THE EQUIVALENCE OF EXOTIC AND BLOCK-EXOTIC FUSION SYSTEMS

Patrick Serwene

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Department of Mathematics

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Declaration

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Abstract

One of the main problems in the theory of fusion systems is whether a fusion system occurs as the fusion system of a finite p-group if and only if it occurs as the fusion system of a p-block of a finite group. It is conjectured that the answer is yes. We present reduction theorems for this problem, reducing it to blocks of quasisimple groups in certain cases. One of these reductions is applied to settle the conjecture for a family of fusion systems discovered by Parker and Semeraro. We state a stronger version of the conjecture for the class of generalised block fusion systems. We show that several key reduction results for block fusion systems carry over to generalised block fusion systems. Finally, we extend a result of Cabanes proving the conjecture for unipotent blocks of finite groups of Lie type to generalised block fusion systems.

List of symbols

$\Gamma_{p'}(\mathcal{F})$ .... free group on the set $\mathrm{Mor}(\mathcal{F}^c)$ modulo the relations induced by composition and by dividing out by $O^{p'}(\mathrm{Aut}(\mathcal{F}^c))$ for all $Q \in \mathcal{F}^c$

$\pi^{-1}(G)$ .... Preimage of G under map π

φ_e .... e-th cyclotomic polynomial

$\varphi(H)$ .... Image of H under φ

A^G .... G-fixed points of A

$A_{\mathbf{G}}(s)$ .... Quotient $C_{\mathbf{G}}(s)/C^{\circ}_{\mathbf{G}}(s)$

A^H_K .... Image of A^K under relative trace map Tr^H_K

$\mathfrak{A}_n$ .... Alternating group on n letters

$A \otimes B$ .... Tensor product of A and B

$\mathrm{Aut}_{\mathcal{C}}(G)$ .... Automorphisms of G in category $\mathcal{C}$

$\mathrm{Aut}_G(H)$ .... Group of automorphisms of H induced by elements in G

$A^{\times}$ .... Units in A

$b_{\mathbf{G}^F}(\mathbf{L}, \zeta)$ .... Block of $\mathbf{G}^F$ corresponding to cuspidal pair $(\mathbf{L}, \zeta)$

$\mathcal{B}(Q)$ .... Blocks of $kC_G(Q)$

$\mathcal{B}(R, b)$ .... Blocks e of $kC_G(R)$ such that (R, e) is a b-Brauer pair

Br^G_Q .... Brauer map to $C_G(Q)$

$b \otimes \theta$ .... Tensor product of b and θ

$C_{\mathcal{F}}(Q)$ .... Centraliser of Q in $\mathcal{F}$

c_g .... Conjugation map with respect to g

$C_G(H)$ .... Centraliser of H in G

$\mathrm{char}\, k$ .... Characteristic of k

C_k .... Cyclic group of order k

D_n .... Dihedral group of size n

$\det(A)$ .... Determinant of A

$\mathrm{diag}(a_1, \cdots, a_n)$. Diagonal matrix

D_n .... Dihedral group of order $2n$

e_χ .... Central idempotent of KG corresponding to χ

$E(G)$ .... Layer of group G

$\mathcal{E}(\mathbf{G}^F, s)$ .... Lusztig series

F .... Frobenius homomorphism

$\mathcal{F}$ .... Fusion system

$\mathcal{F}^c$ .... Subcategory of $\mathcal{F}$ defined on $\mathcal{F}$-centric subgroups

$F(G)$ .... Fitting subgroup of G

$\mathcal{F}_{(P,e_P)}(G, N, b)$. Generalised block fusion system corresponding to (G, N, b)

$\mathcal{F}_{(P,e_P)}(G, b)$ .... Fusion system of block b of group G

$\mathcal{F}_P(G)$ .... Fusion system of G on P

$\mathbb{F}_q$ .... Field with q elements

$F^*(G)$ .... Generalised Fitting subgroup

$\mathbf{G}$ .... Algebraic group

$\mathbf{G}^F$ .... Finite group of Lie type

$[G, G]$ .... Commutator of G

$G \times H$ .... Direct product of G and H

$\mathrm{GL}_n(q)$ .... General Linear Group of size n over $\mathbb{F}_q$

G° .... Connected compontent of G

${}^gP$ .... Image of P under conjugation with g

G_p .... p-part of group G

$G_{p'}$ .... p'-part of group G

$G \rtimes N$ .... Semidirect product of G and N

$\mathbf{G}^*$ .... Dual group to $\mathbf{G}$

$\mathrm{GU}_n(q)$ .... General Unitary Group of size n over $\mathbb{F}_q$

$G \wr N$ .... Wreath product

$\mathrm{Hom}_{\mathcal{C}}(H, K)$ .... Homomorphisms between groups H and K in a category $\mathcal{C}$

$\mathrm{Hom}_G(H, K)$ ... Homomorphisms between groups H and K induced by an element in G

H_c^i .... i-th cohomology group

$[H/K]$ .... Set of representatives of left cosets of K in H

id_Q .... Identity on Q

$I_G(c)$ .... Stabiliser of block c in group G

$ikHi$ .... Source algebra

$\mathrm{Inj}(Q, R)$ .... Injective morphisms between groups Q and R

I_n .... Identity matrix of size n

$\mathrm{Inn}(A)$ .... Inner automorphisms of A

$\mathrm{Irr}(A)$ .... Irreducible characters of A

$\mathrm{Irr}_k(G)$ .... Irreducible characters of kG

$\mathrm{Irr}_k(G, b)$ .... Irreducible characters of kG belonging to block b

$\mathrm{Iso}_{\mathcal{C}}(R, Q)$ ...... Isomorphims between R and Q in category $\mathcal{C}$

$J(\mathcal{O})$ .... Jacobson radical of $\mathcal{O}$

k .... arbitrary field

$\overline{k}$ .... Algebraic closure of k

$\ker(b)$ .... Kernel of block b

$\ker(\varphi)$ .... Kernel of map φ

kG .... Group algebra

kGb .... Block algebra of block b

$(K, \mathcal{O}, k)$ .... p-modular system

$k_\varphi G$ .... Twisted group algebra with respect to φ

$\mathbf{L}$ .... Levi subgroup

$(\mathbf{L}, \zeta)$ .... Cuspidal pair

M .... Monster group

$\max S$ .... Maximum in S

$\mathrm{Mor}(\mathcal{C})$ .... Morphisms in category $\mathcal{C}$

$\mathbf{N}$ .... BDR extension

$N_{\varphi}^{\mathcal{F}}$ .... Elements g of $N_P(R)$ such that ${}^{\varphi}c_g \in \mathrm{Aut}_P(Q)$

$N_{\mathcal{F}}(Q)$ .... Normaliser of Q in $\mathcal{F}$

$N_G(Q, e_Q)$ ...... Normaliser of (Q, e_Q) in G

$N_G(R)$ .... Normaliser of R in G

$\mathcal{O}$ .... Local principal ideal domain

$\mathcal{O}_p(\mathcal{F})$ .... Largest normal subgroup in $\mathcal{F}$

$O^{p'}(G)$ .... smallest normal subgroup of G such that the index is prime to p

$O_p(G)$ .... largest normal p-subgroup of G

$\mathcal{O}^{p'}(\mathcal{F})$ .... Unique minimal fusion system normal in $\mathcal{F}$ of index prime to p

$\mathrm{ord}(g)$ .... Order of element g

$O_{p'}(G)$ .... largest normal subgroup of G whose index is prime to p

$\mathrm{Out}(A)$ .... Outer automorphisms of A

$\mathbf{P}$ .... Parabolic subgroup

p^{1+2n} .... Extraspecial group of order $1 + 2n$

(P, e_P) .... Brauer pair

$\mathrm{PGL}_n(q)$ .... Projective General Linear Group of size n over $\mathbb{F}_q$

$\mathrm{PGU}_n(q)$ .... Projective General Unitary Group of size n over $\mathbb{F}_q$

(p, q) .... Greatest common divisor of p and q

$\mathrm{U}_n(q)$ .... Projective Special Unitary Group of size n over $\mathbb{F}_q$

$\mathrm{Res}_H^G \zeta$ .... Restriction of ζ from G to H

RG .... Group algebra of G over R

$R_{\mathbf{L}}^{\mathbf{G}}$ .... Lusztig induction

$\mathrm{SL}_n(q)$ .... Special Linear Group of size n over $\mathbb{F}_q$

$\mathfrak{S}_n$ .... Symmetric group on n letters

$\mathrm{Stab}_G(a)$ .... Stabiliser of a in G

$\mathrm{SU}_n(q)$ .... Special Unitary Group of size n over $\mathbb{F}_q$

$\mathrm{Syl}_p(G)$ .... Set of Sylow p-subgroups of G

$\mathbf{T}$ .... Torus

$\mathbf{T}_{\phi_n}$ .... ϕ_n-part of torus $\mathbf{T}$

Tr_K^H .... Relative trace map with respect to K in H

$\mathrm{tr}_V(a)$ .... Trace of a over V

$Z(G)$ .... Centre of G

1 Introduction

The main objects of interest in this thesis are saturated fusion systems. For convenience, we will drop the term saturated and mean saturated fusion system whenever we say fusion system. A fusion system $\mathcal{F}$ on a finite p-group P is a category, whose objects are the subgroups of P, with the set $\mathrm{Hom}_{\mathcal{F}}(R,Q)$ of morphisms from R to Q for $R,Q \leq P$, consisting of injective group homomorphisms from R into Q, such that some additional axioms are satisfied. The standard example is $\mathcal{F}_P(G)$, where G is a finite group, P is a Sylow p-subgroup of G and the morphisms are those induced by conjugation in G. The theory of fusion systems is linked to many other areas of algebra, such as group theory, representation theory and algebraic topology.

Local finite group theorists are interested in fusion systems, since methods from local group theory proved to be effective in the study of fusion systems, but also because certain results in finite group theory seem to be easier to prove in the category of fusion systems. One example of this is Aschbacher's Program, see [2] for details: In the proof of the Classification of Finite Simple Groups, so-called 2-local subgroups play an important role. Those are groups of the form $N_G(P)$, where P is a 2-subgroup of G. These 2-local subgroups can have non-trivial normal $2'$-subgroups, which cause some problems. However, one can use fusion systems to eliminate these problems: For $K \leq G$, let $\overline{K}$ be the image under the map $G \to G/O_{2'}(G)$. Instead of studying $\mathcal{F}_P(G)$ one can study the category $\mathcal{F}_{\overline{P}}(\overline{G})$, which is isomorphic to $\mathcal{F}_P(G)$ for a Sylow 2-subgroup P of G.

Homotopy theorists use the theory of fusion systems to provide a formal setting for, and prove results about, the p-completed classifying spaces of finite groups. Objects called p-local finite groups associated to abstract fusion systems were introduced by Broto, Levi and Oliver. These also possess interesting p-completed classifying spaces. Fusion systems arise as well in block theory. Let G be a finite group, b a block of kG, where p is a prime and k is an algebraically closed field of characteristic p and let (P, e_P) be a maximal b-Brauer pair (which means P is a defect group of b and e_P is a p-block of

$C_G(P)$ in correspondence with b). With this setup, to each quadruple (G, b, P, e_P) is associated a fusion system $\mathcal{F}_{(P,e_P)}(G, b)$ on P, called the fusion system of b. If we choose a maximal b-Brauer pair different from (P, e_P), we obtain an isomorphic fusion system. Many results and conjectures in modular representation theory are statements relating the representation theory of b to $\mathcal{F}_{(P,e_P)}(G, b)$. Examples for this are the Alperin and the Ordinary Weight Conjecture, see [31].

This thesis is about the connection between fusion systems of groups and fusion systems of blocks. Not every fusion system needs to be of the form $\mathcal{F}_P(G)$ from above. Fusion systems which are not of this form are called exotic. Similarly, not every fusion system is of the form $\mathcal{F}_{(P,e_P)}(G, b)$ and if a fusion system is not of this form, we call it block-exotic. A consequence of Brauer's Third Main Theorem, see Theorem 2.37, is that every block-exotic fusion system is exotic. This dissertation is about the reverse implication, i.e. whether every exotic fusion system is block-exotic, giving rise to the following

Conjecture 1.1. *Let $\mathcal{F}$ be a fusion system. Then $\mathcal{F}$ is exotic if and only if it is block-exotic.*

This conjecture has been around for nearly 20 years. Proving this conjecture would be very interesting, since it links the local structure fusion systems determine for groups and for blocks. Conjecture 1.1 is difficult to approach, since there seems to be no way to directly use the information that a given fusion system is a block fusion system in order to conclude that it must be the fusion system of a finite group. The only approach so far has been to work with specific classes of exotic systems and show block-exoticity of those. This has been done in [28] for the Solomon fusion systems for $p = 3$, which has been generalised to all primes in [16], and for fusion systems on extra-special p-groups of order 7^3 and exponent 7 in [32]. In this thesis, two new reduction theorems relating to Conjecture 1.1 are presented. The first reduction theorem is the following result. See Chapter 2 for notation.

Theorem 1.2. *Let $\mathcal{F}$ be a reduction simple fusion system on a finite non-abelian p-group P. If $\mathcal{F}$ is block-realisable, then there exists a finite group G possessing an $\mathcal{F}$-block b such that the following holds*
(a) $|G : Z(G)|$ is minimal among all groups G having an $\mathcal{F}$-block,
(b) if $H \trianglelefteq G$ with $P \nsubseteq H$, then H is a central p'-group and $F(G) = Z(G)$,
(c) the number of components of G is bounded by the rank of $Z(P)$.

In particular, when specialising to the case of a defect group with cyclic centre, we can reduce the problem to quasisimple groups. We apply Theorem 1.2 in this version to prove block-exoticity for a family of exotic fusion systems discovered by Parker and Semeraro, see [42], obtaining

Theorem 1.3. *Conjecture* 1.1 *is true for all fusion systems $\mathcal{F}$ on a Sylow p-subgroup of $G_2(p^n)$ or $\mathrm{PSU}_4(p^n)$ for all primes p and $n \in \mathbb{N}$.*

The second main reduction theorem is the following result. Recall that to each fusion system is associated a group $\Gamma_{p'}(\mathcal{F})$ whose subgroups are in 1:1-correspondence with the fusion subsystems of p'-index in $\mathcal{F}$, see [3, Part I, Theorem 7.7]. In particular, $\mathcal{O}^{p'}(\mathcal{F})$ corresponds to the trivial subgroup under this bijection.

Theorem 1.4. *Let $\mathcal{F}$ be a fusion system on a non-abelian p-group P. For any subgroup $H \leq \Gamma_{p'}(\mathcal{F})$, denote by $\mathcal{F}_H$ the subsystem of $\mathcal{F}$ corresponding to H. Assume*
(a) $\mathcal{O}^{p'}(\mathcal{F})$ is reduction simple,
(b) if $\mathcal{G}$ is a fusion system on P containing $\mathcal{O}^{p'}(\mathcal{F})$, then $\mathcal{G} \subseteq \mathcal{F}$,
(c) if $\mathcal{G}$ is a fusion system on P such that $\mathcal{G} \trianglelefteq \mathcal{F}_H$ for some $H \leq \Gamma_{p'}(\mathcal{F})$, then $\mathcal{O}^{p'}(\mathcal{F}) \subseteq \mathcal{G}$.
If there exists a finite group having an $\mathcal{F}_H$-block for some $H \leq \Gamma_{p'}(\mathcal{F})$, then there exists a finite quasisimple group with p'-centre having an $\mathcal{F}_{H'}$-block for some $H' \leq \Gamma_{p'}(\mathcal{F})$.

This result generalises the main result of [32] since the main theorem of [32] covers the cases where $|\Gamma_{p'}(\mathcal{F})| \leq 2$.

A more general category than block fusion systems, which we call generalised block fusion systems, introduced in [32], plays a key role in the main result of [32], as well as in our two Reduction Theorems 1.2 and 1.4. Generalised block fusion systems generalise block fusion systems: They depend on a quintuple (H, G, b, P, e_P), where H is a finite group with $G \trianglelefteq H$, b is an H-stable block of kG and (P, e_P) is called a maximal (b, H)-Brauer pair. See Section 2.3 for details. We make the following conjecture.

Conjecture 1.5. *Let H be a finite group with normal subgroup G having an H-stable p-block b with maximal (b, H)-Brauer pair (P, e_P). Then the generalised block fusion system $\mathcal{F}_{(P,e_P)}(H, G, b)$ is non-exotic.*

When $H = G$, then $\mathcal{F}_{(P,e_P)}(H, G, b)$ is the fusion system of b as defined above. In particular, Conjecture 1.5 implies Conjecture 1.1. We prove three key results for block fusion systems for this more general category, namely Brauer's Third Main Theorem and the First and Second Fong Reduction. The generalised version of Brauer's Third Main Theorem is the following result. If $G = H$, we get the result for block fusion systems, which can be found in [29, Theorem 3.6].

Theorem 1.6. *Let $G \trianglelefteq H$, b the principal block of kG, (P, e_P) a maximal (b, H)-Brauer pair, then $P \in \mathrm{Syl}_p(H)$, e_P is the principal block of $kC_G(P)$ and $\mathcal{F}_{(P,e_P)}(H, G, b) = \mathcal{F}_P(H)$.*

The next result is the Generalised First Fong Reduction. If $G = H$, we obtain the original result for block fusion systems. This special case was proved in [3, Part IV, Proposition 6.3]. Let $G \trianglelefteq H$, c be a block of kG and $I_H(c) = \{h \in H \mid {}^h c = c\}$. If b is a block of kH covering c, then there is a block of $kI_H(c)$ corresponding to b which we call Fong correspondent of b.

Theorem 1.7. *Let $\mathcal{F}$ be a fusion system on a p-group P and let G, H be finite groups such that $G \trianglelefteq H$. Let b be an H-stable block of kG with $\mathcal{F} = \mathcal{F}_{(P,e_P)}(H, G, b)$. Let N be a normal subgroup of H contained in G and c be a block of kN which is covered by b.*

Then $\mathcal{F} = \mathcal{F}_{(P,\widetilde{e_P})}(I_H(c), I_G(c), \tilde{b})$, where $\tilde{b}$ is the Fong correspondent of b in $I_G(c)$ and $(P, \widetilde{e_P})$ is a maximal $(\tilde{b}, I_H(c))$-Brauer pair.

Finally, we state the Generalised Second Fong Reduction. If $G = H$, we obtain the original theorem for block fusion systems. This special case was proved in [3, Part IV, Theorem 6.6].

Theorem 1.8. *Let $M \leq H$ such that $|H : M|_p = 1$ and let A be a normal subgroup of H contained in M. Let c be an H-stable block of kA of defect zero and d be an H-stable block of kM covering c with maximal (d, H)-Brauer pair (P, e_P). Then there exists a p'-central extension $\widetilde{H}$ of H/A and a block $\tilde{d}$ of $\widetilde{M}$ with maximal $(\tilde{d}, \widetilde{H})$-Brauer pair $(\widetilde{P}, e'_P)$, with $\widetilde{P} \cong P$, where $\widetilde{M}$ is the full inverse image of M/A in $\widetilde{H}$ such that $\tilde{d}$ is $\widetilde{H}$-stable and $\mathcal{F}_{(P,e_P)}(H, M, d) \cong \mathcal{F}_{(\widetilde{P},e'_P)}(\widetilde{H}, \widetilde{M}, \tilde{d})$.*

When tackling Conjectures 1.1 and 1.5, a first goal would be to prove these for all finite simple groups. We are interested in the case of finite simple groups of Lie type. For these, Cabanes proved Conjecture 1.1 for unipotent p-blocks, where p is at least 7 and not the natural characteristic of the group. We extend this result with some methods developed by Bonnafé, Dat and Rouquier (henceforth called BDR) in [5] to generalised block fusion systems. Let $\mathbf{G}^F$ be a finite group of Lie type with a block d and $\mathbf{L} \leq \mathbf{G}$ a Levi subgroup. BDR prove that, in many cases, the fusion system of d is equivalent to the fusion system of a block c of a subgroup $\mathbf{N}^F$ of $\mathbf{G}^F$ containing $\mathbf{L}^F$ as a normal subgroup, where c covers a unipotent block b of $\mathbf{L}^F$. We call the generalised block fusion systems in this situation BDR generalised block fusion systems and prove the following result about those, which provides some evidence for Conjecture 1.5.

Theorem 1.9. *With the notation above, assume $p \geq 7$ and let (P, e_P) be a maximal $(b, \mathbf{N}^F)$-Brauer pair. The BDR generalised block fusion system $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{L}^F, b)$ is non-exotic.*

This thesis is organised as follows: In Chapter 2, we recall key concepts and results of fusion systems, block fusion systems and generalised block fusion systems. Then in Section 3.1, we state several well-known reduction theorems for block fusion systems. In Section 3.2 we prove Theorems 1.2 and 1.4 and in Section 3.3 we prove Theorems 1.6, 1.7 and 1.8. Some of those will be applied in Chapter 4 to prove that the exotic Parker–Semeraro systems are block-exotic as well, obtaining Theorem 1.3. In Chapter 5, we tackle Conjectures 1.1 and 1.5 for finite groups of Lie type in non-describing characteristic by extending Cabanes' result to generalised block fusion systems, obtaining Theorem 1.9. Finally, in Chapter 6, we give an overview of all exotic fusion systems that have been discovered and address the state of Conjecture 1.1 for finite simple groups not covered in Chapter 5. We assume that the reader has basic knowledge in algebra and group theory.

2 Fusion Systems

2.1 Introduction to Fusion Systems

In this section, we recall the definition and some key properties of fusion systems. Let p be a prime number. We begin by defining categories on p-groups. Note that by p-group we always mean finite p-group.

Definition 2.1. *A category whose objects consist of the subgroups of a p-group P is called category on P.*

Definition 2.2. *Let G be a finite group.*

(a) Let $g \in G$, then for $x \in G$, c_g is defined to be the conjugation map $c_g : G \to G, x \mapsto gxg^{-1}$.

(b) If $P \leq G$ and P is a p-group, we write $P \leq_p G$.

(c) Let $H, K \leq G$. Note that if ${}^gH \leq K$ for some $g \in G$, c_g induces a map $c_g|_H : H \to K$. By $\mathrm{Hom}_G(H, K)$ we denote the set $\{\varphi : H \to K | \varphi = c_g|_H$ for some $g \in G, {}^g H \leq K\}$ and by $\mathrm{Aut}_G(H) = \mathrm{Hom}_G(H, H)$.

(d) Let $\mathcal{C}$ be a category on a p-group P with $Q, R \leq P$. By $\mathrm{Hom}_{\mathcal{C}}(Q, R)$ we denote the set of morphisms between Q and R in $\mathcal{C}$.

Definition 2.3. *(a) Let p be a prime and P be a p-group. A fusion system is a category $\mathcal{F}$ on P, such that for all $Q, R \leq P$ we have:*

(i) $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R) \subseteq \mathrm{Inj}(Q, R)$, where the latter denotes the set of injective group homomorphisms between Q and R,

(ii) each homomorphism in $\mathcal{F}$ is the composition of an $\mathcal{F}$-isomorphism and an inclusion.

(b) Let $\mathcal{F}$ be a fusion system on a p-group P. Two subgroups $Q, R \leq P$ are called $\mathcal{F}$-conjugate if they are isomorphic as objects of the category $\mathcal{F}$.

(c) A subgroup $Q \leq P$ is called fully automised in $\mathcal{F}$ if $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

(d) A subgroup $Q \leq P$ is called receptive in $\mathcal{F}$ if for each $R \leq P$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(R, Q)$,

φ has an extension to the group $N_\varphi := N_\varphi^{\mathcal{F}} := \{g \in N_P(R) \mid {}^{\varphi}c_g \in \mathrm{Aut}_P(Q)\}$.

(e) A fusion system is called saturated if each subgroup of P is $\mathcal{F}$-conjugate to a subgroup which is fully automised and receptive in $\mathcal{F}$.

In many applications, it is crucial for fusion systems to be saturated, since fusion systems only satisfying part (a) of the previous definition are too general. For convenience, we drop the term saturated, and mean saturated fusion system whenever we say fusion system. In the literature, fusion system means categories satisfying only part (a) from Definition 2.3.

Theorem 2.4. [36, Theorem 2.11] *Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. We denote the category on P with morphisms consisting of homomorphisms induced by conjugation by elements in G by $\mathcal{F}_P(G)$. Then $\mathcal{F}_P(G)$ is a fusion system on P.*

If a fusion system is of the form $\mathcal{F}_P(G)$ for a finite group G and $P \in \mathrm{Syl}_p(G)$, we call it realisable, otherwise we call it exotic. Furthermore, we say that a fusion system on a p-group P is trivial if $\mathcal{F} = \mathcal{F}_1(1)$.

Definition 2.5. *Let $\mathcal{F}$ be a fusion system on a p-group P and $Q \leq P$.*

(a) If $C_P(Q') = Z(Q')$ for each $Q' \leq P$ which is $\mathcal{F}$-conjugate to Q, then Q is called $\mathcal{F}$-centric. Define $\mathcal{F}^c$ to be the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of P.

(b) A proper subgroup H of a finite group G is called strongly p-embedded if H contains a Sylow p-subgroup P of G and $P \neq 1$ but ${}^{x}P \cap H = 1$ for any $x \in G \setminus H$.

(c) A subgroup $Q \leq P$ is called fully $\mathcal{F}$-normalised if $|N_P(R)| \leq |N_P(Q)|$ for any $R \leq P$ with $R \cong Q$ in $\mathcal{F}$.

(d) We call a subgroup $Q \leq P$ $\mathcal{F}$-essential if Q is $\mathcal{F}$-centric and fully normalised in $\mathcal{F}$, and if $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ has a strongly p-embedded subgroup.

Note that if $\mathcal{F}$ is a fusion system on P, an $\mathcal{F}$-essential subgroup of P is always a proper subgroup since $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_P(P)$ is a p'-group by Definition 2.3(c).

Let $n \in \mathbb{N}_{\geq 1}$ and $i \in \{1, \cdots, n\}$. If $\mathcal{F}$ is a fusion system on P and $M_i \subseteq \mathrm{Aut}_{\mathcal{F}}(Q_i)$ for some $Q_i \leq P$, we denote by $\langle M_1, \cdots, M_n \rangle$ the smallest (not necessarily saturated) subsystem of $\mathcal{F}$ on P such that its morphisms contain all the sets M_i for $i = 1, \cdots, n$. The following theorem tells us that the structure of a fusion system $\mathcal{F}$ on P is determined by the automorphisms of $\mathcal{F}$-essential subgroups of P and P itself.

Theorem 2.6. (Alperin's Fusion Theorem) [3, Part I, Theorem 3.5] *Let $\mathcal{F}$ be a fusion system on a p-group P. Then $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q) | Q = P$ or Q is $\mathcal{F}$-essential $\rangle$.*

We can define a substructure similar to normal subgroups for fusion systems.

Definition 2.7. *Let $\mathcal{F}$ be a fusion system on a p-group P and $\mathcal{E} \subseteq \mathcal{F}$ be a subcategory of $\mathcal{F}$ which is a fusion system itself on some subgroup $P' \leq P$.*
(a) A subgroup $Q \leq P$ is called strongly $\mathcal{F}$-closed, if $\varphi(R) \subseteq Q$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ and each $R \leq Q$.
(b) If P' is normal in P and strongly $\mathcal{F}$-closed, ${}^{\alpha}\mathcal{E} = \mathcal{E}$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P')$ and for each $Q \leq P'$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P')$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P')$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(Q, P')$ with $\varphi = \alpha \circ \varphi_0$, then $\mathcal{E}$ is called weakly normal in $\mathcal{F}$, denoted $\mathcal{E} \dot{\trianglelefteq} \mathcal{F}$.
(c) If $\mathcal{E}$ is weakly normal and in addition, we have that each $\alpha \in \mathrm{Aut}_{\mathcal{E}}(P')$ has an extension $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(P'C_P(P'))$ with $[\overline{\alpha}, C_P(P')] \leq Z(P')$, then we call $\mathcal{E}$ normal in $\mathcal{F}$ and write $\mathcal{E} \trianglelefteq \mathcal{F}$.
(d) A fusion system is called simple if it does not contain any non-trivial proper normal subsystem.

Definition 2.8. *Let G be a group and p be a prime. The subgroup $O^{p'}(G)$ is defined to be the smallest normal subgroup of G such that the index $|G : O^{p'}(G)|$ is prime to p.*

Definition 2.9. *Let $\mathcal{F}$ be a fusion system on a p-group P.*
(a) We say that a subsystem $\mathcal{E}$ of $\mathcal{F}$ has index prime to p (or p'-index) if it is also defined over P and $\mathrm{Aut}_{\mathcal{E}}(Q) \geq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$ for each $Q \leq P$.

(b) We define the (not necessarily saturated) fusion system $\mathcal{O}_*^{p'}(\mathcal{F}) := \langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q)) \mid Q \leq P\rangle$ *and the group* $\mathrm{Aut}^0_{\mathcal{F}}(\mathcal{F}) := \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \mid \alpha|_Q \in \mathrm{Hom}_{\mathcal{O}_*^{p'}(\mathcal{F})}(Q, P)$ *for some* $\mathcal{F}$*-centric* $Q \leq P\rangle$.

Note that part (b) of the above definition makes sense since clearly $\mathcal{O}_*^{p'}(\mathcal{F}) \leq \mathcal{F}$. Also note that $\mathrm{Aut}^0_{\mathcal{F}}(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$.

