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Asymptotic and numerical analysis of the optimal investment strategy for an insurer

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Abstract

The asymptotic behaviour of the optimal investment strategy for an insurer is analysed for a number of cash flow processes. The insurer's portfolio consists of a risky stock and a bond and the cash flow is assumed to be either a normal or a compound Poisson process. For a normally distributed cash flow, the asymptotic limits are found in the cases that the stock is very risky or very safe. For a compound Poisson risk process, a composite asymptotic expansion is found for the optimal investment strategy in the case that stock is very risky and the claim size distribution is of an exponential type.

In general for a compound Poisson cash flow, the outer asymptotic limit reduces the integro-differential equation describing the optimal stock allocation to an integral equation, which determines the classical survival probability in ruin theory. The leading order optimal asset allocation is derived from this survival probability through a feedback law. Calculation of the optimal asset allocation leads to a difficult numerical problem because of the boundary layer structure of the solution and the tail properties of the claim size distribution. A second order numerical method is successfully developed to calculate the optimal allocation for light and heavy-tailed claim size distributions.

1 Introduction

The net cash flow for an insurance portfolio from collected premiums and claims paid on insurance is called the risk process. Ruin theory has traditionally been the study of the probability of ruin of an insurer determined directly from the risk process without allowing for explicit investment of the insurer's surplus. Generally, an analytical expression for the ruin probability as a function of the current surplus (the ruin function) is only available for claim size distributions of exponential form. Rolski *et al.* (1999) develop the theory in considerable detail. As analytical results are not available research

has concentrated on the behaviour of the ruin function as the current surplus tends to infinity. Lundberg bounds limit the size of the ruin function as long as the claim size distribution is light-tailed. For sub-exponential claim size distributions the asymptotic behaviour of the ruin function as the surplus tends to infinity is proportional to the integrated tail distribution of the claims.

We consider the optimal investment strategy for an insurer, which has a portfolio of a stock and a bond, where there is a stochastic cash flow arising from the risk process R_t. This problem was first analysed by Browne (1995) who adopted a normally distributed risk process. He found the optimal investment strategy for two different objectives: to minimise the probability of ruin and to maximise the utility of wealth for an exponential utility function. If there is no bond in the portfolio then for both these objectives it is optimal to invest a constant amount of money, independent of the current surplus, in the risky stock.

Hipp & Plum (2000) consider a portfolio consisting of just a risky stock, but used a compound Poisson process to model the risk process R_t. They found that the optimal strategy depends critically on the distribution of the claim sizes. In contrast to the normally distributed cash flow, the optimal strategy is to invest a fraction of the current surplus in the risky stock. Analytical results are available if the claim size distribution is exponential, and even here, an explicit optimal investment strategy can be found only for particular parameter choices. Liu & Yang (2004) extend the Hipp & Plum model to include a bond as well as a stock in the portfolio, and calculated the optimal asset allocation numerically for a number of different claim size distributions. A review of the optimal investment problem, as well as other optimal control problems in insurance, can be found in Hipp (2004).

There are many refinements to the basic model of ruin in the literature. Of relevance here is the work by Frolova, Kabanov & Pergamenshchikov (2002) who consider the effect a *fixed* investment in a risky asset has on the ruin probability. They use an exponential claim size distribution and a Brownian motion for the risky asset and found the behaviour of the ruin probability for small and large volatility of the asset. Recent research for the optimal investment problem for an insurer has mirrored the development of ruin theory. Hipp & Schmidli (2003) determine the asymptotic behaviour of the ruin probability as the current surplus tends to infinity in the small claims case. Gaier, Grandits & Schachermayer (2002) consider a claim size distribution with exponential moments, while Schmidli (2005) has determined the corresponding result for sub-exponentially distributed claim sizes.

Here, we study the asymptotic behaviour of optimal investment decision for an insurer using a small dimensionless parameter of the model rather than as the surplus tends to infinity. We use as this small parameter, the amount of risk in the asset defined by

$$\eta = \frac{\mu - r_0}{\sigma^2}, \tag{1}$$

where μ, σ are the drift and coefficient of volatility in the stock and r_0 is the risk free rate of interest. All these quantities are taken as constants. The drift μ and the risk free rate r_0 are usually quoted in units of percentage per annum. Thus they have dimension

per unit time. If the stock is lognormally distributed, the coefficient of volatility σ is also quoted per annum. This represents the observed standard deviation of the log return of the asset after one year, rather than signifying that this quantity has dimension per unit time. The coefficient of volatility has dimension per square root of unit time in order to be consistent with the dimension of the Brownian motion. Consequently, η is a dimensionless quantity. Notice that with this notation the market price of risk is $\eta\sigma$.

Asymptotic methods have already been successfully applied to the optimal insurance pricing problem. Emms, Haberman & Savoulli (2004) use a perturbation expansion to determine the optimal premium in order to maximise the insurer's expected total wealth. This work represents an extension of those techniques to ordinary differential equations (ODEs) and integro-differential equations arising from the optimal investment problem. Hinch (1991) and Bender & Orszag (1978) develop the general theory of perturbation expansions in a small parameter to give approximate analytical solutions.

In Section 2 we identify the limiting behaviour of the optimal strategy as $\eta \to 0$ for a normally distributed risk process. Using this feature of the model allows us to construct approximate optimal strategies when the parameter is not zero, and so describe the qualitative features of the optimal strategy when analytical results are not available. The cash flow is taken as compound Poisson process in Section 3. We obtain approximate optimal strategies for two light-tailed claim size distributions in Section 4. For these distributions the problem reduces to the solution of an ODE and a system of ODEs. It is easy to integrate these equations numerically so the exact solution can be readily compared with the asymptotic results. For some of the analysis, asymptotic results are only available analytically if there is no bond in the portfolio. We examine this case by setting $r_0 = 0$. Section 5 develops the theory for general claim size distributions and describes three numerical schemes to solve the optimal asset allocation problem when there is no reduction to ODEs. Conclusions are given in Section 6.

2 Normal risk process

First, we illustrate the asymptotic limit by considering the simple insurance model presented by Browne (1995). Consider a financial market consisting of a risk-free bond of price B_t and a stock of price P_t. We suppose the price of the bond evolves according to

$$dB_t = r_0 B_t \, dt.$$

Further, we assume the price of the stock is lognormally distributed:

$$dP_t = \mu P_t \, dt + \sigma P_t \, dW_t,$$

where W_t is a standard Brownian motion. We denote the surplus process by X_t and suppose the insurer invests an amount A_t in the stock and $X_t - A_t$ in the bond. The surplus of the insurer is generated by the risk process R_t representing the increase in wealth due to premium income and the loss of wealth due to claims. Thus the surplus

process of the insurer is governed by

$$dX_t = A_t \frac{dP_t}{P_t} + (X_t - A_t)\frac{dB_t}{B_t} + dR_t,$$

where the current surplus is

$$X_0 = s.$$

Browne (1995) adopts a normal random process for the risk process R_t:

$$dR_t = \alpha\, dt + \beta\, dZ_t, \tag{2}$$

where α and β are constants and Z_t is a standard Brownian motion correlated with W_t so that

$$\mathbb{E}[W_t Z_t] = \rho t,$$

and $0 \leq |\rho| < 1$ is a dimensionless constant. The case that both processes are perfectly positively/negatively correlated is ignored.

The constants α and β in (2) have units of $[s][t]^{-1}$ and $[s][t]^{-1/2}$ respectively where $[s]$ is the monetary scale and $[t]$ is the time scale. The monetary scale is determined by the actual values of α and β since these model the cash flow of the insurer. Without loss of generality we suppose that the monetary unit is chosen so that $\alpha = \beta = O(1)$ if the unit of time is a year: the current surplus s and the stock investment A_t are measured in terms of this monetary unit. The values of the other rate parameters are also chosen to be consistent with a time unit of a year. Henceforth we shall omit units from the discussion.

We aim to find the optimal stock allocation A^* which maximises the insurer's survival probability. Consequently we define the ruin time as

$$\tau(s) = \inf\{t > 0 : X_t < 0, X_0 = s\}$$

and the insurer's survival probability as

$$\delta(s) = \mathbb{P}[\tau(s) = \infty | X_0 = s].$$

The surplus process is a diffusion given by

$$dX_t = (\mu A_t + r_0(X_t - A_t) + \alpha)\, dt + \sigma A_t\, dW_t + \beta\, dZ_t.$$

Consequently, if we interpret $\delta(s)$ as a value function the HJB equation is

$$\sup_A \{(\alpha + (\mu - r_0)A + r_0 s)\delta'(s) + (\sigma^2 A^2 + \beta^2 + 2\rho\sigma\beta A)\delta''(s)\} = 0, \tag{3}$$

following Fleming & Rischel (1975), and where $'$ denotes d/ds.

We suppose the supremum in (3) is given by the first order condition for a maximum so that

$$A^*(s) = -\frac{\delta'(s)\eta}{2\delta''(s)} - \frac{\rho\beta}{\sigma}, \tag{4}$$

where η is defined by (1). If the surplus $s = 0$ then ruin is certain whereas if $s = \infty$ then survival is assured. Consequently the boundary conditions for (3) are

$$\delta(0) = 0, \quad \delta(\infty) = 1. \tag{5}$$

It is useful to introduce the following additional functions of the survival probability $\delta(s)$:

$$u(s) = \delta'(s), \tag{6}$$

$$\sqrt{w(s)} = -\frac{u(s)}{u'(s)}, \tag{7}$$

This notation will also prove useful in the forthcoming sections. Substituting (4) into the argument of the supremum in (3) yields a quadratic in $\sqrt{w}$:

$$-\tfrac{1}{4}\eta^2\sigma^2 w + (\rho\sigma\beta\eta - r_0 s - \alpha)\sqrt{w} + \beta^2(1-\rho^2) = 0.$$

The discriminant of this equation is positive so the positive root gives the optimal stock allocation as

$$A^*(s) = \tfrac{1}{2}\eta\sqrt{w} - \frac{\rho\beta}{\sigma} = \frac{1}{\eta\sigma^2}[((r_0 s + \alpha - \rho\sigma\beta\eta)^2 + \eta^2\sigma^2\beta^2(1-\rho^2))^{1/2} - (r_0 s + \alpha)]. \tag{8}$$

Browne (1995) derives this result by appealing to a theorem proved by Pestien & Sudderth (1985).

To calculate the corresponding survival probability requires we integrate (6) and (7), which given the complexity of (8) appears difficult. Consequently in order to make progress we find an approximate optimal strategy by assuming a parameter of the model is small . Consider an expansion of the optimal stock allocation for

$$\eta \ll 1,$$

corresponding to a very risky stock: if $(\mu - r_0) \ll \sigma^2$ then the return from the stock may be significantly below the return from the bond. We find after some manipulation

$$\sqrt{w_{risky}(s)} = \frac{\beta^2(1-\rho^2)}{r_0 s + \alpha} + O(\eta), \tag{9}$$

$$A^*_{risky}(s) = -\frac{\rho\beta}{\sigma} + \frac{(1-\rho^2)\beta^2\eta}{2(r_0 s + \alpha)} + O(\eta^2). \tag{10}$$

To leading order the optimal strategy is to buy/sell an amount $\rho\beta/\sigma$ of stock if the risk process is negatively/positively correlated with the very risky stock. If we ignore the bond by setting $r_0 = 0$ in (8) then the optimal strategy is to buy/sell a constant amount of the stock. We mean by constant amount that the optimal stock allocation is independent of the current surplus level s.

If we substitute (9) into (7) and integrate we find at leading order

$$u_{risky}(s) = K \exp\left[-\frac{r_0 s^2 + 2\alpha s}{2\beta^2(1-\rho^2)}\right],$$

where K is a constant. Integrating again, completing the square and applying the boundary conditions (5) yields

$$\delta_{risky}(s) = \frac{\Phi\left(\nu\left(\frac{r_0 s}{\alpha}+1\right)\right) - \Phi(\nu)}{1-\Phi(\nu)},$$

where $\Phi(x)$ is the standard normal distribution function and

$$\nu = \alpha\sqrt{\frac{1}{r_0\beta^2(1-\rho^2)}}.$$

Next, consider the case of a relatively safe stock $\eta \gg 1$. It is simple to expand (8) in powers of η^{-1} to obtain

$$\begin{aligned}
\sqrt{w_{safe}} &= \frac{2(1+\rho)}{\eta}\left(\frac{\beta}{\sigma} - \frac{r_0 s+\alpha}{\sigma^2\eta} + O(\eta^{-2})\right), \\
A^*_{safe} &= \frac{1}{\sigma^2}\left(\sigma\beta - (1+\rho)\left(\frac{r_0 s+\alpha}{\eta}\right) + O(\eta^{-2})\right).
\end{aligned}$$

Thus at leading order it is optimal to buy a constant amount β/σ of stock as the stock becomes safer ($\eta \to \infty$). At leading order, we integrate (6) and (7) and apply the boundary conditions to obtain

$$\delta_{safe}(s) = 1 - \exp\left[-\frac{\eta\sigma}{2\beta(1+\rho)}s\right].$$

Thus the leading order survival probability can be obtained analytically for both $\eta \ll 1$ and $\eta \gg 1$.

We have calculated perturbation expansions for the optimal asset allocation $A^*(s)$, which is given by a quadratic for $\sqrt{w}$, and then used the leading order term to derive the leading order survival probabilities. To illustrate how good an approximation the asymptotic results are relative to a numerical solution we set:

$$\alpha = 0.6, \quad \beta = \sqrt{6}, \quad r_0 = 0.04, \quad \sigma = 0.3, \quad \rho = 0.05, \quad \eta = 2/3.$$

These figures are comparable to the parameter set given in Liu & Yang (2004) and Promislow & Young (2005) for a compound Poisson risk process and exponential claims using a normal approximation (see (31)). We vary μ which in effect varies the dimensionless risk parameter η and plot the results in Figure 1. The solid lines show the solution calculated numerically by evaluating (8) and integrating (6) and (7) with a Runge-Kutta method. Figure 1(a) shows the optimal asset allocation superimposed with the asymptotic results, while Figure 1(b) shows the corresponding survival probabilities. As the stock becomes risker ($\eta \to 0$) it is optimal for the insurer to invest less in the stock and the survival probability decreases. Notice that the asymptotic result A^*_{risky} gives a good approximation up to about $\eta \sim 1$ for this parameter set.

3 Compound Poisson risk process

Hipp & Plum (2000) find that the optimal strategy has a different qualitative form if one adopts a compound Poisson process for the risk process. We shall consider that case next. The continuous Cramér-Lundberg model (Rolski *et al.* 1999) for the risk process R_t of an insurer is

$$dR_t = c\,dt - dS_t,$$

where c is the constant premium rate charged by the insurer, and the aggregate claim amount S_t is defined by

$$S_t = \sum_{i=1}^{N_t} Y_i.$$

Here, N_t denotes the number of claims modelled as a Poisson process with intensity λ and Y_i is an i.i.d. sequence representing the size of individual claims. We shall consider a number of different distributions for the claim size Y_i.

