CITY Cass Business School
City of London

Faculty of Actuarial Science and Statistics

Efficient Asset Valuation Methods for Pension Plans.

M. Iqbal Owadally

Actuarial Research Paper No. 148

March 2003

Sir John Cass Business School, 106 Bunhill Row, London EC1Y 8TZ

Telephone +44 (0) 20 7040 8470
www.cass.city.ac.uk

ISBN 1-901615-70-7

Efficient Asset Valuation Methods for Pension Plans

M. Iqbal Owadally*

Cass Business School, City University, London

Abstract

"Smoothed asset value" methods are used by actuaries, when they value pension plan assets, in order to stabilize the contribution rates recommended to plan sponsors. Methods with exponential and arithmetic smoothing are considered within a simple funding model where only asset gains and losses, smoothed exclusively by the asset valuation method, are permitted. It is shown mathematically that (1) excessive smoothing is counterproductive as it results in less stable contribution rates, (2) exponential smoothing is more efficient than arithmetic smoothing in terms of minimizing the volatility of contribution rates as well as that of funding levels. Suitable smoothing parameter values are discussed.

JEL classification: D81; G23

Insurance and Subject Branch Code: B81

Keywords: actuarial asset valuation; smoothing; pension funding

*Faculty of Actuarial Science and Statistics, Cass Business School, City University, Northampton Square, London EC1V 0HB, England. Phone: +44 (0)20 7040 8478. Fax: +44 (0)20 7040 8838. E-mail: iqbal@city.ac.uk.

1 Introduction

Actuarial practitioners employ special "asset valuation methods" when valuing defined benefit pension plans. Asset values are smoothed or averaged over time to remove excessive volatility which is not reflected in liability values. A comprehensive survey of actuarial practice in this respect was undertaken by the Committee on Retirement Systems Research (2001) at the Society of Actuaries.

Actuaries normally assume that longer averaging periods smooth market fluctuations and reduce the volatility of contribution rates. They also assume that longer averaging periods make funding levels more volatile as gains and losses are deferred over longer periods. In this paper, we investigate this by extending the work of Dufresne (1988, 1989), who defines a criterion of efficiency in terms of a tradeoff between the volatility of contribution rates and the volatility of funding levels.

The paper is structured as follows. Smoothed asset values are incorporated in the model of Dufresne (1988, 1989) in section 2. A direct correspondence with the results of Dufresne (1988, 1989) allows us to state the moments of contribution rates and funding levels in section 3. The effect of the choice of the averaging period or smoothing parameter on the volatility of funding levels and contribution rates is investigated in section 4 and efficient choices are discussed. Finally, it is shown in section 5 that exponential smoothing is preferable to arithmetic averaging.

A list of important symbols is included here for ease of reference:

AL	actuarial liability
AV_t	actuarial (smoothed) value of pension plan assets at time t
β_j	$u^j(n-1-j)/n$ for $j \in \{0, n-1\}$
B	benefit paid every year
C_t	pension contribution paid every year

F_t	market value of pension plan assets at time t
i	technical valuation rate of interest equal to $\mathrm{E}r_t$
k	exponential smoothing parameter
λ_j	$u^j(n-j)/n$ for $j \in \{0, n-1\}$
L_t	intervaluation loss
n	arithmetic averaging period
NC	normal cost or normal contribution rate
π_j	u^j/n for $j \in \{0, n-1\}$
r_t	rate of return on assets in year $(t-1,\, t)$
σ^2	$\mathrm{Var}(r_t)$
u	$1+i$
UL_t	unfunded liability $= AL - F_t$
v	$1/(1+i)$

2 Model

A simple model of a defined benefit pension plan is used here to analyze the effects of smoothing asset values. For details of the model, refer to Dufresne (1988, 1989). Its essential features are listed here: (1) The only source of uncertainty is in asset returns. That is, only asset gains and losses occur. (2) The rate of return is independent and identically distributed each year. (3) Demographic factors are projected in accordance with fixed actuarial valuation assumptions. (4) The pension plan population is stationary. (5) The pension benefit upon retirement is fully indexed with wage inflation.

As a consequence of the last feature, only asset returns net of wage inflation need be considered. Indeed all variables may be deflated by wage inflation. The random rate of return on the pension fund, net of wage inflation, is denoted r_t. The benefit B paid out yearly, the actuarial liability AL calculated at yearly actuarial valuations of the pension plan, the normal cost NC under the actuarial cost method employed, and the total payroll are all constant (all deflated by wage inflation). AL and NC are evaluated at a technical valuation interest rate i which is assumed to be equal to the expected rate of return on assets.

The model is an abstraction of reality but it enables us to obtain closed-form expressions for the variance of contribution rates and funding levels [see Dufresne (1988, 1989)]. It captures the variability of asset returns and hence allows us to study the effects of using asset valuation methods.