Definition 2.10. *For any fusion system* $\mathcal{F}$ *over a* p*-group* P*, let* $\Gamma_{p'}(\mathcal{F})$ *be the free group on the set* $\mathrm{Mor}(\mathcal{F}^c)$ *modulo the relations induced by composition and by dividing out by* $O^{p'}(\mathrm{Aut}(\mathcal{F}^c))$ *for all* $Q \in \mathcal{F}^c$*. In particular, there is a canonical map* $\mathrm{Mor}(\mathcal{F}^c) \to \Gamma_{p'}(\mathcal{F})$*.*

It turns out that the group $\Gamma_{p'}(\mathcal{F})$ carries a lot of information about $\mathcal{F}$:

Theorem 2.11. [3, Part I, Theorem 7.7] *For a fusion system* $\mathcal{F}$ *on a* p*-group* P*, let* $\theta : \mathrm{Mor}(\mathcal{F}^c) \to \Gamma_{p'}(\mathcal{F})$ *be the canonically defined map. Then we have* $\Gamma_{p'}(\mathcal{F}) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}^0_{\mathcal{F}}(P)$*.*
Furthermore, there is a one-to-one-correspondence between the saturated subsystems of $\mathcal{F}$ *of index prime to* p *and subgroups of* $\Gamma_{p'}(\mathcal{F})$ *given by defining* $\mathcal{F}_H = \langle \theta^{-1}(H)\rangle$ *for some* $H \leq \Gamma_{p'}(\mathcal{F})$*. This correspondence respects normality.*
In particular, there is a unique minimal fusion system $\mathcal{O}^{p'}(\mathcal{F}) = \langle \theta^{-1}(1)\rangle \trianglelefteq \mathcal{F}$ *of index prime to* p*.*

Definition 2.12. *Let* $\mathcal{F}$ *be a fusion system on a* p*-group* P*.*
(a) Fix $Q \leq P$*. Let* $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}$ *be the (not necessarily saturated) fusion system over* $N_P(Q)$*, where for* $R, S \leq N_P(Q)$*,* $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P, S)$ *if and only if* φ *has an extension* $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(RQ, SQ)$ *with* $\overline{\varphi}|_R = \varphi$ *and* $\overline{\varphi}(Q) = Q$*.*
When Q *is fully* $\mathcal{F}$*-normalised,* $N_{\mathcal{F}}(Q)$ *is a (saturated) fusion system on* $N_P(Q)$ *by [3, Part I, Theorem 5.5].*
(b) We call a group $Q \leq P$ *normal in* $\mathcal{F}$*, denoted* $Q \trianglelefteq \mathcal{F}$*, when* $Q \trianglelefteq P$ *and* $N_{\mathcal{F}}(Q) = \mathcal{F}$*. The largest subgroup of* P *which is normal in* $\mathcal{F}$ *is denoted by* $\mathcal{O}_p(\mathcal{F})$*.*

Note that $\mathcal{O}_p(\mathcal{F}) = 1$ for simple fusion systems.

Lemma 2.13. *Let $\mathcal{F}, \mathcal{G}$ be fusion systems on a p-group P such that $\mathcal{F} \le \mathcal{G}$ and let $Q \trianglelefteq P$. If Q is normal in $\mathcal{G}$, then Q is normal in $\mathcal{F}$.*

Proof. By [3, Part I, Proposition 4.5], this is equivalent to showing that Q is contained in each $\mathcal{F}$-essential subgroup R of P and for each of these R, Q is $\mathrm{Aut}_{\mathcal{F}}(R)$-invariant, as well as $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant. Since Q is normal in $\mathcal{G}$, it is strongly $\mathcal{F}$-closed. In particular, Q is $\mathrm{Aut}_{\mathcal{F}}(R)$-invariant for all $R \le P$ such that $Q \le R$.

Now let $\varphi : R \to T$ be a $\mathcal{G}$-isomorphism. To prove containment in $\mathcal{F}$-essential subgroups, we first claim $N_P(R) \cap Q \le N_\varphi^{\mathcal{G}}$. Indeed, since $Q \trianglelefteq \mathcal{G}$, φ extends to a $\mathcal{G}$-homomorphism $\overline{\varphi} : QR \to QT$ and thus $N_P(R) \cap Q \le N_\varphi^{\mathcal{G}}$ by definition.

Now let $R \le P$ be $\mathcal{F}$-essential and $\beta \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $N_\beta^{\mathcal{F}} = R$ (such a β exists since R is $\mathcal{F}$-essential, see [3, Part I, Proposition 3.3(b)]). One easily verifies $N_\beta^{\mathcal{F}} = N_\beta^{\mathcal{G}}$. So, by the above, we have $N_P(R) \cap Q \le N_\beta^{\mathcal{F}} = R$. Since $Q \trianglelefteq P$, we have $RQ \le P$. By general properties of p-groups, either $RQ = R$ or $R < N_{RQ}(R)$. Since we have $N_{RQ}(R) = RN_Q(R) = R(N_P(R) \cap Q) = R$, we deduce $RQ = R$, so $Q \le R$. This, together with our observations above, implies normality of Q in $\mathcal{F}$. $\square$

Lemma 2.14. *If $\mathcal{F}, \mathcal{G}$ are fusion systems on a p-group P with $\mathcal{G} \dot{\trianglelefteq} \mathcal{F}$, then $O_p(\mathcal{G})$ is normal in $\mathcal{F}$.*

Proof. We have to check that each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ has an extension $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QO_p(\mathcal{G}), P)$ with $\overline{\varphi}(O_p(\mathcal{G})) = O_p(\mathcal{G})$. Since $\mathcal{G} \dot{\trianglelefteq} \mathcal{F}$, φ can be written as $\varphi = \alpha \circ \beta$, where $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $\beta \in \mathrm{Hom}_{\mathcal{G}}(Q, P)$. Given that β is a morphism in $\mathcal{G}$, it extends to $QO_p(\mathcal{G})$ and this extension sends $O_p(\mathcal{G})$ to itself. So we only need to show that $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ preserves $O_p(\mathcal{G})$. Now an $\mathcal{F}$-automorphism of P sends any $\mathcal{G}$-normal subgroup of P to a $\mathcal{G}$-normal subgroup of P. Indeed, if $Q \trianglelefteq \mathcal{G}$ then for any $R, S \le P$ and any $\varphi \in \mathrm{Hom}_{\mathcal{G}}(R, S)$, φ has an extension $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{G}}(RQ, SQ)$. If $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, clearly $\alpha(Q) \trianglelefteq P$ and if $\varphi \in \mathrm{Hom}_{\mathcal{G}}(R, S)$ we can extend it to $\overline{\overline{\varphi}} \in \mathrm{Hom}_{\mathcal{G}}(R\alpha(Q), S\alpha(Q))$

by defining $\overline{\overline{\varphi}}(r\alpha(q)) = \overline{\varphi}(r)\alpha(\overline{\varphi}(q))$ for $r \in R, q \in Q$ and achieve the desired properties. In particular, $\alpha(O_p(\mathcal{G})) \subseteq O_p(\mathcal{G})$. $\square$

Since fusion systems which do not allow many strongly closed subgroups play an important role, we make the following definition:

Definition 2.15. *Let $\mathcal{F}$ be a fusion system on a p-group P. If P does not have any non-trivial proper strongly $\mathcal{F}$-closed subgroups, we call $\mathcal{F}$ reduction simple.*

2.2 Fusion systems of blocks

In the previous section, we have seen that every finite group induces a fusion system. Next, we see that fusion systems can also be induced by blocks of finite groups. We recall some block-theoretic results, which we need to define these categories. Since block algebras play an important role in the setup, we first need to define structural notation for algebras.

Definition 2.16. *Let k be a commutative ring and A a k-algebra. We denote by* $\operatorname{Aut}(A)$ *the group of all k-algebra automorphisms of A. If $u \in A^\times$, the map $a \mapsto uau^{-1}$, for $a \in A$, given by conjugation with u is an algebra automorphism of A. Any algebra automorphism of A arising in this way is called an inner automorphism of A. The set* $\operatorname{Inn}(A)$ *of inner automorphisms of A is a normal subgroup of* $\operatorname{Aut}(A)$*, and the quotient* $\operatorname{Out}(A) = \operatorname{Aut}(A)/\operatorname{Inn}(A)$ *is called the outer automorphism group of A.*

Definition 2.17. *Let G be a finite group. A G-algebra over k is a k-algebra A endowed with an action $G \times A \to A, (x, a) \mapsto {}^x a$, such that the map sending $a \in A$ to ${}^x a$ is an k-algebra automorphism of A for all $x \in G$.*

An important special case of these algebras are group algebras.

Definition 2.18. *Let H be a finite group, $K \leq H$ and A an H-algebra over k. We denote by A^H the H-fixed point subalgebra of A.*

(*a*) *The relative trace map is defined by* $\mathrm{Tr}_K^H : A^K \to A^H, a \mapsto \sum_{x\in[H/K]} {}^x a$, *where* $[H/K]$ *denotes a set of representatives of the right cosets of* K *in* H.

(*b*) *Denote by* A_K^H *the image of* A^K *under the relative trace map* Tr_K^H.

Before we can discuss block theory, we first recall some definitions from character theory.

Definition 2.19. *Let k be a commutative ring, A a k-algebra and V an A-module. Suppose that V is free of finite rank over k. The character of V is the k-linear map* $\chi_V : A \to k, a \mapsto \mathrm{tr}_V(\rho(a))$, *where $\rho(a)$ is the linear endomorphism $\rho(a)$ of V defined by $\rho(a)(v) = av$ for all $v \in V$. A map $\chi : A \to k$ is called a character of A over k if χ is the character of some A-module which is free of finite rank over k.*

Definition 2.20. *Let k be a field and A be a finite-dimensional k-algebra.*

(*a*) *A central function on A is a k-linear map $\tau : A \to k$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$.*

(*b*) *A central function $\chi : A \to k$ is called an irreducible character of A if χ is the character of a simple A-module. We denote by* $\mathrm{Irr}(A)$ *the set of irreducible characters of A. If $A = kG$ for some finite group G, we write* $\mathrm{Irr}_k(G)$ *instead of* $\mathrm{Irr}(kG)$.

Definition 2.21. *Let p be a prime number. A p-modular system is a triple $(K, \mathcal{O}, k)$, where $\mathcal{O}$ is a local principal ideal domain, K is the field of quotients of $\mathcal{O}$ and k is the residue field of $\mathcal{O}$, i.e. $k = \mathcal{O}/J(\mathcal{O})$, such that*

(*a*) *$\mathcal{O}$ is complete with respect to the natural topology induced by $J(\mathcal{O})$,*

(*b*) $\mathrm{char}\, K = 0$,

(*c*) $\mathrm{char}\, k = p$.

If $(K, \mathcal{O}, k)$ is a p-modular system and $\alpha \in \mathcal{O}$, denote by $\overline{\alpha}$ the element $\alpha + J(\mathcal{O})$ in k.

Definition 2.22. *Let $(K, \mathcal{O}, k)$ be a p-modular system and G a finite group. The primitive central idempotents of KG are in bijection with the isomorphism classes of simple KG-modules and the isomorphism class of a simple KG-module V is determined by its* *character χ_V, see [37, Theorem 6.5.3]. Let $\chi \in \mathrm{Irr}_K(G)$. Denote by e_χ the central idempotent of KG corresponding to χ.*

Definition 2.23. *Let $(K, \mathcal{O}, k)$ be a p-modular system and G a finite group.*
(a) Let $R = \mathcal{O}$ or k. A block of RG is a primitive central idempotent of RG.
(b) Let β be a block of $\mathcal{O}G$. The canonical map $\beta \mapsto \overline{\beta}$ is a bijection between the blocks of $\mathcal{O}G$ and kG. Furthermore, β is a central idempotent of KG, but in general not primitive in $Z(KG)$. Let $\mathrm{Irr}_K(G, \beta)$ consist of those elements $\chi \in \mathrm{Irr}_K(G)$ satisfying $\beta e_\chi = e_\chi$. We say that χ belongs to β or that β contains χ. Each element of $\mathrm{Irr}_K(G)$ belongs to a unique block of $\mathcal{O}G$ (and thus also kG) by [37, Proposition 6.5.2(i)].
(c) The principal block of $\mathcal{O}G$ is the block containing the trivial character of KG. The corresponding block of kG is called the principal block of kG.

Fix k to be an algebraically closed field of characteristic p for the rest of this chapter.

Definition 2.24. *Let G be a finite group and b a block of kG. A Brauer pair is a pair (Q, f) where Q is a p-subgroup of G and f is a block of $kC_G(Q)$. We denote the set of blocks of $kC_G(Q)$ for some p-subgroup Q of G by $\mathcal{B}(Q)$.*

Note that G acts on the set of Brauer pairs by conjugation. We recall the Brauer map to see how Brauer pairs form a poset.

Definition 2.25. *Let G be a finite group and $Q \leq G$. For an element $a = \sum\limits_{g \in G} \alpha_g g \in kG$, set $\mathrm{Br}_Q^G(a) := \sum\limits_{g \in C_G(Q)} \alpha_g g$.*

Proposition 2.26. [29, Proposition 2.2] *Let G be a finite group and $Q \leq_p G$. Then for any $a, a' \in (kG)^Q$, $\mathrm{Br}_Q^G(aa') = \mathrm{Br}_Q^G(a)\,\mathrm{Br}_Q^G(a')$. Consequently, the map $\mathrm{Br}_Q^G : (kG)^Q \to kC_G(Q), a \mapsto \mathrm{Br}_Q^G(a)$ is a surjective homomorphism of k-algebras.*

Definition 2.27. *Let G be a finite group with $N \trianglelefteq G$. We call a block b of kN G-stable if ${}^g b = b$ for all $g \in G$.*

Definition 2.28. *Let G be a finite group, $Q, R \leq G$ and (Q, f) and (R, e) be Brauer pairs. Then*
(a) $(Q, f) \trianglelefteq (R, e)$ *if* $Q \trianglelefteq R$*,* f *is* R*-stable and* $\mathrm{Br}_R^G(f)e = e$*,*
(b) $(Q, f) \leq (R, e)$ *if* $Q \leq R$ *and there exist Brauer pairs* $(S_i, d_i), 1 \leq i \leq n$*, such that* $(Q, f) \trianglelefteq (S_1, d_1) \trianglelefteq (S_2, d_2) \trianglelefteq \cdots \trianglelefteq (S_n, d_n) \trianglelefteq (R, e)$.

Let (R, e) be a Brauer pair and let $Q \leq R$. The idempotent f such that $(Q, f) \leq (R, e)$ as in the previous definition, is actually uniquely determined:

Theorem 2.29. (Alperin–Broué) [29, Theorem 2.9] *Let G be a finite group, $R \leq G$ and let (R, e) be a Brauer pair. For any $Q \leq R$, there exists a unique $f \in \mathcal{B}(Q)$ with $(Q, f) \leq (R, e)$. Furthermore, if $Q \trianglelefteq R$, then f is the unique element of $\mathcal{B}(Q)^R$ with $\mathrm{Br}_R^G(f)e = e$. The conjugation action of G on the set of Brauer pairs preserves $\leq$.*

Definition 2.30. *Let G be a finite group and b a block of kG.*
(a) A b-Brauer pair is a Brauer pair (R, e) such that $(1, b) \leq (R, e)$, or equivalently it is a Brauer pair (R, e) such that $\mathrm{Br}_R^G(b)e = e$.
(b) We denote the blocks e of $kC_G(R)$ such that $(1, b) \leq (R, e)$ by $\mathcal{B}(R, b)$.
(c) A defect group of b is a p-subgroup P of G maximal such that $\mathrm{Br}_P^G(b) \neq 0$.

Note that the group G acts by conjugation on the set of b-Brauer pairs. Furthermore, some p-subgroup P of G is a defect group of b if and only if there is a maximal pair (P, e) such that $(1, b) \leq (P, e)$. We refer to such a pair as a maximal b-Brauer pair. The following theorem shows that defect groups of blocks can be detected locally.

Theorem 2.31. (Brauer's First Main Theorem) [29, Theorem 3.6] *Let G be a finite group with $P \leq_p G$. The map Br_P^G induces a bijection between the set of blocks of kG with defect group P and the set of blocks of $kN_G(P)$ with defect group P.*

Definition 2.32. *Let G be a finite group and $N \trianglelefteq G$. Let b be a block of kG and c be a block of kN. We say that b covers c if $bc \neq 0$.*

We record some facts about covered blocks which we need in later chapters.

Lemma 2.33. *Let G be a finite group, $P \leq_p G$, (P, e_P) and (P, f_P) be Brauer pairs such that f_P covers e_P. Suppose that $e_U f_U \neq 0$ for some $U \leq P$. Then $e_Q f_Q \neq 0$ for any $Q \leq U$.*

Proof. This is proven in more generality in the proof of [32, Theorem 3.5]. □

Lemma 2.34. *Let G be a finite group, N a normal subgroup of G, d a block of kG and c an G-stable block of kN. Suppose that there exists a d-Brauer pair (Q, e) such that $N \leq C_G(Q)$ and e covers c. Then d covers c.*

Proof Since e covers c and c is $C_G(Q)$-stable, $ce = e$, see [37, Proposition 6.8.2(ii)]. So, $dce = de$. On the other hand, since (Q, e) is a d-Brauer pair, $\mathrm{Br}_Q^G(de) = \mathrm{Br}_Q^G(d)\,\mathrm{Br}_Q^G(e) = \mathrm{Br}_Q^G(d)e = e \neq 0$. Hence, $dce = de \neq 0$ which implies that $dc \neq 0$. □

Lemma 2.35. [21, Chapter V, Lemma 3.5] *Let G be a finite group, N a normal subgroup of G such that G/N is a p-group. If b is a block of kN, then there is a unique block of kG that covers b.*

Theorem 2.36. [29, Theorem 3.9(i)] *Let b be a block of kG and (P, e_P) be a maximal b-Brauer pair. For a subgroup $Q \leq P$, denote by e_Q the unique block such that $(Q, e_Q) \leq (P, e_P)$. Denote the category on P whose morphisms consist of all injective group homomorphisms $\varphi : Q \to R$ for which there is some $g \in G$ such that $\varphi(x) = {}^g x$ for all $x \in Q$ and ${}^g(Q, e_Q) \leq (R, e_R)$ by $\mathcal{F}_{(P,e_P)}(G, b)$. Then $\mathcal{F}_{(P,e_P)}(G, b)$ is a fusion system on P.*

If a fusion system is of the form defined in the previous theorem, we call it block-realisable, otherwise we call it block-exotic. The following theorem connects exotic and block-exotic fusion systems.

Theorem 2.37. (Brauer's Third Main Theorem) [29, Theorem 7.1] *Let G be a finite group and b the principal block of kG with maximal b-Brauer pair (P, e_P). Then, for* *any* $Q \leq G$*,* $\mathrm{Br}_Q^G(b)$ *is the principal block of* kG*. In particular,* $P \in \mathrm{Syl}_p(G)$ *and* $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_P(G)$*.*

In particular, any realisable fusion system is block-realisable. The converse is still an open problem, as noted in Conjecture 1.1.

Note that if $\mathcal{F}$ is a fusion system on an abelian group, Conjecture 1.1 holds true since there are no exotic fusion systems on abelian groups: Let $\mathcal{F}$ be a fusion system on an abelian p-group P. In this case, no proper subgroup of P can be $\mathcal{F}$-centric. By Theorem 2.6, every morphism in $\mathcal{F}$ is the restriction of some $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$. Since P is fully $\mathcal{F}$-automised, $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$ is a p'-group by Definition 2.3. In particular, $\mathcal{F} = \mathcal{F}_P(P \rtimes \mathrm{Aut}_{\mathcal{F}}(P))$, so we get that $\mathcal{F}$ is realisable.

If $\mathcal{F}$ is a fusion system on an extraspecial group of order p^3, the conjecture is also known to hold: There are exotic fusion systems on such groups, but in [32], it has been proven that all these are block-exotic as well.

We give an overview of some families for which the conjecture has been proven in Section 6.

We prove that block-realisability is insensitive to taking quotients modulo central subgroups.

Definition 2.38. *Let* b *be a block of* kG *and* $N \trianglelefteq G$*. We denote by* $\hat{b}$ *the image of* b *under the canonical surjection* $kG \to k(G/N)$*,* $\sum_{g \in G} k_g g \mapsto \sum_{g \in G} k_g gN$*.*

Definition 2.39. *Let* b *be a block of* kG*. The kernel of* b*, denoted* $\ker(b)$*, is defined to be the intersection of the kernels of the irreducible representations of* G *over* k *belonging to* b*.*

In particular, if Z is a central p-subgroup of a finite group G, then $\hat{b}$ is a block of G/Z for any block b of G. This is being used in the statement below to see that $\widehat{e_Q}$ is a block of $C_G(Q)/Z$ (applied to the group $C_G(Q)$ and the block e_Q).

Theorem 2.40. *Let G be a finite group and b a block of kG with maximal b-Brauer pair (P, e_P). Assume $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_P(H)$ for some finite group H. Let Z be a central subgroup of both G and H such that its p'-part is in* $\ker(b)$*. Denote by $\hat{b}$ the image of b under the canonical surjection $kG \to k(G/Z)$. Then PZ/Z is a defect group for $\hat{b}$. If $(PZ/Z, e_{P/Z})$ is a maximal $\hat{b}$-Brauer pair, we have $\mathcal{F}_{(PZ/Z, e_{PZ/Z})}(G/Z, \hat{b}) = \mathcal{F}_{PZ/Z}(H/Z)$.*

To prove this theorem, we first need to collect some auxiliary results. We split Z into its p- and p'-part. Since we are interested in p-blocks, the case of reducing modulo the p'-part can be dealt with quickly. This stems mainly from the following fact.

Lemma 2.41. *Let G be a finite group, Z be a p'-subgroup of $Z(G)$ and $P \leq_p G$. For $H \leq G$, let $\overline{H}$ be the image of H under the map $G \to G/Z$. Then $\overline{C_G(P)} = C_{\overline{G}}(\overline{P})$.*

Proof. Clearly, $\overline{C_G(P)} \subseteq C_{\overline{G}}(\overline{P})$. Let $\overline{x} \in C_{\overline{G}}(\overline{P})$, then there is some $z \in Z$ with $xux^{-1} = uz$ for all $u \in P$. However, the left hand side is a p-element, which means that since z is central, that $z = 1$. Thus, $x \in C_G(P)$ and in particular $\overline{x} \in \overline{C_G(P)}$. $\square$

We need to put more work into the case of central p-subgroups and recall some concepts.

Definition 2.42. *Let $\mathcal{F}$ be a fusion system on a p-group P and let $Q \leq P$. The centraliser of Q in $\mathcal{F}$ is the category $C_{\mathcal{F}}(Q)$ on $C_P(Q)$ having as morphisms all group homomorphisms $\varphi : R \to S$ for $R, S \leq C_P(Q)$, for which there exists a $\mathcal{F}$-morphism $\psi : QR \to QS$ in $\mathcal{F}$ satisfying $\psi|_Q = id_Q$ and $\psi|_R = \varphi$.*

Definition 2.43. *Let $\mathcal{F}$ be a fusion system on a p-group P with $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some $Q \trianglelefteq P$. Define the category $\mathcal{F}/Q$ on P/Q as follows: for any two subgroups $Q, S \leq P$ containing Q, a group homomorphism $\psi : R/Q \to S/Q$ is a morphism in $\mathcal{F}/Q$, if there exists a morphism $\varphi : R \to S$ in $\mathcal{F}$ satisfying $\varphi(u)Q = \psi(uQ)$ for all $u \in R$.*

Theorem 2.44. [3, Part II, Theorem 2.1] *If $\mathcal{F}$ is a fusion system and Q is fully $\mathcal{F}$-normalised, $C_{\mathcal{F}}(Q)$ is a fusion system.*

Lemma 2.45. *Let G be a finite group and b a block of G with maximal b-Brauer pair (P, e_P) and let $\mathcal{F} := \mathcal{F}_{(P,e_P)}(G, b)$. Then for any central p-subgroup $Z \leq G$, $\mathcal{F} = C_{\mathcal{F}}(Z)$.*

Proof. Since Z is central, it is fully $\mathcal{F}$-normalised. By the previous theorem, $C_{\mathcal{F}}(Z)$ is a fusion system on $C_P(Z)$. But since $C_P(Z) = P$, this means that $\mathcal{F}$ and $C_{\mathcal{F}}(Z)$ have the same objects. Thus, by Theorem 2.6, it suffices to show $\mathrm{Aut}_{\mathcal{F}}(Q) \subseteq \mathrm{Aut}_{C_{\mathcal{F}}(Z)}(Q)$ for all $Q \leq P$. Hence, let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $g \in G$ such that $\varphi(u) = {}^g u$ for all $u \in Q$. By definition, we then also have ${}^g e_Q = e_Q$, where e_Q denotes the unique block such that $(Q, e_Q) \leq (P, e_P)$. Denote the conjugation with g on QZ by ψ. Since Z is central, this is well-defined, restricts to the identity on Z and to φ on Q. So, we need to check ${}^g e_{QZ} = e_{QZ}$. Note that Q is a normal subgroup of QZ containing all its p'-elements. This holds in particular for blocks, thus, a block idempotent of kQZ is in kQ and $e_{QZ} = e_Q$ by Theorem 2.29. Hence, ${}^g e_{QZ} = {}^g e_Q = e_Q = e_{QZ}$. Thus, φ can be extended to an $\mathcal{F}$-morphism $\psi \in \mathrm{Aut}(QZ)$ with all desired properties, which implies the claim. □

Lemma 2.46. *Let $Q \leq_p G$ and $Z \leq_p G$ be central with $Z \subseteq Q$. Then $C_G(Q) \trianglelefteq C_G(\overline{Q})$ and $C_G(\overline{Q})/C_G(Q)$ is a p-group, where $\overline{Q} := Q/Z$.*

Proof. Consider the map $\tau_u : C_G(\overline{Q}) \to Z, y \mapsto yuy^{-1}u^{-1}$. Since Z is central, it defines a group homomorphism with kernel $C_G(u)$. Since $C_G(Q) = \bigcap\limits_{u \in Q} C_G(u)$ this implies $C_G(Q) \trianglelefteq C_G(\overline{Q})$.
Now let $y \in C_G(\overline{Q})$ be a p'-element. We have $C_G(\overline{Q})/\ker \tau_u \leq Z$, so this quotient is a p-group. This implies $y \in \ker(\tau_u)$ for every $u \in Q$, in particular $y \in C_G(Q)$. □

Lemma 2.47. *Let G be a finite group, b a block of kG and Z a central p-subgroup of G. Denote $\overline{Q} = Q/Z$ for all Q with $Z \leq Q \leq_p G$ and for any element a of kG, denote by $\hat{a}$ its image under the map $kG \to k(G/Z)$. Then (Q, e_Q) is a b-Brauer pair if and only if $(\overline{Q}, f_{\overline{Q}})$ is a $\hat{b}$-Brauer pair, where $f_{\overline{Q}} = \mathrm{Tr}_H^{C_{\overline{G}}(\overline{Q})}(\widehat{e_Q})$ and $H = \mathrm{Stab}_{C_{\overline{G}}(\overline{Q})}(\widehat{e_Q})$.*

Proof. First, we show $\widehat{\mathrm{Br}_Q^G(a)} = \mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{a})$ for any $Q \leq_p G$ and any central idempotent a of kG. Write $a = \sum\limits_{g \in G} \alpha_g g$, $\alpha_g \in k$. We use the notation from Definition 2.38. By definition,

$\mathrm{Br}_Q^G(a) = \sum_{g \in C_G(Q)} \alpha_g g$, so $\widehat{\mathrm{Br}_Q^G(a)} = \sum_{g \in C_G(Q)} \alpha_g \overline{g}$. On the other hand, $\hat{a} = \sum_{g \in G} \alpha_g \overline{g}$ and $\mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{a}) = \sum_{\overline{g} \in C_{\overline{G}}(\overline{Q})} \alpha_g \overline{g}$. Both expressions are the same, since for every block idempotent a, some coefficient α_g can only be non-zero if g is a p'-element, and we have that $C_G(Q)$ is a normal subgroup of $C_G(\overline{Q})$ with index p^n for some $n \in \mathbb{N}$ by Lemma 2.46. In particular, for any $g \in G_{p'}$, we have $g \in C_G(Q)$ if and only if $\overline{g} \in C_{\overline{G}}(\overline{Q})$.

Assume (Q, e_Q) is a b-Brauer pair. By definition, e_Q is a block of $kC_G(Q)$, and since Z is a central p-subgroup of $C_G(Q)$, $\widehat{e_Q}$ is a block of $C_G(Q)/Z$. By Lemma 2.46, $C_G(Q)/Z$ is of p-power index in $C_{\overline{G}}(\overline{Q})$ hence the blocks of $C_{\overline{G}}(\overline{Q})$ are precisely the $C_{\overline{G}}(\overline{Q})$-orbit sums of the blocks of $C_G(Q)/Z$. In particular, $f_{\overline{Q}}$ is a block of $kC_{\overline{G}}(\overline{Q})$. Now $\mathrm{Br}_Q^G(b)e_Q \neq 0$, which means $\mathrm{Br}_Q^G(b)e_Q = e_Q$, but then also $\widehat{\mathrm{Br}_Q^G(b)e_Q} = \widehat{e_Q}$. The kernel of the reduction of $kC_G(Q)$ modulo Z can contain no idempotent, since it is contained in $J(kC_G(Q))$ by [29, Proposition 2.3]. Thus, also using the equality from above, we have $\mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{b})\widehat{e_Q} = \widehat{e_Q} \neq 0$. Since $f_{\overline{Q}}$ is a sum of blocks of $C_G(Q)/Z$ one of which is $\widehat{e_Q}$, $f_{\overline{Q}}\widehat{e_Q} = \widehat{e_Q}$. Thus, $0 \neq \mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{b})\widehat{e_Q} = \mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{b})f_{\overline{Q}}\widehat{e_Q}$, which shows that $\mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{b})f_{\overline{Q}} \neq 0$, proving that $(\overline{Q}, f_{\overline{Q}})$ is a $\hat{b}$-Brauer pair.

