**Buckling and Free Vibration Analysis of Non-prismatic Columns using Optimized Shape Functions and Rayleigh Method**

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Abstract

In this paper, optimized shape functions have been used in Rayleigh method to determine the critical buckling load and the fundamental natural frequency of non-prismatic steel-reinforced slender concrete columns. A range of admissible shape functions describing the mode shape in buckling as well as for the fundamental natural frequency of the column are considered and then an optimization strategy is developed to arrive at the optimum shape function. The results obtained from the present method based on the implementation of Rayleigh method through the concept of generalized coordinates are verified and validated by the finite element method. The application of the theory is demonstrated by two illustrative examples, both of which are steel-reinforced concrete towers that are representative of practical structures. Of particular significance is the duality between the free vibration and buckling problems which is captured and fruitfully exploited in the analysis. The effect of an additional mass located at the top of the tower is included in the investigation. Additionally, the impact of the creep behavior of the towers on results due to continuing lapse of time is critically examined and assessed. Finally, significant conclusions are drawn following the discussion of results.

**Keywords:** non-prismatic columns, Rayleigh method, shape functions, optimization, modal analysis, buckling load.

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# Introduction

Designing slender structures has always been a challenge for structural engineers because it involves considerable complexities, particularly the effects of nonlinearities arising from the geometry of the structure and the properties of materials from which they are made of. Even with the advent of modern digital computers together with powerful software tools in providing approximate, but sufficiently accurate numerical results, the search for closed form analytical solution has continued to attract the attention of engineers and scientists. This is because of the elegant nature and uncompromising accuracy of the closed form analytical solution which is often considered as exact. In particular, the elastic buckling of a slender column has been the subject of continuing investigation ever since its inception. The original formulation of the problem was given by Euler [1], who provided a solution, initially based on the static equilibrium of a bent section of the column. It is worth-noting that Euler had some difficulty in his formulation to include the self-weight of the column and the problem was later overcome by Greenhill [2], as pointed out by Timoshenko and Gere [3]. A column is basically a continuous structural member commonly a beam subjected to a compressive load which can be conservative (Euler’s load) or of other types (Uzny [4]). Clearly, the Euler-Greenhill’s form of solutions cannot be generally applied to non-prismatic structures in a simple and straightforward manner as emphasized by Bert [5].

A particular buckling investigation of non-prismatic columns with cross-sections varying continuously as well as for stepped columns was carried out by Nikolic´and Šalinic [6] who divided the column into several elements representing small segments joined together by rotational and translational springs and then applying the potential energy method to arrive at the solution. The influence of nonuniform distribution of axial load on the critical buckling load resulting from the porosity of the column cross-section was investigated by Hamed et. al [7] when they sought buckling solution for functionally graded sandwich beams with porous core. Belkacem et al. [8] determined the energy dissipation capacity for a reinforced concrete column by analyzing the effects of variable axial load and transverse reinforcement ratios to assess the seismic performance of reinforced concrete columns. In many ways, their research was somehow complimentary to the work of Wang et. al. [9]. By contrast, Rahai and Kazemi [10] determined the critical buckling load of non-prismatic columns by splitting the column into various uniform segments and then using the individual mode shape of each segment. They established the existence of similarity between the vibrational mode shape and the buckling mode shape.

Against the above background, one of the most traditional analytical procedure that is continually being used to investigate the buckling and/or vibration problem is that originally developed by Rayleigh [11] who assumed that a system containing infinite degrees of freedom can be understood by representing it as a system with a finite number of degrees of freedom which in essence means that the entire system can be idealised as a sum of many single degree of freedom systems. The fundamental principle used by Rayleigh is indeed the principle of conservation of energy in which the total energy comprising both potential and kinetic energies generated by the vibratory movement is constant for a conservative system. Following Rayleigh’s work, Ritz [12] presented his method for eigen-solution of structures (natural frequencies in vibration problem and critical buckling load in buckling problem) by choosing multiple permissible displacement functions and then minimizing the energy functional that involved both potential and kinetic energies. So, posteriority to Ritz, the method originated by Rayleigh started being justifiably called the “Rayleigh-Ritz method” by subsequent researchers. Oliveri and Milazzo [13] affirm that the Rayleigh-Ritz method appears in scientific literature as one of the most successful approaches used in free vibration and buckling analysis. This was echoed by Muc et al. [14] who highlighted that the variational principle established by Rayleigh and Ritz is a powerful technique to solve structural problem approximately, but sufficiently accurately by introducing a trial function for the structure’s deformed shape. Interestingly, Leissa [15], in the context of Rayleigh whereas Young [16], in the context of Ritz, affirmed that the precision obtained through these methods would very much depend on the functional form assumed to represent the vibration or buckling mode. Fortuitously, if exact shape function is assumed, the corresponding result generated by the Rayleigh-Ritz method, would be exact. In this regard, Kharghani and Soares [17] recorded in their work that when using the Rayleigh–Ritz approximation, it is generally true that continuous functions such as polynomial and trigonometric functions are usually adequate to be a good option for the analysis.

Using the same trend of thought, Mochida and Ilanko [18] stated that if the exact shapes of the vibration modes are used as admissible functions, the solution by the Rayleigh method would be exact, but there would be no reason to do this exercise if the exact modes were already known. It is possible, but not very likely that exact admissible functions can be chosen by accident to obtain exact result. For this reason and for majority of applications, it would be necessary to have some chosen or assumed admissible functions that can be defined as shape functions.

Clearly, the underlying principle in Rayleigh method can be applied to structural systems with finite degrees of freedom as well as to continuous systems, see Temple and Bickley [19]. The objective of the Rayleigh method is essentially to determine the eigenvalues (natural frequencies in free vibration problem and the critical buckling load in buckling problem) with reasonable accuracy as required when solving engineering problems.

# Fundamental premises of the Rayleigh Method and FEM

Rayleigh initially employed his technique to solve an extremization problem, using variational calculus and linear approximation for the basic functions which he chose for the deformation. The objective was to minimize a special class of analytic functions that satisfy the boundary conditions of the problem. It is often overlooked that the Rayleigh method provided the background for the theoretical basis of the finite element method (FEM) that is currently known today which, of course, has now become an indispensable design and analysis tool in many areas of science and engineering disciplines. One of the main advantages of FEM is that it can be applied to structural or other systems having complex geometries. Although Rayleigh method and FEM are both based on the principles of energy minimization, it should be recognized that the shape function in the Rayleigh method is applicable to the entire domain of the problem whereas in FEM, the interpolation or shape functions are valid for the given subdomain of the individual elements. Papadopoulos et al. [20], amongst other researchers noted the obviously expected fact that when structures become quite complex, a finer finite element mesh would be required to ensure reasonably good accuracy of results. Naturally, refined mesh produces very large size mass and stiffness matrices leading to computationally expensive analysis which can be prohibitive, particularly in optimization studies.

At this juncture, it is important to note that in numerical solution of differential equations, as opposed to explicit exact analytical solution, is generally based on some form of approximations, and strictly speaking, such solution is not indicative of exact solution. Furthermore, different differential equations and their numerical solution, even fulfilling the boundary conditions of the problem, can lead to different results. In this respect, it is worth noting that Dirichlet [21] made some interesting observations of such nature about Rayleigh method. It is obviously clear from his and other researchers’ assertions that both Rayleigh method and FEM are approximate, but each one is based on approximations that are determined by their own characteristics and applications.