The actuarial price of insurance according to the expected value principle is

$$c = (1+\theta)\mathbb{E}[S_t] = \lambda(1+\theta)\mathbb{E}[Y_1], \tag{11}$$

where θ is the relative security loading. We assume that the income generated by selling insurance can be continuously invested in the financial markets without regard to dividend payments or transactions costs.

Following Hipp & Plum (2000) and Liu & Yang (2004) the corresponding HJB equation for the survival probability $\delta(s)$ is

$$\sup_A\{\lambda\mathbb{E}[\delta(s-Y)-\delta(s)] + \delta'(s)[c+(\mu-r_0)A(s)+r_0 s] + \frac{1}{2}\delta''(s)\sigma^2 A(s)^2\} = 0, \tag{12}$$

where Y is the claim size and now there is assumed to be no correlation between the risk and investment processes. We assume that the survival probability is a smooth, strictly increasing, concave function so that the supremum is attained when

$$A^*(s) = -\eta\frac{\delta'(s)}{\delta''(s)} = -\eta\frac{u(s)}{u'(s)} = \eta\sqrt{w}, \tag{13}$$

using the notation in (6) and (7). We also suppose $\mu > r_0$ so that η and A^* are non-negative. This is a feedback law: it gives the optimal allocation in the asset as a function of the current surplus. The optimal control in Section 2 is in feedback form only if $r_0 \neq 0$, whereas here A^* is a function of s irrespective of the interest rate.

Substituting (13) into (12) yields the integro-differential equation

$$\tfrac{1}{2}\tilde{R}\frac{u(s)^2}{u'(s)} = (\tilde{r}_0 s + \tilde{c})u(s) + \mathbb{E}[\delta(s-Y)-\delta(s)], \tag{14}$$

where we introduce the parameters

$$\tilde{R} = \left(\frac{\mu - r_0}{\lambda}\right)\eta, \quad \tilde{r}_0 = \frac{r_0}{\lambda}, \quad \tilde{c} = \frac{c}{\lambda},$$

The dimensionless parameter $\tilde{R}$ can be interpreted as the excess return of the stock compared to the frequency of the claims scaled by the market price of risk. The parameter $\tilde{r}_0$ is also dimensionless while $\tilde{c}$ has monetary units and is proportional to the mean claim size. We set the monetary scale by the mean claim size so that units do not appear in the model. The rate parameters use a time scale of years and so their values are chosen appropriately.

If the surplus $s = 0$ then any investment in the risky stock leads to immediate ruin with probability one and so the optimal allocation of stock is $A^*(0) = 0$. If $s = \infty$ then survival is certain with probability one. Using (12), boundary conditions for (14) are then

$$\tilde{c}\delta'(0) = \delta(0), \quad \delta(\infty) = 1.$$

It is simpler to pose an initial value problem by putting $\hat{\delta}(s) = \delta(s)/\delta'(0)$ so that $\hat{\delta}'(0) = 1$. Henceforth we shall use the survival probability $\hat{\delta}$ and drop the hat so that the boundary conditions are

$$\delta(0) = \tilde{c}, \quad \delta'(0) = 1.$$

The actual survival probability can then be found by dividing through by $\delta(\infty)$.

We can write the HJB equation explicitly by introducing the claim size tail distribution function $H(y)$ and integrating by parts to obtain

$$\tfrac{1}{2}\tilde{R}\sqrt{w(s)}u(s) + (\tilde{r_0}s + \tilde{c})u(s) - \tilde{c}H(s) - \int_0^s u(y)H(s-y)\,dy = 0. \tag{15}$$

The derivation of this equation can be found in more detail in Hipp & Plum (2000) and Liu & Yang (2004). Notice that if the delayed kernel of the integral is separable so that $H(s-y) = H_1(s)H_2(y)$ (as it is for an exponential claim size distribution) then one can divide through by $H_1(s)$ and reduce the equation to an ODE.

3.1 Small and large surplus

It is convenient to write the integro-differential equation (15) as a system of two differential equations and one integral equation by differentiating. We obtain after some manipulation

$$\tfrac{1}{4}\tilde{R}w'(s) = \sqrt{w}\left(\tfrac{1}{2}\tilde{R} - \tilde{r}_0 + 1 + \frac{\tilde{c}H'(s)}{u(s)} + m(s)\right) + \tilde{r}_0 s + \tilde{c}, \tag{16}$$

$$u'(s) = -\frac{u}{\sqrt{w}}, \tag{17}$$

$$m(s) = \int_0^s \frac{u(s-y)}{u(s)} dH(y), \tag{18}$$

with initial conditions

$$w(0) = 0, \quad u(0) = 1, \quad m(0) = 0.$$

Liu & Yang (2004) use a system of equations of this form to calculate the optimal asset allocation numerically for the Pareto claim size distribution. We shall use these equations to derive informally the behaviour of A^* as $s \to 0$. This analysis extends that of Hipp & Plum (2000) who found the leading order term when $r_0 = 0$. The asymptotic expressions are used subsequently in Section 5.2 to initiate the numerical schemes.

Balancing the first term on the LHS of (16) with the last term on the RHS yields

$$w(s) \sim f_0 s \quad \text{as } s \to 0, \tag{19}$$

where

$$f_0 = \frac{4\tilde{c}}{\tilde{R}}.$$

If we pose any other balance in the equation then that leads to a contradiction in either the boundary condition or the size of the neglected terms. The leading order behaviour of the optimal strategy is therefore independent of the claim size distribution. In order to determine the nature of this dependence we must go to the next order. Put

$$w_1(s) = w(s) - f_0 s \sim o(s)$$

and substitute into (16). Integrating and neglecting small terms yields

$$w_1(s) = f_1 s^{3/2} = \frac{16\tilde{c}^{1/2}}{3\tilde{R}^{3/2}}(\tfrac{1}{2}\tilde{R} - \tilde{r}_0 + 1 + \tilde{c}H'(0))s^{3/2}.$$

Substituting into (17), integrating and expanding the resulting exponential leads to

$$u(s) \sim 1 - \frac{2s^{1/2}}{f_0^{1/2}} + \frac{s}{f_0}\left(\frac{f_1}{2f_0^{1/2}} + 4\right) + o(s). \tag{20}$$

Consequently, the asymptotic behaviour of the ruin probability and optimal asset allocation strategy is from (6) and (13)

$$\begin{aligned} \delta(s) &= \tilde{c} + s - \frac{4}{3f_0^{1/2}}s^{3/2} + \frac{s^2}{2f_0}\left(\frac{f_1}{2f_0^{1/2}} + 4\right) + o(s^2). \\ A^*(s) &= \eta(f_0 s)^{1/2}\left(1 + \frac{f_1 s^{1/2}}{2f_0}\right) + o(s), \end{aligned}$$

for $s \ll 1$.

For completeness we describe the existing results in the literature for the case that the surplus $s \to \infty$ when there is no bond in the portfolio. If one assumes the exponential moments of the claims distribution exist then Gaier, Grandits & Schachermayer (2003) find an exponential bound on the ruin probability in terms of an adjustment coefficient. Hipp & Schmidli (2003) show that for these light-tailed distributions $A^* \to$const. as $s \to \infty$ where the constant depends on this adjustment coefficient.

Schmidli (2005) finds that $A^* \to \infty$ as $s \to \infty$ for subexponential claims, which are a class of heavy-tailed distributions (Rolski *et al.* 1999). In our notation, he finds that if the hazard rate $-H'(s)/H(s)$ tends to zero then

$$A^* \to \frac{\mu}{\sigma^2} \int_0^s \frac{H(s)}{H(y)} \, dy \quad \text{as } s \to \infty. \tag{21}$$

As yet there are no corresponding results when there is a bond in the portfolio. One can see from Schmidli (2005) that there is no distinguished limit in (15) as $s \to \infty$, that is all the terms balance in the equation in the limit. In ruin theory, the limit of the ruin probability as the surplus tends to infinity is used to approximate the ruin probability for all s often with limited success (Rolski *et al.* 1999). In the next section we find that a good approximation to a numerically computed solution can be found by considering a small parameter in the model.

4 Light-tailed claim size distributions

We aim to study the qualitative form of the optimal strategy for the case

$$\tilde{R} = \left(\frac{\mu - r_0}{\lambda} \right) \eta \ll 1,$$

for a variety of different claim size distributions. Notice that this covers that part of the parameter space for which $\eta = O(1)$ as long as $(\mu - r_0) \ll \lambda$.

Greater progress on the asymptotic expansion can be made for light-tailed claims distributions. These asymptotic expressions can be used for a sensitivity analysis of the optimal asset allocation to the parameters in the model without recourse to a numerical study (Liu & Yang 2004). Furthermore, they substantiate the idea that the asymptotic limit is well-defined for a risky stock and so motivates the general theory in the subsequent section. The tail distribution function $H(y)$ for each claim size distribution is given in Table 1.

4.1 Exponential claim size distribution

For an exponential claim size distribution with parameter k, $H(y) = e^{-ky}$ and the expected value principle gives the premium

$$\tilde{c} = (1 + \theta)k^{-1}.$$

The integro-differential equation (15) reduces to a single ODE with this distribution. If we multiply (15) by e^{ks} and then differentiate we obtain after some manipulation

$$\tilde{R}w'(s) = -2\tilde{R}kw(s) - 4[\tilde{r}_0 - 1 + k(\tilde{r}_0 s + \tilde{c}) - \tfrac{1}{2}\tilde{R}]\sqrt{w(s)} + 4(\tilde{r}_0 s + \tilde{c}), \tag{22}$$

with boundary condition

$$w(0) = 0.$$

For $\tilde{R} \ll 1$ the form of (22) suggests that the optimal strategy has a region of rapid change or boundary layer close to $s = 0$ (Hinch 1991). In the outer region where $s \sim O(1)$ we expect the derivative on the LHS of (22) to be negligible, whereas in a thin inner region $s \sim O(\tilde{R})$ the derivative is significant assuming $w \sim O(1)$. Thus we introduce the outer perturbation expansion

$$w(s) \sim w_0(s) + \tilde{R} w_1(s) + \dots \tag{23}$$

By substituting (23) into (22) we obtain the leading order outer solution

$$w_0(s) = \left(\frac{\tilde{r}_0 s + \tilde{c}}{\tilde{r}_0 - 1 + k(\tilde{r}_0 s + \tilde{c})} \right)^2 = \left(\frac{r_0 s + (1+\theta)\lambda k^{-1}}{r_0 + \theta\lambda + k r_0 s} \right)^2. \tag{24}$$

The next order problem is

$$w_0' = -2k w_0 - 2\sqrt{w_0} + 2(\tilde{r}_0 + k\tilde{r}_0 s + \theta) w_0^{-1/2} w_1.$$

Solving for w_1 yields

$$w_1(s) = \frac{(\tilde{r}_0 s + (1+\theta)k^{-1})^2}{(\tilde{r}_0 + k\tilde{r}_0 s + \theta)^5} \left[\tilde{r}_0(\tilde{r}_0 - 1) + (\tilde{r}_0 + k\tilde{r}_0 s + \theta)(\tilde{r}_0 + 2k\tilde{r}_0 s + 2\theta + 1) \right].$$

We can continue to calculate higher order terms in the expansion but the expressions become increasingly more cumbersome. Therefore the outer optimal stock allocation is given explicitly by

$$A^*(s) = \eta \left(\sqrt{w_0} + \frac{w_1}{2\sqrt{w_0}} \tilde{R} \right) + o(\eta \tilde{R}).$$

It is clear that this expression does not satisfy the boundary condition at $s = 0$ so that we must consider a thin inner region.

We denote the inner variable S by the scaling

$$s = \tilde{R} S,$$

and the inner dependent variable by $W(S)$. The inner problem is therefore

$$\dot{W}(S) = -2\tilde{R} k W(S) - 4[\tilde{r}_0 + k\tilde{r}_0 \tilde{R} S + \theta - \tfrac{1}{2}\tilde{R}]\sqrt{W(S)} + 4(\tilde{r}_0 \tilde{R} S + \tilde{c}),$$

where $\dot{} \equiv d/dS$. If we write the perturbation expansion

$$W(S) \sim W_0(S) + \tilde{R} W_1(S) + \dots$$

we obtain at leading order

$$\dot{W}_0 = -4\zeta\sqrt{W_0} + 4\tilde{c}.$$

where we have written $\zeta = \tilde{r}_0 + \theta$. Integrating and applying the boundary condition leads to the implicit relation

$$\tilde{c} \log\left(1 - \frac{\zeta\sqrt{W_0}}{\tilde{c}}\right) + \zeta\sqrt{W_0} + 2\zeta^2 S = 0. \tag{25}$$

Since the leading order expansion is only given implicitly it is not possible to make further analytical progress with the inner expansion.

Van Dyke's matching principle (1975) requires that

$$W_0(\infty) = w_0(0) = \left(\frac{\tilde{c}}{\zeta}\right)^2.$$

In the matching region the optimal stock allocation is therefore $\tilde{c}\eta/\zeta$. The leading order composite asymptotic expansion is

$$w(s) \sim w_0(s) + W_0(S) - \left(\frac{\tilde{c}}{\zeta}\right)^2 + O(\tilde{R}), \tag{26}$$

where w_0 is given explicitly by (24) and W_0 is given implicitly by (25).

We use the parameter set given in Liu & Yang (2004):

$$\mu = 0.1, \quad r_0 = 0.04, \quad \sigma = 0.3, \quad \theta = 0.2, \quad \lambda = 3, \quad k = 1, \tag{27}$$

which gives

$$\eta = 2/3, \quad \tilde{R} = 0.013.$$

Notice, that with this parameter set we focus on relatively small surpluses comparable to the mean claim size $\mathbb{E}[Y_1] = 1/k = 1$. We shall find that it is critical to understand how the optimal allocation behaves in this region in order to compute its value for much larger surpluses.

We determine how the asymptotic result (26) compares with the numerical solution of (22) for this parameter set. The numerical solution is calculated using an adaptive Runge-Kutta scheme as given in Press, Teukolsky & Flannery (2002). Figure 2(a) shows the composite asymptotic optimal asset allocation from (13) (thick line) superimposed on the numerical solution (thin line). If $A^*(s) > s$ then it is optimal for the insurer to borrow at the risk free rate and invest that money in the risky stock in order to avoid ruin. We can see in Figure 2(a) that if the current surplus $s \lesssim 3$ (comparable to the mean claim size) then borrowing and investing is optimal. Both qualitative and quantitative agreement are good between the asymptotic and numerical solution: if the outer correction w_1 is also added then the quantitative agreement is even better, but for simplicity this is not shown. Figure 2(b) demonstrates the uniform convergence of the composite asymptotic solution to the numerical solution as $\tilde{R} \to 0$ by decreasing μ. As the return on the stock is decreased with the volatility held fixed then the stock becomes increasingly risky, less money is invested in the stock and so borrowing requirements are reduced. Figure 4(a) summarises the structure of the optimal stock allocation for exponentially distributed claim sizes.