The market value of pension plan assets at time t is denoted by F_t. At the start of year $(t-1,\, t)$ contributions C_{t-1} are paid in and benefits B are paid out and, at a rate of return of r_t, the following recurrence relation applies (for $t \geq 1$):

$$F_t = (1 + r_t)[F_{t-1} + C_{t-1} - B]. \tag{1}$$

Pension plan liability is constant so that the following equation of equilibrium holds (Trowbridge, 1952):

$$AL = (1 + i)(AL + NC - B). \tag{2}$$

The unfunded liability in the plan at time t is the excess of liability over assets:

$$UL_t = AL - F_t. \tag{3}$$

The intervaluation loss L_t in year $(t-1,\, t)$ is defined as the difference between the unfunded liability at time t and the anticipated unfunded liability had valuation assumptions been realized during year $(t-1,\, t)$. In the model of Dufresne (1988, 1989), experience differs from actuarial valuation assumptions only in the rate of return on assets. Hence, the unfunded liability at time t as anticipated at time $t-1$ is $AL - (1+i)[F_{t-1} + C_{t-1} - B]$, whereas the actual unfunded liability at time t is given in equation (3). For $t \geq 1$,

$$L_t = (1+i)(F_{t-1} + C_{t-1} - B)] - F_t \tag{4}$$

$$= (i - r_t)[F_{t-1} + C_{t-1} - B], \tag{5}$$

where the last equality follows from equation (1).

For the purposes of funding, that is when recommending a contribution rate, an actuarial asset value AV_t is used. An actuarial deficit of $AL - AV_t$ is evaluated and the contribution recommended by the actuary should make good this deficit. The contribution is equal to the normal cost NC (associated with the actuarial liability AL under a given actuarial cost method) plus a supplementary contribution.

In practice, the supplementary contribution may be a fraction of the unfunded liability (Dufresne, 1988) or may involve the amortization of intervaluation gains and losses (Dufresne, 1989). It is assumed here that the deficit $AL - AV_t$ is paid off immediately and that the supplementary contribution equals the deficit. The contribution paid at time t is therefore:

$$C_t = NC + (AL - AV_t). \tag{6}$$

It is also assumed that any initial unfunded liability (in respect of amendments to plan benefits, for example) is paid off by means of a separate schedule of contribution payments.

As in Dufresne (1989), we set up two equations as follows. The intervaluation loss may be rewritten by using equation (2) and by replacing UL_{t-1} from equation (3) and C_{t-1} from equation (6) into, respectively, equations (4) and (5), yielding

$$UL_t - u\, UL_{t-1} = L_t - u[AL - AV_{t-1}], \tag{7}$$

$$L_t = (r_t - i)[UL_{t-1} + AV_{t-1} - (1+v)AL]. \tag{8}$$

In the above, $t \geq 1$, $u = 1 + i$ and $v = (1+i)^{-1}$.

3 Asset Valuation Methods

Two methods of calculating the actuarial smoothed asset value AV_t are commonly used for funding purposes. For more details about these methods, refer to the survey of the Committee on Retirement Systems Research (2001).

3.1 Exponential Smoothing

The actuarial asset value AV_t at time t is a weighted average of the market value F_t of the fund at time t and the actuarial value of the fund at time t as anticipated at time $t-1$ given the valuation assumptions at time $t-1$:

$$AV_t = kF_t + (1-k)u(AV_{t-1} + C_{t-1} - B), \tag{9}$$

where k is a smoothing parameter and $k \in \mathbb{R}$: $1 - v < k \leq 1$. A smaller value of k means that more weight is placed on the past market values and more smoothing is applied. AV_t is an infinite exponentially weighted average, allowing for interest and cash flows:

$$AV_t = \sum_{j=0}^{\infty} k[u(1-k)]^j F_{t-j} + \sum_{j=1}^{\infty} [u(1-k)]^j [C_{t-j} - B]. \tag{10}$$

Replace C_t from equation (6) into equation (9), and use equation (2) to simplify:

$$AV_t = kF_t + (1-k)AL. \tag{11}$$

Under exponential smoothing, the contribution recommended is (equation (6)):

$$C_t = NC + k(AL - F_t). \tag{12}$$

Equations (1), (2) and (12) are exactly as in Dufresne (1988). In that paper, no special asset valuation method was used but the contribution involved a fraction of $AL - F_t$, as in equation (12). Asset gains and losses were smoothed implicitly through the contribution in Dufresne (1988) whereas here asset gains and losses are smoothed through the asset valuation method only.

The algebraic identity between the work of Dufresne (1988) and actuarial asset valuation by exponential smoothing means that the results of Dufresne (1988) apply here. In particular, if the rates of return $\{r_t\}$ are independent and identically distributed with mean i and variance σ^2 and provided $(u^2+\sigma^2)(1-k)^2 < 1$,

$$\lim_{t\to\infty} \mathrm{Var}F_t = \sigma^2 v^2 AL^2/[1-(u^2+\sigma^2)(1-k)^2], \tag{13}$$

$$\lim_{t\to\infty} \mathrm{Var}C_t = k^2 \lim_{t\to\infty} \mathrm{Var}F_t. \tag{14}$$