For prismatic columns, the application of the Rayleigh method allows one to obtain a closed-form and well-defined equation to calculate the fundamental natural frequency of the column and the associated mode shape (see appendix A), which can be used to determine the critical buckling load of the column. This follows from the fact that a duality exists between the free vibration and buckling problem in that buckling can be considered as a free vibration problem of an axially loaded (compressive) structures at zero frequency. In other words, if the compressive axial load in a structure such as a column is gradually increased, the fundamental natural frequency diminishes and eventually it becomes zero when buckling occurs as a degenerate case of free vibration analysis at zero frequency. Wahrhaftig et al. [22] investigated this problem and validated their theoretical predictions computed applying Rayleigh method by using experimental results as well as by FEM analysis. However, to apply Rayleigh method to non-prismatic columns whose properties vary along the length is not that simple and straightforward. In this regard, there seems to be a gap in the literature. Rayleigh [11] not only conceived his pioneering method which survived the test of time and is still being used by scientists and engineers even to this day, he also significantly advanced the free vibration theories of beams and bars by including the effects of rotatory and transverse inertia, see for examples Banerjee and Jackson [23] and Banerjee et al. [24].

The solution for the buckling and free vibration problem of non-prismatic columns by Rayleigh method requires a novel approach in that the method should be enhanced and suitably adapted to represent the column in parts or components by considering each geometric interval between the parts or components where the properties vary, in an appropriate manner. Of course, the generalized properties need to be calculated for each segment of the structure. The generalized proprieties of interest such as stiffness and mass of the entire structure can be obtained by the superposition of the generalized properties of all the individual elements. This process is essentially an adaptation of the original method of Rayleigh [11] and can be characterized as partitioned Rayleigh method (PRM). The PRM can be applied successfully to compute the critical buckling load and the fundamental natural frequency of structures. Based on these underlying premises, Wahrhaftig and Brasil [25] made a noteworthy contribution when they investigated the free vibration characteristics of mobile phone masts represented by metallic poles. They embarked on both computational modeling as well as experiments and drew many useful conclusions. However, their investigation was rather restrictive, and they relied on just one value of the lumped mass (and force) connected to the pole and there was no suggestion of optimization in their paper to arrive at the best possible deformed shape. If optimization techniques to optimize the shape functions prior to the application of Rayleigh method are used, the accuracy and reliability of results can be significantly enhanced. One of the main purposes of this paper is to address this issue.

With the above pretext, seven shape functions that satisfy the boundary conditions of the problem are used for optimization studies to analyze two similar, but different steel reinforced non-prismatic concrete columns. The two columns are subjected to compressive external loading applied at the top which varied gradually from zero, when the structures are exclusively subjected to their own weight right up to the buckling load that defines the vicinity of the structural collapse due to the loss of balance. Both, distributed and lumped mass, and stiffness representations are used to form the overall mass and stiffness matrices of the column. The use of optimization technique made it possible to take the first mode shape as close as possible to the one given by the finite element method (FEM). It is worth mentioning that optimization technique of this type that seeks practical applications related to buckling behaviour of composite structures can be found in the work of Lindgaard and Lund [26]; and Neves et al. [27].

The optimization technique performed in the present work, seeks to approximate analytical shape functions to those given by computational modeling through FEM by considering a nonlinear formulation that brings together the geometric and material effects. Both the two illustrative structures investigated in this paper are non-prismatic and very slender reinforced concrete poles. Of particular significance is that the effect of the foundation stiffness is included in the analysis. For all solutions, three important nonlinear aspects were given due recognition and emphasis. These are: (i) the geometric nonlinearity due to the slenderness of the system, (ii) the material nonlinearity, and (iii) the creep of the concrete. The last of these three is considered using the Eurocode 2 [28] criteria. For the analytical and computational procedure, the ground around the foundation is modeled by a set of springs distributed along the depth, and the critical buckling load analysis is defined to be a modal time-dependent parameter.

# Partitioned Rayleigh method

As it is well known, Rayleigh method is based on the principle of energy conservation to generate equilibrium equations and boundary conditions for a problem that has a variational indicator which is generally called a functional, i.e., the function of a function or the totality of the function. When the functional is made stationary using the concept of equilibrium, the procedure generates the differential equations of the problem and the associated boundary conditions. A functional is essentially an integral expression that implicitly contains the differential equations that describe the problem. In Ekeland's words [29], “it is normally an integral function that depends on functions of one or more variables, as well as its derivatives of a certain order, subject to or not to restrictions”. Their coefficients can be optimized by different methods, as illustrated by Wang et al. [30].

When developing the Rayleigh method for calculating the critical buckling load and the fundamental natural frequency of a non-prismatic column as is the present case, the Principle of Virtual Work (PVW) is invoked in a similar manner to that used by Sedira et al. [31] who used the concept of generalized coordinate in conjunction with the assumed fundamental mode shape of the structure undergoing undamped free vibration.

The mathematical procedure which constitutes the investigation of the fundamental natural frequency and the critical buckling load of a column of length *L* with its axis coinciding with the *X*-axis and the column is embedded into the ground with a given soil stiffness, is described in detail in Appendix A and Fig. A1. A different, but related aspect of the investigation is the effect of creep which is an important parameter that perpetuates nonlinear characteristics to degrade the behaviour of concrete (see for examples, Shen, Fang and Xia [32]; Gilbert and Ranzi [33]; Geng et al. [34]; Wahrhaftig [35]; Shariff, Saravanan and Menon [36]). Therefore, creep is introduced in the analysis, particularly given the high slenderness ratio of the structure. As a result of the creep consideration, the natural frequency and critical buckling load became temporal, but progressive functions because of the change in the modulus of elasticity over time. Therefore, the natural frequency can be written in terms of time and the mass at the top, and the ensuing expression that emerges from that process is sufficient to calculate the critical buckling load, when the frequency is allowed to go to zero at any arbitrary time during the lifetime of the structure. It is important to mention here that the procedure presented can be adapted to solve any structure that can be idealized by one-dimensional element, such as beams and bars.

To assess the effectiveness of the proposed method, an initial evaluation is made by using different (typical) expressions for normalised shape functions. These are: (i) a trigonometric function, see Eq. (1), (ii) polynomial functions, see Eqs. (2) and (3) and (iii) a potential function (fractional order power against length coordinate) with exponent *g* = 2.27, see Eq. (4). Note that similar shape functions were used in a different context by Wahrhaftig [37]. These functions are perfectly valid and legitimate in the whole domain of the structure as they satisfy the boundary condition for the problem, i.e., at *x* = 0 and at *x* = *L*. The first objective is to verify which of the following curves for the shape function approximates best when compared with that computed from FEM. Once this is established, optimization technique will be applied to the best curve to diminish the error and improve the accuracy of results.

|  |  |
| --- | --- |
| , | (1) |
| , | (2) |
| , | (3) |
| . | (4) |

# Computational modeling by FEM

It should be noted that while the analytical procedure presented in the previous section provides a single functional form for the entire problem domain, the FEM formulation establishes interpolation functions that are applicable to the domain for each of the individual finite elements. According to Oden [38] and Kurrer [39], FEM is a break-through in solid mechanics, but it is essentially based on the PVW. It is well known that FEM is a discretization technique which represents continuous systems and their numerical approximations from differential equations. Understandably FEM has its root in shape functions implied in the variational methods of Rayleigh [11] and Ritz [40] and the residual weighted method of Galerkin [41]. This assertion was noted by Brasil [42], amongst others. Therefore, in many ways, FEM can be thought of as a modification of the Rayleigh-Ritz technique without much loss of generality.