Given $w(s)$ the survival probability can be calculated by integrating (6) and (7). If $r_0 = 0$ then at leading order we find

$$\delta_0'(s) = u_0(s) = \exp\left(\left(\frac{1 - k\tilde{c}}{\tilde{c}}\right) s\right),$$

after applying the boundary condition $u(0) = 1$. It is straightforward to integrate again and obtain

$$\delta_0(s) = \frac{\tilde{c}}{1 - k\tilde{c}} \left(\exp\left(\left(\frac{1 - k\tilde{c}}{\tilde{c}} \right) s \right) - k\tilde{c} \right). \tag{28}$$

If $r_0 \neq 0$ then a similar calculation leads to

$$\delta_0(s) = \frac{e^{k\tilde{c}/\tilde{r}_0}}{k} \left(\frac{\tilde{r}_0}{k\tilde{c}} \right)^{a-1} (\gamma(a, ks + k\tilde{c}/\tilde{r}_0) - \gamma(a, k\tilde{c}/\tilde{r}_0)) + \tilde{c}. \tag{29}$$

where the incomplete gamma function is

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1}\, dt \quad \text{and} \quad a = \frac{1}{\tilde{r}_0}.$$

These expressions are the survival probabilities for the non-investment problem as we shall see in Section 5. For (28), note that $k\tilde{c} > 1$ is the net profit condition, and $\delta_0(\infty) = k\tilde{c}^2/(k\tilde{c} - 1)$. Consequently the ruin probability is

$$\frac{\lambda}{kc} \exp\left(\frac{\lambda}{c} - k \right) s,$$

which corresponds to (5.3.8) in Rolski *et al.* (1999) in our notation.

Again, using the parameter set in Liu & Yang (2004), we plot the asymptotic and numerical survival probabilities in Figure 2(c). The solid line is calculated by integrating (22), (6) and (7) with a Runge-Kutta method, while the dashed lines are calculated from the asymptotic results (28) and (29). The sensitivity of the optimal stock allocation and the survival probability to the parameters in the model can be deduced directly from the outer expansions rather than an exhaustive numerical study.

If we ignore the bond by setting $r_0 = 0$ then at leading order the outer stock allocation is

$$A_e^* = \frac{\eta}{k} \left(\frac{1 + \theta}{\theta} \right) \approx \frac{\eta}{k\theta}, \tag{30}$$

because we expect the safety loading $\theta \ll 1$. Here the subscript e denotes the exponential distributed claims in a compound Poisson risk process. Notice that the allocation of stock is independent of the current surplus, s, as it was for the normal risk process. The normal risk process (2) can be used to approximate a compound Poisson claims process with intensity λ and exponentially-distributed claim sizes Y_i by matching moments (Grandell 1991):

$$\alpha = c - \lambda \mathbb{E}[Y_1] = c - \frac{\lambda}{k} = \frac{\lambda\theta}{k}, \quad \beta^2 = \lambda \mathbb{E}[Y_1^2] = \frac{2\lambda}{k^2}. \tag{31}$$

For a normal risk process, if there is no correlation between asset and claims and no bond then from (10), the approximate optimal stock allocation is

$$A_n^* = \frac{\beta^2 \eta}{2\alpha} \approx \frac{\eta}{k\theta}. \tag{32}$$

using (31) and where the subscript n refers to the normally distributed risk process.

Consequently, the outer optimal stock allocation for exponentially distributed claim sizes (30) and the optimal allocation for a normal risk process (32) are similar for appropriate parameter choices: compare Figure 1(a) for $\eta = 2/3$ (thin solid line) and Figure 2(a) (thin solid line). In the case of a very risky stock $\tilde{R} \ll 1$ the optimal strategies are quantitatively similar for all s. The strategies differ only when the current surplus is close to zero or the loading is very large. The plots in Liu & Yang (2004), Hipp & Plum (2000), and Castillo & Parrocha (2003) further confirm the boundary layer structure of the optimal asset allocation when the claim size is exponentially distributed.

4.2 Erlang claim size distribution

For an Erlang distribution, $\mathrm{Erl}(n,k)$, we have $H(y) = \sum_{i=1}^{n} [(ky)^{i-1}/(i-1)!]e^{-ky}$ and

$$\tilde{c} = (1+\theta)nk^{-1}.$$

This distribution function leads to a system of n ordinary differential equations which must be solved in order to determine the optimal asset allocation.

Substituting $H(y)$ into (15) yields

$$(\tfrac{1}{2}\tilde{R}\sqrt{w}u + (\tilde{r}_0 s + \tilde{c})u)e^{ks} = \sum_{i=0}^{n-1} a_i(s), \tag{33}$$

where the function

$$a_i(s) := \frac{\tilde{c}(ks)^i}{i!} + \int_0^s u(y)e^{ky}\frac{(k(s-y))^i}{i!}\,dy \quad \text{for } i \geq 0. \tag{34}$$

Differentiating (34) yields a recurrence relation for $a_i(s)$:

$$\dot{a}_i = ka_{i-1}, \quad \text{for } i > 0$$

and $\dot{a}_0 = u(s)e^{ks}$. Next we define

$$M_j(s) := \sum_{i=0}^{j-1} \frac{a_i}{\dot{a}_0} \quad \text{for } 0 < j < n$$

and set $M_0 = 0$ for convenience. The integro-differential equation (33) can then be differentiated to obtain

$$\tilde{R}w' = -2k\tilde{R}w - 4\sqrt{w}(k(\tilde{r}_0 s + \tilde{c}) - \tfrac{1}{2}\tilde{R} + \tilde{r}_0 - 1 - kM_{n-1}) + 4(\tilde{r}_0 s + \tilde{c}), \tag{35}$$

while the recurrence relations for M_j are

$$M_j' = 1 + kM_{j-1} - M_j\left(k - \frac{1}{\sqrt{w}}\right) \quad \text{for } 0 < j < n. \tag{36}$$

The equations (35) and (36) form a system of n ODEs which have boundary conditions

$$w(0) = 0, \quad M_j(0) = \tilde{c} \quad \text{for } 0 < j < n,$$

since $\dot{a}_0(0) = u(0) = 1$.

Following the previous section we introduce an inner variable $s = \tilde{R}S$, write $\dot{} \equiv d/dS$, and obtain the system

$$\dot{W} = -2k\tilde{R}W - 4\sqrt{W}(k(\tilde{r}_0\tilde{R}S + \tilde{c}) - \tfrac{1}{2}\tilde{R} + \tilde{r}_0 - 1 - kM_{n-1}) + 4(\tilde{r}_0\tilde{R}S + \tilde{c}), \tag{37}$$

$$\dot{M}_j = \tilde{R}\left(1 + kM_{j-1} - M_j\left(k - \frac{1}{\sqrt{W}}\right)\right), \tag{38}$$

Notice that in the inner region $W \sim 1$ from (19) and $M_j(s) \sim 1$ from (34). Consequently, at leading order $M_{n-1} = \tilde{c}$ from the boundary condition and

$$\tilde{c}\log\left(1 - \frac{\zeta\sqrt{W_0}}{\tilde{c}}\right) + \zeta\sqrt{W_0} + 2\zeta^2 S = 0,$$

where $\zeta = \tilde{r}_0 - 1$. The inner solution is of the same form as (25) differing only in the definition of the constant ζ.

For the parameter range we consider here $\zeta < 0$ which means $W_0 \to \infty$ as $S \to \infty$ and it is not possible to match to the outer region. This suggests that there is an intermediate region with scales

$$s \sim 1, \quad M_j \sim 1, \quad w = \frac{\Omega}{\tilde{R}^2},$$

where $\Omega \sim 1$. In this intermediate region the governing equations have no analytical solution for arbitrary n, so we focus on the case $n = 2$, which was the case solved numerically by Liu & Yang (2004). The aim here is to ascertain how the optimal strategy differs from that obtained with exponentially distributed claims: the case $n = 2$ is a first step in that direction. In this case the intermediate equations reduce to

$$\Omega' = -2k\Omega - 4\sqrt{\Omega}(k(\tilde{r}_0 s + \tilde{c}) - \tfrac{1}{2}\tilde{R} + \tilde{r}_0 - 1 - kM_1) + 4\tilde{R}(\tilde{r}_0 s + \tilde{c}), \tag{39}$$

$$M_1' = 1 - M_1\left(k - \frac{\tilde{R}}{\sqrt{\Omega}}\right). \tag{40}$$

At leading order the second equation integrates immediately to give

$$M_1 = \left(\tilde{c} - \frac{1}{k}\right)e^{-ks} + \frac{1}{k},$$

where the constant of integration has been chosen to match to the inner region. Using this expression, (39) is the Bernoulli equation

$$\Omega' = -2k\Omega - 4\sqrt{\Omega}(k\tilde{c} - 2 + (1 - k\tilde{c})e^{-ks}),$$

where to match to the inner solution we require $\Omega(0) = 0$, and where for the moment we have taken $r_0 = 0$ to make analytical progress. Using the substitution $z = \sqrt{\Omega}$ this equation can be integrated to obtain

$$\sqrt{\Omega} = 2\left(\left(\frac{2}{k} - \tilde{c}\right)(1 - e^{-ks}) + (k\tilde{c} - 1)se^{-ks}\right).$$

The expression for Ω becomes invalid if $\Omega \approx 0$. If this occurs at s_i then we have approximately that

$$1 - e^{-ks_i} = \left(\frac{1 - k\tilde{c}}{2 - k\tilde{c}}\right) ks_i e^{-ks_i}. \tag{41}$$

If the root of this equation exists, and it does for the parameter set considered here, then there is an interior boundary later at $s = s_i$ in the optimal asset allocation strategy. In this region the dominant balance in the governing equations changes from the intermediate to the outer equations.

From (35) and (36) the outer equations are at leading order

$$\sqrt{w_0}(k(\tilde{r}_0 s + \tilde{c}) + \tilde{r}_0 - 1 - kM_{n-1}) = \tilde{r}_0 s + \tilde{c},$$
$$\dot{M}_{n-1} = 1 + kM_{n-2} - \left(\frac{1 - \tilde{r}_0}{\tilde{r}_0 s + \tilde{c}}\right) M_{n-1} - \frac{kM_{n-1}^2}{\tilde{r}_0 s + \tilde{c}}.$$

In general, the outer equations have no analytical solution. Therefore, we consider the case $n = 2$ again. Combining these two expressions leads to a Ricatti equation for M_1 which has an analytical solution if $r_0 = 0$ since then the coefficients of the equation are constant:

$$\tilde{c}\dot{M}_1 = \tilde{c} - M_1 - kM_1^2. \tag{42}$$

The equation is separable and can be integrated to yield

$$M_1 = \frac{e(s)(1 - \Delta(s)) - (1 + \Delta(s))}{2k(1 - e(s))},$$

where we have written the functions

$$\begin{aligned} \Delta(s) &= \sqrt{1 + 4k\tilde{c}}, \\ e(s) &= \left(\frac{2kC + 1 + \Delta(s)}{2kC + 1 - \Delta(s)}\right) \exp\left(\frac{s\Delta(s)}{\tilde{c}}\right). \end{aligned}$$

and C is a constant of integration. Here, $\Delta(s) > 1$ is the discriminant of the RHS of (42).

The composite expansion is now

$$w(s) \sim \begin{cases} W_0(s/\tilde{R}) + \Omega(s) - \dfrac{2s(1 - \tilde{r}_0)}{\tilde{R}} & \text{for } 0 \le s \le s_i \\ w_0(s) & \text{for } s > s_i, \end{cases} \tag{43}$$

where we have just patched rather than matched the asymptotic expansion in the inner boundary layer for simplicity (Hinch 1991).

Again, we compare the asymptotic expressions with a numerical solution of (35) and (36) using the parameter set (27) unless stated otherwise. The numerical solution uses an adaptive Runge Kutta scheme to integrate the system of n ODES. For the case $n = 2$ and $r_0 = 0$ the results are comparable to Figure 2 of Liu & Yang (2004) (where $r_0 \neq 0$) because the optimal strategy is relatively insensitive to the interest rate on the bond.

Figure 3 shows the optimal asset allocation for claim sizes with an Erlang distribution. Graph (a) is a comparison of the composite asymptotic expansion and the numerical solution. Agreement is good especially over the intermediate region, which extends up until $s = s_i = 2.1$ computed from (41). The second graph (b) shows the uniform convergence of the composite asymptotic expansion (43) as $\tilde{R} \to 0$. The position of the inner boundary layer s_i does not change in the graph because at leading order s_i is independent of $\tilde{R}$. The final graph (c) shows the optimal allocation as the number of terms n increases in the tail distribution $H(y)$. The optimal allocation increases in the intermediate region with n because the mean claim size is n/k: more money is placed in the stock to cover the increased mean claim size. Considerable borrowing is required if the surplus is small because even a moderate claim leads to immediate ruin. The sensitivity of the allocation to the model parameters can be deduced from the composite expansion (43). A summary of the qualitative optimal allocation strategy is given in Figure 4(b).

5 General claim size distributions

The structure of the light-tailed distributions in the previous section suggests that the boundary layer structure of the optimal asset allocation for $\tilde{R} \ll 1$ carries over to more general claim size distributions. We shall verify this assertion by numerical integration using a number of different finite difference schemes and claim size distributions.

5.1 General theory

For an arbitrary claim size distribution the inner equations can be obtained from (16)–(18) by rescaling

$$s = \tilde{R}S, \quad y = \tilde{R}Y,$$

which yields

$$\begin{aligned}
\tfrac{1}{4}\dot{W}(S) &= \sqrt{W}\left(\tfrac{1}{2}\tilde{R} - \tilde{r}_0 + 1 + \frac{\tilde{c}\dot{H}(S)}{u(S)} + m(S)\right) + \tilde{r}_0\tilde{R}S + \tilde{c}, && (44)\\
\dot{u}(S) &= -\tilde{R}\frac{u}{\sqrt{W}}, && (45)\\
m(S) &= \tilde{R}\int_0^S \frac{u(Y)}{u(S)}\dot{H}(\tilde{R}(S-Y))\,dY, && (46)
\end{aligned}$$

where $\dot{} \equiv d/dS$. Consequently, to leading order $u_0 = 1$, $m_0 = 0$ and

$$\dot{W}_0 = -4\zeta\sqrt{W_0} + 4\tilde{c},$$

where

$$\zeta = \tilde{r}_0 - 1 - \tilde{c}H'(0). \tag{47}$$

Integrating and applying the boundary condition gives

$$\tilde{c}\log\left(1 - \frac{\zeta\sqrt{W_0}}{\tilde{c}}\right) + \zeta\sqrt{W_0} + 2\zeta^2 S = 0$$

If $\zeta < 0$ then we expect an intermediate region for the optimal asset allocation strategy because $W_0 \to \infty$ as $S \to \infty$. This was the case for the Erlang distribution. Consider the claim size distributions shown in Table 1. The Erlang and Lognormal distributions have $H'(0) = 0$ and so for a reasonable parameter set they lead to an optimal allocation strategy with an intermediate region since $\tilde{r}_0$ is small. In general terms, if the probability density function of the claims distribution is zero at $y = 0$ (corresponding to few small claims) then there is an intermediate region in the optimal asset allocation strategy. If claims are likely to be close to the mean then the insurer must invest more in the risky asset to prevent ruin.