Conversely, assume $\mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{b})f_{\overline{Q}} \neq 0$, but then by definition of $f_{\overline{Q}}$, also $\mathrm{Br}_{\overline{Q}}^{\overline{G}}(\hat{b})\widehat{e_Q} \neq 0$. Thus, by the above considerations, $\widehat{\mathrm{Br}_Q^G(b)e_Q} \neq 0$, which certainly implies $\mathrm{Br}_Q^G(b)e_Q \neq 0$, so (Q, e_Q) is a b-Brauer pair. □

Now we have enough tools to prove Theorem 2.40.

Proof of Theorem 2.40. Since Z is abelian, we can assume $Z = Z_p Z_{p'}$ and deal with the quotient modulo one of these two factors at a time. Let $\overline{R} = RZ/Z$ for $R \leq G$.

First, assume $Z_p = 1$. By [36, Theorem 2.12], we have $\mathcal{F}_P(H) \cong \mathcal{F}_{\overline{P}}(\overline{H})$. Furthermore, by [13, Proposition 17.8(ii)], we get that $\hat{b} \neq 0$, since any $\chi \in \mathrm{Irr}(G, b)$ has all p'-elements in its kernel, i.e. $\chi(Z_{p'}) = \chi(1)$, and that $\hat{b}$ is a block. Let $P_0 \leq G$ be a p-group such that $\overline{P_0}$ is a defect group for $\hat{b}$. Assume $b = \sum_{x \in G} \alpha_x x$, so $\hat{b} = \sum_{x \in G} \alpha_x \overline{x} = \sum_{y \in \overline{G}} \left(\sum_{x | \overline{x} = y} \alpha_x y \right)$. Since $\mathrm{Br}_{\overline{P_0}}^{G}(\hat{b}) \neq 0$, there has to be some $y \in \overline{G}$ such that $\sum_{x | \overline{x} = y} \alpha_x \neq 0$ with $y \in C_{\overline{G}}(\overline{P_0})$. In particular, there exists some $x \in G$ with $\overline{x} = y$, $y \in C_{\overline{G}}(\overline{P_0})$ and $\alpha_x \neq 0$. By Lemma 2.41, $x \in C_G(P_0)$. This implies, if we denote the preimage of $\overline{P_0}$ under the quotient map by P_0, that $\mathrm{Br}_{P_0}^{G}(b) \neq 0$. The group P_0 is thus contained in a defect group P' of b. Since defect groups are conjugate, there is a some $g \in G$ with ${}^{g}P' = P$. Set $P_0 = {}^{g}P_0$, so we may assume $P_0 \subseteq P$. However, we also have by definition of defect:

$$|\overline{P_0}| = \max \left\{ \frac{|\overline{G}|_p}{\rho(1)_p} : \rho \in \mathrm{Irr}_k(\overline{G}, \hat{b}) \right\} = \max \left\{ \frac{|G|_p}{\chi(1)_p} : \chi \in \mathrm{Irr}_k(G, b) \right\},$$

the latter equality holds since $Z_p = 1$. The latter number is the cardinality of a defect group for b. This means $P_0 = P$ and $\overline{P_0} \cong P$. Write $\overline{P_0}$ as $\overline{P}$ henceforth. We thus have

$$\mathcal{F}_{\overline{P}}(\overline{H}) \cong \mathcal{F}_P(H) \cong \mathcal{F}_{(P, e_P)}(G, b) \cong \mathcal{F}_{(\overline{P}, e_{\overline{P}})}(\overline{G}, \hat{b}),$$

where the last isomorphism is defined by the isomorphism of the defect groups $P \to \overline{P}$. Now assume $Z_{p'} = 1$. By [29, Lemma 3.7], $Z_p \leq P$, thus we can apply [36, Proposition 6.6] to obtain $\mathcal{F}_P(H)/Z_p = \mathcal{F}_{\overline{P}}(\overline{H})$. But we also have $\mathcal{F}_P(H)/Z_p = \mathcal{F}_{(P, e_P)}(G, b)/Z_p$ by [36, Theorem 6.4], which is applicable by Lemma 2.45. Finally, we need to show that $\mathcal{F}_{(P, e_P)}(G, b)/Z_p$ is the fusion system of $\hat{b}$. First note that by [39, Chapter 5, Theorem 8.11], the latter is indeed a block of $k\overline{G}$. For convenience, let $\mathcal{F} := \mathcal{F}_{(P, e_P)}(G, b)$ and $\mathcal{F}' := \mathcal{F}_{(\overline{P}, e_{\overline{P}})}(\overline{G}, \hat{b})$. Note that we have

$$|P| = \max \left\{ \frac{|G|_p}{\chi(1)_p} : \chi \in \mathrm{Irr}_k(G, b) \right\} \geq |Z_p| \max \left\{ \frac{|\overline{G}|_p}{\rho(1)_p} : \rho \in \mathrm{Irr}_k(\overline{G}, \hat{b}) \right\},$$

hence, $\overline{P}$ has at least the order of a defect group of $\hat{b}$. This observation, together with Lemma 2.47, implies that the fusion systems $\mathcal{F}/Z_p$ and $\mathcal{F}'$ are categories on the same group with the same objects. It suffices to show that they also have the same morphisms. For this, we use the element $f_{\overline{Q}}$ as defined as in Lemma 2.47 again. We need to show $\mathrm{Aut}_{\mathcal{F}/Z_p}(\overline{Q}) = \mathrm{Aut}_{\mathcal{F}'}(\overline{Q})$ for all $\overline{Q} \leq \overline{P}$. Take some $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}/Z_p}(\overline{Q})$. Then this map is induced by $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and there is some $g \in N_G(Q, e_Q)$ such that conjugation by g is equal to φ. Thus, we have to show that ${}^{\overline{g}}f_{\overline{Q}} = f_{\overline{Q}}$. Indeed,

$$ {}^{\overline{g}}f_{\overline{Q}}f_{\overline{Q}} = {}^{\overline{g}}f_{\overline{Q}}f_{\overline{Q}}\widehat{e_Q}^2 = {}^{\overline{g}}(f_{\overline{Q}}\widehat{e_Q})f_{\overline{Q}}\widehat{e_Q} \neq 0, $$

since $f_{\overline{Q}}\widehat{e_Q} \neq 0$. Thus, $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}'}(\overline{Q})$. On the other hand, take $\psi \in \mathrm{Aut}_{\mathcal{F}'}(\overline{Q})$ and assume this map is induced by conjugation by some $\overline{x} \in N_{\overline{G}}(\overline{Q}, f_{\overline{Q}})$, so ${}^{\overline{x}}f_{\overline{Q}} = f_{\overline{Q}}$. By definition of $f_{\overline{Q}}$, this implies ${}^{\overline{x}}\widehat{e_Q} = {}^{\overline{t}}\widehat{e_Q}$ for some $t \in C_G(\overline{Q})$. In particular, ${}^{\overline{t}^{-1}\overline{x}}\widehat{e_Q} = \widehat{e_Q}$, so $\overline{t^{-1}x} \in N_{\overline{G}}(\overline{Q}, \widehat{e_Q})$. If we define $y := t^{-1}x$, then $y \in N_G(Q, e_Q)$, otherwise we would have $\widehat{e_Q} = 0$, a contradiction since the kernel of the reduction cannot contain idempotents, as in the proof of Lemma 2.47. Denote the inverse image of $N_{\overline{G}}(\overline{Q}, f_{\overline{Q}})$ under the canonical surjection in $N_G(Q)$ by $N_G(\overline{Q}, f_{\overline{Q}})$, we thus have shown $N_G(\overline{Q}, f_{\overline{Q}}) = C_G(\overline{Q})N_G(Q, e_Q)$. In particular, for every $\overline{x} \in N_{\overline{G}}(\overline{Q}, f_{\overline{Q}})$, there exists some $y \in N_G(Q, e_Q)$ such that conjugation by $\overline{x}$ and $\overline{y}$ are the same map on $\overline{Q}$. This implies $\psi \in \mathrm{Aut}_{\mathcal{F}/Z_p}(\overline{Q})$. Application of Theorem 2.6 implies $\mathcal{F}/Z_p = \mathcal{F}'$, which in turn implies the theorem. $\square$

Finally, we finish this section by proving that a fusion system of a block is invariant under the tensor product with a linear character.

Theorem 2.48. *Let G be a finite group, b a block of kG with maximal b-Brauer pair (P, e_P) and $\theta : G \to k^\times$ a linear character. We get an $\mathcal{O}$-linear map $\tilde{\theta} : G \to \mathcal{O}G, g \mapsto \theta(g^{-1})g$ for $g \in G$ and can extend it linearly to $\mathcal{O}G$. Then $\tilde{\theta}(b)$ is also a block of kG with defect group P and the fusion systems $\mathcal{F}_{(P,e_P)}(G, b)$ and $\mathcal{F}_{(P,\tilde{\theta}(e_P))}(G, \tilde{\theta}(b))$ are isomorphic.*

Proof. For a linear character $\theta : G \to k^\times$, we have a unique lift of $\theta(g)$ to $\mathcal{O}$ for all $g \in G$. It is easy to see that $\tilde{\theta}$ is an $\mathcal{O}$-algebra isomorphism, so we get a map $\widehat{G} \to \mathrm{Aut}(\mathcal{O}G), \theta \mapsto \tilde{\theta}$. Since $\tilde{\theta}$ is an automorphism, it permutes the blocks and in particular, $\tilde{\theta}(b)$ is a block.

Furthermore, note that if A is an algebra, $\varphi \in \mathrm{Aut}(A)$ and M is an A-module, then we can define another A-module ${}_{\varphi}M$ by $a * m := \varphi^{-1}(a)m$. Thus, in our case, if we have $A = \mathcal{O}G$ and $\varphi = \tilde{\theta}$, we get ${}_{\tilde{\theta}}\chi = \theta \otimes \chi$ for some character χ. In this setting, if a module M belongs to a block b, then ${}_{\varphi}M$ belongs to $\varphi(b)$. Indeed, M belongs to b if and only if $bm = m$ for all $m \in M$, which is equivalent by definition to $\varphi(b) * m = m$, thus ${}_{\varphi}M$ belongs to $\varphi(b)$.

A general linear character θ can be written as $\theta = \theta_{p'}\theta_p$ and clearly $\tilde{\theta} = \tilde{\theta}_{p'}\tilde{\theta}_p$. By [39, Chapter 3, Theorem 6.24], we have that if two characters χ, χ' belong to the same block, then $\frac{|G|\chi(x)}{|C_G(x)|\chi(1)} \equiv \frac{|G|\chi'(x)}{|C_G(x)|\chi'(1)}(J(\mathcal{O}))$ for every $x \in C \in \mathrm{Cl}(G_{p'})$, where the latter denotes the conjugacy classes of $G_{p'}$. This implies that χ and $\tilde{\theta}_p(\chi)$ both belong to the same block. In particular, for our remaining considerations we may assume that θ is a p'-character, i.e. $|\,\mathrm{ord}(\theta)|_p = 1$ in $\widehat{G}$. In particular, for such an θ we have $\theta(x) = 1$ for every $x \in G_p$. If $Q \leq_p G$, we thus have $\tilde{\theta}(Q) = Q$, i.e. $\tilde{\theta}|_Q = \mathrm{id}_Q$.

If $Q, R \leq_p G$, we claim that $(Q, b_Q) \leq (R, b_R)$ if and only if $(Q, \tilde{\theta}(b_Q)) \leq (R, \tilde{\theta}(b_R))$. This follows if we can show $(Q, b_Q) \trianglelefteq (R, b_R)$ if and only if $(Q, \tilde{\theta}(b_Q)) \trianglelefteq (R, \tilde{\theta}(b_R))$. By symmetry, it suffices to prove one direction. Thus, assume $(Q, b_Q) \trianglelefteq (R, b_R)$. One easily sees ${}^{g}\tilde{\theta}(b) = \tilde{\theta}({}^{g}b)$ for some block b and some $g \in G$, which implies that $\tilde{\theta}(b_Q)$ is R-stable. It is left to show that $\mathrm{Br}_R^G(\tilde{\theta}(b_Q))\tilde{\theta}(b_R) = \tilde{\theta}(b_R)$. This follows from $\mathrm{Br}_R^G(b_Q)b_R = b_R$, since $\tilde{\theta}$ is an homomorphism and the fact, which is easy to see, that it commutes with the Brauer map.

So, we have a bijection $\{b\text{-Brauer pairs}\} \to \{\tilde{\theta}(b)\text{-Brauer pairs}\}$ given by $(Q, b_Q) \mapsto (Q, \tilde{\theta}(b_Q))$, and (P, e_P) is a maximal b-Brauer pair if and only if $(P, \tilde{\theta}(e_P))$ is a maximal $\tilde{\theta}(b)$-Brauer pair. In particular, the objects of the categories are the same. Furthermore, the erstwhile mentioned identity ${}^{g}\tilde{\theta}(b) = \tilde{\theta}({}^{g}b)$ implies $({}^{g}Q, {}^{g}\tilde{\theta}(b_Q)) = ({}^{g}Q, \tilde{\theta}({}^{g}b_Q))$, which means that the morphisms of the categories are also the same. $\square$

2.3 Generalised block fusion systems

We need to introduce more general categories than block fusion systems, since some group theoretic properties are not captured by these: Assume b is a block of kG with maximal b-Brauer pair (P, e_P) and $N \trianglelefteq G$. If c is a block of kN covered by b, i.e. $bc \neq 0$, $P \cap N$ is a defect group for c, see [39, Chapter 5, Theorem 5.16 (iii)]. However, in general $\mathcal{F}_{(P\cap N, e_{P\cap N})}(N, c)$ is not even a subsystem of $\mathcal{F}_{(P,e_P)}(G, b)$:

Example 2.49. [29, Example 7.5] Let $p = 3$ and let P be the cyclic group of order 3 with generator r. Let $T = \langle x, y \mid x^4 = y^4 = 1, xyx^{-1} = y^3\rangle$ be the quaternion group of order 8 acting on P via ${}^x r = r^2$, ${}^y r = r$. Let $G = PT$ and $N := P\langle x\rangle \trianglelefteq G$. Consider $b = \frac{1}{2}(1 - x^2) = \frac{1}{2}(1 - y^2)$. Then b is a block of kG as well as of kN, see e.g. [29, Lemma 3.7]. Now $C_G(P) = P \times \langle y\rangle$. Thus, $\mathrm{Br}_P^G(b)$ is a sum of two blocks, $\frac{1}{4}(1 + iy - y^2 - iy^3)$ and $\frac{1}{4}(1 - iy - y^2 + iy^3)$ of $kC_G(P)$, where i is a primitive fourth root of unity. Set $e_P = \frac{1}{4}(1 + iy - y^2 - iy^3)$ and $\mathcal{F} = \mathcal{F}_{(P,e_P)}(G, b)$, then $\mathrm{Aut}_{\mathcal{F}}(P) = 1$.
On the other hand, $C_N(P) = P \times \langle x^2\rangle$, whence $\mathrm{Br}_P^N(b) = b$ is a block of $kC_N(P)$. Set $\mathcal{F}' = \mathcal{F}_{(P,\mathrm{Br}_P^N(b))}(N, b)$, it is clear that $\mathrm{Aut}_{\mathcal{F}'}(P)$ is cyclic of order 2.

We use a generalised category, introduced in [32], to circumvent this difficulty, which will turn out to be very useful when proving results about block fusion systems.

Definition 2.50. *Let G be a finite group, $N \trianglelefteq G$ and c be a G-stable block of kN. A (c, G)-Brauer pair is a pair (Q, e_Q), where Q is a p-subgroup of G with $\mathrm{Br}_Q^N(c) \neq 0$ and e_Q is a block of $kC_N(Q)$ such that $\mathrm{Br}_Q^N(c)e_Q \neq 0$.*

Let (Q, e_Q) and (R, e_R) be two (c, G)-Brauer pairs. We say that (Q, e_Q) is contained in (R, e_R) and write $(Q, e_Q) \leq (R, e_R)$, if $Q \leq R$ and for any primitive idempotent $i \in (kN)^R$ with $\mathrm{Br}_R^N(i)e_R \neq 0$, we also have $\mathrm{Br}_Q^N(i)e_Q \neq 0$. This defines an order relation on the set of (c, G)-Brauer pairs compatible with the conjugation action of G. We also have that given a (c, G)-Brauer pair and $Q \leq R$ there exists a unique (c, G)-Brauer pair (Q, e_Q) contained in (R, e_R), see [8, Theorem 1.8(i)]. Also, by [8, Theorem 1.14(2)], all maximal (c, G)-Brauer pairs are G-conjugate. If (P, e_P) is a maximal (c, G)-Brauer pair and $G = N$, P is a defect group as defined in the previous section.

Definition 2.51. *Let G be a finite group, $N \trianglelefteq G$ and c be a G-stable block of kN. Let (P, e_P) be a maximal (c, G)-Brauer pair. Let $(Q, e_Q) \leq (P, e_P)$. Denote by $\mathcal{F}_{(P,e_P)}(G, N, c)$ the category on P with morphisms consisting of all injective group homomorphisms $\varphi : Q \to R$ for which there is some $g \in G$ such that $\varphi(x) = {}^g x$ for all $x \in Q$ and ${}^g(Q, e_Q) \leq (R, e_R)$.*

Theorem 2.52. [32, Theorem 3.4] *Let G be a finite group, $N \trianglelefteq G$ and c be a G-stable block of kN. Let (P, e_P) be a maximal (c, G)-Brauer pair. The category $\mathcal{F}_{(P,e_P)}(G, N, c)$ is a fusion system. If (P', e'_P) is another maximal (c, G)-Brauer pair, then $\mathcal{F}_{(P,e_P)}(G, N, c)$ is isomorphic to $\mathcal{F}_{(P',e'_P)}(G, N, c)$.*

We refer to such fusion systems as generalised block fusion systems. If $G = N$, one obtains the usual fusion systems of blocks from Theorem 2.36. In Chapter 3 we see that we need these more general categories to prove reduction theorems for block fusion systems. The following theorem shows how this category is connected to the block fusion systems defined before and how we circumvent the problems from Example 2.49.

Theorem 2.53. [32, Theorem 3.5] *Let G be a finite group with $N \trianglelefteq G$. Let c be a G-stable block of kN covered by a block $b \in kG$. Let (P, e_P) be a maximal b-Brauer pair. Then there exists a maximal (c, G)-Brauer pair (S, e'_S) with $P \leq S$ and $\mathcal{F}_{(P,e_P)}(G, b) \leq \mathcal{F}_{(S,e'_S)}(G, N, c)$. Furthermore, $P \cap N = S \cap N$, $(S \cap N, e'_{S\cap N})$ is a maximal c-Brauer pair and $\mathcal{F}_{(S\cap N,e'_{S\cap N})}(N, N, c) = \mathcal{F}_{(S\cap N,e'_{S\cap N})}(N, c) \trianglelefteq \mathcal{F}_{(S,e'_S)}(G, N, c)$.*

Note that in [32], only weak normality of $\mathcal{F}_{(S\cap N,e'_{S\cap N})}(N, c)$ in $\mathcal{F}_{(S,e'_S)}(G, N, c)$ was proved, but it was improved to normality in [3, Part IV, Theorem 6.4]. We refer to the relations between the three fusion systems in this theorem as “triangle relations”.

These useful relations allow a descent to normal subgroups, which is not possible for block fusion systems by Example 2.49. This gives reason to believe that in order to prove Conjecture 1.1, one must prove Conjecture 1.5.
Note that Conjecture 1.5 implies Conjecture 1.1, since any block fusion system $\mathcal{F}_{(P,e_P)}(G,b)$ can be written as a generalised block fusion system $\mathcal{F}_{(P,e_P)}(G,G,b)$.

We finish this section by proving that when specialising to the case when G/N is a p'-group, the three fusion systems in Theorem 2.53 are categories on the same group.

Proposition 2.54. *Let G be a finite group with normal subgroup N and let c be a block of kG covering a G-stable block b of kN. If G/N is a p'-group, then three fusion systems occurring in Theorem 2.53 are categories on the same group.*

Proof. Let (S, f_S) be a maximal c-Brauer pair. As $bc = c$ and $\mathrm{Br}_S^G(c)f_S = f_S$, there is a central primitive idempotent f'_S of $kC_N(S)$ such that $\mathrm{Br}_S^N(b)f'_S = f'_S$ and f_S covers f'_S. In particular, (S, f'_S) is a (b, G)-Brauer pair. Let (T, f'_T) be a maximal (b, G)-Brauer pair such that $(S, f'_S) \leq (T, f'_T)$. By the proof of [32, Theorem 3.5] we get that $(T \cap N, f'_{T\cap N})$ is a maximal b-Brauer pair. Since G/N is a p'-group we have that $T = T \cap N$ and thus $S \leq T \cap N$. But we also have $|S| \geq |T \cap N|$ which implies $S = T \cap N = T$. $\square$

Note that this proof partially follows the proof of Theorem 2.53.

3 Reduction Theorems

3.1 Overview of Reduction Theorems

In this section, we recall reduction theorems for block fusion systems. We start off with two essential results for the study of such systems. These results are respectively called the First and Second Fong Reduction.

When studying the relations between a group G with a block b and the blocks of a normal subgroup $N \trianglelefteq G$, a certain subgroup of G plays a huge role:

Theorem 3.1. [3, Part IV, Proposition 6.3] *Let G be a finite group with $N \trianglelefteq G$ and let c be a block of kN. Let $I_G(c) = \{g \in G | {}^g c = c\}$, the stabiliser of c in G. The map $e \mapsto \mathrm{Tr}_{I_G(c)}^G(e)$ is a bijection between the set of blocks of $kI_G(c)$ covering c and the set of blocks of kG covering c. Furthermore, if b is a block of kG covering c, then the fusion system of the block b is isomorphic to the fusion system of the block $\tilde{b}$ of $I_G(c)$ with $b = \mathrm{Tr}_{I_G(c)}^G(\tilde{b})$.*

This theorem is called the First Fong Reduction.

Definition 3.2. *Let $N \trianglelefteq G$, c a block of kN covered by $b \in kG$. We call the block $\tilde{b}$ of $I_G(c)$ with $b = \mathrm{Tr}_{I_G(c)}^G(\tilde{b})$ Fong correspondent of b.*

We use Theorem 3.1 often in the following form:

Corollary 3.3. *Let $\mathcal{F}$ be a fusion system and G be a finite group possessing an $\mathcal{F}$-block b such that $|G : Z(G)|$ is minimal among all finite groups having an $\mathcal{F}$-block. Then b is inertial, i.e. it covers only G-stable blocks.*

Proof. Choose N, c as in Theorem 3.1. Since $Z(G) \subseteq I_G(c)$, this theorem and the minimality assumption implies directly that $I_G(c) = G$. Hence b is inertial. □

The following theorem is the Second Fong Reduction:

Theorem 3.4. [3, Part IV, Theorem 6.6] *Let G be a finite group with $N \trianglelefteq G$ and c be a G-stable block of kN with trivial defect. Let b be a block of kG covering c and let (P, e_P) be a maximal b-Brauer pair. Then $N \cap P = 1$ and there exists a central extension $1 \to Z \to \widetilde{G} \to G/N \to 1$, where Z is a cyclic p'-group such that there is a block $\widetilde{b}$ of $k\widetilde{G}$ such that if we identify P with the Sylow p-subgroup of the inverse image of PN/N in $\widetilde{G}$, then there is a maximal $\widetilde{b}$-Brauer pair (P, f_P) such that $\mathcal{F}_{(P,e_P)}(G, b) = \mathcal{F}_{(P,f_P)}(\widetilde{G}, \widetilde{b})$.*

We state the two known reduction theorems with respect to Conjecture 1.1.

Theorem 3.5. [28, Theorem 3.1] *Let $\mathcal{F}$ be a reduction simple fusion system on a p-group P. Assume that $\mathrm{Aut}(P)$ is a p-group. If G is a finite group having an $\mathcal{F}$-block, then there exists a quasisimple group with p'-centre also having an $\mathcal{F}$-block.*

Theorem 3.6. [32, Theorem 4.2] *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems on a p-group P such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Assume that*
(a) $\mathcal{F}_1$ is reduction simple,
(b) if $\mathcal{F}$ is a fusion system on P containing $\mathcal{F}_1$, then $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$,
(c) if $\mathcal{F}$ is a non-trivial weakly normal subsystem of $\mathcal{F}_2$, then $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$.
If there exists a finite group with an $\mathcal{F}_1$ or $\mathcal{F}_2$-block, then there also exists a quasisimple group with p'-centre with an $\mathcal{F}_1$ or $\mathcal{F}_2$-block.

3.2 New Reduction Theorems for block fusion systems

This subsection will be spent proving the Reduction Theorems 1.2 and 1.4. Note that these results have already been published in a paper of mine, see [48]. We recall some group-theoretic notation.

Definition 3.7. *Let G be a finite group.*
(a) *The unique maximal semisimple normal subgroup of G is called the layer of G and denoted by $E(G)$. If $E(G) \neq 1$, the uniquely determined quasisimple factors of $E(G)$ are called the components of G.*

(*b*) *The unique maximal nilpotent normal subgroup of G is called the Fitting subgroup of G and denoted by $F(G)$.*

We start by proving Theorem 1.4. First we need two auxiliary results, one of which we quote directly from [32].

Lemma 3.8. [32, Lemma 6.1] *Let G be a finite group with $N \trianglelefteq G$ and b be a block of kG with defect group P. Then there exists a block c of kN, which is covered by b, having $P \cap N$ as a defect group.*

Lemma 3.9. *Let $\mathcal{F}$ be a reduction simple fusion system and G be a finite group having an $\mathcal{F}$-block b with non-abelian defect group P. If $G = \langle {}^gP : g \in G \rangle$, then there exists a quasisimple group with p'-centre having an $\mathcal{F}$-block.*

Proof. We claim that if $N \trianglelefteq G$ is proper, then N has a block d which is covered by b and of defect zero. Indeed, by Lemma 3.8, we can choose d such that it has $P \cap N$ as defect group. Since N is normal and each morphism in $\mathcal{F}$ is induced by conjugation with an element in G, $P \cap N$ is also strongly $\mathcal{F}$-closed. If $P \cap N \neq 1$, then by reduction simplicity $P \cap N = P$, which would imply that $N = G$, as $G = \langle {}^gP \mid g \in G \rangle$. This contradiction implies $P \cap N = 1$. Note that we can furthermore assume that d is G-stable by Theorem 3.1.

Apply Theorem 3.4 to get a p'-central extension $\widetilde{G}$ of G/N coming from an exact sequence $1 \to Z \to \widetilde{G} \to G/N \to 1$ having a block c that is an $\mathcal{F}$-block. We now construct a quasisimple group L with an $\mathcal{F}$-block. If we choose N to be a maximal normal subgroup, then G/N is either cyclic of prime order or G/N is a non-abelian simple group.

First, we assume that G/N is cyclic, thus let $g \in G/N$ be a generating element and $\tilde{g}$ be a preimage of g in $\widetilde{G}$. Then $\widetilde{G} = \langle Z, \tilde{g} \rangle$. This means $\widetilde{G}$ is abelian, hence so is P, which is a contradiction.

So, we are left with the case that G/N is non-abelian simple. Note that by simplicity of G/N, we necessarily have $Z = Z(\widetilde{G})$ in the extension above. Define $L = [\widetilde{G}, \widetilde{G}]$. We have $LZ/Z \trianglelefteq \widetilde{G}/Z = G/N$. First assume $LZ/Z = 1$, then $LZ = Z$, which means that $L \subseteq Z$. If we identify $\widetilde{G}/Z$ with its image in $\widetilde{G}/L$, we thus get $\widetilde{G}/Z \subseteq \widetilde{G}/L$. This is a contraction since $\widetilde{G}/L$ is abelian, and $\widetilde{G}/Z$, which is a homomorphic image of G/N, is not. So, by simplicity, $LZ/Z = \widetilde{G}/Z$, so $LZ = \widetilde{G}$. Taking commutators of this equation implies $[L, L] = [\widetilde{G}, \widetilde{G}] = L$, so L is perfect. Since we have $\widetilde{G}/Z = LZ/Z \cong {}^{L}/_{L \cap Z}$, L is also a p'-central extension of G/N by $Z \cap L$, and thus quasisimple.

The map $L \times Z \to \widetilde{G} = LZ$ defined by $(l, z) \mapsto lz$, $l \in L, z \in Z$, is a surjective group homomorphism with kernel $K = \{(x, x^{-1}) : x \in Z \cap L\}$, a central p'-group. This means there exists a bijection, preserving the associated fusion systems, between the blocks of $\widetilde{G}$ and the blocks of $L \times Z$ having K in their kernel, see Theorem 2.40. Since Z is cyclic, it is abelian, and thus its blocks are linear characters, see [37, Theorem 3.3.14]. Theorem 2.48 implies that there is a block of kL which is an $\mathcal{F}$-block. □

Proof of Theorem 1.4. Assume G to be of minimal order among the groups possessing an $\mathcal{F}_H$-block b for some $H \leq \Gamma_{p'}(\mathcal{F})$. Let $N \trianglelefteq G$ and c be a block of kN covered by b. By Theorem 3.1, and our assumption, we may assume that $G = I_G(c)$. In particular, for any normal subgroup N of G, we may assume that c is G-stable and the unique block of kN covered by b.

By construction, P is a defect group for b. Consider $M := \langle {}^gP \mid g \in G \rangle \trianglelefteq G$. Let d be the block of kM covered by b. We can apply the first paragraph to $M \trianglelefteq G$ and thus assume d is G-stable. So, we have a homomorphism $G \to \mathrm{Aut}(kMd), g \mapsto c_g$, inducing a map $G \to \mathrm{Out}(kMd)$. Let K be the kernel of this map. Clearly, $M \subseteq K$. We claim $K = G$.