In terms of modal analysis, the relevant eigenvalues and eigenvectors can be obtained by solving the following eigenvalue equation (See Eq. (5) in the usual form below). The problem, therefore, as presented in Eq. (5) is to seek free undamped vibration analysis often termed as characteristic value, or eigenvalue for a set of linear algebraic equations.

|  |  |
| --- | --- |
| , | (5) |

In Eq. (5) and referring to Appendix B, [*M*] is the mass matrix, given by Eq. (B1), and [*K*] is the total stiffness matrix, which includes [], [] and [], where [] is the usual structural stiffness, [] is the geometric stiffness and [] is the spring stiffness resulting from the soil, respectively. For a typical beam bending element, [], [] and [] are given by Eqs. (B2), (B3), and (B4) as shown in Appendix B, also see Ritto et al. [43] and Rao [44]. The spring matrix [] that represents the soil-structure interaction is a 6 × 6 symmetric matrix of the spring coefficients, *kij*, including all the translational and rotational degrees of freedom of a beam element. The components of the [] matrix are principally the spring stiffnesses attached to the nodes.

In a FEM environment, **2 generally represents the eigenvalues and *f* represents the eigenvectors (displacement vectors) or mode shapes. Therefore, the mathematical solution of a vibration problem using FEM with several degrees of freedom boils down to a polynomial equation of degree *p* which contains the variable **2 and is commonly known as the frequency equation. The *p* solutions for *p* are real, positive, and are called the natural frequencies of the system. The smallest frequency is typically denoted as **1, while the largest frequency is denoted as *p*. Thus, *p* modes of vibration can be determined and collected in a modal *p × p* matrix, which contains *p* columns, each representing a mode shape of undamped, normalized free vibration *fp*.

Once a model is constructed by considering distributed mass and lumped masses at nodes (joints) of the structure, the modal mass or the generalized mass *M* associated to the vibration mode can be calculated by using *M* = where and represent the normalized modal displacement and the mass at the joint *j* (Clough and Penzien [45]).

It is important to highlight that when the problem includes the effect of creep in the calculation, the modulus of elasticity will become time dependent. If any lumped mass is added to the system at a joint, the static force arising from the lumped mass should also be considered in the analysis. It should be noted that the FEM solution here is based on the modal analysis carried out by reducing the stiffness of each segment of the system due to the compressive load by using the geometric stiffness matrix (see Wahrhaftig and Brasil [46]) as well due to creep by using the conventional matrix.

To assess the accuracy of the proposed shape functions when carrying out the critical buckling load analysis by the Rayleigh method, the results were compared with those given by the FEM model. The structure under consideration was modeled using beam elements with constant or variable cross-sections, as appropriate. The varying mass was applied to the model together with the corresponding axial forces, in addition to the existing masses and forces present in the system. Suitably chosen spring constants were assigned to the foundation beam elements using a linear distribution. For solving the problem using both Rayleigh method and FEM, the time interval was divided into 400 different instants of interest to calculate the modulus of elasticity of the material. The FEM model was constructed using SAP2000 [47] structural analysis software. When using FEM, the buckling load was determined by assuming an isotropic homogeneous material with Poisson’s ratio 0.2 which is representative of concrete. The interpolation functions used in FEM were third-degree polynomials, as given by Eq. (3).

# Optimization of shape functions

In the context of solving optimization problems, it is worth noting that the techniques generally employed in undertaking practical problems involve significant computational efforts and costs depending on the numbers of degrees of freedom and number of design variables. For relatively simple models, Li et al. [48] implemented a method of global reduced-order basis by combining proper orthogonal decomposition with an iterative process to optimize the design of structures in reducing vibration. For analysis related to dynamic systems, in relation to the vibration frequency of the structure, the investigation of Xu et al. [49] is of special interest, where the authors expended their efforts to maximize the natural frequency of continuous vibrating structures. Following a similar line of approach, Daxini and Prajapati [50] ascertained that in a typical process of structural optimization, the primary structure must be converted into a suitable model that is compatible with the method of analysis to be performed.

It is important to observe that the present optimizing problem is concerned to approximate curves to determine the fundamental natural frequency and buckling load of steel-reinforced concrete columns with their self-weights included in the formulation. The importance and usefulness of the problem were emphasized by Wahrhaftig et al. [51] in a separate independent investigation of different type. In solving optimization problems, Bruyneel and Duysinx [52] demonstrated that the density of the material, especially in relation to the self-weight of the structure must be considered in the solution because it decisively influences the design criteria.

The optimization problem is generally formulated with an objective function for a given configuration with the premise that the design variables offer predictive security and stability of the solution. In the present study, the curves representing the shape functions obtained by finite element modeling are utilized to fulfil this task. After the first evaluation using the partitioned Rayleigh method, the trigonometric cosine function given by Eq. (1) emerged as the best approximation which fits the FEM result. For that reason, the optimization process was applied over the curve given by Eq. (1) to diminish the departure from the finite element results. In order to achieve this, the trigonometric expression given by Eq. (1) was weighted by a linear equation (Eq. (6)), quadratic equation (Eq. (7)) and cubic equation (Eq. (8)) function, respectively. These functions were considered valid in the whole domain of the structure and clearly, they obey the boundary conditions of the problem. These are given by

|  |  |
| --- | --- |
| , | (6) |
| , | (7) |
| . | (8) |

To determine the coefficients *a*, *b*, *c* and *d* in Eqs. (6)-(8), the optimization problem was formulated and subsequently solved in a similar manner to that carried out by Arora [53] and Fu et al. [54]. The procedure is briefly described below.

Let us define a vector ***b*** whose transpose for the three cases is given by

|  |  |  |
| --- | --- | --- |
| , |  | (9) |

Now minimize the objective function defined below,

|  |  |
| --- | --- |
| , | (10) |

where *n* denotes the number of points, or segments, as mentioned earlier, subjected to the following constraint,

|  |  |
| --- | --- |
| . | (11) |

Clearly the objective function shown in Eq. (10) is the error or the departure of the values of the adopted shape function **(**b**, *x*), given by Eqs. (6)-(8) from the FEM shape function . The penalty parameter *r* is used to adjust the order of magnitude of the errors, since the errors are usually small numbers, they needed to be factored. The initial value is the independent term which is equal to 1 and all the other values are equal to 0. It is important to highlight that, in this sense, all the following functions obtained from the optimization process obey the boundary condition of the problem.

A computer program to determine the buckling critical load is developed using Mathcad based on the flow chart schematically shown in Fig. 1. Initial values for were particularly established for each analytical equation by using a first processing with coarse increments to reduce the computational time, prior to using the finer increments at a later stage. For time, *t*, *t*=0 was defined at the start of the calculation. Increments **of 1 kg; *t* of 10 days, and a final value of 4,000 days for *t*, were subsequently used. The processing time by using an Intel CPU (R) 2.70–2.90 GHz, i7 (7th generation), Core (7M) 7500U, running on Windows 10 (64 bits), 8 GB RAM, took approximately around 6 hours for the completion of a typical analysis. All shape functions were considered in analysis, including the optimized ones.



Fig. 1 Flow chart of computer program for the analysis

# Implementation of the theory and numerical results

## Illustrative example 1

The first optimization problem involves calculating the critical buckling load of a slender reinforced concrete pole with variable geometry shown in Fig. 2, where *g* denotes gravitational acceleration; *Gr* means ground; *s* represents each structural segment; *S*, *DS*, and *thS* are respectively the section type, external diameter and wall thickness of the section; and *dbS*, *nbS*, and *cS´* represent the diameter, quantity, and concrete cover of the reinforcing bars. The structure is 46 m tall, including a 40 m superstructure with a hollow, circular section, and a 6 m deep fully circular foundation.