If there is an intermediate region then we introduce the intermediate variable

$$w = \frac{\Omega}{\tilde{R}^2}$$

and the intermediate equations are from (16)–(18)

$$\begin{aligned}
\tfrac{1}{4}\Omega'(s) &= \sqrt{\Omega}\left(\tfrac{1}{2}\tilde{R} - \tilde{r}_0 + 1 + \frac{\tilde{c}H'(s)}{u(s)} + m(s)\right) + \tilde{R}(\tilde{r}_0 s + \tilde{c}), && (48)\\
u'(s) &= -\tilde{R}\frac{u}{\sqrt{\Omega}}, && (49)\\
m(s) &= \int_0^s \frac{u(s-y)}{u(s)}\,dH(y). && (50)
\end{aligned}$$

So at leading order $u_0 \equiv 1$ to match to the inner solution and

$$m_0(s) = H(s) - 1.$$

Substituting into (48) yields a Bernoulli equation which can be integrated to give

$$\sqrt{\Omega} = 2(\tilde{c}(H(s) - 1) + I(s) - \tilde{r}_0)s), \tag{51}$$

where $I(s) = \int_0^s H(y)\,dy$ is the integrated tail distribution of the claim sizes. Using the general equations we appear to be able to determine the intermediate region analytically with $r_0 \neq 0$. There is an interior boundary layer at s_i given by the root of

$$\tilde{r}_0 s_i = \tilde{c}(H(s_i) - 1) + I(s_i).$$

From (15), the outer optimal strategy to leading order is given by

$$(\tilde{r_0}s + \tilde{c})u_0(s) = \tilde{c}H(s) + \int_0^s u_0(y)H(s-y)\,dy, \tag{52}$$

if $\tilde{R} \ll 1$. This is a *linear* Volterra integral equation of the second kind with kernel H and only has an analytical solution for particular claim size distributions. However, this does represent a simplification of the problem: we have replaced a nonlinear integro-differential equation with a linear integral equation. If there is no bond in the portfolio so $\tilde{r}_0 = 0$ then this integral equation is the classical ruin equation in insurance (see Rolski *et al.* 1999 and differentiate equation (5.3.9)). At leading order the survival probability is just that which minimises the probability of ruin for an insurer whose claims have a compound Poisson distribution. It is simple to see why this is the case: when the stock is very risky it is optimal to allocate very little reserve to the stock. The probability of ruin is then just determined by the risk process R_t. At leading order the survival probability determines the leading order optimal asset allocation through the feedback law (13):

$$A_0^*(s) = -\eta \frac{u_0(s)}{u_0'(s)},$$

since $u_0(s) = \delta_0'(s)$.

5.2 General numerics

If the claims distribution does not reduce to a system of ODEs then one is forced to solve the integro-differential equation (15) numerically. We shall use Pareto, Lognormal and Weibull distributions to illustrate the optimal allocation strategy for heavy-tailed claim size distributions. The tail distribution functions are given in Table 1. Three numerical schemes are considered.

The first scheme is a modification of that presented by Liu & Yang (2004). They differentiated (15) to obtain a system of equations similar to (16)–(18). In order to solve this system numerically they used an Euler finite difference step for the derivative in (16) and a simple step function approximation to the integral in (18). If one adopts a similar technique for the original integro-differential equation one finds that the numerical scheme is unstable. However, the integral in equation (31) of Liu & Yang contains a singular kernel, and it is not clear how they have coped with that singularity. We have used the asymptotic expressions for small surplus to initiate the numerical method at $s = s_0$ so that the singularity is avoided. We call this scheme the modified Liu & Yang scheme.

Two computational difficulties are encountered. First, as $s \to 0$ the gradient of u tends to infinity from (20). If we start the numerical scheme at $s_0 \ll 1$ then the error in a finite difference approximation is very large. However, we need to choose a small value of s_0 in order to resolve the boundary layer of the optimal strategy: for $s \sim \tilde{R} \ll 1$ the optimal asset allocation strategy rapidly varies with the current surplus. Second, as $s \to \infty$, $u \to 0$ because it is the derivative of the survival probability. If the distribution

is light-tailed then this decay is exponential and will rapidly become of the order of the truncation error of the finite difference approximation. Therefore we must adopt a grid spacing which leads to a truncation error much smaller than u for large values of s.

We combat the first problem by using the coordinate transformation:

$$s = \psi^2, \tag{53}$$

so that the first derivative of u wrt. ψ is finite at $\psi = 0$. However, the singularity in u is now removable rather than avoided altogether: we must still start the numerical integration at $\psi_0 = \sqrt{s_0}$. We also use the extra terms in the asymptotic expansions at s_0, which were derived in Section 3.1. First order finite differences are used for derivatives and a step function for the integral in (18). We call this the first order scheme.

In order to tackle numerical problems as $s \to \infty$ we introduce a second order scheme. This has a smaller truncation error for given step size and so allows the computation to proceed for larger values of s. Let us rewrite the integro-differential equation (15) as a system of two Volterra integral equations using the transformation (53):

$$u(\psi) \approx \int_{\psi_0}^{\psi} \frac{\tilde{R} Y u^2(Y)}{(\tilde{r}_0 \psi^2 + \tilde{c}) u(Y) - \tilde{c} H(Y^2) - v(Y)} \, dY + u_0(\psi), \tag{54}$$

$$v(\psi) \approx \int_{\psi_0}^{\psi} 2 Y u(Y) H(\psi^2 - Y^2) \, dY + v_0(\psi), \tag{55}$$

where $y = Y^2$ and we have used the asymptotic expressions u_0, v_0 as $s \to 0$ to start the integration at ψ_0. Notice that we are working with the original integro-differential equation rather than its differentiated form.

It is convenient to write the equations in vector form

$$f(\psi) = \int_{\psi_0}^{\psi} K(\psi, Y, f) \, dY + g(\psi_0), \tag{56}$$

where

$$f = \begin{pmatrix} u \\ v \end{pmatrix}, \quad K(\psi, Y, f) = \begin{pmatrix} \dfrac{\tilde{R} Y u(Y)^2}{(\tilde{r}_0 \psi^2 + \tilde{c}) u(Y) - \tilde{c} H(Y^2) - v(Y)} \\ 2 Y u(Y) H(\psi^2 - Y^2) \end{pmatrix}, \quad g = \begin{pmatrix} u_0(\psi) \\ v_0(\psi) \end{pmatrix}.$$

We approximate the integral on the uniform grid given by

$$\psi_i = \psi_0 + ih, \quad \text{for } i = 1, \ldots, N,$$

and use the trapezium rule to approximate the integrals. This leads to an implicit scheme at step i so we define the functions

$$F_i(f_0, \ldots, f_i) := f_i - \tfrac{1}{2} h K_i(f_i) - h \left(\tfrac{1}{2} K_0(f_0) + \sum_{j=1}^{i-1} K_j(f_j) \right) - g = 0, \tag{57}$$

for $0 \leq i \leq N$.

In order to calculate the root of this equation we use Newton's method: if f_i approximates the root then the correction δf_i satisfies

$$J\delta f_i = -F_i(f_0, \ldots, f_i).$$

Here, the Jacobian J is a matrix given by

$$(J^{mn}) = \frac{\partial F_i^{(m)}}{\partial f_i^{(n)}}$$

and superscripts denote vector components. It can be inverted easily for this two equation system to calculate δf_i. The correction to the root is $f_i + \delta f_i$ and this procedure is continued until $||\delta f_i||$ is less than a prescribed tolerance. The numerical method then proceeds to f_{i+1} up until f_N. We call this the second order scheme.

Define the residual of (15) by

$$E(s) := \tfrac{1}{2}\tilde{R}\sqrt{w(s)}u(s) + (\tilde{r_0}s + \tilde{c})u(s) - \tilde{c}H(s) - \int_0^s u(y)H(s-y)\,dy,$$

as evaluated on the finite difference grid. For the modified Liu & Yang and the first order scheme a trapezium rule can be used to calculate the integral. We shall find if one does not resolve the boundary layer of the optimal strategy then this residual may not be small even though the numerical solution appears reasonable. The convergence properties of the numerical solution can be assessed by varying the initial surplus s_0 and the step size h.

We start by validating the new numerical schemes for the Exponential and Erlang ($n = 2$) distributions in Figure 5(a),(b) respectively. For these distributions the exact solution is easy to calculate by integrating the ODEs numerically. In Figure 5(a) the second order scheme is indistinguishable from the integrating the ODE, but the first order scheme shows considerable error as s becomes large. The same behaviour occurs in the second order scheme for $s \gg 10$ and is caused by the rapid decay in the values of u and v to values of the order of the truncation error. For Erlang distributed claims in Figure 5(b) all schemes give good agreement over the integration range because the tail distribution decays less rapidly than the exponential case.

Figure 6 is an attempt to reproduce the optimal asset allocation strategy for the Pareto distribution given in Figure 3 of Liu & Yang (2004) . We have chosen $s_0 = 10^{-2}$ in order to approximately reproduce their results in graph (a) using our modification of their method. However, it is clear by the considerable disagreement between the numerical schemes that convergence has not been achieved in one or more of these schemes. We require that as the step size $h \to 0$ and as the initial surplus $s_0 \to 0$ then the maximum error in the optimal numerical strategy tends to zero. Further we require that the residual $E(s) \to 0$ for all s because it represents the truncation error in the numerical method.

Figure 6(b) shows the residual E from these schemes: only the second order scheme yields small residuals. Graphs (c) and (d) use the same parameter set but reduce the

initial start point to $s_0 = 10^{-4}$. Again, in graph (c) there is considerable disagreement in results and the modified Liu & Yang and the first order scheme demonstrate poor convergence at large s. For both the modified Liu & Yang and the first order schemes integration terminates before the end of the integration range because u and w become negative. If one reduces the step size h drastically then both these methods converge towards the second order scheme, but then computational time becomes prohibitive. A non-uniform grid may yield better performance. It is clear that the second order scheme is the best behaved numerical method as it yields consistent results in graphs 6(a) and 6(c), but it is still not clear what is the optimal strategy.

We confirm in Figure 7 that the second order scheme does give a good approximation to the optimal strategy by examining the limits $s_0 \to 0$ and $h \to 0$. The original graph in Liu & Yang is inaccurate because it fails to adequately resolve the boundary layer near $s = 0$ and fails to converge at large s. For this parameter set the second order scheme displays adequate convergence at $N = 1000$ steps and $s_0 = 10^{-4}$. Figures 8, 9 show the optimal strategy for Lognormal and Weibull distributions with parameters given in the captions. Both distributions are sub-exponential so $A^* \to \infty$ as $s \to \infty$ although this is not demonstrated on the graphs. For the lognormal distribution $\zeta < 0$ so there is an intermediate region in Figure 8, while for Weibull distributions $\zeta = \infty$ so there is no intermediate region in Figure 9. On Figure 8 we have added the composite asymptotic expansion for the inner and intermediate regions to demonstrate the applicability of the asymptotic method. The integrated tail distribution in (51) was calculated using an adaptive Simpson method. Figure 10 is a cautionary example, which further demonstrates that one must check for convergence of the numerical scheme. Here, we have reduced the resolution of the finite difference grid if claims are Weibull distributed, but left the other parameters as in Figure 9. The optimal strategy in Figure 10(a) looks plausible, but in (b) it is clear that the truncation error is large and a much finer grid is required.

The qualitative structure of the optimal allocation strategy is shown in Figure 4(c) for the Pareto claim and Lognormal claim size distributions and in Figure 4(d) for the Weibull distributions.

Define the terms in the integro-differential equation (15) by

$$t_0 = \tfrac{1}{2}\tilde{R}\sqrt{w}u, \quad t_1 = (\tilde{r_0}s + \tilde{c})u(s), \quad t_2 = \tilde{c}H(s), \quad t_3 = \int_0^s u(y)H(s-y)\,dy.$$

In Figures 8,9(b) we plot t_1, t_2, t_3 for the Lognormal and Weibull distributions distributions and the parameters given in the captions. Term t_2 decays rapidly as $s \to \infty$ since it is the tail distribution function. Terms t_1 and t_3 balance for both distributions, which confirms that the outer approximation (52) is valid for $s \sim 1$. Furthermore, the residuals for both Lognormal and Weibull distributions are very small so the boundary layer is adequately resolved for the chosen step size h. These results suggest that the boundary layer structure, which we developed in Section 5.1, does hold for heavy-tailed distributions.

Figure 11 shows how the asymptotic structure of the optimal strategy changes as the stock becomes safer i.e. as $\tilde{R} \to \infty$. We examine this behaviour by decreasing the

volatility of the stock and comparing the optimal strategy with the asymptotic result of Schmidli (21). As the volatility decreases the stock offers a greater return than the bond without a concomitant increase in risk. The optimal strategy is to increase substantially the amount allocated to the stock. For large values of s the second order method accurately computes the optimal strategy since it compares favourably with (21). The intermediate region is still discernable for small surplus values even as the volatility is decreased outside the range of validity of the asymptotic expansions. The terms of (15) are plotted in Figures 11(b),(c) for $\sigma = 0.05, 0.1$. As s increases all the terms in (15) become comparable and very small so that the outer balance (52) breaks down and one should use (21) to approximate the optimal strategy. This also highlights the difficulty in computing the optimal strategy for large s because the error in any finite difference scheme must be smaller than the size of these terms in order to maintain accuracy. This is easier to achieve with a higher order numerical method.

6 Conclusions

There are few analytical solutions for the optimal investment problem for an insurer. Hipp & Plum (2000) found a solution when the risk process is compound Poisson, the interest rate is zero, and the claim size distribution is exponential. Hipp & Plum (2003) found another solution if the income from interest and claims is constant. We have identified the limiting behaviour of the optimal strategy as the asset becomes increasingly risky when the cash flow is normally distributed and when the risk process is a compound Poisson process. If the claim size distribution is exponential we have found a uniformly valid asymptotic expansion for the optimal stock allocation. The optimal strategies for a normal risk process and a risk process with exponentially distributed claims are comparable.

For the classical ruin problem there is an analytical expression for the ruin probability if the claims are distributed exponentially. For more general claims distributions the De Vylder approximation uses moment matching in order to derive an explicit expression for the ruin probability (Rolski *et al.* 1999). It is tempting to use the asymptotic solution for exponential claims and match moments for more general distributions in order to approximate the optimal asset allocation strategy. Given the qualitative differences in the optimal strategy for light and heavy-tailed distributions we can see that this will be a poor approximation over a large range of current surplus values. It is better to calculate the optimal strategy numerically.

