing. *Operations Research* **53:3**, 532-548.

Copeland, A. , W. Dunn, G. Hall. 2005. Prices, Production, and Inventories over the Automotive Model Year. *Working paper*, Board of Governors of the Federal Reserve System.

Darke, Peter R., J. L. Freedman. 1993. Deciding Whether to Seek a Bargain: Effects of Both Amount and Percentage Off. *Journal of Applied Psychology*, **78:6**, 960-965.

DeCroix, G., A. Arreola Risa. 1998. Optimal Production and Inventory Policy for Multiple Products Under Resource Constraints. *Management Science* **44:7**, 950-961.

Elmaghraby, W., P. Keskinocak. 2002. Dynamic Pricing: Research Overview, Current Practices and Future Directions. *Management Science* **49:10**, 1287-1309.

Evans, R. 1967. Inventory Control of a Multiproduct System with a Limited Production Resource. *Naval Research Logistics*. **14:2**, 173-184.

Federgruen, A., A. Heching. 1999. Combined Pricing and Inventory Control under Uncertainty. *Operations Research*. **47**, 454-475.

Fiacco, A. 1976. Sensitivity Analysis for Nonlinear Programming using Penalty Methods. *Mathematical Programming*. **10**, 287-311.

Glasserman, P. 1996. Allocating Production Capacity Among Multiple Products. *Operations Research*. **44:5**, 724-734.

Ha, A. 1997. Optimal Dynamic Scheduling Policy for a Make-to-stock Production System. *Operations Research*. **45**, 42-53

Jedidi, K., C. F. Mela, S. Gupta. 1999. Managing Advertising and Promotion for Long-run Profitability. *Marketing Science*. **18:1**, 1-22.

Kahneman, D., A. Tversky. 1984. Choices, Values and Frames. *American Psychologist* **39**, 341-350.

Li, Q., S. Zheng. 2006. Joint Inventory Replenishment and Pricing Control for Systems with Uncertain Yield and Demand. *Operations Research.* **54:4**, 696-705.

Lus, B., A. Muriel. 2009. Measuring the Impact of Increased Product Substitution on Pricing and Capacity Decisions Under Linear Demand Models. *Production and Operations Management.* **18** 95-113.

Mela, Carl F., S. Gupta, and D. R. Lehmann. 1997. The Long Term Impact of Promotion and Advertising on Consumer Brand Choice. *Journal of Marketing Research.* **34:2**, 248-261.

Mela, C. F., S. Gupta, and K. Jedidi. 1998. Assessing Long Term Promotional Influences on Market Structure. *International Journal of Research in Marketing.* **15:2**, 89-107.

Monroe, K. B. 1973. Buyers' Subjective Perceptions of Price. *Journal of Marketing Research.* **10:1**, 70-80.

Moreno, A., C. Terwiesch. 2010. Pricing and Production Flexibility: An Empirical Analysis of the U.S. Automobile Industry. *Working paper*, The Wharton School, University of Pennsylvania.

Nabben, R., R.S. Varga. 1994. A Linear Algebra Proof that the Inverse of a Strictly Ultrametric Matrix is a Strictly Diagonally Dominant Stieltjes Matrix. *SIAM Journal of Matrix Analysis and Applications.* **15:1**, 107-113.

Rao, A., K. Monroe. 1989. The Effect of Price, Brand Name, and Store Name on Buyers' Perceptions of Product Quality. An Integrative Review. *Journal of Marketing Research.* **26**, 351-57.

Song, J.S., Z. Xue. 2007. Demand Management and Inventory Control for Substitutable Products. *Working Paper*, Duke University.

Tang, C. S., R. Yin, R. 2007. Joint Ordering and Pricing Strategies for Managing Substitutable Products. *Production and Operations Management*, **16** 138-153.

Varga, R.S. 2009. Matrix Iterative Analysis. *Springer Series in Computational Mathematics*, Springer-Verlag Berlin Heidelberg.

Whitin, T. M. 1955. Inventory Control and Price Theory. *Management Science.* **2:1**, 61-68.

Ye, Y. 2008. Joint Pricing and Inventory Decision for Competitive Products. M.S. Thesis, Massachusetts Institute of Technology.

Zhu, K., U. W. Thonemann. 2009. Coordination of Pricing and Inventory Control Across Products. *Naval Research Logistics*, **56:2**, 175-190.

Online Supplement

A. Proofs of Main Results:

Proof of Theorem 1: As described in detail in the proof of Lemma 1 part(b) (provided subsequently in the Proofs of Supplementary Results), for $i=\{1,2\}$, the KKT conditions for the problem may be represented as:

$$\frac{\partial J^t}{\partial p_i^t} = a_{1i}\lambda_1^t + a_{2i}\lambda_2^t - (a_{1i}+a_{2i})\mu^t \tag{9a}$$

$$\frac{\partial J^t}{\partial z_i} = \mu^t - \lambda_i^t \tag{9b}$$

We start by solving the KKT conditions given in (9a) for p_1^t and p_2^t and find $p_i^t = p_{i,L}^t - \frac{1}{2}(\lambda_i^t - \mu^t)$ where we define $p_{i,L}^t$ as the list-price in period t for product i, $i=\{1,2\}$, given by $p_{1,L}^t = \frac{a_{22}b_1 - a_{12}b_2}{2(a_{11}a_{22}-a_{12}^2)} + \frac{c_1}{2}$ and $p_{2,L}^t = \frac{a_{11}b_2 - a_{12}b_1}{2(a_{11}a_{22}-a_{12}^2)} + \frac{c_2}{2}$. Further, (9b) imply implicit functions ϕ_1^t and ϕ_2^t such that $z_1^t = \phi_1^t(\lambda_1^t,\lambda_2^t,\mu^t)$ and $z_2^t = \phi_2^t(\lambda_1^t,\lambda_2^t,\mu^t)$ as stated in Lemma A.1 below.

Lemma A.1. *There exists implicit functions ϕ_1 and ϕ_2 such that $z_1^t = \phi_1(\lambda_1^t,\lambda_2^t,\mu^t)$ and $z_2^t = \phi_2(\lambda_1^t,\lambda_2^t,\mu^t)$. Furthermore, $\phi_1(\lambda_1^t,\lambda_2^t,\mu^t)$ is increasing in λ_1^t, and decreasing in λ_2^t,μ^t whereas $\phi_2(\lambda_1^t,\lambda_2^t,\mu^t)$ is increasing in λ_2^t, and decreasing in λ_1^t,μ^t.*

Proof: Provided in the Proofs of Supplementary Results

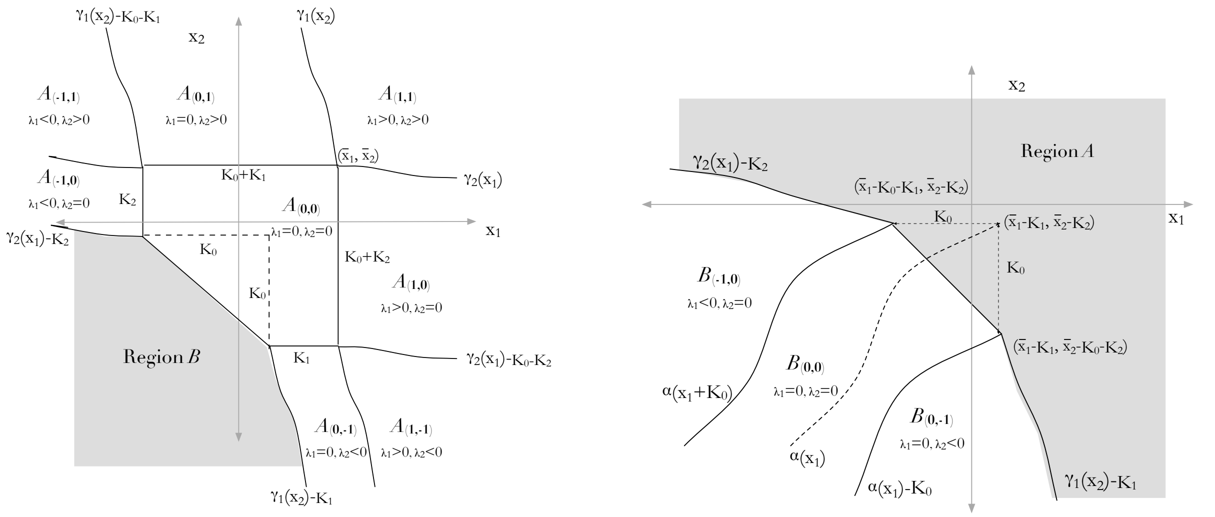
By Lemma A.1, we can rewrite the capacity constraints as follows.

$$x_1^t \le \phi_1^t(\lambda_1^t,\lambda_2^t,\mu^t) - a_{11}p_1^t(\lambda_1^t,\mu^t) - a_{12}p_2^t(\lambda_2^t,\mu^t) + b_1 \le x_1^t + K_0 + K_1 \tag{10a}$$

$$x_2^t \le \phi_2^t(\lambda_1^t,\lambda_2^t,\mu^t) - a_{21}p_1^t(\lambda_1^t,\mu^t) - a_{22}p_2^t(\lambda_2^t,\mu^t) + b_2 \le x_2^t + K_0 + K_2 \tag{10b}$$

$$\begin{aligned}\phi_1^t(\lambda_1^t,\lambda_2^t,\mu^t) + \phi_2^t(\lambda_1^t,\lambda_2^t,\mu^t) - (a_{11}+a_{21})p_1^t(\lambda_1^t,\mu^t)&\\ - (a_{12}+a_{22})p_2^t(\lambda_2^t,\mu^t) + b_1 + b_2 \le x_1^t + x_2^t + K_0 + K_1 + K_2&\end{aligned} \tag{10c}$$