It is important to consider that concrete presents cracks (Branson [55]) when in service, even at relatively low stress values. Cracking reduces the inertia of the sections, which alters structural stiffness and reduces the natural frequency of vibration (Liu et al. [56], Alijanim, Barrera and Bordas [57]). If bent and depending on the stress level to which they are subjected, structural concrete parts can have cross sections with significant loss of inertia, see Lapi et al. [58]. The factored modulus of elasticity which accounts for the crack formation in a bent configuration, adopted for the superstructure and foundation were 19 GPa and 13 GPa, respectively, which were computed by multiplying the product of the flexural stiffness by 0.50. This is a typical parameter for that condition, considering that the column will be bent by lateral forces associated with the wind action when in service. This criterion is recommended by the Brazilian Association for Standardization, ABNT NBR 6118:2014 [59], American Concrete Institute, ACI-209R-08 [60], Mexican Standards for Design and Construction of Concrete Structures NTCC-17 [61], European Normalization, EN 1992‐1‐1:2004 (Eurocode 2) [28], and American Society of Civil Engineering, ASCE FEMA – 356 [62]. The recommendation has also been considered by other investigators such as Colunga et al. [63], Kara and Dundar [64], and Marin and El Debs [65]. The corresponding densities for superstructure and foundation were taken to be 2,600 kg/m3 and 2,500 kg/m3, respectively. The slenderness ratio of the tower is approximately 408. A set of antennas and a platform are usually installed on its top, constituting a concentrated mass whose limiting value based on the criteria arising from the loss of stability for buckling failure needs to be determined. Cables and a ladder are attached along the entire length, adding an additional 40 kg/m distributed mass to the system. The discretization of the structure using FEM is based on 51 beam elements.

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| --- | --- | --- | --- |
| Uma imagem contendo ao ar livre, navio, relógio, luz  Descrição gerada automaticamente  Descrição: Passagem de cabos existente (2)  (a) Photos | (b) Geometry “cm” | (c) Cross-sections | Discretization_Mast RC 46m_NLA (2)  (d) FEM |

Fig. 2 The analyzed system (first case): Gr = ground, *S* = cross-section , *DS* = external diameter, *thS* = thickness, *nbS* number of reinforcement bar; *dbS* = diameters of reinforcement; *cS’* = concrete cover.

For the above proposed problem, the foundation is a relatively deep shaft with a 140 cm bell diameter, 20 cm length, 80 cm shaft diameter, and 580 cm length. The lateral soil resistance is represented by elastic springs with stiffness equal to 2,669 kN/m3. The creep of concrete is considered to occur in accordance with the Eurocode 2 criteria [28]. The homogenizing factors, required for the calculation of the area of reinforcement present in the cross-section, were taken to be 1.026, 1.074, 1.050, 1.061, and 1.064 for sections 1 to 5, respectively. The factors calculated before considering the concrete cracks and using the approach given in [66].

An important point in the assessment of results in the initial stage, is the comparison between the cumulative force obtained though the analytical solution with the corresponding value computed by FEM. This allows us to verify how the load distributions in both solutions match. In this way, the value obtained for the analytical solution and FEM modeling were found to be close to each other and calculated at 349.76 kN, which is equal to the total weigh of the structure (rection in the base). In Table 1, the optimum values of the optimization problem for Eqs. (6), (7) and (8) can be seen for the coefficients *a*, *b*, *c* and *d*. As can be seen in Table 1, the errors are 41.0%, 3.7%, and 1.4%, respectively for the three equations. As expected, the error significantly decreased when the number of coefficients in the equations is increased.

Table 1 Errors in the coefficients of Eqs. (6)-(8) for illustrative example 1.

|  |  |  |
| --- | --- | --- |
| Equation (6) | Equation (7) | Equation (8) |
| r = 100,000 | r = 100,000 | r = 100,000 |
| *a =* 0.0069 | *a =* -0.26161 | *a =* 0.3071 |
| *b =* 0.9931 | *b =* 0.435296 | *b =* -0.9452 |
| *c = 0* | *c =* 0.826314 | *c =* 0.9205 |
| *d = 0* | *d = 0* | *d =* 0.7176 |
| *Error =* 41.0 | *Error =* 3.7 | *Error =* 1.4 |

r = Penalty parameter; a, b, c, d are coefficients that are optimized

Fig. 3(a) shows the values of the critical buckling load, obtained through the nullity of the structural frequency, for the seven shape functions used and from the FEM analysis when *t* = 0, the instant at which the structure was loaded and for which the concrete creep did not produce any effects. The polynomial function given by Eq. (3) leads to a quite different set of result from those computed from other equations. To evaluate the values of the critical load at different times other than the initial one, and to consider possible effects of the creep of the concrete, a period of 4,000 days after the structure entered service was considered in the analysis. Fig. 3(b) shows results for the case after 4,000 days. Table 2 summarizes the values computed for each equation analyzed, considering a standard gravitational acceleration of 9.80665 m/s² and a 70% environmental humidity for creep calculations. It is clear from Table 2 that in relation to the FEM results, Eq. (8) provides the smallest value of the error, when compared to other equations.

Table 2 Critical buckling load

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Eq. | Classification | Error  (%) | Buckling load (kN) | | | |
| *t* = 0 | Dif. FEM | *t* = 4,000 days | Dif. FEM |
| Eq. (1) | Trigonometric | 42.2 | 281.1 | 12.9% | 221.8 | 23.6% |
| Eq. (2) | Polynomial | 474.8 | 311.4 | 25.0% | 256.1 | 42.7% |
| Eq. (3) | Polynomial 2 | 54063.5 | 1,308.1 | 425.2% | 1,107.9 | 517.4% |
| Eq. (4) | Potential  = 2.23 | 9,413.8 | 339.9 | 36.5% | 245.3 | 36.7% |
| **Eq. (5)** | **FEM - Reference** | **0.0** | **249.6** | **0.0%** | **179.5** | **0.0%** |
| Eq. (6) | Eq. (1) opt. with linear eq. | 41.0 | 280.1 | 12.2% | 220.7 | 23.0% |
| Eq. (7) | Eq. (1) opt. with quadratic eq. | 3.7 | 274.9 | 10.4% | 209.7 | 16.8% |
| **Eq.** **(8)** | **Eq. (1) opt. with cubic eq.** | **1.4** | 270.4 | **8.6%** | 203.5 | **13.4%** |

Eq. = eq. = equation; opt. = optimized; Dif. = difference to

|  |  |
| --- | --- |
|  |  |
| (a) *t* = 0 | (b) *t* = 4000 days |

Fig. 3 Variation of the fundamental natural frequency to the force at the top

If Eq. (3) is omitted from the analysis, a better evaluation between the other formulations can be performed as indicated by Fig. 4(a) and 5(b). Fig. 5 presents a comparison between the shape functions and those given by FEM along the length of the column (vertical axis) that are discretized in segments. It is observed that although Eq. (3) obeys the boundary conditions of the problem, its behavior diverges significantly from the reference curve based on FEM. This is clearly obvious from Fig. 5(c).

|  |  |
| --- | --- |
|  |  |
| (a) *t* = 0 | (b) *t* = 4000 days |

Fig. 4 Variation of the fundamental natural frequency with the force at the top without Eq. (3)

|  |  |  |
| --- | --- | --- |
|  |  |  |
| (a) Equation (1) | (b) Equation (2) | (c) Equation (3) |
|  |  |  |
| (d) Equation (4) | (d) Equation (6) | (e) Equation (7) |
|  | Label for graphics |  |
| (f) Equation (8) |  |  |

Fig. 5 Analytical functions and FEM mode shape for illustrative example 1

It is important to note that each expression for the shape function leads to different generalized modal properties, i.e., when Eqs. (1)-(4) are applied in the modal analysis for the period from *t* = 0 to *t* = 4000 days. After applying optimization techniques, Eq. (8) gives the best result for the stiffness and natural frequency during the time interval of *t* between 0 and 4000 days, when compared to FEM results. These differences are equal to 1.06%, 1.22%, for stiffness; 1.24%, 1.32% for the natural frequency. However, for the generalized mass, Eq.(7) presents the best result in relation to FEM, presenting a difference of only 1.31%. For the optimized condition, the generalized stiffness vary in percentages relatively low with respect to the FEM values, as shown in Table 3 with the maximum being 3.70% at time t = 0, and 8.04% at time t = 4,000 days. For the modal frequency these values correspond to 2.71% and 4.99%, respectively.