An understanding of the asymptotic structure of the allocation strategy allows us to interpret the optimal stock allocation and also formulate a good numerical method for its computation. If the stock is risky then there is a thin boundary layer which differentiates the optimal stock allocation for a normal risk process from a compound Poisson risk process. If the current surplus is very small then the latter model says it is optimal to invest very little in the stock. This boundary layer must be resolved, that is, sufficient grid points must be placed across the region of rapid change. Furthermore, as the current surplus becomes large, the tail distribution function decays rapidly, which leads to an

unstable numerical problem. If there are few small claims then an intermediate region is present, which means it is optimal to invest in the stock an amount comparable to the mean claim size.

The parameters of the model and the claim size distribution change the optimal amount invested in the stock, but the overall strategy can be characterised according to the form of the claim size distribution. A qualitative summary of the optimal stock allocation is given in Figure 4. For the parameters we have chosen in the paper, the exponential claim size distribution corresponds to 4(a), Erlang distributed claim sizes lead to 4(b), Pareto & Weibull distributions yield 4(c), while a Lognormal claim size distribution gives the optimal strategy in 4(d).

We have described three numerical methods: the second order method gives greater accuracy for given surplus step but requires more computational time because it is an implicit method. The numerical results reveal the sensitivity of the numerical approximation to the initial surplus value and the resolution of the finite difference grid. We confirmed the convergence of the schemes using step size reduction and examination of the residual of the integro-differential equation. The application of these techniques to other optimal control problems in insurance (Hipp 2004) may yield further insight where analytical solutions are not present.

Acknowledgements

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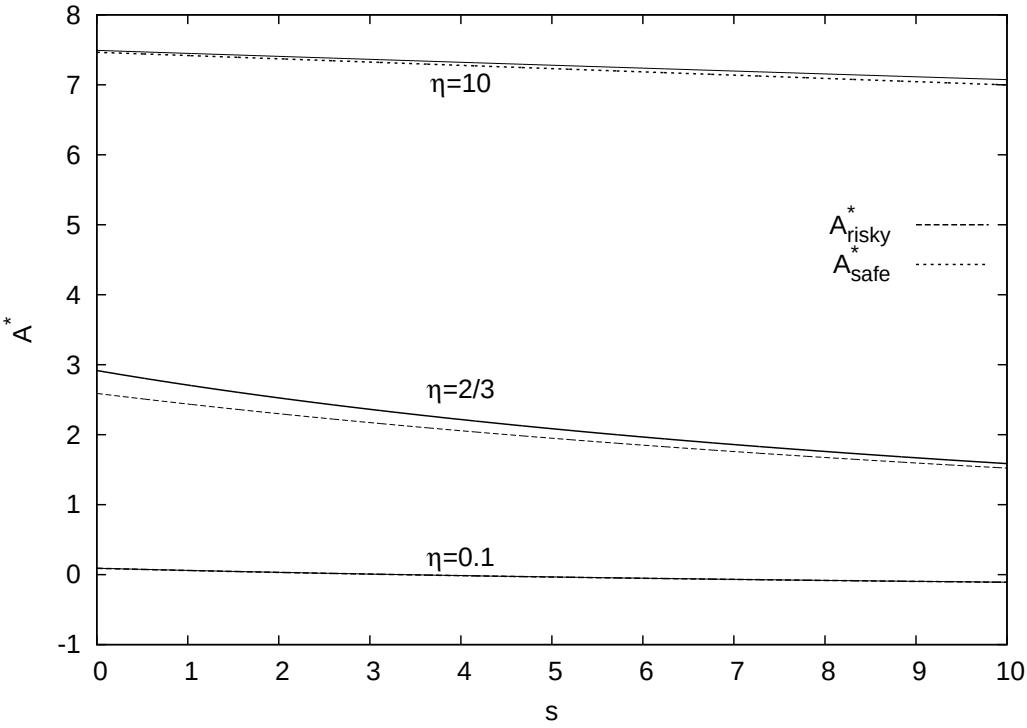
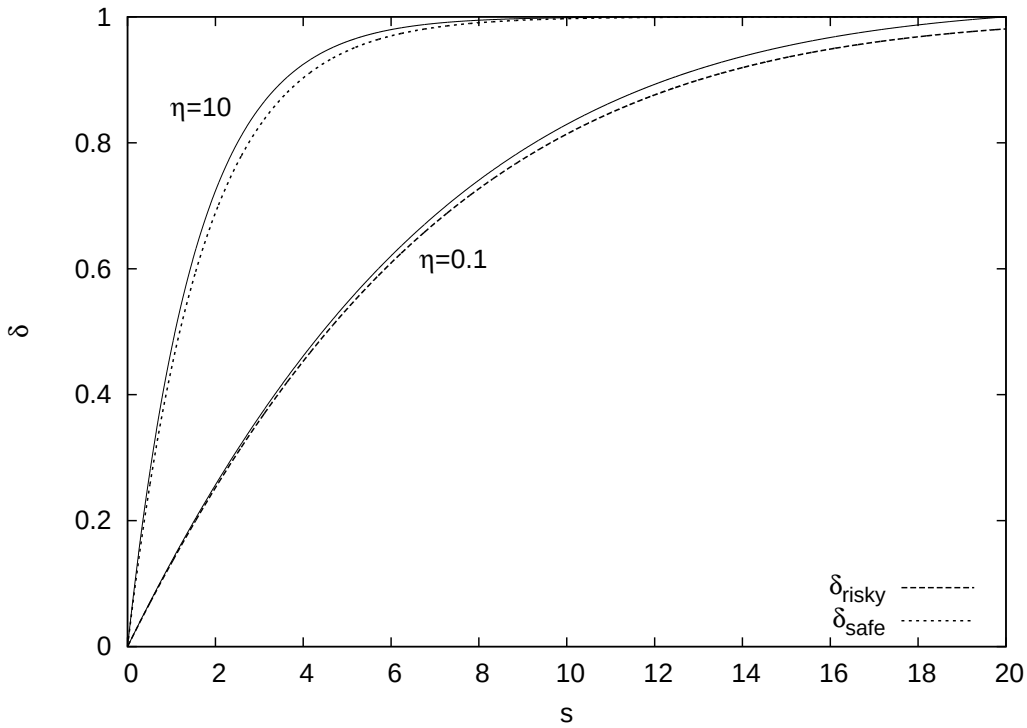
Tables and Figures

Distribution	$H(y)$	$H'(0)$	ζ	Parameters
Exponential	e^{-ky}	$-k$	$\tilde{r}_0 + \theta$	$k > 0$
Erlang	$\sum_{i=1}^{n} [(ky)^{i-1}/(i-1)!]e^{-ky}$	0	$\tilde{r}_0 - 1$	$n > 1$, $k > 0$
Pareto	$(\beta/(y+\beta))^{\alpha}$	$-\dfrac{\alpha}{\beta}$	$\tilde{r}_0 - 1 + \dfrac{\alpha}{\alpha - 1}$	$\alpha > 2$, $\beta > 0$
Lognormal	$1 - \phi\left(\dfrac{\log y - M}{S}\right)$	0	$\tilde{r}_0 - 1$	$M > 0$, $S > 0$
Weibull	e^{-cy^r}	$-\infty$	∞	$c > 0$, $0 < r < 1$

Table 1: Claim size tail distribution functions, $H(y)$, for the Compound Poisson model of the risk process R_t. Here, M and S are the mean and standard deviation of $\log Y_1$ and the standard normal distribution function is ϕ. If $r \geq 1$ then the Weibull distribution is light-tailed. The parameter ζ determines the existence of an intermediate region and is given by (47).

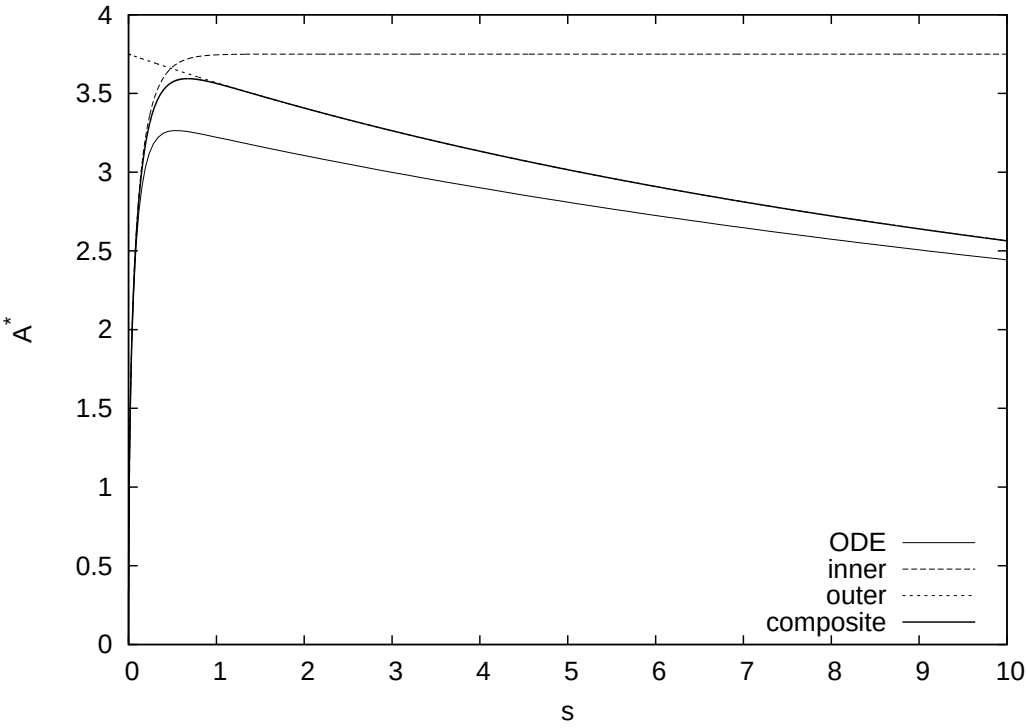


(a)



(b)

Figure 1: Optimal investment strategy with a normally distributed cash flow as a function of the current surplus s. Graph (a) shows the optimal stock allocation $A^*(s)$ while (b) shows the corresponding survival probability $\delta(s)$. The thin solid lines in the figure correspond to varying the risk in the asset: the survival probability is calculated using numerical integration. The broken lines in (a),(b) show the asymptotic asset allocation and survival probability respectively to first order in η given in Section 2.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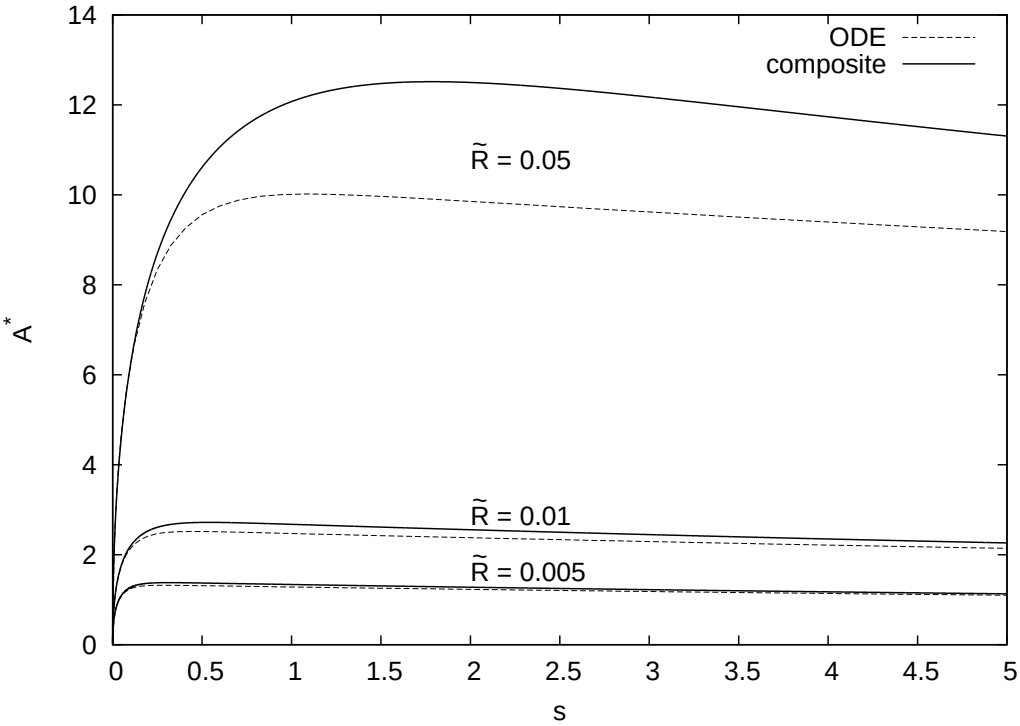
(a)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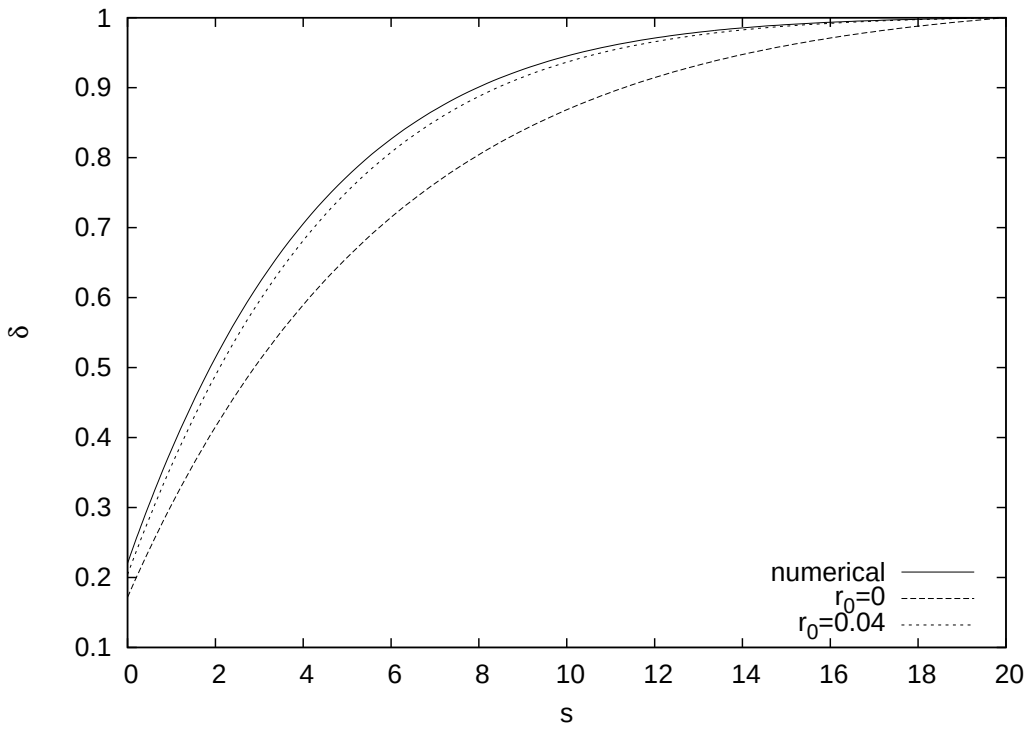
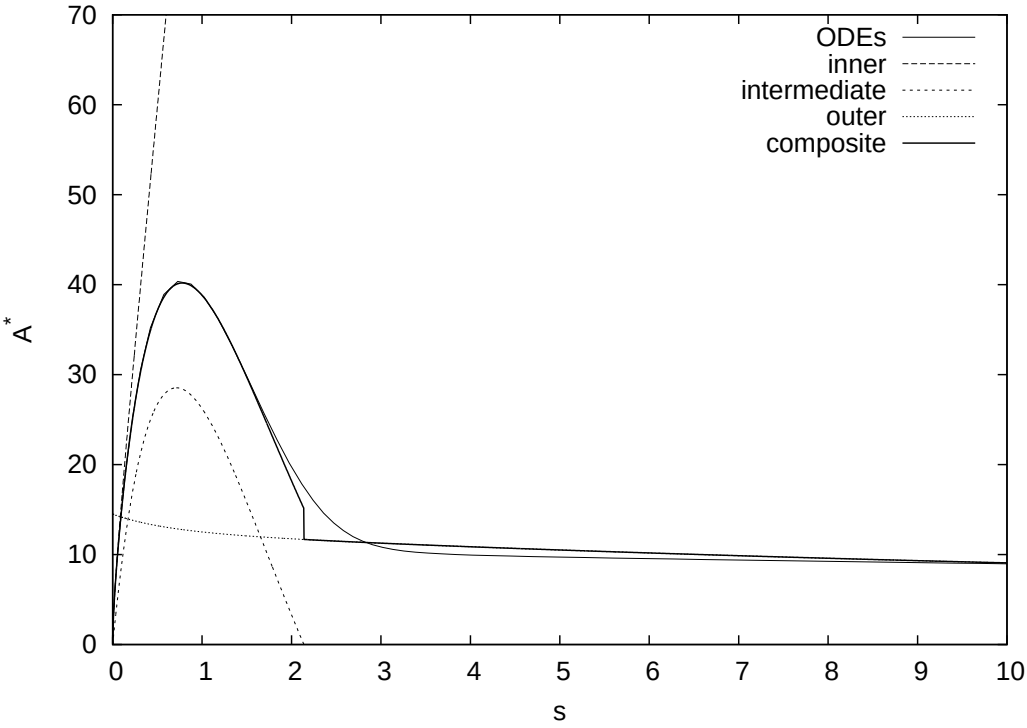
R̃ = 0.05
R̃ = 0.01
R̃ = 0.005