We also note from equation (6) or equation (11) that

$$\lim_{t\to\infty} \mathrm{Var}AV_t = \lim_{t\to\infty} \mathrm{Var}C_t = k^2 \lim_{t\to\infty} \mathrm{Var}F_t. \tag{15}$$

3.2 Arithmetic Smoothing

An alternative method is to define the actuarial asset value AV_t as an arithmetic average of the market values of assets over the past n years, with an explicit adjustment for cash flows and interest:

$$AV_t = \frac{1}{n}\left\{F_t + \sum_{j=1}^{n-1}\left[u^j F_{t-j} + \sum_{k=1}^{j} u^k [C_{t-k} - B]\right]\right\}, \tag{16}$$

where $n \in \mathbb{N} : n \geq 2$. The term in square brackets in equation (16) above represents a written-up value of the fund at time t based on the value at time $t-j$ allowing both for interest and intermediate cash flows in the fund. By definition, $AV_t = F_t$ for $n = 1$.

Since $u(C_t - B) = F_{t+1} + L_{t+1} - uF_t$, from equation (4), it follows that

$$u^j F_{t-j} + \sum_{k=1}^{j} u^k [C_{t-k} - B] = F_t + \sum_{k=0}^{j-1} u^k L_{t-k}, \tag{17}$$

and that

$$AV_t = F_t + \frac{1}{n}\sum_{j=1}^{n-1}\sum_{k=1}^{j-1} u^k L_{t-k} = F_t + \sum_{j=0}^{n-2} \frac{n-j-1}{n} u^j L_{t-j}. \tag{18}$$

Using equation (3), the above may be rewritten as

$$AL - AV_t = UL_t - \sum_{j=0}^{n-1} \frac{n-j-1}{n} u^j L_{t-j}. \tag{19}$$

For $n = 1$, $AV_t = F_t = AL - UL_t$ by definition and the sums on the right hand sides of equations (18) and (19) do not appear.

Equations (7) and (19) give an equation for the unfunded liability UL_t in terms of the intervaluation losses only:

$$UL_t = \sum_{j=0}^{n-1} \lambda_j L_{t-j} \tag{20}$$

where

$$\lambda_j = u^j(n-j)/n, \qquad j \in [0, n-1]. \tag{21}$$

A similar equation is given in Dufresne (1989). In that paper no special asset valuation method was used but the supplementary contribution was calculated so that gains and losses were amortized over a period m. In Dufresne (1989), equation (20) holds with the exception that $\lambda_j = a_{\overline{m-j}|}/\ddot{a}_{\overline{m}|}$.

Substituting equation (20) into equation (19) expresses the actuarial value of assets in terms of the intervaluation losses:

$$AV_t = AL - \sum_{j=0}^{n-1} \pi_j L_{t-j} \tag{22}$$

where

$$\pi_j = u^j/n, \qquad j \in [0, n-1]. \tag{23}$$

From equation (6),

$$C_t = NC + \sum_{j=0}^{n-1} \pi_j L_{t-j}. \tag{24}$$

Equation (24) has an analogue in Dufresne (1989), wherein $\pi_j = 1/\ddot{a}_{\overline{m}|}$.

Finally, a recurrence relation for the intervaluation losses may be obtained by substituting equation (19) into equation (8) to give

$$L_{t+1} = (r_{t+1} - i)\left[\sum_{j=0}^{n-1} \beta_j L_{t-j} - vAL\right], \tag{25}$$

where

$$\beta_j = u^j(n-j-1)/n, \qquad j \in [0, n-1]. \tag{26}$$

Equation (25) also holds in Dufresne (1989) except that $\beta_j = a_{\overline{m-j-1|}}/\ddot{a}_{\overline{m|}}$.

By direct correspondence with the results of Dufresne (1989), the following may be stated regarding the second moments. If the rates of return $\{r_t\}$ are independent and identically distributed with mean i and variance σ^2 and provided that $1 - \sigma^2 \sum \beta_j^2 < 1$,

$$\lim_{t\to\infty} \mathrm{Var} F_t = \sigma^2 v^2 AL^2 {\textstyle\sum} \lambda_j^2 \,/\left[1 - \sigma^2 {\textstyle\sum} \beta_j^2\right], \tag{27}$$

$$\lim_{t\to\infty} \mathrm{Var} C_t = \sigma^2 v^2 AL^2 {\textstyle\sum} \pi_j^2 \,/\left[1 - \sigma^2 {\textstyle\sum} \beta_j^2\right]. \tag{28}$$

From equation (6), it is also clear that

$$\lim_{t\to\infty} \mathrm{Var} AV_t = \lim_{t\to\infty} \mathrm{Var} C_t. \tag{29}$$

When neither arithmetic smoothing is used (put $n = 1$ in equations (27)–(29)) nor exponential smoothing is used (put $k = 1$ in equations (13)–(15)),

$$\lim \mathrm{Var} F_t = \lim \mathrm{Var} C_t = \lim \mathrm{Var} AV_t = \sigma^2 v^2 AL^2. \tag{30}$$

3.3 Relationship between Asset Valuation and Supplementary Contribution

Dufresne (1988, 1989) explores practical actuarial funding methods concerning the determination of supplementary contributions to pay off intervaluation gains and losses over time. Market values of assets are assumed. In this paper, smoothed market values are explored but the deficit or surplus emerging during a valuation is liquidated immediately.