The modal mass shows a maximum variation of 1.88%. In fact*,* the optimization technique applied over Eq.(1) compensates for the stiffness and mass to minimize the difference in results obtained from FEM (see Table 3). It is important to note that there is a small difference in the FEM mode shapes when they are computed at *t* = 0 and *t* = 4000 days, as shown in Fig. 6.

Table 3 Modal parameters (Case 1)

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Eq. | *K*(*m0*,*t*) kN/m | | *w*(*m0*,*t*) rad/s | | *M*(*m0*) Kg |
| *K*(0,0) | *K*(0,4000 days) | *w*(0,0) | *w*(0,4000 days) | *M*(0) |
| Eq. (1) | 7.539523 | 5.948328 | 1.056843 | 0.938720 | 6,750.30 |
| Eq. (2) | 8.122122 | 6.680416 | 1.075592 | 0.975472 | 7,020.60 |
| Eq. (3) | 34.124531 | 28.902457 | 1.753401 | 1.613672 | 11,099.52 |
| Eq. (4) | 10.747526 | 7.753840 | 1.414641 | 1.201574 | 5,370.52 |
| **Eq. (5)** | **7.255823** | **5.450838** | **1.028457** | **0.891404** | **6,859.85** |
| Eq. (6) | 7.524415 | 5.927684 | 1.057104 | 0.938262 | 6,733.44 |
| Eq. (7) | 7.445143 | 5.677041 | 1.048571 | 0.915635 | 6,771.38 |
| Eq. (8) | 7.332938 | 5.518032 | 1.041364 | 0.903349 | 6,761.96 |
| Differences to **Eq. (5)** (**FEM**) | | |  |  |  |
| Eq. (1) | 3.91% | 8.36% | 2.69% | 5.04% | 1.62% |
| Eq. (2) | 11.94% | 18.41% | 4.38% | 8.62% | 2.29% |
| Eq. (3) | 370.31% | 81.14% | 41.35% | 44.76% | 38.20% |
| Eq. (4) | 48.12% | 29.70% | 27.30% | 25.81% | 27.73% |
| Eq. (6) | 3.70% | 8.04% | 2.71% | 4.99% | 1.88% |
| Eq. (7) | 2.61% | 3.98% | 1.92% | 2.65% | **1.31%** |
| **Eq. (8)** | **1.06%** | **1.22%** | **1.24%** | **1.32%** | 1.45% |



Fig. 6 FEM modal shape at *t* = 0 and *t* = 4000days for illustrative example 1

## Illustrative example 2

The illustrative example 2 is a forty-meter high, reinforced concrete pole structure, with a hollow circular cross-section of varying thickness with external diameter 60 cm, as shown in Fig. 7. It is of interest to register that the thicknesses of the structure were collected in field by using a measurement instrument based on radio waves. The pole has a slenderness ratio of approximately 472. The concrete used in the manufacture of the structure has a modulus of elasticity equal to 19 GPa and 15 GPa for the superstructure and foundation, respectively. For each cross-section (*S*), Table 4 presents: the external diameter (*DS*), thickness (*thS*), number (*nbS*) and diameters (*dbS*) of reinforcement bar, the concrete cover (*cS’*), and the homogenizing factors (*FS*) which were calculated by considering the uncracked cross-section and using the same approach as used in the illustrative example 1. The foundation has the same parameters and foundation stiffness as in the previous example. For this second problem, the total normal force in the analytical solution and the FEM solution are both equal which is 280.93 kN, representing the total weigh of the structure.

|  |  |  |
| --- | --- | --- |
|  |  |  |
| (a) Photos | (b) Geometry in “cm” | (b) FEM |

Fig. 7 A column as a pole transition system (second case): Gr = ground, *S* = cross-section , *DS* = external diameter, *thS* = thickness , *nbS* number of reinforcement bar; *dbS* = diameters of reinforcement; *cS’* = concrete cover.

Table 4 Structural properties and homogenizing factors of sections for illustrative example 2.

| Height  *Ls* (m) | External  diameter  *DS* (cm) | Concrete covering *cS’* (mm) | Thickness  *thS* (cm) | Number of  Bar *nbS* | Bar diameter  *dbS* (mm) | Homogenizing factors *FS* |
| --- | --- | --- | --- | --- | --- | --- |
| 40 | 60 | 25 | 10 | 20 | 13 | 1.0963 |
| 39 | 60 | 25 | 10 | 20 | 13 |
| 38 | 60 | 25 | 10 | 20 | 13 |
| 37 | 60 | 25 | 10 | 20 | 13 |
| 36 | 60 | 25 | 10 | 20 | 13 |
| 35 | 60 | 25 | 10 | 20 | 13 |
| 34 | 60 | 25 | 10 | 20 | 13 |
| 33 | 60 | 25 | 10 | 20 | 13 |
| 32 | 60 | 25 | 10 | 20 | 13 |
| 31 | 60 | 25 | 13 | 20 | 13 | 1.0869 |
| 30 | 60 | 25 | 12 | 15 | 16 | 1.0995 |
| 29 | 60 | 25 | 11 | 15 | 16 | 1.1029 |
| 28 | 60 | 25 | 11 | 15 | 16 |
| 27 | 60 | 25 | 11 | 15 | 16 |
| 26 | 60 | 25 | 11 | 15 | 16 |
| 25 | 60 | 25 | 11 | 16 | 16 | 1.1091 |
| 24 | 60 | 25 | 11 | 17 | 16 | 1.1153 |
| 23 | 60 | 25 | 11 | 18 | 16 | 1.1214 |
| 22 | 60 | 25 | 11 | 19 | 16 | 1.1274 |
| 21 | 60 | 25 | 11 | 20 | 16 | 1.1334 |
| 20 | 60 | 25 | 14 | 20 | 16 | 1.1230 |
| 19 | 60 | 25 | 15 | 15 | 20 | 1.1374 |
| 18 | 60 | 25 | 16 | 15 | 20 | 1.1354 |
| 17 | 60 | 25 | 13 | 16 | 20 | 1.1512 |
| 16 | 60 | 25 | 13 | 16 | 20 |
| 15 | 60 | 25 | 13 | 17 | 20 | 1.1594 |
| 14 | 60 | 25 | 13 | 18 | 20 | 1.1675 |
| 13 | 60 | 25 | 13 | 19 | 20 | 1.1755 |
| 12 | 60 | 25 | 13 | 19 | 20 |
| 11 | 60 | 25 | 13 | 20 | 20 | 1.1833 |
| 10 | 60 | 25 | 13 | 22 | 20 | 1.1987 |
| 9 | 60 | 25 | 16 | 22 | 20 | 1.1889 |
| 8 | 60 | 25 | 16 | 15 | 25 | 1.1961 |
| 7 | 60 | 25 | 17 | 15 | 25 | 1.1940 |
| 6 | 60 | 25 | 14 | 16 | 25 | 1.2132 |
| 5 | 60 | 25 | 14 | 16 | 25 |  |
| 4 | 60 | 25 | 14 | 17 | 25 | 1.2241 |
| 3 | 60 | 25 | 14 | 17 | 25 |
| 2 | 60 | 25 | 14 | 17 | 25 |
| 1 | 60 | 25 | 18 | 17 | 25 | 1.2136 |
| 0 | 60 | 25 | 18 | 17 | 25 |

*s* (lowercase) = segment; *S* (uppercase) = cross-section

Table 5 shows the errors associated with the optimized coefficients of the shape functions given in Eqs. (6)-(8) for the second illustrative example when compared with the corresponding values obtained from the FEM analysis. As was the case with the first illustrative example, the error diminishes with increasing number of coefficients, as expected.