The inventory state space may be partitioned into several regions based on the signs of λ_1^t, λ_2^t, and μ^t. (Formal partition is provided in the subsequent Lemmas A.2 and A.3.) In order to clarify the portrayal of state space segmentation, we define two broad regions, region A and region B (illustrated in Figure VIII), corresponding to initial inventory levels for which $\mu^t=0$ and $\mu^t>0$, respectively. In words, region A represents the initial inventory levels for which there remains some resource, either dedicated or flexible, that is not fully utilized. Region B, on the other hand, corresponds to inventory levels for which all resources are fully utilized. A specific point is of certain interest in our partitioning of the state space. When none of the constraints are binding, we have $\lambda_1^t = \lambda_2^t = \mu^t = 0$. Hence, $(\boldsymbol{\phi}^t(0,0,0),\mathbf{p_L^t})$ is the optimal solution to the unconstrained problem of $\max J^t(\mathbf{z}^t,\mathbf{p}^t)$. If we define $\bar{\mathbf{x}}^t$ such that $\bar{x}_1^t = \phi_1^t(0,0,0) - a_{11}p_{1L}^t - a_{12}p_{2L}^t + b_1$ and



(a)Partitioning of region A (b)Partitioning of region B (enlarged for clarification)

Figure VIII Segmentation of the state space

$\bar{x}_2^t = \phi_2^t(0,0,0) - a_{21}p_{1L}^t - a_{22}p_{2L}^t + b_2$, then $(\bar{\mathbf{x}}^t, \mathbf{p}_L^t)$ is the optimal solution for the unconstrained original problem $\max G^t(\mathbf{y}^t, \mathbf{p}^t)$.

Lemma A.2. *The boundaries of the state space Region A are defined by two monotone functions:*

i. $\gamma_1^t(x_2^t): \Re \to \Re$ *with* $\gamma_1^t(x_2^t) = \bar{x}_1^t$ *for* $x_2^t \in [\bar{x}_2^t - K_0 - K_2,\ \bar{x}_2^t]$ *and* $\gamma_1^t(x_2^t)$ *strictly decreasing with respect to* x_2^t *for* $x_2^t \in \Re \setminus [\bar{x}_2^t - K_0 - K_2,\ \bar{x}_2^t]$

ii. $\gamma_2^t(x_1^t): \Re \to \Re$ *with* $\gamma_2^t(x_1^t) = \bar{x}_2^t$ *for* $x_1^t \in [\bar{x}_1^t - K_0 - K_1,\ \bar{x}_1^t]$ *and* $\gamma_2^t(x_1^t)$ *strictly decreasing with respect to* x_1^t *for* $x_1^t \in \Re \setminus [\bar{x}_1^t - K_0 - K_1,\ \bar{x}_1^t]$

that further partitions Region A into the following eight subregions:

- $A_{(0,0)} := \{(x_1^t, x_2^t) : \bar{x}_i^t - K_0 - K_i \le x_i^t < \bar{x}_i^t \ \forall i = 1,2 \ \textit{and} \ \ x_1^t + x_2^t > \bar{x}_1^t + \bar{x}_2^t - K_0 - K_1 - K_2\}$

- $A_{(0,j)} := \left\{ (x_1^t, x_2^t) : \begin{array}{lll} \gamma_1^t(x_2^t) - K_0 - K_1 \le x_1^t < \gamma_1^t(x_2^t) & \textit{and} \quad\quad x_2^t \ge \bar{x}_2^t & \textit{if} \quad j = 1 \\ \gamma_1^t(x_2^t) - K_1 \le x_1^t < \gamma_1^t(x_2^t) & \textit{and} \ \ x_2^t < \bar{x}_2^t - K_0 - K_2 & \textit{if} \ \ j = -1 \end{array} \right\}$

- $A_{(1,j)} := \left\{ (x_1^t, x_2^t) : \begin{array}{lll} x_1^t \ge \bar{x}_1^t & \textit{and} \ \ \gamma_2^t(x_1^t) - K_0 - K_2 \le x_2^t < \gamma_2^t(x_1^t) & \textit{if} \quad j = 0 \\ x_1^t \ge \gamma_1^t(x_2^t) & \textit{and} \quad\quad x_2^t \ge \gamma_2^t(x_1^t) & \textit{if} \quad j = 1 \\ x_1^t \ge \gamma_1^t(x_2^t) & \textit{and} \quad x_2^t < \gamma_2^t(x_1^t) - K_0 - K_2 & \textit{if} \ \ j = -1 \end{array} \right\}$

- $A_{(-1,j)} := \left\{ (x_1^t, x_2^t) : \begin{array}{lll} x_1^t \le \bar{x}_1^t - K_0 - K_1 & \textit{and} \ \ \gamma_2^t(x_1^t) - K_2 \le x_2^t < \gamma_2^t(x_1^t) & \textit{if} \ \ j = 0 \\ x_1^t \le \gamma_1^t(x_2^t) - K_0 - K_1 & \textit{and} \quad\quad x_2^t \ge \gamma_2^t(x_1^t) & \textit{if} \ \ j = 1 \end{array} \right\}$

Proof: The subscripts of A reflect the sign of the Lagrange variables and imply which, if any, of the constraints are binding. As an example, consider the region defined by $A_{(k_1,k_2)}$. Then, we have the index $k_i = 1$ if $\lambda_i^t > 0$, $k_i = 0$ if $\lambda_i^t = 0$, and $k_i = -1$ if $\lambda_i^t < 0$. For brevity, we only provide the results associated with regions $A_{(0,0)}$, and $A_{(0,1)}$ as the analysis for other regions are similar. We first consider region $A_{(0,0)}$ that corresponds to $\lambda_1^t = \lambda_2^t = 0$. Following (10a) - (10c), in this region we have

$$x_1^t < \phi_1^t(0,0,0) - a_{11}p_{1L}^t - a_{12}p_{2L}^t + b_1 < x_1^t + K_0 + K_1 \tag{11a}$$

$$x_2^t < \phi_2^t(0,0,0) - a_{21}p_{1L}^t - a_{22}p_{2L}^t + b_2 < x_2^t + K_0 + K_2 \tag{11b}$$

$$\phi_1^t(0,0,0) + \phi_2^t(0,0,0) - (a_{11} + a_{21})p_{1L}^t - (a_{12} + a_{22})p_{2L}^t + b_1 + b_2 < x_1^t + x_2^t + K_0 + K_1 + K_2 \tag{11c}$$

Thus, by substituting the expressions for $\bar{x}_1^t$ and $\bar{x}_2^t$ into (11a)-(11c), we can define this region as $\{(x_1^t, x_2^t) : \bar{x}_i^t - K_0 - K_i \le x_i^t < \bar{x}_i^t \ \forall i = 1,2 \text{ and } \ x_1^t + x_2^t > \bar{x}_1^t + \bar{x}_2^t - K_0 - K_1 - K_2\}$. Next, we consider region $A_{(0,1)}$ that corresponds to $\lambda_1^t = 0$ and $\lambda_2^t > 0$. Since $\lambda_2^t > 0$, after substituting in the expressions for p_i^t, $\bar{x}_1^t$ and $\bar{x}_2^t$, the constraints (10a) - (10c) reduce to the following:

$$x_1^t < \phi_1^t(0,\lambda_2^t,0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t < x_1^t + K_0 + K_1 \tag{12a}$$

$$x_2^t = \phi_2^t(0,\lambda_2^t,0) + \bar{x}_2^t - \phi_2^t(0,0,0) + \frac{a_{22}}{2}\lambda_2^t \qquad \text{(equality due to } \lambda_2^t > 0\text{)} \tag{12b}$$

We first consider (12b) which defines one boundary for this region resulting in $x_2^t = \phi_2^t(0,\lambda_2^t,0) + \bar{x}_2^t - \phi_2^t(0,0,0) + \frac{a_{22}}{2}\lambda_2^t > \bar{x}_2^t$ (since $\phi_2^t \uparrow \lambda_2^t$ by Lemma A.1 and $a_{22} > 0$) and $\lim_{\lambda_2^t \to 0} x_2^t = \phi_2^t(0,0,0) + \bar{x}_2^t - \phi_2^t(0,0,0) = \bar{x}_2^t$. Furthermore, as $\phi_2^t \uparrow \lambda_2^t$ by Lemma A.1 and $a_{22} > 0$, x_2^t is strictly increasing with respect to λ_2^t in this region (equivalently, λ_2^t is strictly increasing with respect to x_2^t), there is a one-to-one function defining λ_2^t in terms of x_2^t, that is $\lambda_2^t = \lambda_2^t(x_2^t)$. The remaining boundaries are given by the inequalities in (12a). Since $\lambda_1^t = 0$, the constraints are not binding. We have

$$\phi_1^t(0,\lambda_2^t,0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t - K_0 - K_1 < x_1^t < \phi_1^t(0,\lambda_2^t,0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t \tag{13}$$

Temporarily defining a function $\delta_1(\lambda_2^t) := \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \frac{a_{12}}{2}\lambda_2^t$, we can rewrite (13) as $\delta_1^t(\lambda_2^t) + \bar{x}_1^t - K_0 - K_1 < x_1^t < \delta_1^t(\lambda_2^t) + \bar{x}_1^t$. Lemma A.1 and $a_{12} < 0$ yields $\delta_1^t(\lambda_2^t) < 0$ and that $\delta_1^t(\lambda_2^t)$ is strictly decreasing with respect to λ_2^t. If we now define $\gamma_1^t(x_2^t) := \bar{x}_1^t + \delta_1^t\left(\lambda_2^t(x_2^t)\right)$, we can write the boundaries for this region as $\gamma_1^t(x_2^t) - K_0 - K_1 < x_1^t < \gamma_1^t(x_2^t)$. The fact that $\gamma_1^t(x_2^t)$ strictly decreasing with respect to x_2^t follows immediately from $\delta_1^t(\lambda_2^t)$ strictly decreasing with respect to λ_2^t and λ_2^t strictly increasing with respect to x_2^t. $\square$

