Defining an Affine Partition Algebra

Samuel Creedon

A thesis submitted for the degree of
Doctor of Philosophy

City, University of London
Department of Mathematics

March 2022

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Acknowledgements

First and formost I would like to express my sincere gratitude to Dr Maud De Visscher for her continuous support and guidence throughout my studies. Without her assistance at every stage during my PhD, the current thesis would certainly not exist. I am also deeply grateful to the welcoming environment and staff at the mathematics department at City, University of London.

I am thankful to my family for their tremendous understanding and encouragement in the past few years.

In addition, I would like to thank both Dr Anton Cox and Professor Paul Martin for invaluable discussions and their advice regarding the thesis, and for agreeing to be the internal and external examiners for my Viva respectively. I would also like to thank Dr Olalla Castro Alvaredo for moderating the Viva.

Abstract

This thesis is focused on constructing an affine version of the partition algebra and investigating expected properties that such an algebra should admit. In fact we provide two definitions of such an algebra in this thesis. The first definition is given by generators and relations, and comes about by affinizing the partition algebra in a similar manner employed by others on related diagrams algebras. The second definition is obtained by generalising the orbit basis of the partition algebra in a particular manner. Both such algebras give rise to actions on a tensor space which extends the action in Schur-Weyl duality between the partition algebra and group algebra of the symmetric group. We establish a strong connection to one of these affine partition algebras with the Heisenberg category. Namely we prove that a certain endomorphism algebra of a given object in the Heisenberg category is a quotient of the affine partition algebra.

Pursuing the construction of such algebras has also lead to new results regarding both the partition algebra and symmetric group. For the partition algebra we obtain a complete description of the center in the semisimple case, and give an alternative description of the blocks in the non-semisimple case. For the symmetric group, we generalise certain results regarding the centers of the group algebras of the symmetric groups to certain centraliser algebras. From such we are able to provide a centraliser construction of the degenerate affine Hecke algebra, and show that a certain limit of centralizer algebras appears as an endomorphism algebra in the Heisenberg category.

1 Introduction

In this chapter we summarise various results to set the scene and motivate the work of this thesis. We begin by recalling the classical Schur-Weyl duality between the general linear group and the symmetric group, and an extension involving the degenerate affine Hecke algebra. We review some properties of the degenerate affine Hecke algebra and discuss its construction. We describe various generalisations of Schur-Weyl duality and the resulting diagram algebras which emerge. We summarise the construction of new affine versions of some of these diagram algebras. We end the chapter by discussing the main questions investigated within this thesis, summarise the structure of the thesis, and providing a list of the main results.

1.1 Schur-Weyl Duality

For any $n \geq 0$ we let $\mathsf{GL}_n(\mathbb{C})$ denote the group of invertible n-by-n matrices with entries in $\mathbb{C}$. Also let $V = \mathbb{C}^n$ be its fundamental n-dimensional representation. For any $k \geq 0$ we view the k-fold tensor space $V^{\otimes k}$ as a representation of $\mathsf{GL}_n(\mathbb{C})$ given by the diagonal action. The symmetric group $\mathfrak{S}_k$ also acts on this space via permutating the k tensor components. This pair of actions

$$\mathsf{GL}_n(\mathbb{C}) \circlearrowleft V^{\otimes k} \circlearrowright \mathfrak{S}_k \tag{1.1}$$

commute with one another, and the centralizer of one action in $\mathsf{End}_{\mathbb{C}}(V^{\otimes k})$ generates the other. As such they satisfy the double centralizer theorem. This gives the classical Schur-Weyl duality, established first by Schur in [Schur01]. This duality allows information regarding the representation theory of these two groups to flow back and forth, and hence gives a powerful tool in investigating and answering problems for either group. One may restate this result by working with the special linear group $\mathsf{SL}_n(\mathbb{C})$ instead of $\mathsf{GL}_n(\mathbb{C})$, or further still by working with their corresponding Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ or their universal enveloping algebras $U(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_n)$ respectively.

The duality of *Equation* (1.1) has seen numerous generalisations in various directions. A quantum (or q-deformation) of this duality was given by M. Jimbo in [Jimbo86] where the universal enveloping algebras $U(\mathfrak{gl}_n)$ is replaced by its quantum group, and the symmetric group is replaced with the Hecke algebra. An affine counterpart to this quantum duality of M. Jimbo was established by V. Chari and A. Pressley in [CP96], where the quantum group is replaced with the quantum affine group and the Hecke algebra is replaced with the affine Hecke algebra.

Another generalisation of Schur-Weyl duality, which is of particular interest for this thesis, was established by T. Arakawa and T. Suzuki in [AS98]. This involves replacing

$V^{\otimes k}$ by $M \otimes V^{\otimes k}$, where M is any (possible infinite dimensional) module over $U(\mathfrak{gl}_n)$, and replacing $\mathfrak{S}_k$ with the degenerate affine Hecke algebra $\mathcal{H}_k$ introduced by V. Drinfeld in [Dri86]. Put into symbols we have commuting actions

$$U(\mathfrak{gl}_n) \circlearrowleft M \otimes V^{\otimes k} \circlearrowright \mathcal{H}_k. \tag{1.2}$$

For an arbitrary M, these commuting actions are far from satisfying the double centralizer theorem. However, by specialising M and working with quotients of $\mathcal{H}_k$ referred to as *cyclotomic quotients*, then the double centralizer theorem can be shown to hold, and gives rise to the so called *higher Schur-Weyl dualities*. See for example the works of J. Brundan and A. Kleshchev in [BK08].

1.2 Constructing $\mathcal{H}_k$ from $\mathbb{C}\mathfrak{S}_k$

Often the degenerate affine Hecke algebra $\mathcal{H}_k$ is defined by a presentation, where the generators are the simple transpositions of the symmetric group $\mathfrak{S}_k$ alongside new pairwise commuting generators which resemble the *Jucys-Murphy elements* of the group algebra $\mathbb{C}\mathfrak{S}_k$ in a manner we explain below. In this section we briefly summarise the construction of $\mathcal{H}_k$, with greater detail given in *Section 2.1.7* (see also [K05]).

Firstly, the Jucys-Murphy elements $Y_1, \ldots, Y_k$ of the group algebra $\mathbb{C}\mathfrak{S}_k$ are a collection of pairwise commuting elements defined by the sum of certain transpositions (see *Definition 2.11*). These elements play an important role in the representation theory of $\mathbb{C}\mathfrak{S}_k$. In fact, A. Okounkov and A. Vershik in [OV96] used the Jucys-Murphy elements to provide a spectral approach to the representation theory of $\mathbb{C}\mathfrak{S}_n$, recovering its combinatorial features in a natural fashion. Let s_i denote the simple transposition in $\mathfrak{S}_k$ exchanging i and $i+1$ for each $1 \leq i \leq k-1$, which generate the algebra $\mathbb{C}\mathfrak{S}_k$. The Jucys-Murphy elements share the following relations with these generators:

- $s_i Y_j = Y_j s_i$ for all $j \neq i, i+1$.
- $Y_{i+1} = s_i Y_i s_i + s_i$ for all $1 \leq i \leq k-1$.

The second relation demonstrates a recursive structure for the Jucys-Murphy elements, in particular they can be defined from such by setting $Y_1 = 0$. This recursive structure was an important aspect of the work in [OV96].

To go from $\mathbb{C}\mathfrak{S}_k$ to the degenerate affine Hecke algebra $\mathcal{H}_k$, one adjoins new pairwise commuting generators (called affine generators) y_i for each $1 \leq i \leq k$, and imposes relations between these generators and the simple transpositions which are identical to above where one replaces Y_i with y_i (see *Definition 2.23* for the presentation of $\mathcal{H}_k$). Hence the generators y_i resemble the Jucys-Murphy elements Y_i in the sense that they share a handful of analogous relations, in particular an analogous recursive structure. We have a surjective algebra homomorphism $\mathcal{H}_k \to \mathbb{C}\mathfrak{S}_k$ by projecting $y_i \mapsto Y_i$ for each $1 \leq i \leq k$. Also it can be shown that as a vector space $\mathcal{H}_k \cong \mathbb{C}[y_1, \ldots, y_k] \otimes \mathbb{C}\mathfrak{S}_k$, where $\mathbb{C}[y_1, \ldots, y_k]$ is the space of polynomials in k commuting variables. In this manner we have obtained $\mathcal{H}_k$ by adjoining a polynomial algebra to $\mathbb{C}\mathfrak{S}_k$, but have asked the variables to satisfy some non-trivial relations which project down to analogous relations satisfied by the Jucy-Murphy elements. It is worth mentioning that this procedure to construct $\mathcal{H}_k$ from $\mathbb{C}\mathfrak{S}_k$ relied on a choice of what relations to impose on the affine generators. We will discuss more on this matter at the begining of *Chapter 4*.

The action of $\mathcal{H}_k$ on $M \otimes V^{\otimes k}$ in *Equation* (1.2) is obtained by allowing the generators y_j to act in a manner which extends the action of the Jucy-Murphy elements Y_j on $V^{\otimes k}$ in *Equation* (1.1) onto the M component (see *Theorem 2.27*). One can also interpret the algebra $\mathcal{H}_k$ diagrammatically, as described later in *Section 2.1.7*, were the permutations are viewed using string permutation diagrams, and the affine generators y_i are viewed as decorations on the strings. The relations above which the generators y_i are asked to satisfy can be interpreted as local relations encoding how decorations can move along a string and how they interact with crossings of strings.

The properties of the Jucys-Murphy elements of the group algebra of the symmetric group have been abstracted (see for example [GG11] where Jucys-Murphy elements are defined in the general setting of cellular algebras belonging to strongly coherent towers). Hence many algebras possess their own versions of Jucys-Murphy elements. Also the Jucys-Murphy elements are not unique, that is many such collections of Jucy-Murphy elements may be defined for a given algebra.

1.3 Diagram Algebras

In this section we summarise properties of certain examples of diagram algebras, which are algebras with a diagrammatically defined basis where the product is given by the linear extension of diagram concatenation. These algebras come about from generalisations of Schur-Weyl Duality. We also describe the construction of algebras which are to most of these examples of diagram algebras what the degenerate affine Hecke algebra is to the group algebra of the symmetric group.

Firstly, when one replaces the group $\mathsf{GL}_n(\mathbb{C})$ with the subgroup of orthogonal matrices $\mathsf{O}_n(\mathbb{C})$ (or the symplectic subgroup $\mathsf{Sp}_n(\mathbb{C})$ when n is even) in *Equation* (1.1), then the algebra playing the role of $\mathfrak{S}_k$ is the diagram algebra called the Brauer algebra $\mathfrak{B}_k(n)$ (respectively $\mathfrak{B}_k(-n)$) which was first introduced by Brauer in [Br37]. Thus we have commuting actions

$$\mathsf{O}_n(\mathbb{C}) \circlearrowleft V^{\otimes k} \circlearrowright \mathfrak{B}_k(n) \tag{1.3}$$

which satisfy the double centralizer theorem. The Brauer algebra $\mathfrak{B}_k(\delta)$ may be defined for any parameter $\delta \in \mathbb{C}$. The group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_k$ is a subalgebra of $\mathfrak{B}_k(\delta)$, where the diagrammatic basis of $\mathfrak{B}_k(\delta)$ contains the permutation diagrams of $\mathfrak{S}_k$, and new diagrams which introduce 'cups' and 'caps'. When taking the product of two diagrams within $\mathfrak{B}_k(\delta)$, in general 'floating loops' will appear, and these are resolved by replacing each such occurence with the scalar δ.

M. Nazarov defined Jucys-Murphy elements for the Brauer algebras in [N96]. A variety of relations between these Jucys-Murphy elements and a natural generating set of $\mathfrak{B}_k(\delta)$ (including a relation which can recursively define the Jucys-Murphy elements) were proven. M. Nazarov also define a new algebra $\mathcal{W}_k$ constructed from $\mathfrak{B}_k(\delta)$ by adjoining new pairwise commuting affine generators which were asked to satisfy various relations which projected down to relations satisfied by the Jucys-Murphy elements. Hence there is a surjective algebra homomorphism $\mathcal{W}_k \to \mathfrak{B}_k(\delta)$ sending the affine generators onto the corresponding Jucys-Murphy elements. Also the polynomial algebra in the affine generators is a subalgebra of $\mathcal{W}_k$. The algebra $\mathcal{W}_k$ is called the degenerate affine Wenzl algebra or just the affine Wenzl algebra. The affine generators may be interpreted diagrammatically as decorations on the underlying diagrams defining $\mathfrak{B}_k(\delta)$. Also new central generators were added in the construction of $\mathcal{W}_k$ as a means to resolve decorated floating loops which appear within the diagrammatics, a feature which was absent in the diagrammatics of the symmetric group. The algebra $\mathcal{W}_k$ also gives rise to commuting actions

$$U(\mathfrak{o}_n) \circlearrowleft M \otimes V^{\otimes k} \circlearrowright \mathcal{W}_k, \tag{1.4}$$

where $U(\mathfrak{o}_n)$ is the universal enveloping algebra of $\mathfrak{o}_n$, the orthogonal Lie algebra, and M is taken to be any (possible infinite dimensional) module over $U(\mathfrak{o}_n)$. In this sense, the algebra $\mathcal{W}_k$ is to the Brauer algebra $\mathfrak{B}_k(\delta)$ what the degenerate affine Hecke algebra $\mathcal{H}_k$ is to the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_k$. See also the work of Z. Daugherty, A. Ram, and R. Virk in [DVR11] where the algebra $\mathcal{W}_k$ is studied in unison with a non-degenerate version called the affine BMW algebra, with a focus on such commuting actions.

Returning to the classical Schur-Weyl duality of *Equation* (1.1), if one replaces $V^{\otimes k}$ with the mixed tensor space $V^{\otimes s} \otimes (V^*)^{\otimes r}$ where $s, r \in \mathbb{Z}_{\geq 0}$ such that $s + r = k$, and where V^* is the dual representation of V, then the algebra which replaces the symmetric group $\mathfrak{S}_k$ is the diagram algebra called the walled Brauer algebra $\mathfrak{B}_{s,r}(n)$. Hence we have commuting actions

$$\mathsf{GL}_n(\mathbb{C}) \circlearrowleft V^{\otimes s} \otimes (V^*)^{\otimes r} \circlearrowright \mathfrak{B}_{s,r}(n), \tag{1.5}$$

which satisfy the double centralizer theorem. The algebra $\mathfrak{B}_{s,r}(\delta)$ may be defined for any parameter $\delta \in \mathbb{C}$, is a subalgebra of the Brauer algebra $\mathfrak{B}_k(\delta)$, and contains the group algebra $\mathbb{C}(\mathfrak{S}_s \times \mathfrak{S}_r)$. The walled Brauer algebras first appeared independently in [Koi89] and [Tur89], which were partially motivated by such a duality. It was later studied in [BHCLLS94] as a means of decomposing the mixed tensor space into irreducible $\mathsf{GL}_n(\mathbb{C})$ representations. Jucys-Murphy elements have also been defined for the walled Brauer algebras, and new algebras which are to the walled Brauer algebras what $\mathcal{H}_k$ is to $\mathbb{C}\mathfrak{S}_k$ were introduced independently by A. Sartori in [Sar13], and by H. Rui and Y. Su in [RS13]. In the latter these algebras were called the affine walled Brauer algebras, and were denoted by $\mathscr{B}^{\text{aff}}_{s,r}(\omega_0, \omega_1) = \mathscr{B}^{\text{aff}}_{s,r}$. They were constructed in an analogous manner in which M. Nazarov constructed $\mathcal{W}_k$, by introducing new pairwise commuting affine generators to the walled Brauer algebra and imposing relations on such generators which project down to relations satisfied by the Jucys-Murphy elements. As one could expect, these affine generators may be interpreted diagrammatically as decorations added to the diagrammatics of the walled Brauer algebra. A collection of central generators were also added to account for decorated floating components, as was done for $\mathcal{W}_k$. It was shown that the affine walled Brauer algebra gives rise to commuting actions

$$U(\mathfrak{gl}_n) \circlearrowleft M \otimes V^{\otimes s} \otimes (V^*)^{\otimes r} \circlearrowright \mathscr{B}^{\mathrm{aff}}_{s,r}, \tag{1.6}$$

where one has replaced the mixed tensor space $V^{\otimes s} \otimes (V^*)^{\otimes r}$ with $M \otimes V^{\otimes s} \otimes (V^*)^{\otimes r}$, where M is any (possible infinite dimensional) module over $U(\mathfrak{gl}_n)$, and replaces the walled Brauer algebra $\mathfrak{B}_{s,r}(n)$ with $\mathscr{B}^{\mathrm{aff}}_{s,r}$.

Again returning to the classical Schur-Weyl duality of *Equation* (1.1), if one replaces the group $\mathsf{GL}_n(\mathbb{C})$ with the subgroup of permutation matrices, which one may identify with the symmetric group, then the diagram algebra which appears in the duality is the partition algebra $\mathcal{A}_{2k}(n)$. Hence we have commuting actions

$$\mathbb{C}\mathfrak{S}_n \circlearrowleft V^{\otimes k} \circlearrowright \mathcal{A}_{2k}(n), \tag{1.7}$$

which satisfy the double centralizer theorem. This duality was orginally proved by V. Jones in [J94]. Compared to the other dualities we have discussed, this case is unique in the sense that no classical Lie algebra is involved, but rather the group algebra of the symmetric group. The partition algebra $\mathcal{A}_{2k}(\delta)$ may be defined for any parameter $\delta \in \mathbb{C}$, in fact we will often set $\mathcal{A}_{2k} := \mathcal{A}_{2k}(z)$ where z is a free central generator of the algebra (see *Section 2.2* for more details). This algebra was first introduced by P. Martin in the works of [M91] regarding problems in statistical mechanics. It is a diagram algebra and was defined by a diagrammatic basis and structure constants. It was later given a presentation in [HR05] and [East11]. T. Halverson and A. Ram in [HR05] gave the first definition of Jucys-Murphy elements (defined diagrammatically) for the partition algebra $\mathcal{A}_{2k}(\delta)$. Later J. Enynag gave a recursive definition for these Jucys-Murphy elements in [Eny12] and [Eny13] and proved a variety of relations involving these elements.

To the best of the author's knowledge, before starting this thesis there was no algebra which played an analogous role for the partition algebra which $\mathcal{H}_k$ plays for $\mathbb{C}\mathfrak{S}_k$. Although, while writing this thesis such an algebra was defined in the works of J. Brundan and M. Vargas in [BV21], and more details of this algebra and its connection to our work will be given later in *Section 4.3*.

1.4 Aims of the Thesis

The main aim of the thesis is to construct an affine partition algebra, that is an algebra which is to the partition algebra what the degenerate affine Hecke algebra is to the group algebra of the symmetric group, and to investigate some of its structural and representation theoretic properties. However, there is no canonical definition of what it means to be such an algebra. Thus it is important that we make clear what we would consider to be an appropriate definition for an affine partition algebra. We say that A^{aff}_{2k} is an *affinization* of $\mathcal{A}_{2k}$ if the following hold:

1. A^{aff}_{2k} contains both $\mathcal{A}_{2k}$ and the polynomial algebra $\mathbb{C}[x_1, \ldots, x_{2k}]$ in $2k$ commuting variables as subalgebras.

2. The variables x_i should satisfy relations which are analogous to relations in $\mathcal{A}_{2k}$ which are satisfied by Jucys-Murphy elements (for some fixed choice of Jucys-Murphy elements of $\mathcal{A}_{2k}$), including recursive relations for the Jucys-Murphy elements.

3. $\mathcal{A}_{2k}$ is a quotient of A_{2k}^{aff} in such a way that the quotient map $A_{2k}^{\mathrm{aff}} \to \mathcal{A}_{2k}$ projects the variables x_i to corresponding Jucys-Murphy elements.

4. Given any $\mathfrak{S}_n$-module M, there exists an action of A_{2k}^{aff} on the tensor space $M \otimes V^{\otimes k}$ which commutes with the diagonal action of $\mathfrak{S}_n$. This action should extend *Equation* (1.7) in an analogous manner to how *Equation* (1.2) extended *Equation* (1.1).

5. There exists a diagrammatic description of A_{2k}^{aff} which generalises that of $\mathcal{A}_{2k}$ where the variables x_i are interpreted as decorations.

We refer to these five properties as the *affinization properties*. It is worth mentioning that such properties could be abstracted to describe the affinization of any diagram algebra which possess a collection of Jucys-Murphy elements and belongs to some Schur-Weyl duality. In particular the algebras $\mathcal{H}_k$, $\mathcal{W}_k$, and $\mathscr{B}_{s,r}^{\mathrm{aff}}$ satisfy analogous properties in relation to the underlying diagram algebras $\mathbb{C}\mathfrak{S}_k$, $\mathfrak{B}_k(\delta)$, and $\mathfrak{B}_{s,r}(\delta)$ respectively. Hence in our terminology we say that $\mathcal{H}_k$, $\mathcal{W}_k$, and $\mathscr{B}_{s,r}^{\mathrm{aff}}$ are affinizations of $\mathbb{C}\mathfrak{S}_k$, $\mathfrak{B}_k(\delta)$, and $\mathfrak{B}_{s,r}(\delta)$ respectively. Some aspects of these affinization properties are fairly vague, although we give a bit more detail below. We do not expect that abstracting the above properties would give a good notion of affinizing an algebra in a more general sense, and we do not pursue defining such a notion in this thesis. We simply wish to be more concrete in what our main aim of the thesis was implying. So we treat the above description of an affinization of $\mathcal{A}_{2k}$ as a guide/benchmark for our constructions.

We briefly give some additional details to the affinization properties above. Firstly all five such properties certainly seem appropriate to ask since each of the algebras $\mathcal{H}_k$, $\mathcal{W}_k$, and $\mathscr{B}_{s,r}^{\mathrm{aff}}$ satisfy analogous properties, and none of the five properties appears to be asking too much. Affinization property 1 confirms that we at least have a non-trivial superalgebra of the partition algebra $\mathcal{A}_{2k}$ which contains the affine algebra $\mathbb{C}[x_1, \dots, x_{2k}]$. Affinization property 2 enforces the affine generators x_i to share at least some important structure with that of Jucys-Murphy elements. This property is quite vague. Informally we want the affine generators to share as much structure with Jucys-Murphy elements as they can while still satisfying the other affinization properties. In particular, we do not want to add too many relations for the affine generators to satisfy which would prevent the set of monomials $x_1^{n_1} \cdots x_{2k}^{n_{2k}}$ (for $n_i \in \mathbb{Z}_{\geq 0}$ for each $1 \leq i \leq 2k$) from being linearly independent, or worst still which collapses the algebra A_{2k}^{aff} down to $\mathcal{A}_{2k}$. For example if we imposed the relation $y_1 = 0$ in $\mathcal{H}_k$, which satisfies item 2 since $Y_1 = 0$, then the algebra $\mathcal{H}_k$ would collapse to $\mathbb{C}\mathfrak{S}_k$. Affinization property 3 simply means that the affine generators do not satisfy relations not shared by Jucys-Murphy elements. As such, in the sense of satisfying item 1, item 2, and item 3, we refer to the affine generators x_i as affinizations of the Jucys-Murphy elements. Lastly affinization properties 4 and 5 are asking the most of A_{2k}^{aff}, which are the key representation theoretic and structural results, respectively, which we expect to be satisfied when comparing to the algebras $\mathcal{H}_k$, $\mathcal{W}_k$, and $\mathscr{B}^{\mathrm{aff}}_{s,r}$.

1.5 Structure of Thesis

This section closes out the introduction by summarising the structure of the thesis. We first give a discussion of what can be found in each of the proceeding chapters, highlighting the new definitions and new results established. For ease of navigation, we then provide a list of the main features of the thesis in the order in which they appear.

Chapter 2 will recall most of the theory and results regarding both the partition algebra and the symmetric group which we will use throughout the thesis. For the symmetric group, one of the main aspects we wish to focus on is the center of the group algebra of the symmetric group and the works of H. Farahat and G. Higman in [FH59]. Namely we recall a basis of the center by class sums, a polynomial property of the respected structure constants, and how this is used in [FH59] to define an algebra over a polynomial ring which "interpolates" these centers. The other main aspect of the symmetric group we recall is the role of the Jucys-Murphy elements with regard to the representation theory of the symmetric group and in the construction of the degenerate affine Hecke algebra. For the partition algebra, we will also be interested in recalling the Jucys-Murphy elements and their role in the representation theory. We recall the definition of new generators σ_i introduced by J. Enyang in [Eny12], and how they were used to give a new presentation for the partition algebra and a new recursive definition for the Jucys-Murphy elements. We also give details to the Schur-Weyl duality of *Equation* (1.7), and recall a basis of the partition algebra which is particularly well-adapted to this duality called the orbit basis. We end the chapter by describing how one can define the partition algebra from the ground up with a focus on Schur-Weyl duality and the orbit basis.

Chapter 3 focuses on proving two new results regarding the partition algebra. The first result is *Theorem 3.17*, which gives a description of the center of the semisimple partition algebras by supersymmetric polynomials (see *Definition 3.1*) in normalised Jucys-Murphy elements. The centers of the (semisimple) diagram algebras mentioned above have all been shown to equal certain (super)symmetric polynomials in some collection of Jucys-Murphy elements. As such an analogous result for the partition algebra was expected but not yet know. This is what our result gives, and it is worth remarking that such a clean answer was not necessarily expected since the Jucys-Murphy element in the partition algebra are much more complicated than their counterparts in the other aforementioned diagram algebras. Before this description of the center of the semisimple partition algebra, the primitive central idempotents had been constructed by P. Martin and D. Woodcock in [MW99]. In their work they gave a recursive construction of the so-called splitting idempotent associated to a certain exact sequence related to the partition algebra. Then by multiplying this idempotent by the primitive central idempotents of group algebras of the symmetric group sitting inside the partition algebra, one recovers the primitive central idempotents for the partition algebra. They gave a complete and explicit description of the primitive central idempotents for $\mathcal{A}_4(\delta)$ ($k = 2$

case), and highlighted that information about the blocks in the non-semisimple case can be determined by the blow up of certain denominators in the coefficients appearing in such primitive central idempotents. The work of [MW99] was conducted prior to the definition of the Jucys-Muprhy elements for the partition algebra. The second new result of *Chapter 3* is *Corollary 3.25* which closes out the chapter and gives an alternative description of the blocks of the partition algebra $\mathcal{A}_{2k}(\delta)$ in the non-semisimple case. These blocks were already described in an elegant manner by P. Martin in [Martin96] (see also [DW00]) as corresponding to certain chains of Young diagrams which satisfy a particular combinatorial condition involving the parameter δ. On the other hand our result given in *Corollay 3.25* describes these blocks using certain generating functions which appear naturally from the action of certain central elements of the partition algebra on simple modules. The results of this chapter were published in [Cre21], although some of the proofs and exposition have been significantly streamlined in this thesis.

Chapter 4 is where we tackle the main aim of this thesis by introduced an affine partition algebra, which we denote by $\mathcal{A}_{2k}^{\text{aff}}$. This is constructed by employing a procedure to the partition algebra $\mathcal{A}_{2k}(\delta)$ analogous to what was done for the other diagram algebras mentioned above. We give greater details to this procedure at the start of the chapter. In short we begin by establishing a variety of relations involving both the Jucys-Murphy elements and the new generators σ_i of J. Enyang. We then go on to use such relations to define $\mathcal{A}_{2k}^{\text{aff}}$ by a presentation. The recursive relations of the Jucys-Murphy elements of the partition algebra are much more complicated than their counterparts in other diagram algebras, and hence choosing what relations to include in our presentation of $\mathcal{A}_{2k}^{\text{aff}}$ was less clear. In fact we treated the generators σ_i of J. Enyang in a similar manner to the Jucys-Murphy elements in our procedure to construct $\mathcal{A}_{2k}^{\text{aff}}$, that is not only do we introduce affine generators x_i to affinize the Jucys-Murphy elements but we also introduced new generators τ_i to play a similar role for the elements σ_i. This is a significant addition to what has been employed by others, and as such it is less obvious from its definition whether $\mathcal{A}_{2k}^{\text{aff}}$ is indeed an appropriate affine version for the partition algebra. However throughout the remainder of the chapter we show that $\mathcal{A}_{2k}^{\text{aff}}$ satisfies (fully or at least partially) each of the affinization properties 1 to 5 described previously. Most of these properties only hold if one makes the choice of introducing the new generators τ_i as we did. We interpret such as evidence to suggest that this was the correct approach to take in constructing $\mathcal{A}_{2k}^{\text{aff}}$.

The first half of *Chapter 4* investigates structural properties of $\mathcal{A}_{2k}^{\text{aff}}$. We prove many relations between the generators which allows us to establish that both the partition algebra $\mathcal{A}_{2k}$ and polynomial algebra $\mathbb{C}[x_1, \ldots, x_{2k}]$ are subalgebras of $\mathcal{A}_{2k}^{\text{aff}}$. We also prove that a large collection of polynomials in the variables $x_1, \ldots, x_{2k}$ are central, and that such polynomials project down to the center of the semisimple partition algebras via evaluation by normalised Jucys-Murphy elements. We prove in *Theorem 4.24* that we have commuting actions

$$\mathbb{C}\mathfrak{S}_n \circlearrowleft M \otimes V^{\otimes k} \circlearrowright \mathcal{A}_{2k}^{\text{aff}}, \tag{1.8}$$

which extend *Equation* (1.7) in the manner we desired. Such an action would not be possible without working with the new generators τ_i instead of Enyang's generators σ_i.

The last section of the chapter is concerned with invesitgating whether $\mathcal{A}_{2k}^{\text{aff}}$ satisfies affinization poroperty 5. This leads us to establishing connections with the Heisenberg category Heis introduced by M. Khovanov in [Kho14], and to the work of J. Brundan and M. Vargas in [BV21]. We prove in *Theorem 4.53* that $\mathcal{A}_{2k}^{\text{aff}}$ projects onto an endomorphism algebra of a particular object in the Heisenberg category. The morphism spaces of the Heisenberg category are diagrammatically defined, and this projection of $\mathcal{A}_{2k}^{\text{aff}}$ sends our generators to very natural diagrams, with the generators x_i getting mapped to decorations. We suspect that the projection of $\mathcal{A}_{2k}^{\text{aff}}$ onto such an endomorphism space is an isomorphism of algebras, which would show that $\mathcal{A}_{2k}^{\text{aff}}$ satisfies 5. At this moment however we are only able to prove a surjective algebra homomorphism. Recently J. Brundan and M. Vargas in [BV21] defined a subcategory APar of the Heisenberg category generated by one object and a set of morphisms which they called the affine partition category. They used this category to recover results in the representation theory of the partition algebra. One obtains an algebra AP_k, which they also call the affine partition algebra, by taking the endomorphism algebra of a certain object in their category APar. We prove that the affine partition category APar is in fact a full subcategory of the Heisenberg category Heis. As a consequence, we obtain a basis for the morphism spaces of APar, and prove that the algebra AP_k of J. Brundan and M. Vargas is a quotient of our affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$. The results of *Chapter 4* have been submitted for publication (see [CD21]).

Chapter 5 is broken into two sections. The first is concerned with certain centralizer algebras of the group algebra of the symmetric group. Namely, given any subset $X \subset \{1,\ldots,n\}$ let $\mathsf{Stab}_n(X)$ denote the subgroup of $\mathfrak{S}_n$ consisting of permutations fixing X element-wise, then the focus is on the centralizer algebra

$$Z_n(X) := \{g \in \mathbb{C}\mathfrak{S}_n \mid \pi g = g\pi \text{ for all } \pi \in \mathsf{Stab}_n(X)\}.$$

When X is the empty set $\emptyset$ then the algebra $Z_n(X)$ is simply the center of $\mathbb{C}\mathfrak{S}_n$. We generalise many of the results of H. Farahat and G. Higman in [FH59] (which are recalled in *Chapter 2*) to these centralizer algebras. We provide a class sum basis for $Z_n(X)$ and prove a polynomial property regarding their respected structure constants in *Theorem 5.17*. From such we define a new algebra $Z(X)$ over the polynomial ring $\mathbb{C}[z]$ (see *Definition 5.33*) which generalises the Farahat-Higman algebra presented in [FH59, Page 214]. The remainder of the section proves a variety of results regarding this new algebra, many of which generalise known results regarding the Farahat-Higman algebra. A particularly interesting result is *Theorem 5.43* where we establish a connection between $Z(X)$ and the degenerate affine Hecke algebra. As a corollary we realise $Z(X)$ as an endomorphism algebra of a particular object in the Heisenberg category in a natural fashion. This section appears quite disjoint from our previous work regarding the affine partition algebra, however these results came about from investigating the image in $\mathsf{End}_{\mathfrak{S}_k}(M \otimes V^{\otimes k})$ of the action of $\mathcal{A}_{2k}^{\text{aff}}$ given in *Equation* (1.8).

The second section of *Chapter 5* focuses on certain subalgebras $Q_{2k}(M,n)$ of the endomorphism algebras $\mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$, and uses the algebras $Z(X)$ to establish certain spanning sets for $Q_{2k}(M,n)$. When M is a free $\mathbb{C}S_n$-module, we provide in

Proposition 5.54 a basis for $Q_{2k}(M,n)$ which generalises the orbit basis of the partition algebra. In a way we view the algebra $Q_{2k}(M,n)$ as a generalisation of the partition algebra $\mathcal{A}_{2k}(n) \cong \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ with the extension of the tensor space $V^{\otimes k}$ to $M \otimes V^{\otimes k}$ where M is any $\mathbb{C}\mathfrak{S}_n$-module. In *Theorem 5.52* we implicitly describe how the elements of such a basis of $Q_{2k}(M,n)$ multiply, with the main feature being that the structure constants are polynomial in n. This allows us to define a new algebra $\mathcal{Q}_{2k}^{\text{aff}}$ which we obtain from the algebras $Q_{2k}(M,n)$ in a similar manner to how $Z(X)$ was obtained from the centralizer algebras $Z_n(X)$. The algebra $\mathcal{Q}_{2k}^{\text{aff}}$ is our second construction of an affine partition algebra. Unlike with $\mathcal{A}_{2k}^{\text{aff}}$, we do not know a generating set or presentation for $\mathcal{Q}_{2k}^{\text{aff}}$, but we do have a basis for it. Under a certain projection $\mathcal{Q}_{2k}^{\text{aff}} \to Q_{2k}(M,n)$ the basis elements of $\mathcal{Q}_{2k}^{\text{aff}}$ which do not vanish form a spanning set for $Q_{2k}(M,n)$, and this spanning set is a basis whenever M is a free $\mathbb{C}\mathfrak{S}_n$-module. We end the chapter by proving the existence of an algebra homomorphism $\mathcal{A}_{2k}^{\text{aff}} \to \mathcal{Q}_{2k}^{\text{aff}}$. We suspect that such a map is an isomorphism, but this appears very difficult to prove at this point.

In summary the main features of the thesis are as follows:

Chapter 3:

- *Theorem 3.17*: $Z(\mathcal{A}_{2k}(\delta)) = \mathsf{SSym}_\delta[N_1, \dots, N_{2k}]$ for $\delta \notin \{0, 1, \dots .2k-2\}$.
- *Corollary 3.25*: Alterntaive description of the blocks of $\mathcal{A}_{2k}(\delta)$ for any $\delta \in \mathbb{C}$.

Chapter 4:

- *Definition 4.7*: Definition of the Affine Partition Algebra $\mathcal{A}_{2k}^{\text{aff}}$.
- *Theorem 4.18*: $\mathcal{A}_{2k}^{\text{aff}}$ satisfies affinization properties 1, 2, and 3.
- *Theorem 4.24*: $\mathcal{A}_{2k}^{\text{aff}}$ satisfies affinization property 4.
- *Theorem 4.53*: $\mathcal{A}_{2k}^{\text{aff}}$ partially satisfies affinization property 5.

Chapter 5:

- *Theorem 5.17*: Polynomial structure constants for class sum basis of $Z_n(X)$.
- *Definition 5.33*: Definition of the X-marked cycle shape algebra $Z(X)$.
- *Theorem 5.43*: Establishing an isomorphism $Z(X) \cong \mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes |X|})$.
- *Theorem 5.52*: Polynomial structure constants for defining basis of $Q_{2k}(M,n)$.
- *Definition 5.56*: Definition of the Orbit Affine Partition Algebra $\mathcal{Q}_{2k}^{\text{aff}}$.
- *Theorem 5.65*: Construction of an algebra homomorphism $\mathcal{A}_{2k}^{\text{aff}} \to \mathcal{Q}_{2k}^{\text{aff}}$.

2 Background

2.1 Symmetric Group Algebra

2.1.1 Definitions and Presentation

For any bijection $\pi : \mathbb{N} \to \mathbb{N}$ we define $\mathsf{Sup}(\pi) := \{i \in \mathbb{N} \mid \pi(i) \neq i\}$ and refer to such a set as the *support* of π. We denote by $\mathfrak{S}_{\mathbb{N}}$ the set of all such bijections with finite support, and refer to its elements as *permutations*. This is an infinite group under composition of functions, and we write 1 for the identity. For any $\pi \in \mathfrak{S}_{\mathbb{N}}$ we write $||\pi|| := |\mathsf{Sup}(\pi)|$. We say a permutation $\pi \neq 1$ is a *cycle* if it acts transitively on its support. When this is the case we write $\pi = (a_1, a_2, \dots, a_k)$ where $\{a_1, \dots, a_k\} = \mathsf{Sup}(\pi)$ and

$$\pi(a_i) = a_{i+1(\text{mod } k+1)}.$$

The *length* of a cycle is the size of its support, and from our convention all cycles are of length at least 2. By decomposing the support of a permutation π into disjoint orbits, we obtain a unique decomposition $\pi = \pi_1 \pi_2 \cdots \pi_k$ into cycles with pairwise disjoint supports (up to rearrangement of cycles), and we say that π contains a cycle if it is present in this decomposition. Given any $n \in \mathbb{N}$ we let $[n] := \{1, 2, \dots, n\}$ and define the finite subgroup $\mathfrak{S}_n := \{\pi \in \mathfrak{S}_{\mathbb{N}} \mid \mathsf{Sup}(\pi) \subseteq [n]\}$. Naturally $\mathfrak{S}_m \subseteq \mathfrak{S}_n$ for $m \leq n$ and $\mathfrak{S}_{\mathbb{N}} = \cup_{n \geq 1} \mathfrak{S}_n$. We identify $\mathfrak{S}_n$ with the symmetric group of permutations of $[n]$ in the obvious manner, and let $s_i := (i, i+1)$ denote the simple transposition exchanging i and $i+1$. We have the well-known presentation of $\mathfrak{S}_n$ in terms of the simple transpositions as follows:

Theorem 2.1. The group $\mathfrak{S}_n$ has a presentation with generating set $\{s_i \mid i \in [n-1]\}$ and relations

(i) $s_i^2 = 1$, for $i \in [n-1]$.

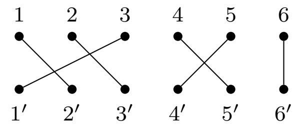
(ii) $s_i s_j = s_j s_i$, for $j \neq i-1, i+1$.

(iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [n-2]$.

$\square$

Let $[n'] := \{1', 2', \dots, n'\}$, then we view $\mathfrak{S}_n$ diagrammatically by associating a given $\pi \in \mathfrak{S}_n$ with the graph whose vertex set is $[n] \cup [n']$ and edge set $\{\{i', \pi(i)\} \mid i \in [n]\}$. We draw $[n]$ and $[n']$ as rows with vertices increasing from left to right and with the former row above the latter.

Example 2.2. Diagrammatically the permutation $\pi = (1,3,2)(4,5)$ viewed in $\mathfrak{S}_6$ may be interpreted by

$\pi =$ 

From this perspective the product of permutations corresponds to stacking diagrams and reading off the resulting pairs of vertices formed from the bottom and top row. This set up will be given a more formal treatment to define the partition algebra in *Section 2.2.1*. Also these diagrammtics will be extended to the degenerate affine Hecke algebra in *Section 2.1.7*.

2.1.2 Conjugacy Classes of $\mathfrak{S}_\mathbb{N}$ and $\mathfrak{S}_n$

In this section we recall the conjugacy classes of $\mathfrak{S}_\mathbb{N}$ and $\mathfrak{S}_n$. We define a graded monoid whose elements index the conjugacy classes of $\mathfrak{S}_\mathbb{N}$, and whose n-filtered components index the conjugacy classes of $\mathfrak{S}_n$. Some of the notation used in this section is non-standard to better compare with the results proven in *Chapter 5*.

The group $\mathfrak{S}_\mathbb{N}$ acts on itself by conjugation, and the *conjugacy classes* of $\mathfrak{S}_\mathbb{N}$ are precisely the orbits of this action. As such for any $\pi \in \mathfrak{S}_\mathbb{N}$ the conjugacy class of $\mathfrak{S}_\mathbb{N}$ containing π is given by $\mathsf{CL}(\pi) := \{\tau \mid \tau = \sigma\pi\sigma^{-1} \text{ for some } \sigma \in \mathfrak{S}_\mathbb{N}\}$. Conjugation has a natural description in cycle notation as shown in the following lemma (see for example [JL93, Proposition 12.13]).

Lemma 2.3. Express $\pi \in \mathfrak{S}_\mathbb{N}$ in cycle form as

$$\pi = (a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}) \cdots (a_{m,1}, a_{m,2}, \ldots, a_{m,n_m})$$

where $\{a_{i,j} \mid 1 \le i \le m, 1 \le j \le n_j\} = \mathsf{Sup}(\pi)$. Then for any $\sigma \in \mathfrak{S}_\mathbb{N}$,

$$\sigma\pi\sigma^{-1} = (\sigma(a_{1,1}), \sigma(a_{1,2}), \ldots, \sigma(a_{1,n_1})) \cdots (\sigma(a_{m,1}), \sigma(a_{m,2}), \ldots, \sigma(a_{m,n_m})).$$

□

From this lemma we see that $\sigma \in \mathsf{CL}(\pi)$ if and only if σ contains the same number of cycles of a given length as π. We will encode this information into a monoid as follows: Let $\mathcal{C}$ denote the free commutative monoid on the infinite set $\{c_i \mid i \in \mathbb{N}\}$. Let $\mathbb{Z}_{\ge 0}^{\mathbb{N}}$ be the set of functions $\boldsymbol{l} : \mathbb{N} \to \mathbb{Z}_{\ge 0}$ with finite support, that is there is only a finite number of elements $i \in \mathbb{N}$ such that $\boldsymbol{l}(i) \ne 0$. Then for any $\boldsymbol{l} \in \mathbb{Z}_{\ge 0}^{\mathbb{N}}$ we write

$$c^{\boldsymbol{l}} := \prod_{i \in \mathbb{N}} c_i^{\boldsymbol{l}(i)},$$

which is well-defined since $\mathcal{C}$ is commutative and $\boldsymbol{l}$ has finite support. Then as a set $\mathcal{C} = \{c^{\boldsymbol{l}} \mid \boldsymbol{l} \in \mathbb{Z}_{\ge 0}^{\mathbb{N}}\}$. We can associated any element $\lambda = c^{\boldsymbol{l}}$ of $\mathcal{C}$ with the conjugacy class of $\mathfrak{S}_\mathbb{N}$ consisting of all permutations which contain $\boldsymbol{l}(i)$ number of cycles of length $i+1$, for each $i \in \mathbb{N}$. We denote such a class by $\mathsf{CL}(\lambda)$. To help with the generalisations we will make later in *Section 5.1.2*, we will define a formal object called a *cycle shape* which we may identify bijectively to the elements of $\mathcal{C}$.

Given a set A, a cycle with entries belonging to A is a tuple $(a_1, \dots, a_m) \in A^{\times m}$ for some $m \in \mathbb{N}$, where we only care about the order of the coordinates up to cyclic shifts.

Definition 2.4. Let $*$ be a formal symbol. We define a *cycle shape* to be a finite collection of cycles with entries belonging to $\{*\}$, where no cycles of length one are present. We write a cycle shape as a formal product of its cycles by juxtaposition, where the order of the cycles is immaterial.

We let $\emptyset$ denote the unique cycle shape consisting of no cycles. It is clear that the set of all cycles shapes indexes the conjugacy classes of $\mathfrak{S}_{\mathbb{N}}$, where the class associated to a cycle shape is all the permutations of $\mathfrak{S}_{\mathbb{N}}$ one can obtain by replacing the symbols $*$ with distinct elements of $\mathbb{N}$. We may also identify $\mathcal{C}$ with the set of cycle shapes by associating an element $\lambda = c^{\boldsymbol{l}}$ with the cycle shape consisting of $\boldsymbol{l}(i)$ cycles of length $i+1$ for each $i \in \mathbb{N}$. We illustrate this by the following example.

Example 2.5. Consider the element $\lambda = c_1^2 c_2^1 c_3^2 \in \mathcal{C}$, we identify λ with the cycle shape

$$\lambda = (*,*)(*,*)(*,*,*)(*,*,*,*)(*,*,*,*),$$

where we have added colours to aid in demonstrating the identification. The corresponding conjugacy class of $\mathfrak{S}_{\mathbb{N}}$ is given by

$$\mathsf{CL}(\lambda) = \{(a_1,a_2)(a_3,a_4)(a_5,a_6,a_7)(a_8,a_9,a_{10},a_{11})(a_{12},a_{13},a_{14},a_{15}) \mid (a_i)_{i=1}^{15} \in \mathbb{N}^{!15}\},$$

where $\mathbb{N}^{!15}$ is the subset of the 15-fold cartesian product of $\mathbb{N}$ consisting of tuples with pairwise distinct entries.

We will not distinguish between an element of the monoid $\mathcal{C}$ and its associated cycle shape. We define a degree function $\mathsf{deg} : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ by

$$\mathsf{deg}(c_1^{l_1} c_2^{l_2} \cdots c_k^{l_k}) := \sum_{i=1}^{\infty} (i+1) l_i.$$

In terms of the cycle shape, the function deg is simply counting the number of symbols $*$ appearing among the cycles. Recall that $||\pi|| = |\mathsf{Sup}(\pi)|$, then given any conjugacy class C of $\mathfrak{S}_{\mathbb{N}}$ and permutations $\pi, \sigma \in \mathsf{C}$, it is clear from *Lemma 2.3* that $||\pi|| = ||\sigma||$. As such it makes sense to set $||\mathsf{C}|| = ||\pi||$ for any $\pi \in \mathsf{C}$. One can note that given $\lambda \in \mathcal{C}$ and $\pi \in \mathsf{CL}(\lambda)$, we have that $\mathsf{deg}(\lambda) = ||\pi|| = |\mathsf{Sup}(\pi)|$. Viewing $\mathbb{Z}_{\geq 0}$ as a monoid under addition, it is clear that deg is a monoid homomorphism, and hence provides a grading for the monoid $\mathcal{C}$. For any $n \in \mathbb{Z}_{\geq 0}$ we let $\mathcal{C}_n := \{\lambda \in \mathcal{C} \mid \mathsf{deg}(\lambda) = n\}$ denote the *n-th graded component* of $\mathcal{C}$. Also let

$$\mathcal{C}_{\leq n} := \{\lambda \in \mathcal{C} \mid \mathsf{deg}(\lambda) \leq n\} = \bigsqcup_{0 \leq m \leq n} \mathcal{C}_m$$

denote the *n-th filtered component* of $\mathcal{C}$. Now for any $n \geq 0$, the group $\mathfrak{S}_n$ acts on itself by conjugation and the conjugacy classes of $\mathfrak{S}_n$ are the orbits of this action. Hence for any $\pi \in \mathfrak{S}_n$, the conjugacy class of $\mathfrak{S}_n$ containing π is given by

$$\mathsf{CL}_n(\pi) := \{\tau \mid \tau = \sigma\pi\sigma^{-1} \text{ for some } \sigma \in \mathfrak{S}_n\}.$$

Given any conjugacy class C of $\mathfrak{S}_{\mathbb{N}}$ and $n \geq 0$, let $\mathsf{C}_n := \mathsf{C} \cap \mathfrak{S}_n$. For the following result see [FH59, Lemma 2.1].