In both cases, gains and losses are deferred. This explains the similarity between supplementary contribution funding methods, as in Dufresne (1988, 1989), and asset valuation methods as described in this paper.

Asset valuation and the calculation of supplementary contribution are nevertheless distinct. The incidence of payments for gains and losses is different when the actuarial asset value involves arithmetic smoothing compared to when gain/loss amortization (Dufresne, 1989) is used. There is no algebraic identity between the results in section 3.2 and those of Dufresne (1989). Furthermore, only asset gains and losses are modelled here. Gains and losses also arise in practice when demographic and economic experience diverges from actuarial valuation assumptions. They are amortized through the supplementary contribution payment and are not smoothed by any asset valuation method.

Further work is required to explore the relationship between funding and asset valuation methods. In this paper, only asset valuation is investigated.

4 Effect of Smoothing Asset Values on Pension Funding

The choice of the smoothing parameter k or averaging period n on the pension funding process is investigated in this section. To this end, define

$$A(k) = 1 - (u^2 + \sigma^2)(1-k)^2, \tag{31}$$

$$B_n = \left[1 - \sigma^2 \textstyle\sum \beta_j^2\right] / \textstyle\sum \lambda_j^2, \tag{32}$$

$$P(k) = [1 - (u^2 + \sigma^2)(1-k)^2]/k^2, \tag{33}$$

$$Q_n = \left[1 - \sigma^2 \textstyle\sum \beta_j^2\right] / \textstyle\sum \pi_j^2, \tag{34}$$

which are proportional to the reciprocals of the variances in equations (13), (27), (14) and (28) respectively.

4.1 Volatility of Funding Level

The funding level or funded ratio of a pension plan is defined as the ratio of assets to liabilities. It provides a measure of the security of pension benefits (McGill *et al.* , 1996, p. 592). One of the desirable properties of a funding method is that the funding level is stable. The variance of the funding level is proportional to the variance of F_t as actuarial liability is constant under the simple model described in section 2.

If smoothed asset values are used, intervaluation gains and losses will not be recognised immediately but will instead be deferred. Losses accumulate and the consequent unfunded liability may not be paid off fast enough. It is sensible that the more asset values are smoothed, the more variable the funding level is. The following proposition makes this concrete.

PROPOSITION 1 *As the amount of smoothing in actuarial asset values increases, the variance of the funding level increases. That is,*

1. $\lim \mathrm{Var} F_t$ *increases as* k *decreases in equation* (13) *(exponential smoothing),*

2. $\lim \mathrm{Var} F_t$ *increases as* n *increases in equation* (27) *(arithmetic smoothing).*

The proof for part 1 of Proposition 1 concerning exponential smoothing is identical to the one given in Dufresne's (1988) Proposition 2 because of the algebraic identity discussed in section 3.1. The proof for part 2 of Proposition 1 concerning arithmetic smoothing is given in the Appendix.

4.2 Volatility of Contribution Rate and Actuarial Asset Value

The contribution rate is the contribution paid as a proportion of total payroll. Since payroll is constant in this model (section 2), the variance of the contribution rate is equal to the variance of C_t.

It is usually assumed that smoother actuarial asset values lead to more stable contribution rates. Stable contribution rates are desirable to plan sponsors since one of the objectives of funding pension benefits in advance is to spread costs (Trowbridge and Farr, 1976).

Dufresne (1988) and Owadally and Haberman (1999) have shown that if gains and losses are deferred beyond a certain period longer spreading or amortization periods lead to more variable contribution rates. The similarity between supplementary contribution methods and asset valuation indicates that excessive smoothing of asset values may have the counterintuitive effect of greater variability in contribution rates. The next proposition confirms this.

PROPOSITION 2 *As the amount of smoothing in actuarial asset values increases, the variance of the contribution rate decreases and then increases. That is,*

1. $\lim \mathrm{Var} C_t$ *has a single minimum wrt. k in equation* (14) *(exponential smoothing),*

2. $\lim \mathrm{Var} C_t$ *has a single minimum wrt. to n in equation* (28) *(arithmetic smoothing).*

Part 1 of Proposition 2 is proven exactly as in Dufresne (1988) because of the identity between exponential smoothing and the spreading of unfunded liability. Part 2 of Proposition 2 is proven in the Appendix. Note that $\lim \mathrm{Var} AV_t = \lim \mathrm{Var} C_t$ from equations (15) and (29).

The preconception described at the beginning of this section that smoother actuarial asset values lead to more stable contribution rates is not therefore always borne out. If the amount of smoothing in asset values is excessive, contribution rate volatility increases with more smoothing.

4.3 Efficient Choices of k and n

Owadally and Haberman (1999) argue that the actuarial objectives of pension benefit security for pension plan members and contribution stability for pension sponsors can be interpreted as a criterion that both the variances of the funding level and of the contribution rate should be minimized.