Lemma A.3. *Together with $\gamma_1^t(x_2^t)$ and $\gamma_2^t(x_1^t)$, a monotone function $\alpha^t(x_1^t) : [-\infty, \bar{x}_1^t - K_1] \to [-\infty, \bar{x}_2^t - K_2]$ with $\alpha^t(\bar{x}_1^t - K_1) = \bar{x}_2^t - K_2$ and $\alpha^t(x_1^t)$ strictly increasing with respect to x_1^t divides Region B into the three subregions:*

- $B_{(0,-1)} := B'_{(0,-1)} \cup B''_{(0,-1)}$ *where*

  $B'_{(0,-1)} := \{(x_1^t, x_2^t) : \bar{x}_1^t - K_1 < x_1^t \le \gamma_1^t(x_2^t) - K_1 \text{ and } \ x_2^t \le \bar{x}_2^t - K_0 - K_2\}$

$B''_{(0,-1)} := \{(x_1^t, x_2^t) : x_1^t \le \bar{x}_1^t - K_1 \text{ and } x_2^t \le \alpha^t(x_1^t) - K_0\}$

- $B_{(-1,0)} := B'_{(-1,0)} \cup B''_{(-1,0)}$ *where*

  $B'_{(-1,0)} := \{(x_1^t, x_2^t) : x_1^t \le \bar{x}_1^t - K_0 - K_1 \text{ and } \bar{x}_2^t - K_2 \le x_2^t \le \gamma_2^t(x_1^t) - K_2\}$

  $B''_{(-1,0)} := \{(x_1^t, x_2^t) : x_1^t \le \bar{x}_1^t - K_0 - K_1 \text{ and } \alpha^t(x_1^t + K_0) < x_2^t \le \bar{x}_2^t - K_2\}$
- $B_{(0,0)} := \Re^2 \backslash \{A \cup (B_{(0,-1)} \cup B_{(-1,0)})\}$.

Proof: For brevity, we only provide the proof for region $B_{(0,-1)}$ as the analysis of $B_{(-1,0)}$ is similar and region $B_{(0,0)}$ is defined by the remaining area in Region B. Region $B_{(0,-1)}$ corresponds to $\lambda_1^t = 0$, $\lambda_2^t < 0$, and $\mu^t > 0$ for which constraints (10a) - (10c) reduce to

$$x_1^t = \phi_1^t(0, \lambda_2^t, \mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1 \tag{14a}$$

$$x_2^t = \phi_2^t(0, \lambda_2^t, \mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t + \frac{a_{22}}{2}\lambda_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_0 - K_2 \tag{14b}$$

The analysis of this region is simpler if we consider the cases where $x_1^t > \bar{x}_1^t - K_1$ and $x_1^t \le \bar{x}_1^t - K_1$ separately corresponding to subregions $B'_{(0,-1)}$ and $B''_{(0,-1)}$, respectively. For subregion $B'_{(0,-1)}$, we first find the feasible values for x_2^t and then show that $\gamma_1^t(x_2^t) - K_1$ defines the remaining boundary for the possible values for x_1^t. For subregion $B''_{(0,-1)}$, we show that a function $\alpha^t(x_1^t)$ that is defined on the domain $x_1^t \le \bar{x}_1^t - K_1$ establishes the boundary for the subregion. We first show that in the subregion $B'_{(0,-1)}$, we have $x_2^t \le \bar{x}_2^t - K_0 - K_2$. For arbitrary $\lambda_2^t < 0$ and $\mu^t > 0$, by (14b), we have $x_2^t = \phi_2^t(0,\lambda_2^t,\mu^t) - \phi_2^t(0,0,0) + \frac{a_{22}}{2}\lambda_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t + \bar{x}_2^t - K_0 - K_2 < \phi_2^t(0,\lambda_2^t,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - K_0 - K_2 < \bar{x}_2^t - K_0 - K_2$ where the first inequality follows from $\lambda_2^t < 0, \mu^t > 0$, and $a_{22} > 0$, $a_{21} + a_{22} > 0$ and the second inequality follows from $\phi_2^t \uparrow \lambda_2^t, \downarrow \mu^t$ and $\lambda_2^t < 0, \mu^t > 0$. We further have $\lim_{\lambda_2^t,\mu^t \to 0} x_2^t = \phi_2^t(0,0,0) - \phi_2^t(0,0,0) + \bar{x}_2^t - K_0 - K_2 = \bar{x}_2^t - K_0 - K_2$. Next, examining the expression for x_1^t given in (14a), we get $x_1^t = \phi_1^t(0,\lambda_2^t,\mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1 = \phi_1^t(0,\lambda_2^t,\mu^t) - \phi_1^t(0,\lambda_2^t,0) - \frac{(a_{11}+a_{12})}{2}\mu^t + \gamma_1^t(x_2^t) - K_1 < \gamma_1^t(x_2^t) - K_1$ where the inequality is due to $\mu^t > 0, \phi_1^t \downarrow \mu^t$ and $a_{11} + a_{12} > 0$. We also have $\lim_{\mu^t \to 0} x_1^t = \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t - K_1 = \gamma_1^t(x_2^t) - K_1$, the left-hand-side boundary for Region $A_{(0,-1)}$. We note that the increasing property of $\gamma_1^t(x_2^t)$ established in the proof of Lemma A.2 ensures that $\bar{x}_1^t - K_1 \le \gamma_1^t(x_2^t) - K_1$. Thus the expressions $\bar{x}_1^t - K_1 \le x_1^t \le \gamma_1^t(x_2^t) - K_1$ and $x_2^t \le \bar{x}_2^t - K_0 - K_2$ defines the states corresponding to $B'_{(0,-1)}$.

For subregion $B''_{(0,-1)}$, we first note that $\lim_{\lambda_2^t \to 0} x_1^t$ defines the boundary between regions $B''_{(0,-1)}$ and $B_{(0,0)}$. Along this boundary, by (14a), we have $x_1^t = \phi_1^t(0,0,\mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1$. Using Lemma A.1, we find that x_1^t is strictly decreasing with respect to μ^t and hence $x_1^t \le \bar{x}_1^t - K_1$. Further x_1^t strictly decreasing with respect to μ^t implies that there is a one-to-one function along the boundary defining x_1^t and μ^t, i.e., $\mu^t(x_1^t)$ where μ^t is strictly decreasing with respect to

x_1^t. By (14b) along the boundary, we have $x_2^t = \phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_0 - K_2$. Now, consider a point in subregion $B_{(0,0)}$, with $\lambda_1^t = 0$, $\lambda_2^t = 0$, and $\mu^t > 0$ for which the constraints (10a) - (10c) result in $\phi_1^t(0,0,\mu^t) - \phi_1^t(0,0,0) + \bar{x}_1^t - \frac{(a_{11}+a_{12})}{2}\mu^t - K_1 = x_1^t + l$, and $\phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_2 = x_2^t + K_0 - l$ for some $0 < l < K_0$. By Lemma A.1, both $x_1^t + l$ and $x_2^t + K_0 - l$ are strictly decreasing with μ^t. There is a one-to-one function that defines $x_1^t + l$ and $x_2^t + K_0 - l$. Consequently, let $\alpha^t(x_1^t + l) = x_2^t + K_0 - l$. Approaching from a point in $B_{(0,0)}$, $\lim_{l\to 0} x_2^t$ defines the boundary between regions $B_{(0,0)}$ and $B''_{(0,-1)}$ which is equivalent to the previously defined boundary $x_2^t = \phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t - K_0 - K_2$. Hence, the boundary can also be expressed as $x_2^t = \alpha^t(x_1^t) - K_0$, for which $x_2^t < \alpha^t(x_1^t) - K_0$ falls in region $B''_{(0,-1)}$. Let us also temporarily define $\sigma_2(\mu^t) = \phi_2^t(0,0,\mu^t) - \phi_2^t(0,0,0) + \bar{x}_2^t - \frac{(a_{21}+a_{22})}{2}\mu^t$. Then, by Lemma A.1, $\sigma_2(\mu^t)$ is strictly decreasing with respect to μ^t. Consequently, as $\mu^t(x_1^t)$ is strictly decreasing with respect to x_1^t, we have $\sigma_2(\mu^t(x_1^t))$ strictly increasing with respect to x_1^t. Since $\sigma_2(\mu^t(x_1^t)) \uparrow x_1^t$, we have $\alpha(x_1^t) \uparrow x_1^t$. Lastly, by (14a), $x_1^t = \bar{x}_1^t - K_1$ implies $\phi_1^t(0,0,\mu^t) - \phi_1^t(0,0,0) - \frac{(a_{21}+a_{22})}{2}\mu^t = 0$ for which the only solution is $\mu^t = 0$. (Note $\phi_1^t \downarrow \mu^t$ and $\mu^t \geq 0$). Hence by (14b), we have $x_2^t = \bar{x}_2^t - K_0 - K_2$. which yields $\alpha(\bar{x}_1^t - K_1) = \bar{x}_2^t - K_2$. $\square$