(b)



(c)

Figure 2: Comparison of the asymptotic and numerical solution for a compound Poission risk process with exponentially distributed claim sizes. Graph (a) shows the asymptotic optimal allocation strategy superimposed on the numerical solution of equation (22). The parameter set chosen is that used in Figure 1 of Liu & Yang (2004) which gives $\tilde{R} = 0.013$. Graph (b) demonstrates the uniform convergence of the asymptotic asset allocation to the numerical solution as $\tilde{R} \to 0$ corresponding to an increasingly risky stock. Graph (c) shows the corresponding results for the survival probability $\delta(s)$.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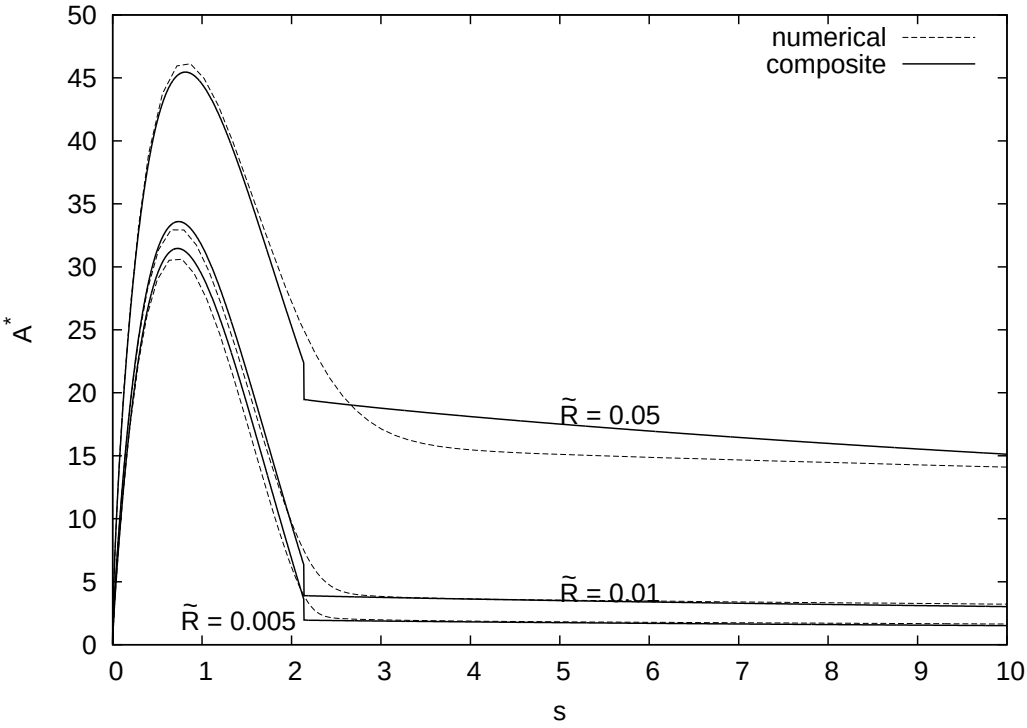
(a)


numerical
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R̃ = 0.01
R̃ = 0.005
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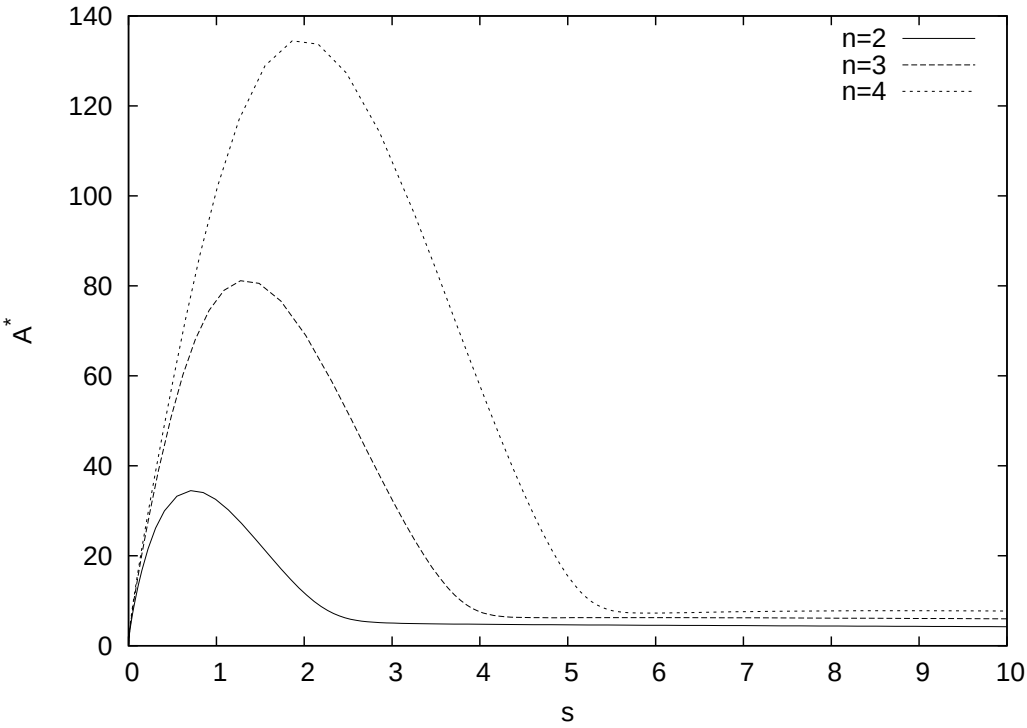
(b)



(c)

Figure 3: Comparison of the asymptotic and numerical solution for a compound Poission risk process for Erlang distributed claim sizes. The parameter set chosen for graph (a) corresponds to that used in Figure 2 of Liu & Yang (2004). In graph (b) we show the uniform convergence of the asymptotic optimal allocation strategy to the numerical solution as $\tilde{R} \to 0$, while in (c) we have increased the number of terms n in the Erlang tail distribution function $H(y)$.

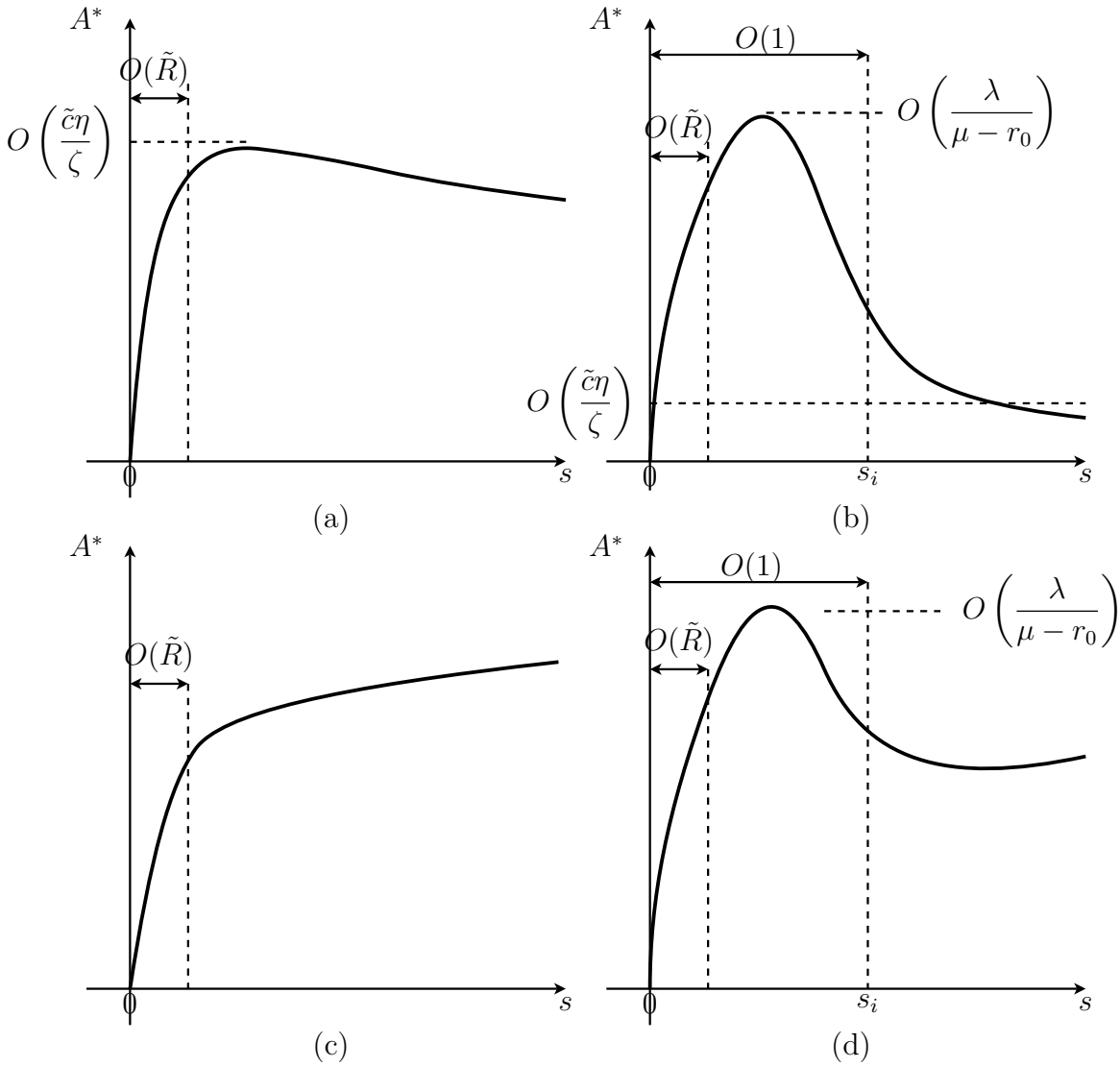
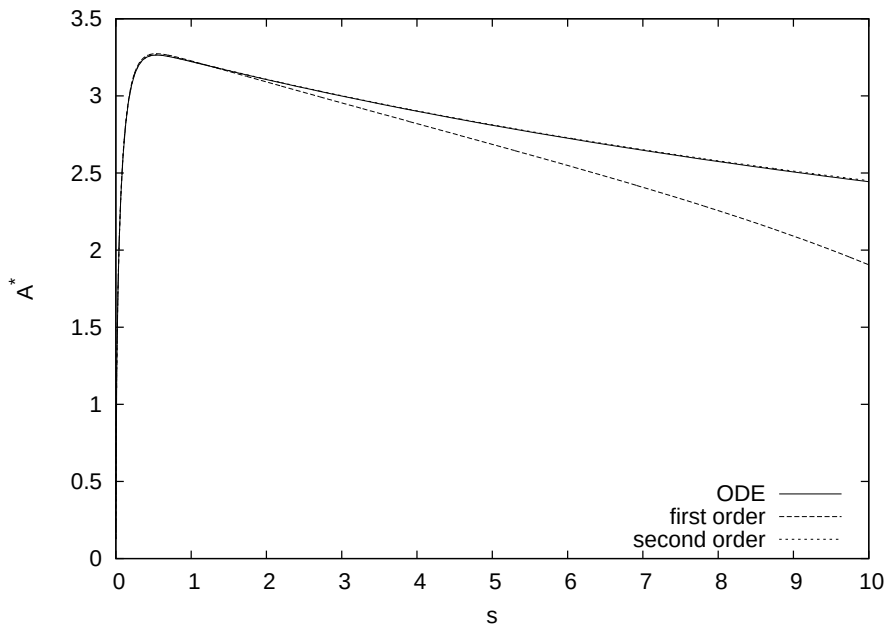
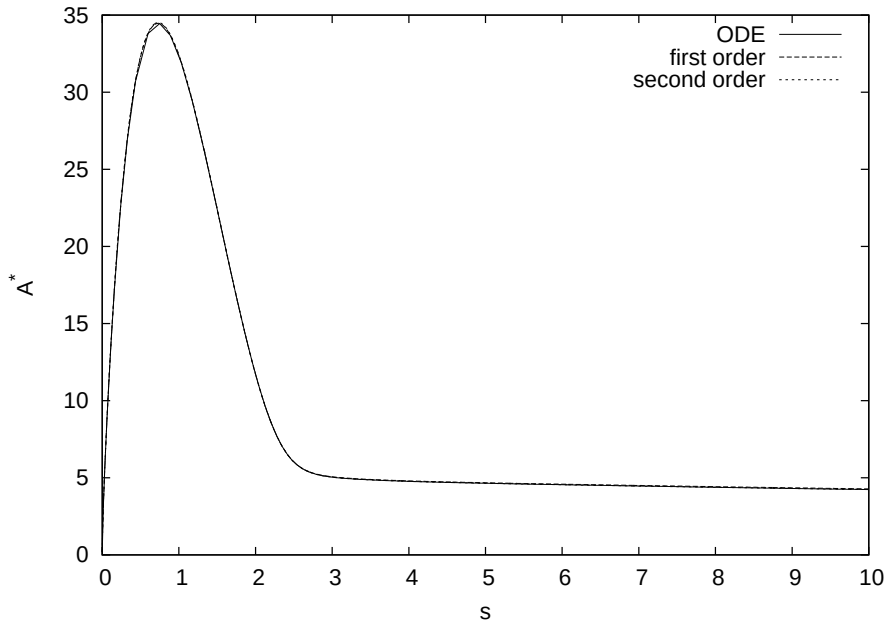


Figure 4: Summary of the qualitative structure of the optimal stock allocation A^* to minimise the ruin probability if $\tilde{R} \ll 1$. The strategy can be categorised according to properties of the claim size distribution: (a) Light-tailed, (b) Light-tailed with few small claims, (c) Heavy-tailed and (d) Heavy-tailed with few small claims .