Table 5 Errors and coefficients obtained for the 2nd optimization problem.

|  |  |  |
| --- | --- | --- |
| Equation (6) | Equation (7) | Equation (8) |
| r = 100,000 | r = 100,000 | r = 100,000 |
| *a =* 0.0149 | *a =* -0.4228 | *a =* 0.9509 |
| *b =* 0.9851 | *b =* 0.6963 | *b =* -2.4952 |
| *c = 0* | *c =* 0.7265 | *c =* 2.1229 |
| *d = 0* | *d = 0* | *d =* 0.4214 |
| *Error =* 160.9% | *Error =* 28.5% | *Error =* 2.0% |

r = Penalty parameter; *a*, *b*, *c*, *d* = Applied coefficients to optimize the equation

Table 6 summarizes the results obtained for Eqs. (1)-(5) and (6)-(8) for the buckling load, including equations of the PRM, before and after applying the optimization technique, and FEM. It can be observed that Eq. (8) followed presenting the smallest error in relation to other equations when the comparison with results obtained by the FEM is done. For this example, although Eq. (8) had shown to be the best approach, the existing differences of its results to these of the FEM are still too large with percentual reaching 153% and 340.6% at the initial time (0 day) and 4,000 days, respectively.

Table 6 Critical buckling load

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Eq. | Classification | Error  (%) | Buckling load (kN) | | | |
| *t* = 0 | Dif. FEM | *t* = 4000 days | Dif. FEM |
| Eq. (1) | Trigonometric | 167.1 | 201.1 | 209.2% | 174.0 | 542.9% |
| Eq. (2) | Polynomial | 769.1 | 244.2 | 275.5% | 219.7 | 711.8% |
| Eq. (3) | Polynomial 2 | 59,811.9 | 1.024.0 | 1474.5% | 932.7 | 3345.7% |
| Eq. (4) | Potential  = 2.23 | 10,117.8 | 185.6 | 185.4% | 138.4 | 411.4% |
| **Eq. (5)** | **FEM - Reference** | **0.0** | **65.0** | **0.0%** | **27.1** | **0.0%** |
| Eq. (6) | Eq.1 opt. with linear eq. | 160.9 | 198.2 | 204.7% | 170.9 | 531.5% |
| Eq. (7) | Eq.1 opt. with quadratic eq. | 28.5 | 174.6 | 168.5% | 142.0 | 424.5% |
| **Eq. (8)** | **Eq.1 opt. with cubic eq.** | **2.0** | 153.0 | **135.3%** | 119.3 | **340.6%** |

Eq. = eq. = equation; opt. = optimized; Dif. = difference to

Fig. 8 shows the values of the critical buckling load for the seven shape functions used using the present theory and FEM, computed at time *t* = 0, see Fig. 8(a), for which no one creep effect is produced; also for 4,000 days after the structure entered service, considering the creep effect, see Fig. 8(b). For a better visualization of the results, Fig. 10 shows results by removing the results using Eq. (3). Although contrastingly a very different theory and approach have been used, interestingly, the behavior of the structure described in this study, closely resembles the findings of Ferretti, D’Annibale and Luongo [67] when they analyzed a 3D building modelled as an equivalent column under compressive tip force and self-weight, and they also considered the soil structure interaction like the present authors.

|  |  |
| --- | --- |
|  |  |
| (a) *t*= 0 | (b) *t* = 4000 days |

Fig. 8 Results of 2nd problem of optimization.

|  |  |
| --- | --- |
|  |  |
| (a) *t*= 0 | (b) *t* = 4000 days |

Fig. 9 Results of 2nd problem of optimization without equation (3)

Fig. 10 illustrates the comparison of all shape functions with those given by FEM, within the domain of analysis. It can be observed that similar to the previous problem analyzed, Eq. (3) diverges significantly from the reference curve generated by FEM, as can be seen in Fig. 10(c). On the other hand, it is worth mentioning that the trigonometric function demonstrated to be the best representation of the vibration mode for both illustrative examples when compared with the FEM analysis.

|  |  |  |
| --- | --- | --- |
|  |  |  |
| (a) Equation (1) | (b) Equation (2) | (c) Equation (3) |
|  |  |  |
| (d) Equation (4) | (d) Equation (6) | (e) Equation (7) |
|  | Label for graphics |  |
| (f) Equation (8) |  |  |

Fig. 10 Analytical functions and FEM modal shape for the 2nd case

It can be observed that Eq. (1) after optimized by a 3rd order polynomial equation produces the smallest variation in frequencies (for 0 and 4,000 days) in relation to FEM. The optimization process introduces changes to the modal stiffness and mass of the original Eq. (1) diminishing differences from the linear to the polynomial. It is sufficient to ascertain that Eq. (1) provides the best solution in terms of the natural frequency. Optimization technique applied on Eq. (1) balances stiffness and mass to simulate the results to FEM most accurately. In this problem, the stiffness from FEM is very low in comparison with the one obtained analytically. The optimization technique appears to cause the analytical stiffness to reduce in comparison to their original values. Even so, the FEM stiffness is only 44% and 24% of the best analytical approach, respectively for 0 and 4,000 days, see Table 7. The best results for stiffness and frequency are given by Eq. (8), whereas for the generalized mass by Eq. (2), i.e., a polyonomy without optimization. The small difference between FEM vibration shapes, at 0 and 4,000 days, shown in Fig. 11, produces a variation by 1.93% to the modal stiffness and mass which is not so significant.

Table 7 Fundamental natural frequency (Case 2)

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Eq. | *K*(*m0*,*t*) kN/m | | *w*(*m0*,*t*) rad/s | | *M*(*m0*) Kg |
| *K*(0,0) | *K*(0,4000 days) | *w*(0,0) | *w*(0,4000 days) | *M*(0) |
| Eq. (1) | 5.392524 | 4.667223 | 1.079895 | 1.004650 | 4,624.12 |
| Eq. (2) | 6.371242 | 5.732468 | 1.150706 | 1.091498 | 4,811.67 |
| Eq. (3) | 26.714263 | 24.332044 | 1.911888 | 1.824652 | 7,308.33 |
| Eq. (4) | 5.868670 | 4.376588 | 1.241993 | 1.072549 | 3,804.53 |
| **Eq. (5)** | **1.832977** | **0.778908** | **0.607661** | **0.396119** | **4,964.04** |
| Eq. (6) | 5.332634 | 4.599394 | 1.074034 | 0.997465 | 4,622.80 |
| Eq. (7) | 4.752242 | 3.864087 | 1.008438 | 0.909334 | 4,673.05 |
| Eq. (8) | 4.177066 | 3.255033 | 0.947435 | 0.836357 | 4,653.42 |
| Differences to **Eq. (5)** (**FEM**) | | |  |  |  |
| Eq. (1) | 66.01% | 83.31% | 43.73% | 60.57% | 7.35% |
| Eq. (2) | 71.23% | 86.41% | 47.19% | 63.71% | **3.17%** |
| Eq. (3) | 93.14% | 96.80% | 68.22% | 78.29% | 32.08% |
| Eq. (4) | 68.77% | 82.20% | 51.07% | 63.07% | 30.48% |
| Eq. (6) | 65.63% | 83.06% | 43.42% | 60.29% | 7.38% |
| Eq. (7) | 61.43% | 79.84% | 39.74% | 56.44% | 6.23% |
| Eq. (8) | **56.12%** | **76.07%** | **35.86%** | **52.64%** | 6.68% |