Lemma 2.6. Given a conjugacy class C of $\mathfrak{S}_{\mathbb{N}}$ and $n \geq 0$, the set C_n is non-empty if and only if $||\mathsf{C}|| \leq n$, and in this case C_n is a conjugacy class of $\mathfrak{S}_n$.

$\square$

This result thus tells us that the n-th filtered component $\mathcal{C}_{\leq n}$ of $\mathcal{C}$ gives an indexing set of the conjugacy classes of $\mathfrak{S}_n$. Given any $\lambda \in \mathcal{C}$, the set $\mathsf{CL}_n(\lambda) := \mathsf{CL}(\lambda) \cap \mathfrak{S}_n$ is non-empty whenever $\mathsf{deg}(\lambda) \leq n$, and in which case it is the conjugacy class of $\mathfrak{S}_n$ consisting of all permutations obtained by replacing the symbols $*$ with distinct elements from $[n]$. It is well-known that the conjugacy classes of $\mathfrak{S}_n$ are in bijection with the partitions of n, and in turn these are in bijection with the n-th filtered component $\mathcal{C}_{\leq n}$ in a natural fashion. We have chosen to focus on $\mathcal{C}_{\leq n}$ as our indexing set for the conjugacy classes of $\mathfrak{S}_n$ since this perspective will generalise better for the results on *Chapter 5*.

2.1.3 The Center of $\mathbb{C}\mathfrak{S}_n$ and the Cycle Shape Algebra Z

In this section we recall the center of the group algebra of $\mathfrak{S}_n$ given in terms of the class sum basis. We also recall the definition of an $\mathbb{C}[z]$-algebra Z first presented in the works of H. Farahat and G. Higman in [FH59], and recall some of its basic properties. What is covered here will be generalised in *Chapter 5* to certain centralizer algebras of the group algebras of the symmetric groups.

We let $\mathbb{C}\mathfrak{S}_n$ denote the group algebra of $\mathfrak{S}_n$ over $\mathbb{C}$. Thus any element of $\mathbb{C}\mathfrak{S}_n$ is a formal $\mathbb{C}$-linear combination of permutations of $[n]$. We let

$$Z_n := Z(\mathbb{C}\mathfrak{S}_n) = \{z \in \mathbb{C}\mathfrak{S}_n \mid z\pi = \pi z \text{ for all } \pi \in \mathfrak{S}_n\}$$

denote the center of $\mathbb{C}\mathfrak{S}_n$.

Definition 2.7. For $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathcal{C}_{\leq n}$, define the *class sum* element $K_n(\lambda) \in \mathbb{C}\mathfrak{S}_n$ by

$$K_n(\lambda) := \sum_{\pi \in \mathsf{CL}_n(\lambda)} \pi.$$

Proposition 2.8. The set of class sums $\{K_n(\lambda) \mid \lambda \in \mathcal{C}_{\leq n}\}$ forms a basis of Z_n.

Proof. We first show that each class sum is central. For any $\sigma \in \mathfrak{S}_n$ and $\lambda \in \mathcal{C}_{\leq n}$, since $\mathsf{CL}_n(\lambda)$ is an orbit with respect to conjugation we have that

$$\sigma K_n(\lambda)\sigma^{-1} = \sum_{\pi \in \mathsf{CL}_n(\lambda)} \sigma\pi\sigma^{-1} = \sum_{\pi \in \mathsf{CL}_n(\lambda)} \pi = K_n(\lambda).$$

Thus $\sigma K_n(\lambda) = K_n(\lambda)\sigma$ for all $\sigma \in \mathfrak{S}_n$, confirming that $K_n(\lambda)$ belongs to Z_n. We now show that any central element is a linear combination of class sums. Let

$$z = \sum_{\pi \in \mathfrak{S}_n} a_\pi \pi$$

belong to the center Z_n. Thus $\sigma z \sigma^{-1} = z$ for all $\sigma \in \mathfrak{S}_n$, and so $a_{\pi_1} = a_{\pi_2}$ for any π_1 and π_2 belonging to the same conjugacy class of $\mathfrak{S}_n$. Hence setting $a_\lambda := a_\pi$ for any $\pi \in \mathsf{CL}_n(\lambda)$, we have that z is the sum of terms $a_\lambda K_n(\lambda)$ for each $\lambda \in \mathcal{C}_{\leq n}$, showing that z is indeed a linear combination of class sums. Lastly since conjugacy classes are disjoint, the class sums are clearly linearly independent.

□

Since the class sums form a basis of the center Z_n, one may express the product of two class sums elements as a $\mathbb{C}$-linear combination of class sum elements. It turns out that the structure constants appearing in such products are polynomial in n. For the following result see [FH59, Theorem 2.2].

Proposition 2.9. Let z be a formal variable. For any $n \in \mathbb{Z}_{\geq 0}$, and each $\lambda, \mu, \tau \in \mathcal{C}$, there exists a unique polynomial $f^\tau_{\lambda,\mu}(z)$ such that in Z_n we have

$$K_n(\lambda)K_n(\mu) = \sum_{\tau \in \mathcal{C}_{\leq n}} f^\tau_{\lambda,\mu}(n) K_n(\tau).$$

□

Note that in general the conjugacy classes of $\mathfrak{S}_\mathbb{N}$ are of infinite size, and hence the class sum elements do not have analogs in the setting of the group algebra $\mathbb{C}\mathfrak{S}_\mathbb{N}$. Moreover, the center of the group algebra $\mathbb{C}\mathfrak{S}_\mathbb{N}$ is trivial. However, one can define a new algebra over the polynomial ring $\mathbb{C}[z]$ by more or less replacing the structure constants above with their corresponding polynomial. This is what was done in [FH59], and the new algebra Z which is formed in some sense plays the role of the the center of $\mathbb{C}\mathfrak{S}_\mathbb{N}$ comparable to the finite cases Z_n. For the following definition see [FH59, Page 214].

Definition 2.10. Let Z be the free $\mathbb{C}[z]$-module with basis $\{K(\lambda) \mid \lambda \in \mathcal{C}\}$. Equip Z with the product given by the $\mathbb{C}[z]$-linear extension of

$$K(\lambda)K(\mu) = \sum_{\tau \in \mathcal{C}} f^\tau_{\lambda,\mu}(z) K(\tau),$$

where $f^\tau_{\lambda,\mu}(z)$ are the polynomials in *Proposition 2.9*.

It is not a given that Z is a $\mathbb{C}[z]$-algebra, but this was proved in [FH59], which we now summarise. Firstly, from the definition of Z, it is certainly a *distributive ring*, that is an object which satisfies all the axioms of a ring except possibly the associativity of the multiplicative product, and the existence of a multiplicative identity. From *Proposition 2.9* it follows that, as distributive rings, we have a surjective homomorphism $\mathsf{pr}_n : Z \to$

Z_n given by $\mathsf{pr}_n(K(\lambda)) = K_n(\lambda)$ and $\mathsf{pr}_n(z) = n$. The kernel $\mathsf{Ker}(\mathsf{pr}_n)$ is generated by the polynomial $z - n$ and the elements $K(\lambda)$ such that $\mathsf{deg}(\lambda) > n$. Hence one can see that we have the trivial intersection

$$\bigcap_{n\geq 0} \mathsf{Ker}(\mathsf{pr}_n) = 0.$$

From such a result, one can show that given any $R_1, R_2 \in Z$, then $R_1 = R_2$ if and only if $\mathsf{pr}_n(R_1) = \mathsf{pr}_n(R_2)$ for all $n \geq 0$. Then since each Z_n has a multiplicative identity and is associative, one can deduce the same for Z.

As a $\mathbb{C}[z]$-algebra Z is of infinite rank, with a basis indexed by the set of cycle shapes $\mathcal{C}$. It is worth remarking that for any $\lambda \in \mathcal{C}$, the basis element $K(\lambda)$ of Z is not a class sum element for any $\mathbb{C}\mathfrak{S}_n$, nor is it an element of $\mathbb{C}\mathfrak{S}_\mathbb{N}$. Such a basis element should be thought of as a formal object which projects down to the class sums $K_n(\lambda)$ of $\mathbb{C}\mathfrak{S}_n$ for any $n \geq 0$.

We end this section with a brief discussion of some structural properties of Z. In *Chapter 5* these properties will be proved as special cases in a more general setting. It can be shown that lifting the degree map deg on the cycle shapes $\mathcal{C}$ to a map acting on Z, in the natural manner, gives a filtration of Z. Moreover, the multiplication of elements in Z exhibit unique leading terms which are encoded by $\mathcal{C}$. That is to say, given any $\lambda, \mu \in \mathcal{C}$ we have that

$$K(\lambda)K(\mu) = c_{\lambda,\mu}K(\lambda\mu) + \sum_{\substack{\tau\in\mathcal{C} \\ \mathsf{deg}(\tau)<\mathsf{deg}(\lambda\mu)}} f^{\tau}_{\lambda,\mu}(z)K(\tau),$$

where $c_{\lambda,\mu} \in \mathbb{N}$ and $\lambda\mu$ is the product of λ and μ in $\mathcal{C}$. Hence $c_{\lambda,\mu}K(\lambda\mu)$ is the term of highest degree appearing in the product $K(\lambda)K(\mu)$ in Z. This leading term result can be used to prove that the set $\{K(c_i) \mid i \in \mathbb{N}\}$ is a generating set for Z, which natural extends the fact that the elements c_i generate the monoid $\mathcal{C}$.

2.1.4 Jucys-Murphy Elements

We will be interested in a family of commuting elements of $\mathbb{C}\mathfrak{S}_n$ called the *Jucys-Murphy elements*. These elements were studied in the works of [Jucys74] and [Murphy81] with regard to giving an alternative description of the center of $\mathbb{C}\mathfrak{S}_n$ and of Young's seminormal form. They have a simple explicit description as a sum of certain transpositions.

Definition 2.11. For $i \in [n]$, the i-th Jucys-Murphy element $Y_i \in \mathbb{C}\mathfrak{S}_n$ is defined by

$$Y_i = \sum_{1\leq j<i} (j, i).$$

It was demonstrated by A. Okounkov and A. Vershik in [OV96] that the Jucys-Murphy elements can be used to provide a new spectral approach to the representation theory of $\mathbb{C}\mathfrak{S}_n$. It was shown that each simple $\mathbb{C}\mathfrak{S}_n$-module possesses a basis which diagonalises the action of the Jucys-Murphy elements. Thus one may associate to any simple module a tuple of eigenvalues which plays the role of a weight from the representation theory of semisimple Lie algebras. Then a combinatorial analysis of such weights recovers many of the classical results regarding the representation theory of $\mathbb{C}\mathfrak{S}_n$. Such results will be summarized in the next section. For this section we recall various algebraic properties of the Jucys-Murphy elements. We start by establishing the commuting and recursive relations.

Lemma 2.12. The following relations are satisfied in $\mathbb{C}\mathfrak{S}_n$:

(i) $Y_iY_j = Y_jY_i$ for all $i, j \in [n]$.

(ii) $s_iY_j = Y_js_i$ for all $j \neq i, i+1$.

(iii) $Y_{i+1} = s_iY_is_i + s_i$ for all $i \in [n-1]$.

Proof. Item (ii) is immediate whenever $j < i$, otherwise conjugating Y_j by s_i simply permutes the transposition (i, j) and $(i+1, j)$ around, leaving the sum invariant, proving (ii). This implies that Y_j commutes with $\mathbb{C}\mathfrak{S}_i$ for all $i < j$ which gives item (i). Lastly conjugating Y_i by s_i gives the sum of transposition $(j, i+1)$ for all $j < i$. This differs to Y_{i+1} by the term s_i, which proves (iii).

□

As mentioned in *Chapter 1*, these relations are precisely the relations which are used to construct the *degenerate affine Hecke algebra* $\mathcal{H}_n$ via a presentation (see *Section 2.1.7*). The relation

$$Y_{i+1} = s_iY_is_i + s_i, \tag{2.1}$$

and its affine counterpart (see (2)(iii) from *Definition 2.23*) were core components in the inductive arguments used in the combinatorial analysis of [OV96] in understanding the weights of simple modules. One may alternatively define the Jucys-Murphy elements by setting $Y_1 = 0$ and constructing Y_{i+1} for $i > 1$ recursively according to *Equation* (2.1).

We end this subsection by giving an alternative description of the center Z_n of the group algebra $\mathbb{C}\mathfrak{S}_n$ using the Jucys-Murphy elements. First let $\mathbb{C}[y_1, \ldots, y_n]$ be the polynomial $\mathbb{C}$-algebra in n commuting variables $y_1, \ldots, y_n$. Then $\mathfrak{S}_n$ acts on $\mathbb{C}[y_1, \ldots, y_n]$ by permuting the variables, that is for any $\pi \in \mathfrak{S}_n$ and $f \in \mathbb{C}[y_1, \ldots, y_n]$ we have that $(\pi \circ f)(y_1, \ldots, y_n) := f(y_{\pi(1)}, \ldots, y_{\pi(n)})$. The subalgebra of symmetric polynomials is given by

$$\mathsf{Sym}[y_1, \ldots, y_n] := \{f \in \mathbb{C}[y_1, \ldots, y_n] \mid \pi \circ f = f,\ \forall \pi \in \mathfrak{S}_n\}.$$

By item (i) of *Lemma 2.12* we have a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}\mathfrak{S}_n$ by letting $y_i \mapsto Y_i$. We denote the image of $\mathsf{Sym}[y_1, \ldots, y_n]$ under this homomorphism by $\mathsf{Sym}[Y_1, \ldots, Y_n]$.

Theorem 2.13 (Theorem 1.9 of [Murphy83])**.** The center of $\mathbb{C}\mathfrak{S}_n$ is given by

$$Z_n = \mathsf{Sym}[Y_1, \ldots, Y_n].$$

□

In *Chapter 3* we will prove a new analogous result regarding the center for semisimple partition algebras.

2.1.5 Representation Theory

In this section we summarise some of the combinatorial features of the representation theory of $\mathbb{C}\mathfrak{S}_n$. Much of the notation presented here and some of the results will be used throughout the thesis. We will give an analogous description of the representation theory of the semisimple partition algebras in *Section 2.2.3*.

For any $n, l \in \mathbb{Z}_{\geq 0}$, a tuple $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ is called a *partition of* n, written $\lambda \vdash n$, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ and $|\lambda| := \lambda_1 + \cdots + \lambda_l = n$. We let $\emptyset$ denote the unique partition of 0, and let $\Lambda_n := \{\lambda \in \mathbb{N}^l \mid l \geq 0, \lambda \vdash n\}$ denote the set of partitions of n. We associate any partition $\lambda = (\lambda_1, \dots, \lambda_l) \in \Lambda_n$ with the set of coordinates

$$\lambda = \{(i, j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

We view such a set pictorially by representing each coordinate (i, j) as a *box* $\square$ positioned in row i and column j, where rows read 1 to l going from top to bottom, and the columns read 1 to λ_1 going from left to right. This pictorial representation of a partition λ is called its associated *Young diagram*. In general we will not distinguish between a partition and its Young diagram. We will let

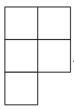
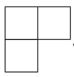
$$\Lambda := \bigsqcup_{n \geq 0} \Lambda_n$$

denote the set of all partitions.

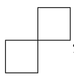
Definition 2.14. Given $\lambda \in \Lambda$ and box $\square = (i, j) \in \lambda$, we let $c(\square) := j - i$. Such a statistic is called the *content* of the box. For any subset of boxes $B \subseteq \lambda$, we let $c(B)$ denote the multiset of contents of the boxes in B.

Given any $\lambda, \mu \in \Lambda$, we write $\mu \subseteq \lambda$ whenever the associated Young diagram of μ is contained within that of λ. In this case we let $\lambda \backslash \mu$ denote the collection of boxes contained in λ but not in μ, which is referred to as a *skew diagram*. We illustrate these definitions with the following example:

Example 2.15. Consider the two partitions $\lambda = (2, 2, 1), \mu = (2, 1) \in \Lambda$. Their associated Young diagrams are given by

$\lambda =$ , and $\mu =$ .

We have that $c(\lambda) = \{1, 0^2, -1, -2\}$ and $c(\mu) = \{1, 0, -1\}$, with superscripts denoting multiplicity. We have $\mu \subseteq \lambda$ and the corresponding skew diagram is given by

$\lambda \backslash \mu =$ ,

where $c(\lambda \backslash \mu) = \{0, -2\}$.

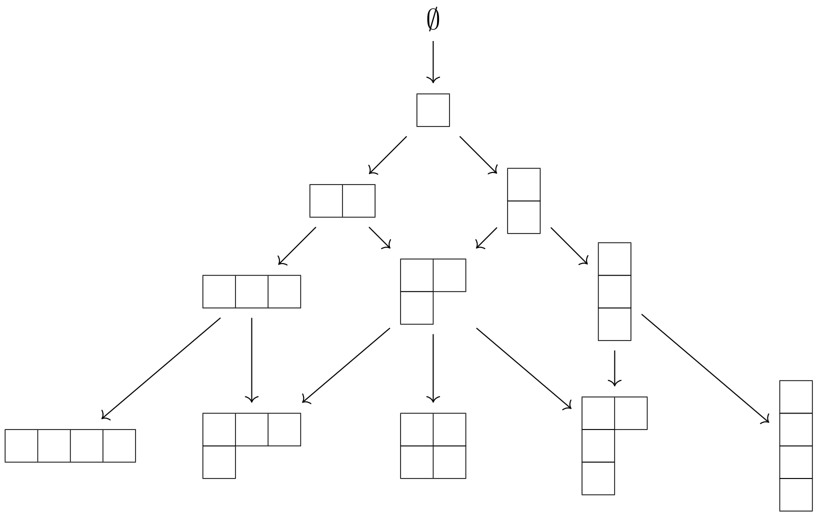
For any $\lambda \in \Lambda$, we call a box $\square = (i,j) \notin \lambda$, for $i, j \geq 1$, an *addable box of* λ if adjoining it to the associated Young diagram of λ yields a new Young diagram for a partition of $|\lambda| + 1$. Similarly, we call a box $\square \in \lambda$ a *removable box of* λ if removing it from the associated Young diagram yields a new Young diagram of a partition of $|\lambda| - 1$. We now have enough notation to describe *Young's lattice*.

Definition 2.16. Young's lattice $\widehat{\mathsf{S}}$ is the graded directed graph with levels indexed by the non-negative integers such that:

(1) Vertices on level n are given by the set Λ_n.

(2) An edge $\mu \to \lambda$ exists if there is an addable box $\square$ of μ such that $\lambda = \mu \cup \square$.

Example 2.17. The first five levels of $\widehat{\mathsf{S}}$, reading top to bottom, are given by



A path in $\widehat{\mathsf{S}}$ is a sequence $\mathsf{T} = (\lambda^{(i)})_{i=m}^{n}$, with $n \geq m$, where $\lambda^{(i)} \to \lambda^{(i+1)}$ is an edge for each $m \leq i \leq n-1$. We let $\mathsf{Path}(\lambda)$ denote the set of paths in $\widehat{\mathsf{S}}$ starting at level 0 and whose terminal vertex is λ.

Young's lattice $\widehat{\mathsf{S}}$ encodes many combinatorial features of the representation theory of all the symmetric groups simultaneously. From definition we have a natural chain of group algebras

$$\mathbb{C} := \mathbb{C}\mathfrak{S}_0 \subset \mathbb{C}\mathfrak{S}_1 \subset \mathbb{C}\mathfrak{S}_2 \subset \cdots . \tag{2.2}$$

It was shown (see for example [OV96, Theorem 2.9]) that this chain is multiplicity-free. That is, given any finite-dimensional simple $\mathbb{C}\mathfrak{S}_n$-module M, any simple constituent in the restriction $\mathsf{Res}_{\mathbb{C}\mathfrak{S}_{n-1}}(M)$ has multiplicity zero or one. The graph $\widehat{\mathsf{S}}$ describes the multiplicity of such simple constituents as one restricts down the chain of *Equation* (2.2), making $\widehat{\mathsf{S}}$ the *Branching Graph* of such a chain. This is summarised as follows (see Theorem 2.2.10 of [K05]):

Theorem 2.18. For any $n \in \mathbb{Z}_{\geq 0}$:

(1) The n-th level of $\widehat{\mathsf{S}}$ gives an indexing set for the isomorphism classes of simple $\mathbb{C}\mathfrak{S}_n$-modules. We let S^λ denote a simple module of class $\lambda \in \Lambda_n$.

(2) For any $\lambda \in \Lambda_n$ we have

$$\mathsf{Res}_{\mathbb{C}\mathfrak{S}_{n-1}}(\mathsf{S}^\lambda) = \bigoplus_{\mu \to \lambda} \mathsf{S}^\mu,$$

where the sum runs over all $\mu \in \Lambda_{n-1}$ such that $\mu \to \lambda$ is an edge in $\widehat{\mathsf{S}}$.

(3) For each $\lambda \in \Lambda_n$ we have $\mathsf{dim}(\mathsf{S}^\lambda) = |\mathsf{Path}(\lambda)|$.

□

One may use item (2) from above to construct a basis for any simple module of the group algebra of the symmetric group. First for any $\lambda \in \Lambda_n$ we have a canonical decomposition

$$\mathsf{Res}_{\mathbb{C}\mathfrak{S}_{n-1}}(\mathsf{S}^\lambda) = \bigoplus_{\mu \to \lambda} \mathsf{S}^\mu.$$

We may iterate this process on the summands to obtain a decomposition of $\mathsf{Res}_{\mathbb{C}\mathfrak{S}_{n-2}}(\mathsf{S}^\lambda)$ into simple $\mathbb{C}\mathfrak{S}_{n-2}$-modules, and such simple modules appearing can be indexed by paths in $\widehat{\mathsf{S}}$ starting at level $n-2$ and terminating at λ. Hence continuing to restrict the summands down the chain of *Equation* (2.2) allows us to obtain a unique decomposition of $\mathsf{Res}_{\mathbb{C}\mathfrak{S}_0}(\mathsf{S}^\lambda)$ into simple $\mathbb{C}\mathfrak{S}_0$-modules, i.e. into 1-dimensional $\mathbb{C}$-vector spaces, which are indexed by the paths in $\mathsf{Path}(\lambda)$. Thus as $\mathbb{C}$-vector spaces we have

$$\mathsf{S}^\lambda = \bigoplus_{\mathsf{T} \in \mathsf{Path}(\lambda)} V_{\mathsf{T}} \tag{2.3}$$

where V_{T} is a 1-dimensional vector space. Note this shows that item (2) implies (3) of *Theorem 2.18*. Picking a non-zero vector $v_{\mathsf{T}} \in V_{\mathsf{T}}$ for each $\mathsf{T} \in \mathsf{Path}(\lambda)$ gives a unique (up to scalars) basis $\{v_{\mathsf{T}} \mid \mathsf{T} \in \mathsf{Path}(\lambda)\}$ for S^λ. Such a basis is called a *Gelfand-Zeitlin basis*, or simply a GZ-basis. We end the section by describing how the Jucys-Murphy elements act on the GZ-basis of any simple module.

Definition 2.19. Let $n \in \mathbb{Z}_{\geq 0}$, $\lambda \in \Lambda_n$, and $\mathsf{T} = (\lambda^{(i)})_{i=0}^n \in \mathsf{Path}(\lambda)$. For any $i \in [n]$ let

$$\mathrm{cont}(\mathsf{T}, i) := c(\square)$$

where $\square$ is the single box belonging to the skew diagram $\lambda^{(i)} \backslash \lambda^{(i-1)}$. We refer to such a statistic as the *contents of* T *at* i.

For the following result see for example Theorem 2.2.10 of [K05].

Proposition 2.20. Let $n \in \mathbb{Z}_{\geq 0}$, $\lambda \in \Lambda_n$, and $\mathsf{T} = (\lambda^{(i)})_{i=0}^n \in \mathsf{Path}(\lambda)$. Let v_{T} be a GZ-basis element of the simple module S^λ, then for each $i \in [n]$ we have that

$$Y_i v_{\mathsf{T}} = \mathrm{cont}(\mathsf{T}, i) v_{\mathsf{T}}.$$

□

Hence the GZ-basis simultaneously diagonalises the action of the Jucys-Murphy elements, and thus *Equation* (2.3) gives the decomposition of S^λ into the shared eigenspaces of the commuting actions of the Jucys-Murphy elements.

2.1.6 Schur-Weyl Duality

Consider an n-dimensional $\mathbb{C}$-vector space V with basis $\{v_1, \ldots, v_n\}$. For $a, b, c \in [n]$, let E^a_b denote the endomorphism of V acting on the basis by $\mathsf{E}^a_b(v_c) = \delta_{b,c} v_a$ where $\delta_{b,c}$ is the Kronecker delta. For $k \geq 0$ let $V^{\otimes k} := V \otimes \cdots \otimes V$ with k components. The set of simple tensors $\{v_{a_1} \otimes \cdots \otimes v_{a_k} \mid 1 \leq i \leq k,\ a_i \in [n]\}$ forms a basis of $V^{\otimes k}$. For any $i \in [k]$ and $a, b \in [n]$ we let $\mathsf{E}^a_b[i]$ denote the endomorphism of $V^{\otimes k}$ acting on simple tensors by

$$\mathsf{E}^a_b[i](v_{c_1} \otimes \cdots \otimes v_{c_k}) = v_{c_1} \otimes \cdots \otimes \mathsf{E}^a_b(v_{c_i}) \otimes \cdots \otimes v_{c_k}.$$

Hence $\mathsf{E}^a_b[i]$ acts on the i-th component by E^a_b. Note $\mathsf{E}^a_b[i]$ and $\mathsf{E}^c_d[j]$ commute for any $a, b, d, c \in [n]$ and $i, j \in [k]$ such that $i \neq j$.

Consider the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms of V with Lie bracket given by the commutator. Let $U(\mathfrak{gl}(V))$ denote the universal enveloping algebra of $\mathfrak{gl}(V)$, which is the algebra generated by the set of elements $\{\mathsf{e}^a_b \mid a, b \in [n]\}$ and satisfying the defining relations $\mathsf{e}^a_b \mathsf{e}^c_d - \mathsf{e}^c_d \mathsf{e}^a_b = \delta_{b,c} \mathsf{e}^a_d - \delta_{a,d} \mathsf{e}^c_b$ for all $a, b, c, d \in [n]$. The space $V^{\otimes k}$ may be regarded as a $U(\mathfrak{gl}(V))$-module with action $U(\mathfrak{gl}(V)) \to \mathsf{End}_{\mathbb{C}}(V^{\otimes k})$ given by

$$\mathsf{e}^a_b \mapsto \sum_{i=1}^{k} \mathsf{E}^a_b[i]. \tag{2.4}$$

We let $\mathsf{End}_{U(\mathfrak{gl}(V))}(V^{\otimes k})$ denote the space of endomorphisms which commute with this action. The space $V^{\otimes k}$ is also an $\mathbb{C}\mathfrak{S}_k$-module by the natural action of permuting the k tensor components. Let $\Phi_{k,n}$ be the corresponding representation, hence

$$\Phi_{k,n}(\pi)(v_{a_1} \otimes \cdots \otimes v_{a_k}) = v_{a_{\pi(1)}} \otimes \cdots \otimes v_{a_{\pi(k)}}$$

for any $\pi \in \mathfrak{S}_k$ and $a_i \in [n]$. It is clear that $\Phi_{k,n}(\pi)\mathsf{E}^a_b[i] = \mathsf{E}^a_b[\pi(i)]\Phi_{k,n}(\pi)$, and thus from *Equation* (2.4) we see that $\Phi_{k,n}(\pi)$ commutes with the action of $\mathcal{U}(\mathfrak{gl}(V))$. Hence we may regard $\Phi_{k,n}$ as an algebra homomorphism $\mathbb{C}\mathfrak{S}_k \to \mathsf{End}_{U(\mathfrak{gl}(V))}(V^{\otimes k})$. We describe this action in terms of the endomorphisms $\mathsf{E}^a_b[i]$ as follows. For $1 \leq i < j \leq k$ let

$$\Omega_{i,j} := \sum_{a,b \in [n]} \mathsf{E}^a_b[i]\mathsf{E}^b_a[j]. \tag{2.5}$$

Then one can observe that $\Omega_{i,j}$ acts on simple tensors by exchanging the i-th and j-th components. As such $\Omega_{i,j}$ is the image of the transposition (i, j) of $\mathfrak{S}_k$ under $\Phi_{k,n}$, in particular $\Phi_{k,n}(s_i) = \Omega_{i,i+1}$. For the following result see for example the Theorem of Section 3 in [CL74].

Theorem 2.21. The algebra homomorphism $\Phi_{k,n} : \mathbb{C}\mathfrak{S}_k \to \mathsf{End}_{U(\mathfrak{gl}(V))}(V^{\otimes k})$ is surjective, and is an isomorphism when $n \geq k$.

□

Schur-Weyl duality provides a much richer collection of results than just the above theorem, but for our purposes such theory will not be required. We are only interested in the morphism $\Phi_{n,k}$ and its various generalisations. In particular we will be describing an extension of this morphism to the degenerate affine Hecke algebra in the next section. To help compare and motivate this extension we end by stating the action of the Jucys-Murphy elements, which follows directly from their definition as a sum of transpositions.

Lemma 2.22. For any $k, n \in \mathbb{Z}_{\geq 0}$, and $i \in [n]$, we have that

$$\Phi_{k,n}(Y_i) = \sum_{1 \leq j < i} \Omega_{j,i}.$$

□

2.1.7 Degenerate Affine Hecke Algebra

In this section we give the definition of the degenerate affine Hecke algebra $\mathcal{H}_n$. This algebra is constructed from the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_n$ by adjoining new pairwise commuting generators and imposing relations on them which are analogous to those presented in *Lemma 2.12*. We present a variety of basic results regarding this algebra with a focus on how it extends certain properties of $\mathbb{C}\mathfrak{S}_n$.

We begin by defining $\mathcal{H}_n$ by generators and relations.

Definition 2.23. The degenerate affine Hecke algebra $\mathcal{H}_n$ is the $\mathbb{C}$-algebra presented with generating set

$$\{s_i, y_j \mid 1 \leq i \leq n-1,\ j \in [n]\}$$

and defining relations

(1) (i) $s_i^2 = 1$, for $i \in [k-1]$.
  (ii) $s_i s_j = s_j s_i$, for $j \neq i-1, i+1$.
  (iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [k-2]$.

(2) (i) $y_i y_j = y_j y_i$ for all $i, j \in [k]$.
  (ii) $s_i y_j = y_j s_i$ for all $j \neq i, i+1$.
  (iii) $y_{i+1} = s_i y_i s_i + s_i$ for all $i \in [k-1]$.

We have overloaded the symbols s_i as elements of $\mathcal{H}_n$ and $\mathbb{C}\mathfrak{S}_n$, however it will be shown that the subalgebra generated by these elements in $\mathcal{H}_n$ gives an isomorphic copy of $\mathbb{C}\mathfrak{S}_n$. Relations (1) above are precisely the relations of the presentation of $\mathfrak{S}_n$ given in *Theorem 2.1*, while relations (2) are obtained from *Lemma 2.12* by replacing the Jucys-Murphy elements Y_i with the generators y_i respectively. Note that the relations present in *Theorem 2.1* and *Lemma 2.12* are invariant under a shift in indices, hence we immediately obtain the following result.

Lemma 2.24. For any $l \in \mathbb{Z}_{\geq 0}$ we have an algebra homomorphism $\rho_l : \mathcal{H}_n \to \mathbb{C}\mathfrak{S}_{n+l}$ defined on the generators by $\rho_l(s_i) = s_{i+l}$ and $\rho_l(y_i) = Y_{i+l}$.

□

For the case $l = 0$, the homomorphism ρ_0 is a surjection of $\mathcal{H}_n$ onto $\mathbb{C}\mathfrak{S}_n$. We also have a homomorphism $\iota : \mathbb{C}\mathfrak{S}_n \to \mathcal{H}_n$ given by $\iota(s_i) = s_i$. Clearly the composition $\rho_0 \circ \iota$ is the identity on $\mathbb{C}\mathfrak{S}_n$, and hence ι has a left inverse. This implies that ι is injective, and thus $\mathbb{C}\mathfrak{S}_n$ is indeed the subalgebra of $\mathcal{H}_n$ generated by the elements s_i for $i \in [n-1]$. So we may interpret any permutation $\pi \in \mathfrak{S}_n$ as an element of $\mathcal{H}_n$. Collectively $\mathbb{C}\mathfrak{S}_n$ is both a subalgebra and quotient of $\mathcal{H}_n$.

We refer to the generators $y_1, \ldots, y_n$ as the *affine generators*. Relation (2)(*iii*) of *Definition 2.23* shows that the affine generators share the same recursive structure as the Jucys-Murphy elements. The base case is taken to be the generator y_1 itself, as opposed to $Y_1 = 0$. As such any affine generator may be expression in terms of the generators s_i and y_1, that is to say $\mathcal{H}_n = \langle s_i, y_1 \mid 1 \leq i \leq n-1 \rangle$. From this one can deduce that $\mathsf{Ker}(\rho_0) = (y_1)$, the two-sided ideal of $\mathcal{H}_n$ generated by y_1.

The algebra $\mathcal{H}_n$ has a natural basis.

Theorem 2.25 (Theorem 3.2.2 of [K05])**.** The set

$$\{\pi y_1^{a_1} \ldots y_n^{a_n} \mid (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n,\ \pi \in \mathfrak{S}_n\}$$

is a basis for $\mathcal{H}_n$.

□

As a corollary the subalgebra of $\mathcal{H}_n$ generated by the affine generators is precisely the polynomial algebra $\mathbb{C}[y_1, \ldots, y_n]$ in n commuting variables. The projection ρ_0 acts on this subalgebra by evaluating the variables at the Jucys-Murphy elements. In this sense the algebra $\mathcal{H}_n$ was obtained from $\mathbb{C}\mathfrak{S}_n$ by lifting the Jucys-Murphy elements to free commuting variables.

Recall from *Theorem 2.13* that the center of the group algebra of the symmetric group is given by symmetric polynomials in the Jucys-Murphy elements. We have an analogous result regarding the center of $\mathcal{H}_n$.

Theorem 2.26 (Theorem 3.3.1 of [K05])**.** The center of $\mathcal{H}_n$ is given by

$$Z(\mathcal{H}_n) = \mathsf{Sym}[y_1, \ldots, y_n],$$

the symmetric polynomials in the affine generators.

□

Restricting the projection ρ_0 down to the center of $\mathcal{H}_n$ thus gives a surjective homomorphism between the centers $\mathsf{Sym}[y_1, \ldots, y_n] \to \mathsf{Sym}[Y_1, \ldots, Y_n]$. The inclusion $\mathsf{Sym}[y_1, \ldots, y_n] \subseteq Z(\mathcal{H}_n)$ can be proven by direct computations, but it is worth mentioning that the reverse inclusion requires the use of *Theorem 2.25*.

We now wish to describe a representation of $\mathcal{H}_k$ which gives a counterpart to the representation $\Phi_{k,n}$ of $\mathbb{C}\mathfrak{S}_k$ given in *Theorem 2.21*. Recall the set up in describing $\Phi_{k,n}$ in *Section 2.1.6*. Let M be any (possibly infinite dimensional) $U(\mathfrak{gl}(V))$-module with basis $\{m_i \mid i \in I\}$ for some indexing set I. Then for any $k \in \mathbb{Z}_{\geq 0}$ we consider the tensor space $M \otimes V^{\otimes k}$ with basis $\{m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_k} \mid (a_0, a_1, \ldots, a_k) \in I \times [n]^k\}$. We refer to the M component of the tensor space $M \otimes V^{\otimes k}$ as the 0-th component. For any $a, b \in [n]$ and $0 \leq i \leq k$, let $\mathsf{E}_b^a[i]$ denote the endomorphism of $M \otimes V^{\otimes k}$ acting on the i-th component by E_b^a (or by e_b^a when $i = 0$). We may regard the operators $\Omega_{i,j}$ given in *Equation* (2.5) as operators of $M \otimes V^{\otimes k}$ in the natural manner. We can extend the definition by setting

$$\Omega_{0,j} := \sum_{a,b\in[n]} \mathsf{E}_b^a[0]\mathsf{E}_a^b[j],$$

For any $j \in \mathbb{Z}_{\geq 0}$. For the following result see [AS98, Section 2.2].

Theorem 2.27. For any $k, n \in \mathbb{Z}_{\geq 0}$ and $U(\mathfrak{gl}(V))$-module M, we have an algebra homomorphism

$$\Phi_{k,n}^{(M)} : \mathcal{H}_k \to \mathsf{End}_{U(\mathfrak{gl}(V))}(M \otimes V^{\otimes k})$$

given on the generators by

$$\Phi_{k,n}^{(M)}(s_i) = \Omega_{i,i+1}, \quad \text{and} \quad \Phi_{k,n}^{(M)}(y_j) = \sum_{0\leq i<j} \Omega_{i,j},$$

for all $i \in [n-1]$ and $j \in [n]$.

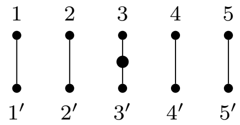
$\square$

Comparing with the action of the Jucys-Murphy element Y_j on $V^{\otimes k}$ displayed in *Proposition 2.61*, the action of the corresponding affine generator y_j on $M \otimes V^{\otimes k}$ differs only in the term $\Omega_{0,j}$. Therefore the representation $\Phi_{k,n}^{(M)}$ has extended that of $\Phi_{k,n}$ in *Theorem 2.21* by allowing the affine generators to act in a way which has generalised the action of the Jucys-Murphy elements onto an additional factor of M. For example, letting $l \geq 0$ and $M = V^{\otimes l}$ (with action given by *Equation* (2.4) with k replaced by l), then one can deduce that the representation

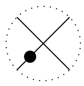
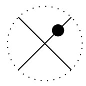
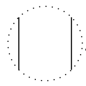
$$\Phi_{k,n}^{(V^{\otimes l})} : \mathcal{H}_k \to \mathsf{End}_{U(\mathfrak{gl}(V))}(V^{\otimes(l+k)})$$

satisfies $\Phi_{k,n}^{(V^{\otimes l})} = \Phi_{k,n} \circ \rho_l$, where ρ_l is the homomorphism given in *Lemma 2.24*. In particular the affine generators y_i act under $\Phi_{k,n}^{(V^{\otimes l})}$ in the same manner that the Jucys-Murphy elements Y_{i+l} act under $\Phi_{k,n}$.

We now end this section by giving a brief description of how the diagrammatics of $\mathbb{C}\mathfrak{S}_n$, mentioned at the end of *Section 2.1.1*, can be extended to $\mathcal{H}_n$ by introducing decorations. This discussion will be an informal one. A more rigorous treatment of these diagrammatics will be postponed to *Section 4.3* of *Chapter 4*. As a diagram, the identity permutation of $\mathfrak{S}_n$ consists precisely of the trivial edges $\{\{i', i\} \mid i \in [n]\}$. We may associate the affine generator y_j of $\mathcal{H}_n$ with the diagram we get from the identity diagram by adding a decoration (black dot) onto the edge $\{i', i\}$. For example with $j = 3$ we have

$y_3 =$ 

as an element of $\mathcal{H}_5$. These dots can move freely along their edge and commute, which allows relation (2)(i) and (2)(ii) of *Definition 2.23* to be satisfied. To capture relation (2)(iii) of *Definition 2.23* within these diagrammatics, a local relation is imposed describing how a dot passes over a crossing of edges. This is given by

 $=$  $-$ . (2.6)

Algebraically this is equivalent to the relation $s_i y_i = y_{i+1} s_i - 1$, which is equivalent to relation (2)(iii) of *Definition 2.23*. Hence a permutation diagram with decorations on its edges represents a word in the generators $\{s_i, y_j \mid i \in [n-1],\ j \in [n]\}$ of $\mathcal{H}_n$ and moving a decoration along its edge corresponds to employing a sequence of the relations (2) of *Definition 2.23*.

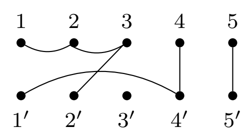
2.2 Partition Algebra

2.2.1 Definitions and Presentation

Given a finite set X recall that a *set partition* $\alpha = \{A_1, \dots, A_n\}$ of X is a collection of non-empty subsets such that $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $X = \bigcup_{i=1}^{n} A_i$. We will write $\alpha \vdash X$ to denote this and refer to the elements A_i as the *blocks* of α. Any graph with vertex set X and whose connected components partition the vertices according to the blocks of α will be called an *associated graph of* α. We will not distinguish between a set partition α and any associated graph of α, in particular we will only care about the connected components of such graphs, not the edges forming the components. For any $x, y \in X$ we write $x \sim_\alpha y$ whenever x and y belong to the same block of α, and write $x \nsim_\alpha y$ otherwise.

Now for any $k \in \mathbb{N}$ recall that $[k] = \{1, \dots, k\}$ and $[k'] = \{1', \dots, k'\}$. Let Π_{2k} denote the set of all set partitions of $[k] \cup [k']$. We call any associated graph of an $\alpha \in \Pi_{2k}$ a *partition diagram*. When drawing such diagrams, we arrange the vertices in two rows with the top row reading 1 to k from left to right, and the bottom row reading $1'$ to k' from left to right. We call the vertices $[k]$ *top vertices* and $[k']$ *bottom vertices*.

Example 2.28. In Π_{10} we have the identification

$\{\{1, 2, 2', 3\}, \{3'\}, \{1', 4, 4'\}, \{5, 5'\}\} =$ .

We now describe an associative product on the set Π_{2k}.

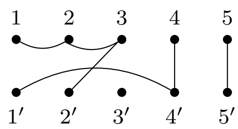
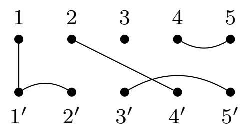
Definition 2.29. Given $\alpha, \beta \in \Pi_{2k}$ we define the product $\alpha \circ \beta \in \Pi_{2k}$ as follows:

(i) Construct $\alpha \star \beta \vdash \{1, \ldots, k, 1', \ldots, k', 1'', \ldots, k''\}$ by identifying, for each $i \in [k]$, the bottom vertex i' of α with the top vertex i of β, then relabel the vertex by i''.

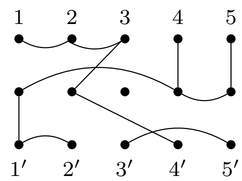
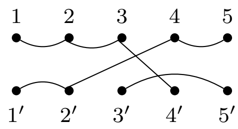
(ii) With $\alpha \star \beta = \{A_1, \ldots, A_n\}$ define $\alpha \circ \beta$ to be the set of all non-empty $A_i \cap ([k] \cup [k'])$.

We refer to the elements of the set $\{1'', \ldots, k''\}$ as the *middle vertices* of $\alpha \star \beta$. Diagrammatically step (i) in *Definition 2.29* corresponds to stacking α on top of β while step (ii) is the action of removing connected components containing only middle vertices, and then reading off the connected components formed from the top row of α and the bottom row of β, see *Example 2.30* below. We refer to the set partition $\alpha \star \beta$ as a *stacked diagram*. Often in the literature both steps are described in unison without the need to define a stacked diagram, however we will find it helpful to have such an object to work with for later results.

Example 2.30. Consider the elements $\alpha, \beta \in \Pi_{10}$ given diagrammatically by

$\alpha =$  and $\beta =$  .

Then the stacked diagram and the product are given by

$\alpha \star \beta =$  and $\alpha \circ \beta =$  .

Here the middle row of vertices of $\alpha \star \beta$ are labelled $1''$ to $5''$ from left to right. To go from $\alpha \star \beta$ to $\alpha \circ \beta$ the middle component $\{3''\}$ has been removed and the remaining middle vertices have been ignored.

This product is easily seen to be associative and it is clear that the set partition $1 = \{\{i, i'\} \mid i \in [k]\} \in \Pi_{2k}$ is an identity element. Thus Π_{2k} is a monoid, and is called the *partition monoid*. When drawing partition diagrams we will often suppress the labels of the vertices since these can be deduced from their relative positions. The symmetric group $\mathfrak{S}_k$ can be regarded as the submonoid of Π_{2k} by associating any permutation π with the partition diagram $\{\{i', \pi(i)\} \mid i \in [k]\}$, which we call a *permutation diagram*. These permutation diagrams are precisely the diagrams described at the end of *Section 2.1.1*. The subgroup $\mathfrak{S}_k$ of Π_{2k} is precisely the subgroup of units of Π_{2k}.

Whenever $k \in \mathbb{Z}_{>0}$ we define Π_{2k-1} to be the subset of Π_{2k} consisting of all set partitions such that k and k' belong to the same block, that is

$$\Pi_{2k-1} := \{\alpha \in \Pi_{2k} \mid k \sim_\alpha k'\}.$$

For instance the set partition α in *Example 2.30* is a member of Π_9, while β is not. It can be seen diagrammatically that Π_{2k-1} is closed under the product $\circ$, and hence Π_{2k-1} is a submonoid of Π_{2k}. Furthermore $\Pi_{2(k-1)}$ can be realised as the submonoid of Π_{2k-1} consisting of all set partitions containing the block $\{k, k'\}$, that is

$$\Pi_{2(k-1)} \cong \{\alpha \in \Pi_{2k-1} \mid \{k, k'\} \in \alpha\}.$$

For instance, under such an isomorphism, we may interpret the set partition α in *Example 2.30* as an element of Π_8 which sits inside Π_9. As for the element β, the smallest partition monoid containing it is Π_{10}, and for example we may view β as an element of Π_{11} by adjoining the block $\{6, 6'\}$. In this manner we obtain a chain of monoids $\emptyset = \Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \dots$, which have a natural diagrammatic interpretation. Any time we write $\Pi_i \subset \Pi_r$ for $i \leq r$, we abide by the identification given in this chain. We also have that $|\Pi_r| = B_r$, where B_r is the r^{th} *Bell number*.