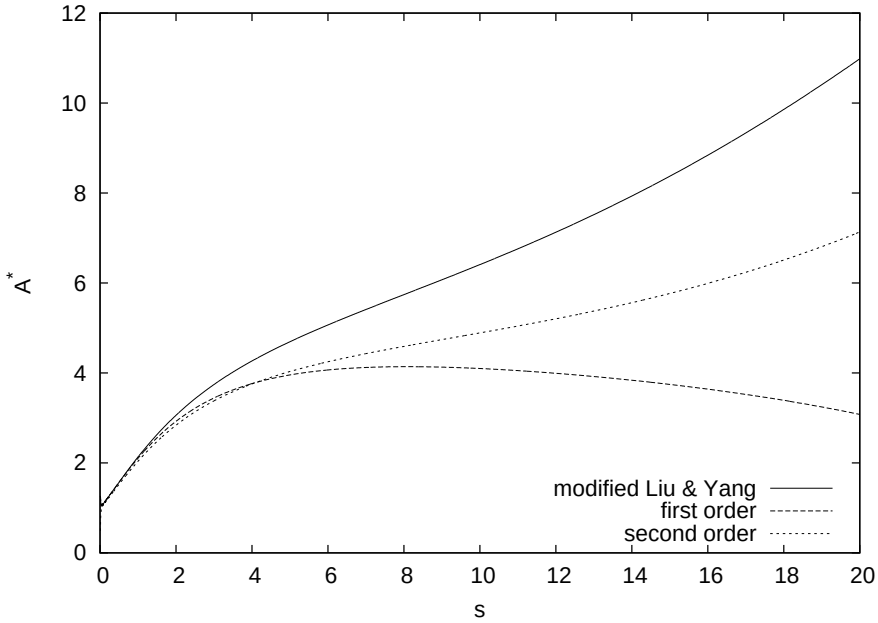


(a)



(b)

Figure 5: Validation of the first and second order numerical schemes to solve the integro-differential equations (16)–(18) using the numerical solution of the ODE (22) and an exponential claim-size distribution. Graphs (a),(b) are a reproduction of Figures 1,2 respectively contained in Liu & Yang (2004) and use the parameter set given in that paper: $\lambda = 3$, $r_0 = 0.04$, $\sigma = 0.3$, $\theta = 0.2$, $\mu = 0.1$, $k = 1$, $n = 2$. Numerical parameters are $s_0 = 10^{-4}$ and $N = 2000$.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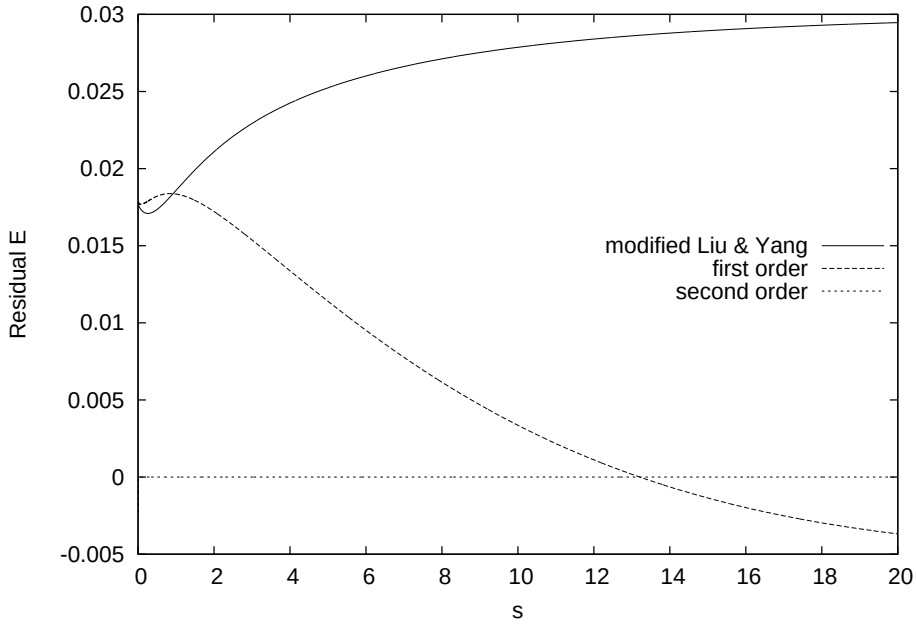
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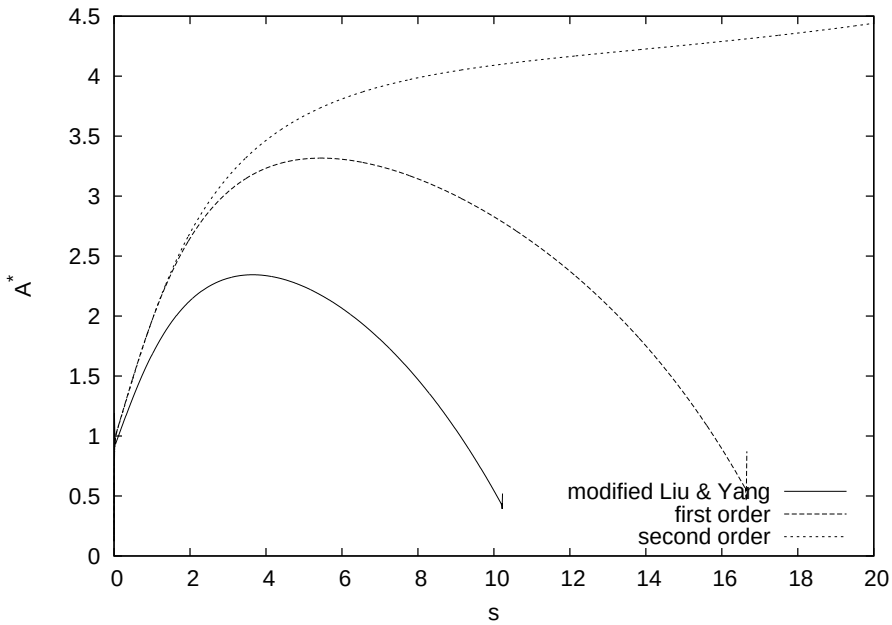

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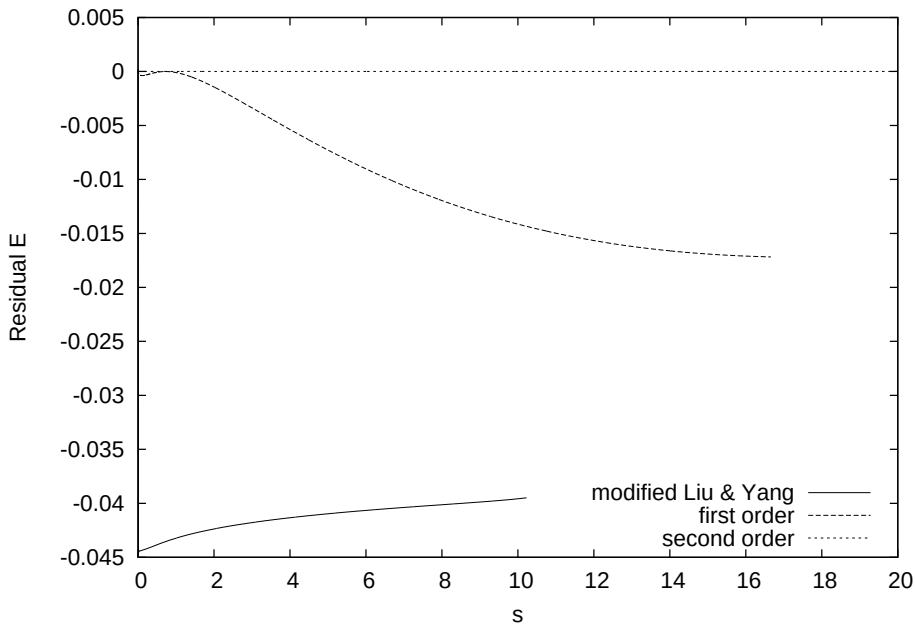
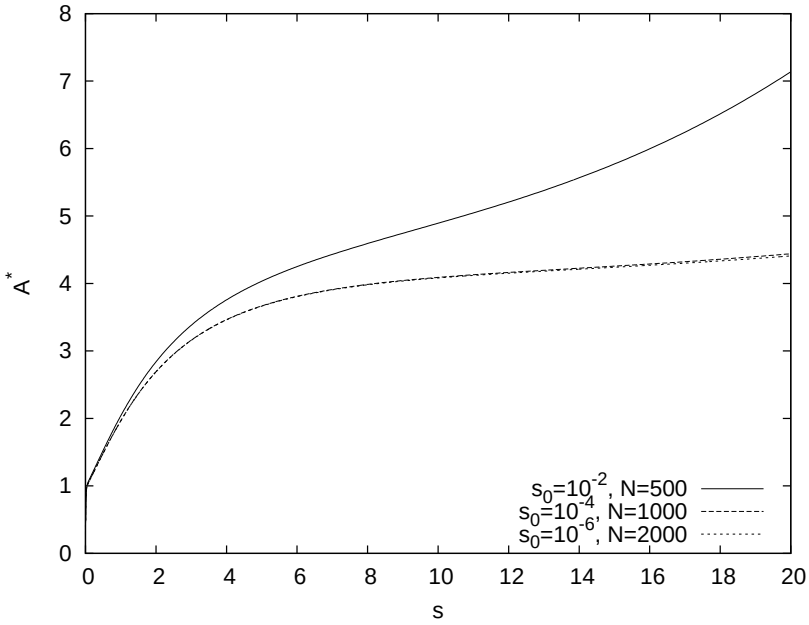


(d)

Figure 6: Comparison of numerical methods for calculating the optimal asset allocation A^* for claim sizes with Pareto distribution. Graphs (a) and (b) show the results for $s_0 = 10^{-2}$, while (c) and (d) show results for $s_0 = 10^{-4}$. In all graphs $N = 2000$ and $\alpha = 3$, $\beta = 2$ corresponding to the choice of parameters chosen in Figure 3 of Liu & Yang (2004). In (b) and (d) the second order method generates residuals of the order of 10^{-16}.



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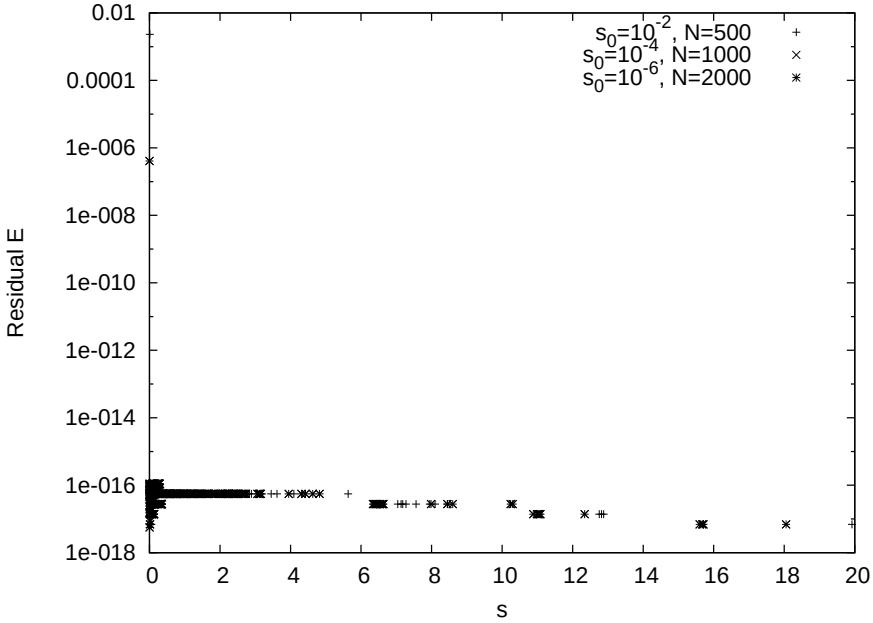
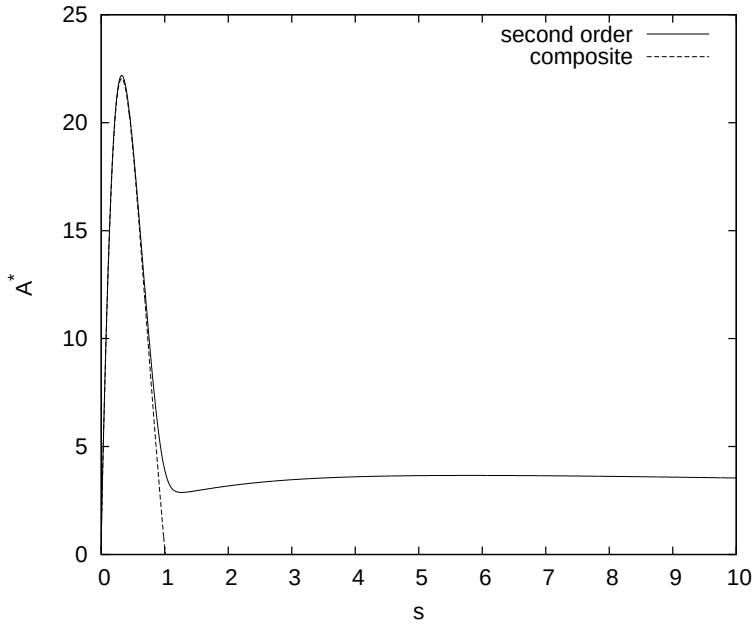


(b)

Figure 7: Behaviour of the second order method as the starting surplus s_0 is decreased and the number of grid points N is increased when the claim sizes have Pareto distribution. The residuals are shown on a logarithmic scale since they are only significant at s_0 when s_0 is relatively large.