Fig. 11 FEM modal shape at 0 and 4000 days for the second illustrative example

# Conclusions

In this paper, an optimization technique has been used to optimize shape functions for buckling and modal analysis of non-prismatic slender columns through an analytical approach based on the Rayleigh method. The proposed theory is used to determine the fundamental natural frequency and critical buckling load of steel reinforced concrete columns with varying geometry. In parallel, a solution based on the finite element method (FEM) was also undertaken as a reference. The analytical procedure was numerically solved for different shape functions while FEM was based on computational modeling with the help of a commercially available software package. Two realistic non-prismatic, slender, reinforced concrete poles were analyzed. Both structures had variations in elastic parameters, density, reinforcement rate, and geometry throughout their length. The structural self-weight was directly considered in the theory and numerical simulation in addition to the effects of distributed mass and forces. The analytical procedure considered all the parameters necessary for a nonlinear calculation, such as the geometry, material, and concrete creep. Seven shape functions were used, all obeying the boundary conditions for the problem. As the fundamental prerequisite of the Rayleigh method is based on the suitable shape of the deformation (for natural frequency or buckling analysis), the investigation has shown that once it is defined properly, other intervening factors could be considered effectively to analyze the structure.

The principal findings of the research are summarized as follows:

* For *t* = 0, the instant at which the structure was loaded, the lowest critical load for the first optimization problem (249.6 kN) was obtained by using the finite element analysis, while the largest critical load of 54,063.5 kN was provided by the polynomial function given by Eq. (3). It was observed that, even by obeying the boundary conditions of the problem, the difference in results between Eq. (3) and the FEM solution was exceptionally large (425.2%). It was highlighted that the closest result from the FEM is provided by Eq. (8), which presents a difference of only 1.4%, showing the effectiveness of the optimization technique used.
* When analyzing for *t* = 4,000 days (and for the first optimization problem), the highest critical load (1,107.9 kN) was found through the polynomial function given by Eq. (3), while the lowest (179.5 kN), was defined by a modal analysis through FEM. The smallest difference to the FEM (13.4%) was obtained by Eq. (8). Therefore, it is possible to conclude that the trigonometric function (Eq. (8)) presented better results in comparison with the finite element method for the proposed time intervals.
* Evaluating the results of the first illustrative example show a decrease of 4.2% in the intensity of the critical buckling load after 4,000 days when using the polynomial of Eq. (2) when compared to the critical buckling load obtained from Eq. (4). This represents an inversion in the behavior of the latter equation once it turns back to Eq. (2) when the force on the top is approximately 215 kN. FEM presented the largest percent difference between the results for 0 and 4,000 days. Equation (3) led to results that were quite different from the other equations. In terms of the fundamental natural frequency Eq. (8) presented the smallest differences with FEM, the differences being 1.24% and 1.32%, for *t* = 0 and 4,000 days, respectively.
* By analyzing the results of the second illustrative example, it could be observed that the values of the critical buckling load that showed the smallest variations in relation to FEM, were: from Eq. (8) (135.3% for *t* = 0, and 119.3% for *t* = 4,000 days). Equation (3) kept generating values completely out of the range of results obtained by the other functions. In terms of analysis of the shape function in comparison with the FEM solution, the trigonometric function, Eq. (1), met the curve requirement obtained by computational FEM modeling quite perfectly, followed by Eq. (2) and then Eq, (4), while equation (3) resulted in a behavior that was completely distinct from both cases. In essence, the curve of Eq.(3) presented to be too distant of the curve of reference as well present two distinct and inverse curvatures along the height.
* Considering the current mathematical approach, FEM by contrast involved many more vectors in the function vector space compared to Rayleigh’s method, which requires only one function for the entire domain. For that reason, FEM tends to be more precise. Different functions result in different generalized stiffness and masses when using Rayleigh’s method. However, it should be noted that the method finds a solution directly in the continuum, while the solution by FEM needs to discretize the medium and is, therefore, sensitive to the discretization adopted. On the other hand, the optimization technique presented in this work can be applied over the shape functions into the FEM environment to make it possible that the domain of the problem will not need to be divided into too many subdomains, especially for nonlinear problems. Therefore, this paper presents contribution in a deep sense the usefulness of optimization of shape functions instead of increasing the discretization of the numerical model by using excessive number of the so-called finite element.
* When comparing Rayleigh’s method with FEM, it can be observed that the structural frequency is conditioned by the modulus of elasticity of the material, i.e., as the structure is flexible the difference with FEM results increases due to the mass present at the generalized coordinate of the system. The Rayleigh method tends to add more stiffness to the system than FEM, but leads to lesser mass than FEM. The results depend on the magnitude of these generalized parameters obtained by different equations and methods. In the Rayleigh method the shape function is not corrected as it is done in FEM when the modulus of elasticity varies with time. Another important aspect related to the comparison between Rayleigh method and FEM is that the former can be used to establish the upper bound whereas the latter can be used to establish the lower bound of the result for the vibration and bucking problems of non-prismatic columns.
* It is important to highlight that the analysis carried out in this article can be extended to many other practical problems that can be modeled by one-dimensional elements. It is worth noting that the results obtained in the present investigation can be compared to similar structures, using an analogy through the slenderness ratio and the reinforcement rate of the cross-sections. However, it is important to note that the results reported are applicable to the specific illustrative examples, although the general procedure can be adopted as a guide to solve similar problems.
* The different sets of results were obtained by two distinct mathematical methods, each one approximated by the characteristics of its own. This can motivate a comparison with other mathematical methods in future works. Correcting the shape function according to longitudinal and transversal deformations to consider the effect of normal force and Poisson’s ratio is also a research topic for future investigations. In accordance with the methodology used in the present article, studies about the application of the optimization technique considering a greater number of cases may provide optimized coefficients which can be representative of a family of structures.

# Appendix A – Rayleigh method for free vibration and buckling analysis of non-prismatic and prismatic columns

A non-prismatic column of length *L* is schematically shown in Fig. A1 in which the *X*-axis coincides with the axis of the column and the origin is chosen to be at the bottom (base) of the column. The *Y*-axis which is perpendicular to the *X*-axis, is from left to right as shown in the figure. The column is split into *n* segments with each segment designated by the letter *s* (lowercase). Therefore, *Ls* and *Ls−1* are the height at the upper and lower limits of a given segment *s*, whose extension is calculated by the difference between these two positions. In the analysis given below *v*(*x*, *t*) is the displacement of the column in the *Y*-direction, **(*x*) the assumed shape function (to be optimized later), *q*(*t*) is the generalized co-ordinate so that the displacement of the column *v*(*x*, *t*) at a distance *x* from the origin is given by *v*(*x*, *t*) = **(*x*)*q*(*t*), with *u*(*t*) representing the vertical shortening of the column due to the lateral movement, where *t* denotes the time. The system has constant as well as variable properties. These include axial compressive forces due to the acceleration *g* of the gravity which takes effect through the distributed mass along the length and the externally located lumped mass at the top. The external diameter, the second moment of area, the modulus of elasticity or Young’s modulus and the mass per unit length of a typical segment *s* are represented by , , and  respectively. The spring stiffness associated with the lateral soil resistance in the *Y*-direction within the foundation part of the column is represented by which varies within the foundation length *Gr* and the springs are abbreviated as *Spr* for brevity.