Given this criterion and given Propositions 1 and 2, the argument of Dufresne (1988) concerning admissible or efficient parameters for pension funding may be reiterated. The argument is briefly restated here in terms of n and k. As the amount of smoothing in asset valuation increases (that is, as n increases or k decreases), $\lim \mathrm{Var} F_t$ increases. As the amount of smoothing increases up to a critical amount, $\lim \mathrm{Var} C_t$ decreases. Beyond that critical amount of smoothing, $\lim \mathrm{Var} C_t$ increases with increased smoothing and the tradeoff between $\lim \mathrm{Var} F_t$ and $\lim \mathrm{Var} C_t$ is broken. Therefore, it is not efficient to smooth asset values beyond that critical amount since a lower $\lim \mathrm{Var} F_t$ is always achievable if asset values are smoothed by less than the critical amount. This is encapsulated in the following proposition.

PROPOSITION 3 *Let n^* and k^* be the $\lim \mathrm{Var} C_t$-minimizing values of the averaging period and smoothing parameter respectively. The efficient range of averaging periods is $[1,\ n^*]$. The efficient range of the smoothing parameter is $k^* \leq k \leq 1$.*

By correspondence with the results of Dufresne (1988), $k^* = 1 - 1/(u^2 + \sigma^2)$. Tables 1 and 2 list n^* and k^* respectively for various values of the moments of the rate of return. The Committee on Retirement Systems Research (2001) reports that a typical value for n is 5 years. We conclude from Table 1 that the typical arithmetic averaging period *is* efficient.

5 Comparison of Exponential and Arithmetic Smoothing in Asset Valuation

It is also possible to show mathematically, in the simple model described in section 2, that exponential smoothing in asset valuation should be preferred to arithmetic smoothing.

PROPOSITION 4 *Asset valuation using exponential smoothing is more efficient than arithmetic smoothing in the sense that, for any combination of* k *and* n *(with* $k < 1$ *and* $n > 1$*) such that* $\lim \mathrm{Var} F_t$ *is equal under exponential and arithmetic smoothing,* $\lim \mathrm{Var} C_t$ *is less under exponential smoothing than under arithmetic smoothing.*

Refer to the Appendix for a proof of Proposition 4. The proof is similar, but not identical, to a proof by Owadally and Haberman (1999) concerning the efficiency of the supplementary contribution method described by Dufresne (1988) over the alternative method in Dufresne (1989).

We conclude from Proposition 4 and Table 2 that efficient actuarial asset valuation requires exponential smoothing with a smoothing parameter (or weight on current market value) greater than 15%.

6 Conclusion

Actuarial asset valuation methods involving exponential and arithmetic smoothing and allowing for interest and cash flows were considered in a simplified model of a defined benefit pension plan described by Dufresne (1988). The analysis of Dufresne (1988, 1989) and Owadally and Haberman (1999) was adapted and the choice of averaging periods and smoothing parameters and its effect on the pension plan funding level and on contribution rates was explored. It was shown that smoother actuarial asset values may *increase* the volatility of contribution rates. Practical smoothing periods were discussed: arithmetic averaging over 5 years and exponential smoothing with more than 15% weight on current market value are both efficient. It was shown, however, that exponential smoothing is preferable to arithmetic smoothing in the sense that less volatile contribution rates and funding levels are achievable with the former than with the latter.

A deficiency in the analysis in this paper pertains to the modelling assumption that only the return on pension plan assets, net of wage inflation, is uncertain. Gains and losses also emerge in practice as a consequence of various uncertain factors such as mortality and disability which were ignored here. The wage inflation-indexed benefit design is also an abstraction of reality. Another deficiency lies in the assumption that plan sponsors pay off any actuarial deficit (that is, the excess of actuarial liability over actuarial asset value) immediately rather than over time. The similarity between the funding methods described by Dufresne (1988, 1989) and smoothed asset value methods deserves more study. Further work to remedy these deficiencies is in progress.

Appendix

Proof of Part 2 of Proposition 1

Showing that $\lim \mathrm{Var}F_t$ in equation (27) strictly increases with n is equivalent to showing that B_n in equation (32) strictly decreases with n. It is easily shown from equations (21) and (26) that

$$\sum \beta_j^2 = v^2 \sum \lambda_j^2 - v^2, \tag{35}$$

so that B_n may be rewritten as:

$$B_n = \left[1 + \sigma^2 v^2 - \sigma^2 v^2 \textstyle\sum \lambda_j^2\right] / \textstyle\sum \lambda_j^2 \,. \tag{36}$$

Now,

$$\nabla_n \left(\sum_{j=0}^{n-1} \lambda_j^2 \right) = \sum_{j=0}^{n-1} u^{2j} \frac{(n-j)^2}{n^2} - \sum_{j=0}^{n-2} u^{2j} \frac{(n-1-j)^2}{(n-1)^2}$$

$$= \frac{u^{2n-2}}{n^2} + \frac{1}{n^2(n-1)^2} \sum_{j=0}^{n-2} u^{2j} [(n-j)(n-1) - (n-1-j)n][(n-j)(n-1) + (n-1-j)n]$$

$$= \frac{u^{2n-2}}{n^2} + \sum_{j=0}^{n-2} u^{2j} j \frac{(n-j)(n-1) + (n-1-j)n}{n^2(n-1)^2} \quad > \quad 0.$$