Indeed, let f be the block of kK covered by b. Let (P, e_P) be a maximal b-Brauer pair and (S, e'_S) a maximal (f, G)-Brauer pair as in Theorem 2.53. By [34, Section 5], G/K is a p'-group. Thus, by Proposition 2.54, $P = S$. Furthermore $\mathcal{F}_H \leq \mathcal{F}_{(P,e'_P)}(G, K, f)$ and $\mathcal{F}_{(P,e'_P)}(K, f) \trianglelefteq \mathcal{F}_{(P,e'_P)}(G, K, f)$ by Theorem 2.53.

Thus, by assumption (b), $\mathcal{F}_{(P,e'_P)}(G,K,f) \subseteq \mathcal{F}$ and by definition it is also of p'-index. Hence, there is some $H' \leq \Gamma_{p'}(\mathcal{F})$ such that $\mathcal{F}_{(P,e'_P)}(G,K,f) = \mathcal{F}_{H'}$. Similarly, by assumption (c), there is also a $J \leq \Gamma_{p'}(\mathcal{F})$ such that $\mathcal{F}_{(P,e'_P)}(K,f) = \mathcal{F}_J$. By the minimality of G, we deduce $G = K$.

By this observation, G acts as inner automorphisms on kMd. Thus, by [33, Theorem 7], kMd and kGb have isomorphic source algebras. By [28, Proposition 2.12], we have that d is an $\mathcal{F}_H$-block as well. Using the minimality once more, we obtain $G = M$.

Note that reduction simplicity of $\mathcal{O}^{p'}(\mathcal{F})$ implies reduction simplicity of $\mathcal{F}$ for a fusion system $\mathcal{F}$, since the non-existence of proper non-trivial strongly $\mathcal{O}^{p'}(\mathcal{F})$-closed subgroups of P implies also the non-existence of proper non-trivial strongly $\mathcal{F}$-closed subgroups of P. In particular, we can apply Lemma 3.9 to deduce the theorem. $\square$

Note that we obtain Theorem 3.6 as a corollary of the theorem we just proved by setting $\Gamma_{p'}(\mathcal{F}) = C_2$.

We finish this section with proving Theorem 1.2, which further restricts the structure of reduction simple fusion systems:

Proof of Theorem 1.2. Let $\widetilde{G}$ be a group having an $\mathcal{F}$-block $\widetilde{b}$ subject to $|\widetilde{G} : Z(\widetilde{G})|$ being minimal and $M \trianglelefteq \widetilde{G}$ be maximal such that $P \not\subseteq M$. By [1, 2.9], each normal p-subgroup is contained in each defect group of a block, so we have that $O_p(\widetilde{G}) \leq P$. Furthermore, $O_p(\widetilde{G})$ is strongly $\mathcal{F}$-closed, so either $O_p(\widetilde{G}) = 1$ or $O_p(\widetilde{G}) = P$. Suppose that $O_p(\widetilde{G}) = P$, then $Z(P) \trianglelefteq \widetilde{G}$, since $Z(P)$ is characteristic in P. In particular, $Z(P)$ is strongly $\mathcal{F}$-closed, which is not possible since P is non-abelian and $\mathcal{F}$ is reduction simple. Thus, $O_p(\widetilde{G}) = 1$.

In particular, $Z(\widetilde{G})$ is a p'-group. By maximality, we must have $Z(\widetilde{G}) \subseteq M$. Since $P \not\subseteq M$ and $P \cap M$ is strongly $\mathcal{F}$-closed, reduction simplicity implies that $P \cap M = 1$.

However, $P \cap M$ is a defect group of a block of kM, which is covered by $\widetilde{b}$ by Lemma 3.8. This block is furthermore $\widetilde{G}$-stable by our assumptions and Corollary 3.3. In particular, there is a central extension $1 \to Z \to G \xrightarrow{\pi} \widetilde{G}/M \to 1$ for some central p'-group Z and G has an $\mathcal{F}$-block b by Theorem 3.4 (with the roles of G and $\widetilde{G}$ interchanged).

We check that G satisfies the claims (a), (b) and (c). To prove claim (a), note that $|G : Z(G)| \leq |G : Z| = |\widetilde{G} : M| \leq |\widetilde{G} : Z(\widetilde{G})|$. Now part (a) follows by our assumption. In particular, by Corollary 3.3, b is inertial.

Let $\varepsilon : \widetilde{G} \twoheadrightarrow \widetilde{G}/M$ be the canonical surjection. The maps ε and π induce a bijection between the set of subgroups of G containing Z and the set of subgroups of $\widetilde{G}$ containing M, which preserves normality, by sending a subgroup $H \leq G$ to $\varepsilon^{-1}(\pi(H)) \leq \widetilde{G}$. Suppose $H \trianglelefteq G$ with $P \nsubseteq H$. We show $H \subseteq Z(G)$. We may assume $Z \subsetneq H$. Now $P \subseteq H$ if and only if $\varepsilon^{-1}(\pi(P)) \subseteq \varepsilon^{-1}(\pi(H))$. Since there is no normal subgroup of $\widetilde{G}$ properly containing M and not containing P, it follows that there is no normal subgroup of G properly containing Z and not containing P, which proves the first part of (b). In particular, $Z(G)$ is a p'-group since it clearly does not contain P.

Note that we have $O_p(G) = 1$ for any G having an $\mathcal{F}$-block. Note that $F(G) = \prod_{q \in \mathbb{P}} \prod_{Q \in \mathrm{Syl}_q(F(G))} Q$. In particular, $\mathrm{Syl}_p(F(G)) = 1$ since these subgroups are characteristic in $F(G)$ and otherwise $O_p(G) \neq 1$. Thus, by the above, $F(G) \subseteq Z(G)$, so in fact $F(G) = Z(G)$.

Now let c be the block of $kE(G)$ which is covered by b. If $E(G) \cap P = 1$, $E(G) \subseteq Z(G)$ again by the above. But then $E(G) = 1$. Since $F(G)$ is central, by Bender's Theorem, see [23, Theorem 4.8], this means $C_G(Z(G)) = G \leq Z(G)$, so G is abelian. Thus, $E(G) \cap P \neq 1$ and $P \subseteq E(G) = L_1 \cdots L_t$, where $\{L_1, \ldots, L_t\}$ are the components of G. We have $E(G) \cong (L_1 \times \cdots \times L_t)/K$ for $K \subseteq Z(L_1 \times \cdots \times L_t) = Z(L_1) \times \cdots \times Z(L_t)$. We claim that K is a p'-group. It suffices to prove $O_p(L_i) = 1$ for each $1 \leq i \leq t$. Indeed, if we assume the contrary, then the group $O_p(L_1) \cdots O_p(L_t)$ is a non-trivial normal subgroup of $E(G)$. In particular, $O_p(E(G)) \neq 1$. However, this is a characteristic subgroup of the layer, which implies $O_p(G) \neq 1$, a contradiction.

Thus, there is a fusion system preserving bijection between the blocks of $E(G)$ and the blocks of $L_1 \times \cdots \times L_t$ which have K in their kernel. In particular, we may assume $P = P_1 \times \cdots \times P_t$ and $c = c_1 \times \cdots \times c_t$, where for $1 \leq i \leq t$, P_i is a defect group of the block c_i, which is a block of kL_i covered by c. If r is the rank of $Z(P)$, then at most r of the blocks c_i can have non-trivial defect. Let $s \leq r$ be such that s of the blocks c_i have non-trivial defect. After possibly reordering, we may assume these are $c_1, \cdots, c_s$. We claim $L_{s+1} \cdots L_t \trianglelefteq G$. Indeed, the conjugation action of G on $E(G)$ induces a group homomorphism $\sigma : G \to \mathrm{Sym}(\{L_1, \ldots, L_t\}) \cong \mathfrak{S}_t$ as follows: $\sigma(x)(i) := j$ iff ${}^xL_i = L_j$. Assume there is an $x \in G$ such that ${}^xL_i = L_j$ for $i \leq s$, $j > s$. Since b is inertial, c is G-stable. This means ${}^xc = c$, so ${}^xc_1 \times \cdots \times {}^xc_t = c_1 \times \cdots \times c_t$, but this implies ${}^xP_i$ is non-trivial. This contradiction implies normality. Now we can apply part (b) to deduce $L_{s+1} \cdots L_t = 1$, which implies claim (c). □

We can further restrict the structure of reduction simple fusion systems by specialising to the case of $Z(P)$ being cyclic:

Theorem 3.10. *Let P be a non-abelian p-group such that $Z(P)$ is cyclic and let $\mathcal{F}$ be a reduction simple fusion system on P. If $\mathcal{F}$ is block-realisable, then there exists a fusion system $\mathcal{F}_0$ on P and a quasisimple group with an $\mathcal{F}_0$-block, where $\mathcal{O}_p(\mathcal{F}_0) = 1$.*

Proof. Assume G is a finite group having an $\mathcal{F}$-block b with defect group P. We may choose G such that the conclusions of Theorem 1.2 hold. Let $L = \langle {}^gP \mid g \in G \rangle$. Thus, since $P \subseteq E(G)$ as in the proof of Theorem 1.2, we have $L \trianglelefteq E(G)$. By Theorem 1.2, the number of components of G is bounded by the rank of $Z(P)$. By cyclicity of that group, $E(G)$ is quasisimple. Furthermore, L is non-central, so we must have $L = E(G)$ is quasisimple.

Let d be the block of kL which is covered by b. Define K to be the kernel of the map $G \to \mathrm{Out}(kLd)$, which is induced by $G \to \mathrm{Aut}(kLd), g \mapsto c_g$, and assume K has a block c which is covered by b. We get the triangle relations $\mathcal{F} \subseteq \mathcal{F}_{(P,e'_P)}(G,K,c)$ and $\mathcal{F}_{(P,e'_P)}(K,c) \trianglelefteq \mathcal{F}_{(P,e'_P)}(G,K,c) =: \widetilde{\mathcal{F}}$ as in the proof of Theorem 1.4. In the same fashion as in the proof of this theorem, application of [34, Theorem 7] and [28, Proposition 2.12] also implies $\mathcal{F}_{(P,e'_P)}(K,c) \cong \mathcal{F}_{(P,f_P)}(L,d) =: \mathcal{F}_0$. We have $\mathcal{O}_p(\mathcal{F}) = 1$ by reduction simplicity. Thus, Lemma 2.13 implies $\mathcal{O}_p(\widetilde{\mathcal{F}}) = 1$ and Lemma 2.14 implies $\mathcal{O}_p(\mathcal{F}_0) = 1$. $\square$

3.3 Reduction Theorems for generalised block fusion systems

In this chapter, we generalise key results for block fusion systems to generalised block fusion systems. We start with Brauer's Third Main Theorem.

Proof of Theorem 1.6. Let d be the principal block of kH. Then $db \neq 0$ and b is H-stable. By Brauer's Third Main Theorem (see Theorem 2.37), $\mathrm{Br}_Q^H(d)$ is the principal block of $kC_H(Q)$ for any $Q \leq_p H$. In particular, there is a maximal d-Brauer pair (P, e_P), where $P \in \mathrm{Syl}_p(H)$ and e_P is the principal block of $kC_H(P)$ and $\mathcal{F}_{(P,e_P)}(H,d) = \mathcal{F}_P(H)$. By Theorem 2.53, there exists a maximal (b,H)-Brauer pair (S, e'_S) such that $\mathcal{F}_P(H) = \mathcal{F}_{(P,e_P)}(H,d) \subseteq \mathcal{F}_{(S,e'_S)}(H,G,b) \subseteq \mathcal{F}_S(H)$. Since $P \in \mathrm{Syl}_p(H)$, it follows that $S = P$ and that $\mathcal{F}_{(S,e'_S)}(H,G,b) = \mathcal{F}_P(H)$.
By the above, $S = P$. For any $Q \leq P$, let e_Q be the unique block of $kC_H(Q)$ such that $(Q, e_Q) \leq (P, e_P)$ as d-Brauer pairs and let e'_Q be the unique block of $kC_G(Q)$ such that $(Q, e'_Q) \leq (P, e'_P)$ as (b,H)-Brauer pairs. By Lemma 2.33, $e_Q e'_Q \neq 0$. In other words, the block e_Q of $kC_H(Q)$ covers the block e'_Q of $kC_G(Q)$. But e_Q is the principal block of $kC_H(Q)$. Since a principal block covers only principal blocks, e'_Q is the principal block of $kC_G(Q)$. $\square$

Note that if $G = H$, Theorem 1.6 becomes the original Brauer's Third Main Theorem (see Theorem 2.37).

Next, we generalise both Fong reductions, starting with the first one. We need some background before proving it. We follow the approach of [37, Chapter 8.7] and directly quote some of the background needed. Fix $(K, \mathcal{O}, k)$ to be a p-modular system.

Definition 3.11. *Let G be a finite group.*

(a) Let A be an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. A point of A is an $A^\times$-conjugacy class α of primitive idempotents in A.

(b) Let A be a G-algebra over $\mathcal{O}$ and let $P \leq_p G$. We set

$$A(P) = A^P/(\sum_{Q<P} A_Q^P + J(\mathcal{O})A^P)$$

and call the canonical map $\mathrm{Br}_P^A : A^P \to A(P)$ Brauer homomorphism.

(c) Let A be a G-algebra over $\mathcal{O}$ and let $P \leq_p G$. A local point of P on A is a point γ of P on A with $\mathrm{Br}_P^A(\gamma) \neq 0$.

Let G, H be finite groups with $G \trianglelefteq H$ and b be an H-stable block of kG with $P \leq_p H$ being maximal such that $\mathrm{Br}_P^G(b) \neq 0$. A source idempotent is a primitive idempotent $i \in (kGb)^P$ satisfying $\mathrm{Br}_P^G(i) \neq 0$ such that for any $Q \leq P$, there is a unique block e_Q with $\mathrm{Br}_Q^G(i)e_Q \neq 0$. The interior P-algebra $A = ikHi$ is called a source algebra of the block b. The pair (P, e_P) is thus a maximal (b, H)-Brauer pair. In particular, any source idempotent i determines a generalised block fusion system $\mathcal{F} = \mathcal{F}_{(P,e_P)}(H, G, b)$ on P and we call $\mathcal{F}$ the fusion system of the source algebra A or the fusion system on P determined by the source idempotent i.

Proposition 3.12. *Let $G \trianglelefteq H$, b an H-stable block of $kG, (P, e_P)$ a maximal (b, H)-Brauer pair, i a source idempotent of $(kGb)^P$ such that $\mathrm{Br}_P^G(i)e_P = \mathrm{Br}_P^G(i)$. Then $\mathcal{F}_{(P,e_P)}(H, G, b)$ is generated by the set of inclusions between subgroups of P and those automorphisms φ of subgroups Q of P such that $\varphi|_{kQ}$ is isomorphic to a direct summand of $ikHi$ as $k[Q \times Q]$-module.*

We need two auxiliary results before proving this proposition.

Proposition 3.13. *Let $G \trianglelefteq H$, b an H-stable block of kG, (P, e_P) a maximal (b, H)-Brauer pair, i a source idempotent of $(kGb)^P$ and $A = ikHi$. Let $\mathcal{F}$ be the fusion system of A on P and $Q \leq P$ fully $\mathcal{F}$-centralised with e_Q being the unique idempotent such that $(Q, e_Q) \leq (P, e_P)$. For any local point δ of Q on $\mathcal{O}Hb$ satisfying $\mathrm{Br}_Q^G(\delta)e_Q \neq \{0\}$, we have $\delta \cap A \neq \emptyset$.*

Proof. By definition, if δ is a local point of Q on $\mathcal{O}Hb$ satisfying $\mathrm{Br}_Q^G(\delta)e_Q \neq \{0\}$, then $\mathrm{Br}_Q^G(\delta)$ is a conjugacy class of primitive idempotents in $kC_H(Q)e_Q$. Since $kC_H(Q)e_Q$ is Morita equivalent to $A(Q) = \mathrm{Br}_Q^G(i)kC_H(Q)\,\mathrm{Br}_Q^G(i)$, see [37, Theorem 6.4.6], it follows that there is $j \in \delta$ such that $\mathrm{Br}_Q^G(j) \in \mathrm{Br}_Q^G(i)kC_H(Q)\,\mathrm{Br}_Q^G(i)$. The lifting theorems for idempotents imply that j can be chosen in $A^Q = i(kH)^Q i$. $\square$

Proposition 3.14. *Keep the assumptions of Proposition 3.12, let $A = ikHi$, $\mathcal{F}$ be the fusion system of A on P and $Q, R \leq P$.*
(i) Every indecomposable direct summand of A as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule is isomorphic to $\mathcal{O}Q \otimes_\varphi \mathcal{O}R$ for some $S \leq Q$ and $\varphi \in \mathrm{Hom}_\mathcal{F}(S, R)$;
(ii) If $\varphi \in \mathrm{Hom}_\mathcal{F}(S, R)$, R fully $\mathcal{F}$-centralised, then ${}_\varphi R$ is isomorphic to a direct summand of A as $\mathcal{O}Q$-$\mathcal{O}R$-bimodule.

Proof. (i) Let Y be an indecomposable direct summand of A as an $\mathcal{O}Q$-$\mathcal{O}R$-bimodule. Then Y has a $Q \times R$-stable $\mathcal{O}$-basis on which Q and R act freely on the left and on the right, respectively. So $Y \cong \mathcal{O}Q \otimes_{\mathcal{O}S} {}_\varphi\mathcal{O}R$ for some $S \leq Q$ and some injective group homomorphism $\varphi : S \to R$. Let $T = \varphi(S)$. Restricting Y to $S \times T$ shows that ${}_\varphi\mathcal{O}T$ is isomorphic to a direct summand of Y, hence of A, as an $\mathcal{O}S$-$\mathcal{O}T$-bimodule. Now A is a direct summand of $\mathcal{O}H$ as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule. In particular, ${}_\varphi\mathcal{O}T$ is isomorphic to a direct summand of $\mathcal{O}H$ as an $\mathcal{O}S$-$\mathcal{O}T$-bimodule, hence isomorphic to $\mathcal{O}Sy^{-1} = y^{-1}\mathcal{O}T$ for some $y \in H$ with ${}^yS = T$ and ${}^ys = \varphi(s)$ for all $s \in S$. Then $\mathcal{O}S$ is isomorphic to a direct summand of $i\mathcal{O}Hiy = i\mathcal{O}Hy^{-1}iy$ as an $\mathcal{O}S$-$\mathcal{O}S$-bimodule. Thus

$\mathrm{Br}_S^G(i\mathcal{O}Hy^{-1}iy) \neq 0$ by [37, Lemma 5.8.8]. Since $\mathrm{Br}_S^G(i) \in kC_H(S)e_S$ this also forces $\mathrm{Br}_S^G(y^{-1}iy)e_S \neq 0$. Conjugating by y yields $\mathrm{Br}_T^G(i)^y e_S \neq 0$. But this means ${}^y e_S = e_T$ because e_T is the unique block of $kC_H(T)$ with the property $\mathrm{Br}_T^G(i)e_T \neq 0$. This shows that $\varphi \in \mathrm{Mor}(\mathcal{F})$, whence (i).

(ii) By definition, there is $x \in H$ such that $\varphi(u) = {}^x u$ for all $u \in Q$ and ${}^x e_Q = e_R$. Let μ be a local point of Q on $\mathcal{O}Hb$ such that $\mu \cap A \neq \emptyset$. Set $\nu = {}^x\mu$, i.e. ${}^x(Q_\delta) = R_\nu$. $\mu \cap A \neq \emptyset$ implies $\mathrm{Br}_Q^G(\mu)e_Q \neq 0$. Conjugating by x gives $\mathrm{Br}_R^G(\mu)e_R \neq 0$. Since R is fully $\mathcal{F}$-centralised, we get from the previous proposition that $\nu \cap A \neq 0$. Let $m \in \mu \cap A$ and $n \in \nu \cap A$. Then ${}^x m$ and n both belong to ν, so they are conjugate in $(A^R)^\times$. Since $\mathrm{Br}_Q^G(m) \neq 0$, we get $(m\mathcal{O}Hm)(Q) \neq \{0\}$. This means $m\mathcal{O}Hm$ has a direct summand isomorphic to $\mathcal{O}Q$ as an $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule. This means $m\mathcal{O}Hmx^{-1} = m\mathcal{O}Hxmx^{-1} \cong m\mathcal{O}Hm = mAn$ has a direct summand isomorphic to $\mathcal{O}Q_{\varphi^{-1}} \cong_\varphi \mathcal{O}R$ as an $\mathcal{O}Q$-$\mathcal{O}R$-bimodule. □

Proof of Proposition 3.12. The result follows from the previous proposition together with Alperin's Fusion Theorem, see Theorem 2.6. □

Proposition 3.15. *Let $N \trianglelefteq G \trianglelefteq H$ with $N \trianglelefteq H$, c be a block of kN covered by $b \in kG$ and let $\tilde{b}$ be the Fong correspondent of b in $I_G(b)$. There exists a subgroup P of $I_H(c)$ and $i \in (kI_G(c)\tilde{b})^P$ such that i is a source idempotent of the $I_H(c)$-algebra $kI_G(c)\tilde{b}$ and of the H-algebra kGb.*

Proof. By Frattini argument, $H = GI_H(c)$ ($G \trianglelefteq H$ and G transitively permutes all H-conjugates of c).

In particular, $H/I_H(c) = GI_H(c)/I_H(c) = G/(G \cap I_H(c)) = G/I_G(c)$. Thus, a system of left coset representatives of $I_G(c)$ in G is also a system of left coset representatives of $I_H(c)$ in H. Since $\tilde{b}$ is $I_H(c)$-stable, we obtain $b = \mathrm{Tr}_{I_G(c)}^G \tilde{b} = \mathrm{Tr}_{I_H(c)}^H \tilde{b}$. Let $x \in H \backslash I_H(c)$, then $\tilde{b}{}^x\tilde{b} = \tilde{b}c{}^x c{}^x\tilde{b} = 0$.

We are viewing kGb as H-algebra and $kI_G(c)\tilde{b}$ as $I_H(c)$-algebra. Let $P \leq H$ be maximal s.t. $\mathrm{Br}_P^G(b) \neq 0$ and $j \in (kGb)^P$ primitive idempotent s.t. $\mathrm{Br}_P^G(j) \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of $(kGb)^P$ not containing j and let $\pi : (kGb)^P \to (kGb)^P/\mathfrak{m}$ be the canonical surjection. Since $\mathrm{Br}_P^G(j) \neq 0$, we have a factorisation, see [49, Lemma 14.4],

$$\begin{tikzcd} (kGb)^P \arrow[r, "\pi"] \arrow[dr, "\mathrm{Br}_P^G"'] & (kGb)^P/\mathfrak{m} \\ & \mathrm{Br}_P^G((kGb)^P) \arrow[u, "\pi_0"'] \end{tikzcd}$$

We have: $0 \neq \pi(j) = \pi(jb) = \pi(j)\pi(b)$. Hence, $\pi(b) \neq 0$. Further $\pi(b) = \pi_0 \circ \mathrm{Br}_P^G(\mathrm{Tr}_{I_H(c)}^H \tilde{b}) = \pi_0(\mathrm{Br}_P^G(\mathrm{Tr}_{I_H(c)}^H \tilde{b}))$. By the Mackey formula, see [37, Proposition 2.5.5], the latter expression is equal to $\pi_0(\mathrm{Br}_P^G(\sum_{x \in P\backslash H/I_H(c)} \mathrm{Tr}_{P\cap^x I_H(c)}^P({}^x\tilde{b})))$, which means that $\pi(b) = \pi_0(\sum_{x\in P\backslash H/I_H(c)} \mathrm{Br}_P^G(\mathrm{Tr}_{P\cap^x I_H(c)}^P({}^x\tilde{b})))$.

However, $\mathrm{Br}_P^G(\mathrm{Tr}_R^P a) = 0$ for any proper subgroup R of P and $a \in A^R$. Thus, the above shows that there is $x \in H$ s.t. $P \cap^x I_H(c) = P$ and $0 \neq \pi_0(\mathrm{Br}_P^G({}^x\tilde{b})) = \pi({}^x\tilde{b})$.

The equation $0 \neq \pi({}^x\tilde{b})$ implies that there is a primitive idempotent i of $(kGb)^P$ s.t. i is $(kGb)^P$-conjugate to j and s.t. $i{}^x\tilde{b}i = i$, i.e. $i \in^x \tilde{b}kG^x\tilde{b}$. In particular, $\pi(i) \neq 0$, hence $\mathrm{Br}_P^G(i) \neq 0$. Setting $i' =^{x^{-1}} i, P' =^{x^{-1}} P$ we get: i' is a primitive idempotent of $(kGb)^{P'}$ with $\mathrm{Br}_{P'}^{I_G(c)}(i') \neq 0$ and $i' \in \tilde{b}kG\tilde{b} = kI_G(c)\tilde{b}$.

Thus, we have shown: If j is a primitive idempotent of $(kGb)^P$ s.t. $\mathrm{Br}_P^G(j) \neq 0$, then there exists $x \in H$ s.t. ${}^{x^{-1}}P \subseteq I_H(c)$ and a $(kGb)^P$-conjugate i of j s.t. ${}^{x^{-1}}i$ is a primitive idempotent of $kI_G(c)\tilde{b}$ with $\mathrm{Br}_{{}^{x^{-1}}P}^{I_G(c)}({}^{x^{-1}}i) \neq 0$.

Replacing (P, i) with $({}^{x^{-1}}P, {}^{x^{-1}}i)$ we obtain: There exists a p-subgroup P of $I_H(c)$ and a primitive idempotent i of $kI_H(c)\tilde{b}$ s.t. $\mathrm{Br}_P^G(i) \neq 0$, i is primitive in kGb and P is maximal among p-subgroups of H s.t. $\mathrm{Br}_P^{I_G(c)}(b) \neq 0$.

Conversely, suppose that $P \leq I_H(c)$ is maximal s.t. $\mathrm{Br}_P^{I_G(c)}(\tilde{b}) \neq 0$. By maximality, $\tilde{b} = \mathrm{Tr}_P^{I_H(c)}(a)$ for some $a \in (kG\tilde{b})^P$. Thus, $b = \mathrm{Tr}_{I_H(c)}^H \tilde{b} = \mathrm{Tr}_P^H(a)$. This shows that P is contained in a maximal p-subgroup Q of H s.t. $\mathrm{Br}_Q^G(b) \neq 0$.

Combining, we obtain: There exists a p-subgroup P of $I_H(c)$ and a primitive idempotent i of $(kI_G(c)\tilde{b})^P$ s.t. P is maximal among subgroups Q of H s.t. $\mathrm{Br}_Q^G(b) \neq 0$, P is maximal among subgroups Q of $I_H(c)$ s.t. $\mathrm{Br}_Q^{I_G(c)}(\tilde{b}) \neq 0$, i is a primitive idempotent of $(kGb)^P$ and $\mathrm{Br}_P^G(i) \neq 0$. $\square$

Proposition 3.16. *Keep the notation from the previous proposition. Then the algebras $ikHi$ and $ikI_H(c)i$ are isomorphic as interior P-algebras.*

Proof. Let P, i be as above, then we claim that also have $ikHi = ikI_H(c)i$. Indeed, clearly $ikI_H(c)i \subseteq ikHi$.
Now suppose $x \in H \setminus I_H(c)$. Then $ixi = i\tilde{b}x\tilde{b}i = i\tilde{b}^x\tilde{b}xi = 0$. This shows that $ikHi \subseteq ikI_H(c)i$. $\square$

Proof of Theorem 1.7. Since $I_G(c) = I_H(c) \cap G$, and G is normal in H, $I_G(c)$ is normal in $I_H(c)$.
Next, we claim that $\tilde{b}$ is $I_H(c)$-stable. Let $x \in I_H(c)$, then

$$b =^x b =^x \mathrm{Tr}_{I_G(c)}^G(\tilde{b}) = \mathrm{Tr}_{^xI_G(c)}^{^xG}(^x\tilde{b}) = \mathrm{Tr}_{I_G(c)}^G(^x\tilde{b}),$$

where the last equality follows from the normality of G in H and $I_G(c)$ in $I_H(c)$. This equation, together with the uniqueness of the Fong correspondent, implies stability. Note that $^x\tilde{b}$ is indeed a block of $kI_G(c)$ covering c since $^x\tilde{b}c{=}^x\tilde{b}^xc{=}^x(\tilde{b}c) \neq 0$.
We apply Propositions 3.15 and 3.16 to obtain a source idempotent i of b and $\tilde{b}$ respectively with $ikHi = ikI_H(c)i$. This observation, together with Proposition 3.12, implies the theorem. $\square$

Theorem 1.7 is a generalisation of the First Fong Reduction, which we obtain from it if G and H coincide. We give an example proving that the assumption $N \trianglelefteq H$ in the theorem is necessary.

Example 3.17. Let b and c be principal blocks, then, by Theorem 1.6, the statement becomes $\mathcal{F}_S(H) = \mathcal{F}_S(I_H(c))$. In particular, since c is principal, we also have $I_H(c) =$

$N_H(N)$, thus $\mathcal{F}_S(H) = \mathcal{F}_S(N_H(N))$ whenever $N \trianglelefteq G \trianglelefteq H$.

Now let $p = 3$, $N = (C_3 \times C_3) \rtimes C_2$, where C_2 acts as a reflection, then $S = C_3 \times C_3 \in \mathrm{Syl}_p(N)$. Let $H = S \rtimes D_8$, $G = S \rtimes (C_2 \times C_2)$ then $N_H(N) = G$, but clearly $\mathcal{F}_S(G) \subsetneq \mathcal{F}_S(H)$.

Finally, we also generalise the Second Fong Reduction to generalised block fusion systems.