To complete the Proof of Theorem 1, we note that part 1(a) follows directly from the definitions of the monotone functions $\gamma_1^t(x_2^t)$ and $\gamma_2^t(x_1^t)$ in Lemma A.2 and the complementary slackness conditions. For example, in region $A_{(-1,1)}$, the binding constraints yield $y_1^t = x_1^t + K_0 + K_1$ and $y_2^t = x_2^t$. In region $A_{(0,1)}$, we have $y_1^t = \gamma_1^t(x_2^t)$ since $y_1^t = z_1^t + \bar{d}_1^t = \phi_1^t(0,\lambda_2^t,0) + b_1 - a_{11}p_1^t - a_{12}p_2^t = \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \phi_1^t(0,0,0) + b_1 - a_{11}p_{1L}^t - a_{12}p_{2L}^t + \frac{a_{12}}{2}\lambda_2^t = \phi_1^t(0,\lambda_2^t,0) - \phi_1^t(0,0,0) + \bar{x}_1^t + \frac{a_{12}}{2}\lambda_2^t = \gamma_1^t(x_2^t)$. For part 1(b), in Regions $A_{(0,j)}$, the optimal order-up-to level for product 1 is independent of its own starting inventory x_1^t and by Lemma A.2 and part (a), it is non-increasing with x_2^t. In Region $A_{(0,0)}$, it is independent of x_2^t and in regions $A_{(0,1)}$ and $A_{(0,-1)}$, it is strictly decreasing with the inventory position of x_2^t. In Regions $A_{(1,j)}$, by part (a), we have $y_1^t = x_1^t$. For regions $A_{(-1,0)}$ and $A_{(-1,1)}$, again by part (a), we have $y_1^t = x_1^t + K_0 + K_1$, thus the order-up-to level of product 1 is increasing with x_1^t and independent of x_2^t. Symmetric arguments hold for product 2.

The proofs of part 2 (a) and (b) are due to Lemma A.3. Suppose $l^t(x_1^t, x_2^t)$ denotes the optimal amount of flexible capacity allocated to product 1. Since in Region B, the complementary slackness conditions imply full utilization of each resource, $K_0 - l^t(x_1^t, x_2^t)$ will be the amount of flexible capacity allocated to product 2. After the allocation of the flexible resource and employing the dedicated resources, the optimal production policy brings inventories of products 1 and 2 to

$x_1^t+K_1+l^t(x_1^t,x_2^t)$ and $x_2^t+K_2+K_0-l^t(x_1^t,x_2^t)$, respectively. Specifically, in region $B_{(-1,0)}$, complementary slackness yields $y_1^t-x_1^t=K_0+K_1$ and $y_2^t-x_2^t=K_2$, thus $l^t(x_1^t,x_2^t)=K_0$. Similarly, in region $B_{(0,-1)}$, complementary slackness conditions give $y_1^t-x_1^t=K_1$ and $y_2^t-x_2^t=K_0+K_2$, hence $l^t(x_1^t,x_2^t)=0$. For region $B_{(0,0)}$, the definition of $l^t(x_1^t,x_2^t)$ yields $y_2^t-x_2^t=K_0+K_2-l^t(x_1^t,x_2^t)$ which leads to $\phi_2^t(0,0,\mu^t)-\phi_2^t(0,0,0)+\bar{x}_2^t-\frac{(a_{21}+a_{22})}{2}\mu^t-x_2^t=K_0+K_2-l^t(x_1^t,x_2^t)$ and therefore the optimal production policy satisfies $x_2^t+K_0-l^t(x_1^t,x_2^t)=\alpha^t(x_1^t+l^t(x_1^t,x_2^t))$. Furthermore, the complementary slackness condition (10c) yields μ^t to be a function of x_1^t and x_2^t only through their sum $x_1^t+x_2^t$. Thus, the optimal modified base stock levels for products 1 and 2 are identical for starting inventory positions for which the total inventory level, $x_1^t+x_2^t$, is identical.

For part 2(c), first let $\alpha'^t(x_1^t+l^t)$ denote the derivative of $\alpha^t(x_1^t+l^t)$ with respect to its argument. By Lemma A.3, α^t is increasing, thus $\alpha'^t(x_1^t+l^t)>0$. Next, differentiating both sides of $l^t(x_1^t,x_2^t)+\alpha^t(x_1^t+l^t(x_1^t,x_2^t))=x_2^t+K_0$ with respect to x_1^t, we get $\frac{\partial l^t}{\partial x_1^t}=-\frac{\alpha'^t(x_1^t+l^t)}{1+\alpha'^t(x_1^t+l^t)}<0$. Thus, l^t is decreasing with respect to x_1^t. Similarly, differentiating both sides of $l^t(x_1^t,x_2^t)+\alpha^t(x_1^t+l^t(x_1^t,x_2^t))=x_2^t+K_0$ with respect to x_2^t, we get $\frac{\partial l^t}{\partial x_2^t}=\frac{1}{1+\alpha'^t(x_1^t+l^t)}>0$. Hence, l^t is increasing with respect to x_2^t. Finally, for part 2(d), the order-up-to level for product 1 is $x_1^t+l^t(x_1^t,x_2^t)+K_1$. Differentiating it with respect to x_1^t and with respect to x_2^t and using the expressions for $\frac{\partial l^t}{\partial x_1^t}$ and $\frac{\partial l^t}{\partial x_2^t}$, we get $\frac{\partial x_1^t+l^t(x_1^t,x_2^t)+K_1}{\partial x_1^t}=\frac{1}{1+\alpha'^t(x_1^t+l^t)}>0$ and $\frac{\partial x_1^t+l^t(x_1^t,x_2^t)+K_1}{\partial x_2^t}=\frac{1}{1+\alpha'^t(x_1^t+l^t)}>0$. The order-up-to level for product 2 is $x_2^t+K_0-l^t(x_1^t,x_2^t)+K_2$. Again, differentiating it both with respect to x_1^t and x_2^t, we get $\frac{\partial x_2^t+K_0-l^t(x_1^t,x_2^t)+K_2}{\partial x_1^t}=\frac{\alpha'^t(x_1^t+l^t)}{1+\alpha'^t(x_1^t+l^t)}>0$ and $\frac{\partial x_2^t+K_0-l^t(x_1^t,x_2^t)+K_2}{\partial x_2^t}=\frac{\alpha'^t(x_1^t+l^t)}{1+\alpha'^t(x_1^t+l^t)}>0$. Hence, the order-up-to level for both products is increasing with respect to either starting inventory position.□

Proof of Theorem 2: The proof follows from the expressions for p_1^t and p_2^t given in the Proof of Theorem 1, i.e. $p_i^t=p_{i,L}^t-\frac{1}{2}(\lambda_i^t-\mu^t)$ where $p_{i,L}^t$ is defined as the list-price in period t for product i, $i=\{1,2\}$, given by $p_{1,L}^t=\frac{a_{22}b_1-a_{12}b_2}{2(a_{11}a_{22}-a_{12}^2)}+\frac{c_1}{2}$ and $p_{2,L}^t=\frac{a_{11}b_2-a_{12}b_1}{2(a_{11}a_{22}-a_{12}^2)}+\frac{c_2}{2}$. For part (a), corresponding to Region A, $\mu^t=0$, therefore using (5) we have $m_1^t(x_1^t,x_2^t)=-\frac{1}{2}\lambda_1^t(x_1^t,x_2^t)$ and $m_2^t(x_1^t,x_2^t)=-\frac{1}{2}\lambda_2^t(x_1^t,x_2^t)$. Thus, we have $m_1^t(x_1^t,x_2^t)>0$ for $\lambda_1^t<0$, $m_1^t(x_1^t,x_2^t)=0$ for $\lambda_1^t=0$, and $m_1^t(x_1^t,x_2^t)<0$ for $\lambda_1^t>0$. Following the state space segmentation set forth in Lemma A.2, $\lambda_1^t<0$, $\lambda_1^t=0$, and $\lambda_1^t>0$ correspond to item 1 being critically understocked, moderately understocked, and overstocked, respectively. Hence, a price surcharge is applied if the item is critically understocked, list price is charged if the item is moderately understocked and a discount is given if the item is overstocked. Similar arguments yield the results corresponding to product 2.

For part (b), corresponding to Region B, we have $m_1^t(x_1^t,x_2^t)=-\frac{1}{2}\left(\lambda_1^t(x_1^t,x_2^t)-\mu^t(x_1^t,x_2^t)\right)$ and $m_2^t(x_1^t,x_2^t)=-\frac{1}{2}\left(\lambda_2^t(x_1^t,x_2^t)-\mu^t(x_1^t,x_2^t)\right)$. Since region B is defined as the states corresponding to $\mu^t>0$ and non-positive λ_1^t and λ_2^t, we have $m_1^t(x_1^t,x_2^t)>0$ and $m_2^t(x_1^t,x_2^t)>0$ indicating markups for both items. In the states that correspond to $B_{(0,0)}$, we have $m_1^t(x_1^t, x_2^t) = \frac{1}{2}\mu^t(x_1^t, x_2^t)$ and $m_2^t(x_1^t, x_2^t) = \frac{1}{2}\mu^t(x_1^t, x_2^t)$. Therefore, $m_1^t(x_1^t, x_2^t) = m_2^t(x_1^t, x_2^t)$. Further, (5) then yields $p_2^t(x_1^t, x_2^t) = p_1^t(x_1^t, x_2^t) + C^t$ where $C^t = p_{2L}^t - p_{1L}^t$. The fact that $m_i^t(x_1^t, x_2^t)$ is a function of x_1^t and x_2^t through their sum follows from (10c) which for this region implies that μ^t is a function of $x_1^t + x_2^t$.