Let z be a formal variable and $\mathbb{C}[z]$ the polynomial ring in z over the field $\mathbb{C}$. We now present the definition for the partition algebra which is an algebra over the ring $\mathbb{C}[z]$.

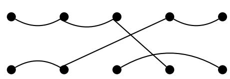
Definition 2.31. For any $k \in \mathbb{Z}_{\geq 0}$, the partition algebra $\mathcal{A}_{2k}(z)$ is the $\mathbb{C}[z]$-algebra with $\{D(\alpha) \mid \alpha \in \Pi_{2k}\}$ as a $\mathbb{C}[z]$-basis whose product is given by

$$D(\alpha)D(\beta) = z^{m(\alpha,\beta)} D(\alpha \circ \beta)$$

for all $\alpha, \beta \in \Pi_{2k}$, extended linearly over $\mathbb{C}[z]$, where $m(\alpha, \beta)$ is the number of blocks of $\alpha \star \beta$ containing only middle vertices.

As was the case for the partition monoids, for any $i \leq 2k$ we may regard $\mathcal{A}_i(z)$ as the subalgebra of $\mathcal{A}_{2k}(z)$ generated as a $\mathbb{C}[z]$-algebra by $\{D(\alpha) \mid \alpha \in \Pi_i\}$. We refer to the set $\{D(\alpha) \mid \alpha \in \Pi_r\}$ as the *diagrammatic basis* of $\mathcal{A}_r(z)$. We may regard $\mathbb{C}[z]\mathfrak{S}_k$ as the subalgebra of $\mathcal{A}_{2k}(z)$ generated by $D(\pi)$ for all permutation diagrams $\pi \in \Pi_{2k}$. We abuse notation a little and use partition diagrams to also represent elements of the diagrammatic basis. In this manner an arbitrary element of $\mathcal{A}_r(z)$ is a $\mathbb{C}[z]$-linear combination of partition diagrams. The product of $\mathcal{A}_r(z)$ is a deformation of the monoid product of Π_r. More formally, as described in [W07], the algebra $\mathcal{A}_r(z)$ is the *twisted semigroup algebra* of Π_r with *twisting* $t : \Pi_r \times \Pi_r \to \mathbb{C}[z]$ given by $t(\alpha, \beta) = z^{m(\alpha,\beta)}$.

Example 2.32. Consider the set partitions $\alpha, \beta \in \Pi_{10}$ given in *Example 2.30*. The stacked diagram $\alpha \star \beta$ contains only one block consisting only of middle vertices, namely the block $\{3''\}$. Hence $m(\alpha, \beta) = 1$ and so

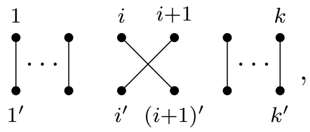
$D(\alpha)D(\beta) = z$ 

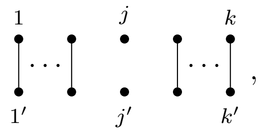
We can view $\mathcal{A}_r(z)$ as an algebra over $\mathbb{C}$ with basis $\{z^n D(\alpha) \mid n \in \mathbb{Z}_{\geq 0}, \alpha \in \Pi_r\}$. When we want to stress this perspective we will use the notation $\mathcal{A}_r$ instead. For any $\delta \in \mathbb{C}$ we define the $\mathbb{C}$-epimorphism $\mathsf{ev}_\delta : \mathbb{C}[z] \to \mathbb{C}$ by $\mathsf{ev}_\delta(z) = \delta$. Then by extension of scalars we may define the $\mathbb{C}$-algebras

$$\mathcal{A}_r(\delta) := \mathbb{C} \otimes_{\mathsf{ev}_\delta} \mathcal{A}_r(z).$$

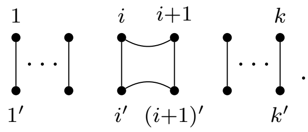
We let $D_\delta(\alpha) := 1 \otimes D(\alpha) \in \mathcal{A}_r(\delta)$ and so $\{D_\delta(\alpha) \mid \alpha \in \Pi_r\}$ gives a diagrammatic basis for $\mathcal{A}_r(\delta)$. Thus the algebra $\mathcal{A}_r(\delta)$ is finite dimensional with $\dim_{\mathbb{C}}(\mathcal{A}_r(\delta)) = B_r$. We have a canonical projection of rings $\mathsf{pr}_\delta : \mathcal{A}_r(z) \to \mathcal{A}_r(\delta)$ defined by evaluating polymonials $f(z) \mapsto f(\delta)$ and $D(\alpha) \mapsto D_\delta(\alpha)$. One may define the algebra $\mathcal{A}_r(\delta)$ in a completely analgous manner to $\mathcal{A}_r(z)$ with the role of z instead being played by δ. When $\delta = 1$ the algebra $\mathcal{A}_r(1)$ is precisely the monoid algebra of Π_r.

We end this subsection by recalling a presentation of the partition algebra $\mathcal{A}_{2k}(z)$. For $i \in [k-1]$ and $j \in [k]$, we define the following elements of $\mathcal{A}_{2k}(z)$:

$s_i =$ 

, $e_{2j-1} =$ 

,

$e_{2i} =$ 

.

One can deduce that these elements generate the algebra $\mathcal{A}_{2k}(z)$. We use the same notation for these generating elements for their projections to $\mathcal{A}_{2k}(\delta)$. The following presentation for the partition algebra is given in terms of these generators and was first established by T. Halverson and A. Ram in [HR05], see also [East11].

Theorem 2.33 (Theorem 1.11 of [HR05])**.** The partition algebra $\mathcal{A}_{2k}(z)$ has a presentation with generating set

$$\{s_i, e_j \mid i \in [k-1], j \in [2k-1]\}$$

and relations

(HR1) (Coxeter relations)

- (i) $s_i^2 = 1$, for $i \in [k-1]$.
- (ii) $s_i s_j = s_j s_i$, for $j \neq i-1, i+1$.
- (iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [k-2]$.

(HR2) (Idempotent relations)

- (i) $e_{2i-1}^2 = z e_{2i-1}$, for $i \in [k]$.
- (ii) $e_{2i}^2 = e_{2i}$, for $i \in [k-1]$.
- (iii) $s_i e_{2i} = e_{2i} s_i = e_{2i}$, for $i \in [k-1]$.
- (iv) $s_i e_{2i-1} e_{2i+1} = e_{2i-1} e_{2i+1} s_i = e_{2i-1} e_{2i+1}$, for $i \in [k-1]$.

(HR3) (Commutation relations)

- (i) $e_{2i-1} e_{2j-1} = e_{2j-1} e_{2i-1}$, for $i, j \in [k]$.

(ii) $e_{2i}e_{2j} = e_{2j}e_{2i}$, for $i, j \in [k-1]$.

(iii) $e_{2i-1}e_{2j} = e_{2j}e_{2i-1}$, for $j \neq i-1, i$.

(iv) $s_i e_{2j-1} = e_{2j-1}s_i$, for $j \neq i, i+1$.

(v) $s_i e_{2j} = e_{2j}s_i$, for $j \neq i-1, i+1$.

(vi) $s_i e_{2i-1}s_i = e_{2i+1}$, for $i \in [k-1]$.

(vii) $s_i e_{2i-2}s_i = s_{i-1}e_{2i}s_{i-1}$, for $i \in [k-1]$.

(HR4) (Contraction relations)

(i) $e_i e_{i+1} e_i = e_i$ for $i \in [2k-2]$.

(ii) $e_{i+1}e_i e_{i+1} = e_{i+1}$, for $i \in [2k-2]$.

□

The presentation above extends to one for the $\mathbb{C}$-algebra $\mathcal{A}_{2k}$ by simply adding z as a central generator. The $\mathbb{C}$-algebra $\mathcal{A}_{2k}(\delta)$ has a presentation identical to above with the exception of replacing z with δ in relation *(HR2)(i)*. From this presentation, one can easily check that for any $\delta \in \mathbb{C}$ we have a surjective $\mathbb{C}$-algebra homomorphism $f_\delta : \mathcal{A}_{2k} \to \mathbb{C}\mathfrak{S}_k$ given by $f_\delta(z) = \delta$, $f_\delta(e_i) = 0$, and $f_\delta(s_i) = s_i$. Thus the group algebra $\mathbb{C}\mathfrak{S}_k$ is not only a subalgebra of $\mathcal{A}_{2k}$ but also a quotient.

From the symmetry of the above presentation, one can deduce that we have an anti-involution $* : \mathcal{A}_{2k}(z) \to \mathcal{A}_{2k}(z)$ which fixes the generators s_i and e_j. Diagrammatically this map corresponds to flipping a partition diagram up-side-down (i.e. swapping vertices i and i' around for each $i \in [k]$), and extending linearly over $\mathbb{C}[z]$. We denote the image of an element $a \in \mathcal{A}_{2k}(z)$ under this anti-involution by a^*.

2.2.2 Jucys-Murphy Elements and Enyang's Presentation

A collection of Jucys-Murphy elements $L_1, \ldots, L_{2k}$ for the partition algebra $\mathcal{A}_{2k}(z)$ were first defined in [HR05] where they gave a diagrammatic description. They were later given a recursive definition in [Eny12]. This recursive definition introduced new elements $\sigma_i \in \mathcal{A}_{i+1}(z)$ which resemble the simple transpositions, and we will refer to these as *Enyang's generators*. We now give the definition of such Jucys-Murphy elements and Enyang's generators as presented in [Eny13] (with some typos corrected).

Definition 2.34. Let $L_1 = 0, L_2 = e_1, \sigma_2 = 1$, and $\sigma_3 = s_1$. Then for $i = 1, 2, \ldots,$ define

$$L_{2i+2} = s_i L_{2i} s_i - s_i L_{2i} e_{2i} - e_{2i} L_{2i} s_i + e_{2i} L_{2i} e_{2i+1} e_{2i} + \sigma_{2i+1},$$

where, for $i = 2, 3, \ldots,$ we have

$$\begin{aligned}\sigma_{2i+1} = s_{i-1}s_i\sigma_{2i-1}s_i s_{i-1} + s_i e_{2i-2}L_{2i-2}s_i e_{2i-2}s_i + e_{2i-2}L_{2i-2}s_i e_{2i-2} \\ - s_i e_{2i-2}L_{2i-2}s_{i-1}e_{2i}e_{2i-1}e_{2i-2} - e_{2i-2}e_{2i-1}e_{2i}s_{i-1}L_{2i-2}e_{2i-2}s_i.\end{aligned}$$

Also for $i = 1, 2, \dots,$ define

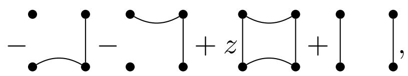
$$L_{2i+1} = s_i L_{2i-1} s_i - L_{2i} e_{2i} - e_{2i} L_{2i} + (z - L_{2i-1}) e_{2i} + \sigma_{2i},$$

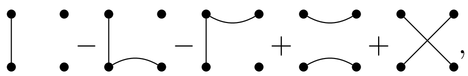
where, for $i = 2, 3, \dots,$ we have

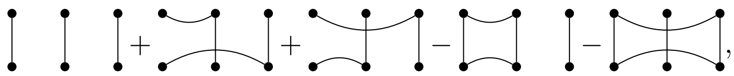
$$\begin{aligned}\sigma_{2i} = s_{i-1} s_i \sigma_{2i-2} s_i s_{i-1} + e_{2i-2} L_{2i-2} s_i e_{2i-2} s_i + s_i e_{2i-2} L_{2i-2} s_i e_{2i-2} \\ - e_{2i-2} L_{2i-2} s_{i-1} e_{2i} e_{2i-1} e_{2i-2} - s_i e_{2i-2} e_{2i-1} e_{2i} s_{i-1} L_{2i-2} e_{2i-2} s_i.\end{aligned}$$

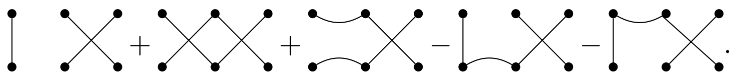
A proof by induction can confirm that $L_i \in \mathcal{A}_i(z)$ and $\sigma_i \in \mathcal{A}_{i+1}(z)$. We use the same notation to denote the corresponding elements in $\mathcal{A}_r(\delta)$. The two recursive expressions for the Jucys-Murphy elements L_{2i+2} and L_{2i+1} above are the partition algebra counterparts to *Equation* (2.1) of the Jucys-Murphy elements of $\mathbb{C}\mathfrak{S}_n$.

Example 2.35. The first few non-trivial examples are

$L_3 =$ 

$L_4 =$ 

$\sigma_4 =$ 

$\sigma_5 =$ 

Even though the recursive definitions of the Jucys-Murphy elements are quite involved, J. Enyang managed to prove a variety of relations within both [Eny12] and [Eny13] involving the Jucys-Murphy elements and Enyang's generators. We will be employing many such relations throughout this thesis, most of which will not be recalled here but rather referenced when needed.

Remark 2.36. The change of notation between [Eny12] and [Eny13] is given respectively by $p_i \sim e_{2i-1}$, $p_{i+\frac{1}{2}} \sim e_{2i}$, $\sigma_i \sim \sigma_{2i-1}$, $\sigma_{i+\frac{1}{2}} \sim \sigma_{2i}$, $L_i \sim L_{2i}$, and $L_{i+\frac{1}{2}} \sim L_{2i+1}$.

We will however recall some properties proved in [Eny12] which will be used frequently.

Proposition 2.37. The following relations hold for all $i \geq 1$:

(i) $L_i^* = L_i$ and $\sigma_i^* = \sigma_i$.

(ii) L_i commutes with $\mathcal{A}_{i-1}(z)$.

(iii) σ_{i+1} commutes with $\mathcal{A}_{i-1}(z)$.

Proof. Items (ii) and (iii) are given in [Eny12, Theorem 3.8]. For item (i) we have that $L_{2i-1}^* = L_{2i-1}$ by [Eny12, Proposition 3.3 (2)], and we have that $L_{2i}^* = L_{2i}$ by definition and noting that L_{2i} commutes with e_{2i+1}. We have that $\sigma_{2i}^* = \sigma_{2i}$ by [Eny12, Proposition 3.3 (1)], and it follows that $\sigma_{2i+1}^* = \sigma_{2i+1}$ by [Eny12, Proposition 3.4]. □

We will often use the properties of the above proposition without reference when clear to do so. Note that since $L_i \in \mathcal{A}_i(z)$, item (ii) above implies that the Jucys-Murphy elements pairwise commute. We now recall an alternative presentation of the partition algebra involving Enyang's generators.

Theorem 2.38 (Theorem 4.1 of [Eny12])**.** The partition algebra $\mathcal{A}_{2k}(z)$ has a presentation with generating set

$$\{\sigma_i, e_j \mid 3 \leq i \leq 2k-1,\ j \in [2k-1]\}$$

and relations:

(E1) (Involution)

- (i) $\sigma_{2i+2}^2 = 1$ for $i \in [k-2]$.
- (ii) $\sigma_{2i+1}^2 = 1$ for $i \in [k-1]$.

(E2) (Braid relations)

- (i) $\sigma_{2i+1}\sigma_{2j} = \sigma_{2j}\sigma_{2i+1}$ for $j \neq i+1$.
- (ii) $\sigma_{2i+1}\sigma_{2j+1} = \sigma_{2j+1}\sigma_{2i+1}$ for $j \neq i \pm 1$.
- (iii) $\sigma_{2i}\sigma_{2j} = \sigma_{2j}\sigma_{2i}$ for $j \neq i \pm 1$.
- (iv) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [k-2]$, where $s_j = \begin{cases} \sigma_3, & j = 1 \\ \sigma_{2j}\sigma_{2j+1}, & j > 1 \end{cases}$

(E3) (Idempotent relations)

- (i) $e_{2i-1}^2 = z e_{2i-1}$ for $i \in [k]$.
- (ii) $e_{2i}^2 = e_{2i}$ for $2 \geq i \leq k-1$.
- (iii) $\sigma_{2i+1}e_{2i} = e_{2i}\sigma_{2i+1} = e_{2i}$ for $i \in [k-1]$.
- (iv) $\sigma_{2i}e_{2i} = e_{2i}\sigma_{2i} = e_{2i}$ for $2 \leq i \leq k-1$.
- (v) $\sigma_{2i}e_{2i-1}e_{2i+1} = \sigma_{2i+1}e_{2i-1}e_{2i+1}$ for $2 \leq i \leq k-1$.
- (vi) $e_{2i+1}e_{2i-1}\sigma_{2i} = e_{2i+1}e_{2i-1}\sigma_{2i+1}$ for $2 \leq i \leq k-1$.

(E4) (Commutation relations)

- (i) $e_i e_j = e_j e_i$, if $|i-j| \geq 2$.
- (ii) $\sigma_{2i-1}e_{2j-1} = e_{2j-1}\sigma_{2i-1}$, if $j \neq i-1, i$.
- (iii) $\sigma_{2i-1}e_{2j} = e_{2j}\sigma_{2i-1}$, if $j \neq i$.

(iv) $\sigma_{2i}e_{2j-1} = e_{2j-1}\sigma_{2i}$, if $j \neq i, i+1$.

(v) $\sigma_{2i}e_{2j} = e_{2j}\sigma_{2i}$, if $j \neq i-1$.

(E5) (Contractions)

(i) $e_ie_{i+1}e_i = e_i$ and $e_{i+1}e_ie_{i+1} = e_{i+1}$, for $i \in [2k-2]$.

(ii) $\sigma_{2i}e_{2i-1}\sigma_{2i} = \sigma_{2i+1}e_{2i+1}\sigma_{2i+1}$, for $2 \leq i \leq k-1$.

(iii) $\sigma_{2i}e_{2i-2}\sigma_{2i} = \sigma_{2i-1}e_{2i}\sigma_{2i-1}$, for $2 \leq i \leq k-1$.

□

Only the elements σ_i for $i \geq 3$ were involved in the above presentation since the even indexed base case $\sigma_2 = 1$ is trivial. The elements s_j in the above presentation are precisely the simple transpositions of the subalgebra of $\mathbb{C}[z]\mathfrak{S}_k$.

From *(ii)* and *(iii)* of *Proposition 2.37*, one can deduce that L_i and σ_j commute whenever $j \neq i-1, i, i+1$. We end this subsection by providing relations which tell us how the Jucys-Murphy elements interact with Enyang's generators when they do not commute.

Proposition 2.39. The following relations hold for any $i \geq 1$:

(i) $L_{2i+1} = \sigma_{2i}L_{2i-1}\sigma_{2i} - e_{2i}e_{2i-1}\sigma_{2i} - \sigma_{2i}e_{2i-1}e_{2i} + e_{2i}e_{2i+1}\sigma_{2i}e_{2i+1}e_{2i} + \sigma_{2i}$.

(ii) $L_{2i+2} = \sigma_{2i+1}L_{2i}\sigma_{2i+1} - e_{2i}e_{2i+1} - e_{2i+1}e_{2i} + e_{2i}e_{2i+1}\sigma_{2i+1}e_{2i+1}e_{2i} + \sigma_{2i+1}$.

(iii) $L_{2i} = \sigma_{2i}L_{2i}\sigma_{2i} + e_{2i}e_{2i-1}\sigma_{2i} + \sigma_{2i}e_{2i-1}e_{2i} - e_{2i}e_{2i+1} - e_{2i+1}e_{2i}$.

(iv) $L_{2i+1} = \sigma_{2i+1}L_{2i+1}\sigma_{2i+1} - e_{2i}e_{2i+1}\sigma_{2i+1} - \sigma_{2i+1}e_{2i+1}e_{2i} + e_{2i}e_{2i+1} + e_{2i+1}e_{2i}$.

Proof. (i): By definition,

$$L_{2i+1} = s_iL_{2i-1}s_i - L_{2i}e_{2i} - e_{2i}L_{2i} + (z - L_{2i-1})e_{2i} + \sigma_{2i}. \tag{2.7}$$

We examine the right hand side term by term. For the first term we have

$$s_iL_{2i-1}s_i = \sigma_{2i}\sigma_{2i+1}L_{2i-1}\sigma_{2i+1}\sigma_{2i} = \sigma_{2i}\sigma_{2i+1}^2L_{2i-1}\sigma_{2i} = \sigma_{2i}L_{2i-1}\sigma_{2i}.$$

For the second term, multiplying [Eny12, Proposition 3.2 (3)] on the left by s_i we get $\sigma_{2i}e_{2i-1}e_{2i} = L_{2i}e_{2i}$. Acting by the anti-automorphism $*$ yields $e_{2i}e_{2i-1}\sigma_{2i} = e_{2i}L_{2i}$ for the third term. Lastly for the forth term

$$\begin{aligned}(z - L_{2i-1})e_{2i} &= (z - L_{2i-1})e_{2i}e_{2i+1}e_{2i} \\ &= e_{2i}(z - L_{2i-1})e_{2i+1}e_{2i} \\ &= e_{2i}e_{2i+1}\sigma_{2i}e_{2i+1}e_{2i}\end{aligned}$$

where the last equality follows by [Eny12, Proposition 4.3 (2)]. Substituting these terms back into *Equation* (2.7) yields *(i)*.

(ii): By definition,

$$L_{2i+2} = s_i L_{2i} s_i - s_i L_{2i} e_{2i} - e_{2i} L_{2i} s_i + e_{2i} L_{2i} e_{2i+1} e_{2i} + \sigma_{2i+1}. \tag{2.8}$$

Multiplying this equation on the left and right by σ_{2i} gives

$$\begin{aligned} L_{2i+2} &= \sigma_{2i+1} L_{2i} \sigma_{2i+1} - \sigma_{2i+1} L_{2i} e_{2i} - e_{2i} L_{2i} \sigma_{2i+1} + e_{2i} L_{2i} e_{2i+1} e_{2i} + \sigma_{2i+1} \\ &= \sigma_{2i+1} L_{2i} \sigma_{2i+1} - \sigma_{2i+1}^2 e_{2i+1} e_{2i} - e_{2i} e_{2i+1} \sigma_{2i+1}^2 + e_{2i} e_{2i+1} \sigma_{2i+1} e_{2i+1} e_{2i} + \sigma_{2i+1} \\ &= \sigma_{2i+1} L_{2i} \sigma_{2i+1} - e_{2i+1} e_{2i} - e_{2i} e_{2i+1} + e_{2i} e_{2i+1} \sigma_{2i+1} e_{2i+1} e_{2i} + \sigma_{2i+1} \end{aligned}$$

which gives *(ii)*, where the second equality follows by relation $\sigma_{2i+1} e_{2i+1} e_{2i} = L_{2i} e_{2i}$ and its dual $e_{2i} e_{2i+1} \sigma_{2i+1} = e_{2i} L_{2i}$, which follows from [Eny12, Proposition 3.2 (3)].

(iii): It was shown in [Eny12, Proposition 3.10] that the element $L_1 + L_2 + \cdots + L_r$ belongs to the center of $\mathcal{A}_r(z)$. From this, and the fact that L_i and σ_j commute whenever $j \neq i-1, i, i+1$, one may deduce that

$$\sigma_{2i}(L_{2i-1} + L_{2i} + L_{2i+1})\sigma_{2i} = L_{2i-1} + L_{2i} + L_{2i+1}.$$

Rearranging gives

$$L_{2i} = \sigma_{2i} L_{2i} \sigma_{2i} + (\sigma_{2i} L_{2i-1} \sigma_{2i} - L_{2i+1}) + (\sigma_{2i} L_{2i+1} \sigma_{2i} - L_{2i-1}). \tag{2.9}$$

We examine the bracketed terms in *Equation* (2.9). Rearranging *(i)* gives the first bracketed term as

$$\sigma_{2i} L_{2i-1} \sigma_{2i} - L_{2i+1} = e_{2i} e_{2i-1} \sigma_{2i} + \sigma_{2i} e_{2i-1} e_{2i} - e_{2i} e_{2i+1} \sigma_{2i} e_{2i+1} e_{2i} - \sigma_{2i}.$$

Multiplying this on the left and right by σ_{2i}, and then rearranging gives the second bracketed term

$$\sigma_{2i} L_{2i+1} \sigma_{2i} - L_{2i-1} = -e_{2i} e_{2i-1} - e_{2i-1} e_{2i} + e_{2i} e_{2i+1} \sigma_{2i} e_{2i+1} e_{2i} + \sigma_{2i}.$$

Substituting these back into equation *Equation* (2.9) yields *(iii)*.

(iv): Analogously to the previous case, one can deduce that

$$\sigma_{2i+1}(L_{2i} + L_{2i+1} + L_{2i+2})\sigma_{2i+1} = L_{2i} + L_{2i+1} + L_{2i+2}.$$

Rearranging gives

$$L_{2i+1} = \sigma_{2i+1} L_{2i+1} \sigma_{2i+1} + (\sigma_{2i+1} L_{2i} \sigma_{2i+1} - L_{2i+2}) + (\sigma_{2i+1} L_{2i+2} \sigma_{2i+1} - L_{2i}). \tag{2.10}$$

We examine the bracketed terms in *Equation* (2.10). Rearranging *(2)(ii)* gives the first bracketed term as

$$\sigma_{2i+1} L_{2i} \sigma_{2i+1} - L_{2i+2} = e_{2i} e_{2i+1} + e_{2i+1} e_{2i} - e_{2i} e_{2i+1} \sigma_{2i+1} e_{2i+1} e_{2i} - \sigma_{2i+1}.$$

Multiplying this on the left and right by σ_{2i+1}, and then rearranging gives the second bracketed term

$$\sigma_{2i+1} L_{2i+2} \sigma_{2i+1} - L_{2i} = -e_{2i} e_{2i+1} \sigma_{2i+1} - \sigma_{2i+1} e_{2i+1} e_{2i} + e_{2i} e_{2i+1} \sigma_{2i+1} e_{2i+1} e_{2i} + \sigma_{2i+1}.$$

Substituting these back into equation *Equation* (2.10) yields *(iv)*.

□

2.2.3 Representation Theory

In *Chapter 3* we will provide a description of the center of the partition algebra $\mathcal{A}_{2k}(z)$ and as a result obtain a spectral condition which describes the blocks of $\mathcal{A}_{2k}(\delta)$ for any $\delta \in \mathbb{C}$. To do so we will utilise certain results regarding the representation theory of the partition algebras. This subsection seeks to summarise all such results. We first recall some semisimple results analogous to those presented in *Section 2.1*; describing the branching graph of the partition algebras, the *Gelfand-Zetlin basis* for simple modules, and the corresponding diagonal action of the Jucys-Murphy elements. All such results can be found in [HR05]. Next we give the combinatorial description of the blocks of the partition algebras first presented in [Martin96]. Lastly we mention a basis for the *cell modules* of the partition algebras and recall the upper triangular action of the Jucys-Murphy elements on such a basis as described in [Eny13].

Semisimple Case

Here we focus on the $\mathbb{C}$-algebras $\mathcal{A}_r(\delta)$ for parameters $\delta \in \mathbb{C}$. The partition algebra $\mathcal{A}_r(\delta)$ is semisimple for all but a finite number of choices of δ. The following result is due to [MS94] (see also [MS93], [Martin96], and [HR05]).

Theorem 2.40. For $r \in \mathbb{Z}_{\geq 0}$, the partition algebra $\mathcal{A}_r(\delta)$ is semisimple if and only if $\delta \notin \{0, 1, \dots, 2\lfloor \frac{r}{2} \rfloor - 2\}$.

□

Using the natural diagrammatic embeddings of the partition monoids described in *Section 2.2.1*, we have a chain of $\mathbb{C}$-algebras

$$\mathbb{C} = \mathcal{A}_0(\delta) \subset \mathcal{A}_1(\delta) \subset \mathcal{A}_2(\delta) \subset \cdots \subset \mathcal{A}_r(\delta). \tag{2.11}$$

It was shown in [MR98, Proposition 1] that, when $\delta \notin \{0, 1, \dots, 2\lfloor \frac{r}{2} \rfloor - 2\}$, this chain is multiplicity-free. That is, given $1 \leq i \leq r$ and any simple $\mathcal{A}_i(\delta)$-module M, the multiplicity of any simple $\mathcal{A}_{i-1}(\delta)$-module in $\mathsf{Res}_{\mathcal{A}_{i-1}(\delta)}(M)$ is at most one. As was the case for the chain of group algebras of the symmetric group, the description of the simple constituents arising from restriction over the above chain may be encoded in a combinatorial manner by a *branching graph*, which we describe below. Recall from *Section 2.1* that Λ_n denotes the set of partitions of n. Then we define the set

$$\Lambda_{\leq r} := \bigcup_{i=0}^{r} \Lambda_i.$$

Hence any element $\lambda \in \Lambda_{\leq r}$ is a partition $\lambda \vdash i$ for some $i = 0, 1, \dots, r$.

Definition 2.41. We let $\widehat{\mathsf{A}}$ denote the graded directed graph with levels indexed by the non-negative integers such that:

(1) Vertices on level r are given by the set $\Lambda_{\leq \lfloor \frac{r}{2} \rfloor} \times \{r\}$.

(2) For r even, an edge $(\lambda, r) \to (\mu, r+1)$ exists if either

(a) $\lambda = \mu$, or

(b) μ is obtained from λ by removing a box.

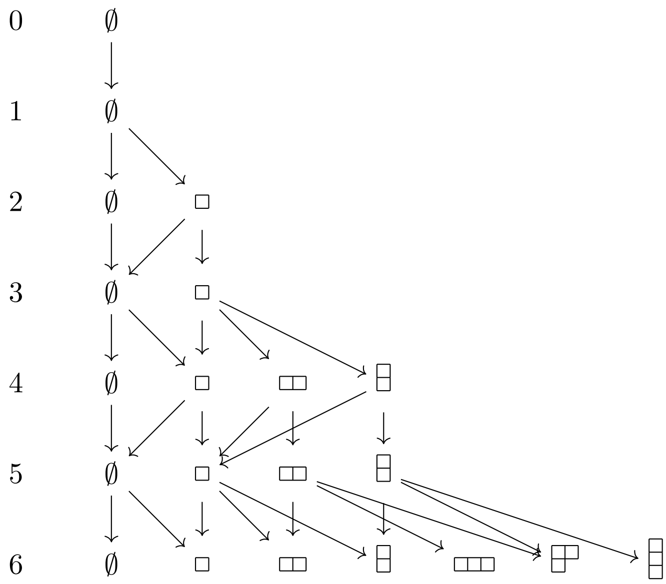
(3) For r odd, an edge $(\lambda, r) \to (\mu, r+1)$ exists if either

(a) $\lambda = \mu$, or

(b) μ is obtained from λ by adding a box.

The second coordinate of a vertex of $\widehat{\mathsf{A}}$ simply records the level it belongs to. We have included this as the partition in the first coordinate will appear in multiple levels, and we wish to distinguish between them easily.

Example 2.42. The first seven levels of $\widehat{\mathsf{A}}$ (where we have dropped the second coordinate of vertices and instead recorded the level to the left) are given by



A path in $\widehat{\mathsf{A}}$ is a sequence $\mathsf{T} = ((\lambda^{(i)}, i))_{i=s}^{r}$ for some $r \geq s$ where $(\lambda^{(i)}, i) \to (\lambda^{(i+1)}, i+1)$ is an edge. We alternatively write

$$\mathsf{T} = \lambda^{(s)} \to \lambda^{(s+1)} \to \cdots \to \lambda^{(r)}.$$

We let $\mathsf{Path}(\lambda, r)$ denote the set of paths in $\widehat{\mathsf{A}}$ starting at level 0 and whose terminal vertex is (λ, r). Truncating the graph $\widehat{\mathsf{A}}$ up to level r gives the branching graph associated with the multiplicity-free chain of *Equation* (2.11). We summarise this in the following theorem (see [HR05, Theorem 2.24] and [Martin00, Proposition 7]).

Theorem 2.43. Let $r \in \mathbb{Z}_{\geq 0}$ and $\delta \notin \{0, 1, \ldots, 2\lfloor \frac{r}{2} \rfloor - 2\}$. Then for all $0 \leq i \leq r$:

(1) The i-th level of $\widehat{\mathsf{A}}$ gives an indexing set for the isomorphism classes of simple $\mathcal{A}_i(\delta)$-modules. Let $\mathsf{A}^{(\lambda,i)}$ denote a simple module of class $(\lambda,i) \in \Lambda_{\leq \lfloor \frac{i}{2} \rfloor} \times \{i\}$.

(2) For $i \geq 1$ and $(\lambda, i) \in \widehat{\mathsf{A}}$, we have

$$\mathsf{Res}_{\mathcal{A}_{i-1}(\delta)}(\mathsf{A}^{(\lambda,i)}) = \bigoplus_{(\mu,i-1)\to(\lambda,i)} \mathsf{A}^{(\mu,i-1)},$$

where the sum runs over all edges $(\mu, i-1) \to (\lambda, i)$ in $\widehat{\mathsf{A}}$.

(3) For each $(\lambda, i) \in \widehat{\mathsf{A}}$ we have $\mathsf{dim}(\mathsf{A}^{(\lambda,i)}) = |\mathsf{Path}(\lambda, i)|$.

$\square$

As was the case for the group algebra of the symmetric group given in *Section 2.1.5*, we may employ item (2) of the above theorem to obtain a canconical decomposition of any simple $\mathcal{A}_r(\delta)$-module $\mathsf{A}^{(\lambda,r)}$ into one dimensional vector spaces indexed by paths. That is

$$\mathsf{A}^{(\lambda,r)} = \bigoplus_{\mathsf{T}\in\mathsf{Path}(\lambda,r)} V_{\mathsf{T}},$$

where V_{T} is 1-dimensional. Picking a non-zero vector $v_{\mathsf{T}} \in V_{\mathsf{T}}$ for each $\mathsf{T} \in \mathsf{Path}(\lambda, r)$ gives a unique (up to scalars) basis $\{v_{\mathsf{T}} \mid \mathsf{T} \in \mathsf{Path}(\lambda, r)\}$ for the simple $\mathcal{A}_r(\delta)$-module $\mathsf{A}^{(\lambda,r)}$. Again we refer to such a basis as the *Gelfand-Zetlin basis*, or GZ-basis for short.

Lastly, T. Halverson and A. Ram showed that the GZ-basis of a given simple module $\mathsf{A}^{(\lambda,r)}$ diagonalises the action of the Jucys-Murphy elements, and they gave a description of the corresponding eigenvalues. To present this we first define the following.

Definition 2.44. For $(\lambda, r) \in \widehat{\mathsf{A}}$, path $\mathsf{T} = ((\lambda^{(i)}, i))_{i=0}^{r} \in \mathsf{Path}(\lambda, r)$, and $0 \leq i \leq r$, when i is even we set

$$\mathrm{cont}(\mathsf{T}, i) := \begin{cases} z - |\lambda^{(i)}|, & \text{if } \lambda^{(i)} = \lambda^{(i-1)} \\ c(\square), & \text{if } \lambda^{(i)} = \lambda^{(i-1)} \cup \{\square\} \end{cases}$$

and for i odd we set

$$\mathrm{cont}(\mathsf{T}, i) := \begin{cases} |\lambda^{(i)}|, & \text{if } \lambda^{(i)} = \lambda^{(i-1)} \\ z - c(\square), & \text{if } \lambda^{(i)} = \lambda^{(i-1)} \backslash \{\square\} \end{cases}.$$

We refer to such a value as the *contents of path* T *at* i. For any $\delta \in \mathbb{C}$ we let $\mathrm{cont}_\delta(\mathsf{T}, i)$ denote the element of $\mathbb{C}$ obtained by replacing z with δ in the corresponding definition.

For the semisimple partition algebras, these values above provide statistics to the paths in $\widehat{\mathsf{A}}$ which generalises the contents of paths in $\widehat{\mathsf{S}}$ given in *Definition 2.19*.

Theorem 2.45 (Theorem 3.37 (b) of [HR05])**.** Let $r \in \mathbb{Z}_{\geq 0}$ and $\delta \notin \{0, 1, \ldots, 2\lfloor \frac{r}{2} \rfloor - 2\}$. Let v_{T} be a GZ-basis element of the simple $\mathcal{A}_r(\delta)$-module $\mathsf{A}^{(\lambda,r)}$ for some $(\lambda, r) \in \widehat{\mathsf{A}}$ and $\mathsf{T} \in \mathsf{Path}(\lambda, r)$. Then for any $i \in [2k]$ we have that

$$L_i v_{\mathsf{T}} = \mathrm{cont}_\delta(\mathsf{T}, i) v_{\mathsf{T}}$$

$\square$

Description of Blocks

In this section we will present a combinatorial condition describing the blocks of $\mathcal{A}_{2k}(\delta)$ for arbitrary $\delta \in \mathbb{C}$. This condition was first given by P. Martin in [Martin96] for the case $\delta \neq 0$, and later extended by D. Wales and W. Doran in [DW00] to include the case $\delta = 0$. We begin by briefly recalling some definitions and notation.

Let A be any finite dimensional $\mathbb{C}$-algebra, and let Λ be an indexing set for the isomorphism classes of simple A-modules. The algebra A has a unique decomposition as a direct sum of indecomposable ideals

$$A = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_n,$$

where $1 = e_1 + e_2 + \cdots + e_n$ is a decomposition of unity as a sum of primitive central idempotents $e_i \in A$. The direct summands in the above decomposition are called the *blocks* of A. We say that an A-module M belongs to the block Ae_i if $e_iM = M$ and $e_jM = 0$ for all $j \neq i$. Any simple module of A belongs to a particular block. Also one can show that M belongs to a given block if and only if all its composition factors do also. For any $\lambda, \mu \in \Lambda$ we write $\lambda \sim \mu$ whenever the corresponding simples modules belong to the same block. This equips Λ with an equivalence relation. We let $\mathcal{B}_A(\lambda)$ be the equivalence class of λ in Λ with respect to this equivalence relation and refer to it as a *block* of Λ. Whenever A is semisimple, then $\mathcal{B}_A(\lambda) = \{\lambda\}$ for all $\lambda \in \Lambda$.

As described in the previous subsection, whenever $\delta \notin \{0, 1, \ldots, 2k-2\}$, the $\mathbb{C}$-algebra $\mathcal{A}_{2k}(\delta)$ is semisimple and the set $\Lambda_{\leq k} \times \{2k\}$ indexes the isomorphism classes of simple modules. For this section we drop the second coordinate and just work with the set $\Lambda_{\leq k}$. It was shown in [Martin96] that the set $\Lambda_{\leq k}$ also indexes the set of simple $\mathcal{A}_{2k}(\delta)$-modules whenever $\delta \neq 0$. When $\delta = 0$, the indexing set is $\Lambda_{\leq k} \backslash \{\emptyset\}$ (see for example [DW00, Corollary 2.3]). With this in mind we set

$$\Lambda^{(\delta)}_{\leq k} := \begin{cases} \Lambda_{\leq k}, & \delta \neq 0, \\ \Lambda_{\leq k} \backslash \{\emptyset\}, & \delta = 0. \end{cases}$$

For any $\lambda \in \Lambda^{(\delta)}_{\leq k}$ the block $\mathcal{B}_{\mathcal{A}_{2k}(\delta)}(\lambda) \subset \Lambda^{(\delta)}_{\leq k}$ consists of all partitions which belong to a chain of partitions satisfying a combinatorial criteria involving the parameter δ. To describe this first recall for any partitions $\mu \subset \lambda$ we have a skew diagram $\lambda \backslash \mu$ consisting of all boxes in λ which do not belong to μ. Then we call a skew diagram a *horizontal strip* if all its boxes belong to the same row.

Definition 2.46. Given $\mu, \lambda \in \Lambda^{(\delta)}_{\leq k}$ such that $\mu \subset \lambda$, we call the pair (μ, λ) a δ-pair if $\lambda \backslash \mu$ is a horizontal strip where the last (right-most) box has content $\delta - |\mu|$.

Example 2.47. Let $\delta = 1$ and $k \geq 3$, then consider $\mu = (2), \lambda = (2,1) \in \Lambda^{(1)}_{\leq k}$. Inscribing the contents within the boxes we have

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline \end{array} \subset \begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \lambda \backslash \mu = \begin{array}{|c|}\hline -1 \\ \hline \end{array}.$$

Since $-1 = \delta - |\mu| = 1 - 2$ the pair $(\square\square, \begin{smallmatrix}\square\square\\ \square\,\,\,\end{smallmatrix})$ is a 1-pair.

Let $r \in \mathbb{N}$ and suppose we have a chain of partitions

$$\tau^{(1)} \subset \tau^{(2)} \subset \cdots \subset \tau^{(r)}$$

belonging to $\Lambda^{(\delta)}_{\leq k}$ such that each consecutive pair $(\tau^{(i-1)}, \tau^{(i)})$ is a δ-pair. Then we call such a chain a δ-chain, and we say it is maximal if no other partition can be adjoined to form a longer δ-chain. Now given $\lambda \in \Lambda^{(\delta)}_{\leq k}$, let R_i denote the maximal horizontal strip which can be removed from the i-th row of λ and still form a partition. Let $c(R_i)$ denote the set of contents of the boxes in R_i, then one can note that $c(R_i) \cap c(R_j)$ is empty whenever $i \neq j$. As such if there exists a partition μ such that (μ, λ) is a δ-pair, then μ is the unique partition to do so. From this one can deduce that the maximal δ-chains give a set partition of $\Lambda^{(\delta)}_{\leq k}$, and so any λ belongs to a unique maximal δ-chain. We let $\mathcal{C}_{2k,\delta}(\lambda)$ denote the set of partitions belonging to the same maximal δ-chain as λ.

Example 2.48. For $k = 4$ and $\delta = 2$, there are two non-trivial maximal 2-chains with partitions belonging to $\Lambda^{(2)}_{\leq 4}$ given by

$$\emptyset \subset \begin{ytableau} 0 & 1 & 2 \end{ytableau} \subset \begin{ytableau} 0 & 1 & 2 \\ \text{-1} \end{ytableau}, \qquad \begin{ytableau} 0 \end{ytableau} \subset \begin{ytableau} 0 & 1 \end{ytableau} \subset \begin{ytableau} 0 & 1 \\ \text{-1} & 0 \end{ytableau}.$$

The other maximal 2-chains are of length one. These chains partition the set $\Lambda^{(2)}_{\leq 4}$ into distinct subsets $\mathcal{C}_{8,2}(\lambda)$, where for example $\mathcal{C}_{8,2}(\emptyset) = \{\emptyset, \ydiagram{3}, \ydiagram{3,1}\}$.

For the following result see [Martin96, Proposition 9] and [DW00, Theorem 1.1].

Proposition 2.49. Let $\lambda \in \Lambda^{(\delta)}_{\leq k}$, then $\mathcal{B}_{\mathcal{A}_{2k}(\delta)}(\lambda) = \mathcal{C}_{2k,\delta}(\lambda)$.

$\square$

In *Chapter 3* we will provide an alternative spectral description of the blocks of $\Lambda^{(\delta)}_{\leq k}$. We will associate to each partition a certain generating function which will encode the action of a central subalgebra of the partition algebra. Then the block structure appears by proving that two partitions belong to the same maximal δ-chain if and only if they have the same associated generating functions.

Cellularity

Cellular algebras were first introduced by Graham and Lehrer in [GL96]. Roughly speaking, given a commutative ring R, an R-algebra A is cellular if it come with a distinguished basis called a cellular basis which is particularly well adapted to studying its representation theory. The cellular basis induces a basis for what are called cell modules. These modules come equiped with a bilinear form, and when R is a field, quotienting the cell modules by the radical of the form gives zero or a simple module. All simple A-modules appear in this manner. It was first shown that the partition algebra $\mathcal{A}_{2k}(\delta)$ is a cellular algebra in [Xi99], where an explicit construction of a cellular basis was given. Later an alternative cellular basis was constructed for the partition algebra $\mathcal{A}_r(z)$ in [EG17, Theorem 6.30], and was called a *Murphy type* cellular basis. For our purposes, we do not require a detailed description of cellular algebras or of the cellular structure of the partition algebra, we simply want to recall some results given in [Eny13] regarding the action of the Jucys-Murphy elements on the cell modules.

To begin, for any $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$ there exists a cell module $\Delta_{2k}^{(\lambda,2k)}$ for the partition algebra $\mathcal{A}_{2k}(z)$ (see [Eny13, Definition 3.6]). This module has a basis $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathsf{Path}(\lambda, 2k)\}$ indexed by the paths in $\mathsf{Path}(\lambda, 2k)$. There is a natural partial order $\prec$ one can impose on the set of paths $\mathsf{Path}(\lambda, 2k)$ defined in [Eny13, Definition 3.8], and such an ordering allows one to describe the action of Jucys-Murphy elements on the cell modules as upper triangular.

Proposition 2.50 (Proposition 3.15 of [Eny13])**.** Let $k \in \mathbb{Z}_{\geq 0}$, $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, $\mathsf{T} \in \mathsf{Path}(\lambda, 2k)$, and m_{T} be the corresponding basis element of the cell module $\Delta_{2k}^{(\lambda,2k)}$ of $\mathcal{A}_{2k}(z)$. Then for any $i \in [2k]$,

$$L_i m_{\mathsf{T}} = \mathrm{cont}(\mathsf{T}, i) m_{\mathsf{T}} + \sum_{\mathsf{T} \prec \mathsf{S}} u_{\mathsf{S}}(\mathsf{T}, i) m_{\mathsf{S}}$$

where $u_{\mathsf{S}}(\mathsf{T}, i) \in \mathbb{C}[z]$, and $\mathrm{cont}(\mathsf{T}, i)$ is the content of T at i defined in *Definition 2.44*.

□

Remark 2.51. We are considering the cell modules as left modules instead of right as was presented in [Eny13].

The cellular structure of a cellular algebra is preserved under the extension of scalars of the ground ring. As such the $\mathbb{C}$-algebra $\mathcal{A}_{2k}(\delta) = \mathbb{C} \otimes_{\mathsf{ev}_\delta} \mathcal{A}_{2k}(z)$ is also cellular for any $\delta \in \mathbb{C}$, and in particular the $\mathcal{A}_{2k}(\delta)$-modules given by

$$\Delta_{2k,\delta}^{(\lambda,2k)} := \mathbb{C} \otimes_{\mathsf{ev}_\delta} \Delta_{2k}^{(\lambda,2k)}$$

provide a complete collection of cell modules for $\mathcal{A}_{2k}(\delta)$. Now working over a field the theory of cellular algebras tells us that each of the simple $\mathcal{A}_{2k}(\delta)$-modules appears as the head of some cell module. For any $\mathsf{T} \in \mathsf{Path}(\lambda, 2k)$, we abuse notation a little and write m_{T} to denote the basis element $1 \otimes m_{\mathsf{T}}$ of the cell module $\Delta_{2k,\delta}^{(\lambda,2k)} := \mathbb{C} \otimes_{\mathsf{ev}_\delta} \Delta_{2k}^{(\lambda,2k)}$. Then from the above proposition we immediately obtain the following results.