Since $\sum \lambda_j^2$ increases strictly with n, it follows from equation (36) that B_n decreases strictly with n. $\square$

Proof of Part 2 of Proposition 2

Part 1 of Proposition 2 states that $\lim \mathrm{Var}C_t$ under exponential smoothing has a minimum over k, as proven by Dufresne (1988). From equation (33), $P(k)$ is proportional to the reciprocal of $\lim \mathrm{Var}C_t$ under exponential smoothing and therefore has a maximum over k. Proving that $\lim \mathrm{Var}C_t$ in the arithmetic smoothing case has a minimum with n is equivalent to proving that Q_n has a maximum with n (see equation (34)). To do this, we use the approach in Owadally and Haberman (1999) of comparing $P(k)$ with Q_n.

Assume that k is discretized and $k = 1/n$, $n \in \mathbb{N}$, and denote $P(k)$ by P_n. From equation (33), $P_n = n^2 - (u^2 + \sigma^2)(n-1)^2$. It is shown in the following that

$$Q_n \sum_{j=0}^{n-1} u^{2j} = \sum_{j=1}^{n} P_j u^{2(n-j)}. \tag{37}$$

By comparison, Owadally and Haberman (1999) discretize k such that $k = 1/\ddot{a}_{\overline{m}|}$ and then establish that $\beta_a(m)m = \sum_{j=1}^{m} \beta_s(i)$, in their notation.

Observe that $P_j u^{2(n-j)} = [j^2 - (u^2+\sigma^2)(j-1)^2]u^{2(n-j)} = \nabla_j \left(j^2 u^{2(n-j)}\right) - (j-1)^2\sigma^2 u^{2(n-j)}$ so that the right hand side of equation (37) is

$$\sum_{j=1}^{n} P_j u^{2(n-j)} = n^2 - \sigma^2 \sum_{j=1}^{n} (j-1)^2 u^{2(n-j)} = n^2 - \sigma^2 \sum_{j=0}^{n-1} (n-1-j)^2 u^{2j} = Q_n \sum_{j=0}^{n-1} u^{2j}$$

where Q_n is given in equation (34). This proves equation (37).

Now, apply the lag operator once on both sides of equation (37), multiply both sides by u^2, and subtract the resulting equation from equation (37) to yield $Q_n + (\nabla_n Q_n) \sum_{j=1}^{n-1} u^{2j} = P_n$ which may be rewritten as

$$\nabla Q_n = (P_n - Q_n) \bigg/ \sum_{j=1}^{n-1} u^{2j}. \tag{38}$$

Furthermore, the quotient rule immediately yields

$$\nabla^2 Q_n = \left[(\nabla P_n - \nabla Q_n) \sum_{j=1}^{n-1} u^{2j} - (P_n - Q_n) u^{2(n-1)} \right] \bigg/ \left[\sum_{l=1}^{n-1} u^{2l} \sum_{j=1}^{n-2} u^{2j} \right]. \tag{39}$$

When there is no smoothing ($n = k = 1$), $P_1 = Q_1 = 1$. Consider the variation of P_n and Q_n against $n \in \mathbb{N}$. P_n has a single maximum. From equation (37), Q_n is a weighted average of $\{P_j\}_1^n$ with positive weights $u^{2(n-j)}/\sum u^{2j}$ that sum to unity. As n increases, P_n initially increases and, likewise, Q_n must initially increase; P_n eventually reaches a maximum and decreases; and Q_n continues to increase and then also decreases. Equation (38) shows that Q_n has a stationary point ($\nabla Q_n \approx 0$) when Q_n intersects P_n ($P_n \approx Q_n$). Equation (39) shows that, at that stationary point, $\nabla^2 Q_n = (\nabla P_n)/\sum_{j=1}^{n-2} u^{2j}$.

The stationary point in Q_n occurs when P_n is decreasing and at that point $\nabla^2 Q_n < 0$, giving rise to a single maximum in Q_n, that is, a single minimum in $\lim \mathrm{Var} C_t$ under arithmetic smoothing. $\square$

Proof of Proposition 4

We wish to show that $Q_n < P(k)$ when $B_n = A(k)$, with $n > 1$ and $k < 1$.