For part (c), we only show the proof for product 1, as similar arguments yield the desired monotonicity results for product 2. In regions $A_{(0,0)}$, $A_{(0,1)}$, and $A_{(0,-1)}$, we have $p_1^t = p_{1L}^t$ and hence p_1^t is independent of both x_1^t and x_2^t. In region $A_{(1,0)}$, we have $p_1^t = p_{1L}^t - \frac{1}{2}\lambda_1^t$. Based on Lemma A.2, in this region λ_1^t increases with x_1^t and is independent of x_2^t, hence p_1^t decreases with x_1^t and is independent of x_2^t. With a similar analysis, we also find that p_1^t decreases with x_1^t and is independent of x_2^t in region $A_{(-1,0)}$ as well. In region $A_{(1,1)}$, we have $x_1^t = \phi_1^t(\lambda_1^t, \lambda_2^t, 0) + \bar{x}_1^t - \phi_1^t(0,0,0) + \frac{a_{11}}{2}\lambda_1^t + \frac{a_{12}}{2}\lambda_2^t$ and $x_2^t = \phi_2^t(\lambda_1^t, \lambda_2^t, 0) + \bar{x}_2^t - \phi_2^t(0,0,0) + \frac{a_{21}}{2}\lambda_1^t + \frac{a_{22}}{2}\lambda_2^t$. By differentiating these two expressions with respect to x_1^t, we find that both λ_1^t and λ_2^t are increasing with respect to x_1^t. Similarly, λ_1^t and λ_2^t are increasing with respect to x_2^t. Since, in this region p_1^t is given by $p_1^t = p_{1L}^t - \frac{1}{2}\lambda_1^t$, p_1^t decreases with respect to both x_1^t and x_2^t. Similar analysis yield λ_1^t to be independent of x_1^t and x_2^t in Regions $B_{(0,-1)}$ and $B_{(0,0)}$ and be increasing with respect to x_1^t and x_2^t in Region $B_{(-1,0)}$. Likewise, we find λ_2^t to be independent of x_1^t and x_2^t in Regions $B_{(-1,0)}$ and $B_{(0,0)}$ and be increasing with respect to x_1^t and x_2^t in Region $B_{(0,-1)}$. Lastly, we find μ^t to be increasing with respect to x_1^t and x_2^t in regions $B_{(-1,0)}, B_{(0,-1)}$ and $B_{(0,0)}$. Therefore, the desired monotonicity results follow immediately form the definitions of p_1^t in these regions. □

Proof of Theorem 3: The result follows from a similar methodology given in the proofs of Theorems 1 and 2 and hence omitted for brevity.

Proof of Theorem 4: We first note that $J^t(\mathbf{z^t}, \mathbf{p^t})$ is strictly concave. This follows from similar arguments as in the proof of Lemma 1(a) and due to the fact that the matrix $\mathbf{A}$ is positive definite, since $\mathbf{A}$ is a symmetric, strictly diagonally dominant matrix with positive diagonal elements. We construct the following KKT conditions:

$$\frac{\partial J^t}{\partial p_i^t} = \sum_{j \in \mathcal{N}} a_{ij}\lambda_j^t \quad \forall i \in \mathcal{N} \tag{15a}$$

$$\frac{\partial J^t}{\partial z_i^t} = -\lambda_i^t \quad \forall i \in \mathcal{N} \tag{15b}$$

where $\lambda_i^t > 0$ implies that product i is overstocked and $z_i^t + b_i - \sum_{j=1:N} a_{ij}p_j = x_i^t$; $\lambda_i^t = 0$ implies that product i is moderately understocked and $x_i^t + K_i > z_i^t + b_i - \sum_{j=1:N} a_{ij}p_j > x_i^t$; and $\lambda_i^t < 0$ implies that product i critically understocked with $z_i^t + b_i - \sum_{j=1:N} a_{ij}p_j = x_i^t + K_i$.

Using (8) and (15a), we can solve for $\mathbf{p}^t$ to get $p_i^t = p_{iL}^t - \lambda_i^t/2$ where $p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b}+\mathbf{c})_i/2$. By (15b), we have $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial \mathbf{z}^t}\right) = -\left(\frac{\partial^2 J^t}{\partial z_i^t z_j^t}\right)$. Since $J(\mathbf{z^t},\mathbf{p^t})$ is strictly concave, $\left(\frac{\partial^2 J^t}{\partial z_i^t z_j^t}\right)$ is invertible. By the implicit function theorem, we can write $\mathbf{z^t}$ as a function of $\boldsymbol{\lambda}^t$ as $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) = -\left(\frac{\partial^2 J^t}{\partial z_i^t z_j^t}\right)^{-1}$.

The below definition and lemmas describe a certain property on the objective function that enables us to characterize the structure of the optimal policy and its monotonicity with respect to starting inventory levels.

Definition 2. A Stieltjes matrix (symmetric M-matrix) is a real symmetric and positive definite matrix $\mathbf{M} = [m_{i,j}]$ in $R^{n,n}$ for which $m_{i,j} \leq 0$ for all $i \neq j$.

Lemma A.4. *(*Nabben and Varga 1994*) The inverse of a Stieltjes matrix is a real nonsingular and symmetric matrix with all of its entries nonnegative.*

Lemma A.5. $-\left(\frac{\partial J^t}{\partial z_i^t \partial z_j^t}\right)^{-1}$ *is a strictly diagonally dominant Stieltjes matrix.*

Proof: Provided in the Proofs of Supplementary Results.

By Lemmas A.4 and A.5, we also have $\frac{\partial \lambda_j^t}{\partial x_i^t} \geq 0 \; \forall i,j \in \mathcal{N}$. To see why, first consider the cases where $\lambda_j^t \neq 0 \; \forall j$. Then, we have $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) = \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1} e_i$. By Lemma A.4, $\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}$ is positive, hence $\frac{\partial \lambda_i^t}{\partial x_j^t} \geq 0$. Now consider the case where $\exists j$ s.t. $\lambda_j^t = 0$. In this case, we have $\frac{\partial \lambda_j^t}{\partial x_i^t} = 0$ and $\frac{\partial \lambda_i^t}{\partial x_j^t} = 0 \; \forall i$ where the second relationship follows from $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right)$ being symmetric. Differentiating the active constraints with respect to x_i^t and solving for $\frac{\partial \lambda_k^t}{\partial x_i^t}$, we get $\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}$ to be of the form $\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t} = \mathbf{S}^{-1} e_i$ where $\mathbf{S}$ is a principle submatrix of $\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)$ and is also a strictly diagonal dominant Stieltjes matrix with a nonnegative inverse. Thus, $\frac{\partial \lambda_k^t}{\partial x_i^t} \geq 0$.

We can now show the structure of the optimal policy. For the optimal production policy described in part (a), consider a product $i \in \mathcal{P}$, i.e. $\lambda_i^t \leq 0$ and it is optimal to order product i. If $\lambda_i^t = 0$, then product i is moderately understocked and we have $x_i^t < z_i^t + b_i - \sum_{j=1:N} a_{ij} p_j < x_i^t + K_i$. The optimal base stock level is given by the expression $y_i^t = z_i^t + b_i - \sum_{k=1:N} a_{ik} p_k^t$ which can be equivalently written as $y_i^t = z_i^t + b_i - \sum_{k=1:N} a_{ik} p_{kL}^t + \sum_{k=1:N} a_{ik} \frac{\lambda_k^t}{2}$. Differentiating with respect to $x_j^t, j \neq i$, we get $\frac{\partial y_i^t}{\partial x_j^t} = \sum_{k=1:N} \frac{\partial z_i^t}{\partial \lambda_k^t} \frac{\partial \lambda_k^t}{\partial x_j^t} + \frac{a_{ik}}{2} \frac{\partial \lambda_k^t}{\partial x_j^t}$. Since $\lambda_i^t = 0$, we have $\frac{\partial y_i^t}{\partial x_j^t} = \sum_{k=1:N, k\neq i} \left(\frac{\partial z_i^t}{\partial \lambda_k^t} + \frac{a_{ik}}{2}\right) \frac{\partial \lambda_k^t}{\partial x_j^t}$. Since the term in the parenthesis is a Stieltjes matrix and $k \neq i$, it is nonpositive. Further the term $\frac{\partial \lambda_k^t}{\partial x_j^t}$ is nonnegative. Therefore, $\frac{\partial y_i^t}{\partial x_j^t} \leq 0$. Now, consider $\lambda_i^t < 0$. Then $y_i^t = x_i^t + K_i$ and hence y_i^t is independent of all x_j^t for $j \neq i$.