Proof of Theorem 1.8. First, we apply the second Fong reduction to $A \trianglelefteq H$. We use the notation from [28, Proof of Theorem 3.1] and are recalling some key steps. Let S be a p-subgroup of H containing P, maximal such that $\mathrm{Br}_S^A(c) \neq 0$. For each $h \in H$ there is an element $i_h \in (kAc)^\times$ such that $c_{i_h} = c_h$ on $(kAc)^\times$. We can choose the elements i_h such that $i_{ha} = i_h ac$ for $h \in H, a \in A$ and such that

$$S \to (kAc)^\times, s \mapsto i_s$$

is a homomorphism. Now define a 2-cocyle α on H via $i_g i_h = \alpha(g,h) i_{gh}$ for $g, h \in H$. Denote by $k_{\overline{\alpha}^{-1}} H/A$ the twisted group algebra corresponding to $\overline{\alpha}^{-1}$, i.e. the free module on $\{\widehat{\overline{h}} | \overline{h} \in H/A\}$ with multiplication given by $\widehat{\overline{h}}\widehat{\overline{g}} = \alpha^{-1}(h,g)\widehat{\overline{hg}}$. Define a function $\phi : kAc \otimes k_{\overline{\alpha}^{-1}} \widehat{H/A} \to kHc$ via the k-linear extension of the map $x \otimes \widehat{\overline{h}} \mapsto x i_h^{-1} h$. This gives a central p'-extension

$$1 \to Z \to \widetilde{H} \to H/A \to 1$$

and by our choice of the elements i_h also a p'-extension

$$1 \to Z \to \widetilde{M} \to M/A \to 1,$$

where $\widetilde{M}$ is the full inverse image of M/A in $\widetilde{H}$. Furthermore, we get an idempotent

e of kZ and an algebra isomorphism $\tau : k\widetilde{H}e \to k_{\overline{\alpha}^{-1}}\widehat{H/A}$ with $\tau(\tilde{h}e) = \alpha_{\tilde{h}}\widehat{\widehat{h}}$ for some $\alpha_{\tilde{h}} \in k^{\times}$, see [49, Proposition 10.8]. Denote by τ_M the restriction to $k\widetilde{M}e$. If we define α_M to be the restriction of α to $M \times M$, then τ_M becomes an algebra isomorphism from $k\widetilde{M}e$ to $k_{\overline{\alpha_M}^{-1}}\widehat{M/A}$.

For each $s \in S$, let $\tilde{s}$ denote the unique lift of s in $\widetilde{H}$ which is also a p-element and for $Q \leq S$ define $\widetilde{Q} = \{\tilde{q} \mid q \in Q\}$. Note that the groups $S/A, \widetilde{S}$ and S are isomorphic and we identify them henceforth. In particular, $\widetilde{P} \cong P$. If we consider $kAc \otimes k\widetilde{H}e$ as interior S-algebras via $s \mapsto i_s \otimes se$, we obtain an S-algebra isomorphism

$$\psi : kAc \otimes k\widetilde{H}e \to kHc, x \otimes y \mapsto \phi(x \otimes \tau(y)).$$

Again, we denote the restriction to $kAc \otimes k\widetilde{M}e$, which is also an S-algebra isomorphism to kMc, by ψ_M. Since kAc and $k\widetilde{H}e$ are p-permutation algebras, see [49, Proposition 28.3], ψ induces algebra isomorphisms $\psi_Q : kAc(Q) \otimes k\widetilde{H}e(Q) \to kHc(Q)$ and $\psi_{Q,M}$: $kAc(Q) \otimes k\widetilde{M}e(Q) \to kMc(Q)$ satisfying $\psi_{Q,M}(\mathrm{Br}_Q^M(x) \otimes \mathrm{Br}_Q^M(y)) = \mathrm{Br}_Q^M(\psi_M(x \otimes y))$. Since kAc is a matrix algebra, we get bijections between the blocks of H covering c and the blocks of $\widetilde{H}$ covering e as well as between the blocks of M covering c and the blocks of $\widetilde{M}$ covering e. For the same reason, $\psi_{Q,M}$ induces a bijection between the blocks of $kMc(Q)$ and $k\widetilde{M}e(Q)$ via $f \mapsto \psi_{Q,M}(1 \otimes f)$. Thus, $(Q, f) \mapsto (Q, \psi_{Q,M}(1 \otimes f))$ provides a bijection between the set of Brauer pairs associated to blocks of $\widetilde{M}$ covering e and the set of Brauer pairs associated to blocks of M covering c with first components respectively contained in S.

Let $\tilde{d}$ be the block of $k\widetilde{M}$ corresponding to d under ψ_M, i.e. $d = \psi_M(c \otimes \tilde{d})$. Note that d is H-stable if and only if $d \in Z(kHc)$. Since ψ_M is an k-algebra isomorphism, this is if and only if $c \otimes \tilde{d} \in Z(k\widetilde{H}\tilde{c} \otimes kAc)$. The latter expression is equal to $Z(k\widetilde{H}\tilde{c}) \otimes k$. Since $d = \psi_M(c \otimes \tilde{d})$, we get that d is H-stable if and only if $\tilde{d}$ is $\widetilde{H}$-stable.

Define $\widetilde{e_P}$ by $\psi_{M,P}(c \otimes \widetilde{e_P}) = e_P$. Note that by the description above, $\psi_{M,P}$ is an inclusion-preserving map of Brauer pairs. Since $P \leq M$ we thus get that $(P, \widetilde{e_P})$ is a maximal $(\tilde{d}, \widetilde{H})$-Brauer pair. We can thus define the category $\mathcal{F}_{(P,\widetilde{e_P})}(\widetilde{H}, \widetilde{M}, \tilde{d})$. Now we define a map from $\mathcal{F}_{(P,e_P)}(H, M, d)$ to $\mathcal{F}_{(P,\widetilde{e_P})}(\widetilde{H}, \widetilde{M}, \tilde{d})$ by $(Q, e_Q) \mapsto (Q, \psi_{M,Q}(1 \otimes e_Q))$. The objects of the generalised block fusion systems $\mathcal{F}_{(P,e_P)}(H, M, d)$ and $\mathcal{F}_{(P,\widetilde{e_P})}(\widetilde{H}, \widetilde{M}, \tilde{d})$ are the same by the above. For the morphisms we want to prove that $N_H(Q, e_Q)$ and $N_{\widetilde{H}}(Q, \psi_{M,Q}(1 \otimes e_Q))$ have the same image in $\mathrm{Aut}(Q)$. For this, we need to prove that $\psi_{M,Q}$ is H-equivariant. Note that we can apply [28, 3.4] to get that for any $h \in H$ with lift $\tilde{h}$ of hA in $N_{\widetilde{H}}(Q)$, there exists some $a \in A$ such that $n := ah \in N_H(Q)$ and $\widetilde{{}^n x} =^{\tilde{h}} \tilde{x}$ for all $x \in Q$. Let $\widetilde{e_Q} := \psi_{M,Q}(1 \otimes e_Q)$ and $\widetilde{f_Q} \in (k\widetilde{M}e)^Q$ such that $\mathrm{Br}_Q^M(\widetilde{f_Q}) = \widetilde{e_Q}$. If we define $f_Q := \psi_M(1 \otimes \widetilde{f_Q})$, then $\mathrm{Br}_Q^M(f_Q) = e_Q$ by the observations about ψ_M above. Let $h \in N_H(Q)$ with $\tilde{h} \in \widetilde{H}$ a lift of $\overline{h}$. If we define $t := \psi^{-1}(hc)$, then $t = i_h \otimes \alpha\tilde{h}e$ for some $\alpha \in k^\times$. We get:

$$
\begin{aligned}
{}^h e_Q =^h \mathrm{Br}_Q^M(f_Q) = \mathrm{Br}_Q^M({}^h f_Q) = \mathrm{Br}_Q^M(\psi({}^t(1 \otimes \widetilde{f_Q}))) = \mathrm{Br}_Q^M(\psi(1 \otimes^{\tilde{h}} \widetilde{f_Q})) \\
= \mathrm{Br}_Q^M(\psi_M(1 \otimes^{\tilde{h}} \widetilde{f_Q})) = \psi_{M,Q}(1 \otimes \mathrm{Br}_Q^M({}^{\tilde{h}}\widetilde{f_Q})) = \psi_{M,Q}(1 \otimes^{\tilde{h}} \mathrm{Br}_Q^M(\widetilde{f_Q})) = \psi_{M,Q}(1 \otimes^{\tilde{h}} \widetilde{e_Q})
\end{aligned}
$$

In particular, h stabilises e_Q if and only if $\tilde{h}$ stabilises $\widetilde{e_Q}$. Theorem 2.6 implies our claim. $\square$

If $H = M$, we obtain the original second Fong reduction. Note that the First and Second Fong Reductions for generalised block fusion systems might be used to generalise Theorems 1.2 and 1.4 from block fusion systems to generalised block fusion systems.

4 Parker–Semeraro systems

In this section, we provide further evidence for Conjecture 1.1. The results in this chapter have already been published in a paper of mine, see [48]. The main result of this section is Theorem 1.3. The fusion systems on a Sylow p-subgroup of $G_2(p)$ for odd p and $\mathcal{O}_p(\mathcal{F}) = 1$ have been classified by Parker and Semeraro in [42] and thus we refer to them as Parker–Semeraro systems. For $p \neq 7$, all Parker–Semeraro systems are realised by finite groups, whereas for $p = 7$, there are 29 Parker–Semeraro systems, of which 27 are exotic. In [51], the classification has been extended to any fusion system on a Sylow p-subgroup of $G_2(p^n)$ and $\mathrm{PSU}_4(p^n)$ for any prime p and $n \in \mathbb{N}$. It remains that the 27 systems found by Parker and Semeraro are the only exotic systems on such groups. In this section, we prove block-exoticity of these. We give an overview of these systems in the Table 1. The groups W, R and Q mentioned in this table are essential subgroups, see [42, 5] for details and explanation of the different cases. Fix M to be the Monster group for the rest of this section.

The relevance of the systems on groups as described above stems from the fact that one wants to classify all fusion systems over maximal unipotent subgroups of finite groups of Lie type of small rank. Note that the Solomon systems belong to this class of fusion systems as well. Furthermore, another important factor is that 7 is a good prime and thus, many results for groups of Lie type are applicable.

Also note that neither of the Reduction Theorems 3.5 and 3.6 can be applied to the exotic Parker–Semeraro system $\mathcal{F}_7^1(6)$ on a Sylow 7-subgroup S of $G_2(7)$: Since subsystems of $\mathcal{F}_7^1(6)$ are in correspondence to the subgroups of C_6, it is not possible to fit each subsystem into a pair fulfilling the assumptions of Theorem 3.6. Since $\mathrm{Aut}(S)$ is not a 7-group, Theorem 3.5 is also not applicable.

In what follows, we use the notation of [42], where these systems have been classified. We start by proving that it is not possible for most finite quasisimple groups to have a block with $S \in \mathrm{Syl}_7(G_2(7))$ as a defect group.

$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(W)$	$\mathrm{Out}_{\mathcal{F}}(R)$	$\mathrm{Out}_{\mathcal{F}}(Q)$	$\mathrm{Out}_{\mathcal{F}}(S)$	$\Gamma_{p'}(\mathcal{F})$	Group
$\mathcal{F}_7^0$	-	$\mathrm{GL}_2(7)$	$3 \times 2\mathfrak{S}_7$	6×6	1	-
$\mathcal{F}_7^1(1_1)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(2_1)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(2_2)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(2_3)$	$\mathrm{SL}_2(7)$	-	-	6×2	2	-
$\mathcal{F}_7^1(3_1)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(3_2)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(3_3)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(3_4)$	$\mathrm{SL}_2(7)$	-	-	6×3	3	-
$\mathcal{F}_7^1(4_1)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(4_2)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(4_3)$	$\mathrm{SL}_2(7)$	-	-	6×2	2	-
$\mathcal{F}_7^1(5)$	$\mathrm{SL}_2(7)$	-	-	6	1	-
$\mathcal{F}_7^1(6)$	$\mathrm{SL}_2(7)$	-	-	6×6	6	-
$\mathcal{F}_7^2(1)$	$\mathrm{SL}_2(7)$	$\mathrm{SL}_2(7).2$	-	6×2	1	-
$\mathcal{F}_7^2(2)$	$\mathrm{SL}_2(7)$	$\mathrm{SL}_2(7).2$	-	6×2	1	-
$\mathcal{F}_7^2(3)$	$\mathrm{SL}_2(7)$	$\mathrm{GL}_2(7)$	-	6×6	3	-
$\mathcal{F}_7^3$	$\mathrm{SL}_2(7)$	-	$\mathrm{GL}_2(7)$	6×6	1	-
$\mathcal{F}_7^4$	$\mathrm{SL}_2(7)$	-	$3 \times 2\mathfrak{S}_7$	6×6	1	-
$\mathcal{F}_7^5$	$\mathrm{SL}_2(7)$	$\mathrm{GL}_2(7)$	$\mathrm{GL}_2(7)$	6×6	1	-
$\mathcal{F}_7^6$	$\mathrm{SL}_2(7)$	$\mathrm{GL}_2(7)$	$3 \times 2\mathfrak{S}_7$	6×6	1	M

Table 1: Parker–Semeraro systems $\mathcal{F}$, $\mathcal{F} \neq \mathcal{F}_S(G_2(7))$

Proposition 4.1. *Let $S \in \mathrm{Syl}_7(G_2(7))$. Assume G is a finite quasisimple group having a block with defect group S. Then either $G = M$ or $G = G_2(7)$.*

Most work has to be done to deal with groups of Lie type. We follow a similar approach as in [32] and are going to restate a lemma from that paper, which is very useful to deal with these groups.

Lemma 4.2. [32, Lemma 6.2] *Let $H = LP$ be a finite group such that $L \trianglelefteq H$ and P is a p-group. Furthermore, let c be a P-stable block of kL with defect group $P \cap L$ and $\mathrm{Br}_P^H(c) \neq 0$ and let P' be such that $P \cap L \leq P' \leq P$. Then,*
(a) c is a block of kLP' with defect group P',
(b) if P' acts on L as elements of $\mathrm{Inn}(L)$, then $P' = (P' \cap L)C_{P'}(L)$.

Proposition 4.3. *Let G be a quasisimple finite group and denote the quotient $G/Z(G)$ by $\overline{G}$. Suppose $\overline{G} = G(q)$ is a finite group of Lie type and let p be a prime number ≥ 7, $(p, q) = 1$. Let D be a p-group such that $Z(D)$ is cyclic of order p and $Z(D) \subseteq [D, D]$. If D is a defect group of a block of kG, then there are $n, k \in \mathbb{N}$ and a finite group H with $\mathrm{SL}_n(q^k) \leq H \leq \mathrm{GL}_n(q^k)$ (or $\mathrm{SU}_n(q^k) \leq H \leq \mathrm{GU}_n(q^k)$) such that there is a block c of H with non-abelian defect group D' such that D'/Z is of order $|D/Z(D)|$ for some $Z \leq D' \cap Z(H)$.*

Proof. Suppose G has a block with defect group D. Then the Sylow p-subgroups of $\overline{G}$ cannot be abelian, which implies that the Weyl group of the algebraic group corresponding to $\overline{G}$ has an order divisible by p, see [24, Theorem 4.10.2(a)]. This implies that the exceptional part of the Schur multiplier of $\overline{G}$ is trivial, see [24, Table 6.1.3]. Thus, there is a simple simply connected algebraic group $\overline{K}$ defined over $\overline{\mathbb{F}_q}$ and a Frobenius morphism $F : \overline{K} \to \overline{K}$ such that $\overline{K}^F$ is a central extension of G. If $\overline{K}$ is of type A, set $H := \overline{K}^F$ and c to be the block whose image has defect group D under the algebra homomorphism $kH \to kG$ induced by $H \twoheadrightarrow G$.
Thus, assume $\overline{K}$ is not of type A. Since the kernel of $K^F \to \overline{K}^F$ is a p'-group, we have

$|\overline{K}^F|_p = |K^F|_p$, so we may assume $G = \overline{K}^F$.

Denote a generator of $Z(D)$ by z. By Theorem 2.31, we may assume the group $C_G(z)$ has a block b with defect group D. Since $\overline{K}$ is simply connected, $C_{\overline{K}}(z)$ is a Levi subgroup of $\overline{K}$. If we denote $\overline{Z} := Z(C_{\overline{K}}(z))^\circ$, it is a well-known fact that $C_{\overline{K}}(z) = [C_{\overline{K}}(z), C_{\overline{K}}(z)]\overline{Z}$. The latter commutator is simply connected and thus a direct product of its components, which are simply connected as well and permuted by F. In particular, we have a decomposition $[C_{\overline{K}}(z), C_{\overline{K}}(z)] = \prod_{i=1}^{t}\prod_{j=1}^{r_i} \overline{L}_{ij}$, where each $\overline{L}_{ij}$ is simply connected simple, and the set of these groups for a fixed i lie in the same F-orbit.

Set $L_i = (\prod_{j=1}^{r_i} \overline{L}_{ij})^F$. Then we have $C_G(z) = (L_1 \times \cdots \times L_t)T^F$, for an F-stable maximal torus $T \leq C_{\overline{K}}(z)$.

Now T^F is abelian and we have $D \big/ D \cap (L_1 \times \cdots \times L_t) \cong T^F \big/ T^F \cap (L_1 \times \cdots \times L_t)$. So $[D, D] \leq D \cap (L_1 \times \cdots \times L_t) \neq 1$. By Lemma 3.8, the latter group is furthermore a defect group of a block of $kL_1 \times \cdots \times L_t$. Now defect groups respect direct products and $Z(D) \cong C_p$. Thus, we may assume $D \cap (L_1 \times \cdots \times L_t) \leq L_1$. In particular, $[D, D] \leq L_1$ and we may assume $Z(D) \leq L_1$. Since $Z(D)$ is central in $C_G(z)$, each $\overline{L}_{1j}$ is of type A and Lie rank at least p, so L_1 is isomorphic to either $\mathrm{SL}_n(q^k)$ or $\mathrm{SU}_n(q^k)$.

Let $x \in D \setminus Z(D)$. We want to show that x does not centralise L_1. First note, if $\overline{L} = \mathrm{SL}_n(\overline{\mathbb{F}_q})$ and σ a Frobenius morphism then $\overline{L}^\sigma$ is either $\mathrm{SL}_n(q^k)$ or $\mathrm{SU}_n(q^k)$ for some k and we have $C_{\overline{L}}(\overline{L}^\sigma) \leq Z(\overline{L})$. Using the decomposition from above, we may write $x = (\prod_{i=1}^{t}\prod_{j=1}^{r_i} x_{ij})t_x$ for $x_{ij} \in \overline{L}_{ij}, t_x \in Z(\overline{L})$. Let $y \in D$ such that $[x, y] \neq 1$ and write $y = (\prod_{i=1}^{t}\prod_{j=1}^{r_i} y_{ij})t_y$ for $y_{ij} \in \overline{L}_{ij}, t_y \in Z(\overline{L})$. We have $[D, D] \leq L_1$, which means $[x_{11}, y_{11}] \neq 1$, so x_{11} does not centralise $\overline{L}_{11}$ as well as $\overline{L}_{11}^{F^{r_1}}$ by the above and thus also not L_1.

Now let c be a block of kL_1 covered by b, then we may assume that c is D-stable, $\mathrm{Br}_D^{L_1 D}(c) \neq 0$ and $D \cap L_1$ is a defect group of c, see Lemma 3.8. Let D_0 be the kernel of the map $D \to \mathrm{Out}(L_1)$. Then $Z(D) \leq (D \cap L_1) \leq D_0$. Now if we apply Lemma 4.2, we obtain $D_0 = (D_0 \cap L_1)C_{D_0}(L_1)$. But we have seen that $C_D(L_1) = Z(D) \leq L_1$. So

$D_0 \leq L_1$.

If $D_0 = D$, we can take $H = L_1, D' = D$ and $Z = 1$ and the claim holds. Thus, assume $D \neq D_0$. The elements of T^F induce diagonal automorphisms of L_1. For special linear or unitary groups, the group of diagonal automorphisms modulo inner automorphisms is cyclic. In particular, D/D_0 is cyclic. Let $D/D_0 = \langle \overline{y} \rangle$. Let $\eta \in \mathrm{GL}_n(q^k)_p$ (respectively $\mathrm{GU}_n(q^k)_p$) such that ${}^{\eta}u = {}^{y}u$ for $u \in L_1$. In particular, η stabilises c. Define $H := L_1\langle \eta \rangle$ to obtain H as in the claim. Furthermore, define $D' := \langle D_0, \eta \rangle \leq H$. We also have $C_{L_1}(D) = C_{L_1}(D')$, so $\mathrm{Br}_{D'}^{H}(c) \neq 0$. Now $H = L_1 D'$ and $D_0 (\cong D' \cap L_1$ by construction) is a defect group of c as a block of kL_1. Thus, we can apply Lemma 4.2 to obtain that c is a block of kH with defect group D'.

We have $D' = \langle D_0, \eta \rangle$ and $D = \langle D_0, y \rangle$. The canonical maps $D' \to \mathrm{Aut}(L_1)$ and $D \to \mathrm{Aut}(L_1)$ have the same image. Thus, $D' \big/ C_{D'}(L_1) \cong D \big/ C_D(L_1) = D/Z(D)$. Define $Z := Z(\mathrm{GL}_n(q^k)) \cap D'$ or $Z := Z(\mathrm{GU}_n(q^k)) \cap D'$, so $Z = C_{D'}(L_1)$, which gives $D'/Z \cong D/Z(D)$. In particular, D' is non-abelian since y acts non-trivially on D_0. $\square$

Proposition 4.4. *If G is as in the previous proposition, then G has no blocks with defect groups isomorphic to a Sylow 7-subgroup of $G_2(7)$.*

Proof. Assume $P \in \mathrm{Syl}_7(G_2(7))$, in particular $|P| = 7^6$ and $Z(P) \cong C_7$. Let H, P' be as in the assertion of the previous proposition with $p = 7$. Assume first $H \leq \mathrm{GL}_n(q^k)$ and let a be such that $|q^k - 1|_7 = 7^a$. Then, since $\mathrm{SL}_n(q^k) \leq H$, we have $|P'| = |P/Z(P)| \cdot |Z| = 7^5|Z| \leq 7^5|Z(H)| \leq 7^5|Z(\mathrm{SL}_n(q^k))| \leq 7^{5+a}$. Now the block of $k\,\mathrm{GL}_n(q^k)$ covering c has a defect group of order at most 7^{2a+5}. But it is a well-known fact, that (non-abelian) defect groups of $\mathrm{GL}_n(q^k)$ have order at least 7^{7a+1}, see [22, Theorem 3C]. Thus, $7^{7a+1} \leq 7^{2a+5}$, which is a contradiction. The case $H \leq \mathrm{GU}_n(q^k)$ can be shown in the same fashion by considering the 7-part of $q^k + 1$ instead of $q^k - 1$. $\square$

We use this observation to prove Proposition 4.1.

Proof of Proposition 4.1. By the previous proposition, we may assume G is not a group of Lie type in characteristic not equal to 7.

First, assume $G/Z(G)$ is an alternating group $\mathfrak{A}_m$. Then S is isomorphic to a Sylow 7-subgroup of some symmetric group $\mathfrak{S}_{7w}$ with $w \leq 6$. Define the cycle $\sigma_i = ((i-1)7+1, \ldots, i7)$ and the subgroup $S' = \langle \sigma_1, \ldots, \sigma_6 \rangle \leq \mathfrak{A}_m$. Then $S' \in \mathrm{Syl}_7(\mathfrak{A}_m)$. But this group is abelian, which means that $S \notin \mathrm{Syl}_7(\mathfrak{A}_m)$.

Next, assume G is a group of Lie type over a field of characteristic $p = 7$. In this case, $Z(G)$ is a $7'$-group, and we may assume that $G = \mathbf{G}^F$, where $\mathbf{G}$ is simple and simply connected, F is a Frobenius morphism. Furthermore, $S \in \mathrm{Syl}_7(G)$ by [13, Theorem 6.18]. We first deal with the classical groups. First consider type B_n, here we have $|G|_p = q^{n^2}$, which can be equal to 7^6 only if $n = 1$, which is not possible since $n > 1$ for these groups. The case is the same for the groups of type C_n. For type D_n and ${}^2D_n$, we have $|G|_p = q^{n(n-1)}$. Since $n > 3$, p^6 is also no possibility here. Finally, consider types A_n and ${}^2A_n$, here we have $|G|_p = q^{\frac{1}{2}(n+1)n}$. If $n = 1$, these groups have abelian Sylow p-subgroups and if $n \geq 4$, the order is too big. Thus, the only possibilities are $n = 2$ or $n = 3$. However, in these cases we obtain Sylow 7-subgroups which are conjugate to the groups of upper unitary triangle matrices. Hence, these groups have nilpotency class 2 respectively 3. However, the nilpotency class of S is 5. This leaves us with the exceptional groups of Lie type. Looking at their orders, we can directly exclude the exceptional Steinberg groups, the Suzuki, Ree and Tits groups. For the exceptional Chevalley groups, $G_2(7)$ is the only possibility.

Finally, if $G/Z(G)$ is sporadic, the monster M is the only group with a 7-part big enough to contain S, which implies our claim. $\square$

This result can be used to achieve a reduction specifically for the Parker–Semeraro systems.

Lemma 4.5. *Let $\mathcal{F}$ be an exotic Parker–Semeraro system. Then $\mathcal{F}$ is reduction simple.*

Proof. Assume $1 \neq N \leq S$ is strongly $\mathcal{F}$-closed. In particular $N \trianglelefteq S$, which implies $Z(S) \leq N$. Thus, as in the proof of [42, Theorem 6.2], we obtain $N = S$. $\square$

We use the reduction simplicity of the Parker–Semeraro systems in the following theorem.

Theorem 4.6. *Suppose there is an exotic Parker–Semeraro system $\mathcal{F}$ which is block-realisable. Then there is an exotic Parker–Semeraro system $\mathcal{F}_0$ which is block-realisable by a block of a quasisimple group.*

Proof. Assume G is a finite group having an $\mathcal{F}$-block b. We may choose G such that the conclusions of Theorem 1.2 hold, with $S \in \mathrm{Syl}_7(G_2(7))$ in the role of P. Define $L = \langle {}^gS \mid g \in G\rangle \trianglelefteq G$. Since $Z(S)$ is cyclic of order p, $\mathcal{F}$ satisfies the hypothesis of Theorem 3.10 with $P = S$. Arguing as in the proof of that theorem, $L = E(G)$ is quasisimple, and there is a block of kL with defect group S and fusion system $\mathcal{F}_0$ such that $O_p(\mathcal{F}_0) = 1$. So, $\mathcal{F}_0$ is a Parker–Semeraro system.
If $\mathcal{F}_0$ is exotic, we are done. Suppose $\mathcal{F}_0$ is realisable. So, either $\mathcal{F}_0 = \mathcal{F}_S(M)$ or $\mathcal{F}_0 = \mathcal{F}_S(G_2(7))$. By Proposition 4.1, $L = M$ or $L = G_2(7)$ and hence L is simple.
We claim $G = L$. Indeed, consider the map $\varphi : G \to \mathrm{Out}(F^*(G))$ and let $g \in \ker(\varphi)$. Then there exists $x \in F^*(G)$ such that $y \in F^*(G)$ with $gyg^{-1} = xyx^{-1}$, i.e. $x^{-1}g \in C_G(F^*(G)) = Z(F^*(G))$. This implies $\ker\varphi = F^*(G)$ and thus $G/F^*(G) \cong \mathrm{Out}_G(F^*(G))$. But in our case, we have $F^*(G) = Z(G)E(G)$, so $\mathrm{Out}_G(F^*(G)) = \mathrm{Out}_G(E(G))$. However, this group needs to be trivial since $\mathrm{Out}(M) = \mathrm{Out}(G_2(7)) = 1$. This implies $G = Z(G)E(G)$ and in either case $G = Z(G) \times L$. By choice of G, the claim follows.
We now claim that in either of these cases, b needs to be the principal block. For $G_2(7)$ this is a well known fact, see e.g. [30, Example 3.8]. So, assume $G = M$. We want to compute $C_{N_M(P)}(S)$, where P is a subgroup of order 7 of the monster. We know that $C_{N_M(P)}(S) = Z(S) \times O_{7'}(C_{N_M(P)}(S))$. Furthermore, by [44, Theorem 1.1], $N_M(P)$ is 7-constrained, i.e. $C_{N_M(P)}(O_7(N_M(P))) \subseteq O_7(N_M(P))$. Also, by [44, Theorem 1.1],

$N_M(P) = 7^{1+4}(2\mathfrak{S}_7 \times 3)$, i.e. $O_7(N_M(P)) = 7^{1+4}$ and thus $C_{N_M(P)}(S) \cong C_p$. But this means $C_M(S)/Z(S)$ is trivial and since the $N_M(S)$-classes of irreducible characters of this group are in 1:1-correspondence with blocks of M with defect group S, the claim follows. However, this implies that $\mathcal{F}$ cannot be exotic, which is a contradiction. $\square$

We use this to deduce block-exoticity of the Parker–Semeraro systems.

Proof of Theorem 1.3. By [51, Main Theorem], the only exotic fusion systems on such groups are the 27 exotic Parker–Semeraro systems. Let S be a Sylow 7-subgroup of $G_2(7)$ and let $\mathcal{F}$ be one of these exotic systems. Assume G is a group with an $\mathcal{F}$-block. By the previous theorem, we may assume that G is quasisimple. Let A be the simple quotient of G. By Proposition 4.1, we may assume either $A = G_2(7)$ or $A = M$, and thus, since $\mathrm{Out}(A) = 1$ in both these cases, also either $G = G_2(7)$ or $G = M$. But as in the proof of the previous theorem, this means that the $\mathcal{F}$-block is principal, which is not possible for an exotic fusion system. $\square$

5 On fusion systems of finite groups of Lie type in non-describing characteristic

5.1 Background on block theory of finite groups of Lie type

After proving Conjecture 1.1 for a given family of exotic fusion systems in the previous chapter, we are tackling the conjecture for fusion systems of blocks of finite groups of Lie type in non-describing characteristic. We start by giving some background on character and block theory of such groups.