Corollary 2.52. Let $k \in \mathbb{Z}_{\geq 0}$, $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, $\mathsf{T} \in \mathsf{Path}(\lambda, 2k)$, and m_{T} be the corresponding basis element of the cell module $\Delta_{2k,\delta}^{(\lambda,2k)}$ of $\mathcal{A}_{2k}(\delta)$. Then for any $i \in [2k]$,

$$L_i m_{\mathsf{T}} = \mathrm{cont}_\delta(\mathsf{T}, i) m_{\mathsf{T}} + \sum_{\mathsf{T} \prec \mathsf{S}} v_{\mathsf{S}}(\mathsf{T}, i) m_{\mathsf{S}}$$

where $v_{\mathsf{S}}(\mathsf{T}, i) \in \mathbb{C}$, and $\mathrm{cont}_\delta(\mathsf{T}, i)$ is the content of T at i defined in *Definition 2.44*.

□

2.2.4 Schur-Weyl Duality

The partition algebra $\mathcal{A}_{2k}(n)$ and the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_n$ are in Schur-Weyl duality with respect to their actions on a given tensor space. In this section we summarise features of such a duality which we extend in *Chapter 4* and *Chapter 5*. We begin by recalling the orbit basis of the partition algebra, and then describe types of colourings of partition diagrams, both of which help in defining this duality and extending it later. Lastly we present the duality itself and describe how the Jucys-Murphy elements and Enyang's generators act. Most results in this section can be found in [BH18] and [HR05].

Orbit Basis

We can define a partial order on the set Π_r as follows: Given any $\alpha, \beta \in \Pi_r$ we let $\alpha \preceq \beta$ whenever each block of α is contained within a block of β. We then say that α is a *refinement* of β, and that β is a *coarsening* of α.

Definition 2.53. The orbit basis $\{O(\alpha) \mid \alpha \in \Pi_r\}$ of the partition algebra $\mathcal{A}_r(z)$ is defined according to the equation

$$D(\alpha) = \sum_{\alpha \preceq \beta} O(\beta), \tag{2.12}$$

and we let $O_\delta(\alpha) := 1 \otimes O(\alpha)$ in $\mathcal{A}_{2k}(\delta)$ for any $\delta \in \mathbb{C}$.

It is clear that the set $\{O(\alpha) \mid \alpha \in \Pi_r\}$ is indeed a basis since the transition matrix determined by *Equation* (2.12) is unitriangular, and hence invertible, with respect to any extension of the partial order $\preceq$ to a total order. One can essentially "invert" the summation of *Equation* (2.12) to express an orbit basis element as a sum of diagrammatic basis elements. This comes about from the Möbius inversion formula (see for example [S97, Proposition 3.7.1]) which gives

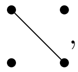
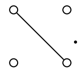
$$O(\alpha) = \sum_{\alpha \preceq \beta} \mu(\beta, \alpha) D(\beta) \tag{2.13}$$

where $\mu : \Pi_r \times \Pi_r \to \mathbb{Z}$ is called the Möbius function of Π_r. The function μ may be calculated as follows: Let $\alpha = \{A_1, \ldots, A_{|\alpha|}\}$ and $\beta = \{B_1, \ldots, B_{|\beta|}\}$ and suppose $\alpha \preceq \beta$. Then for each block of β we have that $B_i = A_{i_1} \cup \cdots \cup A_{i_{n_i}}$ for some $n_i \in \mathbb{Z}_{\geq 0}$ with $n_1 + \cdots + n_{|\beta|} = |\alpha|$. Then from [S97, Example 3.10.4],

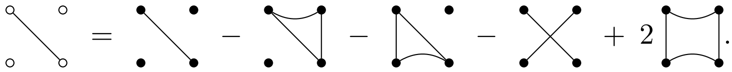
$$\mu(\beta, \alpha) = \prod_{i=1}^{|\beta|} (-1)^{n_i - 1}(n_i - 1)!.$$

We will also represent orbit basis elements diagrammatically as partition diagrams as we did for the diagrammatic basis of the partition algebras. However, to distinguish the orbit and diagrammatic basis, we will adopt the conventions of [BH18] and draw the vertices of the partition diagram representing an orbit basis as clear nodes as opposed to the solid black nodes used up until now.

Example 2.54. Let $\alpha = \{\{1,2'\},\{2\},\{1'\}\} \in \Pi_4$, then in $\mathcal{A}_4(z)$ we have

$D(\alpha) =$  , $\qquad O(\alpha) =$  .

From Equation (2.13) we have



Definition 2.55. Let $k \in \mathbb{N}$. For any subset $B \subseteq [k]$, let B' denote the corresponding subset of $[k']$ obtained by taking the primed elements of B. Similarly, for any set partition $S = \{B_1, \dots, B_{|S|}\}$ of $[k]$ let $S' = \{B_1', \dots, B_{|S|}'\}$ denote the corresponding set partition of $[k']$. We define $I(S)$ to be the set partition of $[k] \cup [k']$ given by $\{B_1 \cup B_1', \dots, B_{|S|} \cup B_{|S|}'\}$.

It is worth mentioning that for the partition diagram $1 = \{\{i, i'\} \mid i \in [k]\} \in \Pi_{2k}$, the element $D(1)$ is the identity of $\mathcal{A}_{2k}(z)$, but $O(1)$ is not. The identity $D(1)$ is equal to the sum of orbit basis elements $O(I(S))$ where S runs over all set partitions of $[k]$.

Colourings of Partition Diagrams

Let $n, k \in \mathbb{Z}_{\geq 0}$. We view any tuple $\boldsymbol{a} = (a_1, \dots, a_k) \in [n]^k$ as a function $[k] \to [n]$ by setting $\boldsymbol{a}(i) = a_i$ for any $i \in [k]$. Similarly, given tuples $\boldsymbol{a} = (a_1, \dots, a_k), \boldsymbol{b} = (b_1, \dots, b_k) \in [n]^k$, we view the pair $(\boldsymbol{a}, \boldsymbol{b})$ as a function $[k] \cup [k'] \to [n]$ by setting $(\boldsymbol{a}, \boldsymbol{b})(i) := a_i$ and $(\boldsymbol{a}, \boldsymbol{b})(i') := b_i$ for any $i \in [k]$.

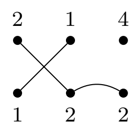
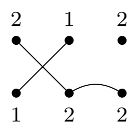
Definition 2.56. Let $\alpha \in \Pi_{2k}$ and $\boldsymbol{a}, \boldsymbol{b} \in [n]^k$. We let $\alpha_{\boldsymbol{b}}^{\boldsymbol{a}}$ denote the coloured partition diagram where vertex $i \in [k] \cup [k']$ in α has been assigned the colour $(\boldsymbol{a}, \boldsymbol{b})(i)$. We say for any $i, j \in [k] \cup [k']$ that

(1) $(\boldsymbol{a}, \boldsymbol{b})$ is a *good colouring* of α if $i \sim_\alpha j$ implies $(\boldsymbol{a}, \boldsymbol{b})(i) = (\boldsymbol{a}, \boldsymbol{b})(j)$. We write $(\boldsymbol{a}, \boldsymbol{b}) \to \alpha$ and let $\mathsf{GC}_n(\alpha) = \{(\boldsymbol{a}, \boldsymbol{b}) \in [n]^k \times [n]^k \mid (\boldsymbol{a}, \boldsymbol{b}) \to \alpha\}$.

(2) $(\boldsymbol{a}, \boldsymbol{b})$ is a *perfect colouring* of α if $i \sim_\alpha j$ if and only if $(\boldsymbol{a}, \boldsymbol{b})(i) = (\boldsymbol{a}, \boldsymbol{b})(j)$. We write $(\boldsymbol{a}, \boldsymbol{b}) \hookrightarrow \alpha$ and let $\mathsf{PC}_n(\alpha) = \{(\boldsymbol{a}, \boldsymbol{b}) \in [n]^k \times [n]^k \mid (\boldsymbol{a}, \boldsymbol{b}) \hookrightarrow \alpha\}$.

Note that all perfect colourings are good colourings.

Example 2.57. Let $k = 3$ and $\alpha = \{\{1,2',3'\},\{2,1'\},\{3\}\} \in \Pi_6$. Consider the tuples $\boldsymbol{a} = (2,1,4)$, $\boldsymbol{b} = (2,1,2)$, and $\boldsymbol{c} = (1,2,2)$. Then $(\boldsymbol{a}, \boldsymbol{c})$ is a perfect colouring of α while $(\boldsymbol{b}, \boldsymbol{c})$ is only a good colouring. The corresponding coloured partition diagrams are

$\alpha_{\boldsymbol{c}}^{\boldsymbol{a}} =$  and $\alpha_{\boldsymbol{c}}^{\boldsymbol{b}} =$  .

For any $n \geq 0$ we have the sets

$$\mathsf{GC}_n(\alpha) = \{((x,y,z),(y,x,x)) \mid (x,y,z) \in [n]^3\},$$

$$\mathsf{PC}_n(\alpha) = \{((x,y,z),(y,x,x)) \mid (x,y,z) \in [n]^{!3}\},$$

where $[n]^{!3}$ is the subset of $[n]^3$ consisting of all tuples with pairwise distinct entries. In particular, whenever $n \leq 2$ we have that $[n]^{!3} = \emptyset$, and thus $\mathsf{PC}_n(\alpha) = \emptyset$.

Put another way, a good colouring of a partition diagram is the same information as assigning a colour to each block, with perfect colourings meaning blocks have been assigned distinct colours. As such if $n < |\alpha|$ then $\mathsf{PC}_n(\alpha) = \emptyset$. For any $\alpha, \beta \in \Pi_{2k}$, it is clear from the definition that $\mathsf{PC}_n(\alpha) \cap \mathsf{PC}_n(\beta) = \emptyset$ whenever $\alpha \neq \beta$. Moreover one can deduce that we have the disjoint unions

$$[n]^k \times [n]^k = \bigsqcup_{\alpha \in \Pi_{2k}} \mathsf{PC}_n(\alpha), \qquad \text{and} \qquad \mathsf{GC}_n(\alpha) = \bigsqcup_{\alpha \preceq \beta} \mathsf{PC}_n(\beta) \tag{2.14}$$

The former disjoint union tells us that for any tuple $(\boldsymbol{a}, \boldsymbol{b}) \in [n]^k \times [n]^k$ there exists a unique set partition $\alpha \in \Pi_{2k}$ such that $(\boldsymbol{a}, \boldsymbol{b}) \hookrightarrow \alpha$. Now the symmetric group $\mathfrak{S}_n$ acts on $[n]^k$ coordinate-wise, and we denote this action by $\pi\boldsymbol{a}$ for any $\pi \in \mathfrak{S}_n$ and $\boldsymbol{a} \in [n]^k$. This action extends to $[n]^k \times [n]^k$ component-wise. For $(\boldsymbol{a}, \boldsymbol{b}) \in [n]^k \times [n]^k$ we denote the orbit by $\mathsf{Orbit}_{\mathfrak{S}_n}(\boldsymbol{a}, \boldsymbol{b}) = \{(\pi\boldsymbol{a}, \pi\boldsymbol{b}) \mid \pi \in \mathfrak{S}_n\}$. It is simple to deduce that

$$\mathsf{Orbit}_{\mathfrak{S}_n}(\boldsymbol{a}, \boldsymbol{b}) = \mathsf{PC}_n(\alpha)$$

where α is the unique set partition of Π_{2k} which is perfectly coloured by $(\boldsymbol{a}, \boldsymbol{b})$. In particular, both of the disjoint unions of *Equation* (2.14) are decompositions of $\mathfrak{S}_n$-action sets into $\mathfrak{S}_n$-orbits.

Schur-Weyl Duality

Let $V = \mathrm{Span}_{\mathbb{C}}\{v_1, \ldots, v_n\}$ be the n-dimensional permutation module for the group algebra $\mathbb{C}\mathfrak{S}_n$ of the symmetric group. Thus $\pi v_i = v_{\pi(i)}$ for any $\pi \in \mathfrak{S}_n$ and $i \in [n]$. For any $k \in \mathbb{Z}_{\geq 0}$ we let $V^{\otimes k} := V \otimes \cdots \otimes V$ with k tensor components. Then the tensor space $V^{\otimes k}$ may be regarded as a $\mathbb{C}\mathfrak{S}_n$-module by extending the action of V diagonally. For any $\boldsymbol{a} = (a_1, \ldots, a_k) \in [n]^k$ we set $v_{\boldsymbol{a}} = v_{a_1} \otimes \cdots \otimes v_{a_k}$. Then we have that

$$V^{\otimes k} = \mathrm{Span}_{\mathbb{C}}\{v_{\boldsymbol{a}} \mid \boldsymbol{a} \in [n]^k\},$$

and the diagonal action is given by $\pi v_{\boldsymbol{a}} = v_{\pi\boldsymbol{a}}$ for all $\pi \in \mathfrak{S}_n$ and $\boldsymbol{a} \in [n]^k$. For any $\boldsymbol{a}, \boldsymbol{b} \in [n]^k$ we let $\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}} \in \mathsf{End}_{\mathbb{C}}(V^{\otimes k})$ be the endomorphism defined on the basis by $\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}}(v_{\boldsymbol{c}}) = \delta_{\boldsymbol{b},\boldsymbol{c}} v_{\boldsymbol{a}}$ where $\delta_{\boldsymbol{b},\boldsymbol{c}}$ is the Kronecker delta. We let $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ denote the space of endomorphisms which commute with the diagonal action of $\mathbb{C}\mathfrak{S}_n$. For the following result see for example [HR05, Theorem 3.6].

Theorem 2.58. For any $n, k \in \mathbb{Z}_{\geq 0}$, there exists a surjective homomorphism of $\mathbb{C}$-algebras

$$\Psi_{2k,n} : \mathcal{A}_{2k}(n) \to \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$$

with the following properties:

(1) The map $\Psi_{2k,n}$ acts on the diagrammatic basis by the $\mathbb{C}$-linear extension of

$$\Psi_{2k,n}(D_n(\alpha)) = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{GC}_n(\alpha)} \mathsf{E}_{\boldsymbol{b}}^{\boldsymbol{a}}$$

(2) The map $\Psi_{2k,n}$ acts on the orbit basis by the $\mathbb{C}$-linear extension of

$$\Psi_{2k,n}(O_n(\alpha)) = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)} \mathsf{E}_{\boldsymbol{b}}^{\boldsymbol{a}}$$

(3) The image and kernel of $\Psi_{2k,n}$ are given by

$$\begin{aligned}\mathsf{Im}(\Psi_{2k,n}) = \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k}) &= \mathrm{Span}_{\mathbb{C}}\{O_n(\alpha) \mid \alpha \in \Pi_{2k},\ |\alpha| \leq n\},\\ \mathsf{Ker}(\Psi_{2k,n}) &= \mathrm{Span}_{\mathbb{C}}\{O_n(\alpha) \mid \alpha \in \Pi_{2k},\ |\alpha| > n\}.\end{aligned}$$

Whenever $n > 2k$ then $\mathsf{Ker}(\Psi_{2k,n}) = 0$, and thus $\Psi_{2k,n}$ is an isomorphism witnessing $\mathcal{A}_{2k}(n) \cong \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$.

□

In *Chapter 4* and *Chapter 5* we extend the map $\Psi_{2k,n}$ to affine versions of the partition algebra, in other words construct an action which satisfies affinization property 4 from *Section 1.4*. In *Chapter 5* we will be able to directly generalise items (2) and (3) of *Theorem 2.58*. However it proves difficult to generalises (1) in an analogous manner since we lack an affine counterpart for the diagrammatic basis. In *Chapter 4* the extension will generalise the action of $\Psi_{2k,n}$ on the generating set given in *Theorem 2.38* and the Jucys-Murphy elements, as described below.

Due to the fact that $\Psi_{2k,n}$ is an isomorphism whenever $2k > n$, this action can by used to verify relations in the partition algebra $\mathcal{A}_{2k}(z)$ as follows:

Lemma 2.59. Let $R_1, R_2 \in \mathcal{A}_{2k}(z)$. Then $R_1 = R_2$ if and only if for all $n \geq 0$,

$$(\Psi_{2k,n} \circ \mathsf{pr}_n)(R_1) = (\Psi_{2k,n} \circ \mathsf{pr}_n)(R_2)$$

Proof. The forward implication is immediate. Assuming the latter equations we have

$$R_1 - R_2 \in \bigcap_{n\geq 0} \mathsf{Ker}(\Psi_{2k,n} \circ \mathsf{pr}_n) \subset \bigcap_{n>2k} (z-n) = 0,$$

showing that $R_1 = R_2$, where we used the facts that for all $n > 2k$ we have $\mathsf{Ker}(\Psi_{2k,n}) = 0$ and $\mathsf{Ker}(\mathsf{pr}_n) = (z-n)$, the two-sided ideal of $\mathcal{A}_{2k}(z)$ generated by $z - n \in \mathbb{C}[z]$.

□

We close this subsection by recalling how both Enyang's generators and the Jucys-Murphy elements act under $\Psi_{2k,n}$. Recall for any distinct $a, b \in [n]$ that $(a, b) \in \mathfrak{S}_n$ denotes the transposition exchanging a and b. We take the convention that $(a, b) = 1$ whenever $a = b$. The following can be found in [Eny12, Section 5].

Proposition 2.60. Let $n, k \in \mathbb{Z}_{\geq 0}$. For any $\boldsymbol{a} = (a_1, \dots, a_k) \in [n]^k$ and $i \in [k-1]$,

$$\Psi_{2k,n}(\sigma_{2i})(v_{\boldsymbol{a}}) = (a_i, a_{i+1})(v_{a_1} \otimes \cdots \otimes v_{a_{i-1}}) \otimes v_{a_i} \otimes \cdots \otimes v_{a_k}$$
$$\Psi_{2k,n}(\sigma_{2i+1})(v_{\boldsymbol{a}}) = (a_i, a_{i+1})(v_{a_1} \otimes \cdots \otimes v_{a_{i+1}}) \otimes v_{a_{i+2}} \otimes \cdots \otimes v_{a_k}$$

□

Proposition 2.61. Let $n, k \in \mathbb{Z}_{\geq 0}$. For any $\boldsymbol{a} = (a_1, \dots, a_k) \in [n]^k$ and $i \in [k-1]$,

$$\Psi_{2k,n}(L_{2i-1})(v_{\boldsymbol{a}}) = n v_{\boldsymbol{a}} - \sum_{b=1}^{n} (a_i, b)(v_{a_1} \otimes \cdots \otimes v_{a_{i-1}}) \otimes v_{a_i} \otimes \cdots \otimes v_{a_k}$$
$$\Psi_{2k,n}(L_{2i})(v_{\boldsymbol{a}}) = \sum_{b=1}^{n} (a_i, b)(v_{a_1} \otimes \cdots \otimes v_{a_i}) \otimes v_{a_{i+1}} \otimes \cdots \otimes v_{a_k}$$

□

2.2.5 Constructing $\mathcal{A}_{2k}(z)$ via the Orbit Basis

The definition of the partition algebra presented in *Section 2.2.1* is the typical construction given in the literature. For this section we provide a construction from the perspective of the orbit basis and the algebra $\mathrm{Im}(\Psi_{2k,n}) = \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$. We recover items (2) and (3) of *Theorem 2.58* and describe how the orbit basis elements multiply, all of which can be found in [BH18]. We have included these results and proofs of such results since, although they will not be directly applied in later chapters, the overall process in constructing the partition algebra in this manner will be generalised in *Chapter 5* and the details of the proofs here will resemble later results.

Firstly for $k \in \mathbb{Z}_{\geq 0}$ and any $\alpha \in \Pi_{2k}$, let

$$\overline{O}_n(\alpha) := \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}} \in \mathsf{End}_{\mathbb{C}}(V^{\otimes k}) = \mathrm{Span}_{\mathbb{C}}\{\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}} \mid (\boldsymbol{a}, \boldsymbol{b}) \in [n]^k \times [n]^k\}.$$

It is easy to check that as operators on $V^{\otimes k}$ we have $\pi \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}} \pi^{-1} = \mathsf{E}^{\pi\boldsymbol{a}}_{\pi\boldsymbol{b}}$, and as such (since $\mathsf{PC}_n(\alpha)$ is $\mathfrak{S}_n$-invariant) we have $\pi \overline{O}_n(\alpha) \pi^{-1} = \overline{O}_n(\alpha)$. This tells us that the operators $\overline{O}_n(\alpha)$ belong to $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$. Moreover these operators provide a spanning set for $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ which can be shown as follows: Let

$$E = \sum_{(\boldsymbol{a},\boldsymbol{b}) \in [n]^k \times [n]^k} c_{(\boldsymbol{a},\boldsymbol{b})} \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}}$$

be an arbitrary element of $\mathsf{End}_{\mathbb{C}}(V^{\otimes k})$, where $c_{(\boldsymbol{a},\boldsymbol{b})} \in \mathbb{C}$. Then E belongs to $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ if and only if $\pi E \pi^{-1} = E$ or all $\pi \in \mathfrak{S}_n$. This implies that $c_{(\boldsymbol{a},\boldsymbol{b})} = c_{(\pi\boldsymbol{a},\pi\boldsymbol{b})}$ for all $\pi \in \mathfrak{S}_n$. As such for each $\alpha \in \Pi_{2k}$ we can set $c_\alpha := c_{(\boldsymbol{a},\boldsymbol{b})}$ for any perfect colouring $(\boldsymbol{a}, \boldsymbol{b}) \hookrightarrow \alpha$, and thus E is the sum of terms $c_\alpha \overline{O}_n(\alpha)$ as α runs over Π_{2k}. Therefore E belongs to $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ if and only if E belongs to $\mathrm{Span}_{\mathbb{C}}\{\overline{O}_n(\alpha) \mid \alpha \in \Pi_{2k}\}$. Also from definition we have that $\overline{O}_n(\alpha) = 0$ whenever $|\alpha| > n$, and that the set $\{\overline{O}_n(\alpha) \mid \alpha \in \Pi_{2k}, |\alpha| \leq n\}$ is linearly independent, and hence gives a basis for $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$.

We now wish to "lift" the $\mathbb{C}$-algebra $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ to an algebra over $\mathbb{C}[z]$, which will coincide with the partition algebra $\mathcal{A}_{2k}(z)$. To do so we need to understand how the composition of a pair of elements $\overline{O}_n(\alpha)$ and $\overline{O}_n(\beta)$ decomposes as a linear combination of elements of the same kind. We first set up some notation.

Definition 2.62. Given any $\alpha, \beta \in \Pi_{2k}$ we define the following:

(1) $\mathsf{Top}(\alpha) := \{T \mid T \in \alpha,\ T \subseteq [k]\}$,

(2) $\mathsf{Bot}(\alpha) := \{B \mid B \in \alpha,\ B \subseteq [k']\}$,

(3) $\mathsf{Mid}(\alpha \star \beta) := \{M \mid M \in \alpha \star \beta,\ M \subseteq [k'']\}$.

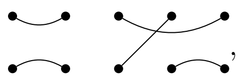
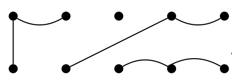
We refer to elements in $\mathsf{Top}(\alpha)$ and $\mathsf{Bot}(\alpha)$ as the top and bottom blocks of α respectively, and we refer to $\mathsf{Mid}(\alpha \star \beta)$ as the middle blocks of $\alpha \star \beta$.

Recalling *Definition 2.31* we have that $m(\alpha, \beta) = |\mathsf{Mid}(\alpha \star \beta)|$. Also one can see that $\mathsf{Top}(\alpha) \subseteq \mathsf{Top}(\alpha \circ \beta)$ and $\mathsf{Bot}(\beta) \subseteq \mathsf{Bot}(\alpha \circ \beta)$, but in general these inclusions are strict.

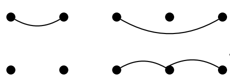
Definition 2.63. Let T and B be two finite sets. A partial bijection from T to B is a triple (θ, X, Y) where $X \subset T$, $Y \subset B$, $|X| = |Y|$, and $\theta : X \to Y$ is a bijection. We will often supress the domain and codomain and just write θ.

Definition 2.64. Let $\alpha, \beta \in \Pi_{2k}$. We say that $\gamma \in \Pi_{2k}$ is a *top-bottom coarsening* of the pair (α, β) if there exists a partial bijection θ from $\mathsf{Top}(\alpha)$ to $\mathsf{Bot}(\beta)$ such that γ is obtained from $\alpha \circ \beta$ by merging the blocks T and $\theta(T)$ for each T in the domain of θ. We let $\mathsf{TBC}(\alpha, \beta)$ denote the set of top-bottom coarsenings of (α, β).

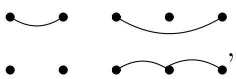
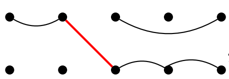
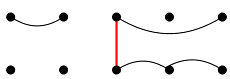
Example 2.65. Consider $\alpha, \beta \in \Pi_{10}$ given diagrammatically by

$\alpha =$  $, \quad \beta =$  .

We have that $\mathsf{Top}(\alpha) = \{\{1,2\}, \{3,5\}\}$ and $\mathsf{Bot}(\beta) = \{\{3', 4', 5'\}\}$. There are three partial bijections from $\mathsf{Top}(\alpha)$ to $\mathsf{Bot}(\beta)$: the empty partial bijection $\theta_1 : \emptyset \to \emptyset$, $\theta_2 : \{1,2\} \mapsto \{3', 4', 5'\}$, and $\theta_3 : \{3,5\} \mapsto \{3', 4', 5'\}$. We have that

$\alpha \circ \beta =$  .

Therefore the set $\mathsf{TBC}(\alpha, \beta)$ is given by

$\Big\{ \gamma_1 =$  $, \quad \gamma_2 =$  $, \quad \gamma_3 =$  $\Big\}$.

Each γ_i is associated with the partial bijection θ_i respectively, and we have added edges in red to highlight the blocks which were merged. For example γ_2 has been obtained from $\alpha \circ \beta$ by merging blocks $\{1,2\}$ and $\theta_2(\{1,2\}) = \{3',4',5'\}$. Note that the top block $\{4\}$ of $\alpha \circ \beta$ never gets merged to a bottom block for any $\gamma \in \mathsf{TBC}(\alpha,\beta)$ since, although it is a top block of $\alpha \circ \beta$, it does not belong to $\mathsf{Top}(\alpha)$, in other words it was a top block formed in the process of taking the product of α and β and not a top block of α originally. Similarly, the bottom blocks $\{1'\}$ and $\{2'\}$ of $\alpha \circ \beta$ are not merged with any top blocks since they do not belong to $\mathsf{Bot}(\beta)$.

Definition 2.66. For any $\alpha, \beta \in \Pi_{2k}$, we will say that the pair (α, β) *matches in the middle* whenever $i \sim_\beta j \iff i' \sim_\alpha j'$ for all $i, j \in [k]$.

When (α, β) matches in the middle, it means that the set partition of $[k']$ induced from the bottom row of α is the prime counterpart to the set partition of $[k]$ induced from the top row of β. For example, given α and β as in *Example 2.65*, the pair (α, β) matches in the middle while (β, α) does not.

The following result was given in [BH18, Lemma 4.2].

Proposition 2.67. For any $k \in \mathbb{Z}_{\geq 0}$ and partition diagrams $\alpha, \beta, \gamma \in \Pi_{2k}$, there exists a polynomial $p^{\gamma}_{\alpha,\beta}(z) \in \mathbb{Z}[z]$ such that

$$\overline{O}_n(\alpha)\overline{O}_n(\beta) = \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(n)\overline{O}_n(\gamma).$$

Proof. Firstly note that $\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}}\mathsf{E}^{\boldsymbol{d}}_{\boldsymbol{c}} = \delta_{\boldsymbol{b},\boldsymbol{d}}\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}}$ for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in [n]^k \times [n]^k$. Then by definition

$$\begin{aligned}
\overline{O}_n(\alpha)\overline{O}_n(\beta) &= \left(\sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{b}} \right) \left(\sum_{(\boldsymbol{d},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)} \mathsf{E}^{\boldsymbol{c}}_{\boldsymbol{d}} \right), \\
&= \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} \sum_{(\boldsymbol{d},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)} \delta_{\boldsymbol{b},\boldsymbol{d}}\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}}.
\end{aligned}$$

For this sum to be non-zero we require that there exists a tuple $\boldsymbol{b} = \boldsymbol{d} \in [n]^k$ which perfectly colours both the bottom row of α and the top row of β. This occurs if and only if the pair (α, β) matches in the middle. Hence we set $p^{\gamma}_{\alpha,\beta}(z) = 0$ whenever (α, β) does not match in the middle, and thus the lemma holds for all such pairs. Assume now that (α, β) does match in the middle, then from above we obtain

$$\overline{O}_n(\alpha)\overline{O}_n(\beta) = \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} \sum_{(\boldsymbol{b},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)} \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}}.$$

Let $\mathsf{C}_n(\alpha, \beta)$ be the set consisting of the pairs of tuples $(\boldsymbol{a}, \boldsymbol{c}) \in [n]^k \times [n]^k$ such that there exists a tuple $\boldsymbol{b} \in [n]^k$ where $(\boldsymbol{a}, \boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ and $(\boldsymbol{b}, \boldsymbol{c}) \in \mathsf{PC}_n(\beta)$. Also, for any $(\boldsymbol{a}, \boldsymbol{c}) \in \mathsf{C}_n(\alpha, \beta)$, we let $\mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha, \beta)$ denote the set of $\boldsymbol{b} \in [n]^k$ such that $(\boldsymbol{a}, \boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ and $(\boldsymbol{b}, \boldsymbol{c}) \in \mathsf{PC}_n(\beta)$. Thus we may rewrite the double summation above as

$$\begin{aligned}
\overline{O}_n(\alpha)\overline{O}_n(\beta) &= \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{C}_n(\alpha,\beta)} \sum_{\boldsymbol{b} \in \mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha,\beta)} \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}}, \\
&= \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{C}_n(\alpha,\beta)} |\mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha,\beta)|\mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}}.
\end{aligned}$$

Note that any pair $(\boldsymbol{a}, \boldsymbol{c}) \in \mathsf{C}_n(\alpha, \beta)$ gives a good colouring of $\alpha \circ \beta$. Also from the definition of the set $\mathsf{C}_n(\alpha, \beta)$, if $(\boldsymbol{a}, \boldsymbol{c})$ is not a perfect colouring of $\alpha \circ \beta$, then the blocks of $(\alpha \circ \beta)^{\boldsymbol{a}}_{\boldsymbol{c}}$ which share a colour must appear in pairs between the sets of blocks $\mathsf{Top}(\alpha) \subset \mathsf{Top}(\alpha \circ \beta)$ and $\mathsf{Bot}(\beta) \subset \mathsf{Bot}(\alpha, \circ\beta)$. Hence $(\boldsymbol{a}, \boldsymbol{c})$ belongs to $\mathsf{C}_n(\alpha, \beta)$ if and only if there exists $\gamma \in \mathsf{TBC}(\alpha, \beta)$ such that $(\boldsymbol{a}, \boldsymbol{c}) \in \mathsf{PC}_n(\gamma)$. Moreover, it is simple to check that $\mathsf{C}_n(\alpha, \beta)$ is $\mathfrak{S}_n$-invariant, hence we must have the disjoint union

$$\mathsf{C}_n(\alpha, \beta) = \bigsqcup_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \mathsf{PC}_n(\gamma).$$

As for the set $\mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta)$, its size equals the number of tuples $\boldsymbol{b}$ which perfectly colour the middle row of $\alpha \star \beta$ given that $(\boldsymbol{a}, \boldsymbol{b}) \hookrightarrow \alpha$ and $(\boldsymbol{b}, \boldsymbol{c}) \hookrightarrow \beta$. Hence it equals the number of ways to assign colours to the middle blocks $\mathsf{Mid}(\alpha \star \beta)$ which are distinct from the colours appearing in the tuples $\boldsymbol{a}$ and $\boldsymbol{c}$. The number of colours appearing within the tuples $\boldsymbol{a}$ and $\boldsymbol{c}$ equals the number of blocks in γ, where γ is the unique element of $\mathsf{TBC}(\alpha, \beta)$ such that $(\boldsymbol{a}, \boldsymbol{c}) \hookrightarrow \gamma$. Thus

$$|\mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta)| = (n - |\gamma|)(n - |\gamma| - 1) \cdots (n - |\gamma| - m(\alpha, \beta)),$$

which is polynomial in n and depends only on α, β, and γ, but not on the particular perfect colouring $(\boldsymbol{a}, \boldsymbol{c})$. Thus setting $p^{\gamma}_{\alpha,\beta}(z) := (z - |\gamma|)(z - |\gamma| - 1) \cdots (z - |\gamma| - m(\alpha, \beta))$ whenever $\gamma \in \mathsf{TBC}(\alpha, \beta)$ and 0 otherwise, we have

$$\begin{aligned}
\overline{O}_n(\alpha)\overline{O}_n(\beta) &= \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{C}_n(\alpha,\beta)} |\mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta)| \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}}, \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} p^{\gamma}_{\alpha,\beta}(n) \mathsf{E}^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(n) \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} \mathsf{E}^{\boldsymbol{c}}_{\boldsymbol{c}} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(n) \overline{O}_n(\gamma).
\end{aligned}$$

□

Whenever $n > 2k$, the elements $\{\overline{O}_n(\alpha) | \alpha \in \Pi_{2k}\}$ form a basis of $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$, and thus the values $p^{\gamma}_{\alpha,\beta}(n)$ are precisely the structure constants. The fact that they are polynomial in n as described in the above lemma, allows us to define a $\mathbb{C}[z]$-algebra which possesses a projection down to $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$.

Definition 2.68. Let $\mathcal{Q}_{2k}(z)$ denote the free $\mathbb{C}[z]$-module with basis $\{O(\alpha) | \alpha \in \Pi_{2k}\}$ equiped with the product given by the $\mathbb{C}[z]$-linear extension of

$$O(\alpha)O(\beta) = \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(z) O(\gamma)$$

for any $\alpha, \beta, \gamma \in \Pi_{2k}$ and where $p^{\gamma}_{\alpha,\beta}(z)$ are the polynomials given in *Proposition 2.67*.

The construction of $\mathcal{Q}_{2k}(z)$ from the algebras $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ should be compared to how the Farahat-Higman algebra Z was constructed from the centers $Z_n = Z(\mathbb{C}\mathfrak{S}_n)$, as was described in *Section 2.1.3*.

As we will show shortly, the algebra $\mathcal{Q}_{2k}(z)$ is exactly the partition algebra $\mathcal{A}_{2k}(z)$, and the basis elements $O(\alpha)$ are precisely the orbit basis elements, which is why we have used the same symbols to denote them. This abuse of notation should hopefully cause no confusion. However we have used different notation to denote the algebras themselves to stress the distinct manners in which they were constructed. This will also help compare the two affine counterparts to the partition algebras which we construct in *Chapter 4* and *Chapter 5*.

The object $\mathcal{Q}_{2k}(z)$ is indeed an algebra, although this is something that needs to be proved as it is not immediate that the product is associative or that a multiplicative identity exists. Recall that a *distributive ring* is a ring where we do not require a multiplicative identity or for the product to be associative. By definition $\mathcal{Q}_{2k}(z)$ is certainly a distributive ring. Also by *Proposition 2.67*, for any $k, n \in \mathbb{Z}_{\geq 0}$ there exists a surjective homomorphism of distributive rings

$$\psi_{2k,n} : \mathcal{Q}_{2k}(z) \to \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$$

given by the extension of $\psi_{2k,n}(O(\alpha)) = \overline{O}_n(\alpha)$ and $\psi_{2k,n}(z) = n$. Moreover, whenever $n > 2k$ we have $\mathrm{Ker}(\psi_{2k,n}) = (z - n)$. Thus employing the same arguments as in *Lemma 2.59* gives the following result.

Lemma 2.69. Let $R_1, R_2 \in \mathcal{Q}_{2k}(z)$. Then $R_1 = R_2$ if and only if for all $n \geq 0$

$$\psi_{2k,n}(R_1) = \psi_{2k,n}(R_2).$$

□

This allows us to deduce that $\mathcal{Q}_{2k}(z)$ is indeed a $\mathbb{C}[z]$-algebra.

Proposition 2.70. The $\mathbb{C}[z]$-module $\mathcal{Q}_{2k}(z)$ is a unital associative algebra over $\mathbb{C}[z]$ with the product described in *Definition 2.68*.

Proof. Let $1 = \{\{i, i'\} \mid i \in [k]\} \in \Pi_{2k}$, then it follows from *Lemma 2.69* that

$$\sum_{S \vdash [k]} O(I(S))$$

is the identity of $\mathcal{Q}_{2k}(z)$ since its image under $\psi_{2k,n}$ is the identity of $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ for all $n \in \mathbb{Z}_{\geq 0}$. Similarly for any $A, B, C \in \mathcal{Q}_{2k}(z)$ let $[A, B, C] := (AB)C - A(BC)$. Then for all $n \in \mathbb{Z}_{\geq 0}$ we have that $\psi_{2k,n}([A, B, C]) = 0$ since $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is an associative algebra. Thus by *Lemma 2.69* we have $[A, B, C] = 0$ showing that $\mathcal{Q}_{2k}(z)$ is also associative.

□

We end this section by proving that $\mathcal{Q}_{2k}(z)$ and $\mathcal{A}_{2k}(z)$ isomorphic.

Proposition 2.71. We have an isomorphism of $\mathbb{C}[z]$-algebras $\mathcal{Q}_{2k}(z) \cong \mathcal{A}_{2k}(z)$.

Proof. By item (2) of *Theorem 2.58* we have that $(\Psi_{2k,n} \circ \mathsf{pr}_n)(O(\alpha)) = \overline{O}_n(\alpha)$. Hence by *Lemma 2.59* and *Proposition 2.67* we see that the orbit basis elements of $\mathcal{A}_{2k}(z)$ satisfy the equations

$$O(\alpha)O(\beta) = \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(z) O(\gamma)$$

where $p^{\gamma}_{\alpha,\beta}(z)$ are the polynomials given in *Proposition 2.67*. Thus we have a $\mathbb{C}[z]$-algebra homomorphism $\mathcal{A}_{2k}(z) \to \mathcal{Q}_{2k}(z)$ defined by $O(\alpha) \mapsto O(\alpha)$ which sends basis to basis, and hence is clearly both injective and surjective.

□

3 Center of the Partition Algebra

This chapter is broken into three sections. The first section recalls the definition of supersymmetric polynomials and describes some generating sets for the algebra of supersymmetric polynomials. The second section proves that the center of the semisimple partition algebras is given by the supersymmetric polynomials in normalised Jucys-Murphy elements. The last section uses this description of the center to prove an alterative criteria for the blocks of the partition algebra using certain generating functions which appear in the theory of supersymmetric polynomials.

3.1 Supersymmetric Polynomials

This section is a brief recap on supersymmetric polynomials and a result of J. Stembridge in [Stem85] regarding natural generating sets for the algebra of supersymmetric polynomials. We remodel the definitions a little to better align with our situation.

Let $k \in \mathbb{Z}_{\geq 0}$ and consider the polynomial algebra $\mathbb{C}[x_1, \ldots, x_{2k}]$ in $2k$ commuting variables. The symmetric group $\mathfrak{S}_k$ acts on this algebra by permuting variables of the same parity, that is $\pi \circ x_{2i-1} := x_{2\pi(i)-1}$ and $\pi \circ x_{2i} := x_{2\pi(i)}$ for any $\pi \in \mathfrak{S}_k$ and $i \in [k]$.

Definition 3.1. Let $p \in \mathbb{C}[x_1, \ldots, x_{2k}]$, then we say that p is *supersymmetric* if

(1) p is *parity symmetric*: $\pi \circ p = p$ for all $\pi \in \mathfrak{S}_k$.

(2) p satisfies the *cancellation property*: substituting $x_1 = -x_2 = y$ yields a polynomial in $x_3, x_4, \ldots, x_r$ which is independent of y.

We denote by $\mathsf{SSym}[x_1, \ldots, x_{2k}]$ the set of all supersymmetric polynomials in $\mathbb{C}[x_1, \ldots, x_{2k}]$.

It is simple to check that the two defining properties above are respected under addition and multiplication of polynomials, and thus $\mathsf{SSym}[x_1, \ldots, x_{2k}]$ is a subalgebra of $\mathbb{C}[x_1, \ldots, x_{2k}]$. We will be interested in two types of supersymmetric polynomials, both of which generate all supersymmetric polynomials.

Definition 3.2. For any $n, k \in \mathbb{Z}_{\geq 0}$, the n-th power sum supersymmetric polynomial in $\mathbb{C}[x_1, \ldots, x_{2k}]$ is given by

$$q_n(x_1, \ldots, x_{2k}) := x_1^n + x_3^n + \cdots + x_{2k-1}^n + (-1)^{n+1}(x_2^n + x_4^n + \cdots + x_{2k}^n).$$

These polynomails are the supersymmetric counterparts to the usual power sum symmetric polynomials. It is clear that permuting the odd indexed (respectively even indexed) variables around leaves $q_n(x_1, \ldots, x_{2k})$ invariant, hence they are parity symmetric. Also, the sign $(-1)^{n+1}$ which appears means that the cancellation property is upheld, thus these polynomials are indeed supersymmetric. When the number of variables in play is clear, we will drop the variable arguments and write q_n instead of $q_n(x_1, \dots, x_{2k})$.

Definition 3.3. For t a formal variable, and $n, k \in \mathbb{Z}_{\geq 0}$, the n-th elementary supersymmetric polynomial is defined to be the coefficient of t^n in the generating function

$$\sum_{n=0}^{\infty} l_n(x_1, \dots, x_{2k}) t^n = \frac{\prod_{i=1}^{k}(1 + x_{2i-1}t)}{\prod_{i=1}^{k}(1 - x_{2i}t)}.$$

These polynomials are the supersymmetric counterparts to the regular elementary symmetric polynomials, and it is clear from the generating function defining them that they are indeed supersymmetric. As above, when the number of variables in play is clear, we will drop the variable arguments and write l_n instead of $l_n(x_1, \dots, x_{2k})$.

The core result we will need is by J. Stembridge, and is the fact that the algebra of supersymmetric polynomials is generated by either the power sum supersymmetric polynomials or the elementary supersymmetric polynomials.

Theorem 3.4 (*Theorem 2; Corollary of [Stem85]*)**.** As algebras we have that

$$\mathsf{SSym}[x_1, \dots, x_{2k}] = \langle q_n \mid n \in \mathbb{Z}_{\geq 0} \rangle = \langle l_n \mid n \in \mathbb{Z}_{\geq 0} \rangle.$$

□

3.2 Center of the Semisimple Partition Algebra

In this section we will prove that the center of the partition algebra $\mathcal{A}_{2k}(\delta)$, whenever $\delta \notin \{0, 1, \dots, 2k-2\}$ (i.e. whenever $\mathcal{A}_{2k}(\delta)$ is semisimple by *Theorem 2.40*), is given by the subalgebra of supersymmetric polynomials in (normalised) Jucys-Murphy elements. This result easily extends to one for $\mathcal{A}_{2k}(z)$. We approach this result in a similar manner to what was done in [JK17], where they gave a description of the center of the walled Brauer algebras.

Recall the Jucys-Murphy elements $L_1, \dots, L_{2k}$ of the partition algebra $\mathcal{A}_{2k}(z)$ given in *Definition 2.34*. To help with future computations and results, for this chapter we will work with the normalisation of the Jucys-Murphy elements given by

$$N_i := L_i - \frac{z}{2} \tag{3.1}$$

for each $i \in [2k]$. These normalisations were considered in [Eny13, Section 6] in demonstrating central element recursions analogous to ones established in [N96] for the Brauer algebras. Given any $\delta \in \mathbb{C}$, we let N_i also denote the corresponding elements $1 \otimes N_i$ in the finite dimensional partition algebra $\mathcal{A}_{2k}(\delta)$, which are simply given by *Equation* (3.1) with z replaced by δ.

We will let $\mathsf{SSym}[N_1, \dots, N_{2k}]$ denote the $\mathbb{C}[z]$-subalgebra of $\mathcal{A}_{2k}(z)$ generated by the supersymmetric polynomials evaluated at the normalised Jucys-Murphy elements. Then by *Theorem 3.4* we have that

$$\mathsf{SSym}[N_1, \dots, N_{2k}] = \langle q_n(N_1, \dots, N_{2k}) \mid n \in \mathbb{Z}_{\geq 0} \rangle = \langle l_n(N_1, \dots, N_{2k}) \mid n \in \mathbb{Z}_{\geq 0} \rangle.$$

Similarly let $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}]$ denote the corresponding image in $\mathcal{A}_{2k}(\delta)$ for any $\delta \in \mathbb{C}$. We seek to show that $\mathsf{SSym}[N_1, \dots, N_{2k}]$ gives the center $Z(\mathcal{A}_{2k}(z))$ of $\mathcal{A}_{2k}(z)$ and that $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}]$ gives the center $Z(\mathcal{A}_{2k}(\delta))$ of $\mathcal{A}_{2k}(\delta)$ for any $\delta \notin \{0, 1, \dots, 2k-2\}$.

We delay the proof of the following result to *Chapter 4*, were a more general result will be proven regarding the *affine partition algebra* defined in the same chapter. Alternatively one can find a proof in [Cre21, Theorem 3.2.6].

Theorem 3.5. The supersymmetric polynomials in the normalised Jucys-Murphy elements are central in $\mathcal{A}_{2k}(z)$, that is to say $\mathsf{SSym}[N_1, \dots, N_{2k}] \subseteq Z(\mathcal{A}_{2k}(z))$.

□

Of course this theorem implies that $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}] \subseteq \mathcal{A}_{2k}(\delta)$ for any $\delta \in \mathbb{C}$. Assume for this section that $\delta \notin \{0, 1, \dots, 2k-2\}$, hence $\mathcal{A}_{2k}(\delta)$ is semisimple by *Theorem 2.40*. It is well known, say from the Weddernburn-Artin theorem, that the dimension of the center of a semisimple algebra equals the number of isomorphism classes of simple modules. Recall from *Theorem 2.43* that $\Lambda_{\leq k} \times \{2k\}$ is an indexing set for the simple $\mathcal{A}_{2k}(\delta)$-modules. Thus we have that

$$\mathsf{dim}_{\mathbb{C}}(Z(\mathcal{A}_{2k}(\delta))) = |\Lambda_{\leq k} \times \{2k\}|. \tag{3.2}$$

Our plan to show that $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}]$ is the center of $\mathcal{A}_{2k}(\delta)$ is to confirm that its dimension equals $|\Lambda_{\leq k} \times \{2k\}| = |\Lambda_{\leq k}|$. To do this we will utilize the action of the Jucys-Murphy elements on the simple modules to implicitly show the existence of the right number of linearly independent supersymmetric polynomials in the normalised Jucys-Murphy elements.