For the pricing policy given in part (b), the optimal price to charge for product i in period t was given by $p_i^t = p_{iL}^t - \frac{\lambda_i^t}{2}$. Consider $i \in \mathcal{P}$, i.e. $\lambda_i^t \leq 0$. If $\lambda_i^t = 0$, the product i is moderately understocked and $p_i^t = p_{iL}^t$ where $p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b}+\mathbf{c})_i/2$ as given earlier. If $\lambda_i^t < 0$, then the product is critically understocked and it is optimal to mark up the price the of product i. If the product is overstocked, i.e., for product j, $j \in \mathcal{N}\backslash\mathcal{P}$, then $\lambda_j^t > 0$ and hence, it is optimal to give a discount on product j. Regarding the monotonocity of the optimal price policy, we have $\frac{\partial p_i^t}{\partial x_k^t} = -\frac{1}{2}\frac{\partial \lambda_i^t}{\partial x_k^t} \leq 0$. Therefore, the price of an item is decreasing with respect to the inventory positions of all other items. □

Proof of Theorem 5: As in the proof of Theorem 5, it can be verified that $J^t(\mathbf{z^t}, \mathbf{p^t})$ is strictly concave. A similar construction of KKT conditions yields:

$$\frac{\partial J^t}{\partial p_i^t} = \sum_{n=1,\ldots,N} a_{in}\lambda_n^t \quad \forall i \leq k, i \in \mathcal{N} \tag{16a}$$

$$\frac{\partial J^t}{\partial p_j^t} = \sum_{n=1,\ldots,N} a_{in}(\lambda_n^t - \mu) \quad \forall j > k,\, j \in \mathcal{N} \tag{16b}$$

$$\frac{\partial J^t}{\partial z_i^t} = -\lambda_i^t \quad \forall i \leq k,\, i \in \mathcal{N} \tag{16c}$$

$$\frac{\partial J^t}{\partial z_j^t} = -\lambda_j^t + \mu \quad \forall j > k,\, j \in \mathcal{N} \tag{16d}$$

where for product i, $i \leq k$, $\lambda_i^t > 0$ implies the product is overstocked and $z_i^t + b_i - \sum_{n=1:N} a_{in}p_n = x_i^t$, $\lambda_i^t = 0$ implies the product is moderately understocked and $x_i^t + K_i > z_i^t + b_i - \sum_{n=1:N} a_{in}p_n > x_i^t$; and $\lambda_i^t < 0$ implies that the product is critically understocked with $z_i^t + b_i - \sum_{n=1:N} a_{in}p_n = x_i^t + K_i$. Similarly, for product j, $j > k$, $\lambda_j^t > 0$ implies that the product is overstocked and $z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n = x_j^t$ while $\lambda_j^t = 0$ implies that the product is understocked and $z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n > x_j^t$. Further, $\mu = 0$ implies that the available flexible capacity is not fully utilized, i.e., $\sum_j \left(z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n\right) < \sum_j x_j^t + K_0$ whereas $\mu > 0$ implies the capacity is entirely used with the corresponding active constraint $\sum_j \left(z_j^t + b_j - \sum_{n=1:N} a_{jn}p_n\right) = \sum_j x_j^t + K_0$.

Using (8), (16a) and (16b), we can solve for prices to get $p_i^t = p_{iL}^t - \lambda_i^t/2$ where $p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$ and $p_j^t = p_{jL}^t - \lambda_j^t/2 + \mu^t/2$ where $p_{jL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_j/2$. For part (a), consider product $i \leq k$, $i \in \mathcal{P}$, If $\lambda_i^t = 0$, then $p_i^t = p_{iL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_i/2$. If $\lambda_i^t < 0$, then $p_i^t = p_{iL}^t - \lambda_i^t/2$ implies it is optimal to markup the price of item i. If, on the other hand, $i \leq k$, $i \in \mathcal{N}\backslash\mathcal{P}$, then $\lambda_i^t > 0$ and it is optimal to give a price discount on product i. For part (b), if the flexible capacity is not fully utilized, we have $\mu_t = 0$. Considering a product $j > k$, if $j \in \mathcal{P}$, then $\lambda_j^t = 0$ and $p_j^t = p_{jL}^t = (\mathbf{A}^{-1}\mathbf{b} + \mathbf{c})_j/2$. If, however, $j \in \mathcal{N}\backslash\mathcal{P}$, then $\lambda_j^t > 0$ and $p_j^t = p_{jL}^t - \lambda_j^t$. Hence it is optimal to give a discount to product j. If the flexible capacity is fully utilized, i.e. $\mu^t > 0$. For each product $j > k$, $j \in \mathcal{P}$, we then have $p_j^t = p_{jL}^t + \mu^t/2$. Therefore it is optimal to give the same price surcharge for each item that is produced.

B. Proofs of Supplementary Results:

Proof of Lemma 1: Part (a): The proof is by induction. It can be verified that $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$ is strictly concave due to the assumptions on the demand parameters (i.e., $a_{ii} > 0$ and $a_{ii} > |a_{ij}|$) and that the terms associated with holding and backorder costs are strictly concave in (z_1^1, z_2^1). Since $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$ is formed by the addition of strictly concave and linear functions, itself is strictly concave. Next, note that the capacity constraints result in a convex domain over which the maximization is performed. Since concavity is preserved under maximization in a convex domain, we have $V^1(x_1^1, x_2^1)$ concave. Now, assume that $J^t(z_1^t, z_2^t, p_1^t, p_2^t)$ is strictly concave which yields $V^t(x_1^t, x_2^t)$ to be concave. Then, we have $J^{t+1}(z_1^{t+1}, z_2^{t+1}, p_1^{t+1}, p_2^{t+1})$ strictly concave since it is formed by the addition of a strictly concave term in $(z_1^{t+1}, z_2^{t+1}, p_1^{t+1}, p_2^{t+1})$ and a concave function in (z_1^{t+1}, z_2^{t+1}).

Parts (b) and (c): We first construct the KKT conditions and introduce Lagrange multipliers $\lambda_{ij}^t > 0$ for $i, j = \{1, 2\}$ and $\mu^t > 0$ where $\lambda_{i1}^t > 0$ and $\lambda_{i2}^t > 0$ are associated with constraints $z_i^t + b_i - a_{i1}p_1^t - a_{i2}p_2^t \geq x_i^t$ and $z_i^t + b_i - a_{i1}p_1^t - a_{i2}p_2^t \leq x_i^t + K_0 + K_i$, respectively and μ^t corresponds to the constraint $z_1^t + z_2^t + b_1 + b_2 - (a_{11} + a_{21})p_1^t - (a_{12} + a_{22})p_2^t \leq x_1^t + x_2^t + K_0 + K_1 + K_2$. Together with the complementary slackness conditions, we then have for i={1,2}, $\frac{\partial J^t}{\partial p_i^t} = a_{1i}(\lambda_{11}^t - \lambda_{12}^t) + a_{2i}(\lambda_{21}^t - \lambda_{22}^t) - (a_{1i} + a_{2i})\mu^t$ and $\frac{\partial J^t}{\partial z_i} = \mu^t - (\lambda_{i1}^t - \lambda_{i2}^t)$. Several pairs of constraints form "box constraints" and may not be simultaneously active for positive capacity parameters. As the following observation suggests, we can exploit this special structure of constraints to represent the first-order optimality conditions in simpler notation.

OBSERVATION 1: For $i = 1, 2$, let λ_i^t be defined such that $\lambda_i^t := \lambda_{i1}^t - \lambda_{i2}^t$. Then, λ_i^t uniquely determines λ_{ij}^t for $j = 1, 2$ where (a) $\lambda_i^t < 0$ implies $\lambda_{i1}^t = 0$ and $\lambda_{i2}^t > 0$, (b) $\lambda_i^t > 0$ implies $\lambda_{i1}^t > 0$ and $\lambda_{i2}^t = 0$; and (c) $\lambda_i^t = 0$ implies $\lambda_{i1}^t = \lambda_{i2}^t = 0$. In addition, for $i = \{1, 2\}$, the KKT conditions may be represented as:

$$\frac{\partial J^t}{\partial p_i^t} = a_{1i}\lambda_1^t + a_{2i}\lambda_2^t - (a_{1i} + a_{2i})\mu^t \tag{17a}$$

$$\frac{\partial J^t}{\partial z_i} = \mu^t - \lambda_i^t \tag{17b}$$

Proof: We first observe that having $\lambda_{11}^t > 0$ and $\lambda_{12}^t > 0$ simultaneously, implies that both $z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t - x_1^t = 0$ and $z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t - x_1^t - K_0 - K_1 = 0$. Since this is not possible for any $K_0 + K_1 > 0$ (as there must be some capacity to produce product 1), we conclude that λ_{11}^t and λ_{12}^t cannot be simultaneously positive. Thus, if we define $\lambda_1^t := \lambda_{11}^t - \lambda_{12}^t$, any value of λ_1^t uniquely determines the values of λ_{11}^t and λ_{12}^t. We note that with this definition, λ_1^t is no longer sign restricted. Specifically, we have $\lambda_1^t < 0$ for the case where $\lambda_{11}^t = 0$, $\lambda_{12}^t > 0$, and we have $\lambda_1^t > 0$ for the case where $\lambda_{11}^t > 0$ and $\lambda_{12}^t = 0$. For the case where $\lambda_{11}^t = \lambda_{12}^t = 0$, we have $\lambda_1^t = 0$. An analogous argument holds for λ_{21}^t and λ_{22}^t, hence a corresponding $\lambda_2^t := \lambda_{21}^t - \lambda_{22}^t$ can be similarly defined. $\square$

Following the observation, λ_i^t is no longer sign restricted and is associated with two constraints where its sign - negative, positive or zero - identifies which of the corresponding constraints, if any, is binding. We can now continue with the proof of parts (b) and (c) which are by induction. To simplify the notation, recalling that J^t is separable in (z_1^t, z_2^t) and (p_1^t, p_2^t), we let $J_{ij}^t := \frac{\partial^2 J^t}{\partial z_i^t \partial z_j^t}$. Similarly, we also let $V_{ij}^t = \frac{\partial^2 V^t}{\partial x_i^t \partial x_j^t}$. For $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$, both cross partials are zero, thus $J_{12}^1 = J_{22}^1 = 0$ and part (b) follows. Part (c) results from $J^1(z_1^1, z_2^1, p_1^1, p_2^1)$ being strictly concave. We now assume that the Lemma holds for period t and show that it continues to hold for $t+1$. Due to the strictly concave and separable additional terms on holding and backorder costs, it is sufficient to show that $\mathbb{E}V^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)$ preserves these properties with weak inequalities. It can be verified recursively that the first and second derivatives of $V^t(x_1^t, x_2^t)$ are bounded. Through the interchangeability of differentiation and expectation, it is then sufficient to show that $V^t(x_1^t, x_2^t)$ has the required properties. From Envelope Theorem, we have $\frac{\partial V^t(x_1^t, x_2^t)}{\partial x_1^t} = \frac{\partial J^t}{\partial x_1^t} - \lambda_1^t + \mu^t = c_1 - \lambda_1^t + \mu^t$ and $\frac{\partial V^t(x_1^t, x_2^t)}{\partial x_2^t} = \frac{\partial J^t}{\partial x_2^t} - \lambda_2^t + \mu^t = c_2 - \lambda_2^t + \mu^t$. At this point, it is helpful to partition the state space in two broad regions: Region A where $\mu^t = 0$ and Region B where $\mu^t > 0$.