For the rest of this chapter, let $\mathbf{G}$ be a connected, algebraic group defined over $\overline{\mathbb{F}}_q$ where q is a prime power and let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius morphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $\mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Recall that $R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\operatorname{Irr}(\mathbf{L}^F) \to \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F), [M] \mapsto \sum_{i\in\mathbb{Z}}(-1)^i[H_c^i(\mathbf{Y_P}) \otimes_{\mathbb{C}\mathbf{L}^F} M]$, where $[M]$ is the class of a $\mathbb{C}\mathbf{L}^F$-module M in $\operatorname{Irr}(\mathbf{L}^F)$, is the Lusztig induction map. This is defined using a parabolic subgroup $\mathbf{P}$ of which $\mathbf{L}$ is a Levi. However, in all cases we consider, $R_{\mathbf{L}}^{\mathbf{G}}$ is independent of the choice of $\mathbf{P}$ and hence we will suppress it from the notation. We use this construction in the case where $\mathbf{L}$ is a maximal torus to parametrise the irreducible characters of $\mathbf{G}^F$.

If we fix an F-stable torus of $\mathbf{G}$, we can define $\mathbf{G}^*$ to be a group in duality with $\mathbf{G}$ with respect to this torus, with corresponding Frobenius again denoted by F. Then there is a bijection $\{\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)\} \leftrightarrow \{{\mathbf{G}^*}^F$-classes of pairs $(\mathbf{T}^*, s)$ where $\mathbf{T}^*$ is an F-stable maximal torus of $\mathbf{G}^*$ and $s \in {\mathbf{T}^*}^F\}$.

Theorem 5.1. (Deligne–Lusztig). [10, Theorem 4.7] *For $s \in {\mathbf{G}^*}^F$ a semisimple element, one defines $\mathcal{E}(\mathbf{G}^F, s)$ to be the set of irreducible components of generalised characters $R_{\mathbf{T}}^{\mathbf{G}}\theta$ for $(\mathbf{T}, \theta)$ corresponding to some $(\mathbf{T}^*, s)$ through the above correspondence. One* *gets a partition* $\operatorname{Irr}\mathbf{G}^F = \bigsqcup_s \mathcal{E}(\mathbf{G}^F, s)$*, where* s *ranges over semisimple classes of* $\mathbf{G}^{*F}$*.*

This theorem plays an important role when studying blocks of groups of Lie type. The blocks obtained when specialising to $s = 1$ are the starting point for many constructions.

Definition 5.2. *Keep the notation of the previous theorem. The elements of* $\mathcal{E}(\mathbf{G}^F, 1)$ *are called unipotent characters. A block* b *of* $\mathbf{G}^F$ *such that* $\operatorname{Irr}(\mathbf{G}^F, b) \cap \mathcal{E}(\mathbf{G}^F, 1) \neq \emptyset$ *is called a unipotent block of* $\mathbf{G}^F$*.*

If χ is a unipotent character of $\mathbf{L}^F$, then all constituents of $R_{\mathbf{L}}^{\mathbf{G}}\chi$ are unipotent. We recall an important result related to the parametrisation by Deligne–Lusztig.

Theorem 5.3. (Broué–Michel, Hiss). [13, Theorem 9.12] *Let* $s \in \mathbf{G}^*$ *be a semisimple* p'*-element, define* $b(\mathbf{G}^F, s) = \sum_{\chi \in \mathcal{E}(\mathbf{G}^F, s)} e_\chi$ *and* $\mathcal{E}_p(\mathbf{G}^F, s)$ *as the union of rational series* $\mathcal{E}(\mathbf{G}^F, t)$ *such that* $s = t_{p'}$*.*

(*a*) *The set* $\mathcal{E}_p(\mathbf{G}^F, s)$ *is a union of blocks* $\operatorname{Irr}(\mathbf{G}^F, b_i)$*, i.e.* $b(\mathbf{G}^F, s) \in \mathcal{O}\mathbf{G}^F$*.*

(*b*) *For each block* b *of* $k\mathbf{G}^F$ *with* $\operatorname{Irr}(\mathbf{G}^F, b) \subseteq \mathcal{E}_p(\mathbf{G}^F, s)$*, one has* $\operatorname{Irr}(\mathbf{G}^F, b) \cap \mathcal{E}(\mathbf{G}^F, s) \neq \emptyset$*.*

If $e \geq 1$, recall that $\phi_e(x) \in \mathbb{Z}[x]$ denotes the e-th cyclotomic polynomial, whose complex roots are the roots of unity of order e. Any F-stable torus $\mathbf{S}$ of $\mathbf{G}$ has so-called polynomial order $P_{\mathbf{S},F} \in \mathbb{Z}[x]$ defined by $|\mathbf{S}^{F^m}| = P_{\mathbf{S},F}(q^m)$ for some $a \geq 1$ and any $m \in 1 + a\mathbb{N}$. Moreover, $P_{\mathbf{S},F}$ is a product of cyclotomic polynomials $P_{\mathbf{S},F} = \prod_{e \geq 1} \phi_e^{n_e}, n_e \geq 0$. A ϕ_e-torus of $\mathbf{G}$ is an F-stable torus whose polynomial order is a power of ϕ_e. An e-split Levi subgroup is any $C_{\mathbf{G}}(\mathbf{S})$, where $\mathbf{S}$ is a ϕ_e-torus of $\mathbf{G}$.

Let $\mathbf{L}_i, i = 1, 2$, be an e-split Levi subgroup in $\mathbf{G}$ and $\zeta_i \in \mathcal{E}(\mathbf{L}_i^F, 1)$. We write $(\mathbf{L}_1, \zeta_1) \leq_e (\mathbf{L}_2, \zeta_2)$ if ζ_2 is a component of $R_{\mathbf{L}_1}^{\mathbf{L}_2}(\zeta_1)$, i.e. $\langle R_{\mathbf{L}_1}^{\mathbf{L}_2}(\zeta_1), \zeta_2 \rangle \neq 0$.

Definition 5.4. *A unipotent character* $\chi \in \mathcal{E}(\mathbf{G}^F, 1)$ *is said to be e-cuspidal if a relation* $(\mathbf{G}, \chi) \geq_e (\mathbf{L}, \zeta)$ *is only possible with* $\mathbf{L} = \mathbf{G}$*. A pair* $(\mathbf{L}, \zeta)$ *with* $\mathbf{L}$ *an e-split Levi subgroup and an e-cuspidal* $\zeta \in \mathcal{E}(\mathbf{L}^F, 1)$ *is called a unipotent e-cuspidal pair of* $\mathbf{G}^F$*.*

Theorem 5.5. (Cabanes–Enguehard). [10, Theorem 6.10] *Assume $p \geq 7$ is a prime not dividing q. Let e be the multiplicative order of q mod p. If $(\mathbf{L}, \zeta)$ is an e-cuspidal pair of $\mathbf{G}$, then there exists a block $b_{\mathbf{G}^F}(\mathbf{L}, \zeta)$ such that all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\zeta)$ belong to $b_{\mathbf{G}^F}(\mathbf{L}, \zeta)$. The map $(\mathbf{L}, \zeta) \mapsto b_{\mathbf{G}^F}(\mathbf{L}, \zeta)$ gives a bijection between $\mathbf{G}^F$-classes of unipotent e-cuspidal pairs of $\mathbf{G}^F$ and unipotent blocks of $\mathbf{G}^F$.*

A starting point to prove Conjecture 1.1 for groups of Lie type in non-describing characteristic is the following theorem by Cabanes, which proves the conjecture for almost all primes if the underlying block is unipotent:

Theorem 5.6. (Cabanes) [10, Theorem 7.11] *Let $p \geq 7$ a prime with $(p, q) = 1$ and b a unipotent block of $k\mathbf{G}^F$ with maximal b-Brauer pair (P, e_P). Then the fusion system $\mathcal{F}_{(P,e_P)}(\mathbf{G}^F, b)$ is non-exotic.*

Let $\mathbf{G}^F$ be a finite group of Lie type with block d and $\mathbf{L} \leq \mathbf{G}$ a Levi subgroup. In [5], Bonnafé, Dat and Rouquier(BDR) prove that, in many cases, the fusion system of d is equivalent to the fusion system of a block c of a subgroup of $\mathbf{G}^F$ containing $\mathbf{L}^F$ as a normal subgroup, where c covers a unipotent block of $\mathbf{L}^F$. We want to combine this result with Theorem 5.6 to tackle Conjectures 1.1 and 1.5.

We need to set up some of the notation from [5].

Remark 5.7. *The duality between $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F)$ gives a correspondence $\mathbf{L} \mapsto \mathbf{L}^*$ between $\mathbf{G}$-classes of Levi subgroups of $\mathbf{G}$ and $\mathbf{G}^*$-classes of Levi subgroups of $\mathbf{G}^*$ and isomorphisms $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} \cong N_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ for such Levi subgroups in duality. Moreover, $\mathbf{L}$ is F-stable if and only if $\mathbf{L}^*$ is F-stable.*

Definition 5.8. *Let $s \in \mathbf{G}^{*F}$ be a semisimple p'-element and $\mathbf{L}^* \leq \mathbf{G}^*$ an F-stable Levi subgroup minimal with respect to the property of containing $C^\circ_{\mathbf{G}^*}(s)$. Let $\mathbf{N}^*$ be such that $\mathbf{L}^* \leq \mathbf{N}^* \leq N_{\mathbf{G}^*}(\mathbf{L}^*)$. Define $\mathbf{N}$ as the subgroup of $\mathbf{G}$ containing $\mathbf{L}$ such that $\mathbf{N}/\mathbf{L}$*

corresponds to $\mathbf{N}^*/\mathbf{L}^*$ *under the isomorphism between* $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ *and* $N_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$. *We refer to such* $\mathbf{L} \trianglelefteq \mathbf{N}$ *as a BDR-extension in* $\mathbf{G}$.

We state the theorem by Bonnafé, Dat and Rouquier relating the fusion systems in a simplified version.

Theorem 5.9. [5, Theorem 7.7+Example 7.9] *Let* $\mathbf{G}$ *be a connected, reductive group with Frobenius* F *in duality with* $(\mathbf{G}^*, F)$. *Let* s *be a semisimple* p'*-element of* $\mathbf{G}^*$ *and* d *a block of* $k\mathbf{G}^F$ *such that* $\mathrm{Irr}(\mathbf{G}^F, d) \cap \mathcal{E}(\mathbf{G}^F, s) \neq \emptyset$. *Let* $\mathbf{L} \trianglelefteq \mathbf{N}$ *be the BDR-extension defined by setting* $\mathbf{N}^* := C_{\mathbf{G}^*}(s)^F \cdot \mathbf{L}^*$. *If* $\mathbf{N}/\mathbf{L}$ *is cyclic, then there is a block* c *of* $k\mathbf{N}^F$ *covering a unipotent block of* $k\mathbf{L}^F$ *such that the fusion system of* c *is equivalent to the fusion system of* d.

We state the triangle relations from Theorem 2.53 in the case of a BDR-extension and fix notation for the occurring fusion systems.

Definition 5.10. *Let* $\mathbf{L} \trianglelefteq \mathbf{N}$ *be a BDR-extension in* $\mathbf{G}$. *Let* b *be an* $\mathbf{N}^F$*-stable unipotent block of* $k\mathbf{L}^F$ *covered by a block* $c \in k\mathbf{N}^F$. *Let* (P, e_P) *be a maximal* c*-Brauer pair and let* (S, e'_S) *be a maximal* $(b, \mathbf{N}^F)$*-Brauer pair with* $P \leq S$ *and* $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, c) \leq \mathcal{F}_{(S,e'_S)}(\mathbf{N}^F, \mathbf{L}^F, b)$. *We have* $P \cap \mathbf{L}^F = S \cap \mathbf{L}^F$ *and* $(S \cap \mathbf{L}^F, e'_{S\cap\mathbf{L}^F})$ *is a maximal* b*-Brauer pair with* $\mathcal{F}_{(S\cap\mathbf{L}^F, e'_{S\cap\mathbf{L}^F})}(\mathbf{L}^F, b) \trianglelefteq \mathcal{F}_{(S,e'_S)}(\mathbf{N}^F, \mathbf{L}^F, b)$.
We call $\mathcal{F}_{(S\cap\mathbf{L}^F, e'_{S\cap\mathbf{L}^F})}(\mathbf{L}^F, b)$ *a unipotent block fusion system,* $\mathcal{F}_{(S,e'_S)}(\mathbf{N}^F, \mathbf{L}^F, b)$ *BDR generalised block fusion system and* $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, c)$ *BDR block fusion system.*

5.2 Extension of Cabanes' Theorem

In this section, we extend Cabanes' Theorem to BDR generalised block fusion systems. For this section, fix $\mathbf{X}$ to be a connected, reductive group with Frobenius F defining an $\mathbb{F}_q$-structure on $\mathbf{X}$, $\mathbf{G} \trianglelefteq \mathbf{N}$ a BDR-extension in $\mathbf{X}$, where $\mathbf{G} \leq \mathbf{X}$ is a Levi subgroup. Let c be a block of $k\mathbf{N}^F$ covering a unipotent block b of $k\mathbf{G}^F$ with $b = b_{\mathbf{G}^F}(\mathbf{L}, \zeta)$ for an

e-cuspidal pair $(\mathbf{L}, \zeta)$ of $\mathbf{G}^F$.

Let (Q, b_Q) be a Brauer pair of a finite group H. Then it is called centric if and only if b_Q has defect group $Z(Q)$ in $C_H(Q)$. Note that this is equivalent to Q being centric in the fusion system of a block b of kH such that $(1, b) \leq (Q, b_Q)$. Then there is a single $\zeta \in \mathrm{Irr}(b_Q)$ with $Z(Q)$ in its kernel, we call it the canonical character of the centric subpair.

We assume for the rest of this section that $p \geq 7$. Let $Z = Z^\circ(\mathbf{L})_p^F$. By [11, Proposition 2.2], $Z = Z(\mathbf{L})_p^F$, $\mathbf{L} = C^\circ_{\mathbf{G}}(Z)$ and $\mathbf{L}^F = C^\circ_{\mathbf{G}}(Z)^F = C_{\mathbf{G}^F}(Z)$.
Let e_Z be the block of $\mathbf{L}^F$ containing the character ζ. Since $\mathbf{L}^F = C_{\mathbf{G}^F}(Z)$, (Z, e_Z) is a Brauer pair for $\mathbf{G}^F$. By [11, Lemma 4.5], (Z, e_Z) is a b-Brauer pair, (Z, e_Z) is a centric Brauer pair and ζ is the canonical character of e_Z. Further, there exists a maximal b-Brauer pair (P, e_P) such that $(Z, e_Z) \trianglelefteq (P, e_P)$ and such that the canonical character of e_P is $\mathrm{Res}^{\mathbf{L}^F}_{C_{\mathbf{G}^F}(P)}(\zeta)$. Here we note that if (Q, e) is a centric Brauer pair, then for any Brauer pair (R, f) such that $(Q, e) \leq (R, f)$, we also have that (R, f) is also centric.
Note that P is a Sylow p-subgroup of $(C^\circ_{\mathbf{G}}([\mathbf{L}, \mathbf{L}]))^F$. Furthermore, note that by Theorem 3.1, we can always choose the BDR-extension $\mathbf{N}$ such that b is $\mathbf{N}^F$-stable and such that $\mathbf{N}^F/\mathbf{G}^F$ is a p'-group by [5, 7.A]. If we thus consider the fusion systems occurring in Definition 5.10, we can assume that all of them are defined on the same p-group by Proposition 2.54.

Recall that for a semisimple, algebraic group $\mathbf{X}$, there exist natural isogenies $\mathbf{X}_{\mathrm{sc}} \to \mathbf{X} \to \mathbf{X}_{\mathrm{ad}}$, where $\mathbf{X}_{\mathrm{sc}}$ is simply connected and $\mathbf{X}_{\mathrm{ad}}$ is of adjoint type. There is a decomposition of $[\mathbf{X}, \mathbf{X}] = \mathbf{X}_1 \cdots \mathbf{X}_m$ as a central product of F-stable, closed subgroups, see [13, Definition 22.4] for more details. Define $\mathbf{X}_{\mathbf{a}} = Z^\circ(\mathbf{X})\mathbf{X}'_{\mathbf{a}}$, where $\mathbf{X}'_{\mathbf{a}}$ is the subgroup generated by the $\mathbf{X}_i$ with $(\mathbf{X}_i)^F_{\mathrm{ad}} \cong \mathrm{PGL}_{n_i}(\epsilon_i q^{m_i})$ and p dividing $q^{m_i} - \epsilon_i$. Let $\mathbf{X}_{\mathbf{b}}$ be generated by the remaining $\mathbf{X}_i$. We thus have a decomposition $\mathbf{X} = \mathbf{X_a X_b}$.

Let $M = N_{\mathbf{G}^F}([\mathbf{L},\mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$. As noted above, $(Z, e_Z) \trianglelefteq (P, e_P)$. Thus, P normalises $\mathbf{L} = C_{\mathbf{G}^\circ}(Z)$ and consequently P normalises $[\mathbf{L},\mathbf{L}]$ and $[\mathbf{L},\mathbf{L}]^F$. Again, since $(Z, e_Z) \trianglelefteq (P, e_P)$, ${}^x e_Z = e_Z$ for all $x \in P$. Since ζ is the canonical character of e_Z, ${}^x\zeta = \zeta$ for all $x \in P$. Since restriction induces a bijection between $\mathcal{E}(\mathbf{L}^F, 1)$ and $\mathcal{E}([\mathbf{L},\mathbf{L}]^F, 1)$, see [19, Proposition 13.20], ${}^x \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta = \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta$ for all $x \in P$. Thus, $P \leq M$. Also, $C_{\mathbf{G}^F}(P) \leq C_{\mathbf{G}^F}(Z) = \mathbf{L}^F \leq M$. Since $PC_{\mathbf{G}^F}(P) \leq M$, (P, e_P) is a Brauer pair for a block of M.

We recall some details which can be found in the proof of [10, Theorem 7.11]: Let $Q \leq P$ be $\mathcal{F}_{(P,e_P)}(\mathbf{G}^F, b)$-centric. We have $C^\circ_{\mathbf{G}}(Q)_{\mathbf{b}} = [\mathbf{L},\mathbf{L}]$. Let ζ_Q be the canonical character of e_Q. Then $\zeta^\circ_Q := \operatorname{Res}^{C_{\mathbf{G}^F}(Q)}_{C^\circ_{\mathbf{G}}(Q)^F} \zeta_Q$ is the unique unipotent character of $C^\circ_{\mathbf{G}}(Q)^F$ whose restriction to $[\mathbf{L},\mathbf{L}]^F$ is $\operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta$.

Proposition 5.11. *Let $H := N_{\mathbf{N}^F}([\mathbf{L},\mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta, d)$, where d is the unique block of kM, where $M := N_{\mathbf{G}^F}([\mathbf{L},\mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$, with (P, e_P) a d-Brauer pair. We have $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{G}^F, b) = \mathcal{F}_{(P,e_P)}(H, M, d)$.*

Proof. Let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{G}^F, b)$, $\mathcal{G} = \mathcal{F}_{(P,e_P)}(H, M, d)$ and $\mathcal{F}_0 = \mathcal{F}_{(P,e_P)}(\mathbf{G}^F, b)$.
Assume $(X, e_X) \leq (P, e_P)$ is $\mathcal{F}_0$-centric. Let $x \in N_{\mathbf{N}^F}(X, e_X)$, then x normalises $C_{\mathbf{G}}(X)$, hence, also $C_{\mathbf{G}}(X)_{\mathbf{b}} = [\mathbf{L},\mathbf{L}]$. Also, x stabilises ζ_X and thus also $\zeta_X|_{[\mathbf{L},\mathbf{L}]^F} = \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta$. We thus have $x \in N_{\mathbf{N}^F}([\mathbf{L},\mathbf{L}]^F, \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$.
Furthermore, $C_{\mathbf{G}^F}(X) \leq M$, so e_X is a block of $kC_M(X)$ and hence (X, e_X) is the unique d-Brauer pair with first component X such that $(X, e_X) \leq (P, e_P)$. In particular, b-Brauer pairs and d-Brauer pairs, for which the first component is a $\mathcal{F}_0$-centric subgroup of P, are the same.
Thus, if x stabilises (X, e_X), it also stabilises d, and we get $N_{\mathbf{N}^F}(X, e_X) \leq H$ for all $X \leq P$ which are $\mathcal{F}_0$-centric. Now if $X \leq P$ is $\mathcal{F}$-centric, it is in particular $\mathcal{F}_0$-centric,

so $\mathcal{F} \subseteq \mathcal{G}$.

Conversely, suppose that $(X, e_X) \leq (P, e_P)$ is $\mathcal{G}$-centric. In particular, by the previous paragraph, X is $\mathcal{F}$- and thus also $\mathcal{F}_0$-centric. But again, since e_X is a block of $kC_M(X)$, (X, e_X) is an (d, H)-Brauer pair if and only if it is an $(b, \mathbf{N}^F)$-Brauer pair and hence $\mathrm{Aut}_{\mathcal{G}}(X) \leq \mathrm{Aut}_{\mathcal{F}}(X)$. This shows $\mathcal{G} \subseteq \mathcal{F}$ and hence $\mathcal{G} = \mathcal{F}$. □

Fix H as in the previous proposition for the rest of this section.

Lemma 5.12. *For a p-group Q, the group $C_{\mathbf{G}^F}(Q)/C^\circ_{\mathbf{G}}(Q)^F$ is also a p-group.*

Proof. By [20, Proposition 2.1.6(e)], $C_{\mathbf{G}}(Q)/C^\circ_{\mathbf{G}}(Q)$ is a p-group. Now compose the inclusion $C_{\mathbf{G}^F}(Q) \to C_{\mathbf{G}}(Q)$ with the natural surjection $C_{\mathbf{G}^F}(Q) \to C_{\mathbf{G}^F}(Q)/C^\circ_{\mathbf{G}}(Q)^F$ and denote this morphism by φ. Then $\ker\varphi = C^\circ_{\mathbf{G}}(Q)^F$. Thus $C_{\mathbf{G}^F}(Q)/C^\circ_{\mathbf{G}}(Q)^F$ is isomorphic to a subgroup of $C_{\mathbf{G}}(Q)/C^\circ_{\mathbf{G}}(Q)$ and in particular also a p-group. □

Lemma 5.13. *We have $N_{\mathbf{N}^F}(Q, e_Q) = N_H(Q)$ for any Q which is $\mathcal{F}_{(P,e_P)}(\mathbf{G}^F, b)$-centric.*

Proof. In the proof of Proposition 5.11, we proved $N_{\mathbf{N}^F}(Q, e_Q) \leq H$. Let $x \in N_H(Q)$. We need to prove that x stabilises e_Q. First, we prove that x preserves the set of unipotent characters of $C^\circ_{\mathbf{G}}(Q)^F$. Note that $\mathbf{N} \subseteq N_{\mathbf{X}}(\mathbf{G})$. Further, $N_{\mathbf{N}}(Q) \subseteq N_{\mathbf{X}}(\mathbf{G}) \cap N_{\mathbf{X}}(Q) \subseteq N_{\mathbf{X}}(\mathbf{G}) \cap N_{\mathbf{X}}(C_{\mathbf{X}}(Q)) \subseteq N_{\mathbf{X}}(C_{\mathbf{G}}(Q)) \subseteq N_{\mathbf{X}}(C^\circ_{\mathbf{G}}(Q))$, whereas the last inclusion follows from the fact that $\mathbf{X}$ is an algebraic group. Now x normalises $[\mathbf{L}, \mathbf{L}]$, $\mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta$ and $C^\circ_{\mathbf{G}}(Q)$ and thus, by the remarks before Proposition 5.11 together with the observation about unipotent characters, also ζ°_Q. In particular, it also stabilises the block of $kC^\circ_{\mathbf{G}}(Q)^F$ containing ζ°_Q. Further, by Lemmas 5.12 and 2.35, e_Q is the unique block of $kC_{\mathbf{G}^F}(Q)$ covering the block of $kC^\circ_{\mathbf{G}}(Q)^F$ containing ζ°_Q. Thus, x stabilises e_Q, so $N_H(Q) \subseteq N_{\mathbf{N}^F}(Q, e_Q)$. □

Lemma 5.14. *Let $A = [\mathbf{L}, \mathbf{L}]^F$. The group PA/A is in $\mathrm{Syl}_p(H/A)$.*

Proof. Since $|H:M|_p = 1$, it suffices to prove that PA/A is a Sylow p-subgroup of M/A. Now P is a Sylow p-subgroup of $(C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}]))^F$. Hence, it suffices to prove that p does not divide $|M:(C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}]))^F A|$. Since $C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])$ and $[\mathbf{L},\mathbf{L}]$ commute element wise and $C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}]) \cap [\mathbf{L},\mathbf{L}] \leq Z([\mathbf{L},\mathbf{L}])$, a standard application of the Lang–Steinberg theorem, see [19, 3.10], gives

$$|(C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])[\mathbf{L},\mathbf{L}])|^F = |C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])|^F|[\mathbf{L},\mathbf{L}]|^F = |C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])|^F|A|.$$

On the other hand, $C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])^F \cap A \leq Z([\mathbf{L},\mathbf{L}]^F) = Z([\mathbf{L},\mathbf{L}])^F$ is a p'-group by [12, Proposition 4]. Hence, the p-part of $|C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])^F||A| = \frac{|C^\circ_{\mathbf{G}}\mathbf{L},\mathbf{L}])^F||A|}{|C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}]^F \cap A|}$ equals the p-part of $|C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])^F||A|$. Thus, by the above displayed equation, it suffices to prove that p does not divide $|M:(C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}])[\mathbf{L},\mathbf{L}])^F|$. This follows from [12, Proposition 6] $\square$

Proof of Theorem 1.9. Let $H := N_{\mathbf{N}^F}([\mathbf{L},\mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta, d)$, where d is the unique block of kM, where $M := N_{\mathbf{G}^F}([\mathbf{L},\mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta)$, with (P, e_P) a d-Brauer pair. By Proposition 5.11, we have $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{G}^F, b) \cong \mathcal{F}_{(P,e_P)}(H, M, d)$.

Next we apply the Generalised Second Fong Reduction to $\mathcal{F}_{(P,e_P)}(H, M, d)$. For this, let $A := [\mathbf{L},\mathbf{L}]^F$. In particular, we have normal inclusions $A \leq M \leq H$ with $A \trianglelefteq H$ and $|H:M|_p = 1$. Let c be the block of A containing $\mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta$. Apply [13, Theorem 22.9(ii)] to $\mathbf{L}$ to see that $\mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta$ is a defect zero character and thus c is of defect zero. Clearly, c is M-stable, so we can apply Lemma 2.34 with $Z = Z(\mathbf{L})^F_p$ and $e = e_Z$ to see that c is covered by d. Thus, we can apply Theorem 1.8 to this situation and get a fusion system $\mathcal{F}_{(P,e'_P)}(\widetilde{H}, \widetilde{M}, \tilde{d})$ such that $\widetilde{H}$ is a p'-central extension of H/A, $\widetilde{M}$ is the full inverse image of M/A in $\widetilde{H}$ and $\tilde{d}$ is an $\widetilde{H}$-stable block of $k\widetilde{M}$ with maximal $(\tilde{d}, \widetilde{H})$-Brauer pair (P, e'_P).

The final step is now to prove that $\mathcal{F}_{(P,e'_P)}(\widetilde{H}, \widetilde{M}, \tilde{d})$ is non-exotic. Let $X \leq P$ be $\mathcal{F}_{(P,e'_P)}(\widetilde{H}, \widetilde{M}, \tilde{d})$-centric, then by Lemma 5.13 we have that the normaliser of (X, e_X) is the normaliser of X. Furthermore, $P \in \mathrm{Syl}_p(\widetilde{H})$ by Lemma 5.14, which together with the observation about the normalisers implies $\mathcal{F}_{(P,e'_P)}(\widetilde{H}, \widetilde{M}, \tilde{d}) = \mathcal{F}_P(\widetilde{H})$. $\square$

5.3 Alternative proof of Theorem 1.9 in special cases using control subgroup

The proof of Theorem 5.6 uses a concept called control subgroups, which are defined in [12]. The following main theorem of this section uses these control subgroups to prove non-exoticity of certain generalised BDR block fusion systems.

Theorem 5.15. *Let $(\mathbf{L}, \zeta)$ be a unipotent e-cuspidal pair corresponding to a block b of $\mathbf{G}^F$, $\mathbf{N}$ be a BDR-extension of $\mathbf{G}$ such that b is $\mathbf{N}^F$-stable and let (P, e_P) be a maximal $(b, \mathbf{N}^F)$-Brauer pair. If there is a group $\widetilde{H} \leq N_{\mathbf{N}^F}([\mathbf{L},\mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta)$ with $\widetilde{H}[\mathbf{L},\mathbf{L}]^F = N_{\mathbf{N}^F}([\mathbf{L},\mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta)$ and P is a Sylow p-subgroup of $\widetilde{H}$, then the BDR-generalised block fusion system $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{L}^F, b)$ is isomorphic to $\mathcal{F}_P(\widetilde{H})$.*

The purpose of this section is to show how far we can go by using control subgroups. We use the above theorem to prove non-exoticity for generalised block fusion systems of special linear groups, achieving an alternative proof of Theorem 1.9 in this case. We will see that these methods only work for type A.
The proof of Theorem 5.15 follows some of the steps as in Section 5.2. We give all these steps again for convenience.

