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A block theoretic proof of Thompson's $A \times B$-lemma

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Abstract. We show that Thompson's $A \times B$-lemma can be obtained as a consequence of Brauer's third main theorem.

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Let k be an algebraically closed field of prime characteristic p. Brauer's third main theorem [2, Theorem 3], rephrased using Brauer pairs (cf. [1, Theorem 3.13] or [7, Theorem 6.3.14], or also [5, Theorem 7] for a different proof), states that if b is the principal block idempotent of a finite group algebra kG, then $\mathrm{Br}_Q(b)$ is the principal block idempotent of $kC_G(Q)$ for any p-subgroup Q of G. Here $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ denotes the Brauer homomorphism (cf. [3, §1.2] or [6, Theorem 5.4.1]). In particular, if kG has a unique block, then $kC_G(Q)$ has a unique block. Moreover, if kG has a unique block, then $O_{p'}(G) = 1$ because otherwise $\frac{1}{|O_{p'}(G)|} \sum_{g \in O_{p'}(G)} g$ would be a central idempotent in kG different from 1. Also, it is well-known that if G has a self-centralising normal p-subgroup, then kG has a unique block (cf. [7, Corollary 6.2.8]). We use these facts to give a proof of the following result.

Theorem 1 (Thompson's $A \times B$-lemma; cf. [4, Chapter 5, Theorem 3.4]). *Let $A \times B$ be a subgroup of the automorphism group of a finite p-group P, with A a p'-group and B a p-group. If A acts trivially on $C_P(B)$, then $A = 1$.*

Proof. Consider the group $G = P \rtimes (A \times B)$, where the notation is as in the statement, and suppose that A acts trivially on $C_P(B)$. Note that $S = P \rtimes B$ is the unique Sylow p-subgroup of G, and that A can be regarded as a p'-subgroup of the automorphism group of S. Thus S is self-centralising and normal in G, and hence kG has a unique block.

By the assumptions, A acts trivially on the group $Q = C_P(B) \times B$. That is, we have $A \leq C_G(Q)$, and hence $C_G(Q) = C_S(Q) \rtimes A$. Since kG has a unique block, $kC_G(Q)$ has a unique block. We now show that Q is self-centralising in

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S. Let $x \in C_S(Q)$. Write $x = yu$ for some $y \in P$ and $u \in B$. Since $u \in B$, it follows that conjugation by u preserves the decomposition $Q = C_P(B) \times B$. Thus conjugation by y preserves this decomposition as well. In particular, y normalises B. By elementary group theory, it follows that y centralises B. Indeed, if $u \in B$, then $yuy^{-1}u^{-1} \in P \cap B = 1$. This shows that $C_S(Q) \leq Q$. Since $Q \leq C_G(A)$, it follows that $C_G(Q) = C_S(Q) \times A$. By the above, $kC_G(Q)$ has a single block, and hence $A = O_{p'}(C_G(Q)) = 1$. □

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