<u>*Region A*</u>: We first treat the cases associated with $\mu^t = 0$. For these cases, we have $V_{12}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_1^t}{\partial x_2^t}$. From the KKT conditions, we further have $-\frac{\partial \lambda_1^t}{\partial x_2^t} = \frac{\partial}{\partial x_2^t}\left(\frac{\partial J^t}{\partial z_1^t}\right)$ Therefore,

$$V_{12}^t(x_1^t, x_2^t) = \frac{\partial}{\partial x_2^t}\left(\frac{\partial J^t}{\partial z_1^t}\right) = J_{11}^t \frac{\partial z_1^t}{\partial x_2^t} + J_{12}^t \frac{\partial z_2^t}{\partial x_2^t} \tag{18}$$

We implicitly assume $V_{12}^t(x_1^t, x_2^t) = V_{21}^t(x_1^t, x_2^t)$ which requires continuity of the second partial derivatives. This is fulfilled since J^t is strictly concave and twice continuously differentiable in (z_1^t, z_2^t) and (z_1^t, z_2^t) are differentiable in (x_1^t, x_2^t). There are four cases to consider: (1) $\lambda_1^t = 0$ or $\lambda_2^t = 0$, (2) $\lambda_1^t > 0$ and $\lambda_2^t > 0$, (3) $\lambda_1^t > 0$ and $\lambda_2^t < 0$, and (4) $\lambda_1^t < 0$ and $\lambda_2^t > 0$. (Note that the case $\lambda_1^t < 0$ and $\lambda_2^t < 0$ is not feasible as the flexible capacity may not be utilized in full for each product individually.)

Case 1: When $\lambda_1^t = 0$, we have $\frac{\partial \lambda_1^t}{\partial x_2^t} = 0$. Thus $V_{12}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_1^t}{\partial x_2^t} = 0$. A similar argument for $\lambda_2^t = 0$ also yields $V_{12}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_2^t}{\partial x_1^t} = 0$. This establishes the result for part (b), i.e., that $V^t(x_1^t, x_2^t)$ is submodular. For part (c), since $V^t(x_1^t, x_2^t)$ is concave, we have $V_{11}^t(x_1^t, x_2^t) \le 0$, hence $V_{11}^t(x_1^t, x_2^t) \le V_{12}^t(x_1^t, x_2^t)$. The result for $V_{22}^t(x_1^t, x_2^t)$ is similar.

Case 2: When $\lambda_1^t > 0$ and $\lambda_2^t > 0$, from KKT conditions we have $p_1^t = p_{1L}^t + \frac{1}{2}\frac{\partial J^t}{\partial z_1^t}$ and $p_2^t = p_{2L}^t + \frac{1}{2}\frac{\partial J^t}{\partial z_2^t}$ where $p_{1L}^t = \frac{a_{22}b_1 - a_{12}b_2}{2(a_{11}a_{22} - a_{12}^2)} + \frac{c_1}{2}$ and $p_{2L}^t = \frac{a_{11}b_2 - a_{12}b_1}{2(a_{11}a_{22} - a_{12}^2)} + \frac{c_2}{2}$. Complementary slackness yields $x_1^t = z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t$ and $x_2^t = z_2^t + b_2 - a_{21}p_1^t - a_{22}p_2^t$. Combining these we get $x_1^t = z_1^t + b_1 - a_{11}p_{1L}^t - a_{12}p_{2L}^t - \frac{a_{11}}{2}\frac{\partial J^t}{\partial z_1^t} - \frac{a_{12}}{2}\frac{\partial J^t}{\partial z_2^t}$ and $x_2^t = z_2^t + b_2 - a_{21}p_{1L}^t - a_{22}p_{2L}^t - \frac{a_{21}}{2}\frac{\partial J^t}{\partial z_1^t} - \frac{a_{22}}{2}\frac{\partial J^t}{\partial z_2^t}$. Taking partial derivatives with respect to x_2^t and solving for $\frac{\partial z_1^t}{\partial x_2^t}$ and $\frac{\partial z_2^t}{\partial x_2^t}$, we get $\frac{\partial z_1^t}{\partial x_2^t} = \frac{1}{\Lambda}\left(a_{11}J_{12}^t + a_{12}J_{22}^t\right)$ and $\frac{\partial z_2^t}{\partial x_2^t} = \frac{1}{\Lambda}\left(2 - a_{11}J_{11}^t - a_{12}J_{21}^t\right)$ where $\Lambda = 2 - (a_{11}J_{11}^t + 2a_{12}J_{12}^t + a_{22}J_{22}^t) + \frac{1}{2}\left[(a_{11}a_{22} - a_{12}^2)(J_{11}^t J_{22}^t - J_{12}^{t\,2})\right]$. We note that $\Lambda > 0$ by first observing that the terms in the brackets are strictly positive since $a_{11}a_{22} - a_{12}^2 > 0$ by the assumptions on demand parameters and $J_{11}^t J_{22}^t - J_{12}^{t\,2} \ge 0$. We only need to show that $a_{11}J_{11}^t + 2a_{12}J_{12}^t + a_{22}J_{22}^t \le 0$. We have $a_{11}J_{11}^t + 2a_{12}J_{12}^t + a_{22}J_{22}^t \le (a_{11} + 2a_{12} + a_{22})\, J_{12}^t \le 0$ where the first inequality is due to diagonal dominance and the second is due to $a_{11} + a_{12} > 0, a_{12} + a_{22} > 0$ and $J_{12}^t \le 0$. Substituting the expressions for $\frac{\partial z_1^t}{\partial x_2^t}$ and $\frac{\partial z_2^t}{\partial x_2^t}$ into (18) establishes submodularity as follows: $V_{12}^t(x_1^t, x_2^t) = \frac{1}{\Lambda}\left(J_{11}^t\left(a_{11}J_{12}^t + a_{12}J_{22}^t\right) + J_{12}^t\left(2 - a_{11}J_{11}^t - a_{12}J_{21}^t\right)\right) =$

$\frac{1}{\Lambda}\left(a_{12}\left(J_{11}^t J_{22}^t - J_{12}^{t\,2}\right) + 2J_{12}^t\right) \leq 0$ where the inequality is due to $J_{12}^t \leq 0$, $J_{11}^t J_{22}^t - J_{12}^{t\,2} > 0$, and $a_{12} \leq 0$. Part (c) may be shown similarly by evaluating the expressions for $V_{11}^t(x_1^t, x_2^t)$ and $V_{22}^t(x_1^t, x_2^t)$, $\frac{\partial z_1^t}{\partial x_1^t}$ and $\frac{\partial z_2^t}{\partial x_1^t}$. The analysis for Cases 3 and 4 are very similar to the analysis of Case 2 and are omitted for brevity.

Region B: We now consider the region corresponding to $\mu^t > 0$. By the definition of the multipliers and their relationships among each other, this region is subdivided into three subregions such that (1) $\mu^t > 0$, $\lambda_1^t < 0$, $\lambda_2^t = 0$; (2) $\mu^t > 0$, $\lambda_1^t = 0$, $\lambda_2^t = 0$; (3) $\mu^t > 0$, $\lambda_1^t = 0$, $\lambda_2^t < 0$.

Case 1 corresponds to the regions where the flexible capacity is used solely and fully to produce item 1. Once again, the Envelope Theorem yields $V_{12}^t(x_1^t, x_2^t) = \frac{\partial}{\partial x_2^t}\left(\frac{\partial J^t}{\partial z_1^t}\right) = J_{11}^t \frac{\partial z_1^t}{\partial x_2^t} + J_{12}^t \frac{\partial z_2^t}{\partial x_2^t}$. Complementary slackness conditions yield $x_1^t = z_1^t + b_1 - a_{11}p_1^t - a_{12}p_2^t - K_0 - K_1$ and $x_2^t = z_2^t + b_2 - a_{21}p_1^t - a_{22}p_2^t - K_2$. Combining these, we get $x_1^t = z_1^t + b_1 - a_{11}p_{1L}^t - a_{12}p_{2L}^t - K_0 - K_1 - \frac{a_{11}}{2}\frac{\partial J^t}{\partial z_1^t} - \frac{a_{12}}{2}\frac{\partial J^t}{\partial z_2^t}$ and $x_2^t = z_2^t + b_2 - a_{21}p_{1L}^t - a_{22}p_{2L}^t - K_2 - \frac{a_{21}}{2}\frac{\partial J^t}{\partial z_1^t} - \frac{a_{22}}{2}\frac{\partial J^t}{\partial z_2^t}$. The same arguments as presented in the analysis of the previous case yields the desired result. Further, the analysis for Case 3 is also symmetric to the analysis of Case 1 and hence omitted.