Proof of Theorem 5.15. Define a map $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{G}^F, b) \to \mathcal{F}_P(\widetilde{H}), (Q, e_Q) \mapsto Q$ for $Q \leq P$. By Theorem 2.6, it suffices to prove $\mathrm{Aut}_{\mathcal{F}_{(P,e_P)}(\mathbf{N}^F,\mathbf{G}^F,b)}(Q) = \mathrm{Aut}_{\mathcal{F}_P(\widetilde{H})}(Q)$, which is equivalent to proving $N_{\mathbf{N}^F}(Q, e_Q)/C_{\mathbf{N}^F}(Q, e_Q) = N_{\widetilde{H}}(Q)/C_{\widetilde{H}}(Q)$. Thus, we need to prove $N_{\mathbf{N}^F}(Q, e_Q) = N_{\widetilde{H}}(Q)C_{\mathbf{N}^F}(Q, e_Q)$, since $\widetilde{H} \leq \mathbf{N}^F$.
Since $N_{\mathbf{G}^F}(Q, e_Q)$ acts by algebraic automorphisms of $C^\circ_{\mathbf{G}}(Q)$ commuting with F, and $\mathbf{N}^F$ acts on $\mathbf{G}$ by automorphisms, $N_{\mathbf{N}^F}(Q, e_Q)$ normalises $([\mathbf{L},\mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta)$. Together with our assumption and [12, Proposition 5] we get $N_{\mathbf{N}^F}(Q, e_Q) \leq \widetilde{H}[\mathbf{L},\mathbf{L}]^F \leq$

$\widetilde{H}C_{\mathbf{G}^F}(Q) = \widetilde{H}C_{\mathbf{G}^F}(Q, e_Q) \leq \widetilde{H}C_{\mathbf{N}^F}(Q, e_Q)$. Now let $g \in N_{\mathbf{N}^F}(Q, e_Q)$ and write $g = ht$ for $h \in \widetilde{H}, t \in C_{\mathbf{N}^F}(Q, e_Q)$. Note that $t \in C_{\mathbf{N}^F}(Q, e_Q) \leq N_{\mathbf{N}^F}(Q)$ and thus $h = gt^{-1} \in N_{\mathbf{N}^F}(Q) \cap \widetilde{H} = N_{\widetilde{H}}(Q)$ and so $g \in N_{\widetilde{H}}(Q)C_{\mathbf{N}^F}(Q, e_Q)$.

For the other inclusion, we use the decomposition $\mathbf{G} = \mathbf{G_a}\mathbf{G_b}$, associated to a pair $(\mathbf{G}, F)$, and define $\mathbf{K} = \mathbf{G_a}C^\circ_{\mathbf{G}}(Z(P))$. We have $\mathcal{E}(\mathbf{K}^F, 1) \cong \mathcal{E}(\mathbf{K}^F_{\mathbf{a}}, 1) \times \mathcal{E}(\mathbf{L}^F, 1)$, see [19, Proposition 13.20]. Let $\widetilde{\zeta} \in \mathcal{E}(\mathbf{K}^F, 1)$ be the character corresponding to $(1, \zeta)$ under this bijection and define $\zeta_Q = \mathrm{Res}^{\mathbf{K}^F}_{C_{\mathbf{G}^F}(Q)} \widetilde{\zeta}$. This character is the canonical character of e_Q since it has $Z(Q)$ in its kernel. Moreover, $C^\circ_{\mathbf{G}}(Q) = C^\circ_{\mathbf{G_a}}(Q)\mathbf{K_b}$. So, $C^\circ_{\mathbf{G}}(Q)_{\mathbf{b}} = \mathbf{K}_b = [\mathbf{L}, \mathbf{L}]$. Thus, the restriction of $\widetilde{\zeta}$ to $C^\circ_{\mathbf{G}}(Q)^F$ is a unipotent character, it is the unique one whose restriction to $[\mathbf{L}, \mathbf{L}]^F$ is the restriction of ζ. We thus get that $\mathrm{Res}^{C_{\mathbf{G}^F}(Q)}_{C^\circ_{\mathbf{G}}(Q)^F} \zeta_Q$ is the only unipotent character $\zeta^\circ_Q \in \mathcal{E}(C^\circ_{\mathbf{G}}(Q)^F, 1)$ such that $\mathrm{Res}^{C^\circ_{\mathbf{G}}(Q)^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta^\circ_Q = \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta$.

Now assume $h \in \widetilde{H}$ normalises Q and thus by assumption also $([\mathbf{L}, \mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$. Furthermore, h normalises $C^\circ_{\mathbf{G}}(Q)$ and sends $\mathrm{Res}^{C_{\mathbf{G}}(Q)^F}_{C^\circ_{\mathbf{G}}(Q)^F} \zeta_Q$ to a unipotent character whose restriction to $[\mathbf{L}, \mathbf{L}]^F$ is ${}^h\mathrm{Res}^{C_{\mathbf{G}}(Q)^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta_Q = {}^h\mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta = \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta$ by the above. This means h fixes $\mathrm{Res}^{C_{\mathbf{G}}(Q)^F}_{C^\circ_{\mathbf{G}}(Q)^F} \zeta_Q$. By Lemmas 5.12 and 2.35, e_Q is the unique block covering $e_{C^\circ_{\mathbf{G}}(Q)^F}(\mathrm{Res}^{C_{\mathbf{G}}(Q)^F}_{C^\circ_{\mathbf{G}}(Q)^F} \zeta_Q)$. Thus, e_Q is fixed by h which implies $N_{\widetilde{H}}(Q) \leq N_{\mathbf{N}^F}(Q, e_Q)$. □

Before specialising to special linear groups, we recall the definition of control subgroups.

Definition 5.16. *Let $\mathcal{F}$ be a fusion system on a p-group P. We call a group H with $P \leq H$ control group for $\mathcal{F}$ if every homomorphism in $\mathcal{F}$ is induced by an element in H. If $\mathcal{F}$ is the fusion system of a block b, we call H b-control subgroup.*

For fusion systems $\mathcal{F}$ that are induced by a group G, or a block b of a group G, we are often interested in groups $H \leq G$ that are control groups for $\mathcal{F}$ and call these control subgroups or control b-subgroups.

In [12], Cabanes and Enguehard construct such a subgroup for a unipotent block of a finite group of Lie type $\mathbf{G}^F$. Cabanes later proved in [10] that the fusion system of this group realises the fusion system of this unipotent block. Note that the $\widetilde{H}$ constructed in the previous section is not a subgroup of $\mathbf{N}^F$ like this, but we construct such a control subgroup for some cases in this chapter.

Lemma 5.17. *Let $\varphi : G \to H$ be a surjective morphism of affine algebraic groups. Suppose that H and $\ker\varphi$ are connected, then G is connected.*

Proof. By [27, 7.4, Proposition B], $\varphi(G^\circ) = \varphi(G)^\circ = H^\circ = H$. Thus $G = G^\circ \ker\varphi$ and $G/G^\circ = \ker\varphi \big/ (\ker\varphi \cap G^\circ)$. By, [27, 7.3, Proposition], G/G° is a finite group. So $\ker\varphi \cap G^\circ$ is of finite index in $\ker\varphi$ and $\ker\varphi \cap G^\circ$ is a closed subgroup of $\ker\varphi$. Hence again by [27, 7.3, Proposition], $\ker\varphi \cap G^\circ$ contains $(\ker\varphi)^\circ = \ker\varphi$. So, $\ker\varphi \subseteq G^\circ$, hence $G = G^\circ \ker\varphi = G^\circ$. $\square$

Corollary 5.18. *Let $X = \{(A, B) \mid A = \alpha I_r, B \in \mathrm{GL}_{m-r}(\overline{\mathbb{F}}_q), \alpha \in \overline{\mathbb{F}}_q^\times, \det(AB) = 1\} \subseteq \mathrm{GL}_{m-r}(\overline{\mathbb{F}}_q) \times \mathrm{GL}_r(\overline{\mathbb{F}}_q) \subseteq \mathrm{GL}_m(\overline{\mathbb{F}}_q)$. Then X is connected.*

Proof. Define the map $d : X \to \overline{\mathbb{F}}_q^\times$ by $d(A, B) = \det B$. Then d is surjective, $\ker d = \{(I_r, B) \mid B \in \mathrm{SL}_{m-r}(\overline{\mathbb{F}}_q)\}$ is connected and $\overline{\mathbb{F}}_q^\times$ is connected. By the previous lemma, X is connected. $\square$

We specialise now to the case of special linear groups. Fix the following notation for the rest of this section. Let $\mathbf{X} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and $\mathbf{G} = \prod_{i=1}^{k} \mathrm{GL}_{m_i}(\overline{\mathbb{F}}_q)) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, where $\sum_{i=1}^{k} m_i = n$, F the standard Frobenius and let p be an odd prime which is good for $\mathbf{G}$ with $(p, q) = 1$. Then $\mathbf{G}$ is a Levi subgroup of $\mathbf{X}$. Let b be a unipotent block of $k\mathbf{G}^F$ with $b = b_{\mathbf{G}^F}(\mathbf{L}, \zeta)$ for an e-cuspidal pair $(\mathbf{L}, \zeta)$ of $\mathbf{G}^F$. Further, let $\mathbf{G} \trianglelefteq \mathbf{N}$ be a BDR-extension in $\mathbf{X}$.

Theorem 5.19. *The group $\mathbf{T}_{p'}^F C_{\mathbf{G}}([\mathbf{L}, \mathbf{L}])^F$ is a b-control subgroup of $\mathbf{G}^F$, where $\mathbf{T}$ is a maximally split torus of $[\mathbf{L}, \mathbf{L}] C_{\mathbf{G}}^\circ([\mathbf{L}, \mathbf{L}])$.*

Proof. In [12, Theorem 7], a group H, which is a b-control subgroup of $\mathbf{G}^F$, is constructed. It is shown there that one can define $H = (N_V^F) S C_{\mathbf{G}}^\circ([\mathbf{L}, \mathbf{L}])^F$. We recall the groups that make up this control subgroup. Let $\mathbf{K} = [\mathbf{L},\mathbf{L}]\ C_{\mathbf{G}}([\mathbf{L},\mathbf{L}])$. The group V is the stabiliser of an F-stable basis of the root system of $\mathbf{K}$ with respect to $\mathbf{T}$ in $N_M(\mathbf{T})/\mathbf{T}^F$, where $M = N_{\mathbf{G}^F}([\mathbf{L},\mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F}\zeta)$. Now N is a subgroup of $N_{\mathbf{G}}(\mathbf{T})$ such that $N \cap \mathbf{T} = \mathbf{T}_{\phi_2}$ and $N\mathbf{T} = N_{\mathbf{G}}(\mathbf{T})$. The group N_V is defined to be the inverse image of V in N under the quotient map by $\mathbf{T}$. The group $S' \leq \mathbf{T}^F_{p'}$ is characterised by the equation $\mathbf{T}^F = S'(\mathbf{T} \cap [\mathbf{L},\mathbf{L}]^F C_{\mathbf{G}}([\mathbf{L},\mathbf{L}])^F)$. Finally, define $S = S'[S',V]$. We analyse these groups which make up H under our assumptions.

Let $\widetilde{\mathbf{G}} = \prod_{i=1}^{k} \mathrm{GL}_{m_i}(\overline{\mathbb{F}}_q)$, then, by [26, 6.2], $(Z(\widetilde{\mathbf{G}})\mathbf{L},\zeta)$ is a unipotent e-cuspidal pair for $\widetilde{\mathbf{G}}$. Let $\widetilde{\mathbf{L}} = Z(\widetilde{\mathbf{G}})\mathbf{L}$, then again by [26, 6.2] we can choose $\widetilde{\mathbf{L}} = (\prod_{i=1}^{k} \mathrm{GL}_{s_i}(\overline{\mathbb{F}}_q) \times R_i)$, for tori R_i with $|R_i^F| = (q^e-1)^{a_i}$ such that $s_i + ea_i = m_i$. Now $\mathbf{L} = \widetilde{\mathbf{L}} \cap \mathbf{G} = (\prod_{i=1}^{k} \mathrm{GL}_{s_i}(\overline{\mathbb{F}}_q) \times R_i) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, $[\mathbf{L},\mathbf{L}] = (\prod_{i=1}^{k} \mathrm{SL}_{s_i}(\overline{\mathbb{F}}_q) \times 1)$ and $C^\circ_{\mathbf{G}}([\mathbf{L},\mathbf{L}]) = C_{\mathbf{G}}([\mathbf{L},\mathbf{L}]) = (\prod_{i=1}^{k} \overline{\mathbb{F}}_q^\times \times \mathrm{GL}_{m_i-s_i}(\overline{\mathbb{F}}_q)) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ by the previous corollary.

We claim $\mathbf{K} = (\prod_{i=1}^{k}(\mathrm{GL}_{s_i}(\overline{\mathbb{F}}_q) \times \mathrm{GL}_{m_i-s_i}(\overline{\mathbb{F}}_q))) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$. Clearly, $\mathbf{K} \subseteq (\prod_{i=1}^{k}(\mathrm{GL}_{s_i}(\overline{\mathbb{F}}_q) \times \mathrm{GL}_{m_i-s_i}(\overline{\mathbb{F}}_q))) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$. For the other inclusion, let $(A_1,B_1) \times \cdots \times (A_k,B_k) \in (\prod_{i=1}^{k}(\mathrm{GL}_{s_i}(\overline{\mathbb{F}}_q) \times \mathrm{GL}_{m_i-s_i}(\overline{\mathbb{F}}_q))) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$. If $a_i = \det A_i$ and $\alpha_i = a_i^{1/s}$, then $A_i = A_{i_0}Z_i$ for $Z_i = \alpha_i I_{s_i}$ and A_{i_0} is a matrix with determinant 1, i.e. $A_{1_0} \times \cdots \times A_{k_0} \in [\mathbf{L},\mathbf{L}]$. Furthermore, $\det(Z_1B_1 \times \cdots \times Z_kB_k) = 1$, so $Z_1B_1 \times \cdots \times Z_kB_k \in C_{\mathbf{G}}([\mathbf{L},\mathbf{L}])$, which proves the equality.

Since $N_{\mathbf{G}}([\mathbf{L},\mathbf{L}])$ must normalise $C_{\mathbf{G}}([\mathbf{L},\mathbf{L}])$, we have $M \subseteq N_{\mathbf{G}}([\mathbf{L},\mathbf{L}]C_{\mathbf{G}}([\mathbf{L},\mathbf{L}]))$. By the previous arguments, we get $M \subseteq (\prod_{i=1}^{k}(\mathrm{GL}_{s_i}(\overline{\mathbb{F}}_q) \times \mathrm{GL}_{m_i-s_i}(\overline{\mathbb{F}}_q))) \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$. In our case we can choose the root system given by the basis $\bigcup_{i=1}^{k}\{(e_{M_i+i} - e_{M_i+i+1}), \cdots, (e_{M_i+s_i} - e_{M_i+s_i}), (e_{M_i+s_i+1} - e_{M_i+s_i+2}), \cdots, (e_{M_i+m_i-1} - e_{M_i+m_i})\}$, for $M_i = m_1 + \cdots + m_{i-1}$. This basis has trivial stabiliser and thus $V = 1$. Note that we may assume $\mathbf{T}_{\phi_2} = 1$, since we can choose $\mathbf{T}$ to be the torus consisting of diagonal matrices in $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$. Since V is trivial, we have $N_V \subseteq N \cap \mathbf{T} = \mathbf{T}_{\phi_2} = 1$. This means, we may assume that $SC_{\mathbf{G}}([\mathbf{L},\mathbf{L}])^F$ controls fusion.

Since $V = 1$, we obtain $S = S'$. The equation clearly holds if we choose $S = S' = \mathbf{T}^F_{p'}$. This choice implies our claim about the control subgroup. $\square$

Proposition 5.20. *The group $\mathbf{N}^F$ normalises the control subgroup $\mathbf{T}^F_{p'} C_{\mathbf{G}}([\mathbf{L}, \mathbf{L}])^F$ from the previous theorem.*

Proof. The group $\mathbf{G}^F$ acts transitively on all unipotent e-cuspidal pairs for b. By Theorem 3.1, we can assume that b is $\mathbf{N}^F$-stable, thus $\mathbf{N}^F$ acts on the set of unipotent e-cuspidal pairs for b. Now we can apply Frattini's argument to see that $\mathbf{N}^F = \mathbf{G}^F \operatorname{Stab}_{\mathbf{N}^F}((\mathbf{L}, \zeta))$. Now note that $\mathbf{T} \leq [\mathbf{L}, \mathbf{L}] C_{\mathbf{G}}([\mathbf{L}, \mathbf{L}])$ is normalised by $\operatorname{Stab}_{\mathbf{N}^F}((\mathbf{L}, \zeta))$. Furthermore, note that all 1-split F-stable maximal tori of $\mathbf{K}$ are $\mathbf{K}^F$-conjugate. Again, by Frattini's argument, we obtain $\operatorname{Stab}_{\mathbf{N}^F}((\mathbf{L}, \zeta)) = \mathbf{T}^F \operatorname{Stab}_{\mathbf{N}^F}(\mathbf{T})$.
In conclusion, we get $\mathbf{N}^F = \mathbf{G}^F \operatorname{Stab}_{\operatorname{Stab}_{\mathbf{N}^F}((\mathbf{L},\zeta))}(\mathbf{T}) = \mathbf{G}^F \operatorname{Stab}_{\mathbf{N}^F}((\mathbf{L}, \zeta) \cap \operatorname{Stab}_{\mathbf{N}^F}(\mathbf{T}))$ which implies our claim. $\square$

Keep the assumptions of Proposition 5.20 for the rest of this chapter and let x be a generator of $\mathbf{N}^F/\mathbf{G}^F$ and define $\widetilde{H} = \langle \mathbf{T}^F_{p'} C_{\mathbf{G}}([\mathbf{L}, \mathbf{L}])^F, x\rangle$.

Lemma 5.21. *We have $\widetilde{H}[\mathbf{L}, \mathbf{L}]^F = N_{\mathbf{N}^F}([\mathbf{L}, \mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$.*

Proof. The proof of the previous proposition gives $\mathbf{N}^F = \mathbf{G}^F \operatorname{Stab}_{\mathbf{N}^F}((L, \zeta) \cap \operatorname{Stab}_{\mathbf{N}^F}(\mathbf{T}))$. Thus, $\mathbf{N}^F/\mathbf{G}^F = \operatorname{Stab}_{\mathbf{N}^F}((L, \zeta) \cap \operatorname{Stab}_{\mathbf{N}^F}(\mathbf{T})) \Big/ (\mathbf{G}^F \cap \operatorname{Stab}_{\mathbf{N}^F}((L, \zeta) \cap \operatorname{Stab}_{\mathbf{N}^F}(\mathbf{T})))$. So we can choose x generating $\mathbf{N}^F/\mathbf{G}^F$ to be in $\operatorname{Stab}_{\mathbf{N}^F}((L, \zeta) \cap \operatorname{Stab}_{\mathbf{N}^F}(\mathbf{T}))$. In particular, this x stabilises $([\mathbf{L}, \mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$ and thus we have the equality $N_{\mathbf{N}^F}([\mathbf{L}, \mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$ $= N_{\mathbf{G}^F}([\mathbf{L}, \mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)\langle x\rangle$. Now by [12, Theorem 7] we know $\mathbf{T}^F_{p'} C_{\mathbf{G}}([\mathbf{L}, \mathbf{L}])^F [\mathbf{L}, \mathbf{L}]^F = N_{\mathbf{G}^F}([\mathbf{L}, \mathbf{L}], \operatorname{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$ which implies our claim together with the previous observation.
$\square$

We prove now that $\widetilde{H}$ is a control subgroup for the corresponding BDR-generalised block fusion system.

Proposition 5.22. *Let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{L}^F, b)$, where b is a unipotent block of $\mathbf{G}^F$ defined by a unipotent e-cuspidal pair $(\mathbf{L}, \zeta)$ with maximal $(b, \mathbf{N}^F)$-Brauer pair (P, e_P). We have $P \in \mathrm{Syl}_p(\widetilde{H})$ and $\widetilde{H}$ is a control subgroup for $\mathcal{F}$.*

Proof. We have that P is a Sylow p-subgroup of $C_{\mathbf{N}}([\mathbf{L}, \mathbf{L}])^F$ since it is one of $C_{\mathbf{G}}([\mathbf{L}, \mathbf{L}])^F$ by [12, Proposition 5] and we can choose the p'-element x generating $\mathbf{N}^F/\mathbf{G}^F$ to stabilise $[\mathbf{L}, \mathbf{L}]$ in similar fashion as in Lemma 5.21. Let (Q, e_Q) be an $\mathcal{F}$-centric Brauer pair with $Q \leq P$. We need to prove $N_{\mathbf{N}^F}(Q, e_Q) \leq \widetilde{H} C_{\mathbf{N}^F}(Q)$. Since $N_{\mathbf{G}^F}(Q, e_Q)$ acts by algebraic automorphisms of $C^\circ_{\mathbf{G}}(Q)$ commuting with F, and $\mathbf{N}^F$ acts on $\mathbf{G}$ by automorphisms, we have that $N_{\mathbf{N}^F}(Q, e_Q)$ normalises the pair $([\mathbf{L}, \mathbf{L}], \mathrm{Res}^{\mathbf{L}^F}_{[\mathbf{L},\mathbf{L}]^F} \zeta)$. Together with Lemma 5.21 and [12, Proposition 5] we get $N_{\mathbf{N}^F}(Q, e_Q) \leq \widetilde{H}[\mathbf{L}, \mathbf{L}]^F \leq \widetilde{H} C_{\mathbf{G}^F}(Q) = \widetilde{H} C_{\mathbf{G}^F}(Q, e_Q) \leq \widetilde{H} C_{\mathbf{N}^F}(Q, e_Q)$. □

We can apply this proposition together with the main result of this section to prove non-exoticity of the generalised block fusion system.

Corollary 5.23. *Let (P, e_P) be a maximal $(b, \mathbf{N}^F)$-Brauer pair. The BDR-generalised block fusion system $\mathcal{F}_{(P,e_P)}(\mathbf{N}^F, \mathbf{L}^F, b)$ is non-exotic.*

Proof. Follows immediately from combining Lemma 5.21, Proposition 5.22 and Theorem 5.15. □

It is not clear currently how to acquire a similar result when dropping the assumption that the group is of type A, since in this case it is not clear whether a basis of the root system in the proof of Theorem 5.19 has trivial stabiliser. We would need to apply Frattini-esque arguments to the group N, which boils down to the uniqueness up to conjugacy of the Tits extension, see [50, 4.4], which is not answered. However, the proof of Theorem 1.9 circumvents using control subgroups and Tits' theorem.

5.4 Non-exoticity of BDR block fusion system in special cases

In many cases for type A, Theorem 5.6 easily implies the non-exoticity of fusion systems of blocks:

Theorem 5.24. *If $p \geq 7$, q a prime power coprime to p, $n \in \mathbb{N}_{\geq 2}$ and G one of $\mathrm{GL}_n(q), \mathrm{PGL}_n(q), \mathrm{GU}_n(q)$ or $\mathrm{PGU}_n(q)$, then fusion systems of blocks of kG are non-exotic.*

Proof. For a connected, reductive group $\mathbf{G}$, with Steinberg morphism F and a block d of $\mathbf{G}^F$, Theorem 5.9 gives an equivalence between the fusion system of d and the fusion system of a block of the group $\mathbf{N}^F$, which covers a unipotent block. If $\mathbf{G}^*$ is among $\mathrm{GL}_n(\overline{\mathbb{F}}_q), \mathrm{SL}_n(\overline{\mathbb{F}}_q), \mathrm{GU}_n(\overline{\mathbb{F}}_q)$ or $\mathrm{SU}_n(\overline{\mathbb{F}}_q)$, we have $C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s)$ for a semisimple element s of ${\mathbf{G}^*}^F$, and $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$, where $\mathbf{L}$ is an F-stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$. In particular, in this case $\mathbf{N}^F = \mathbf{L}^F$ and by [5, Example 7.9], we get that for each block b of $\mathbf{G}^F$, there exists a unipotent block of $k\mathbf{L}^F$ which differs from b only by tensoring with a linear character. Thus, combining these observations with Theorem 5.6 and Theorem 2.48 implies the claim. □

Remark 5.25. To apply [5, Example 7.9], one needs cyclicity of the quotient $C := C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$. While the equality $C^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*$ holds for all groups of type A, the relation $C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s)$ does not hold for $\mathbf{G}^* = \mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ or $\mathbf{G}^* = \mathrm{PGU}_n(\overline{\mathbb{F}}_q)$. Nonetheless, the group C is still cyclic, since it is isomorphic to a subgroup of $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ and $Z(\mathbf{G})$ is cyclic, see [19, Lemma 13.14]. However, by these observations we get that $\mathbf{N}^F \neq \mathbf{L}^F$ if $\mathbf{G}$ is equal to $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$ or $\mathrm{SU}_n(\overline{\mathbb{F}}_q)$.

In the following two subsections, we study some special cases in which we can also prove non-exoticity for block fusion systems of special linear groups. We can only prove these cases under very specific assumptions, which provides some further evidence that it might be better to work with Conjecture 1.5 rather than Conjecture 1.1.

We recall some definitions and results we need for these special cases.

Definition 5.26. *Let $\mathbf{G}$ be a connected reductive group and s a semisimple element in $\mathbf{G}$.*

(a) The element s is called $\mathbf{G}$-quasi-isolated, if $C_{\mathbf{G}}(s)$ is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$.

(b) We define $A_{\mathbf{G}}(s)$ to be the quotient $C_{\mathbf{G}}(s)/C^{\circ}_{\mathbf{G}}(s)$.

Proposition 5.27. *Let $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ with Levi subgroup $\mathbf{L}$. If s is $\mathbf{G}^*$-quasi-isolated, there is a BDR extension $\mathbf{N}$ in $\mathbf{G}$, such that $\mathbf{N}^F/\mathbf{L}^F \cong C_r$ for some $r \mid n$.*

Proof. In [4, Proposition 5.2], it is shown that $A_{\mathbf{G}^*}(s) \cong C_k$ for some $k \mid n$.
Define $\mathbf{N}^* = C_{\mathbf{G}^*}(s)^F\mathbf{L}^*$, then this is a BDR extension, and we get the equality ${\mathbf{N}^*}^F/{\mathbf{L}^*}^F = C_{\mathbf{G}^*}(s)^F/C^{\circ}_{\mathbf{G}^*}(s)^F$. The latter group is a subgroup of $A_{\mathbf{G}^*}(s)$ and thus there is a $r \mid k$ such that $\mathbf{N}^F/\mathbf{L}^F \cong C_r$. Since r divides n, the result follows. □

Definition 5.28. *Let G be a finite group, b a block of kG and P a defect group of b. We say that b is of principal type if $\mathrm{Br}^G_Q(b)$ is a block of $kC_G(Q)$ for every $Q \leq P$.*

5.4.1 Structural assumptions on Levi

We prove the non-exoticity of the BDR block fusion system in a special case.
Fix the following notation for this section. Let $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, F the standard Frobenius such that $(\mathbf{G}, F)$ is in duality with $(\mathbf{G}^*, F)$. Let s a $\mathbf{G}^*$-quasi-isolated semisimple element and $\mathbf{L}^* = C^{\circ}_{\mathbf{G}^*}(s)$. By Theorem 5.9, the fusion system of a block d of $k\mathbf{G}^F$ with $\mathrm{Irr}(\mathbf{G}^F, d) \cap \mathcal{E}(\mathbf{G}^F, s) \neq \varnothing$ is equivalent to the fusion system of a block of $k\mathbf{N}^F$ covering a block of $k\mathbf{L}^F$ in $\mathcal{E}(\mathbf{L}^F, s)$ for a BDR extension $\mathbf{N}$ in $\mathbf{G}$. Since s is central in $\mathbf{L}^*$, the fusion system of d is equivalent to the fusion system of a block c of $\mathbf{N}^F$ covering a unipotent block b of $\mathbf{L}^F$, see also [5, Example 7.9]. Thus, we can prove that d induces a non-exotic fusion system by proving that the BDR block fusion system of c is non-exotic. Denote

the BDR block fusion system of c by $\mathcal{H}$ and the generalised BDR block fusion system of b and the extension $\mathbf{L}^F \trianglelefteq \mathbf{N}^F$ by $\mathcal{G}$. Let $\widetilde{\mathbf{L}} \subseteq \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ such that $\widetilde{\mathbf{L}} \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q) = \mathbf{L}$

Theorem 5.29. *Let $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, F the standard Frobenius and s a $\mathbf{G}^*$-quasi-isolated semisimple element. Assume $A_{\mathbf{G}^*}(s) = \mathbf{N}^F/\mathbf{L}^F$ and thus $\widetilde{\mathbf{L}}^F = \prod_{i=1}^{k/d} \mathrm{GL}_m(q^d)$. If we furthermore assume that $d = 1$ and $\widetilde{\mathbf{L}}^F \cap \mathbf{G} = \widetilde{\mathbf{L}}^F$, then the BDR block fusion system $\mathcal{H}$ and the BDR generalised block fusion system $\mathcal{G}$ coincide. In particular, $\mathcal{H}$ is non-exotic.*

To prove the result in the theorem, one needs a series of intermediate results, that are dealt with in the following five lemmas and propositions.