Recall the notation of *Section 2.2.3*. Let $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$ and $\mathsf{T} \in \mathsf{Path}(\lambda, 2k)$, and recall in particular the contents $\mathrm{cont}_\delta(\mathsf{T}, i)$ defined in *Definition 2.44* for any $i \in [2k]$. By *Theorem 2.45* and *Equation* (3.1) we have the following:

Lemma 3.6. Let $k \in \mathbb{Z}_{\geq 0}$, $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, and $\{v_\mathsf{T} \mid \mathsf{T} \in \mathsf{Path}(\lambda, 2k)\}$ be a GZ-basis of the simple $\mathcal{A}_{2k}(\delta)$-module $\mathsf{A}^{(\lambda, 2k)}$. Then for any $i \in [2k]$ we have that

$$N_i v_\mathsf{T} = \left(\mathrm{cont}_\delta(\mathsf{T}, i) - \frac{\delta}{2}\right) v_\mathsf{T}.$$

□

Let $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, $\mathsf{T} \in \mathsf{Path}(\lambda, 2k)$, and $p \in \mathbb{C}[x_1, \dots, x_{2k}]$. From the above lemma, the action of $p(N_1, \dots, N_{2k})$ on the GZ-basis element v_T is given as scaling by

$$p\left(\mathrm{cont}_\delta(\mathsf{T}, 1) - \frac{\delta}{2}, \dots, \mathrm{cont}_\delta(\mathsf{T}, 2k) - \frac{\delta}{2}\right),$$

the evaluation of p at $x_i = \mathrm{cont}_\delta(\mathsf{T}, i) - \delta/2$. Since the supersymmetric polynomials in the normalised Jucys-Murphy elements are central by *Theorem 3.5*, we show below that such an evaluation of supersymmetric polynomials is independent of the path taken.

Lemma 3.7. Let $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$. Then for any paths $\mathsf{T}, \mathsf{S} \in \mathsf{Path}(\lambda, 2k)$ and any supersymmetric polynomial $p \in \mathsf{SSym}[x_1, \ldots, x_{2k}]$, we have that

$$p\left(\mathrm{cont}_\delta(\mathsf{T}, 1) - \frac{\delta}{2}, \ldots, \mathrm{cont}_\delta(\mathsf{T}, 2k) - \frac{\delta}{2}\right) = p\left(\mathrm{cont}_\delta(\mathsf{S}, 1) - \frac{\delta}{2}, \ldots, \mathrm{cont}_\delta(\mathsf{S}, 2k) - \frac{\delta}{2}\right).$$

Proof. Given the GZ-basis elements v_T and v_S, from *Lemma 3.6* we have that

$$p(N_1, \ldots, N_{2k})v_\mathsf{T} = p\left(\mathrm{cont}_\delta(\mathsf{T}, 1) - \frac{\delta}{2}, \ldots, \mathrm{cont}_\delta(\mathsf{T}, 2k) - \frac{\delta}{2}\right) v_\mathsf{T},$$
$$p(N_1, \ldots, N_{2k})v_\mathsf{S} = p\left(\mathrm{cont}_\delta(\mathsf{S}, 1) - \frac{\delta}{2}, \ldots, \mathrm{cont}_\delta(\mathsf{S}, 2k) - \frac{\delta}{2}\right) v_\mathsf{S}.$$

By Schur's Lemma, any central element z of $\mathcal{A}_{2k}(\delta)$ acts on $\mathsf{A}^{(\lambda,2k)}$ by scaling by a certain constant. Since $p(N_1, \ldots, N_{2k})$ is central by *Theorem 3.5*, it must act on v_T and v_S by the same constant (since $v_\mathsf{T}, v_\mathsf{S} \in \mathsf{A}^{(\lambda,2k)}$), thus the result follows.

$\square$

This result means that we may chose any path to work with when evaluating the action of $p(N_1, \ldots, N_{2k})$ on $\mathsf{A}^{(\lambda,2k)}$ for any supersymmetric polynomials $p \in \mathsf{SSym}[x_1, \ldots, x_{2k}]$. With this in mind, we are going to fix a particular path to work with in $\mathsf{Path}(\lambda, 2k)$ for each $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, also see [Eny13, Lemma 3.9].

Definition 3.8. Let $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$. The *standard path* $\mathsf{T}^{(\lambda,2k)} = ((\lambda^{(i)}, i))_{i=0}^{2k}$ of $\mathsf{Path}(\lambda, 2k)$ is defined as follows:

(i) $\lambda^{(2i)} = \lambda^{(2i+1)}$ for all $0 \leq i \leq k - 1$.

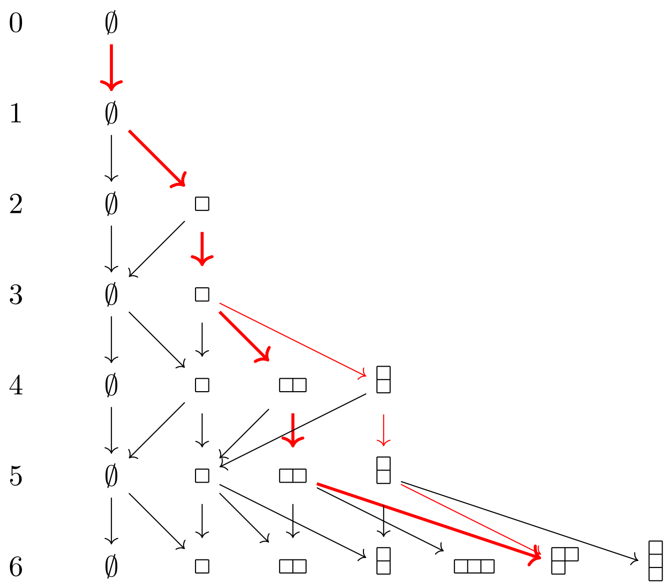
(ii) $\lambda^{(2i)} = \emptyset$ for all $0 \leq i \leq k - |\lambda|$.

(iii) $\lambda^{(2i+2)} = \lambda^{(2i)} \cup \{a\}$ for all $k - |\lambda| \leq i \leq k - 1$, where a is an addable box of $\lambda^{(2i)}$ with minimal row index.

The path $T^{(\lambda,2k)}$ is the one which never removes any boxes, only adds a box when it must, and prioritises adding boxes in the highest row, i.e. lowest row index. Recall from *Section 2.2.3*, in particular *Corollary 2.52*, that J. Enyang defined a partial order $\prec$ on the set of paths $\mathsf{Path}(\lambda, 2k)$ which is compatible with the action of the Jucys-Murphy elements on the cell modules. It was proved in [Eny13, Lemma 3.9] that the standard path $\mathsf{T}^{(\lambda,2k)}$ is a maximal element with respect to this partial order.

Example 3.9. Consider the vertex $(\yng(2,1), 6)$ in $\widehat{\mathsf{A}}$. There are only two paths within $\mathsf{Path}(\yng(2,1), 6)$, both presented below in red sitting inside the truncation of $\widehat{\mathsf{A}}$ at level six.

The standard path $\mathsf{T}^{(\yng(2,1),6)}$ is given in bold:



Now given any $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$ we let

$$\mathsf{ct}_i(\lambda, \delta) := \mathrm{cont}_\delta(\mathsf{T}^{(\lambda,2k)}, i) - \frac{\delta}{2}. \tag{3.3}$$

Also, for any polynomial $p \in \mathbb{C}[x_1, \dots, x_{2k}]$ we will let

$$p(\lambda, \delta) := p\left(\mathsf{ct}_1(\lambda, \delta), \dots, \mathsf{ct}_{2k}(\lambda, \delta)\right). \tag{3.4}$$

From *Lemma 3.7* and *Lemma 3.6*, the action of $p(N_1, \dots, N_{2k})$ on the simple $\mathcal{A}_{2k}(\delta)$-module $\mathsf{A}^{(\lambda,2k)}$ is given by scaling by $p(\lambda, \delta)$ for any $p \in \mathsf{SSym}[x_1, \dots, x_{2k}]$ and for any $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$. We may record this information in a generating function by evaluating the elementary supersymmetric polynomials at $x_i = \mathsf{ct}_i(\lambda, \delta)$ in the generating function given in *Definition 3.3*. Thus for any $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, and formal variable t, consider the generating function

$$\mathsf{L}(\lambda, \delta) := \sum_{n=0}^{\infty} l_n(\lambda, \delta) t^n = \frac{\prod_{i=1}^{k}(1 + \mathsf{ct}_{2i-1}(\lambda, \delta)t)}{\prod_{i=1}^{k}(1 - \mathsf{ct}_{2i}(\lambda, \delta)t)}. \tag{3.5}$$

Hence the coefficient of t^n in $\mathsf{L}(\lambda, \delta)$ records the action of $l_n(N_1, \dots, N_{2k})$ on $\mathsf{A}^{(\lambda,2k)}$. We seek to better understand the rational polynomial in t given by the right hand side of *Equation* (3.5). First we want to understand the multiset $\{\mathsf{ct}_i(\lambda, \delta) \mid i \in [2k]\}$, and to do so we introduce some notation regarding integer partitions.

Definition 3.10. Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n \in \mathbb{N}$. We say that the *height* of λ, written $h(\lambda)$, is the number of rows of λ minus 1. Similarly we say the width of λ, written $w(\lambda)$, is the number of columns of λ minus 1.

Recall from *Definition 2.14* that given a box $\square = (i, j) \in \lambda$ the content is $c(\square) = j - i$, the column index minus the row index. Also recall that $c(\lambda)$ is the multiset of content of the boxes of λ. It can be seen that two boxes $a, b \in \lambda$ belong to the same diagonal (top-left to bottom-right) in the corresponding Young diagram if and only if their contents agree, that is $c(a) = c(b)$. Thus we can index the diagonals of λ by the underlying set of $c(\lambda)$. It is clear that given a box $\square \in \lambda$, then $-h(\lambda) \leq c(\square) \leq w(\lambda)$. Moreover, for any value c such that $-h(\lambda) \leq c \leq w(\lambda)$, there exists a box $\square \in \lambda$ such that $c(\square) = c$. Hence the set $\{-h(\lambda), \dots, w(\lambda)\}$ is precisely the underlying set of $c(\lambda)$. It is clear that knowning λ is the same as knowing the set $\{-h(\lambda), \dots, w(\lambda)\}$ and the number of boxes whose content equals c for each $c \in \{-h(\lambda), \dots, w(\lambda)\}$. With this in mind we introduce the following definition.

Definition 3.11. Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n \in \mathbb{N}$. We define the *diagonal datum* of λ to be the pair $(D(\lambda), m_\lambda)$ where $D(\lambda) = \{-h(\lambda), -h(\lambda)+1, \dots, w(\lambda)\}$ and $m_\lambda : D(\lambda) \to \mathbb{N}$, given by $m_\lambda(c) = |\{\square \in \lambda \mid c(\square) = c\}|$.

Example 3.12. Consider the partition $\lambda = (7, 5, 4, 3) \vdash 19$. The corresponding Young diagram (where each box $a \in \lambda$ has its content inscribed within it) is given by

0	1	2	3	4	5	6
-1	0	1	2	3		
-2	-1	0	1			
-3	-2	-1				

The height and width of λ are given by $h(\lambda) = 3$ and $w(\lambda) = 6$. We have that

$$D(\lambda) = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\},$$

and for example $m_\lambda(-1) = 3$.

The diagonal datum $(D(\lambda), m_\lambda)$ of an integer partition λ will lend itself better for future computations.

Proposition 3.13. Let $(\lambda, 2k) \in \Lambda_{\leq k} \times \{2k\}$, then as multisets

$$\{\mathsf{ct}_{2i+1}(\lambda, \delta) \mid 0 \leq i \leq k-1\} = \left\{ \left(-\frac{\delta}{2}\right)^{k-|\lambda|}, \ i - \frac{\delta}{2} \ \middle| \ 0 \leq i \leq |\lambda| - 1 \right\}$$

$$\{\mathsf{ct}_{2i}(\lambda, \delta) \mid 1 \leq i \leq k\} = \left\{ \left(\frac{\delta}{2}\right)^{k-|\lambda|}, \ c(\square) - \frac{\delta}{2} \ \middle| \ \square \in \lambda \right\}$$

$$= \left\{ \left(\frac{\delta}{2}\right)^{k-|\lambda|}, \ \left(i - \frac{\delta}{2}\right)^{m_\lambda(i)} \ \middle| \ i \in D(\lambda) \right\}$$

where the superscript denotes multiplicity in the multisets.

Proof. Let $\mathsf{T}^{(\lambda,2k)} = (\lambda^{(i)})_{i=0}^{2k}$ be the standard path of $(\lambda, 2k)$. We begin with the set $\{c_{2i+1}(\lambda,\delta) \mid 0 \le i \le k-1\}$. By definition of $T^{(\lambda,2k)}$ we have that $\lambda_{2i} = \lambda_{2i+1}$ for all $0 \le i \le k-1$. Hence from *Definition 2.44* we have that

$$\mathsf{ct}_{2i+1}(\lambda,\delta) = |\lambda^{(2i+1)}| - \frac{\delta}{2}.$$

From item (i) and (ii) of *Definition 3.8*, we have that $\lambda^{(2i)} = \lambda^{(2i+1)} = \emptyset$ for any i in the range $0 \le i \le k - |\lambda| - 1$. Thus

$$\{\mathsf{ct}_{2i+1}(\lambda,\delta) \mid 0 \le i \le k-|\lambda|-1\} = \left\{ \left(-\frac{\delta}{2}\right)^{k-|\lambda|} \right\}.$$

Now note that $\lambda^{(2i+2)} = \lambda^{(2i)} \cup \{\Box\}$ for all $k-|\lambda| \le i \le k-1$, with $\Box$ an addable box of $\lambda^{(2i)}$. Hence $|\lambda^{(2i+2)}| = |\lambda^{(2i)}| + 1$ for each $k-|\lambda| \le i \le k-1$, and so item (i) of *Definition 3.8* implies that $|\lambda^{(2i+3)}| = |\lambda^{(2i+1)}| + 1$ for all $k-|\lambda| \le i \le k-2$. Thus the quantities $|\lambda^{(2i+1)}|$ increase by one each time as i runs from $k-|\lambda|$ to $k-1$, starting at 0 and ending with $k-1-(k-|\lambda|) = |\lambda|-1$. As such we have

$$\begin{aligned}\{\mathsf{ct}_{2i+1}(\lambda,\delta) \mid k-|\lambda| \le i \le k-1\} &= \left\{ |\lambda^{(2i+1)}| - \frac{\delta}{2} \;\middle|\; k-|\lambda| \le i \le k-1 \right\} \\ &= \left\{ i - \frac{\delta}{2} \;\middle|\; 0 \le i \le |\lambda|-1 \right\}.\end{aligned}$$

Therefore collectively we have

$$\{\mathsf{ct}_{2i+1}(\lambda,\delta) \mid 0 \le i \le k-1\} = \left\{ \left(-\frac{\delta}{2}\right)^{k-|\lambda|}, i - \frac{\delta}{2} \;\middle|\; 0 \le i \le |\lambda|-1 \right\}$$

as desired. We now focus on the set $\{\mathsf{ct}_{2i}(\lambda,\delta) \mid 1 \le i \le k\}$. By item (ii) of *Definition 3.8*, we have that $\lambda^{(2i)} = \emptyset$ for all $0 \le i \le k-|\lambda|$. Hence from *Definition 2.44* we have that $c_{2i}(\lambda,\delta) = \delta/2$, and so

$$\{\mathsf{ct}_{2i}(\lambda,\delta) \mid 1 \le i \le k-|\lambda|\} = \left\{ \left(\frac{\delta}{2}\right)^{k-|\lambda|} \right\}.$$

For the cases $k-|\lambda|+1 \le i \le k$ we have that $\lambda_{2i} = \lambda_{2i-2} \cup \{\Box_{2i}\}$ for some addable box $\Box_{2i}$ of $\lambda^{(2i)}$. As such

$$\mathsf{ct}_{2i}(\lambda,\delta) = c(\Box_{2i}) - \frac{\delta}{2}$$

for each $k-|\lambda|+1 \le i \le k$. All the boxes of λ are added during these steps since $\lambda_{2(k-|\lambda|)} = \lambda_{2(k-|\lambda|)+1} = \emptyset$, that is $\Box_{2i}$ runs over all the boxes of λ as i runs from $k-|\lambda|+1$ to k. Therefore

$$\{\mathsf{ct}_{2i}(\lambda,\delta) \mid k-|\lambda|+1 \le i \le k\} = \left\{ c(\Box) - \frac{\delta}{2} \;\middle|\; \Box \in \lambda \right\} = \left\{ \left(i - \frac{\delta}{2}\right)^{m_\lambda(i)} \;\middle|\; i \in D(\lambda) \right\}.$$

Thus altogether we have that

$$\begin{aligned}\{\mathsf{ct}_{2i}(\lambda,\delta)\mid 1\le i\le k\} &= \left\{\left(\frac{\delta}{2}\right)^{k-|\lambda|},\ c(\square)-\frac{\delta}{2}\ \middle|\ \square\in\lambda\right\}\\ &= \left\{\left(\frac{\delta}{2}\right)^{k-|\lambda|},\ \left(i-\frac{\delta}{2}\right)^{m_\lambda(i)}\ \middle|\ i\in D(\lambda)\right\}.\end{aligned}$$

□

We now wish to express the generating function $\mathsf{L}(\lambda,\delta)$ defined by *Equation* (3.5) as a rational polynomial in reduced form, i.e. where the numerator and demonerator share no common factors. Given the diagonal datum $(D(\lambda), m_\lambda)$ we let $D(\lambda)_{\le\delta} := D(\lambda)\cap\mathbb{Z}_{\le\delta}$ and $D(\lambda)_{>\delta} := D(\lambda)\cap\mathbb{Z}_{>\delta}$ whenever δ is an integer.

Lemma 3.14. Let $(\lambda, 2k)\in\Lambda_{\le k}\times\{2k\}$, then we have the following two cases:

(1) Suppose $\delta\notin\{-h(\lambda),\dots,2k-2\}$. Then

$$\mathsf{L}(\lambda,\delta)=\frac{\prod_{i=0}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}},$$

where the rational polynomial in t on the right is reduced.

(2) Suppose $h(\lambda)\ge 1$ and $\delta\in\{-h(\lambda),\dots,-1\}$. Then

$$\mathsf{L}(\lambda,\delta)=\frac{\prod_{i=\delta+h(\lambda)+1}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)_{\le\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)-1}\prod_{j\in D(\lambda)_{>\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}},$$

where the rational polynomial in t on the right is reduced.

Proof. Firstly, by *Proposition 3.13* and *Equation* (3.5) we have that

$$\begin{aligned}\mathsf{L}(\lambda,\delta) &= \frac{\prod_{i=1}^{k}(1+\mathsf{ct}_{2i-1}(\lambda,\delta)t)}{\prod_{i=1}^{k}(1-\mathsf{ct}_{2i}(\lambda,\delta)t)} = \frac{(1-\frac{\delta}{2}t)^{k-|\lambda|}\prod_{i=0}^{|\lambda|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right)}{(1-\frac{\delta}{2}t)^{k-|\lambda|}\prod_{j\in D(\lambda)}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)}}\\ &= \frac{\prod_{i=0}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}}\end{aligned}$$

(1): We seek to show that the polynomials

$$\prod_{i=0}^{|\lambda|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right)\quad\text{and}\quad\prod_{j\in D(\lambda)}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)}$$

share no common factors. Assume for contradiction that this is not the case. Then for some $0\le i\le|\lambda|-1$ and $j\in D(\lambda)=\{-h(\lambda),\dots,w(\lambda)\}$ we have that $i-\delta/2=\delta/2-j$, and so $\delta = i + j$. Thus immediately we see that if $\delta \notin \mathbb{Z}$ then the fraction is reduced. Furthermore by consider the range of values i and j can take, we have that

$$-h(\lambda) \leq \delta \leq w(\lambda) + |\lambda| - 1 \leq 2(|\lambda| - 1) \leq 2k - 2,$$

which contradicts the assumption that $\delta \notin \{-h(\lambda), \ldots, 2k-2\}$.

(2): We now seek to understand what factors of

$$\frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{j\in D(\lambda)}(1 + (\frac{\delta}{2} - j)t)^{m_\lambda(j)}}, \tag{3.6}$$

cancel out when $h(\lambda) \geq 1$ and $\delta \in \{-h(\lambda), \ldots, -1\}$. As described in the previous case, numerator and denominator share a common factor if $\delta = i + j$ for some $0 \leq i \leq |\lambda| - 1$ and $j \in D(\lambda)$. Let $P(\delta) = \{(i, j) \mid \delta = i + j,\ 0 \leq i \leq |\lambda| - 1,\ j \in D(\lambda)\}$. So $|P(\delta)|$ is the number of pairs of common factors between the numerator and denomerator of *Equation* (3.6). One can deduce that we must have

$$P(\delta) = \{(0, \delta), (1, \delta - 1), \ldots, (h(\lambda) + \delta, -h(\lambda))\},$$

and so $|P(\delta)| = h(\lambda) + \delta + 1$. Therefore the factors $(1 + (i - \frac{\delta}{2})t)$ in the numerator corresponding to $i = 0, 1, \ldots, h(\lambda) + \delta$ cancel with one of the factors $(1 + (\frac{\delta}{2} - j)t)$ in the denominator corresponding to $j = \delta, \delta - 1, \ldots, -h(\lambda)$ respectively. Hence we obtain

$$\mathsf{L}(\lambda, \delta) = \frac{\prod_{i=\delta+h(\lambda)+1}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{j\in D(\lambda)_{\leq\delta}}(1 + (\frac{\delta}{2} - j)t)^{m_\lambda(j)-1}\prod_{j\in D(\lambda)_{>\delta}}(1 + (\frac{\delta}{2} - j)t)^{m_\lambda(j)}},$$

where the rational polynomial in t on the right is reduced.

□

We can now show that the action of the central subalgebra $\mathsf{SSym}_\delta[N_1, \ldots, N_{2k}]$ of $\mathcal{A}_{2k}(\delta)$ (whenever $\delta \notin \{0, 1, \ldots, 2k-2\}$) can distinguish between the simple modules.

Proposition 3.15. Let $k \in \mathbb{Z}_{\geq 0}$ and $\delta \notin \{0, 1, \ldots, 2k-2\}$. Let $(\lambda, 2k), (\mu, 2k) \in \Lambda_{\leq k} \times \{2k\}$ such that $\lambda \neq \mu$. Then there exists a supersymmetric polynomial $p \in \mathsf{SSym}[x_1, \ldots, x_{2k}]$ such that $p(\lambda, \delta) \neq p(\mu, \delta)$.

Proof. We will prove this by showing the contrapositive, that is if $p(\lambda, \delta) = p(\mu, \delta)$ for all $p \in \mathsf{SSym}[x_1, \ldots, x_{2k}]$, then $\lambda = \mu$. Now since the elementary supersymmetric polynomials generate all supersymmetric polynomails by *Theorem 3.4*, we have that $p(\lambda, \delta) = p(\mu, \delta)$ for all $p \in \mathsf{SSym}[x_1, \ldots, x_{2k}]$ if and only if $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$, i.e. if and only if

$$\frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{j\in D(\lambda)}(1 + (\frac{\delta}{2} - j)t)^{m_\lambda(j)}} = \frac{\prod_{i=0}^{|\mu|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{j\in D(\mu)}(1 + (\frac{\delta}{2} - j)t)^{m_\mu(j)}}.$$

Using *Lemma 3.14* we will break the above equality into four cases, and for each we will show that either $(\lambda, 2k) = (\mu, 2k)$, or the case is impossible. The four cases to consider are the following:

(C1) $\delta \notin \{-h(\lambda), \ldots, 2k-2\} \cup \{-h(\mu), \ldots, 2k-2\}$.

(C2) $\delta \notin \{-h(\mu), \ldots, 2k-2\}$ but $\delta \in \{-h(\lambda), \ldots, -1\}$ with $h(\lambda) \geq 1$.

(C3) $\delta \notin \{-h(\lambda), \ldots, 2k-2\}$ but $\delta \in \{-h(\mu), \ldots, -1\}$ with $h(\mu) \geq 1$.

(C4) $\delta \in \{-h(\mu), \ldots, -1\} \cap \{-h(\lambda), \ldots, -1\}$.

(C1): Since $\delta \notin \{-h(\lambda), \ldots, 2k-2\} \cup \{-h(\mu), \ldots, 2k-2\}$, *Lemma 3.14* (1) implies

$$\frac{\prod_{i=0}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}} = \frac{\prod_{i=0}^{|\mu|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\mu)}(1+(\frac{\delta}{2}-j)t)^{m_\mu(j)}}.$$

where both sides are reduced. Since they are reduced, we may equate the numerators and denominators. Equating the numerators gives

$$\prod_{i=0}^{|\lambda|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right) = \prod_{i=0}^{|\mu|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right). \tag{Eq4}$$

Assume one of the factors on the left hand side is trivial, that is $i = \delta/2$ for some $0 \leq i \leq |\lambda|-1$. This would imply that $0 \leq \delta \leq 2(|\lambda|-1)$, which contradicts the assumption $\delta \notin \{-h(\lambda), \ldots, 2k-2\}$. As such no factor on the left hand side of (Eq4) is trivial, similarly no factor on the right is trivial. Therefore (Eq4) implies that $|\lambda| = |\mu|$. Now equating the denominators gives

$$\prod_{j\in D(\lambda)}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)} = \prod_{j\in D(\mu)}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\mu(j)}.$$

This implies $D(\lambda)\backslash\{\delta/2\} = D(\mu)\backslash\{\delta/2\}$ and that $m_\lambda(j) = m_\mu(j)$ for all $j \in D(\lambda)\backslash\{\delta/2\}$. This means that the Young diagrams λ and μ can only differ in the diagonal indexed by $\delta/2$. However since $|\lambda| = |\mu|$, no such difference is present, hence $(\lambda, 2k) = (\mu, 2k)$.

(C2): Since $\delta \notin \{-h(\mu), \ldots, 2k-2\}$ but $\delta \in \{-h(\lambda), \ldots, -1\}$ with $h(\lambda) \geq 1$, *Lemma 3.14* (1) and (2) tell us that

$$\frac{\prod_{i=\delta+h(\lambda)+1}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)_{\leq\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)-1}\prod_{j\in D(\lambda)_{>\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}} = \frac{\prod_{i=0}^{|\mu|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\mu)}(1+(\frac{\delta}{2}-j)t)^{m_\mu(j)}}.$$

As these are reduced we may equate the numerators and denominators. Equating the numerators gives

$$\prod_{i=\delta+h(\lambda)+1}^{|\lambda|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right) = \prod_{i=0}^{|\mu|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right). \tag{Eq5}$$

From the previous case we know that the right hand side of (Eq5) has no trivial factors. Assume the left hand side has a trivial factor, that is $i = \delta/2$ for some $\delta+h(\lambda)+1 \leq i \leq$

$|\lambda|-1$. This implies that $2(\delta+h(\lambda)+1) \leq \delta$, which gives the inequality $\delta \leq -2h(\lambda)-2$. However this contradicts the assumption $\delta \in \{-h(\lambda), \ldots, -1\}$. Hence none of the factors on the left hand side of (Eq5) are trivial. Therefore (Eq5) implies that $\delta + h(\lambda) + 1 = 0$, and so $\delta = -h(\lambda) - 1$, but this contradicts the assumption $\delta \in \{-h(\lambda), \ldots, -1\}$. Thus this equality can never hold, i.e. this case is impossible. By symmetry, the same can be said for (C3).

(C4): Since $\delta \in \{-h(\mu), \ldots, -1\} \cap \{-h(\lambda), \ldots, -1\}$ with $h(\lambda), h(\mu) \geq 1$, *Lemma 3.14* (2) implies

$$\frac{\prod_{i=\delta+h(\lambda)+1}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)_{\leq\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)-1}\prod_{j\in D(\lambda)_{>\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}}$$
$$= \frac{\prod_{i=\delta+h(\mu)+1}^{|\mu|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\mu)_{\leq\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\mu(j)-1}\prod_{j\in D(\mu)_{>\delta}}(1+(\frac{\delta}{2}-j)t)^{m_\mu(j)}}.$$

Since both sides are reduced, we may equate the numerators and denominators. Equating numerators gives

$$\prod_{i=\delta+h(\lambda)+1}^{|\lambda|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right) = \prod_{i=\delta+h(\mu)+1}^{|\mu|-1}\left(1+\left(i-\frac{\delta}{2}\right)t\right).$$

Arguing as in case (2), none of the factors in the above equality are trivial. As such we must have that both $\delta + h(\lambda) + 1 = \delta + h(\mu) + 1$ and $|\lambda| - 1 = |\mu| - 1$, hence $h(\lambda) = h(\mu)$ and $|\lambda| = |\mu|$. By assumption $-h(\lambda) = -h(\mu) \leq \delta \leq -1$, hence we have that $D(\lambda)_{\leq\delta} = D(\mu)_{\leq\delta}$. Now equating the denominators gives

$$\prod_{j\in D(\lambda)_{\leq\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)-1}\prod_{j\in D(\lambda)_{>\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)}$$
$$= \prod_{j\in D(\mu)_{\leq\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\mu(j)-1}\prod_{j\in D(\mu)_{>\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\mu(j)}.$$

Since $D(\tau_1)_{\leq\delta} \cap D(\tau_2)_{>\delta} = \emptyset$ for any $\tau_1, \tau_2 \in \{\lambda, \mu\}$, we must have

$$\prod_{j\in D(\lambda)_{\leq\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)-1} = \prod_{j\in D(\mu)_{\leq\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\mu(j)-1}, \tag{Eq6}$$

and

$$\prod_{j\in D(\lambda)_{>\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\lambda(j)} = \prod_{j\in D(\mu)_{>\delta}}\left(1+\left(\frac{\delta}{2}-j\right)t\right)^{m_\mu(j)}. \tag{Eq7}$$

Since $\delta/2 \notin D(\lambda)_{\leq\delta} = D(\mu)_{\leq\delta}$ by definition, there must be no trivial factors in (Eq6). As such the multiplicities in (Eq6) must agree, that is $m_\lambda(j) = m_\mu(j)$ for all $j \in D(\lambda)_{\leq\delta}$.

Now (Eq7) tells us that $D(\lambda)_{>\delta}\backslash\{\delta/2\} = D(\mu)_{>\delta}\backslash\{\delta/2\}$ and that the multiplicity functions m_λ and m_μ agree on this set. Hence together, (Eq6) and (Eq7) tell us that $D(\lambda)\backslash\{\delta/2\} = D(\mu)\backslash\{\delta/2\}$ and their multiplicity functions m_λ and m_μ agree on this set. As was the situation in $(C1)$, this implies that the Young diagrams λ and μ can only differ in the diagonal indexed by $\delta/2$, but since $|\lambda| = |\mu|$, no difference is present, showing that $(\lambda, 2k) = (\mu, 2k)$.

□

We now have enough information to prove that $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}]$ is the center of $\mathcal{A}_{2k}(\delta)$ whenever $\delta \notin \{0, 1, \dots, 2k-2\}$. We will employ the following result, whose proof can be found in [JK17, Lemma 4.4].

Lemma 3.16. Let A be a $\mathbb{C}$-subalgebra of $\mathbb{C}[x_1, \dots, x_n]$ and let

$$(c_{11}, \dots, c_{1n}), \dots, (c_{m1}, \dots, c_{mn})$$

be m n-tuples in $\mathbb{C}^n$ for some $m \in \mathbb{Z}_{>0}$. Suppose that for each $1 \le i \ne j \le m$, there exists a polynomial $p \in A$ such that $p(i) \ne p(j)$, where $p(i) := p(c_{i1}, \dots, c_{in})$. Then there exists a family of polynomials $p_1, \dots, p_m \in A$ such that

$$\begin{vmatrix} p_1(1) & p_1(2) & \dots & p_1(m) \\ p_2(1) & p_2(2) & \dots & p_2(m) \\ \dots & \dots & \dots & \dots \\ p_m(1) & p_m(2) & \dots & p_m(m) \end{vmatrix} \ne 0.$$

□

Theorem 3.17. Let $k \in \mathbb{Z}_{\ge 0}$ and $\delta \notin \{0, 1, \dots, 2k-2\}$. Then the center of $\mathcal{A}_{2k}(\delta)$ is given by the supersymmetric polynomials in $N_1, \dots, N_{2k}$, that is $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}] = Z(\mathcal{A}_{2k}(\delta))$.

Proof. By *Proposition 3.15*, we can apply *Lemma 3.16* to the case $A = \mathsf{SSym}[x_1, \dots, x_{2k}]$ and where $\{(c_{11}, \dots, c_{1n}), \dots, (c_{m1}, \dots, c_{mn})\} = \{(\mathsf{ct}_1(\lambda, \delta), \dots, \mathsf{ct}_{2k}(\lambda, \delta)) \mid \lambda \in \Lambda_{\le k}\}$. So we have that $n = 2k$ and $m = |\Lambda_{\le k}|$. Hence *Lemma 3.16* tells us that there exists a family of supersymmetric polynomials $\{p_\lambda : \lambda \in \Lambda_{\le k}\} \subset \mathsf{SSym}[x_1, \dots, x_{2k}]$ such that the matrix $(p_\lambda(\mu, \delta))_{\lambda, \mu \in \Lambda_{\le k}}$ in $\mathbb{C}^{m \times m}$ is invertible, recalling that

$$p_\lambda(\mu, \delta) := p_\lambda(\mathsf{ct}_1(\mu, \delta), \dots, \mathsf{ct}_{2k}(\mu, \delta)).$$

We will now show that the corresponding elements $p_\lambda(N_1, \dots, N_{2k})$ in $\mathsf{SSym}_\delta[N_1, \dots, N_{2k}]$ are also linearly independent. Assume that

$$P = \sum_{\lambda \in \Lambda_{\le k}} c_\lambda p_\lambda(N_1, \dots, N_{2k}) = 0,$$

where $c_\lambda \in \mathbb{C}$ for each $\lambda \in \Lambda_{\le k}$. We seek to show that $c_\lambda = 0$ for each $\lambda \in \Lambda_{\le k}$. For any $(\mu, 2k) \in \Lambda_{\le k} \times \{2k\}$, the element P acts on the simple $\mathcal{A}_{2k}(\delta)$-module $\mathsf{A}^{(\mu, 2k)}$ by 0. From *Lemma 3.6* and *Lemma 3.7* this means that

$$\sum_{\lambda \in \Lambda_{\le k}} c_\lambda p_\lambda(\mu, \delta) = 0,$$

for any $(\mu, 2k) \in \Lambda_{\leq k} \times \{2k\}$. However, since the column vectors of $(p_\lambda(\mu, \delta))_{\lambda,\mu\in\Lambda_{\leq k}}$ are linearly independent, we must have that $c_\lambda = 0$ for all $\lambda \in \Lambda_{\leq k}$. Therefore the set

$$\{p_\lambda(N_1, \ldots, N_{2k}) : \lambda \in \Lambda_{\leq k}\}$$

is linearly independent in $\mathsf{SSym}_\delta[N_1, \ldots, N_{2k}]$. Since $\mathcal{A}_{2k}(\delta)$ is semisimple, we know by the Weddernburn-Artin theorem that the dimension of the center $Z(\mathcal{A}_{2k}(\delta))$ equals $|\Lambda_{\leq k} \times \{2k\}| = |\Lambda_{\leq k}|$, which equals the size of the above linearly independent set. Hence by *Theorem 3.5* we must have that $\mathsf{SSym}_\delta[N_1, \ldots, N_{2k}] = Z(\mathcal{A}_{2k}(\delta))$. □

Corollary 3.18. We have that $\mathsf{SSym}[N_1, \ldots, N_{2k}] = Z(\mathcal{A}_{2k}(z))$.

Proof. This follows from *Theorem 3.17* and then applying *Lemma 2.59*. □

Remark 3.19. As mentioned in *Theorem 2.13*, the center of the group algebra of the symmetric group can be described as the subalgebra of symmetric polynomials in the Jucys-Murphy elements. Analogous descriptions of the centers of the Brauer algebras and walled Brauer algebras (in the semisimple settings) have also been shown in [N96] and [JK17] respectively. Both *Theorem 3.17* and *Corollary 3.18* provide analogous results for the partition algebras.

Remark 3.20. Before the Jucys-Murphy elements of the partition algebra were defined in [HR05], the central idempotents of the partition algebra were described in [MW99]. These central idempotents were obtained by taking the product of the central idempotents in the underlying group algebra of the symmetric group by a certain recursively defined splitting idempotent associated to the partition algebra. In this sense the center of the partition algebra is understood via the splitting idempotent and central idempotents in the group algebra of the symmetric group. Our result above gives us an understanding of the center of the partition algebra via the Jucys-Murphy elements. In [MW99] they described how their central idempotents can give information on the blocks of the partition algebra, and in the next section we show how our description of the center can do the same.

3.3 Alternative Description of the Blocks

Recall from *Section 2.2.3* that the blocks of the partition algebra $\mathcal{A}_{2k}(\delta)$ were characterised by P. Martin as maximal δ-chains of $\Lambda^{(\delta)}_{\leq k}$ for any $\delta \in \mathbb{C}$. In this section we will present an alternative characterisation of the blocks by utilising the action of the central subalgebra $\mathsf{SSym}_\delta[N_1, \ldots, N_{2k}]$ of $\mathcal{A}_{2k}(\delta)$.

Let A be any finite dimensional $\mathbb{C}$-algebra and let Λ be an indexing set for the isomorphism classes of the simple A-modules. Recall that the set of blocks $\{\mathcal{B}_A(\lambda) \mid \lambda \in \Lambda\}$ give a set partition of Λ. We will write A^λ to denote a simple A-module belonging to the class $\lambda \in \Lambda$. Let z belong to the center $Z(A)$ of A, then by Schur's lemma z acts by a scalar on A^λ. Let $\chi_\lambda(z) \in \mathbb{C}$ denote this scalar. Then we obtain a $\mathbb{C}$-algebra homomorphism $\chi_\lambda : Z(A) \to \mathbb{C}$ which is referred to as the *central character induced by* λ. It is well known that λ and μ belong to the same block of Λ if and only if the central characters χ_λ and χ_μ are equal. In this sense, the center $Z(A)$ can distinguish between the blocks of A. Now for the partition algebra $\mathcal{A}_{2k}(\delta)$ recall that

$$\Lambda^{(\delta)}_{\leq k} := \begin{cases} \Lambda_{\leq k}, & \delta \neq 0, \\ \Lambda_{\leq k} \backslash \{\emptyset\}, & \delta = 0. \end{cases}$$

index the isomorphism classes of simple $\mathcal{A}_{2k}(\delta)$-modules. Consider the central characters

$$\chi_\lambda : Z(\mathcal{A}_{2k}(\delta)) \to \mathbb{C}$$

for $\lambda \in \Lambda^{(\delta)}_{\leq k}$, then $\mathcal{B}_{\mathcal{A}_{2k}(\delta)}(\lambda) = \{\mu \in \Lambda_{\leq k} \mid \chi_\mu = \chi_\lambda\}$. When $\mathcal{A}_{2k}(\delta)$ is semisimple then the blocks are trivial, and by *Theorem 3.17* we know that $Z(\mathcal{A}_{2k}(\delta)) = \mathsf{SSym}_\delta[N_1, \ldots, N_{2k}]$. From *Lemma 3.7* and *Lemma 3.6* we know that the central character χ_λ acts on any $p \in \mathsf{SSym}_\delta[N_1, \ldots, N_{2k}]$ by $\chi_\lambda(p) = p(\lambda, \delta)$, i.e. it evaluates the polynomial at the tuple $(\mathsf{ct}_1(\lambda, \delta), \ldots, \mathsf{ct}_{2k}(\lambda, \delta))$. We now show that even in the non-semisimple case, the central characters still act on the supersymmetric polynomials in the normalised Jucys-Murphy elements by evaluating them at the contents of standard paths.

Proposition 3.21. For any $\delta \in \mathbb{C}$ and $\lambda \in \Lambda^{(\delta)}_{\leq k}$, we have that

$$\chi_\lambda(p) = p(\lambda, \delta)$$

for any $p \in \mathsf{SSym}_\delta[N_1, \ldots, N_{2k}]$.

Proof. By *Corollary 2.52* we know for any $\lambda \in \Lambda^{(\delta)}_{\leq k}$ there exists a cell module $\Delta^\lambda_{2k,\delta}$ of $\mathcal{A}_{2k}(\delta)$ which possesses a basis $\{m_\mathsf{T} \mid \mathsf{T} \in \mathsf{Path}(\lambda, 2k)\}$ such that

$$N_i m^\lambda_\mathsf{T} = \left(\mathrm{cont}_\delta(\mathsf{T}, i) - \frac{\delta}{2}\right) m^\lambda_\mathsf{T} + \sum_{\substack{\mathsf{S} \in \mathsf{Path}(\lambda, 2k) \\ \mathsf{T} \prec \mathsf{S}}} v_\mathsf{S}(\mathsf{T}, i) m^\lambda_\mathsf{S} \tag{3.7}$$

where $\prec$ is the partial ordering on the set of paths $\mathsf{Path}(\lambda, 2k)$ defined in [Eny13, Definition 3.8]. As mentioned previously, the standard path $\mathsf{T}^{(\lambda,2k)} \in \mathsf{Path}(\lambda, 2k)$ is a maximal element with respect to this partial ordering (see [Eny13, Lemma 3.9]), then for any supersymmetric polynomial $p \in \mathsf{SSym}[x_1, \ldots, x_{2k}]$, by *Equation* (3.7) we must have that

$$p(N_1, \ldots, N_{2k}) m^\lambda_{\mathsf{T}^{(\lambda,2k)}} = p(\lambda, \delta) m^\lambda_{\mathsf{T}^{(\lambda,2k)}}.$$

Since $\Delta^\lambda_{2k,\delta}$ is a cell module, there exists a maximal submodule $N \subset \Delta^\lambda_{2k,\delta}$ such that

$$D^\lambda_{2k,\delta} := \Delta^\lambda_{\mathcal{A}_{2k}(\delta)} / N$$

is a simple module of the class λ. Then

$$p(N_1, \ldots, N_{2k})(m^\lambda_{\mathsf{T}^{(\lambda,2k)}} + N) = p(\lambda, \delta)(m^\lambda_{\mathsf{T}^{(\lambda,2k)}} + N).$$

Since $\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}] \subseteq Z(\mathcal{A}_{2k}(\delta))$, then Schur's lemma tells us that $p(N_1,\ldots,N_{2k})$ acts on all of D_{2k}^λ by a certain constant, and from above this constant must be $p(\lambda,\delta)$. Therefore we must have that $\chi_\lambda(p) = p(\lambda,\delta)$.

□

Now recall the generating functions $\mathsf{L}(\lambda,\delta)$ of *Equation* (3.5) whose coefficient of the n-th degree term is $l_n(\lambda,\delta)$, the evaluation of the elementary supersymmetric polynomial at the contents of the standard path. This can be defined for any $\delta \in \mathbb{C}$. From *Proposition 3.21* and *Proposition 3.13* we have that

$$\mathsf{L}(\lambda,\delta) = \sum_{n=0}^{\infty} \chi_\lambda(l_n)t^n = \frac{\prod_{i=0}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{j\in D(\lambda)}(1+(\frac{\delta}{2}-j)t)^{m_\lambda(j)}},$$

for any $\delta \in \mathbb{C}$ and $\lambda \in \Lambda_{\leq k}^{(\delta)}$. Since the elementary supersymmetric polynomials generate the algebra of supersymmetric polynomials by *Theorem 3.4*, the generating function $\mathsf{L}(\lambda,\delta)$ contains the same information as the action of the central character χ_λ on the subalgebra $\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]$ of the center $Z(\mathcal{A}_{2k}(\delta))$. In particular, for any $\lambda,\mu \in \Lambda_{\leq k}^{(\delta)}$, we have that $\mathsf{L}(\lambda,\delta) = \mathsf{L}(\mu,\delta)$ if and only if

$$\chi_\lambda|_{\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]} = \chi_\mu|_{\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]}.$$

When $\mathcal{A}_{2k}(\delta)$ is semisimple, then we know by *Theorem 3.17* that $\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]$ is precisely the center of $\mathcal{A}_{2k}(\delta)$, thus $\chi_\lambda = \chi_\mu$ if and only if $\lambda = \mu$. This was shown in *Proposition 3.15* to be the same as asking that $\mathsf{L}(\lambda,\delta) = \mathsf{L}(\mu,\delta)$. We end this section by showing that even in the non-semisimple case, these generating functions still determine the blocks, that is to say that the action of the subalgebra $\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]$ of the center $Z(\mathcal{A}_{2k}(\delta))$ can distinguish between the blocks of $\mathcal{A}_{2k}(\delta)$. We will prove this by showing that $\mathsf{L}(\lambda,\delta) = \mathsf{L}(\mu,\delta)$ if and only if μ and λ belong to the same maximal δ-chain.

Lemma 3.22. Let $\lambda,\mu \in \Lambda_{\leq k}^{(\delta)}$. If (μ,λ) is a δ-pair, then $\mathsf{L}(\mu,\delta) = \mathsf{L}(\lambda,\delta)$.

Proof. We have that $\lambda\backslash\mu = R$ where R is a horizontal strip. Let $R = \{b_1,\ldots,b_n\}$ where the boxes b_i run from left to right as i runs from 1 to n. Since (μ,λ) is a δ-pair we have that $c(b_n) = \delta - |\mu|$. As such,

$$\mathsf{L}(\lambda,\delta) = \frac{\prod_{i=0}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{a\in\lambda}(1+(\frac{\delta}{2}-c(a))t)} = \frac{\prod_{i=0}^{|\mu|-1}(1+(i-\frac{\delta}{2})t)\prod_{i=|\mu|}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{a\in\lambda/R}(1+(\frac{\delta}{2}-c(a))t)\prod_{1\leq i\leq n}(1+(\frac{\delta}{2}-c(b_i))t)}.$$

We have that $|\lambda| = |\mu| + n$, then reindexing gives

$$\prod_{i=|\mu|}^{|\lambda|-1}(1+(i-\delta/2)t) = \prod_{i=1}^{n}(1+(|\lambda|-i-\delta/2)t). \tag{3.8}$$

As $R = \{b_1,\ldots,b_n\}$ consists of consecutive boxes in the same row, and $c(b_n) = \delta - |\mu|$, we see that $c(b_i) = c(b_n) - (n-i) = \delta - |\mu| - n + i = \delta - |\lambda| + i$. Thus

$$\prod_{1\leq i\leq n}(1+(\delta/2-c(b_i))t) = \prod_{1\leq i\leq n}(1+(\delta/2-(\delta-|\lambda|+i))t) = \prod_{1\leq i\leq n}(1+(|\lambda|-i-\delta/2)t). \tag{3.9}$$

Thus *Equation* (3.8) and *Equation* (3.9) agree, and so these factors cancel in $\mathsf{L}(\lambda, \delta)$. Hence, since $\lambda/R = \mu$, we see that

$$\mathsf{L}(\lambda, \delta) = \frac{\prod_{i=0}^{|\mu|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \lambda/R}(1 + (\frac{\delta}{2} - c(a))t)} = \mathsf{L}(\mu, \delta).$$

□

Recall that $\mathcal{C}_{2k,\delta}(\lambda)$ denotes the set of partitions in $\Lambda^{(\delta)}_{\leq 2k}$ which belong to the same maximal δ-chain as λ, which we know equals the block $\mathcal{B}_{\mathcal{A}_{2k}(\delta)}(\lambda)$. From the above lemma, one can immediately see that if $\tau^{(1)} \subset \cdots \subset \tau^{(r)}$ is a maximal δ-chain, then $\mathsf{L}(\tau^{(i)}, \delta) = \mathsf{L}(\tau^{(j)}, \delta)$ for any $1 \leq i, j \leq r$. This tells us that $\mu \in \mathcal{C}_{2k,\delta}(\lambda)$ implies that $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$. The other direction will following from the next two lemmas.