For the condition described, the non-prismatic column of Fig. A1 is under the action of gravitational forces, originating from distributed masses, including the self-weight and a an externally applied concentrated mass, , at the upper end, whose limiting value is determined in terms of the critical buckling load. In this assumption the mass is obtained by dividing the buckling load by the acceleration due to gravity. It is important to note here that buckled configuration occurs in the neighborhood of the unbuckled state and all values of loadings are evaluated near the critical values.

Diagrama

Descrição gerada automaticamente

Fig. A1. Model of an isolated column in an undamped free vibration

The shape function **(*x*) below in Eq. (A1), describing the displacement *v*(*x*, *t*) of the column at a distance *x* through the use of the generalised coordinate *q*(*t*) can be expressed as

|  |  |
| --- | --- |
|  | (A1) |

It is assumed that the free vibration of the column occurs in the neighbourhood of the undeformed configuration. So, the external virtual work done by the forces of inertia, *WE*, can be expressed as:

|  |  |
| --- | --- |
| , | (A2) |

where an over dot represents differentiation with respect to time. The internal virtual work performed by the bent deformation, *WI*(*t*), is given by:

|  |  |
| --- | --- |
| , | (A3) |

where a prime denotes differentiation with respect to *x*. The work performed by the foundation represented by springs, *WSpr*, is given by:

|  |  |
| --- | --- |
| . | (A4) |

To calculate the work done by the vertical force, whose direction and amplitude remained unchanged during the movement, it is necessary to evaluate the vertical component of the motion of the column tip. The total compressive displacement *u*(*t*) along the column in the *X*-direction is given by:

|  |  |
| --- | --- |
| or | (A5) |

Thus, the potential energy of the axial load including the concentrated mass at the top, the acceleration due to gravity of the variable mass and axial shortening of the column can be represented by the following expression.

|  |  |
| --- | --- |
| , | (A6) |

where:

|  |  |
| --- | --- |
| , , with , | (A7) |

in which is the lumped mass at the position of the generalized coordinate, represents the cross-sectional area, is the density of the material, and is an external additional distributed mass of the segment *s*. Therefore, the external and internal virtual work obeys the following relationships:

|  |  |
| --- | --- |
| , | (A8) |

which leads to the equation given below

|  |  |  |
| --- | --- | --- |
|  |  | (A9) |

where:

|  |  |
| --- | --- |
| , , . | (A10) |

Therefore, the real and virtual displacements and their derivatives can now be expressed as functions of generalized coordinates *q*(*t*) and shape function *f*(*x*) using the following equations.

|  |  |  |
| --- | --- | --- |
|  |  | (A11) |

Substituting appropriately each term from Eq. (A11) into Eq. (A9) and collecting terms for the common parameters, it is now possible to arrive at the following equation.

|  |  |
| --- | --- |
|  | (A12) |

Because the variation, *q*(*t*), can be completely arbitrary, the term within the square bracket must be zero. In this way, for undamped free natural vibration, Eq. (A12) can be written in simplified notation to give the following familiar form.

|  |  |
| --- | --- |
| , | (A13) |

where the total generalized mass, *M*(), of the system, including the mass at the tip, is given by

|  |  |
| --- | --- |
| . | (A14) |

By considering the normal force, i.e., the compressive force, as positive, the total structural stiffness *K*(*t*, ) is essentially a function of two variables, namely and *t* and the stiffness arising from the soil resistance of the foundation. Thus,

|  |  |
| --- | --- |
| , | (A15) |

where the first term , is the usual generalized stiffness resulting from elastic deformation given by:

|  |  |
| --- | --- |
| , | (A16) |

in which, for a segment *s* of the column, , is the time-dependent elastic modulus of the column material, is the variable second moment of area of the cross-section along the segment *s*.

The second term in Eq. (A15), , is the geometric stiffness, which appears as a function of the axial load, including the self-weight contribution. can be expressed as:

|  |  |
| --- | --- |
| , | (A17) |

In accordance with Eq. (A7), is actually the concentrated force at the top, which is dependent on the mass located at the top, and is the normal force from the upper segments, where is the mass per unit length representing the segment weight when integrated in the segment length. The third term in Eq. (A15), is the elastic stiffness of the spring which characterize the soil resistance, which must be integrated within limits of the segment defined to foundation part. is given by

|  |  |
| --- | --- |
| . | (A18) |

Consider that the elastic stiffness of a segment *s* assumes the following form

|  |  |
| --- | --- |
| , | (A19) |

where the total stiffness of the soil is basically an elastic characteristic of the soil, comprising the sum of stiffnesses of all individual components along the foundation depth, which will of course, depend on the geometry of the foundation denoted here by and the elastic soil parameter denoted here by . In Eq. (A6), the summation is extended over *n* where *n* is the number of segments along the height of the column within the foundation. Therefore, for circular or angular natural frequency *w*, as a function of time and the mass at the tip, the following expression is valid.

|  |  |
| --- | --- |
| . | (A20) |

Making the mass at the top of the pole to vary, the force acting on the pole will naturally vary and so does the natural frequency of the structure in accordance with Eq. (A20). Thus, the critical buckling load is defined as the compressive normal force at zero frequency to give

|  |  |
| --- | --- |
| . | (A21) |

In the special case when the column is of prismatic cross-section, as indicated in Fig. A2, closed-from equations for the frequency and critical buckling load can be obtained for each basic shape function proposed in Eqs. (1) to (4) of the main text. In the analysis, the column presents lateral movement according to *Y*-direction in relation to the assumed shape function **(*x*). Once the column has constant parameters along its height, *L*, the external dimension, the moment of inertia of the cross section, the modulus of elasticity and the mass per unit length are valid to entire structure and are represented by *D*, *I*, *E* and respectively. The modulus of elasticity can assume the form *E*(*t*) to represent eventual viscoelastic behavior of the material and *g* is the acceleration due to gravity.

Diagrama

Descrição gerada automaticamente

Fig. A2. A prismatic column model with constant parameters

For obtaining closed-form solution, it is necessary to solve the integrals represented by the PRM considering the interval from 0 to *L* for the generalized properties of the structure and between 0 to *aL* for the stiffness of the soil. Therefore, *a* represents the relative value of the buried part of the foundation, which can vary from 0, when the column is laterally free, up to 1, when the column is completely buried. Therefore, *n*(*a*) represents the influence of the stiffness of the soil, where is the spring, *Spr*, coefficient, which depends on the type of soil, and which must be applied within the foundation length, *Gr*. For the shape function given by Eq. (1), the frequency and critical buckling load are:

|  |  |
| --- | --- |
| , | (A22) |
| , with | (A23) |
| . | (A24) |

For Eq. (2), they are:

|  |  |
| --- | --- |
|  | (A25) |
| , with | (A26) |
| . | (A27) |

For Eq. (3):

|  |  |
| --- | --- |
| , | (A28) |
| , with | (A29) |
| . | (A30) |

And for Eq. (4):

|  |  |
| --- | --- |
| , | (A31) |
| , with | (A32) |
| . | (A33) |

# Appendix B – Mass, structural stiffness and geometric stiffness and soil stiffness matrices in the usual notation for FEM application

|  |  |
| --- | --- |
| , | (B1) |

|  |  |
| --- | --- |
| , | (B2) |

|  |  |
| --- | --- |
| . | (B3) |

|  |  |
| --- | --- |
| . | (B4) |

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