Case 2 defines the only remaining region and it corresponds to $\mu^t > 0$, $\lambda_1^t = 0$, $\lambda_2^t = 0$, where the flexible capacity is used fully to produce both products simultaneously. In this region we have $x_1^t + x_2^t = z_1^t + z_2^t + b_1 + b_2 - (a_{11} + a_{21})p_1^t - (a_{12} + a_{22})p_2^t - K_0 - K_1 - K_2$. Differentiating with respect to x_2^t, we get:

$$1 = \left(1 - \frac{(a_{11}+a_{21})}{2}J_{11}^t - \frac{(a_{12}+a_{22})}{2}J_{21}^t\right)\frac{\partial z_1^t}{\partial x_2^t} + \left(1 - \frac{(a_{11}+a_{21})}{2}J_{12}^t - \frac{(a_{12}+a_{22})}{2}J_{22}^t\right)\frac{\partial z_2^t}{\partial x_2^t} \tag{19}$$

Through the KKT conditions, in this region we also have $J_1^t = J_2^t$, hence differentiating with respect to x_2^t we get

$$J_{11}^t \frac{\partial z_1^t}{\partial x_2^t} + J_{12}^t \frac{\partial z_2^t}{\partial x_2^t} = J_{21}^t \frac{\partial z_1^t}{\partial x_2^t} + J_{22}^t \frac{\partial z_2^t}{\partial x_2^t} \tag{20}$$

Combining (19) and (20), we get $\frac{\partial z_1^t}{\partial x_2^t} = \frac{1}{\Lambda'}(J_{12}^t - J_{22}^t)$ and $\frac{\partial z_2^t}{\partial x_2^t} = \frac{1}{\Lambda'}(J_{12}^t - J_{11}^t)$. where $\Lambda' = -J_{12}^t + 2J_{12}^t - J_{12}^t + (a_{11} + a_{12})(J_{11}^t J_{22}^t - J_{12}^{t\,2})/2 + (a_{22} + a_{12})(J_{11}^t J_{22}^t - J_{12}^{t\,2})/2$. As in the previous discussion for Λ, it can easily be shown that $\Lambda' > 0$. Substituting $\frac{\partial z_1^t}{\partial x_2^t}$ and $\frac{\partial z_2^t}{\partial x_2^t}$ into (18), we get $V_{12}^t(x_1^t, x_2^t) = \frac{1}{\Lambda}\left(J_{11}^t\left(J_{12}^t - J_{22}^t\right) + J_{12}^t\left(J_{12}^t - J_{11}^t\right)\right) = \frac{1}{\Lambda}\left(J_{12}^{t\,2} - J_{11}^t J_{22}^t\right) \leq 0$. Similar steps verify part (c). □

Proof of Lemma A.1: We first introduce two functions $F_1(\mathbf{L}^t, \mathbf{z}^t)$ and $F_2(\mathbf{L}^t, \mathbf{z}^t)$ where $\mathbf{L}^t = (\lambda_1^t, \lambda_2^t, \mu^t)$ and $\mathbf{z}^t = (z_1^t, z_2^t)$. We define these functions to represent KKT conditions (17b).

$$F_1(\mathbf{L}^t, \mathbf{z}^t) = J_1^t(z_1^t, z_2^t) + \lambda_1^t - \mu^t \tag{21a}$$

$$F_2(\mathbf{L}^t, \mathbf{z}^t) = J_2^t(z_1^t, z_2^t) + \lambda_2^t - \mu^t \tag{21b}$$

Differentiating (21a) and (21b), and letting $D_{\mathbf{L}}F$ and $D_{\mathbf{z}}F$ to denote partial Jacobians, we have

$$D_{\mathbf{L}}F = \begin{bmatrix} \frac{\partial F_1}{\partial \lambda_1} & \frac{\partial F_1}{\partial \lambda_2} & \frac{\partial F_1}{\partial \mu} \\ \frac{\partial F_2}{\partial \lambda_1} & \frac{\partial F_2}{\partial \lambda_2} & \frac{\partial F_2}{\partial \mu} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad D_{\mathbf{z}}F = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} \\ \frac{\partial F_2}{\partial z_2} & \frac{\partial F_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} J_{11}^t & J_{12}^t \\ J_{21}^t & J_{22}^t \end{bmatrix}$$

Since $J^t(z_1^t, z_2^t)$ is strictly concave by Lemma 1, $D_{\mathbf{z}}F$ is invertible. Thus, there exists implicit functions ϕ_1 and ϕ_2 such that $z_1^t = \phi_1(\lambda_1^t, \lambda_2^t, \mu^t)$ and $z_2^t = \phi_2(\lambda_1^t, \lambda_2^t, \mu^t)$. Moreover, by the Implicit Function Theorem, we have $D\phi = -D_{\mathbf{z}}F^{-1}D_{\mathbf{L}}F$, that is,

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial \lambda_1} & \frac{\partial \phi_1}{\partial \lambda_2} & \frac{\partial \phi_1}{\partial \mu} \\ \frac{\partial \phi_2}{\partial \lambda_1} & \frac{\partial \phi_2}{\partial \lambda_2} & \frac{\partial \phi_2}{\partial \mu} \end{bmatrix} = -\begin{bmatrix} J_{11}^t & J_{12}^t \\ J_{21}^t & J_{22}^t \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{J_{11}^t J_{22}^t - (J_{12}^t)^2} \begin{bmatrix} -J_{22}^t & J_{12}^t & J_{22}^t - J_{12}^t \\ J_{12}^t & -J_{11}^t & J_{11}^t - J_{12}^t \end{bmatrix}$$

The strict concavity established in Lemma 1 yields $J_{11}^t < 0, J_{22}^t < 0$, and $J_{11}^t J_{22}^t - (J_{12}^t)^2 > 0$. The submodularity and diagonal dominance properties in Lemma 1, gives $J_{12}^t \le 0, J_{11}^t - J_{12}^t < 0, J_{22}^t - J_{12}^t < 0$. Therefore, the monotonicity results follow immediately. □

Proof of Lemma A.5: The proof is by induction. For t=1, $-\left(\frac{\partial J^t}{\partial z_i^t \partial z_j^t}\right)^{-1}$ is a diagonal matrix with positive diagonal elements (due to the strict concavity of expected holding and shortage costs) and hence is a strictly diagonally dominant Stieltjes matrix. Now assume that the result holds for period t. By Envelope Theorem, we have $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right) = -\left(\frac{\partial \lambda_i^t}{\partial x_j^t}\right)$. Several cases arise since each λ_i^t may be greater than, less than, or equal to zero.

We first consider the case where $\lambda_i^t \neq 0 \; \forall i \in \mathcal{N}$. Similar to the steps in the proof of Lemma 1, differentiating all active constraints with respect to each x_i^t, we get $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) \cdot \left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) + \frac{\mathbf{A}}{2} \cdot \left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) = e_i$. Hence, we have $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial x_i^t}\right) = \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1} e_i$, where it can be verified by inspection that both $\mathbf{A}$ and the sum $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}$ are also strictly diagonally dominant Stieltjes matrices, thus invertible by Lemma 4. We therefore have $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right) = -\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}$. If we let $\mathbf{D} := diag(h_1'', h_2'', ..., h_N'')$ denote a diagonal matrix with positive diagonal elements which refers to the partial derivatives of the expected holding and shortage costs, then, $\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right) = -\mathbf{D} - \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}$. We note that if a matrix $\mathbf{M}$ is a strictly diagonally dominant Stieltjes matrix, and $\mathbf{D}$ is a positive diagonal matrix, then $(\mathbf{D} + \mathbf{M}^{-1})^{-1}$ is a strictly diagonally dominant Stieltjes matrix. (The proof of this claim can be found in Lemma A.3 in Ye (2008) where their definition of competitive matrix refers to a diagonally dominant Stieltjes matrix and their analysis to show that diagonal dominance holds is extendable to show that strict diagonal dominance holds as well.) Therefore, $-\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right)^{-1} = \left(\mathbf{D} + \left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)^{-1}\right)^{-1}$ is a strictly diagonally dominant Stieltjes matrix.

Consider now a case for which there exists some $\lambda_k^t = 0,\; k \in \mathcal{N}$. For representation purposes and without loss of generality, assume $\lambda_N^t = 0$. Then, differentiating the active constraints for $x_i^t,\; i \neq N$ and noting that $\frac{\partial \lambda_N^t}{\partial x_i^t} = \frac{\partial \lambda_i^t}{\partial x_N^t} = 0$, we find $\left(\frac{\partial V^t}{\partial x_i^t \partial x_j^t}\right) = \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$ where $\mathbf{S}$ is a principle submatrix of $\left(\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) + \frac{\mathbf{A}}{2}\right)$ of dimension $N-1$ x $N-1$ and is also a strictly diagonal dominant Stieltjes matrix (Varga 2009). Then, $-\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right)^{-1} = \left(\mathbf{D} + \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}\right)^{-1} = \left(\mathbf{D}' + \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \delta_N \end{bmatrix}\right)^{-1} = \left(\mathbf{D}' + \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\delta_N} \end{bmatrix}^{-1}\right)^{-1}$ where $\mathbf{D}' = diag(h_1'', h_2'', ..., h_N'' - \delta_N)$ for $\delta_N > 0$ sufficiently small such that $h_N'' - \delta_N > 0$. It can be verified that $\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\delta_N} \end{bmatrix}$ is also a Stieltjes matrix, hence by the same argument as in the previous case, we have $-\left(\frac{\partial J^{t+1}}{\partial z_i^{t+1} \partial z_j^{t+1}}\right)^{-1}$ a strictly diagonally dominant Stieltjes matrix. The analysis for other cases where more than one $\lambda_k^t = 0$ are similar. □