Using equation (35), $A(k) = B_n$ may be rewritten as

$$1 - (u^2 + \sigma^2)(1-k)^2 = \left[1 - \sigma^2 \textstyle\sum \beta_j^2\right] / \textstyle\sum \lambda_j^2 \,.$$

From equation (35), $1 = \sum \lambda_j^2 - u^2 \sum \beta_j^2$ and the preceding equation may be rewritten as

$$1 - (u^2 + \sigma^2)(1-k)^2 = \left[\textstyle\sum \lambda_j^2 - (u^2 + \sigma^2) \textstyle\sum \beta_j^2\right] / \textstyle\sum \lambda_j^2 = 1 - (u^2 + \sigma^2) \textstyle\sum \beta_j^2 / \textstyle\sum \lambda_j^2.$$

Hence, $(1-k)^2 = \sum \beta_j^2 / \sum \lambda_j^2$ and

$$k^2 \textstyle\sum \lambda_j^2 \;=\; \left[\left(\textstyle\sum \lambda_j^2\right)^{1/2} - \left(\textstyle\sum \beta_j^2\right)^{1/2}\right]^2 \;=\; \textstyle\sum \lambda_j^2 + \textstyle\sum \beta_j^2 - 2 \left(\textstyle\sum \lambda_j^2\right)^{1/2} \left(\textstyle\sum \beta_j^2\right)^{1/2}.$$

Now, $\lambda_j > \beta_j \geq 0$ and

$$\begin{aligned} \textstyle\sum \lambda_j^2 / \textstyle\sum (\lambda_j \beta_j) \;&>\; 1 \;>\; \textstyle\sum (\lambda_j \beta_j) / \textstyle\sum \beta_j^2 \\ \Rightarrow \; \left(\textstyle\sum \lambda_j^2\right)\left(\textstyle\sum \beta_j^2\right) \;&>\; \left[\textstyle\sum (\lambda_j \beta_j)\right]^2. \end{aligned}$$

Hence,

$$k^2 \textstyle\sum \lambda_j^2 < \; \textstyle\sum \lambda_j^2 + \textstyle\sum \beta_j^2 - 2 \textstyle\sum \lambda_j \beta_j \;=\; \textstyle\sum (\lambda_j - \beta_j)^2 \;=\; \textstyle\sum \pi_j^2.$$

Since $A(k) = k^2 P(k)$ from equations (31) and (33) and since $B_n = Q_n \sum \pi_j^2 / \sum \lambda_j^2$ from equations (32) and (34), we can rewrite $A(k) = B_n$ as $P(k) k^2 \sum \lambda_j^2 = Q_n \sum \pi_j^2$. We have shown that $k^2 \sum \lambda_j^2 < \sum \pi_j^2$. It follows that $Q_n < P(k)$. $\square$

References

Committee on Retirement Systems Research (2001). Survey of asset valuation methods for defined benefit pension plans. *Pension Forum*, 13(1), 1–49. Society of Actuaries, Schaumburg, Illinois.

Dufresne, D. (1988). Moments of pension contributions and fund levels when rates of return are random. *Journal of the Institute of Actuaries*, 115, 535–544.

Dufresne, D. (1989). Stability of pension systems when rates of return are random. *Insurance: Mathematics and Economics*, 8, 71–76.

McGill, D.M., Brown, K.N., Haley, J.J. and Schieber, S.J. (1996). *Fundamentals of Private Pensions*, 7th ed. University of Pennsylvania Press, Philadelphia, Pennsylvania.

Owadally, M.I. and Haberman, S. (1999). Pension fund dynamics and gains/losses due to random rates of investment return. *North American Actuarial Journal*, 3(3), 105–117.

Trowbridge, C.L. and Farr, C.E. (1976). *The Theory and Practice of Pension Funding*. Richard D. Irwin, Homewood, Illinois.

Cass Business School
City of London

FACULTY OF ACTUARIAL SCIENCE AND STATISTICS

Actuarial Research Papers since 2001

---

135. Renshaw A. E. and Haberman S. On the Forecasting of Mortality Reduction Factors. February 2001.

ISBN 1 901615 56 1

136. Haberman S., Butt Z. & Rickayzen B. D. Multiple State Models, Simulation and Insurer Insolvency. February 2001. 27 pages.

ISBN 1 901615 57 X

137. Khorasanee M.Z. A Cash-Flow Approach to Pension Funding. September 2001. 34 pages.

ISBN 1 901615 58 8

138. England P.D. Addendum to "Analytic and Bootstrap Estimates of Prediction Errors in Claims Reserving". November 2001. 17 pages.

ISBN 1 901615 59 6

139. Verrall R.J. A Bayesian Generalised Linear Model for the Bornhuetter-Ferguson Method of Claims Reserving. November 2001. 10 pages.

ISBN 1 901615 62 6

140. Renshaw A.E. and Haberman. S. Lee-Carter Mortality Forecasting, a Parallel GLM Approach, England and Wales Mortality Projections. January 2002. 38 pages.

ISBN 1 901615 63 4

141. Ballotta L. and Haberman S. Valuation of Guaranteed Annuity Conversion Options. January 2002. 25 pages.

ISBN 1 901615 64 2

142. Butt Z. and Haberman S. Application of Frailty-Based Mortality Models to Insurance Data. April 2002. 65 pages.

ISBN 1 901615 65 0

143. Gerrard R.J. and Glass C.A. Optimal Premium Pricing in Motor Insurance: A Discrete Approximation. (Will be available 2003).