Lemma 5.30. *Let $\pi : \widetilde{\mathbf{G}} \twoheadrightarrow \mathbf{G}^*$ be the canonical projection, then we have $\mathbf{L}^{*F} = \pi(C_{\widetilde{\mathbf{G}}}(\widetilde{s})^F)$.*

Proof. Note $\mathbf{L}^* = C_{\mathbf{G}^*}(s)^\circ$ since we are in type A_n. Since $\pi(\widetilde{\mathbf{G}}^F) \subseteq \mathbf{G}^{*F}$, we have $\pi(C_{\widetilde{\mathbf{G}}}(\widetilde{s})^F) \subseteq (\pi(C_{\widetilde{\mathbf{G}}}(\widetilde{s})))^F = (C_{\mathbf{G}^*}(s)^\circ)^F = \mathbf{L}^{*F}$.
Suppose $y \in (C_{\mathbf{G}^*}(s)^\circ)^F$. Let $\widetilde{y} \in C_{\widetilde{\mathbf{G}}}(\widetilde{s})$ with $\pi(\widetilde{y}) = y$. We have $\pi(F(\widetilde{y})) = F(\pi(\widetilde{y})) = F(y) = y = \pi(\widetilde{y})$, so $\widetilde{y}^{-1}F(\widetilde{y}) \in \ker \pi = Z(\widetilde{\mathbf{G}})$. Now $Z(\widetilde{\mathbf{G}})$ is connected. By the Lang–Steinberg theorem, see [19, 3.10], we have $\widetilde{y}^{-1}F(\widetilde{y}) = z^{-1}F(z)$ for some $z \in Z(\widetilde{\mathbf{G}})$. We thus get $\widetilde{y}z^{-1} \in \widetilde{\mathbf{G}}^F$ with $\pi(\widetilde{y}z^{-1}) = \pi(\widetilde{y}) = y$. $\square$

Proposition 5.31. *Let s be $\mathbf{G}^*$-quasi-isolated and $|A_{\mathbf{G}^*}(s)| = k$. We have $\widetilde{\mathbf{L}}^F = \prod_{i=1}^{t} \mathrm{GL}_m(q^{d_i})$ for some $1 \leq t, d_i \leq k$ such that $\sum_{i=1}^{t} d_i = k$ and $mk = n$.*

Proof. Since s is quasi-isolated, by [4, Proposition 5.2] we have $s =^{\mathbf{G}^*} \pi(\widetilde{s})$ where $\widetilde{s} = I_m \otimes J_k$, $J_k = \mathrm{diag}(1, \zeta, \ldots, \zeta^{k-1})$ for a primitive k-th root of unity ζ and $mk = n$. It is easy to see that $C_{\widetilde{\mathbf{G}}}(\widetilde{s}) = \prod_{i=1}^{k} \mathrm{GL}_m(\overline{\mathbb{F}}_q)$. Note that $\widetilde{\mathbf{G}} = \widetilde{\mathbf{G}}^*$. Let $\mathbf{L} = C_{\widetilde{\mathbf{G}}}(\widetilde{s}) \cap \mathbf{G}$. If $\widetilde{\mathbf{L}}^* = C_{\widetilde{\mathbf{G}}^*}(\widetilde{s})$, we have $\mathbf{L}^* = C^\circ_{\mathbf{G}^*}(s) = \pi(\widetilde{\mathbf{L}}^*)$. Let $\mathbf{M} := \mathbf{L}Z(\widetilde{\mathbf{G}})$. Then $\mathbf{M}$ is a Levi of $\widetilde{\mathbf{G}}$ corresponding to $\widetilde{\mathbf{L}}^*$ under the identification of $\widetilde{\mathbf{G}}^*$ and $\widetilde{\mathbf{G}}$. Now let $\widetilde{\mathbf{L}} = C_{\widetilde{\mathbf{G}}}(\widetilde{s})$ and write $\widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}_1 \times \widetilde{\mathbf{L}}_2 \times \cdots \times \widetilde{\mathbf{L}}_k$, where $\widetilde{\mathbf{L}}_i = \mathrm{GL}_m(\overline{\mathbb{F}}_q)$ for $1 \leq i \leq k$, $mk = n$.
Since $\widetilde{\mathbf{L}}$ is F-stable, F acts on $\widetilde{\mathbf{L}}$ by permuting the factors $\widetilde{\mathbf{L}}_1, \ldots, \widetilde{\mathbf{L}}_k$ i.e. for all i we

have $F(\widetilde{\mathbf{L}}_i) = \widetilde{\mathbf{L}}_j$ for some $1 \leq j \leq k$. Let $O_1, \ldots, O_t$ be the orbits of the F-action on $\{\widetilde{\mathbf{L}}_1, \ldots, \widetilde{\mathbf{L}}_k\}$. If $d_i = |O_i|$ for $1 \leq i \leq t$, then $\widetilde{\mathbf{L}}^F = \prod_{i=1}^{t} \mathrm{GL}_m(q^{d_i})$. $\square$

Lemma 5.32. *Let s be $\mathbf{G}^*$-quasi-isolated, $r = |\mathbf{N}^F/\mathbf{L}^F|, k = |A_{\mathbf{G}^*}(s)|$. We have natural maps $\mathbf{N}^F/\mathbf{L}^F \hookrightarrow A_{\mathbf{G}^*}(s) \to \mathfrak{S}_k$ given by the action of these groups on the components of $\widetilde{\mathbf{L}}$. If T_0 is the image of $\mathbf{N}^F/\mathbf{L}^F$ in $\mathfrak{S}_k$, then $C_{\mathfrak{S}_k}(T_0) = C_r \wr C_{k_0}$ with $k = rk_0$.*

Proof. Since s is quasi-isolated, we get an action of $A_{\mathbf{G}^*}(s)$ on$\{\widetilde{\mathbf{L}}_1, \ldots, \widetilde{\mathbf{L}}_k\}$ induced by permutation of the eigenvalues, see [4, Proposition 5.2]. This proposition also implies that if $T \leq \mathfrak{S}_k$ is the image of $A_{\mathbf{G}^*}(s)$ under the homomorphism $A_{\mathbf{G}^*}(s) \to \mathfrak{S}_k$, we get $T = \langle (1, 2, \ldots, k) \rangle$. If $T_0 \leq T$ is the image of $\mathbf{N}^F/\mathbf{L}^F$ in $\mathfrak{S}_k$, then it is generated (up to reordering of the indices $1, 2, \ldots, k$) by the permutation $(1, 2, \ldots, r)(r + 1, \ldots, 2r) \ldots ((k_0 - 1)r + 1, \ldots, k)$ for some k_0 such that $k = rk_0$. Let $\tau = (1, 2, \ldots, r)(r + 1, \ldots, 2r) \ldots ((k_0 - 1)r + 1, \ldots, k)$. Then $C_{\mathfrak{S}_k}(\tau) = (\langle (1, 2, \ldots, r) \rangle \times \langle (r + 1, \ldots, 2r) \rangle \times \cdots \times \langle ((k_0 - 1)r + 1, \ldots, k) \rangle)H$, where $H = \langle (1, r + 1, \ldots, (k_0 - 1)r + 1)(2, r + 2, \ldots, (k_0 - 1)r + 2) \ldots (r, 2r, \ldots, k_0 r) \rangle \cong C_{k_0}$. In other words, $C_{\mathfrak{S}_k}(\tau) \cong C_r \wr C_{k_0}$, see also [38, Theorem 2]. $\square$

We now specialise Lemma 5.31 to the case where there is no gap between $\mathbf{N}^F/\mathbf{L}^F$ and $A_{\mathbf{G}^*}(s)$, in which case all the d_i coincide.

Proposition 5.33. *If s is $\mathbf{G}^*$-quasi-isolated and $\mathbf{N}^F/\mathbf{L}^F = A_{\mathbf{G}^*}(s) \cong C_k$, then $\widetilde{\mathbf{L}}^F = \prod_{i=1}^{k/d} \mathrm{GL}_m(q^d)$ for some $d \mid k$ and $mk = n$.*

Proof. Let $\sigma, \tau \in \mathfrak{S}_k$ correspond to the action of F respectively $\mathbf{N}^F/\mathbf{L}^F$ on $\{\widetilde{\mathbf{L}}_1, \ldots, \widetilde{\mathbf{L}}_k\}$. Since the action of F and $\mathbf{N}^F/\mathbf{L}^F$ on $\{\widetilde{\mathbf{L}}_1, \ldots, \widetilde{\mathbf{L}}_k\}$ commute, $\sigma \in C_{\mathfrak{S}_k}(\tau)$. So by the previous lemma, we have $\sigma = (1, 2, \ldots, r)^{i_1}(r + 1, \ldots, 2r)^{i_2} \ldots ((k_0 - 1)r + 1, \ldots, k)^{i_{k_0}} h^j$ where $h = (1, r + 1, \ldots, (k_0 - 1)r + 1)(2, r + 2, \ldots, (k_0 - 1)r + 2) \ldots (r, 2r, \ldots, k)$ and $0 \leq i_j \leq r - 1, 0 \leq j \leq k_0 - 1$.

First assume $j = 0$. In this case, each of the sets $\{1, 2, \ldots, r\}, \{r + 1, \ldots, 2r\}, \ldots, \{(k_0 -$

$1)r+1,\dots,k\}$ is a union of $\langle\sigma\rangle$-orbits and the orbits of $\langle\sigma\rangle$ on $\{i+1,\dots,i+r\}$ for any $i\in\{1,r+1,\dots,(k_0-1)r+1\}$ are of equal length, say d_i and there are r/d_i of these. So we obtain $C_{\widetilde{\mathbf{G}}}(\widetilde{s})^F=\prod_{i\in\{1,r+1,\dots,(k_0-1)r+1\}}\mathrm{GL}_m(q^{d_i})^{r/d_i}$ and $\langle\tau\rangle$ transitively permutes the r/d_i factors $\mathrm{GL}_m(q^{d_i})$ for each i.

The assumption $\mathbf{N}^F/\mathbf{L}^F=A_{\mathbf{G}^*}(s)$ implies $\tau=(1,2,\dots,k)$ and thus $C_{\mathfrak{S}_k}(\tau)\cong C_k$ and so indeed $j=0$. We furthermore have $k_0=1$ which implies the claim about $\widetilde{\mathbf{L}}^F$ with the above. $\square$

Proposition 5.34. *Let $N\trianglelefteq G$ and $S=S_1\times\ldots\times S_k\leq N$. Assume that $g\in N_G(S)$ acts on S such that ${}^gS_i=S_j$ and furthermore that if $u\in S_i$ and $n\in N$ are such that ${}^nu\in S$, then ${}^nu\in S_i$. Define $\varphi:N_G(S)\to\mathfrak{S}_k$ by the action of $N_G(S)$ on the components of S. Suppose that $N_N(S)=\ker\varphi$. Then for all $Q\leq S$ such that $C_S(Q)\subseteq Q$, we have that $C_G(Q)=C_N(Q)$.*

Proof. Since $C_S(Q)\subseteq Q$, we have $Z(S_i)\subseteq Q$ for all $1\leq i\leq k$. If $g\in C_G(Q)$, we can use Frattini's argument to write it as $g=nx$ for $n\in N,x\in N_G(S)$. Let $1\neq u_i\in Z(S_i)$, then $u_i={}^gu_i={}^{nx}u_i$. We have ${}^xu_i\in S_{\varphi(x)(i)}$. By assumption, we get $\varphi(x)(i)=i$ for every i, which means $x\in\ker(\varphi)\subseteq N$ and thus $g\in C_N(Q)$. $\square$

We can now prove the theorem.

Proof of Theorem 5.29. The claim $\widetilde{\mathbf{L}}^F=\prod_{i=1}^{k/d}\mathrm{GL}_m(q^d)$ follows from Proposition 5.33.
Let P be a defect group for b. By Theorem 2.6, it suffices to prove $\mathrm{Aut}_{\mathcal{G}}(Q)=\mathrm{Aut}_{\mathcal{H}}(Q)$ for all subgroups $Q\leq P$ with $C_P(Q)\subseteq Q$. We have $\mathrm{Aut}_{\mathcal{G}}(Q)=N_{\mathbf{N}^F}(Q,e_Q)/QC_{\mathbf{N}^F}(Q,e_Q)$ and $\mathrm{Aut}_{\mathcal{H}}(Q)=N_{\mathbf{N}^F}(Q,f_Q)/QC_{\mathbf{N}^F}(Q)$. If we prove $C_{\mathbf{N}^F}(Q)=C_{\mathbf{L}^F}(Q)$, this implies $e_Q=f_Q$ by Lemma 2.33. In particular, the equality of these centralisers implies our claim.

We have $P\in\mathrm{Syl}_p(C^\circ_{\mathbf{L}}([\mathbf{M},\mathbf{M}])^F)$ for a Levi subgroup $\mathbf{M}\leq\mathbf{L}$. Since $C^\circ_{\mathbf{L}}([\mathbf{M},\mathbf{M}])^F\subseteq\mathbf{L}^F$, we can apply Proposition 5.33 together with our assumption $d=1$ to see that P is of the form as in the previous proposition. Now we have $\mathbf{N}^F/\mathbf{L}^F \cong C_k$ acting by permutation of the components and thus we see that $\ker\varphi = N_{\mathbf{L}^F}(P)$ with the notation from the previous proposition. Thus, the previous proposition with $G = \mathbf{N}^F$ and $N = \mathbf{L}^F$ gives us $C_G(Q) = C_N(Q)$.

The claim about non-exoticity follows from Theorem 1.9. □

5.4.2 Block of principal type and stable defect group

We identify another situation where the different side of the triangle collapses. The following is the main result of this section.

Theorem 5.35. *Let $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, F the Standard Frobenius, $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}^* = C_{\mathbf{G}^*}(s)^\circ$ for some quasi-isolated semisimple $s \in \mathbf{G}^*$ and let $\mathbf{N}$ be a BDR extension in $\mathbf{G}$ corresponding to s. Let b be a unipotent block of $\mathbf{L}^F$ with defect group P. If b is of principal type and every element in P is $\mathbf{N}^F/\mathbf{L}^F$-stable, then the unipotent block fusion system $\mathcal{F}$ of b and the BDR generalised block fusion system $\mathcal{G}$ of b and the extension $\mathbf{L}^F \trianglelefteq \mathbf{N}^F$ coincide.*

First, we prove the following:

Lemma 5.36. *Suppose $\mathbf{G}$ is a connected, reductive group, $F : \mathbf{G} \to \mathbf{G}$ a Steinberg morphism, $s \in \mathbf{G}^F$ semisimple and $\mathbf{G}$-conjugate to t. Suppose that $C_{\mathbf{G}}(s)$ is connected and $w \in \mathbf{G}$ such that ${}^{wF}t = t$. Then there exists $h \in \mathbf{G}$ such that $t = hsh^{-1}$ and $C_{\mathbf{G}^F}(s) = h^{-1}C_{\mathbf{G}^{wF}}(t)h$.*

Proof. Let $t = gsg^{-1}$ for $g \in \mathbf{G}$. We have ${}^{wF}t = t$ if and only if ${}^{wF}(gsg^{-1}) = gsg^{-1}$ which is the case if and only if $w \in gC_{\mathbf{G}}(s)F(g^{-1})$. Since $C_{\mathbf{G}}(s)$ is connected, $w = guF(u^{-1})F(g^{-1}) = guF(g^{-1}u^{-1})$ for some $u \in C_{\mathbf{G}}(s)$. Set $h = gu$. Then $t = gsg^{-1} = gusu^{-1}g^{-1} = hsh^{-1}$. Let $x \in \mathbf{G}$, then $y := hxh^{-1} \in C_{\mathbf{G}}(t)$ and as above ${}^{w}F(y) = y$ if and only if $F(x) = x$. Hence, $C_{\mathbf{G}^F}(s) = h^{-1}C_{\mathbf{G}^{wF}}(t)h$. □

We construct such a t for the quasi-isolated s defining $\mathbf{L}$ and want to use this lemma to construct a BDR extension $\mathbf{N}$ in this situation.

Construction 5.37. *Let $\widetilde{\mathbf{G}} = \widetilde{\mathbf{G}}^* = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and F be the standard Frobenius. Let $\zeta \in \mathbb{F}_q^\times$ be a primitive k-th root of unity and let $\eta \in \overline{\mathbb{F}_q^\times}$ be such that $\eta^{q-1} = \zeta$. Then for all $i \in \mathbb{Z}/k\mathbb{Z}$, $(\eta\zeta^i)^q = \eta\zeta^{i+1}$. In particular, $\{\eta, \eta\zeta, \cdots, \eta\zeta^{k-1}\}$ is an orbit of $\mathbb{F}_q^\times$ under $\alpha \mapsto \alpha^q$. Let $\widetilde{t_0} = I_m \otimes J_k$ where $J_k = \mathrm{diag}(1, \zeta, \cdots, \zeta^{k-1})$. Let $\widetilde{t} = \eta\widetilde{t_0}$ and $t := \pi(\widetilde{t}) = \pi(\widetilde{t_0})$. Then t is quasi-isolated in $\mathbf{G}^*$. Let $w := (1, m+1, \cdots, (k-1)m+1)(2, m+2, \cdots, (k-1)m+2)\cdots(m, 2m, \cdots, km) \in \mathfrak{S}_n \leq \widetilde{\mathbf{G}}^*$. By the above observation, ${}^{wF}\widetilde{t} = \widetilde{t}$.*

Hence, by the previous lemma, there is an $\widetilde{s} \in \widetilde{\mathbf{G}}^{*F}$ such that $\widetilde{s}$ and $\widetilde{t}$ are $\widetilde{\mathbf{G}}$-conjugate, so $\widetilde{s} = \widetilde{h}^{-1}\widetilde{t}\widetilde{h}$ with $w = \widetilde{h}F(\widetilde{h}^{-1})$ for some $\widetilde{h} \in \widetilde{\mathbf{G}}^*$. Also, $C^\circ_{\mathbf{G}^*}(s) = \pi(C_{\widetilde{\mathbf{G}}^*}(\widetilde{s})) = \pi(h)^{-1}\pi(C_{\widetilde{\mathbf{G}}^*}(\widetilde{t}))\pi(h)$, $C^\circ_{\mathbf{G}^*}(s)^F = \pi(h)^{-1}\pi(C_{\widetilde{\mathbf{G}}^{*wF}}(\widetilde{t}))\pi(h)$, $C_{\mathbf{G}^*}(s) = \pi(h^{-1})C_{\mathbf{G}^*}(t)\pi(h)$ and $C_{\mathbf{G}^*}(s)^F = \pi(h^{-1})C_{\mathbf{G}^{*wF}}(t)\pi(h)$.

Lemma 5.38. *If w is as in the previous construction, we have $\widetilde{\mathbf{L}}^{wF} \cong \mathrm{GL}_m(q^k)$ is a Levi of $\widetilde{\mathbf{G}}$ and $\widetilde{\mathbf{N}}^F \cong \mathrm{GL}_m(q^k) \rtimes \langle\sigma\rangle$, where σ is the field automorphism of order k, is a BDR extension in $\widetilde{\mathbf{G}}$.*

Proof. We calculate $\widetilde{\mathbf{L}}$ as in the proof of Proposition 5.31 with t taking the role of a $\widetilde{\mathbf{G}}^*$-quasi-isolated element. Together with the previous construction, we obtain $\widetilde{\mathbf{L}}^* = \widetilde{h}^{-1}C_{\widetilde{\mathbf{G}}^*}(\widetilde{t})\widetilde{h}$, $\widetilde{\mathbf{N}}^* = \widetilde{h}^{-1}(C_{\widetilde{\mathbf{G}}^*}(\widetilde{t})C_{\widetilde{\mathbf{G}}^*}(t)^{wF})\widetilde{h}$ and $\widetilde{\mathbf{N}}^F = \widetilde{h}^{-1}C_{\widetilde{\mathbf{G}}^*}(t)^{wF}\widetilde{h}$ for some $\widetilde{h} \in \widetilde{\mathbf{G}}^*$. Identifying $\widetilde{\mathbf{G}}$ with $\widetilde{\mathbf{G}}^*$, $\widetilde{\mathbf{L}}$ with $\widetilde{\mathbf{L}}^*$ and $\widetilde{\mathbf{N}}$ with $\widetilde{\mathbf{N}}^*$ we obtain $\mathbf{N} = \widetilde{\mathbf{N}} \cap \mathbf{G}$ hence $\mathbf{N}^F = \widetilde{\mathbf{N}}^F \cap \mathbf{G}$, i.e. $\mathbf{N} = (\widetilde{h}^{-1}C_{\widetilde{\mathbf{G}}}(\widetilde{t})C_{\widetilde{\mathbf{G}}}(t)^{wF}\widetilde{h}) \cap \mathbf{G}$ and $\mathbf{N}^F = (\widetilde{h}^{-1}C_{\widetilde{\mathbf{G}}}(t)^{wF}\widetilde{h}) \cap \mathbf{G}$.

Now $C_{\widetilde{\mathbf{G}}}(t) = \langle\widetilde{\mathbf{L}}, w\rangle$ and $\widetilde{\mathbf{L}} = \prod_{i=1}^{k} \mathrm{GL}_m(\overline{\mathbb{F}}_q)$. Since $F(w) = w, w \in C_{\widetilde{\mathbf{G}}}(t)^{wF}$, hence $C_{\widetilde{\mathbf{G}}}(t)^{wF} = \langle\widetilde{\mathbf{L}}^{wF}, w\rangle = \widetilde{\mathbf{L}}^{wF} \rtimes \langle w\rangle$.

For the claim about $\widetilde{\mathbf{L}}^{wF}$, let $x = (x_1, \cdots, x_k) \in \widetilde{\mathbf{L}}, x_i \in \mathrm{GL}_m(\mathbb{F}_q)$. We have ${}^{wF}x = (x_k^q, x_1^q, \cdots, x_{k-1}^q)$. Hence ${}^{wF}x = x$ if and only if $x = (x_1, x_1^q, \cdots, x_1^{q^{m-1}}), x_1^{q^m} = x_1$, thus $x_1 \in \mathrm{GL}_m(q^k)$ and we get $\widetilde{\mathbf{L}}^{wF} \cong \mathrm{GL}_m(q^k)$ via the isomorphism $(x_1, x_1^q, \cdots, x_1^{q^{m-1}}) \mapsto$

x_1.

Under this isomorphism, the action of $\langle w\rangle$ on $\widetilde{\mathbf{L}}^{wF}$ is transported to the action of $\langle w\rangle$ on $\mathrm{GL}_m(q^k)$ given by ${}^w x_1 = x_1^{q^{m-1}} = x_1^{-q}$. Thus, w acts as Frobenius $x \mapsto x^q$ on $\mathrm{GL}_m(q^k)$. Thus $\widetilde{\mathbf{N}}^F \cong \widetilde{\mathbf{L}}^{wF} \rtimes \langle w\rangle \cong \mathrm{GL}_m(q^k) \rtimes \langle\sigma\rangle$ where σ is the field automorphism of order k.
□

Proposition 5.39. *Let P be a defect group for the unipotent block b of $\mathbf{L}^F$. If we have the assumptions of the previous lemma, then every element in P is w-stable.*

Proof. We have $\widetilde{\mathbf{L}}^F = \mathrm{GL}_m(q^k)$ and $\mathbf{L}^F = \mathrm{GL}_m(q^k) \cap \mathrm{SL}_n(q)$, where $mk = n$. Furthermore, $P \in \mathrm{Syl}_p(C^\circ_{\mathbf{L}}([\mathbf{M},\mathbf{M}]^F))$ where $\mathbf{M} \leq \mathbf{L}$ is a Levi subgroup. If $\widetilde{\mathbf{M}}$ is a Levi of $\mathrm{GL}_m(\overline{\mathbb{F}}_q)$, then we may assume $\widetilde{\mathbf{M}} = T \times \mathrm{GL}_r$ for some torus T. Thus, $\mathbf{M} = (T \times \mathrm{GL}_r(\overline{\mathbb{F}}_q)) \cap \mathrm{SL}_m(\overline{\mathbb{F}}_q)$, $[\mathbf{M},\mathbf{M}] = 1 \times \mathrm{SL}_r(\overline{\mathbb{F}}_q)$ and $C^\circ_{\mathbf{L}}([\mathbf{M},\mathbf{M}]) = (\mathrm{GL}_{m-r}(\overline{\mathbb{F}}_q) \times (\overline{\mathbb{F}}_q)^r) \cap \mathrm{SL}_m(\overline{\mathbb{F}}_q)$. So we may assume $P \in \mathrm{Syl}_p(\mathrm{SL}_{m-r}(q^k))$. Now let $x \in P$, we have ${}^w x = x$ if and only if $x \in \mathrm{SL}_{m-r}(q)$. Since we have $(p,q) = 1$ the p-parts of the groups $\mathrm{SL}_{m-r}(q)$ and $\mathrm{SL}_{m-r}(q^k)$ coincide. □

Proof of Theorem 5.35. Let $Q \leq P$ be centric. Since b is of principal type, there is a unique block e_Q of $C_{\mathbf{L}^F}(Q)$ such that (Q, e_Q) is a b-Brauer pair. Thus, we have $N_{\mathbf{N}^F}(Q, e_Q) = N_{\mathbf{N}^F}(Q)$ and $N_{\mathbf{L}^F}(Q, e_Q) = N_{\mathbf{L}^F}(Q)$ which also implies $C_{\mathbf{N}^F}(Q, e_Q) = C_{\mathbf{N}^F}(Q)$. By assumption, $C_{\mathbf{N}^F}(Q) = C_{\mathbf{L}^F}(Q) \rtimes x$, where $\langle x\rangle = \mathbf{N}^F/\mathbf{L}^F$ and thus also, $N_{\mathbf{N}^F}(Q) = N_{\mathbf{L}^F}(Q) \rtimes x$. We get $N_{\mathbf{L}^F}(Q, e_Q) C_{\mathbf{N}^F}(Q, e_Q) = N_{\mathbf{L}^F}(Q) C_{\mathbf{N}^F}(Q) = N_{\mathbf{L}^F}(Q)(C_{\mathbf{L}^F}(Q) \rtimes x) = N_{\mathbf{N}^F}(Q) = N_{\mathbf{N}^F}(Q, e_Q)$ which is equivalent to $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{G}}(Q)$. Theorem 2.6 implies the claim. □

6 Conclusion

6.1 Overview of exotic fusion systems

We give an overview of all exotic fusion systems which have been discovered, point out for which ones block-exoticity has been proven, and make some remarks. The order in this table aligns with the bibliography. See the respective references for details.

S	#	Reference	Remarks
$S \in \mathrm{Syl}_p(\mathbb{Z}/p\mathbb{Z})^{p-1} \rtimes (C_{p-1} \rtimes \Sigma_p)$	∞	[6]	f.s. reduction simple, $Z(S)$ cyclic
S group of order p^4, p odd	∞	[7]	Proposition gives fusion systems, Exotic examples given
$S \in \mathrm{Syl}_p(S(n,p))$	∞	[14]	f.s. reduction simple, $\mathrm{Aut}(S)$ no p-group
S Sylow p-subgroup of amalgamated product of matrix groups	∞	[15]	Two families of exotic systems
non-abelian p-group S with unique abelian subgroup A of index p	∞	[17]	A essential, elementary abelian
3-group of rank 2	∞	[18]	generalises systems from [47], only exotic systems on odd rank 2-group
$S \in \mathrm{Syl}_p(\mathrm{PSp}_4(p^a))$	∞	[25]	-
$S \in \mathrm{Syl}_2(\mathrm{Spin}_z(q))$, $q \geq 3$	∞	[35]	block-exoticity proven in [28],[16], only known exotic systems for $p = 2$
non-abelian p-group S with unique abelian subgroup A of index p	∞	[40]	A not essential
non-abelian p-group S with unique abelian subgroup A of index p	∞	[41]	A essential, not elementary abelian
$S \in \mathrm{Syl}_7(G_2(7))$	27	[42]	block-exoticity proven in Chapter 4
$S \in \mathrm{Syl}_p(QL), p \geq 5$	∞	[43]	Q extra-special of order p^{p-2}, $L \cong \mathrm{GL}_2(p)$, $Z(S)$ cyclic
$S \in \mathrm{Syl}_p(\mathrm{GL}_n(q)), q$ prime power with $(q,p) = 1$, p odd	$d(e) - 1$	[46]	subsystems of realisable system, $e := \mathrm{ord}(q) \mod p > 2, n \geq ep$
7_+^{1+2}	3	[47]	block-exoticity proven in [32]

Table 2: Exotic fusion systems

If we say that the number of fusion systems is ∞, we mean that it is an infinite family of exotic fusion systems. Note that the proof of block-exoticity of the exotic fusion systems on 7_+^{1+2} answers Conjecture 1.1 for all extraspecial groups of order p^3, since it is shown in [47] that these are the only exotic fusion systems on such groups. This is the only family of non-abelian groups for which the conjecture is proven. Furthermore, note that it is conjectured that the exotic fusion systems on $\mathrm{Syl}_2(\mathrm{Spin}_z(q))$, $q \geq 3$ are the only exotic systems on 2-groups.

6.2 Overview for finite quasisimple groups not covered in Chapter 5

In this final section, we give an overview on where we stand with Conjecture 1.1 for fusion systems on finite quasisimple groups other than the ones studied in Chapter 5. First of all, note that we can restrict ourselves to non-abelian quasisimple groups, since the conjecture holds for all fusion systems on abelian groups. This is due to non-existence of exotic systems on such groups, as discussed in Chapter 2. For alternating groups and their covers, the conjecture is also known to hold. It can be deduced from [9] together with [45]. For sporadic groups, the conjecture has been proven in [16]:

Theorem 6.1. [16, Theorem 9.22] *Let G be sporadic simple group. Let k be a field of characteristic p and b a block of kG, then the fusion system of b is non-exotic.*

We are left with groups of Lie type. For these, we see that the conjecture holds true if the underlying characteristic is p.

Theorem 6.2. *Let $\mathcal{F}$ be a fusion system realised by a p-block of a finite quasisimple group of Lie type over characteristic p, then $\mathcal{F}$ is non-exotic.*

Proof. Immediately follows from [13, Theorem 6.18]. $\square$

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