Lemma 3.23. Let $\lambda, \mu \in \Lambda^{(\delta)}_{\leq k}$ such that $\mu \subset \lambda$. If $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$, then there exists a δ-chain $\tau^{(1)} \subset \cdots \subset \tau^{(r)}$ for some $r \in \mathbb{N}$ such that $\mu = \tau^{(1)}$ and $\lambda = \tau^{(r)}$.

Proof. We will prove the result by induction on the number of horizontal strips which μ and λ differ by (which is well-defined since $\mu \subset \lambda$). For the base case, assume that μ and λ differ in a single row, that is $\lambda/\mu = R := \{b_1, \ldots, b_n\}$ for some $n \in \mathbb{N}$, and we assume that the boxes b_i run left to right as i runs from 1 to n. Since $\mu \cup R = \lambda$, we have

$$\mathsf{L}(\lambda, \delta) = \frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \lambda}(1 + (\frac{\delta}{2} - c(a))t)} = \frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \mu}(1 + (\frac{\delta}{2} - c(a))t) \prod_{1 \leq i \leq n}(1 + (\frac{\delta}{2} - c(b_i))t)},$$

and by definition we have

$$\mathsf{L}(\mu, \delta) = \frac{\prod_{i=0}^{|\mu|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \mu}(1 + (\frac{\delta}{2} - c(a))t)}.$$

By assumption we have that $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$, and so we can deduce that

$$\frac{\prod_{i=|\mu|}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{1 \leq i \leq n}(1 + (\frac{\delta}{2} - c(b_i))t)} = 1.$$

Note that there is $n = |\lambda| - |\mu|$ irreducible factors in the numerator and denominator of above. Since the fraction equals 1, the factors in the numerator must match up one-to-one with the factors in the denominator. The box b_n has the largest content of all the boxes in R, and so the factor $(1 + (\frac{\delta}{2} - c(b_n))t)$ has the smallest coefficient of t out of all the factors in the denominator. As such, this factor must cancel out with the factor $(1 + (|\mu| - \frac{\delta}{2})t)$ in the numerator, since this has the smallest coefficient of t among the factors in the numerator. Equating these coefficients yields $c(b_n) = \delta - |\mu|$. Hence $\mu \hookrightarrow_\delta \lambda$ proving the base case. Now assume the result holds if λ differs from μ by $r - 1 > 1$ horizontal strips, we seek to prove the r case. So suppose that $\mu \cup_{i \in [r]} R^{(i)} = \lambda$ where $R^{(i)} = \{b^{(i)}_1, \ldots, b^{(i)}_{n_i}\}$ is a horizontal strip of boxes in λ and $n_i \in \mathbb{N}$. Assume that

$R^{(i)}$ is in a lower row of λ that $R^{(j)}$ whenever $i > j$, hence $R^{(r)}$ is found lower in λ than any other $R^{(i)}$ for $i < r$. We have that

$$\mathsf{L}(\lambda, \delta) = \frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \mu}(1 + (\frac{\delta}{2} - c(a))t) \prod_{i \in I} \prod_{1 \le j \le n_i}(1 + (\frac{\delta}{2} - c(b_j^{(i)}))t)}.$$

By assumption we have that $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$, and so one can deduce that

$$\frac{\prod_{i=|\mu|}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{i \in I} \prod_{1 \le j \le n_i}(1 + (\frac{\delta}{2} - c(b_j^{(i)}))t)} = 1.$$

Note, as was the case previously, the number of irreducible factors in the numerator agrees with that of the denominator. Since the fraction equals 1, the factors in the numerator must match up one-to-one with the factors in the denominator. For the horizontal strip $R^{(r)} = \{b_1^{(r)}, \ldots, b_{n_r}^{(r)}\}$ assume that $b_i^{(r)}$ is to the left of $b_j^{(r)}$ whenever $i < j$. Thus since $R^{(r)}$ is lower than any other horizontal strip $R^{(i)}$, the box $b_1^{(r)}$ has the smallest content among all the boxes in $\cup_{i \in [r]} R^{(i)}$. As such the factor $(1 + (\delta/2 - c(b_1^{(r)})t)$ has the largest coefficient of t among the irreducible factors in the denominator, hence this factor must cancel out with $(1 + (|\lambda| - 1 - \frac{\delta}{2})t)$, since this is the irreducible factor with the largest coefficient of t in the numerator. Therefore we must have that

$$c(b_1^{(r)}) = \delta - |\lambda| + 1.$$

The last box $b_{n_r}^{(r)}$ in $R^{(r)}$ has content $c(b_{n_r}^{(r)}) = c(b_1^{(r)}) + n_r - 1$, and so we have that

$$c(b_{n_r}^{(r)}) = \delta - |\lambda| + 1 + n_r - 1 = \delta - |\lambda| + n_r = \delta - |\lambda/R^{(r)}|.$$

Therefore $\lambda/R^{(r)} \hookrightarrow_\delta \lambda$, and thus by *Lemma 3.22* we have that $\mathsf{L}(\lambda, \delta) = \mathsf{L}((\lambda/R^{(r)}), \delta)$. Thus $\mathsf{L}(\mu, \delta) = \mathsf{L}((\lambda/R^{(r)}), \delta)$ and $\mu \subset \lambda/R^{(r)}$, and so by the inductive hypothesis there exists a δ-chain $\tau^{(1)} \subset \cdots \subset \tau^{(r-1)}$ such that $\tau^{(1)} = \mu$ and $\tau^{(r-1)} = \lambda/R^{(r)}$. Since $\lambda/R^{(r)} \hookrightarrow_\delta \lambda$, we can extend this chain by adding $\tau^{(r)} = \lambda$, which completes the proof. □

Lemma 3.24. Let $\lambda, \mu \in \Lambda_{\le k}^{(\delta)}$. If $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$ then $\mu \subset \lambda$ or $\lambda \subset \mu$.

Proof. Assume for contradiction that $\mathsf{L}(\lambda, \delta) = \mathsf{L}(\mu, \delta)$ but $\lambda \not\subset \mu$ and $\mu \not\subset \lambda$. Consider the Young diagram $\tau = \lambda \cap \mu$, then we have

$$\mathsf{L}(\lambda, \delta) = \frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \lambda}(1 + (\frac{\delta}{2} - c(a))t)} = \frac{\prod_{i=0}^{|\lambda|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \tau}(1 + (\frac{\delta}{2} - c(a))t) \prod_{a \in \lambda/\tau}(1 + (\frac{\delta}{2} - c(a))t)},$$

and similarly

$$\mu(t) = \frac{\prod_{i=0}^{|\mu|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \mu}(1 + (\frac{\delta}{2} - c(a))t)} = \frac{\prod_{i=0}^{|\mu|-1}(1 + (i - \frac{\delta}{2})t)}{\prod_{a \in \tau}(1 + (\frac{\delta}{2} - c(a))t) \prod_{a \in \mu/\tau}(1 + (\frac{\delta}{2} - c(a))t)}.$$

Without loss of generality assume that $|\mu| \le |\lambda|$. Since $\mathsf{L}(\lambda,\delta) = \mathsf{L}(\mu,\delta)$ we have that

$$\frac{\prod_{i=|\mu|}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{a\in\lambda/\tau}(1+(\frac{\delta}{2}-c(a))t)} = \frac{1}{\prod_{a\in\mu/\tau}(1+(\frac{\delta}{2}-c(a))t)}, \tag{3.10}$$

which implies

$$\prod_{i=|\mu|}^{|\lambda|-1}(1+(i-\delta/2)t) = \frac{\prod_{a\in\lambda/\tau}(1+(\frac{\delta}{2}-c(a))t)}{\prod_{a\in\mu/\tau}(1+(\frac{\delta}{2}-c(a))t)}. \tag{3.11}$$

This tells us that all the irreducible factors in the denominator of the right hand side of *Equation* (3.11) must cancel out with factors in the numerator, or be trivial. This means that the multiset of contents which do not equal $\delta/2$ of the skew diagram μ/τ is contained in the multiset of contents which do not equal $\delta/2$ of the skew diagram λ/τ. Since $(\lambda/\tau)\cap(\mu/\tau) = \emptyset$, these multisets are distinct, and so μ/τ must consists only of boxes with content $\delta/2$ and λ/τ has no boxes with content $\delta/2$. The only way μ/τ can consist solely of boxes with content $\delta/2$ is if $|\mu/\tau| = 1$. So let $\mu/\tau = \{a\}$ where $c(a) = \delta/2$. Then from *Equation* (3.10) above,

$$\frac{\prod_{i=|\mu|}^{|\lambda|-1}(1+(i-\frac{\delta}{2})t)}{\prod_{a\in\lambda/\tau}(1+(\frac{\delta}{2}-c(a))t)} = 1. \tag{3.12}$$

Note, the number of irreducible factors in the numerator of *Equation* (3.12) equals $|\lambda|-|\mu|$, while there is $|\lambda/\tau| = |\lambda|-|\mu|+1$ (since $|\mu| = |\tau|+1$) irreducible factors in the denominator. Thus for the equality of *Equation* (3.12) to hold, one of the factors in the denominator must equal 1, i.e. there must exist some $a \in \lambda/\tau$ such that $c(a) = \delta/2$. However, as mentioned above this cannot occur, giving the desired contradiction.

□

Corollary 3.25. Given $\lambda,\mu \in \Lambda^{(\delta)}_{\le k}$, then λ and μ belong to the same block if and only if $\mathsf{L}(\lambda,\delta) = \mathsf{L}(\mu,\delta)$. As such we have that

$$\mathcal{B}_{2k}(\lambda) = \left\{\mu \in \Lambda^{(\delta)}_{\le k} \;\middle|\; \chi_\lambda|_{\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]} = \chi_\mu|_{\mathsf{SSym}_\delta[N_1,\ldots,N_{2k}]}\right\}.$$

□

Remark 3.26. If $\delta \in \{0,1,\ldots,2k-2\}$, i.e. when $\mathcal{A}_{2k}(\delta)$ is non-semisimple, then we do not know whether the subalgebra $\mathsf{SSym}[N_1,\ldots,N_{2k}]$ is the entire center $Z(\mathcal{A}_{2k}(\delta))$ or not. However, we know that the action of the center $Z(\mathcal{A}_{2k}(\delta))$ can distinguish the blocks of $\mathcal{A}_{2k}(\delta)$, and *Corollary 3.25* tells us that the central subalgebra $\mathsf{SSym}[N_1,\ldots,N_{2k}]$ can do the same. We believe this gives some evidence to suggest that the algebra $\mathsf{SSym}[N_1,\ldots,N_{2k}]$ is possibly the entire center even in the non-semisimple case.

Remark 3.27. As already summarised in *Section 2.2.3*, the blocks of the partition algebra $\mathcal{A}_{2k}(\delta)$ have already been understood from the works of P. Martin in [Martin96] and of D. Wales and W. Doran in [DW00] as maximal δ-chains. Our result in *Corollary*

3.25 has simply re-expressed such information in the form of a generating function. One reason why this is interesting is that is provides an analogous theory for understanding the blocks of the partition algebra which has been described for various other algebras such as the Brauer and walled Brauer algebras. That is using Jucys-Murphy elements and thier action on simple modules has provided a more uniform approach to analysing the blocks of a given algebra, and we have demonstrated this for the partition algebra. It is also worth mentioning that comparing whether two generating functions $L(\lambda, \delta)$ and $L(\mu, \delta)$ agree or not is a very simple task, as one does not need to "unravel" the generating functions, but instead just needs to treat them as rational functions in the variable t, and simply cancel out common factors.

4 Affine Partition Algebra

This chapter will provide a definition of an affine version of the partition algebra which we denote by $\mathcal{A}_{2k}^{\mathrm{aff}}$ called the *affine partition algebra*, and prove a variety of results regarding it including the five affinization proerties *1* to *5* described in *Section 1.4*. For the first section of this chapter, to help motivate the definition of our affine partition algebra via a presentation, we start by summarising the process employed by others to construct analogous algebras as highlighted in *Section 1.3*. We prove and collect various relations in the partition algebra, then define the affine partition algebra by generators and a presentation, and lastly focus on proving the first three affinization properties *1* to *3*. The second section of this chapter describes an action of $\mathcal{A}_{2k}^{\mathrm{aff}}$ on the tensor space $M \otimes V^{\otimes k}$, where M is any $\mathbb{C}\mathfrak{S}_n$-module, which generalises the action $\Psi_{2k,n}$ described in *Theorem 2.58*. As such we prove that our affine partition algebra satisfies affinization property *4*. The third and last section of this chapter establishes connections between our affine partition algebra and the Heisenberg category defined in [Kho14]. Namely we prove that a certain endomorphism algebra of an object in the Heisenberg category is a quotient of our affine partition algebra, and via this quotient the affine generators get mapped onto decorations as one would hope. This shows that our affine partition algebra partially satisfies the last affinization property *5*. We end the chapter by recalling a subcategory of the Heisenberg category called the affine partition category which was defined in [BV21] and is denoted by APar. This category is generated by a single object and a collection of morphisms, and an algebra also called the affine partition algebra AP_k is defined in [BV21] as the endomorphism algebra of a certain object in APar. We prove that the category APar is a full subcategory of the Heisenberg category, which as a result gives us a basis for the morphism spaces of APar, and shows that the algebra AP_k is a quotient of our affine partition algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$.

4.1 Defining the Affine Partition Algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$

In this section we will prove various relations within the partition algebra, use such to define an affine partition algebra, and prove many structural results for this new algebra including the affinization properties *1* to *3*. We will define the affine partition algebra by employing an analogous procedure to what has been done for other affine counterparts of diagram algebras within the literature. This procedure however is very vague and often only the outcome of such is given with the procedure itself only implicitly present. As such we wish to give some structure to this procedure to help motivate our definition and ease the readability of the chapter. We seek to define an affine partition algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$ by taking the following steps:

1. Fix a generating set G for the partition algebra $\mathcal{A}_{2k}$.
2. Fix a family of Jucys-Murphy elements $\mathsf{X} = \{X_1, \ldots, X_{2k}\}$ for $\mathcal{A}_{2k}$ and a collection of central elements $\mathsf{W} \subset Z(\mathcal{A}_{2k})$ related to X.
3. Fix a collection of relations R involving the elements G, X, and W, which contains a presentation for $\mathcal{A}_{2k}$.
4. Define $\mathcal{A}_{2k}^{\text{aff}}$ with generating set $\mathsf{G}^{\text{aff}} \sqcup \mathsf{X}^{\text{aff}} \sqcup \mathsf{W}^{\text{aff}}$ and defining relations R^{aff} where:
   - The sets G^{aff}, X^{aff}, and W^{aff} are considered to be formal symbols which are in bijection with G, X, and W respectively.
   - The relations R^{aff} are obtained from R by replacing each of the elements from $\mathsf{G} \sqcup \mathsf{X} \sqcup \mathsf{W}$ with their bijective counterparts in $\mathsf{G}^{\text{aff}} \sqcup \mathsf{X}^{\text{aff}} \sqcup \mathsf{W}^{\text{aff}}$.

This series of steps is presented to simply give a broad idea behind how we defined the affine partition algebra. A variety of choices need to be made, with the most significant choice being that for the set of relations R. No conditions for such choices are given, which of course means that following these steps in general would lead to numerous algebras which would be very undesirable candidates for an affine partition algebra. To refine these steps to provide an algorithm which produces desirable algebras appears to be a very difficult task, and we do not attempt to do such here. Instead, we will model our choices to be as analogous as possible to the choices made in other affine diagram algebras, and by keeping in mind that we want the resulting algebra to satisfy the affinization properties *1* to *5*. With this being said, recall the diagram algebras and their affine counterparts discussed in *Section 1.2* and *Section 1.3*, we now explain how analogous steps to those presented above were taking in such settings.

For the setting of the degenerate affine Hecke algebra $\mathcal{H}_k$, we have $\mathsf{G} = \{s_1, \ldots, s_{k-1}\}$, the set of simple transpositions in $\mathfrak{S}_k$, and $\mathsf{X} = \{Y_1, \ldots, Y_k\}$, the family of Jucys-Murphy elements defined in *Definition 2.11*. The set of central elements W is to help account for certain floating components in the diagrammatics, which are absent in this setting and so we have $\mathsf{W} = \emptyset$. The set of relations R is taken to be the relations in *Theorem 2.1* along with the relations in *Lemma 2.12*. Then $\mathcal{H}_k$ is obtained by applying an analogous step to *Step 4* above. Due to the choice in relations R, the elements G^{aff} generate in $\mathcal{H}_k$ a subalgebra isomorphic to the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_k$, and hence we really have that $\mathsf{G}^{\text{aff}} = \mathsf{G}$. However, the elements X^{aff} have provided meaningfully new generators as discussed in *Section 2.1.7*.

The setting for the affine Wenzl algebra $\mathcal{W}_k$ was presented in [N96]. To summarise, let $\delta \in \mathbb{C}$, then $\mathsf{G} = \{s_1, \ldots, s_{k-1}, \overline{s}_1, \ldots, \overline{s}_{k-1}\}$, where s_i are the simple transpositions of $\mathfrak{S}_k$ sitting inside the Brauer algebra $\mathfrak{B}_k(\delta)$, and $\overline{s}_i$ are the generators corresponding to the diagrams $\{\{i, i+1\}, \{i', (i+1)'\}, \{j, j'\} \mid j \in [k] \backslash \{i, i+1\}\}$ (i.e. when viewing $\mathfrak{B}_k(\delta)$ as a subalgebra of $\mathcal{A}_{2k}(\delta)$, then $\overline{s}_i = e_{2i}e_{2i-1}e_{2i+1}e_{2i}$). The family of Jucys-Murphy elements $\mathsf{X} = \{x_1, \ldots, x_k\}$ are taken to be those defined in [N96, Equation (2.2)]. The set of central elements are taken to be set of constants

$$\mathsf{W} = \left\{ z_1^{(n)} := \frac{\delta^n(\delta - 1)}{2} \mid n \in \mathbb{Z}_{\geq 0} \right\}.$$

The motivation behind this choice of central elements is since they satisfy the relation $\overline{s}_1 x_1^n \overline{s}_1 = z_1^{(n)} \overline{s}_1$. Hence thinking ahead, we will want to interpret the affine version of the element x_1 (i.e. its counterpart in X^{aff}) as a decoration on the first string, thus the affine version of the expression $\overline{s}_1 x_1^n \overline{s}_1$ would be viewed diagrammtically as $\overline{s}_1$ plus a floating loop with n decorations on it. As such the affine version of the equation $\overline{s}_1 x_1^n \overline{s}_1 = z_1^{(n)} \overline{s}_1$ will allow the affine version of the central element $z_1^{(n)}$ to play the role of a floating loop with n decorations. The set of relations R are chosen to be those in the presentation of $\mathfrak{B}_k(\delta)$ given in [N96, Proposition 1.1] alongside the relations $\overline{s}_1 x_1^n \overline{s}_1 = z_1^{(n)} \overline{s}_1$, various commuting relations, and some non-commuting relations between the Jucys-Murphy elements and the generators in G (which include recursive relations analogous to item (*iii*) in *Lemma 2.12*). Then $\mathcal{W}_k$ is obtained by applying an analogous step to *Step 4* above. Due to the choice in relations R, the elements G^{aff} generate in $\mathcal{W}_k$ a subalgebra isomorphic to the Brauer algebra $\mathfrak{B}_k(\omega_0)$ (viewed as an algebra over the ring $\mathbb{C}[\omega_0]$, with ω_0 being the affine version of $z_1^{(0)} = (\delta - 1)\backslash 2$ belonging to W^{aff}). Hence we really have that $\mathsf{G}^{\text{aff}} = \mathsf{G}$. However, the generators $\mathsf{X}^{\text{aff}} = \{y_1, \dots, y_k\}$ and $\mathsf{W}^{\text{aff}} = \{\omega_n \mid n \in \mathbb{Z}_{\geq 0}\}$ (adopting the notation of [N96]) have become meaningfully new elements, which in some sense have "freed up" the Jucys-Murphy elements and the central elements $z_1^{(n)}$. These relations of R were very much a choice M. Nazarov made, and although little is said in [N96] regarding the motivation for such choices, they are certainly quite analogous to the relations chosen in the setting of $\mathcal{H}_k$, and are very natural when considering the corresponding diagrammatics that will be produced, in other words for the resulting algebra to satisfy affinization property *5*.

The setting for the affine walled Brauer algebras follows very analogously to that of the affine Wenzl algebra, and the reader may find such details in [Sar13] and [RS13].

The choices we will make to construct our affine partition $\mathcal{A}_{2k}^{\text{aff}}$ are very much guided by what has been done by others summarised above. However there will be numerious unique features with our construction which we will remark on as they emerge. It is worth mentioning now that our choice for the set of relations R was motivated by a mixture of what was done by M. Nazarov in [N96], and in trying to produce an algebra which satisfies the affinization properties *1* to *5*, with the last two such properties being the most helpful in determining appropriate relations to pick.

4.1.1 Making our choices for G, X, W, and R

In this subsection we establish the sets G, X, W, and R involved in *Steps 1* to *3*. We define normalised versions of both the Jucys-Murphy elements and Enyang's generators, which will be easier to work with. Many of the relations in R will need to be proved, and will contain an alternative presentation of $\mathcal{A}_{2k}$, which is simply the presentation *Theorem 2.38* given by J. Enyang with the exception of replacing Enyang's generators with the normalised versions.

Recall the Jucys-Murphy elements $L_1, \dots, L_{2k}$ and Enyang's generators $\sigma_2, \dots, \sigma_{2k-1}$ for the partition algebra $\mathcal{A}_{2k}$ given in *Definition 2.34*. We define the following normalisations of such elements:

Definition 4.1. In $\mathcal{A}_{2k}$ define for any $i \in [k-1]$ the elements

$$t_{2i} := \sigma_{2i} - e_{2i}, \quad t_{2i+1} := \sigma_{2i+1} - e_{2i}.$$

Also for any $i \in [2k]$ define

$$X_i := \begin{cases} z - 1 - L_i, & \text{if } i \text{ odd} \\ L_i - 1, & \text{if } i \text{ even} \end{cases}.$$

We use the same symbols to denote the corresponding elements in $\mathcal{A}_{2k}(\delta)$ for any $\delta \in \mathbb{C}$.

Throughout this chapter we will refer to the elements X_i also as the Jucys-Murphy elements, and the elements t_i also as Enyang's generators. By definition one can see that $t_i \in \mathcal{A}_{i+1}$ and $X_i \in \mathcal{A}_i$, also these elements are invariant under the anti-automorphism $*$. By $(E2)(iv)$ of *Theorem 2.38* one can check that $s_i t_{2i} = t_{2i} s_i = t_{2i+1}$. We briefly collect some simple relations to ease the proof of the following proposition.

Lemma 4.2. The following relations hold:

(i) $e_{2i+1} t_{2i} e_{2i+1} = X_{2i-1} e_{2i+1}$

(ii) $t_{2i} e_{2i-1} e_{2i} = X_{2i} e_{2i}$, and $e_{2i} e_{2i-1} t_{2i} = e_{2i} X_{2i}$

(iii) $t_{2i+1} e_{2i+1} e_{2i} = X_{2i} e_{2i}$, and $e_{2i} e_{2i+1} t_{2i+1} = e_{2i} X_{2i}$

Proof. (i): We have that

$$\begin{aligned} e_{2i+1} t_{2i} e_{2i+1} &= e_{2i+1}(\sigma_{2i} - e_{2i}) e_{2i+1} \\ &= e_{2i+1} \sigma_{2i} e_{2i+1} - e_{2i+1} && \text{by } \textit{(E5)} \text{ of } \textit{Theorem 2.38} \\ &= (z - L_{2i-1}) e_{2i+1} - e_{2i+1} && \text{by [Eny12, Proposition 4.3 (2)]} \\ &= (X_{2i-1} + 1) e_{2i+1} - e_{2i+1} \\ &= X_{2i-1} e_{2i+1} \end{aligned}$$

(ii): We have that

$$\begin{aligned} t_{2i} e_{2i-1} e_{2i} &= (\sigma_{2i} - e_{2i}) e_{2i-1} e_{2i} \\ &= \sigma_{2i} e_{2i-1} e_{2i} - e_{2i} && \text{by } \textit{(E5)} \text{ of } \textit{Theorem 2.38} \\ &= L_{2i} e_{2i} - e_{2i} && \text{by [Eny12, Proposition 3.2 (3)]} \\ &= (X_{2i} + 1) e_{2i} - e_{2i} \\ &= X_{2i} e_{2i} \end{aligned}$$

The relation $e_{2i} e_{2i-1} t_{2i} = e_{2i} X_{2i}$ is obtained by acting by $*$.

(iii): We have $t_{2i+1} e_{2i+1} e_{2i} = t_{2i} s_i e_{2i+1} e_{2i} = t_{2i} e_{2i-1} e_{2i} = X_{2i} e_{2i}$. Again the relation $e_{2i} e_{2i+1} t_{2i+1} = e_{2i} X_{2i}$ is obtained by acting by $*$.

□

The following proposition contains all the relations R we seek for our construction of the affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$, as such some are identical to relations which can be found in *Section 2.2*. It includes a presentation of the partition algebra $\mathcal{A}_{2k}(z)$ which is simply Enyang's presentation of *Theorem 2.38* except working with the generators t_i instead of σ_i. For those relations we have adopted the same naming conventions given in *Theorem 2.38* even though their meaning does not always correspond directly to the given relation.

Proposition 4.3. The partition algebra $\mathcal{A}_{2k}(z)$ is generated by the set

$$\{t_i, e_j \mid 2 \leq i \leq 2k-1,\ j \in [2k-1]\},$$

and the following relations are satisfied:

(1) (Involutions)
  (i) $t_{2i}^2 = 1 - e_{2i}$, for $i \in [k-1]$.
  (ii) $t_{2i+1}^2 = 1 - e_{2i}$, for $i \in [k-1]$.

(2) (Braid relations)
  (i) $t_{2i+1}t_{2j} = t_{2j}t_{2i+1}$ for $j \neq i+1$.
  (ii) $t_{2i+1}t_{2j+1} = t_{2j+1}t_{2i+1}$ for $j \neq i \pm 1$.
  (iii) $t_{2i}t_{2j} = t_{2j}t_{2i}$ for $j \neq i \pm 1$.
  (iv) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [k-2]$, where $s_i = t_{2i}t_{2i+1} + e_{2i}$.

(3) (Idempotent relations)
  (i) $e_{2i-1}^2 = ze_{2i-1}$ for $i \in [k]$.
  (ii) $e_{2i}^2 = e_{2i}$ for $i \in [k-1]$.
  (iii) $t_{2i+1}e_{2i} = e_{2i}t_{2i+1} = 0$ for $i \in [k-1]$.
  (iv) $t_{2i}e_{2i} = e_{2i}t_{2i} = 0$ for $1 \leq i \leq k-1$.
  (v) $t_{2i}e_{2i-1}e_{2i+1} = t_{2i+1}e_{2i-1}e_{2i+1}$ for $1 \leq i \leq k-1$.
  (vi) $e_{2i+1}e_{2i-1}t_{2i} = e_{2i+1}e_{2i-1}t_{2i+1}$ for $1 \leq i \leq k-1$.

(4) (Commutation relations)
  (i) $e_i e_j = e_j e_i$, if $|i-j| \geq 2$.
  (ii) $t_{2i-1}e_{2j-1} = e_{2j-1}t_{2i-1}$, if $j \neq i-1, i$.
  (iii) $t_{2i-1}e_{2j} = e_{2j}t_{2i-1}$, if $j \neq i$.
  (iv) $t_{2i}e_{2j-1} = e_{2j-1}t_{2i}$, if $j \neq i, i+1$.
  (v) $t_{2i}e_{2j} = e_{2j}t_{2i}$, if $j \neq i-1$.

(5) (Contractions)
  (i) $e_i e_{i+1} e_i = e_i$ and $e_{i+1} e_i e_{i+1} = e_{i+1}$, for $i \in [2k-2]$.

(ii) $t_{2i}e_{2i-1}t_{2i} = t_{2i+1}e_{2i+1}t_{2i+1}$, for $i \in [k-1]$.

(iii) $t_{2i}e_{2i-2}t_{2i} = t_{2i-1}e_{2i}t_{2i-1}$, for $2 \leq i \leq k-1$.

Furthermore, the following relations involving the Jucys-Murphy elements and Enyang's generators are satisfied, whenever the indices make sense if not stated:

(6) (JM Commutation Relations)

(i) $X_iX_j = X_jX_i$ for all $i, j \in [2k]$

(ii) $t_iX_j = X_jt_i$ for $j \neq i-1, i, i+1$

(iii) $e_iX_j = X_je_i$ for $j \neq i, i+1$

(7) (Braid-like Relations)

(i) $t_{2i-2}t_{2i}t_{2i-2} = t_{2i}t_{2i-2}t_{2i}(1 - e_{2i-2})$

(ii) $t_{2i+1}t_{2i-1}t_{2i+1} = t_{2i-1}t_{2i+1}t_{2i-1}(1 - e_{2i})$

(iii) $t_{2i-1}t_{2i}t_{2i-1} = t_{2i} - e_{2i-2}t_{2i} - t_{2i}e_{2i-2}$

(iv) $t_{2i}t_{2i-1}t_{2i} = t_{2i-1} - e_{2i}t_{2i-1} - t_{2i-1}e_{2i}$

(8) (Skein-like Relations)

(i) $X_{2i+1} = t_{2i}X_{2i-1}t_{2i} + e_{2i}e_{2i-1}t_{2i} + t_{2i}e_{2i-1}e_{2i} - t_{2i}$.

(ii) $X_{2i+2} = t_{2i+1}X_{2i}t_{2i+1} + e_{2i}e_{2i+1}t_{2i+1}e_{2i+1}e_{2i} + t_{2i+1}$.

(iii) $X_{2i} = t_{2i}X_{2i}t_{2i} + e_{2i}e_{2i-1}t_{2i} + t_{2i}e_{2i-1}e_{2i}$.

(iv) $X_{2i+1} = t_{2i+1}X_{2i+1}t_{2i+1} + e_{2i}e_{2i+1}t_{2i+1} + t_{2i+1}e_{2i+1}e_{2i}$.

(9) (Anti-symmetry Relations)

(i) $e_i(X_i - X_{i+1}) = 0$ for $i \in [2k-1]$.

(ii) $(X_i - X_{i+1})e_i = 0$ for $i \in [2k-1]$.

(10) (Bubble Relations)

(i) $e_1X_1^le_1 = z(z-1)^le_1$, for all $l \in \mathbb{Z}_{\geq 0}$.

Proof. Although lengthy, it is simple to check that relations (1) to (5) are satisfied since we merely exchanged the elements σ_i with t_i from Enyang's presentation given in *Theorem 2.38*. In particular they certainly generate the algebra $\mathcal{A}_{2k}(z)$.
(6): Follows from items *(ii)* and *(iii)* of *Proposition 2.37*, and since e_i commutes with $\mathcal{A}_{i-1}$ for all $i \in [2k-1]$.
(7): These relations will be proven seperately in *Lemma 4.6* below.
(9): Follows from [Eny12, Proposition 3.9] (1) and (2).
(10): We have that $X_1 = z - 1 - L_1 = z - 1$. Thus for any $l \in \mathbb{N}$,

$$e_1X_1^le_1 = (z-1)^le_1^2 = z(z-1)^le_1.$$

(8)(i): From *Proposition 2.39 (i)* we have

$$L_{2i+1} = \sigma_{2i} L_{2i-1} \sigma_{2i} - e_{2i} e_{2i-1} \sigma_{2i} - \sigma_{2i} e_{2i-1} e_{2i} + e_{2i} e_{2i+1} \sigma_{2i} e_{2i+1} e_{2i} + \sigma_{2i}. \tag{4.1}$$

Examining the right hand side term by term: For the first term,

$$\begin{aligned}
\sigma_{2i} L_{2i-1} \sigma_{2i} &= (t_{2i} + e_{2i})(-X_{2i-1})(t_{2i} + e_{2i}) + (z-1) \\
&= -t_{2i} X_{2i-1} t_{2i} - t_{2i} X_{2i-1} e_{2i} - e_{2i} X_{2i-1} t_{2i} - e_{2i} X_{2i-1} e_{2i} + (z-1) \\
&= -t_{2i} X_{2i-1} t_{2i} - X_{2i-1} e_{2i} + (z-1)
\end{aligned}$$

where the last equality follows since X_{2i-1} commutes with e_{2i} and $t_{2i} e_{2i} = e_{2i} t_{2i} = 0$. For the second and third term of *Equation* (4.1), we have

$$-e_{2i} e_{2i-1} \sigma_{2i} = -e_{2i} - e_{2i} e_{2i-1} t_{2i}, \quad \text{and} \quad -\sigma_{2i} e_{2i-1} e_{2i} = -t_{2i} e_{2i-1} e_{2i} - e_{2i}.$$

For the forth term of *Equation* (4.1),

$$\begin{aligned}
e_{2i} e_{2i+1} \sigma_{2i} e_{2i+1} e_{2i} &= e_{2i} e_{2i+1} t_{2i} e_{2i+1} e_{2i} + e_{2i} e_{2i+1} e_{2i} e_{2i+1} e_{2i} \\
&= e_{2i} e_{2i+1} t_{2i} e_{2i+1} e_{2i} + e_{2i} \\
&= e_{2i} e_{2i-1} t_{2i+1} e_{2i+1} e_{2i} + e_{2i} && \text{by } t_{2i} = s_i t_{2i+1} \\
&= e_{2i} e_{2i-1} X_{2i+1} e_{2i} + e_{2i} && \text{by } \textit{Lemma 4.2 (iii)} \\
&= e_{2i} e_{2i-1} e_{2i} X_{2i-1} + e_{2i} && \text{by } \textit{(9)(i), (ii)} \\
&= e_{2i} X_{2i-1} + e_{2i}.
\end{aligned}$$

Substituting all these back into *Equation* (4.1) yields

$$\begin{aligned}
z - 1 - X_{2i+1} &= -t_{2i} X_{2i-1} t_{2i} - X_{2i-1} e_{2i} + (z-1) - e_{2i} - e_{2i} e_{2i-1} t_{2i} - t_{2i} e_{2i-1} e_{2i} - e_{2i} \\
&\quad + e_{2i} X_{2i-1} + e_{2i} + t_{2i} + e_{2i} \\
\iff X_{2i+1} &= t_{2i} X_{2i-1} t_{2i} + e_{2i} e_{2i-1} t_{2i} + t_{2i} e_{2i-1} e_{2i} - t_{2i}
\end{aligned}$$

giving *(8)(i)*.
(8)(ii): From *Proposition 2.39 (ii)* we have

$$L_{2i+2} = \sigma_{2i+1} L_{2i} \sigma_{2i+1} - e_{2i} e_{2i+1} - e_{2i+1} e_{2i} + e_{2i} e_{2i+1} \sigma_{2i+1} e_{2i+1} e_{2i} + \sigma_{2i+1}. \tag{4.2}$$

We examine two terms on the right hand side: The first term gives

$$\begin{aligned}
\sigma_{2i+1} L_{2i} \sigma_{2i+1} &= (t_{2i+1} + e_{2i})(X_{2i} + 1)(t_{2i+1} + e_{2i}) \\
&= t_{2i+1} X_{2i} t_{2i+1} + t_{2i+1} X_{2i} e_{2i} + e_{2i} X_{2i} t_{2i+1} + e_{2i} X_{2i} e_{2i} + 1 \\
&= t_{2i+1} X_{2i} t_{2i+1} + t_{2i+1}^2 e_{2i+1} e_{2i} + e_{2i} e_{2i+1} t_{2i+1}^2 + 1 \\
&= t_{2i+1} X_{2i} t_{2i+1} + e_{2i+1} e_{2i} + e_{2i} e_{2i+1} - 2e_{2i} + 1
\end{aligned}$$

where the second equality follows since $(t_{2i+1} + e_{2i})^2 = 1$, and the third from *Lemma 4.2 (iii)* and since $e_{2i} X_{2i} e_{2i} = e_{2i} e_{2i-1} t_{2i} e_{2i} = 0$. The forth term in *Equation* (4.2) gives

$$\begin{aligned}
e_{2i} e_{2i+1} \sigma_{2i+1} e_{2i+1} e_{2i} &= e_{2i} e_{2i+1} t_{2i+1} e_{2i+1} e_{2i} + e_{2i} e_{2i+1} e_{2i} e_{2i+1} e_{2i} \\
&= e_{2i} e_{2i+1} t_{2i+1} e_{2i+1} e_{2i} + e_{2i}.
\end{aligned}$$

Substituting these back into *Equation* (4.2) yields

$$\begin{aligned} X_{2i+2}+1 &= t_{2i+1}X_{2i}t_{2i+1}+e_{2i+1}e_{2i}+e_{2i}e_{2i+1}-2e_{2i}+1 \\ &\quad -e_{2i}e_{2i+1}-e_{2i+1}e_{2i}+e_{2i}e_{2i+1}t_{2i+1}e_{2i+1}e_{2i}+e_{2i}+t_{2i+1}+e_{2i} \\ \iff X_{2i+2} &= t_{2i+1}X_{2i}t_{2i+1}+e_{2i}e_{2i+1}t_{2i+1}e_{2i+1}e_{2i}+t_{2i+1} \end{aligned}$$

giving *(8)(ii)*.
(8)(iii): From *Proposition* 2.39 *(iii)* we have

$$L_{2i}=\sigma_{2i}L_{2i}\sigma_{2i}+e_{2i}e_{2i-1}\sigma_{2i}+\sigma_{2i}e_{2i-1}e_{2i}-e_{2i}e_{2i+1}-e_{2i+1}e_{2i}. \tag{4.3}$$

We have that

$$\begin{aligned} \sigma_{2i}L_{2i}\sigma_{2i} &= (t_{2i}+e_{2i})(X_{2i}+1)(t_{2i}+e_{2i}) \\ &= t_{2i}X_{2i}t_{2i}+t_{2i}X_{2i}e_{2i}+e_{2i}X_{2i}t_{2i}+e_{2i}X_{2i}e_{2i}+1 \\ &= t_{2i}X_{2i}t_{2i}+t_{2i}^2e_{2i-1}e_{2i}+e_{2i}e_{2i-1}t_{2i}^2+1 \\ &= t_{2i}X_{2i}t_{2i}+e_{2i-1}e_{2i}+e_{2i}e_{2i-1}-2e_{2i}+1 \end{aligned}$$

where the second equality follows since $(t_{2i}+e_{2i})^2=1$, and the third equality from *Lemma* 4.2 *(ii)* and the since $t_{2i}e_{2i}=e_{2i}t_{2i}=0$. Substituting this, and relations

$$e_{2i}e_{2i-1}\sigma_{2i}=e_{2i}e_{2i-1}t_{2i}+e_{2i} \quad \text{and} \quad \sigma_{2i}e_{2i-1}e_{2i}=t_{2i}e_{2i-1}e_{2i}+e_{2i},$$

back into *Equation* (4.3) yields

$$\begin{aligned} X_{2i}+1 &= t_{2i}X_{2i}t_{2i}+e_{2i-1}e_{2i}+e_{2i}e_{2i-1}-2e_{2i}+1+e_{2i}e_{2i-1}t_{2i}+e_{2i} \\ &\quad +t_{2i}e_{2i-1}e_{2i}+e_{2i}-e_{2i}e_{2i+1}-e_{2i+1}e_{2i} \\ \iff X_{2i} &= t_{2i}X_{2i}t_{2i}+e_{2i}e_{2i-1}t_{2i}+t_{2i}e_{2i-1}e_{2i}+1 \end{aligned}$$

giving *(8)(iii)*.
(8)(iv): From *Proposition* 2.39 *(iv)* we have

$$L_{2i+1}=\sigma_{2i+1}L_{2i+1}\sigma_{2i+1}-e_{2i}e_{2i+1}\sigma_{2i+1}-\sigma_{2i+1}e_{2i+1}e_{2i}+e_{2i}e_{2i+1}+e_{2i+1}e_{2i}. \tag{4.4}$$

We have that

$$\begin{aligned} \sigma_{2i+1}L_{2i+1}\sigma_{2i+1} &= (t_{2i+1}+e_{2i})(-X_{2i+1})(t_{2i+1}+e_{2i})+(z-1) \\ &= -t_{2i+1}X_{2i+1}t_{2i+1}-t_{2i+1}X_{2i+1}e_{2i}-e_{2i}X_{2i+1}t_{2i+1}-e_{2i}X_{2i+1}e_{2i}+(z-1) \\ &= -t_{2i+1}X_{2i+1}t_{2i+1}-t_{2i+1}^2e_{2i+1}e_{2i}-e_{2i}e_{2i+1}t_{2i+1}^2+(z-1) \\ &= -t_{2i+1}X_{2i+1}t_{2i+1}-e_{2i+1}e_{2i}-e_{2i}e_{2i+1}+2e_{2i}+(z-1) \end{aligned}$$

where the third equality follows from *Lemma* 4.2 *(iii)*, and noting that $e_{2i}X_{2i+1}e_{2i}=e_{2i}X_{2i}e_{2i}=e_{2i}e_{2i-1}t_{2i}e_{2i}=0$. Substituting this, and the equations

$$-e_{2i}e_{2i+1}\sigma_{2i+1}=-e_{2i}e_{2i+1}t_{2i+1}-e_{2i} \quad \text{and} \quad \sigma_{2i}e_{2i-1}e_{2i}=-t_{2i+1}e_{2i+1}e_{2i}-e_{2i},$$

back into *Equation* (4.4) yields

$$
\begin{aligned}
(z-1) - X_{2i+1} &= -t_{2i+1}X_{2i+1}t_{2i+1} - e_{2i+1}e_{2i} - e_{2i}e_{2i+1} + 2e_{2i} + (z-1) - e_{2i}e_{2i+1}t_{2i+1} \\
&\quad - e_{2i} - t_{2i+1}e_{2i+1}e_{2i} - e_{2i} + e_{2i}e_{2i+1} + e_{2i+1}e_{2i} \\
\iff X_{2i+1} &= t_{2i+1}X_{2i+1}t_{2i+1} + t_{2i+1}e_{2i+1}e_{2i} + e_{2i}e_{2i+1}t_{2i+1}
\end{aligned}
$$

giving *(8)(iv)*.

□

Remark 4.4. Although tedious, one can check that each of the relations given in *Proposition 4.3* above is invariant under the shift of indices given by $2i-1 \mapsto 2(i+m)-1$ and $2i \mapsto 2(i+m)$, for any $m \in \mathbb{Z}_{\geq 0}$.

Remark 4.5. To summarise *Proposition 4.3* in regard to *Steps 1* to *4* discussed at the start of this section, we have picked our generating set to be

$$\mathsf{G} = \{t_i, e_j \mid 2 \leq i \leq 2k-1,\ j \in [2k-1]\},$$

the family of Jucys-Murphy elements X to be the normalisations defined in *Definition 4.1*, and the central elements to be the polynomials $\mathsf{W} = \{z(z-1)^l \mid l \in \mathbb{Z}_{\geq 0}\}$. Also the relations R are all those present in *Proposition 4.3*.

A subtle but important aspect of the choice of generators G is that we have included the element $t_2 = 1 - e_2$ (noting that $\sigma_2 = 1$ is absent in the presentation of the partition algebra given in *Theorem 2.38* since it is clearly redundant). As such, when applying *Step 4* in the next section, it will turn out that $\mathsf{G}^{\text{aff}} \neq \mathsf{G}$, which is a unique difference when compared to the other affine diagram algebras discussed previously. Although the generators e_i will agree with their affine versions in G^{aff}, the generators t_i will not, and we will introduce new notation τ_i to represent their affine versions. One of the reasons this is done is that it allows us to give a clean presentation comparable to other affine diagrams algebras. Also, it will allow us to define an action of $\mathcal{A}_{2k}^{\text{aff}}$ on the tensor space $M \otimes V^{\otimes k}$ which satisfies affinization property *4* (see *Theorem 4.24*), which would not be obtainable otherwise. It appears that this is a good choice to make since it will allow for very natural diagrammtics to come into play, with the new generators τ_i having particularly simple descriptions (see *Proposition 4.43*).

As for the set of relations R, we picked relations (1) through (5) since they provide a presentation of the partition algebra (if one removes t_2 and all relations involving t_2). As for relations (6) through (10), all except (7) are comparable to the relations in [N96, Section 4] which were chosen as the defining relations for $\mathcal{W}_k$. We included relations (7) since they appear to be very natural relations when considering the diagrammatics which will be given in *Proposition 4.43*, and their inclusion allows us to recover affine versions of the recursive relations for Enyang's generators (see *Lemma 4.17*).

We now complete the proof of the above proposition:

Lemma 4.6. The relations

(7) (Braid-like relations)

(i) $t_{2i-2}t_{2i}t_{2i-2} = t_{2i}t_{2i-2}t_{2i}(1 - e_{2i-2})$

(ii) $t_{2i+1}t_{2i-1}t_{2i+1} = t_{2i-1}t_{2i+1}t_{2i-1}(1 - e_{2i})$

(iii) $t_{2i-1}t_{2i}t_{2i-1} = t_{2i} - e_{2i-2}t_{2i} - t_{2i}e_{2i-2}$

(iv) $t_{2i}t_{2i-1}t_{2i} = t_{2i-1} - e_{2i}t_{2i-1} - t_{2i-1}e_{2i}$

are satisfied in $\mathcal{A}_{2k}$, thus completing the proof of *Proposition 4.3*.