144. Mayhew, L. The Neighbourhood Health Economy. A systematic approach to the examination of health and social risks at neighbourhood level. December 2002. 43 pages.

ISBN 1 901615 66 9

145. Ballotta L. and Haberman S. The Fair Valuation Problem of Guaranteed Annuity Options: The Stochastic Mortality Environment Case. January 2003. 25 pages.

ISBN 1 901615 67 7

146. Haberman S., Ballotta L. and Wang N. Modelling and Valuation of Guarantees in With-Profit and Unitised With-Profit Life Insurance Contracts. February 2003. 26 pages.

ISBN 1 901615 68 5

147. Ignatov Z.G., Kaishev V.K and Krachunov R.S. Optimal Retention Levels, Given the Joint Survival of Cedent and Reinsurer. March 2003. 36 pages.

ISBN 1 901615 69 3

148. Owadally M.I. Efficient Asset Valuation Methods for Pension Plans. March 2003. 20 pages.

ISBN 1 901615 70 7

Statistical Research Papers

1. Sebastiani P. Some Results on the Derivatives of Matrix Functions. December 1995. 17 Pages.

ISBN 1 874 770 83 2

2. Dawid A.P. and Sebastiani P. Coherent Criteria for Optimal Experimental Design. March 1996. 35 Pages.

ISBN 1 874 770 86 7

3. Sebastiani P. and Wynn H.P. Maximum Entropy Sampling and Optimal Bayesian Experimental Design. March 1996. 22 Pages.

ISBN 1 874 770 87 5

4. Sebastiani P. and Settimi R. A Note on D-optimal Designs for a Logistic Regression Model. May 1996. 12 Pages.

ISBN 1 874 770 92 1

5. Sebastiani P. and Settimi R. First-order Optimal Designs for Non Linear Models. August 1996. 28 Pages.

ISBN 1 874 770 95 6

6. Newby M. A Business Process Approach to Maintenance: Measurement, Decision and Control. September 1996. 12 Pages.

ISBN 1 874 770 96 4

7. Newby M. Moments and Generating Functions for the Absorption Distribution and its Negative Binomial Analogue. September 1996. 16 Pages.

ISBN 1 874 770 97 2

8. Cowell R.G. Mixture Reduction via Predictive Scores. November 1996. 17 Pages.

ISBN 1 874 770 98 0

9. Sebastiani P. and Ramoni M. Robust Parameter Learning in Bayesian Networks with Missing Data. March 1997. 9 Pages.

ISBN 1 901615 00 6

10. Newby M.J. and Coolen F.P.A. Guidelines for Corrective Replacement Based on Low Stochastic Structure Assumptions. March 1997. 9 Pages.

ISBN 1 901615 01 4.

11. Newby M.J. Approximations for the Absorption Distribution and its Negative Binomial Analogue. March 1997. 6 Pages.

ISBN 1 901615 02 2

12. Ramoni M. and Sebastiani P. The Use of Exogenous Knowledge to Learn Bayesian Networks from Incomplete Databases. June 1997. 11 Pages.

ISBN 1 901615 10 3

13. Ramoni M. and Sebastiani P. Learning Bayesian Networks from Incomplete Databases.

June 1997. 14 Pages.
ISBN 1 901615 11 1

14. Sebastiani P. and Wynn H.P. Risk Based Optimal Designs. June 1997. 10 Pages.
ISBN 1 901615 13 8

15. Cowell R. Sampling without Replacement in Junction Trees. June 1997. 10 Pages.
ISBN 1 901615 14 6

16. Dagg R.A. and Newby M.J. Optimal Overhaul Intervals with Imperfect Inspection and Repair. July 1997. 11 Pages. ISBN 1 901615 15 4

17. Sebastiani P. and Wynn H.P. Bayesian Experimental Design and Shannon Information. October 1997. 11 Pages. ISBN 1 901615 17 0

18. Wolstenholme L.C. A Characterisation of Phase Type Distributions. November 1997. 11 Pages. ISBN 1 901615 18 9

19. Wolstenholme L.C. A Comparison of Models for Probability of Detection (POD) Curves. December 1997. 23 Pages. ISBN 1 901615 21 9

20. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy I: Principles. February 1999. 19 Pages. ISBN 1 901615 37 5

21. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy II: Application to Bayesian Networks. November 1999. 12 Pages ISBN 1 901615 40 5

22. Cowell R.G. FINEX : Forensic Identification by Network Expert Systems. March 2001. 10 pages. ISBN 1 901615 60X

23. Cowell R.G. When Learning Bayesian Networks from Data, using Conditional Independence Tests is Equivalant to a Scoring Metric. March 2001. 11 pages. ISBN 1 901615 61 8

Faculty of Actuarial Science and Statistics

Actuarial Research Club

The support of the corporate members

CGNU Assurance
Computer Sciences Corporation
English Matthews Brockman
Government Actuary's Department
HCM Consultants (UK) Ltd
KPMG
PricewaterhouseCoopers
Swiss Reinsurance
Watson Wyatt Partners

is gratefully acknowledged.

ISBN 1-901615-70-7