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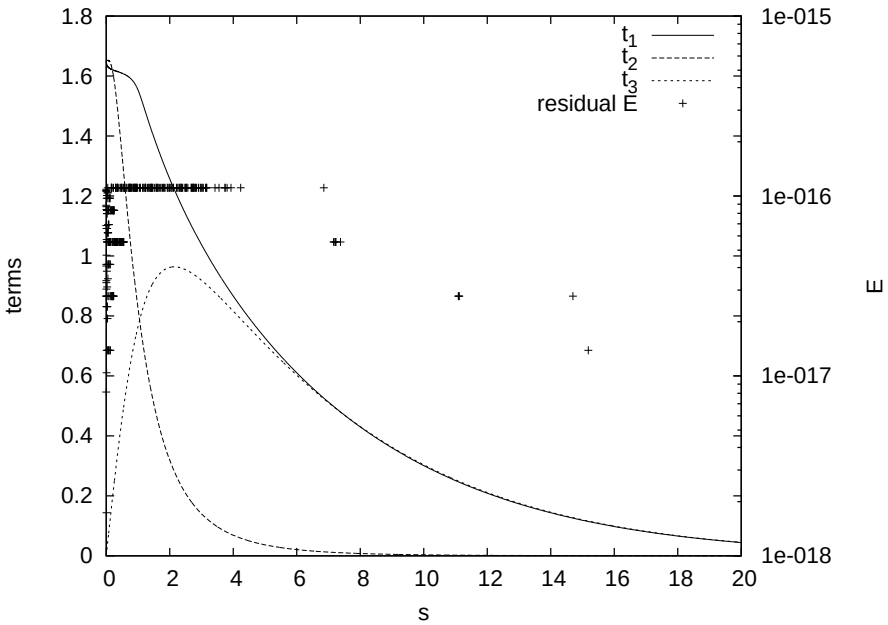
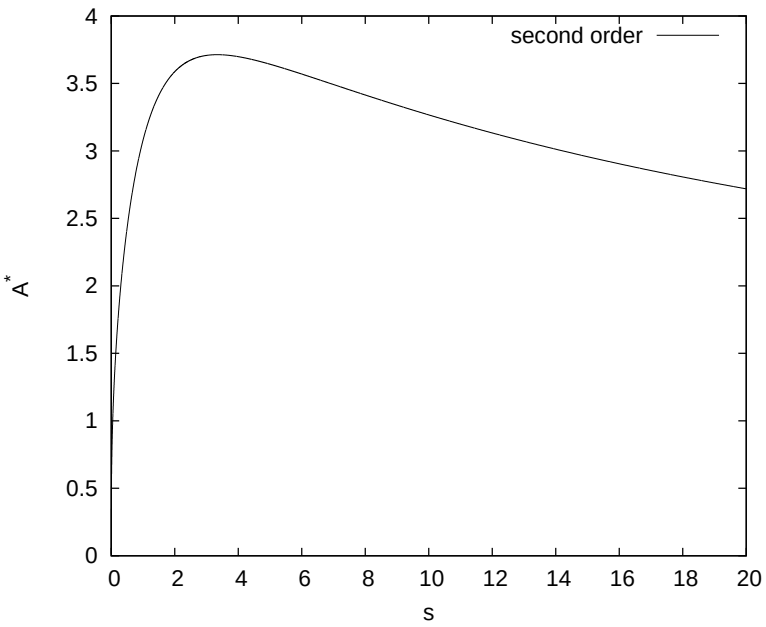
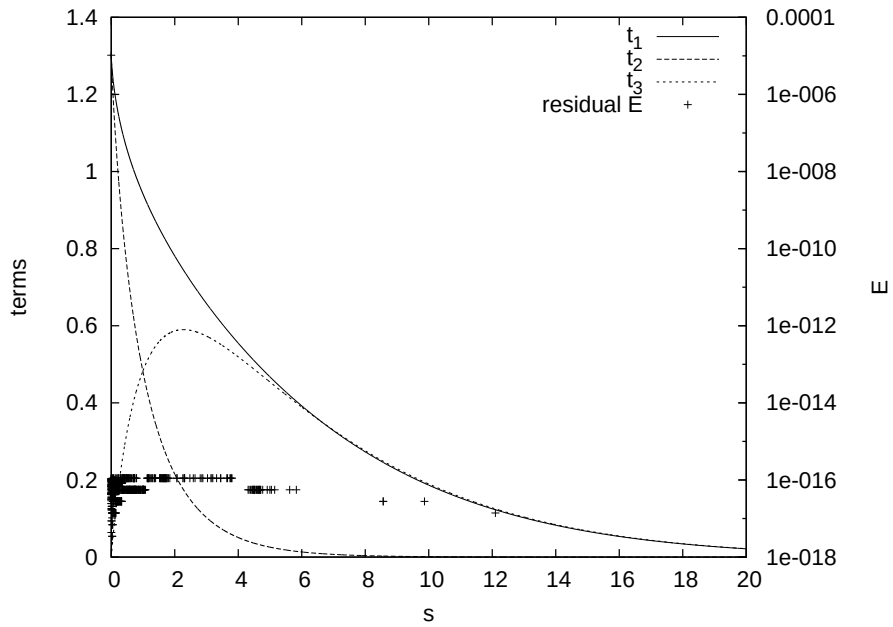


(b)

Figure 8: Optimal asset allocation if the claims are lognormally distributed. The model parameters are $M = 0$ and $S = 0.8$ and the remaining parameter are identical for those used to compute the Exponential distribution. The numerical parameters are $N = 2000$, $s_0 = 10^{-4}$. The residual E given by the right-hand scale of the graph in (b).

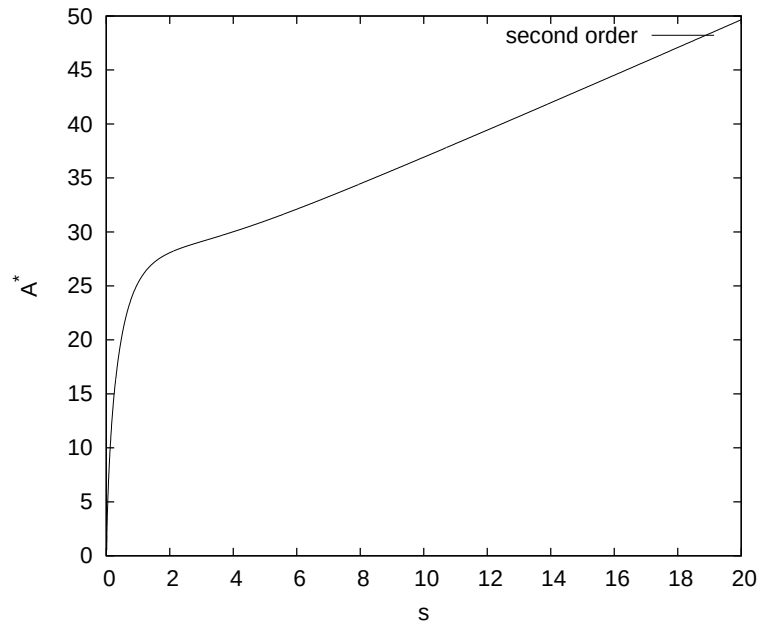


(a)



(b)

Figure 9: Optimal asset allocation if the claims are Weibull distributed with $r = 0.85$, $c = 1$. The second order scheme has been used with $s_0 = 10^{-6}$ and $N = 4000$. The residual E given by the right-hand scale of the graph in (b).



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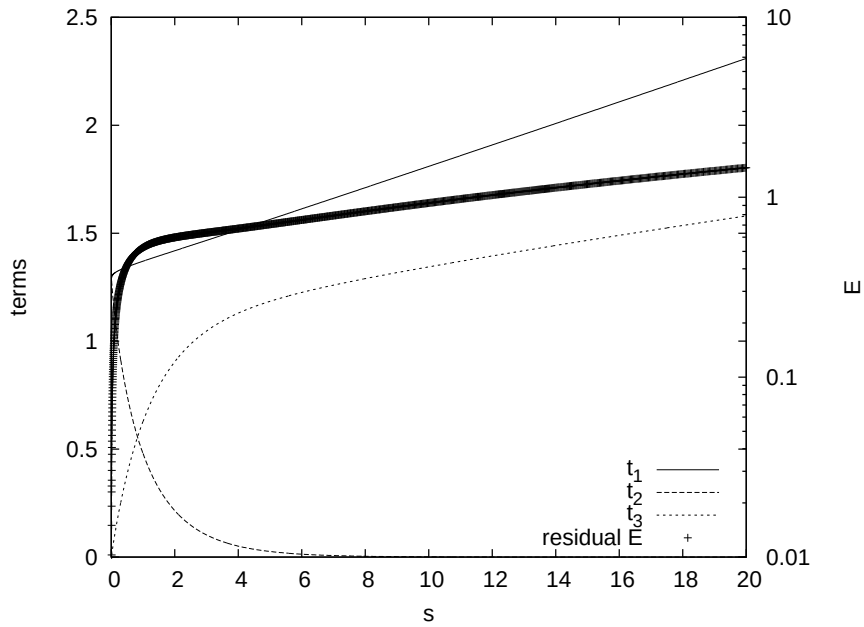
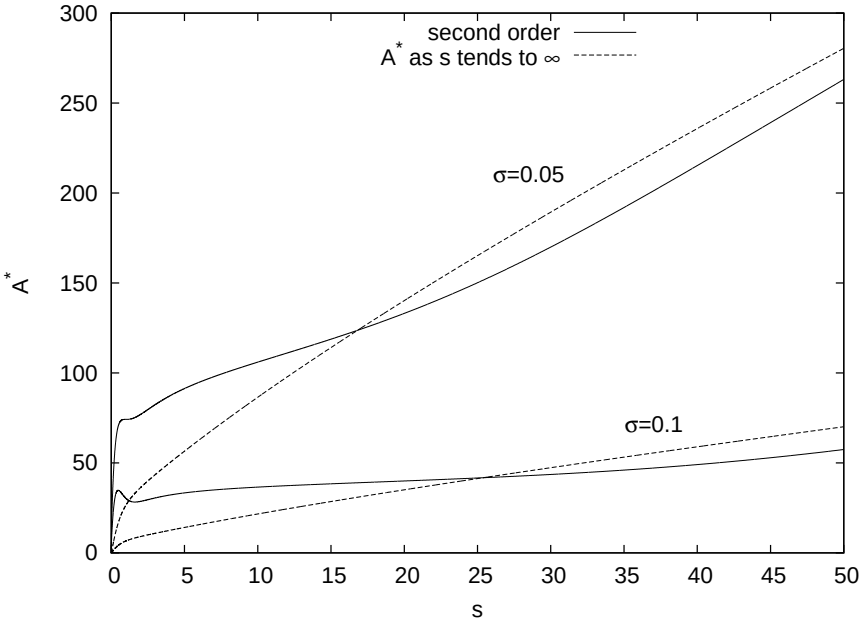


(b)

Figure 10: Optimal asset allocation if the claims are Weibull distributed with $r = 0.85$, $c = 1$. The second order scheme has been used with $s_0 = 10^{-2}$ and $N = 1000$. The residual E given by the right-hand scale of the graph in (b).


second order
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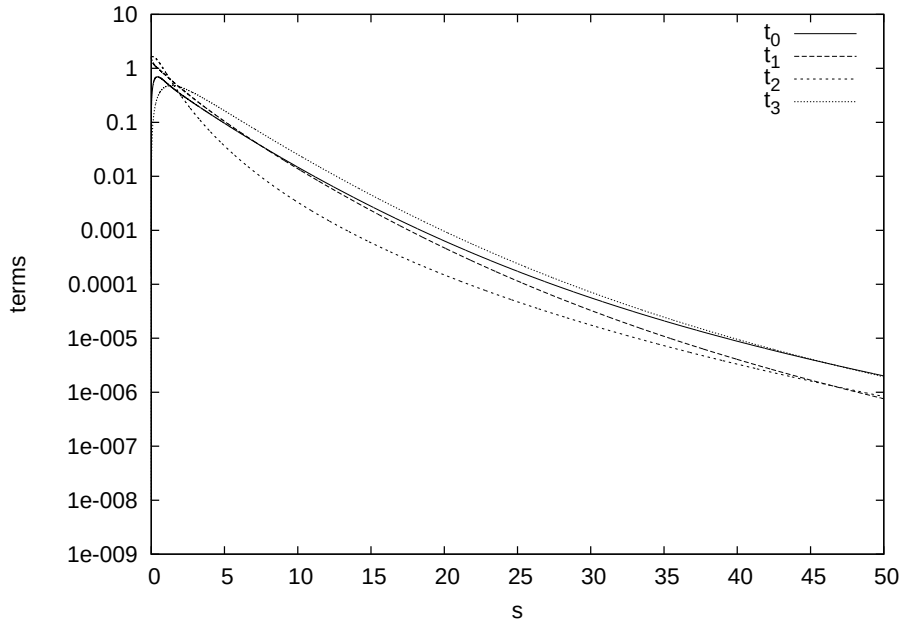
(a)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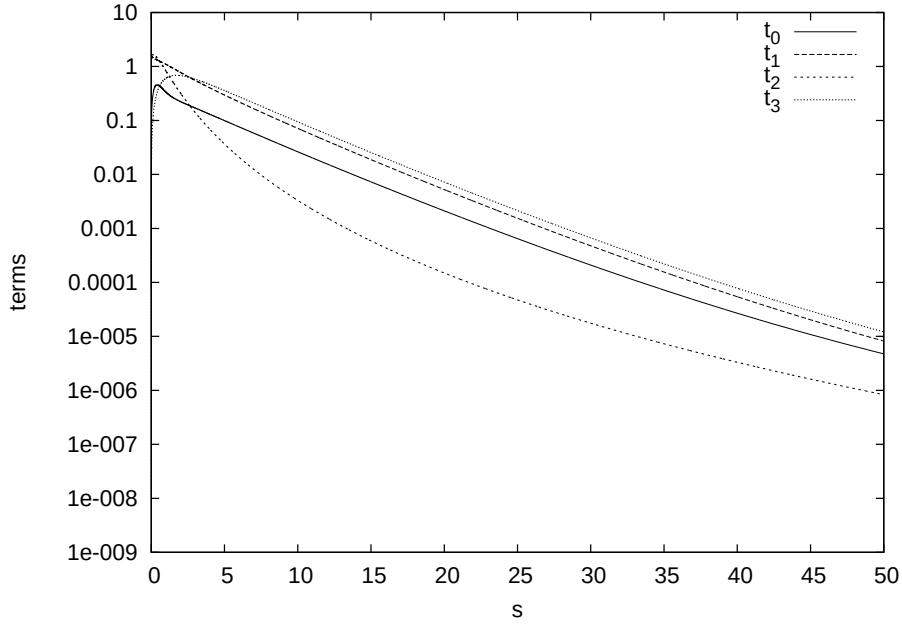
(b)



(c)

Figure 11: Optimal asset allocation if the claims are lognormally distributed with parameters as given in Figure 8 except that a safe stock is chosen with $r_0 = 0$. Graph (a) shows the optimal allocation given to the stock with comparison to the asymptotic expression A^*_∞, while (b) and (c) show the magnitude of the terms in (15) for $\sigma = 0.05$ and $\sigma = 0.1$ respectively.

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