Proof. We will prove these relations by showing that they hold under the homomorphism $\Psi_{2k,n}$ given in *Theorem 2.58*, for all $n \geq 0$, and then employ *Lemma 2.59*. To ease notation, for any tuple $\boldsymbol{a} = (a_1, \ldots, a_k) \in [n]^k$, we represent a simple tensor in $V^{\otimes k}$ by a word in the entries of $\boldsymbol{a}$, that is $a_1 \cdots a_k := v_{a_1} \otimes \cdots \otimes v_{a_k}$. For each relation we will have to consider different cases based on the relative values of the entries a_{i-1}, a_i, and a_{i+1}, although most cases are trivial. Also note that $\psi_{n,k}(1 - e_{2i})(\boldsymbol{a}) = \varepsilon_{a_i,a_{i+1}}\boldsymbol{a}$ where $\varepsilon_{a_i,a_{i+1}} = 1 - \delta_{a_i,a_{i+1}}$, with $\delta_{a_i,a_{i+1}}$ the Kronecker Delta.

(7)(i): If $a_{i-1} = a_i$ or $a_i = a_{i+1}$, then it is easy to check that both $t_{2i-2}t_{2i}t_{2i-2}$ and $t_{2i}t_{2i-2}t_{2i}(1 - e_{2i-2})$ will act on $\boldsymbol{a}$ by 0. Assume that $a_i \neq a_{i-1} = a_{i+1}$, then

$$\begin{aligned}\psi_{n,k}(t_{2i-2}t_{2i}t_{2i-2})(\boldsymbol{a}) &= \psi_{n,k}(t_{2i-2}t_{2i})\Big((a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \psi_{n,k}(t_{2i-2})\Big((a_i, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_i a_i \cdots a_k\Big) = 0.\end{aligned}$$

Similarly one can show that $\psi_{n,k}(t_{2i}t_{2i-2}t_{2i}(1 - e_{2i-2}))(\boldsymbol{a}) = 0$ when $a_i \neq a_{i-1} = a_{i+1}$. Lastly assume that a_{i-1}, a_i, and a_{i+1} are pairwise distinct, in particular $\varepsilon_{a,b} = 1$ for any $a, b \in \{a_{i-1}, a_i, a_{i+1}\}$. Then

$$\begin{aligned}\psi_{n,k}(t_{2i}t_{2i-2}t_{2i}(1 - e_{2i-2}))(\boldsymbol{a}) &= \psi_{n,k}(t_{2i}t_{2i-2}t_{2i})(\boldsymbol{a})\\ &= \psi_{n,k}(t_{2i}t_{2i-2})\Big((a_i, a_{i+1})(a_1 \cdots a_{i-1})a_i \cdots a_k\Big)\\ &= \psi_{n,k}(t_{2i}t_{2i-2})\Big((a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \psi_{n,k}(t_{2i})\Big((a_{i-1}, a_i)(a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \Big((a_i, a_{i+1})(a_{i-1}, a_i)(a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \Big((a_{i-1}, a_i)(a_i, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \psi_{n,k}(t_{2i-2})\Big((a_i, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \psi_{n,k}(t_{2i-2}t_{2i})\Big((a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k\Big)\\ &= \psi_{n,k}(t_{2i-2}t_{2i}t_{2i-2})(\boldsymbol{a})\end{aligned}$$

(7)(ii): If $a_i = a_{i+1}$ then its clear that both $t_{2i+1}t_{2i-1}t_{2i+1}$ and $t_{2i-1}t_{2i+1}t_{2i-1}(1 - e_{2i})$

act on $\boldsymbol{a}$ by 0. Assume that $a_i \neq a_{i+1}$ and $a_{i-1} \in \{a_i, a_{i+1}\}$, then

$$\begin{aligned}
\psi_{n,k}(t_{2i+1}t_{2i-1}t_{2i+1})(\boldsymbol{a}) &= \psi_{n,k}(t_{2i+1}t_{2i-1})\Big((a_i, a_{i+1})(a_1 \cdots a_{i-1})a_{i+1}a_ia_{i+2}\cdots a_k\Big) \\
&= \varepsilon_{b,a_{i+1}}\psi_{n,k}(t_{2i+1})\Big((b, a_{i+1})(a_i, a_{i+1})(a_1 \cdots a_{i-1})ba_ia_{i+2}\cdots a_k\Big) \\
&= \varepsilon_{b,a_i}\varepsilon_{b,a_{i+1}}\Big((b, a_i)(b, a_{i+1})(a_i, a_{i+1})(a_1 \cdots a_{i-1})a_iba_{i+2}\cdots a_k\Big)
\end{aligned}$$

where $b = (a_i, a_{i+1})(a_{i-1})$. Since $a_{i-1} \in \{a_i, a_{i+1}\}$, we have that $\varepsilon_{b,a_i}\varepsilon_{b,a_{i+1}} = 0$, and so $t_{2i+1}t_{2i-1}t_{2i+1}$ acts on $\boldsymbol{a}$ by 0. Similarly one can check that $t_{2i-1}t_{2i+1}t_{2i-1}(1 - e_{2i})$ also acts on $\boldsymbol{a}$ by 0. Lastly assume that a_{i-1}, a_i, and a_{i+1} are pairwise distinct. Then

$$\begin{aligned}
\psi_{n,k}(t_{2i-1}t_{2i+1}t_{2i-1}(1 - e_{2i}))(\boldsymbol{a}) &= \psi_{n,k}(t_{2i-1}t_{2i+1}t_{2i-1})(\boldsymbol{a}) \\
&= \psi_{n,k}(t_{2i-1}t_{2i+1})\Big((a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_ia_{i-1}a_{i+1}\cdots a_k\Big) \\
&= \psi_{n,k}(t_{2i-1})\Big((a_{i-1}, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_ia_{i+1}a_{i-1}a_{i+2}\cdots a_k\Big) \\
&= (a_i, a_{i+1})(a_{i-1}, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i+1}a_ia_{i-1}a_{i+2}\cdots a_k \\
&= (a_{i-1}, a_i)(a_{i-1}, a_{i+1})(a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i+1}a_ia_{i-1}a_{i+2}\cdots a_k \\
&= \psi_{n,k}(t_{2i+1})\Big((a_{i-1}, a_{i+1})(a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i+1}a_{i-1}a_ia_{i+2}\cdots a_k\Big) \\
&= \psi_{n,k}(t_{2i+1}t_{2i-1})\Big((a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i-1}a_{i+1}a_ia_{i+2}\cdots a_k\Big) \\
&= \psi_{n,k}(t_{2i+1}t_{2i-1}t_{2i+1})(\boldsymbol{a})
\end{aligned}$$

(7)(iii): Assume $a_i = a_{i+1}$, then it is easy to check that $t_{2i} - e_{2i-2}t_{2i} - t_{2i}e_{2i-2}$ acts on $\boldsymbol{a}$ by 0. Similarly

$$\begin{aligned}
\psi_{n,k}(t_{2i-1}t_{2i}t_{2i-1})(\boldsymbol{a}) &= \varepsilon_{a_{i-1},a_i}\psi_{n,k}(t_{2i-1}t_{2i})\Big((a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_ia_{i-1}a_{i+1}\cdots a_k\Big) \\
&= \varepsilon_{a_{i-1},a_{i+1}}\varepsilon_{a_{i-1},a_i}\psi_{n,k}(t_{2i-1})\Big((a_{i-1}, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i-1}a_{i-1}a_{i+1}\cdots a_k\Big) \\
&= 0.
\end{aligned}$$

Now assume $a_i \neq a_{i+1}$ and $a_{i-1} \in \{a_i, a_{i+1}\}$. Then

$$\psi_{n,k}(t_{2i} - e_{2i-2}t_{2i} - t_{2i}e_{2i-2})(\boldsymbol{a}) = (1 - \delta_{(a_i,a_{i+1})(a_{i-1}),a_i})(a_i, a_{i+1})(a_1 \cdots a_{i-1})a_i \cdots a_k.$$

In either case for $a_{i-1} = a_i$ or $a_{i-1} = a_{i+1}$, we have $\psi_{n,k}(t_{2i} - e_{2i-2}t_{2i} - t_{2i}e_{2i-2})(\boldsymbol{a}) = 0$. Also, from above we see that $\psi_{n,k}(t_{2i-1}t_{2i}t_{2i-1})(\boldsymbol{a}) = 0$ since the factor $\varepsilon_{a_{i-1},a_{i+1}}\varepsilon_{a_{i-1},a_i}$ comes into play. Lastly, assume that a_{i-1}, a_i, and a_{i+1} are pairwise distinct, then it is easy to check that $\psi_{n,k}(e_{2i-2}t_{2i})(\boldsymbol{a}) = \psi_{n,k}(t_{2i}e_{2i-2})(\boldsymbol{a}) = 0$. Also,

$$\begin{aligned}
\psi_{n,k}(t_{2i-1}t_{2i}t_{2i-1})(\boldsymbol{a}) &= \psi_{n,k}(t_{2i-1}t_{2i})\Big((a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_i a_{i-1} a_{i+1} \cdots a_k\Big) \\
&= \psi_{n,k}(t_{2i-1})\Big((a_{i-1}, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_i a_{i-1} a_{i+1} \cdots a_k\Big) \\
&= (a_{i-1}, a_i)(a_{i-1}, a_{i+1})(a_{i-1}, a_i)(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k \\
&= (a_i, a_{i+1})(a_1 \cdots a_{i-2})a_{i-1} \cdots a_k \\
&= \psi_{n,k}(t_{2i})(\boldsymbol{a}) = \psi_{n,k}(t_{2i} - e_{2i-2}t_{2i} - t_{2i}e_{2i-2})(\boldsymbol{a}).
\end{aligned}$$

(7)(iv): This relation can be proved by analogous computations to *(7)(iii)* above.

□

4.1.2 Definition and Basic Results of $\mathcal{A}_{2k}^{\text{aff}}$

In this section we give the definition of the affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$ by generators and relations. As mentioned in *Chapter 1*, the algebra $\mathcal{A}_{2k}^{\text{aff}}$ is to play an analogous role for the partition algebra $\mathcal{A}_{2k}$ as that of the degenerate affine Hecke algebra $\mathcal{H}_k$ for the group algebra of the symmetric group $\mathfrak{S}_k$. We prove some basic properties about $\mathcal{A}_{2k}^{\text{aff}}$ including affinization properties *1* to *3*. We also show that $\mathcal{H}_k \otimes \mathcal{H}_k$ is a quotient. We will prove a variety of relations in $\mathcal{A}_{2k}^{\text{aff}}$ including counterparts to the recursive definition of both the Jucys-Murphy elements and Enyang's generators.

It is worth mentioning that the definition for the affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$ given below is defined as an algebra over the field of complex numbers $\mathbb{C}$. However, one may easily define this algebra over any commutative ring, and many of the structural properties would still be upheld. We have chosen to work over $\mathbb{C}$ for simplicity, and to allow the results of *Section 4.2* and *Section 4.3* to hold concerning the affinization properties *4* and *5* respectively.

Definition 4.7. Let $k \in \mathbb{Z}_{\geq 0}$, we define the *affine partition algebra* $\mathcal{A}_{2k}^{\text{aff}}$ to be the associative unitial $\mathbb{C}$-algebra with set of generators

$$\{e_j, \tau_i, x_r, z_l \mid 2 \leq i \leq 2k-1, 1 \leq j \leq 2k-1, r \in [2k], l \in \mathbb{Z}_{\geq 0}\}$$

and defining relations

(1) (Involutions)
- (i) $\tau_{2i}^2 = 1 - e_{2i}$, for $i \in [k-1]$.
- (ii) $\tau_{2i+1}^2 = 1 - e_{2i}$, for $i \in [k-1]$.

(2) (Braid relations)
- (i) $\tau_{2i+1}\tau_{2j} = \tau_{2j}\tau_{2i+1}$ for $j \neq i+1$.
- (ii) $\tau_{2i+1}\tau_{2j+1} = \tau_{2j+1}\tau_{2i+1}$ for $j \neq i \pm 1$.
- (iii) $\tau_{2i}\tau_{2j} = \tau_{2j}\tau_{2i}$ for $j \neq i \pm 1$.

(iv) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i \in [k-2]$, where $s_j := \tau_{2j}\tau_{2j+1} + e_{2j}$.

(3) (Idempotent relations)

(i) $e_{2i-1}^2 = z_0 e_{2i-1}$ for $i \in [k]$.

(ii) $e_{2i}^2 = e_{2i}$ for $i \in [k-1]$.

(iii) $\tau_{2i+1} e_{2i} = e_{2i}\tau_{2i+1} = 0$ for $i \in [k-1]$.

(iv) $\tau_{2i} e_{2i} = e_{2i}\tau_{2i} = 0$ for $i \in [k-1]$.

(v) $\tau_{2i} e_{2i-1} e_{2i+1} = \tau_{2i+1} e_{2i-1} e_{2i+1}$ for $i \in [k-1]$.

(vi) $e_{2i+1} e_{2i-1} \tau_{2i} = e_{2i+1} e_{2i-1} \tau_{2i+1}$ for $i \in [k-1]$.

(4) (Commutation relations)

(i) $e_i e_j = e_j e_i$, if $|i-j| \geq 2$.

(ii) $\tau_{2i-1} e_{2j-1} = e_{2j-1}\tau_{2i-1}$, if $j \neq i-1, i$.

(iii) $\tau_{2i-1} e_{2j} = e_{2j}\tau_{2i-1}$, if $j \neq i$.

(iv) $\tau_{2i} e_{2j-1} = e_{2j-1}\tau_{2i}$, if $j \neq i, i+1$.

(v) $\tau_{2i} e_{2j} = e_{2j}\tau_{2i}$, if $j \neq i-1$.

(5) (Contractions)

(i) $e_i e_{i+1} e_i = e_i$ and $e_{i+1} e_i e_{i+1} = e_{i+1}$, for $i \in [2n-2]$.

(ii) $\tau_{2i} e_{2i-1} \tau_{2i} = \tau_{2i+1} e_{2i+1} \tau_{2i+1}$, for $i \in [k-1]$.

(iii) $\tau_{2i} e_{2i-2} \tau_{2i} = \tau_{2i-1} e_{2i} \tau_{2i-1}$, for $2 \leq i \leq k-1$.

(6) (Affine Commuting Relations)

(i) $x_i x_j = x_j x_i$ for all $i, j \in [2k]$

(ii) $\tau_i x_j = x_j \tau_i$ for $j \neq i-1, i, i+1$

(iii) $e_i x_j = x_j e_i$ for $j \neq i, i+1$

(7) (Braid-like relations)

(i) $\tau_{2i-2}\tau_{2i}\tau_{2i-2} = \tau_{2i}\tau_{2i-2}\tau_{2i}(1 - e_{2i-2})$.

(ii) $\tau_{2i+1}\tau_{2i-1}\tau_{2i+1} = \tau_{2i-1}\tau_{2i+1}\tau_{2i-1}(1 - e_{2i})$.

(iii) $\tau_{2i-1}\tau_{2i}\tau_{2i-1} = \tau_{2i} - e_{2i-2}\tau_{2i} - \tau_{2i} e_{2i-2}$.

(iv) $\tau_{2i}\tau_{2i-1}\tau_{2i} = \tau_{2i-1} - e_{2i}\tau_{2i-1} - \tau_{2i-1} e_{2i}$.

(8) (Skein-like Relations)

(i) $x_{2i+1} = \tau_{2i} x_{2i-1} \tau_{2i} + e_{2i} e_{2i-1} \tau_{2i} + \tau_{2i} e_{2i-1} e_{2i} - \tau_{2i}$.

(ii) $x_{2i+2} = \tau_{2i+1} x_{2i} \tau_{2i+1} + e_{2i} e_{2i+1} \tau_{2i+1} e_{2i+1} e_{2i} + \tau_{2i+1}$.

(iii) $x_{2i} = \tau_{2i} x_{2i} \tau_{2i} + e_{2i} e_{2i-1} \tau_{2i} + \tau_{2i} e_{2i-1} e_{2i}$.

(iv) $x_{2i+1} = \tau_{2i+1} x_{2i+1} \tau_{2i+1} + e_{2i} e_{2i+1} \tau_{2i+1} + \tau_{2i+1} e_{2i+1} e_{2i}$.

(9) (Anti-symmetry Relations)

(i) $e_i(x_i - x_{i+1}) = 0$ for $i \in [2k-1]$.

(ii) $(x_i - x_{i+1})e_i = 0$ for $i \in [2k-1]$.

(10) (Bubble Relations)

(i) $e_1 x_1^l e_1 = z_l e_1$, for all $l \in \mathbb{N}$.

(ii) z_l is central for all $l \in \mathbb{Z}_{\geq 0}$.

Note we have overloaded the symbols e_i and s_j as elements in $\mathcal{A}_{2k}$ and $\mathcal{A}_{2k}^{\text{aff}}$, however we will show shortly that the mapping $\mathcal{A}_{2k} \to \mathcal{A}_{2k}^{\text{aff}}$ via $z \mapsto z_0$, $e_i \mapsto e_i$, and $s_j \mapsto s_j$ realises the subalgebra $\langle e_i, s_j, z_0 \rangle$ of $\mathcal{A}_{2k}^{\text{aff}}$ as an isomorphic copy of the partition algebra $\mathcal{A}_{2k}$. As discussed in *Remark 4.5*, the defining relations above are those present in *Proposition 4.3*, except where the Jucys-Murphy elements X_i have been replaced with the affine generators x_i, Enyang's generators t_j have been replaced by new generators τ_j, and the polynomials $z(z-1)^l$ have been replaced by central generators z_l. Also, it is worth mentioning that the map $\mathcal{A}_{2k} \to \mathcal{A}_{2k}^{\text{aff}}$ given by $z \mapsto z_0$, $e_i \mapsto e_i$, and $\sigma_j \mapsto \tau_j + e_{2i}$ does not realise an algebra homomorphism. This is since τ_2 is a non-trivial generator in $\mathcal{A}_{2k}^{\text{aff}}$ (as mentioned in *Remark 4.5*), while σ_2 is absent in the presentation of *Theorem 2.38*, as such the braid relation *(E2)(iv)* is not respected under such a map. The subalgebra $\langle e_i, \tau_j, z_0 \rangle$ of $\mathcal{A}_{2k}^{\text{aff}}$ is not isomorphic to the partition algebra, and in fact one can show that this subalgebra is infinite dimensional as an $\mathbb{C}[z_0]$-module (see *Corollary 4.26*).

The Skein-like relations (8) tell us how the affine generators x_i interact with the generators τ_j when they do not commute. We interpret these relations as affine partition algebra counterparts to the defining relations $y_{i+1} = s_i y_i s_i + s_i$ of the degenerate affine Hecke algebra $\mathcal{H}_k$. In the next section we provide a projection of $\mathcal{A}_{2k}^{\text{aff}}$ onto a diagram algebra living within the Heisenberg category. Under this projection the Skein-like relations will correspond to moving a decoration over crossings. As mentioned in *Remark 4.5*, we have also chosen to replace the generators t_j with new generators τ_j. We will show that these elements are not needed to generate the algebra, that is $\mathcal{A}_{2k}^{\text{aff}} = \langle e_i, s_i, x_i, z_l \rangle$. Hence to go from $\mathcal{A}_{2k}$ to $\mathcal{A}_{2k}^{\text{aff}}$ we have indeed just adjoined the new generators $\mathsf{X}^{\text{aff}} = \{x_1, \ldots, x_{2k}\}$ and $\mathsf{W}^{\text{aff}} = \{z_l \mid l \in \mathbb{Z}_{\geq 0}\}$. However, as previously discussed, letting the elements τ_j play the role of generators allows us to give a presentation which is more comparable to its counterparts within the literature.

We begin by showing that the partition algebra is a quotient of the affine partition algebra. This follows naturally from its construction.

Lemma 4.8. We have a surjective $\mathbb{C}$-algebra homomorphism $\rho : \mathcal{A}_{2k}^{\text{aff}} \to \mathcal{A}_{2k}$, given on the generators by $\rho(\tau_i) = t_i$, $\rho(e_i) = e_i$, $\rho(x_i) = X_i$, and $\rho(z_l) = z(z-1)^l$.

Proof. This follows by *Proposition 4.3* and since

$$\mathcal{A}_{2k} = \langle t_i, e_j, z \rangle = \langle \rho(\tau_i), \rho(e_i), \rho(z_0) \rangle.$$

□

Similar to the partition algebra, the affine partition algebra has a corresponding anti-automorphism which fixes the generators.

Lemma 4.9. The mapping $* : \mathcal{A}_{2k}^{\text{aff}} \to \mathcal{A}_{2k}^{\text{aff}}$ which fixes the generators, extended $\mathbb{C}$-linearly, gives an anti-automorphism.

Proof. All defining relations of *Definition 4.7* are symmetric in the generators except relations (7)(i) and (7)(ii). Thus it is clear that the result holds if we can show that e_{2i-2} and $\tau_{2i}\tau_{2i-2}\tau_{2i}$ commute, and that e_{2i} and $\tau_{2i-1}\tau_{2i+1}\tau_{2i-1}$ commute. For the former,

$$\begin{aligned}
\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} &= \tau_{2i}\tau_{2i-2}\tau_{2i-1}e_{2i}\tau_{2i-1}\tau_{2i} \\
&= \tau_{2i}\tau_{2i-1}e_{2i}\tau_{2i-1}\tau_{2i-2}\tau_{2i} \\
&= e_{2i-2}\tau_{2i}\tau_{2i-2}\tau_{2i}
\end{aligned}$$

where the first equaltiy can be deduced from relation (5)(iii) of *Definition 4.7*, the second equality follows since τ_{2i-2} commutes with τ_{2i-1} and e_{2i}, then the last equality again is deducable from relation (5)(iii). Showing that e_{2i} and $\tau_{2i-1}\tau_{2i+1}\tau_{2i-1}$ commute follows in a similar manner.

□

We now seek to show that $\mathcal{A}_{2k}$ is isomorphic to the subalgebra $\langle s_i, e_j, z_0\rangle$ of $\mathcal{A}_{2k}^{\text{aff}}$. We first prove a few helpful relations.

Lemma 4.10. The following relations hold in $\mathcal{A}_{2k}^{\text{aff}}$:

(i) $e_{2i}x_{2i} = e_{2i}e_{2i-1}\tau_{2i}$, and $x_{2i}e_{2i} = \tau_{2i}e_{2i-1}e_{2i}$

(ii) $e_{2i}x_{2i+1} = e_{2i}e_{2i+1}\tau_{2i+1}$, and $x_{2i+1}e_{2i} = \tau_{2i+1}e_{2i+1}e_{2i}$

(ii) $e_{2i}e_{2i-1}\tau_{2i} = e_{2i}e_{2i+1}\tau_{2i+1}$, and $\tau_{2i}e_{2i-1}e_{2i} = \tau_{2i+1}e_{2i+1}e_{2i}$

Proof. *(i)*: Multiplying *(8)(iii)* of *Definition 4.7* on the left by e_{2i} gives

$$e_{2i}x_{2i} = e_{2i}\tau_{2i}x_{2i}\tau_{2i} + e_{2i}e_{2i}e_{2i-1}\tau_{2i} + e_{2i}\tau_{2i}e_{2i-1}e_{2i} = e_{2i}e_{2i-1}\tau_{2i}$$

since $e_{2i}\tau_{2i} = 0$ and $e_{2i}e_{2i} = e_{2i}$. The relation $x_{2i}e_{2i} = \tau_{2i}e_{2i-1}e_{2i}$ follows by $*$.

(ii): Multiplying *(8)(iv)* of *Definition 4.7* on the left by e_{2i} gives

$$e_{2i}x_{2i+1} = e_{2i}\tau_{2i+1}x_{2i+1}\tau_{2i+1} + e_{2i}e_{2i}e_{2i+1}\tau_{2i+1} + e_{2i}\tau_{2i+1}e_{2i+1}e_{2i} = e_{2i}e_{2i+1}\tau_{2i+1}$$

since $e_{2i}\tau_{2i+1} = 0$ and $e_{2i}e_{2i} = e_{2i}$. The relation $x_{2i+1}e_{2i} = \tau_{2i+1}e_{2i+1}e_{2i}$ follows by $*$.

(iii): By *(9)(i), (ii)* of *Definition 4.7*, $e_{2i}x_{2i} = e_{2i}x_{2i+1}$ and $x_{2i}e_{2i} = x_{2i+1}e_{2i}$. So *(i)* and *(ii)* imply *(iii)*.

□

Proposition 4.11. We have a injective $\mathbb{C}$-algebra homomorphism $\iota : \mathcal{A}_{2k} \to \mathcal{A}_{2k}^{\text{aff}}$ given on the generators by $\iota(z) = z_0$, $\iota(s_i) = \tau_{2i}\tau_{2i+1} + e_{2i}$, and $\iota(e_i) = e_i$.

Proof. We first prove that ι is a homomorphism. To do this we show that each of the defining relations of $\mathcal{A}_{2k}$ given in *Theorem 2.33* is respected under ι. We only check the relations involving s_i since the others are accounted for in the definition of $\mathcal{A}_{2k}^{\mathrm{aff}}$.

(HR1)(i):

$$\iota(s_i^2) = (\tau_{2i}\tau_{2i+1}+e_{2i})(\tau_{2i}\tau_{2i+1}+e_{2i}) = \tau_{2i}^2\tau_{2i+1}^2+e_{2i} = (1-e_{2i})(1-e_{2i})+e_{2i} = 1-2e_{2i}+2e_{2i} = 1$$

where we used *(1)*, *(2)(i)*, *(3)(ii)*, *(3)(iii)*, and *(3)(iv)*.

(HR1)(ii): This holds by relations *(2)(i)*, *(2)(ii)*, *(2)(iii)* and *(4)*.

(HR1)(iii): This is precisely *(2)(iv)*.

(HR2)(iii):

$$\iota(e_{2i}s_i) = e_{2i}(\tau_{2i}\tau_{2i+1} + e_{2i}) = e_{2i} = \iota(e_{2i})$$

where we used *(3)(iii)* and *(3)(ii)*. Similarly we have $\iota(s_ie_{2i}) = \iota(e_{2i})$.

(HR2)(iv):

$$\begin{aligned}
\iota(s_ie_{2i-1}e_{2i+1}) &= (\tau_{2i}\tau_{2i+1} + e_{2i})e_{2i-1}e_{2i+1} \\
&= \tau_{2i}\tau_{2i+1}e_{2i-1}e_{2i+1} + e_{2i}e_{2i-1}e_{2i+1} \\
&= \tau_{2i}^2e_{2i-1}e_{2i+1} + e_{2i}e_{2i-1}e_{2i+1} \\
&= e_{2i-1}e_{2i+1} - e_{2i}e_{2i-1}e_{2i+1} + e_{2i}e_{2i-1}e_{2i+1} \\
&= e_{2i-1}e_{2i+1} = \iota(e_{2i-1}e_{2i+1})
\end{aligned}$$

where the third equality follows from *(3)(v)* and the forth from *(1)(i)*. Similarly we have $\iota(e_{2i-1}e_{2i+1}s_i) = \iota(e_{2i-1}e_{2i+1})$.

(HR3)(iv): Follows from commuting relations *(4)(i)*, *(4)(ii)*, and *(4)(iv)*.

(HR3)(v): Follows from commuting relations *(4)(i)*, *(4)(iii)*, and *(4)(v)*.

(HR3)(vi):

$$\begin{aligned}
\iota(s_ie_{2i-1}s_i) &= (\tau_{2i}\tau_{2i+1} + e_{2i})e_{2i-1}(\tau_{2i}\tau_{2i+1} + e_{2i}) \\
&= \tau_{2i+1}\tau_{2i}e_{2i-1}\tau_{2i}\tau_{2i+1} + \tau_{2i+1}\tau_{2i}e_{2i-1}e_{2i} + e_{2i}e_{2i-1}\tau_{2i}\tau_{2+1} + e_{2i} \\
&= \tau_{2i+1}^2e_{2i+1}\tau_{2i+1}^2 + \tau_{2i+1}^2e_{2i+1}e_{2i} + e_{2i}e_{2i+1}\tau_{2i+1}^2 + e_{2i} \\
&= (1 - e_{2i})e_{2i+1}(1 - e_{2i}) + e_{2i+1}e_{2i} - e_{2i} + e_{2i}e_{2i+1} - e_{2i} + e_{2i} \\
&= e_{2i+1} - e_{2i}e_{2i+1} - e_{2i+1}e_{2i} + e_{2i} + e_{2i+1}e_{2i} - e_{2i} + e_{2i}e_{2i+1} \\
&= e_{2i+1} = \iota(e_{2i+1})
\end{aligned}$$

where the third equality follows by *Lemma 4.10 (iii)* and *(5)(ii)*, and the forth from $\tau_{2i+1}^2 = 1 - e_{2i}$.

(HR3)(vii):

$$\begin{aligned}\iota(s_ie_{2i-2}s_i) &= (\tau_{2i+1}\tau_{2i}+e_{2i})e_{2i-2}(\tau_{2i}\tau_{2i+1}+e_{2i})\\ &= \tau_{2i}\tau_{2i+1}e_{2i-2}\tau_{2i+1}\tau_{2i}+\tau_{2i+1}\tau_{2i}e_{2i-2}e_{2i}+e_{2i}e_{2i-2}\tau_{2i}\tau_{2i+1}+e_{2i}e_{2i-2}e_{2i}\\ &= \tau_{2i}\tau_{2i+1}^2e_{2i-2}\tau_{2i}+e_{2i}e_{2i-2}\\ &= \tau_{2i}e_{2i-2}\tau_{2i}+e_{2i}e_{2i-2}\\ &= \tau_{2i-1}e_{2i}\tau_{2i-1}+e_{2i}e_{2i-2}\end{aligned}$$

where the third equality follows since τ_{2i+1} and e_{2i} commute with e_{2i-2}, $e_{2i}^2 = e_{2i}$, and $e_{2i}\tau_{2i} = \tau_{2i}e_{2i} = 0$. We also have

$$\begin{aligned}\iota(s_{i-1}e_{2i}s_{i-1}) &= (\tau_{2i-2}\tau_{2i-1}+e_{2i-2})e_{2i}(\tau_{2i-2}\tau_{2i-1}+e_{2i-2})\\ &= \tau_{2i-1}\tau_{2i-2}e_{2i}\tau_{2i-2}\tau_{2i-1}+\tau_{2i-1}\tau_{2i-2}e_{2i}e_{2i-2}+e_{2i-2}e_{2i}\tau_{2i-2}\tau_{2i-1}+e_{2i-2}e_{2i}e_{2i-2}\\ &= \tau_{2i-1}\tau_{2i-2}^2e_{2i}\tau_{2i-1}+e_{2i}e_{2i-2}\\ &= \tau_{2i-1}e_{2i}\tau_{2i-1}-\tau_{2i-1}e_{2i-2}e_{2i}\tau_{2i-1}+e_{2i}e_{2i-2}\\ &= \tau_{2i-1}e_{2i}\tau_{2i-1}+e_{2i}e_{2i-2}\end{aligned}$$

where the third equality follows since τ_{2i-2} and e_{2i-2} commute with e_{2i}, $e_{2i-2}^2 = e_{2i-2}$, and $e_{2i-2}\tau_{2i-2} = \tau_{2i-2}e_{2i-2} = 0$. The forth equality follows since $\tau_{2i-1}e_{2i-2} = 0$. Comparing to above, we see that $\iota(s_ie_{2i-2}s_i) = \iota(s_{i-1}e_{2i}s_{i-1})$.

Hence we have shown that ι is indeed an algebra homomorphism. For injectivity, note that $\rho\circ\iota = \mathsf{id}$ where $\mathsf{id} : \mathcal{A}_{2k} \to \mathcal{A}_{2k}$ is the identity morphism. Thus ι has a left inverse, and so is injective.

$\square$

Therefore the partition algebra $\mathcal{A}_{2k}$ is both a subalgebra and quotient of the affine partition algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$. Also note that restricting $*$ down to the partition algebra coincides with the anti-automorphism of flipping a diagram. We now seek to give affine counterparts to the recursive definition of the Jucys-Murphy elements in *Definition 2.34*.

Lemma 4.12. The following relations hold in $\mathcal{A}_{2k}^{\mathrm{aff}}$:

(i) $x_{2i+1} = s_ix_{2i-1}s_i + x_{2i}e_{2i} + e_{2i}x_{2i} - x_{2i-1}e_{2i} - \tau_{2i}$

(ii) $x_{2i+2} = s_ix_{2i}s_i - s_ix_{2i}e_{2i} - e_{2i}x_{2i}s_i + e_{2i}x_{2i}e_{2i+1}e_{2i} + \tau_{2i+1}$

Proof. *(i)*: Multiplying on the left and right of *(8)(i)* of *Definition 4.7* by τ_{2i+1} gives

$$\begin{aligned}\tau_{2i+1}x_{2i+1}\tau_{2i+1} &= \tau_{2i+1}\tau_{2i}x_{2i-1}\tau_{2i}\tau_{2i+1} - \tau_{2i+1}\tau_{2i}\tau_{2i+1}\\ &= (s_i-e_{2i})x_{2i-1}(s_i-e_{2i}) - (s_i-e_{2i})\tau_{2i+1}\\ &= s_ix_{2i-1}s_i - e_{2i}x_{2i-1}s_i - s_ix_{2i-1}e_{2i} + x_{2i+1} - \tau_{2i}\\ &= s_ix_{2i-1}s_i - x_{2i-1}e_{2i} - \tau_{2i}\end{aligned}$$

where, in the first equality we used the fact that $\tau_{2i+1}e_{2i} = e_{2i}\tau_{2i+1} = 0$, the second equality we used the substitution $\tau_{2i}\tau_{2i+1} = \tau_{2i+1}\tau_{2i} = s_i - e_{2i}$, and the last equality we used the fact that e_{2i} and x_{2i-1} commute. Now applying *(8)(iv)* of *Definition 4.7* to the left hand side of above, we obtain

$$x_{2i+1} - e_{2i}e_{2i+1}\tau_{2i+1} - \tau_{2i+1}e_{2i+1}e_{2i} = s_i x_{2i-1} s_i - x_{2i-1}e_{2i} - \tau_{2i}.$$

By applying *Lemma 4.10 (ii)* and rearranging, we arrive at *(i)*. Item *(ii)* is proved in an analogous manner were we instead employ relations *(8)(ii)* and *(8)(iii)* of *Definition 4.7*.

□

By rearranging the relations in the above lemma in terms of the generators τ_{2i} and τ_{2i+1}, we immediately obtain the following:

Corollary 4.13. We have that $\mathcal{A}_{2k}^{\mathrm{aff}} = \langle e_i, s_j, x_k, z_l \rangle_{i,j,k,l}$.

□

We will now show that $\mathcal{H}_k \otimes \mathcal{H}_k$ is a quotient of $\mathcal{A}_{2k}^{\mathrm{aff}}$. It is worth mentioning that this result would not be obtainable if we did not replace Enyang's generators t_i with the new generators τ_i when defining $\mathcal{A}_{2k}^{\mathrm{aff}}$.

Proposition 4.14. Let $\boldsymbol{\lambda} = (\lambda_l)_{l=0}^{\infty}$ be any sequence of constants in $\mathbb{C}$. Then we have a surjective $\mathbb{C}$-algebra homomorphism $f_{\boldsymbol{\lambda}} : \mathcal{A}_{2k}^{\mathrm{aff}} \to \mathcal{H}_k \otimes \mathcal{H}_k$ given on the generators by

$$\begin{aligned} f_{\boldsymbol{\lambda}}(\tau_{2i+1}) &= s_i \otimes 1, & f_{\boldsymbol{\lambda}}(x_{2i-1}) &= -1 \otimes y_i, \\ f_{\boldsymbol{\lambda}}(\tau_{2i}) &= 1 \otimes s_i, & f_{\boldsymbol{\lambda}}(x_{2i}) &= y_i \otimes 1, \\ f_{\boldsymbol{\lambda}}(e_i) &= 0, & f_{\boldsymbol{\lambda}}(z_l) &= \lambda_l. \end{aligned}$$

Proof. We show that each of the defining relations of $\mathcal{A}_{2k}^{\mathrm{aff}}$ are upheld under $f_{\boldsymbol{\lambda}}$. Since $f_{\boldsymbol{\lambda}}(e_i) = 0$, one may observe that most of the defining relations involving generators e_i are trivially upheld.

(1)(i): $f_{\boldsymbol{\lambda}}(\tau_{2i}^2) = (1 \otimes s_i)(1 \otimes s_i) = 1 \otimes s_i^2 = 1 = f_{\boldsymbol{\lambda}}(1 - e_{2i})$.

(1)(ii): Similar to *(1)(i)* above.

(2)(i): For any $j \neq i+1$, $f_{\boldsymbol{\lambda}}(\tau_{2i+1}\tau_{2j}) = (s_i \otimes 1)(1 \otimes s_j) = (1 \otimes s_j)(s_i \otimes 1) = f_{\boldsymbol{\lambda}}(\tau_{2j}\tau_{2i+1})$.

(2)(ii): For any $j \neq i \pm 1$,

$$f(\tau_{2i+1}\tau_{2j+1}) = (s_i \otimes 1)(s_j \otimes 1) = s_i s_j \otimes 1 = s_j s_i \otimes 1 = (s_j \otimes 1)(s_i \otimes 1) = f(\tau_{2j+1}\tau_{2i+1}).$$

(2)(iii): Similar to *(2)(ii)* above.

(2)(iv): Noting that $f_{\boldsymbol{\lambda}}(s_i) = f_{\boldsymbol{\lambda}}(\tau_{2i}\tau_{2i+1} + e_{2i}) = f_{\boldsymbol{\lambda}}(\tau_{2i})f_{\boldsymbol{\lambda}}(\tau_{2i+1}) = s_i \otimes s_i$, then

$$f_{\boldsymbol{\lambda}}(s_i s_{i+1} s_i) = s_i s_{i+1} s_i \otimes s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \otimes s_{i+1} s_i s_{i+1} = f_{\boldsymbol{\lambda}}(s_{i+1} s_i s_{i+1}).$$

(6)(i): Follows since $y_1, \ldots, y_k$ pairwise commute.

(6)(ii): Follows since $s_i y_j = y_j s_i$ whenever $j \neq i, i+1$.

(7)(i):

$$f_{\boldsymbol{\lambda}}(\tau_{2i-2}\tau_{2i}\tau_{2i-2}) = 1 \otimes s_{i-1}s_i s_{i-1} = 1 \otimes s_i s_{i-1} s_i = f_{\boldsymbol{\lambda}}(\tau_{2i}\tau_{2i-2}\tau_{2i}) = f_{\boldsymbol{\lambda}}(\tau_{2i}\tau_{2i-2}\tau_{2i}(1 - e_{2i-2}))$$

(7)(ii): Similar to *(7)(i)*.

(7)(iii): $f_{\boldsymbol{\lambda}}(\tau_{2i-1}\tau_{2i}\tau_{2i-1}) = s_{i-1}^2 \otimes s_i = 1 \otimes s_i = f_{\boldsymbol{\lambda}}(\tau_{2i}) = f_{\boldsymbol{\lambda}}(\tau_{2i} - e_{2i-2}\tau_{2i} - \tau_{2i}e_{2i-2})$.

(7)(iv): Similar to *(7)(iii)*.

(8)(i):

$$\begin{aligned}
f_{\boldsymbol{\lambda}}(\tau_{2i}x_{2i-1}\tau_{2i} + e_{2i}e_{2i-1}\tau_{2i} + \tau_{2i}e_{2i-1}e_{2i} - \tau_{2i}) &= f_{\boldsymbol{\lambda}}(\tau_{2i}x_{2i-1}\tau_{2i}) - f_{\boldsymbol{\lambda}}(\tau_{2i}) \\
&= (1 \otimes s_i)(-1 \otimes y_i)(1 \otimes s_i) - 1 \otimes s_i \\
&= -1 \otimes s_i y_i s_i - 1 \otimes s_i \\
&= -1 \otimes (y_{i+1} - s_i) - 1 \otimes s_i \\
&= -1 \otimes y_{i+1} \\
&= f_{\boldsymbol{\lambda}}(x_{2i+1})
\end{aligned}$$

where the fourth equality follows since $s_i y_i s_i = y_{i+1} - s_i$ in $\mathcal{H}_k$.

(8)(ii):

$$\begin{aligned}
f_{\boldsymbol{\lambda}}(\tau_{2i+1}x_{2i}\tau_{2i+1} + e_{2i}e_{2i+1}\tau_{2i+1}e_{2i+1}e_{2i} + \tau_{2i+1}) &= f_{\boldsymbol{\lambda}}(\tau_{2i+1}x_{2i}\tau_{2i+1}) + f_{\boldsymbol{\lambda}}(\tau_{2i+1}) \\
&= (s_i \otimes 1)(y_i \otimes 1)(s_i \otimes 1) + s_i \otimes 1 \\
&= (s_i y_i s_i + s_i) \otimes 1 \\
&= y_{i+1} \otimes 1 \\
&= f_{\boldsymbol{\lambda}}(x_{2i+2})
\end{aligned}$$

where the fourth equality follows since $y_{i+1} = s_i y_i s_i + s_i$ in $\mathcal{H}_k$.

(8)(iii):

$$\begin{aligned}
f_{\boldsymbol{\lambda}}(\tau_{2i}x_{2i}\tau_{2i} + e_{2i}e_{2i-1}\tau_{2i-1} + \tau_{2i-1}e_{2i-1}e_{2i}) &= f_{\boldsymbol{\lambda}}(\tau_{2i}x_{2i}\tau_{2i}), \\
&= (1 \otimes s_i)(y_i \otimes 1)(1 \otimes s_i), \\
&= y_i \otimes 1, \\
&= f_{\boldsymbol{\lambda}}(x_{2i}).
\end{aligned}$$

(8)(iv):

$$\begin{aligned}
f_{\boldsymbol{\lambda}}(\tau_{2i+1}x_{2i+1}\tau_{2i+1} + e_{2i}e_{2i+1}\tau_{2i+1} + \tau_{2i+1}e_{2i+1}e_{2i}) &= f_{\boldsymbol{\lambda}}(\tau_{2i+1}x_{2i+1}\tau_{2i+1}), \\
&= (s_i \otimes 1)(-1 \otimes y_i)(s_i \otimes 1), \\
&= -1 \otimes y_i, \\
&= f_{\boldsymbol{\lambda}}(x_{2i+1}).
\end{aligned}$$

(10)(i) and *(10)(ii)*: Immediate.

Thus $f_{\boldsymbol{\lambda}}$ is a homomorphism. Surjectivity follows as $\langle f_{\boldsymbol{\lambda}}(\tau_i), f_{\boldsymbol{\lambda}}(x_j)\rangle_{i,j} = \mathcal{H}_k \otimes \mathcal{H}_k$. □

Corollary 4.15. The polynomial algebra $\mathbb{C}[x_1, \ldots, x_{2k}]$ is a subalgebra of $\mathcal{A}_{2k}^{\text{aff}}$.

Proof. This is equivalent to proving that all monomials in the generators of the subalgebra $\langle x_1, \ldots, x_{2k}\rangle$ of $\mathcal{A}_{2k}^{\text{aff}}$ are linearly independent, which follows since their images under the algebra homomorphism $f_{\boldsymbol{\lambda}}$ are. □

To end this section we establish a counterpart to the recursive relations of Enyang's generators. To do so, we collect the more technical relations needed into the following lemma:

Lemma 4.16. The following relations hold in $\mathcal{A}_{2k}^{\text{aff}}$:

(i) $e_{2i}x_{2i}e_{2i} = 0$

(ii) $e_{2i}\tau_{2i-1}e_{2i} = 0$

(iii) $e_{2i-2}\tau_{2i}e_{2i-2} = 0$

(iv) $e_{2i-2}\tau_{2i} = e_{2i-2}x_{2i-2}s_ie_{2i-2}s_i$

(v) $\tau_{2i}e_{2i-2} = s_ie_{2i-2}s_ix_{2i-2}e_{2i-2}$

(vi) $\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} = e_{2i-2}x_{2i-2}s_{i-1}e_{2i}e_{2i-1}e_{2i-2}$

(vii) $\tau_{2i-1}e_{2i}s_{i-1} = s_ie_{2i-2}e_{2i-1}e_{2i}s_{i-1}x_{2i-2}e_{2i-2}s_i$

(viii) $\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} = e_{2i-2}\tau_{2i}\tau_{2i-2}\tau_{2i}$

Proof. (i): We have $e_{2i}x_{2i}e_{2i} = e_{2i}e_{2i-1}\tau_{2i}e_{2i} = 0$, by employing *Lemma 4.10 (i)* and *Definition 4.7 (3)(iv)*.

(ii): By rearranging *(7)(iv)* of *Definition 4.7* in terms of τ_{2i-1}, we have that

$$e_{2i}\tau_{2i-1}e_{2i} = e_{2i}(\tau_{2i}\tau_{2i-1}\tau_{2i} + e_{2i}\tau_{2i-1} + \tau_{2i-1}e_{2i})e_{2i} = e_{2i}\tau_{2i-1}e_{2i} + e_{2i}\tau_{2i-1}e_{2i},$$

where we used relation $e_{2i}\tau_{2i} = 0$. Rearranging gives $e_{2i}\tau_{2i-1}e_{2i} = 0$. Item *(iii)* follows in a similar manner.

(iv): We have

$$\begin{aligned}
e_{2i-2}x_{2i-2}s_ie_{2i-2}s_i &= e_{2i-2}x_{2i-1}s_ie_{2i-2}s_i \\
&= e_{2i-2}(s_ix_{2i+1} - s_ix_{2i}e_{2i} - e_{2i}x_{2i} + x_{2i-1}e_{2i} + \tau_{2i+1})e_{2i-2}s_i \\
&= e_{2i-2}s_ix_{2i+1}e_{2i-2}s_i - e_{2i-2}s_ix_{2i}e_{2i}e_{2i-2}s_i - e_{2i-2}e_{2i}x_{2i}e_{2i-2}s_i \\
&\quad + e_{2i-2}x_{2i-1}e_{2i}e_{2i-2}s_i + e_{2i-2}\tau_{2i+1}e_{2i-2}s_i
\end{aligned}$$

where the first equality follows from *(9)(i)* of *Definition 4.7*, and the second from *Lemma 4.12 (i)*. We examine the five terms above:

(1) $e_{2i-2}s_ix_{2i+1}e_{2i-2}s_i = e_{2i-2}s_ie_{2i-2}x_{2i+1}s_i = e_{2i-2}e_{2i}x_{2i+1}s_i$,

(2) $-e_{2i-2}s_ix_{2i}e_{2i}e_{2i-2}s_i = -e_{2i-2}s_ie_{2i-2}x_{2i}e_{2i} = -e_{2i-2}e_{2i}x_{2i}e_{2i} = 0$,

(3) $-e_{2i-2}e_{2i}x_{2i}e_{2i-2}s_i = -e_{2i-2}e_{2i}x_{2i}s_i = -e_{2i-2}e_{2i}x_{2i+1}s_i$,

(4) $e_{2i-2}x_{2i-1}e_{2i}e_{2i-2}s_i = e_{2i-2}x_{2i-1}e_{2i-2}e_{2i}s_i = 0$,

(5) $e_{2i-2}\tau_{2i+1}e_{2i-2}s_i = e_{2i-2}\tau_{2i+1}s_i = e_{2i-2}\tau_{2i}$.

Substituting back into the above equation gives $e_{2i-2}x_{2i-2}s_ie_{2i-2}s_i = e_{2i-2}\tau_{2i}$ as desired.

(v): This follows by applying the anti-automorphism $*$ to *(iv)*.

(vi):

$$\begin{aligned}
\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} &= \tau_{2i}\tau_{2i-2}(s_ie_{2i-2}s_ix_{2i-2}e_{2i-2}),\\
&= \tau_{2i}\tau_{2i-2}s_{i-1}e_{2i}s_{i-1}x_{2i-2}e_{2i-2},\\
&= \tau_{2i}\tau_{2i-1}e_{2i}s_{i-1}x_{2i-2}e_{2i-2},\\
&= \tau_{2i}(\tau_{2i}e_{2i-2}\tau_{2i}\tau_{2i-1})s_{i-1}x_{2i-2}e_{2i-2},\\
&= (1-e_{2i})e_{2i-2}\tau_{2i}\tau_{2i-2}x_{2i-2}e_{2i-2},\\
&= e_{2i-2}\tau_{2i}\tau_{2i-2}x_{2i-2}e_{2i-2},\\
&= e_{2i-2}\tau_{2i}(x_{2i-2}\tau_{2i-2}+e_{2i-3}e_{2i-2}-e_{2i-2}e_{2i-3})e_{2i-2},\\
&= e_{2i-2}\tau_{2i}e_{2i-3}e_{2i-2},\\
&= (e_{2i-2}x_{2i-2}s_ie_{2i-2}s_i)e_{2i-3}e_{2i-2},\\
&= e_{2i-2}x_{2i-2}s_{i-1}e_{2i}e_{2i-1}e_{2i-2}.
\end{aligned}$$

The first equality follows by *(v)*, the fourth from *(5)(iii)* of *Definition 4.7*, the sixth since $e_{2i}\tau_{2i} = 0$, the seventh from *(8)(iii)* of *Definition 4.7*, the ninth from $\tau_{2i-2}e_{2i-2} = 0$ and *(iii)*, and the tenth from *(iv)*.

(vii):

$$\begin{aligned}
s_ie_{2i-2}e_{2i-1}e_{2i}s_{i-1}x_{2i-2}e_{2i-2}s_i &= s_ie_{2i-2}e_{2i-1}s_{i-1}s_ie_{2i-2}s_ix_{2i-2}e_{2i-2}s_i\\
&= s_ie_{2i-2}s_ie_{2i-3}e_{2i-2}x_{2i-2}s_ie_{2i-2}s_i\\
&= s_{i-1}e_{2i}e_{2i-1}e_{2i-2}x_{2i-2}s_{i-1}e_{2i}s_{i-1}\\
&= s_{i-1}e_{2i}e_{2i-1}e_{2i-2}e_{2i-1}\tau_{2i-1}s_{i-1}e_{2i}s_{i-1}\\
&= s_{i-1}e_{2i}e_{2i-1}\tau_{2i-2}e_{2i}s_{i-1}\\
&= s_{i-1}e_{2i}e_{2i-1}e_{2i}\tau_{2i-1}\\
&= s_{i-1}e_{2i}\tau_{2i-1}\\
&= s_{i-1}e_{2i}\tau_{2i-2}s_{i-1}\\
&= s_{i-1}\tau_{2i-2}e_{2i}s_{i-1}\\
&= \tau_{2i-1}e_{2i}s_{i-1}
\end{aligned}$$

where the fourth equality follows from *Lemma 4.10 (i)*.

(viii):

$$\begin{aligned}
\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} &= \tau_{2i}\tau_{2i-2}(\tau_{2i-1}e_{2i}\tau_{2i-1}\tau_{2i}) \\
&= \tau_{2i}(s_{i-1} - e_{2i-2})e_{2i}\tau_{2i-1}\tau_{2i} \\
&= \tau_{2i}s_{i-1}e_{2i}\tau_{2i-1}\tau_{2i} \\
&= \tau_{2i}s_i e_{2i-2}s_i s_{i-1}\tau_{2i-1}\tau_{2i} \\
&= \tau_{2i+1}e_{2i-2}s_i\tau_{2i-2}\tau_{2i} \\
&= e_{2i-2}\tau_{2i}\tau_{2i-2}\tau_{2i}
\end{aligned}$$

where the first equality follows from *(5)(iii)* of *Definition 4.7*, the second since $s_{i-1} = \tau_{2i-1}\tau_{2i-2} + e_{2i-2}$, the third since $e_{2i-2}\tau_{2i-1} = 0$, and the sixth since τ_{2i+1} and e_{2i-2} commute.

□

Lemma 4.17. The following relations hold in $\mathcal{A}_{2k}^{\mathrm{aff}}$:

$$\begin{aligned}
\tau_{2i} = s_{i-1}s_i\tau_{2i-2}s_i s_{i-1} + e_{2i-2}x_{2i-2}s_i e_{2i-2}s_i + s_i e_{2i-2}x_{2i-2}s_i e_{2i-2} \\
- e_{2i-2}x_{2i-2}s_{i-1}e_{2i}e_{2i-1}e_{2i-2} - s_i e_{2i-2}e_{2i-1}e_{2i}s_{i-1}x_{2i-2}e_{2i-2}s_i.
\end{aligned}$$

and

$$\begin{aligned}
\tau_{2i+1} = s_{i-1}s_i\tau_{2i-1}s_i s_{i-1} + s_i e_{2i-2}x_{2i-2}s_i e_{2i-2}s_i + e_{2i-2}x_{2i-2}s_i e_{2i-2} \\
- s_i e_{2i-2}x_{2i-2}s_{i-1}e_{2i}e_{2i-1}e_{2i-2} - e_{2i-2}e_{2i-1}e_{2i}s_{i-1}x_{2i-2}e_{2i-2}s_i.
\end{aligned}$$

Proof. We prove the first relation, the second follows from by multiplying on the left by s_i. We have that

$$\begin{aligned}
s_i\tau_{2i-2}s_i &= (\tau_{2i}\tau_{2i+1} + e_{2i})\tau_{2i-2}(\tau_{2i+1}\tau_{2i} + e_{2i}) \\
&= \tau_{2i}\tau_{2i+1}^2\tau_{2i-2}\tau_{2i} + \tau_{2i-2}e_{2i} \\
&= \tau_{2i}\tau_{2i-2}\tau_{2i} + \tau_{2i-2}e_{2i}
\end{aligned}$$

where the second equality follows since τ_{2i-2} commutes with τ_{2i+1} and $e_{2i}\tau_{2i} = \tau_{2i}e_{2i} = 0$. Substituting the above we get

$$s_{i-1}s_i\tau_{2i-2}s_i s_{i-1} = s_{i-1}\tau_{2i}\tau_{2i-2}\tau_{2i}s_{i-1} + \tau_{2i-1}e_{2i}s_{i-1}. \tag{4.5}$$

For the first term in equation *(10)* we have

$$\begin{aligned}
s_{i-1}\tau_{2i}\tau_{2i-2}\tau_{2i}s_{i-1} &= s_{i-1}(\tau_{2i-2}\tau_{2i}\tau_{2i-2} + \tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2})s_{i-1} \\
&= \tau_{2i-1}\tau_{2i}\tau_{2i-1} + s_{i-1}\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} \\
&= \tau_{2i-1}\tau_{2i}\tau_{2i-1} + \tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} \\
&= \tau_{2i} - e_{2i-2}\tau_{2i} - \tau_{2i}e_{2i-2} + \tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2}
\end{aligned}$$

where the first equality follows by *(7)(i)* of *Definition 4.7*, the second from $s_{i-1}\tau_{2i-2} = \tau_{2i-2}s_{i-1} = \tau_{2i-1}$, the third from *Lemma 4.16 (viii)*, and the fourth from *(7)(iii)* of *Definition 4.7*. Substituting this back into equation *(10)*, and rearranging yields

$$\tau_{2i} = s_{i-1}s_i\tau_{2i-2}s_is_{i-1} + e_{2i-2}\tau_{2i} + \tau_{2i}e_{2i-2} - \tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2} - \tau_{2i-1}e_{2i}s_{i-1}.$$

The desired relation is obtained by applying relations *(iv)* to *(vii)* of *Lemma 4.16*.

□

In summary this subsection has proved that the affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$ satisfies the affinization properties *1* to *3*. We summarises this as a theorem for readablilty purposes, and to stress that the definition of the affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$ presented in *Definition 4.7* has indeed given us something non-trivial with desirable structural properties.

Theorem 4.18. The affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$ presented in *Definition 4.7* satisfies affinization properties *1* to *3*.

Proof. Affinization property *1* is satisfied by *Proposition 4.11* and *Corollary 4.15*. Affinization property *2* is satisfied by comparing *Definition 4.7* with *Proposition 4.3*, and noting that the recursive relations for the Jucys-Murphy elements and Enyang's generators have counterparts in $\mathcal{A}_{2k}^{\text{aff}}$ by *Lemma 4.12* and *Lemma 4.16* respectively. Affinization property *3* is satisfied by *Lemma 4.8*.

□

4.1.3 A Central Subalgebra

In this section we describe a collection of central elements of $\mathcal{A}_{2k}^{\text{aff}}$, and as a result we give a proof of *Theorem 3.5* stated in the previous chapter. We end the section with a conjecture describing the center of $\mathcal{A}_{2k}^{\text{aff}}$.

We begin by proving relations which resemble the relations $s_iy_{i+1} = y_is_i + 1$ in $\mathcal{H}_k$.

Lemma 4.19. The following relations hold:

(i) $\tau_{2i}x_{2i+1} = x_{2i-1}\tau_{2i} + e_{2i-1}e_{2i} - 1$.

(ii) $\tau_{2i+1}x_{2i+2} = x_{2i}\tau_{2i+1} - e_{2i}e_{2i+1} + 1$.

(iii) $\tau_{2i}x_{2i} = x_{2i}\tau_{2i} + e_{2i-1}e_{2i} - e_{2i}e_{2i-1}$.

(iv) $\tau_{2i+1}x_{2i+1} = x_{2i+1}\tau_{2i+1} - e_{2i}e_{2i+1} + e_{2i+1}e_{2i}$.

Proof. *(i)*: Multiplying *(8)(i)* of *Definition 4.7* on the left by τ_{2i} gives

$$\begin{aligned}
\tau_{2i}x_{2i+1} &= \tau_{2i}^2x_{2i-1}\tau_{2i} + \tau_{2i}e_{2i}e_{2i-1}\tau_{2i} + \tau_{2i}^2e_{2i-1}e_{2i} - \tau_{2i}^2 \\
&= (1-e_{2i})x_{2i-1}\tau_{2i} + (1-e_{2i})e_{2i-1}e_{2i} - (1-e_{2i}) \\
&= x_{2i-1}\tau_{2i} + x_{2i-1}e_{2i}\tau_{2i} + e_{2i-1}e_{2i} - e_{2i} - 1 + e_{2i} \\
&= x_{2i-1}\tau_{2i} + e_{2i-1}e_{2i} - 1
\end{aligned}$$

where the second equality follows as $\tau_{2i}^2 = 1 - e_{2i}$ and $t_{2i}e_{2i} = 0$, and the third since x_{2i-1} and e_{2i} commute.

(ii): Multiplying *(8)(ii)* of *Definition 4.7* on the left by τ_{2i+1} gives

$$\begin{aligned}
\tau_{2i+1}x_{2i+2} &= \tau_{2i+1}^2 x_{2i}\tau_{2i+1} + \tau_{2i+1}e_{2i}e_{2i+1}\tau_{2i+1}e_{2i+1}e_{2i} + \tau_{2i+1}^2 \\
&= (1 - e_{2i})x_{2i}\tau_{2i+1} + 1 - e_{2i} \\
&= x_{2i}\tau_{2i+1} - e_{2i}x_{2i}\tau_{2i+1} + 1 - e_{2i} \\
&= x_{2i}\tau_{2i+1} - e_{2i}x_{2i+1}\tau_{2i+1} + 1 - e_{2i} \\
&= x_{2i}\tau_{2i+1} - e_{2i}e_{2i-1}\tau_{2i+1}^2 + 1 - e_{2i} \\
&= x_{2i}\tau_{2i+1} - e_{2i}e_{2i-1} + e_{2i} + 1 - e_{2i} \\
&= x_{2i}\tau_{2i+1} - e_{2i}e_{2i-1} + 1
\end{aligned}$$

where the second equality follows since $\tau_{2i+1}e_{2i} = 0$ and $\tau_{2i+1}^2 = 1 - e_{2i}$, the fourth equality follows since $e_{2i}x_{2i} = e_{2i}x_{2i+1}$, and the fifth equality follows since $e_{2i}x_{2i} = e_{2i}e_{2i-1}\tau_{2i+1}$ (by *Lemma 4.10 (ii)* and *(iii)*).

(iii): Multiplying *(8)(iii)* of *Definition 4.7* on the left by τ_{2i} gives

$$\begin{aligned}
\tau_{2i}x_{2i} &= \tau_{2i}^2 x_{2i}\tau_{2i} + \tau_{2i}e_{2i}e_{2i-1}\tau_{2i} + \tau_{2i}^2 e_{2i-1}e_{2i} \\
&= (1 - e_{2i})x_{2i}\tau_{2i} + (1 - e_{2i})e_{2i-1}e_{2i} \\
&= x_{2i}\tau_{2i} - e_{2i}x_{2i}\tau_{2i} + e_{2i-1}e_{2i} - e_{2i} \\
&= x_{2i}\tau_{2i} - e_{2i}e_{2i-1}\tau_{2i}^2 + e_{2i-1}e_{2i} - e_{2i} \\
&= x_{2i}\tau_{2i} - e_{2i}e_{2i-1} + e_{2i} + e_{2i-1}e_{2i} - e_{2i} \\
&= x_{2i}\tau_{2i} - e_{2i}e_{2i-1} + e_{2i-1}e_{2i}
\end{aligned}$$

where the second equality follows since $\tau_{2i}e_{2i} = 0$ and $\tau_{2i+1}^2 = 1 - e_{2i}$, and the fourth equality follows since $e_{2i}x_{2i} = e_{2i}e_{2i-1}\tau_{2i}$ (by *Lemma 4.10 (i)*).

(iv): Multiplying *(8)(iv)* of *Definition 4.7* on the left by τ_{2i+1} gives

$$\begin{aligned}
\tau_{2i+1}x_{2i+1} &= \tau_{2i+1}^2 x_{2i+1}\tau_{2i+1} + \tau_{2i+1}e_{2i}e_{2i+1}\tau_{2i+1} + \tau_{2i+1}^2 e_{2i+1}e_{2i} \\
&= (1 - e_{2i})x_{2i+1}\tau_{2i+1} + (1 - e_{2i})e_{2i+1}e_{2i} \\
&= x_{2i+1}\tau_{2i+1} - e_{2i}x_{2i+1}\tau_{2i+1} + e_{2i+1}e_{2i} - e_{2i} \\
&= x_{2i+1}\tau_{2i+1} - e_{2i}e_{2i+1}\tau_{2i+1}^2 + e_{2i+1}e_{2i} - e_{2i} \\
&= x_{2i}\tau_{2i} - e_{2i}e_{2i+1} + e_{2i} + e_{2i+1}e_{2i} - e_{2i} \\
&= x_{2i}\tau_{2i} - e_{2i}e_{2i+1} + e_{2i+1}e_{2i}
\end{aligned}$$

where the second equality follows since $\tau_{2i+1}e_{2i} = 0$ and $\tau_{2i+1}^2 = 1 - e_{2i}$, and the fourth equality follows since $e_{2i}x_{2i+1} = e_{2i}e_{2i+1}\tau_{2i+1}$ (by *Lemma 4.10 (ii)*).

□

Lemma 4.20. For any $n \geq 1$, the following relations hold:

(i) $\tau_{2i}x_{2i+1}^n = x_{2i-1}^n\tau_{2i} + \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i-1}^a(e_{2i-1}e_{2i} - 1)x_{2i+1}^b.$

(ii) $\tau_{2i}x_{2i}^n = x_{2i}^n\tau_{2i} + \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i}^a(e_{2i-1}e_{2i} - e_{2i}e_{2i-1})x_{2i}^b.$

(iii) $\tau_{2i+1}x_{2i+2}^n = x_{2i}^n\tau_{2i+1} + \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i}^a(-e_{2i}e_{2i+1} + 1)x_{2i+2}^b.$

(iv) $\tau_{2i+1}x_{2i+1}^n = x_{2i+1}^n\tau_{2i+1} + \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i+1}^a(-e_{2i}e_{2i+1} + e_{2i+1}e_{2i})x_{2i+1}^b.$

Proof. This follows from *Lemma 4.19* by induction on n.

□

We now use the above lemma to prove that a family of polynomials in the affine generators are central. For any $n \in \mathbb{N}$ define

$$p_n = p_n(x_1, \dots, x_{2k}) := x_1^n + x_3^n + \cdots + x_{2k-1} - (x_2^n + x_4^n + \cdots + x_{2k}^n) \tag{4.6}$$

which belongs to the polynomial algebra $\mathbb{C}[x_1, \dots, x_{2k}]$ in $2k$ commuting variables. Comparing to the supersymmetric power sum polynomials q_n of *Definition 3.2*, we have that

$$p_n(x_1, -x_2, \dots, x_{2k-1}, -x_{2k}) = q_n(x_1, \dots, x_{2k}).$$

We also denote by p_n the corresponding polynomial in the affine generators x_i of $\mathcal{A}_{2k}^{\mathrm{aff}}$.

Proposition 4.21. The polynomial p_n is central in $\mathcal{A}_{2k}^{\mathrm{aff}}$ for each $n \in \mathbb{N}$.

Proof. We simply show that each generator of $\mathcal{A}_{2k}^{\mathrm{aff}}$ commutes with p_n. It is immediate that the generators z_l and x_i commute with p_n by *(10)(ii)* and *(6)(i)* of *Definition 4.7*. Let $[-,-]$ denote the commutator bracket.

For the generators e_{2i} we have

$$[p_n, e_{2i}] = (-x_{2i}^n + x_{2i+1}^n)e_{2i} - e_{2i}(-x_{2i}^n + x_{2i+1}^n) = (-x_{2i}^n + x_{2i}^n)e_{2i} - e_{2i}(-x_{2i}^n + x_{2i}^n) = 0,$$

where the first equality follows from the commuting relation *(6)(iii)* of *Definition 4.7*, and the second equality follows since $x_{2i+1}e_{2i} = x_{2i}e_{2i}$ and $e_{2i}x_{2i+1} = e_{2i}x_{2i}$ by *(9)(ii)* and *(9)(i)* of *Definition 4.7*. Similarly we have $[p_n, e_{2i-1}] = 0$.

For the generator τ_{2i}, the commuting relation *(6)(ii)* of *Definition 4.7* tells us that

$$[\tau_{2i}, p_n] = \tau_{2i}(x_{2i-1}^n - x_{2i}^n + x_{2i+1}^n) - (x_{2i-1}^n - x_{2i}^n + x_{2i+1}^n)\tau_{2i}.$$

By acting on relation *(i)* of *Lemma 4.20* by the anti-automorphism $*$, and rearranging, we obtain

$$\tau_{2i}x_{2i-1}^n = x_{2i+1}^n\tau_{2i} - \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i+1}^a(e_{2i}e_{2i-1} - 1)x_{2i-1}^b.$$

Employing this and relations *(i)* and *(ii)* of *Lemma 4.20*, we have

$$\tau_{2i}(x_{2i-1}^n - x_{2i}^n + x_{2i+1}^n) = (x_{2i-1}^n - x_{2i}^n + x_{2i+1}^n)\tau_{2i} + \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i-1}^a(e_{2i-1}e_{2i} - 1)x_{2i+1}^b$$
$$- \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i}^a(e_{2i-1}e_{2i} - e_{2i}e_{2i-1})x_{2i}^b - \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i+1}^a(e_{2i}e_{2i-1} - 1)x_{2i-1}^b$$

Hence showing that $[\tau_{2i}, p_n] = 0$ is equivalent to showing that the three summations above sum to zero. This follows by changing the second summation accordingly:

$$- \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i}^a(e_{2i-1}e_{2i} - e_{2i}e_{2i-1})x_{2i}^b = - \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i}^a e_{2i-1}e_{2i}x_{2i}^b - x_{2i}^a e_{2i}e_{2i-1}x_{2i}^b$$
$$= - \sum_{\substack{a+b=n-1\\a,b\geq 0}} x_{2i-1}^a e_{2i-1}e_{2i}x_{2i+1}^b - x_{2i+1}^a e_{2i}e_{2i-1}x_{2i-1}^b$$

by repeat application of relations *(9)(i)* and *(9)(ii)* of *Definition 4.7*. One shows $[\tau_{2i+1}, p_n] = 0$ analogously.

□

Recall *Theorem 3.5* in the previous chapter which stated that the supersymmetric polynomials in the normalised Jucys-Murphy elements are central in $\mathcal{A}_{2k}(z)$. The above result allows us to prove such a theorem.

Proof of Theorem 3.5. Since $\rho : \mathcal{A}_{2k}^{\text{aff}} \to \mathcal{A}_{2k}$ of *Lemma 4.8* is surjective and $\rho(x_i) = X_i$, then *Proposition 4.21* tells us that the polynomials $p_n(X_1, \ldots, X_{2k})$ belong to the center of $\mathcal{A}_{2k}(z)$, hence

$$\langle p_n(X_1, \ldots, X_{2k}), z \mid n \geq 1\rangle \subseteq Z(\mathcal{A}_{2k}(z)). \tag{4.7}$$

By *Theorem 3.4*, the subalgebra $\mathsf{SSym}[N_1, \ldots, N_{2k}]$ is generated by the supersymmetric powersum polynomials $q_n(N_1, \ldots, N_{2k})$. Comparing *Definition 4.1* with the normalisation of N_i given in *Equation* (3.1), we have

$$N_i = \begin{cases} \frac{z}{2} - 1 - X_i, & \text{if } i \text{ odd,} \\ X_i - \frac{z}{2} + 1, & \text{if } i \text{ even.} \end{cases}.$$

Let $h := \frac{z}{2} - 1$, thus we have that

$$\begin{aligned}
q_n(N_1, \ldots, N_{2k}) &= \sum_{i=1}^{k}(h - X_{2i-1})^n + (-1)^{n+1}\sum_{i=1}^{k}(X_{2i} - h)^n \\
&= \sum_{m=0}^{n}\sum_{i=1}^{k}(-1)^m h^{n-m} X_{2i-1}^m + (-1)^{n+1}\sum_{m=0}^{n}\sum_{i=1}^{k}(-1)^{n-m} h^{n-m} X_{2i}^m \\
&= \sum_{m=0}^{n}\sum_{i=1}^{k} h^{n-m}\left((-1)^m X_{2i-1}^m + (-1)^{2n-m+1} X_{2i}^m\right) \\
&= \sum_{m=0}^{n} h^{n-m}(-1)^m p_n(X_1, \ldots, X_{2k})
\end{aligned}$$

where the fourth equality follows since $(-1)^{2n-m+1} = (-1)^{m+1}$ for any $0 \leq m \leq n$. Thus we have shown that $q_n(N_1, \ldots, N_{2k}) \in \langle p_n(X_1, \ldots, X_{2k}), z \mid n \geq 1\rangle$, and so the result follows from *Theorem 3.4* and *Equation* (4.7).

□

Comparing the centers of other affine counterparts to diagram algebras, see for example [N96, Corollary 4.10] and [DVR11, Theorem 4.2], we have a natural conjecture for the center of $\mathcal{A}_{2k}^{\mathrm{aff}}$:

Conjecture 4.22. $Z(\mathcal{A}_{2k}^{\mathrm{aff}}) = \langle z_l, p_n \mid l, n \in \mathbb{Z}_{\geq 0}\rangle$.

4.2 Extending Schur-Weyl Duality

Recall *Section 2.2.4*, in particular the epimorphism $\Psi_{2k,n} : \mathcal{A}_{2k}(n) \to \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ given in *Theorem 2.58*. In this section we seek to generalise this action to one of $\mathcal{A}_{2k}^{\mathrm{aff}}$ onto the tensor space $M \otimes V^{\otimes k}$ where M is any $\mathbb{C}\mathfrak{S}_n$-module, hence give an affine partition algebra counterpart to *Theorem 2.27* for the degenerate affine Hecke algebra $\mathcal{H}_k$. We end the section by showing that the $\mathbb{C}[z_0]$-subalgebra of $\mathcal{A}_{2k}^{\mathrm{aff}}$ generated by the elements e_i and τ_j is of infinite rank over $\mathbb{C}[z_0]$, which we remarked on earlier within this chapter.

We begin by describing some elements of the group algebra $\mathbb{C}\mathfrak{S}_n$.

Definition 4.23. For any $b \in [n]$ let

$$T_{n,b} = \sum_{a \in [n]\setminus\{b\}} (a, b),$$

the sum of transpositions which act on b non-trivially. Moreover, for any $l \in \mathbb{Z}_{\geq 0}$ let

$$Z_{n,l} = \sum_{b \in [n]} T_{n,b}^l,$$

the l-power sum in the elements $T_{n,b}$ as b runs from 1 to n, where $Z_{n,0} = n$.

One can check that $\pi T_{n,b} = T_{n,\pi(b)}\pi$ for any $\pi \in \mathfrak{S}_n$, and hence see that $Z_{n,l}$ belongs to the center $Z(\mathbb{C}\mathfrak{S}_n)$.

Theorem 4.24. Let $k, n \in \mathbb{Z}_{\geq 0}$, and $\mathcal{A}_{2k}^{\text{aff}}$ be as in *Definition 4.7*. Let $V = \mathbb{C}^n$ the natural representation for $\mathbb{C}\mathfrak{S}_n$, and let M be any $\mathbb{C}\mathfrak{S}_n$-module with basis $\{m_i \mid i \in I\}$ where I is some (possibly infinite) indexing set. Then we have a $\mathbb{C}$-algebra homomorphism

$$\Psi_{2k,n}^{(M)} : \mathcal{A}_{2k}^{\text{aff}} \to \mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$$

defined on the generators by

$$\begin{aligned}
\Psi_{2k,n}^{(M)}(e_{2i-1})(m_{a_0} \otimes v_{\boldsymbol{a}}) &= \sum_{b=1}^{n} m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_{i-1}} \otimes v_b \otimes v_{a_{i+1}} \otimes \cdots \otimes v_{a_k}, \\
\Psi_{2k,n}^{(M)}(e_{2i})(m_{a_0} \otimes v_{\boldsymbol{a}}) &= \delta_{a_i,a_{i+1}} m_{a_0} \otimes v_{\boldsymbol{a}}, \\
\Psi_{2k,n}^{(M)}(\tau_{2i})(m_{a_0} \otimes v_{\boldsymbol{a}}) &= \varepsilon_{a_i,a_{i+1}}(a_i, a_{i+1})(m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_{i-1}}) \otimes v_{a_i} \otimes \cdots \otimes v_{a_k}, \\
\Psi_{2k,n}^{(M)}(\tau_{2i+1})(m_{a_0} \otimes v_{\boldsymbol{a}}) &= \varepsilon_{a_i,a_{i+1}}(a_i, a_{i+1})(m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_{i+1}}) \otimes v_{a_{i+2}} \otimes \cdots \otimes v_{a_k}, \\
\Psi_{2k,n}^{(M)}(x_{2i-1})(m_{a_0} \otimes v_{\boldsymbol{a}}) &= \sum_{\substack{b=1 \\ b \neq a_i}}^{n} (b, a_i)(m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_{i-1}}) \otimes v_{a_i} \otimes \cdots \otimes v_{a_k}, \\
\Psi_{2k,n}^{(M)}(x_{2i})(m_{a_0} \otimes v_{\boldsymbol{a}}) &= \sum_{\substack{b=1 \\ b \neq a_i}}^{n} (b, a_i)(m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_i}) \otimes v_{a_{i+1}} \otimes \cdots \otimes v_{a_k}, \\
\Psi_{2k,n}^{(M)}(z_l)(m_{a_0} \otimes v_{\boldsymbol{a}}) &= (Z_{n,l} m_{a_0}) \otimes v_{\boldsymbol{a}},
\end{aligned}$$

for all $a_0 \in I$ and $\boldsymbol{a} = (a_1, \ldots, a_k) \in [n]^k$, extended $\mathbb{C}$-linearly across $M \otimes V^{\otimes k}$, and where $\varepsilon_{a,b} = 1 - \delta_{a,b}$ with $\delta_{a,b}$ the Kronecker Delta.

Proof. This can been shown by direct computations, much of which are fairly simple but lengthy. To ease notation, for any tuple $\boldsymbol{a} = (a_0, a_1, \ldots, a_k) \in I \times [n]^k$, we represent a simple tensor in $M \otimes V^{\otimes k}$ by $\boldsymbol{a}$ itself, or a word in the entries of $\boldsymbol{a}$, that is to say we write

$$\boldsymbol{a} = a_0 a_1 \ldots a_k := m_{a_0} \otimes v_{a_1} \otimes \cdots \otimes v_{a_k}.$$

We begin by showing that $\Psi_{2k,n}^{(M)}$ is well-defined, that is to confirm that the endomorphisms in the image commute with the diagonal action of $\mathfrak{S}_n$. We do this by showing for any $\pi \in \mathfrak{S}_n$, that $\pi \Psi_{2k,n}^{(M)}(g)\pi^{-1} = \Psi_{2k,n}^{(M)}(g)$ for each generator g of $\mathcal{A}_{2k}^{\text{aff}}$.

Note that the endomorphisms $\Psi_{2k,n}^{(M)}(e_i)$ act trivially on the M component of $M \otimes V^{\otimes k}$, and hence their action of $V^{\otimes k}$ is given by $\Psi_{2k,n}(e_i)$ as described in (1) of *Theorem 2.58*. As such it is clear that $\pi \Psi_{2k,n}^{(M)}(e_i)\pi^{-1} = \Psi_{2k,n}^{(M)}(e_i)$ for any $\pi \in \mathfrak{S}_n$.

For the generators τ_{2i}, given any $\boldsymbol{a} \in I \times [n]^k$ we have

$$\begin{aligned}
\pi\Psi^{(M)}_{2k,n}(\tau_{2i})\pi^{-1}(\boldsymbol{a}) &= \pi\Psi^{(M)}_{2k,n}(\tau_{2i})\Big(\pi^{-1}(a_0a_1\dots a_k)\Big) \\
&= \varepsilon_{a_i,a_{i+1}}\pi\Big((\pi^{-1}(a_i),\pi^{-1}(a_{i+1}))\pi^{-1}(a_0a_1\dots a_{i-1})\pi^{-1}(a_i\dots a_k)\Big) \\
&= \varepsilon_{a_i,a_{i+1}}\pi(\pi^{-1}(a_i),\pi^{-1}(a_{i+1}))\pi^{-1}(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \varepsilon_{a_i,a_{i+1}}(a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \Psi^{(M)}_{2k,n}(\tau_{2i})(\boldsymbol{a})
\end{aligned}$$

noting $\varepsilon_{\pi^{-1}(a_i),\pi^{-1}(a_{i+1})} = \varepsilon_{a_i,a_{i+1}}$. One can show $\pi\Psi^{(M)}_{2k,n}(\tau_{2i+1})\pi^{-1} = \Psi_{2k,n}(\tau_{2i+1})$ in a similar manner.

For the generators x_{2i-1}, given any $\boldsymbol{a} \in I \times [n]^k$ we have

$$\begin{aligned}
\pi\Psi^{(M)}_{2k,n}(x_{2i-1})\pi^{-1}(\boldsymbol{a}) &= \pi\Psi^{(M)}_{2k,n}(x_{2i-1})\Big(\pi^{-1}(a_0a_1\dots a_k)\Big) \\
&= \pi\left(\sum_{\substack{b\in[n]\\ b\neq\pi^{-1}(a_i)}}(b,\pi^{-1}(a_i))\pi^{-1}(a_0a_1\dots a_{i-1})\pi^{-1}(a_i\dots a_k)\right) \\
&= \sum_{\substack{b\in[n]\\ b\neq\pi^{-1}(a_i)}}\pi(b,\pi^{-1}(a_i))\pi^{-1}(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \sum_{\substack{b\in[n]\\ b\neq\pi^{-1}(a_i)}}(\pi(b),a_i)(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \sum_{\substack{b'\in[n]\\ b'\neq a_i}}(b',a_i)(a_0a_1\dots a_{i-1})a_i\dots a_k = \Psi^{(M)}_{2k,n}(x_{2i-1})(\boldsymbol{a})
\end{aligned}$$

by the substitution $b' = \pi(b)$. One can show $\pi\Psi^{(M)}_{2k,n}(x_{2i})\pi^{-1} = \Psi_{2k,n}(x_{2i})$ similarly.

Lastly $\pi\Psi^{(M)}_{2k,n}(z_l)\pi^{-1} = \Psi_{2k,n}(z_l)$ can be seen since $Z_{n,l}$ are central in $\mathbb{C}\mathfrak{S}_n$.

One now needs to confirm that the defining relations of $\mathcal{A}^{\text{aff}}_{2k}$ in *Definition 4.7* are upheld under $\Psi^{(M)}_{n,k}$. As mentioned, these can be shown by direct, but lengthy computations. With this in mind, we will only give details to some of the more difficult relations, namely relations *(8)* through *(10)*. Note that the Braid-like relations *(7)* follow in an analogous manner to the proof of *Lemma 4.6* as the M component does not complicate the argument.

(8)(i): We seek to show that

$$\Psi^{(M)}_{2k,n}(x_{2i+1}) = \Psi^{(M)}_{2k,n}(\tau_{2i}x_{2i-1}\tau_{2i} + e_{2i}e_{2i-1}\tau_{2i} + \tau_{2i}e_{2i-1}e_{2i} - \tau_{2i}).$$

To show this we examine how each term on the hand right side acts on the simple tensor $\boldsymbol{a}$, and show that the sum recovers the action of x_{2i+1}. It proves easier to do this by tackling two cases, when $a_i \neq a_{i+1}$ and when $a_i = a_{i+1}$.

(Case 1): Assume $a_i \neq a_{i+1}$, then for the first term we have

$$\begin{aligned}
\Psi_{2k,n}(\tau_{2i}x_{2i-1}\tau_{2i})(\boldsymbol{a}) &= \Psi_{2k,n}(\tau_{2i}x_{2i-1})\Big((a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k\Big) \\
&= \Psi_{2k,n}(\tau_{2i})\left(\sum_{\substack{b\in[n]\\ b\neq a_i}}(b,a_i)(a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k\right) \\
&= \sum_{\substack{b\in[n]\\ b\neq a_i}}(a_i,a_{i+1})(b,a_i)(a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \sum_{\substack{b\in[n]\\ b\neq a_i}}((a_i,a_{i+1})(b),a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \sum_{\substack{c\in[n]\\ c\neq a_{i+1}}}(c,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&= \sum_{\substack{c\in[n]\\ c\neq a_{i+1}}}(c,a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k + (a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k \\
&\qquad\qquad - (a_i,a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k \\
&= \Psi^{(M)}_{2k,n}(x_{2i+1})(\boldsymbol{a}) + \Psi^{(M)}_{2k,n}(\tau_{2i})(\boldsymbol{a}) - (a_i,a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k
\end{aligned}$$

where we employed the substitution $c = (a_i, a_{i+1})(b)$. For the second term,

$$\begin{aligned}
\Psi^{(M)}_{2k,n}(e_{2i}e_{2i-1}\tau_{2i})(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(e_{2i}e_{2i-1})\Big((a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_i\dots a_k\Big) \\
&= \Psi^{(M)}_{2k,n}(e_{2i})\left(\sum_{b=1}^{n}(a_i,a_{i+1})(a_0a_1\dots a_{i-1})ba_{i+1}\dots a_k\right) \\
&= (a_i,a_{i+1})(a_0a_1\dots a_{i-1})a_{i+1}a_{i+1}\dots a_k \\
&= (a_i,a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k.
\end{aligned}$$

For the third term $\Psi^{(M)}_{2k,n}(\tau_{2i}e_{2i-1}e_{2i})(\boldsymbol{a}) = 0$ since $a_i \neq a_{i+1}$. Thus collectively,

$$
\begin{aligned}
&\Psi^{(M)}_{2k,n}(\tau_{2i}x_{2i-1}\tau_{2i} + e_{2i}e_{2i-1}\tau_{2i} + \tau_{2i}e_{2i-1}e_{2i} - \tau_{2i})(\boldsymbol{a}) \\
&= \Psi^{(M)}_{2k,n}(\tau_{2i}x_{2i-1}\tau_{2i})(\boldsymbol{a}) + \Psi^{(M)}_{2k,n}(e_{2i}e_{2i-1}\tau_{2i})(\boldsymbol{a}) + \Psi^{(M)}_{2k,n}(\tau_{2i}e_{2i-1}e_{2i})(\boldsymbol{a}) - \Psi^{(M)}_{2k,n}(\tau_{2i})(\boldsymbol{a}) \\
&= \Psi^{(M)}_{2k,n}(x_{2i+1})(\boldsymbol{a}) + \Psi^{(M)}_{2k,n}(\tau_{2i})(\boldsymbol{a}) - (a_i, a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k \\
&\qquad\qquad + (a_i, a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k - \Psi^{(M)}_{2k,n}(\tau_{2i})(\boldsymbol{a}) \\
&= \Psi^{(M)}_{2k,n}(x_{2i+1})(\boldsymbol{a}).
\end{aligned}
$$

(Case 2): Assume $a_i = a_{i+1}$. Then $\Psi^{(M)}_{2k,n}(\tau_{2i})(\boldsymbol{a}) = 0$, and so

$$\Psi^{(M)}_{2k,n}(\tau_{2i}x_{2i-1}\tau_{2i} + e_{2i}e_{2i-1}\tau_{2i} + \tau_{2i}e_{2i-1}e_{2i} - \tau_{2i}) = \Psi^{(M)}_{2k,n}(\tau_{2i}e_{2i-1}e_{2i}).$$

Hence we just need to confirm that $\Psi^{(M)}_{2k,n}(x_{2i+1})(\boldsymbol{a}) = \Psi^{(M)}_{2k,n}(\tau_{2i}e_{2i-1}e_{2i})(\boldsymbol{a})$. Well,

$$
\begin{aligned}
\Psi^{(M)}_{2k,n}(\tau_{2i}e_{2i-1}e_{2i})(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(\tau_{2i})\left(\sum_{b=1}^{n} a_0a_1\dots a_{i-1}ba_{i+1}\dots a_k\right) \\
&= \sum_{b=1}^{n}(b, a_{i+1})(a_0a_1\dots a_{i-1})ba_{i+1}\dots a_k \\
&= \sum_{b=1}^{n}(b, a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k = \Psi^{(M)}_{2k,n}(x_{2i+1})(\boldsymbol{a}).
\end{aligned}
$$

The remaining *Skein-like* relations follow by employing similar arguments.

(9)(i): We seek to show $\Psi^{(M)}_{2k,n}(e_ix_i) = \Psi^{(M)}_{2k,n}(e_ix_{i+1})$. We show this first when working with e_{2i}, then with e_{2i-1}. Assume $a_i \neq a_{i+1}$, then

$$
\begin{aligned}
\Psi^{(M)}_{2k,n}(e_{2i}x_{2i})(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(e_{2i})\left(\sum_{\substack{b=1\\ b\neq a_i}}(b, a_i)(a_0a_1\dots a_i)a_{i+1}\dots a_k\right) \\
&= (a_i, a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k,
\end{aligned}
$$

$$
\begin{aligned}
\Psi^{(M)}_{2k,n}(e_{2i}x_{2i+1})(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(e_{2i})\left(\sum_{\substack{b=1\\ b\neq a_{i+1}}}(b, a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k\right) \\
&= (a_i, a_{i+1})(a_0a_1\dots a_i)a_{i+1}\dots a_k.
\end{aligned}
$$

When $a_i = a_{i+1}$ one can check that $\Psi^{(M)}_{2k,n}(e_{2i}x_{2i})(\boldsymbol{a}) = \Psi^{(M)}_{2k,n}(e_{2i}x_{2i+1})(\boldsymbol{a}) = 0$, and thus we have $\Psi^{(M)}_{2k,n}(e_{2i}x_{2i}) = \Psi^{(M)}_{2k,n}(e_{2i}x_{2i+1})$. For odd indices we have

$$\begin{aligned}\Psi^{(M)}_{2k,n}(e_{2i-1}x_{2i-1})(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(e_{2i-1})\left(\sum_{\substack{b=1\\ b\neq a_i}}(b,a_i)(a_0a_1\dots a_{i-1})a_i\dots a_k\right)\\ &= \sum_{c=1}^{n}\sum_{\substack{b=1\\ b\neq a_i}}(b,a_i)(a_0a_1\dots a_{i-1})ca_{i+1}\dots a_k,\end{aligned}$$

$$\begin{aligned}\Psi^{(M)}_{2k,n}(e_{2i-1}x_{2i})(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(e_{2i-1})\left(\sum_{\substack{b=1\\ b\neq a_i}}(b,a_i)(a_0a_1\dots a_i)a_{i+1}\dots a_k\right)\\ &= \sum_{c=1}^{n}\sum_{\substack{b=1\\ b\neq a_i}}(b,a_i)(a_0a_1\dots a_{i-1})ca_{i+1}\dots a_k.\end{aligned}$$

Thus $\Psi^{(M)}_{2k,n}(e_ix_i) = \Psi^{(M)}_{2k,n}(e_ix_{i+1})$. Relation *(9)(ii)* may be shown in a similar manner. *(10)(i)*:

$$\begin{aligned}\Psi^{(M)}_{2k,n}(e_1x_1^le_1)(\boldsymbol{a}) &= \Psi^{(M)}_{2k,n}(e_1x_1^l)\left(\sum_{b=1}^{n}a_0ba_2\dots a_k\right) = \Psi^{(M)}_{2k,n}(e_1)\left(\sum_{b=1}^{n}(T^l_{n,b}a_0)ba_2\dots a_k\right)\\ &= \sum_{c=1}^{n}\left(\sum_{b=1}^{n}T^l_{n,b}a_0\right)ca_2\dots a_k = \sum_{c=1}^{n}(Z_{n,l}a_0)\,ca_2\dots a_k\\ &= \Psi^{(M)}_{2k,n}(z_l)\left(\sum_{c=1}^{n}a_0ca_2\dots a_k\right) = \Psi^{(M)}_{2k,n}(z_l)\left(\Psi^{(M)}_{2k,n}(e_1)(a_0a_1a_2\dots a_k)\right)\\ &= \Psi^{(M)}_{2k,n}(z_le_1)(\boldsymbol{a}).\end{aligned}$$

Lastly relation *(10)(ii)* is simple to check since $Z_{n,l}$ belongs to the center of $\mathbb{C}\mathfrak{S}_n$.

$\square$

Remark 4.25. When one takes $M = V^{\otimes m}$, for some $m \geq 1$, then the image of $\Psi^{(V^{\otimes m})}_{2k,n}$ belongs to $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes(k+m)})$. By comparing the actions of the affine generators x_i and the generators τ_j to that described in *Proposition 2.61* and *Proposition 2.60*, then one can see that

$$\begin{aligned}\Psi^{(V^{\otimes m})}_{2k,n}(x_{2i-1}) &= \Psi_{2k,n}(X_{2(i+m)-1}), & \Psi^{(V^{\otimes m})}_{2k,n}(\tau_{2i-1}) &= \Psi_{2k,n}(t_{2(i+m)-1}),\\ \Psi^{(V^{\otimes m})}_{2k,n}(x_{2i}) &= \Psi_{2k,n}(X_{2(i+m)}), & \Psi^{(V^{\otimes m})}_{2k,n}(\tau_{2i}) &= \Psi_{2k,n}(t_{2(i+m)}).\end{aligned}$$

Hence the action $\Psi_{2k,n}^{(M)}$ has extended the action of the Jucys-Murphy elements and Enyang's generators onto the M component in a manner comparable to the situation of the degenerate affine Hecke algebra $\mathcal{H}_k$ in the classical Schur-Weyl duality. When $n > 2(k+m)$ then by (3) of *Theorem 2.58* we know that

$$\mathcal{A}_{2(k+m)}(n) \cong \mathsf{End}_{\mathfrak{S}_n}(V^{\otimes(k+m)}),$$

and hence $\Psi_{2k,n}^{(V^{\otimes m})}$ sends $x_{2i-1} \mapsto X_{2(i+m)-1}$ and $\tau_{2i-1} \mapsto t_{2(i+m)-1}$, and similarly for the even indexed generators of the same kind. Thus we obtain a homomorphism $\mathcal{A}_{2k}^{\text{aff}} \to \mathcal{A}_{2(k+m)}(n)$ analogous to $\rho_m : \mathcal{H}_k \to \mathbb{C}\mathfrak{S}_{k+m}$ given in *Lemma 2.24*. Under such a homomorphism, the central generators z_l get send to central elements in $\mathcal{A}_{2k}$ similar to the ones examined in [Eny13, Equation 6.3].

Corollary 4.26. The $\mathbb{C}[z_0]$-subalgebra of $\mathcal{A}_{2k}^{\text{aff}}$ generated by

$$\{\tau_i, e_j \mid 2 \leq i \leq 2k-1, j \in [2k-1]\}$$

has infinite rank over $\mathbb{C}[z_0]$.

Proof. Set $d_m := x_1^m e_2 e_1$ for all $m \in \mathbb{N}$. We first show that $d_m \in \langle \tau_i, e_j \rangle$ by induction on m. By *Lemma 4.10* (i) we see that $e_2 x_2 \in \langle \tau_i, e_j \rangle$. Then multiplying on the right by e_1 yeilds $e_2 x_2 e_1 = e_2 x_1 e_1 = x_1 e_2 e_1 = d_1$, where the first equality follows from (9)(ii) of *Definition 4.7*, and the second from (6)(iii). Thus we have the base case $d_1 \in \langle \tau_i, e_j \rangle$. Assume $d_{m'} \in \langle \tau_i, e_j \rangle$ for all $m' < m$ with $m \geq 2$, we seek to show that $d_m \in \langle \tau_i, e_j \rangle$. Well we have that

$$d_{m-1}\tau_2 e_1 = x_1^{m-1} e_2 e_1 \tau_2 e_1 = x_1^{m-1} e_2 x_2 e_1 = x_1^{m-1} e_2 x_1 e_1 = x_1^m e_2 e_1 = d_n,$$

where the second equality follows from *Lemma 4.10* (i), and the remaining equalities follow in the same manner as the base case. Hence $d_m \in \langle \tau_i, e_j \rangle$ completing induction. We now seek to show that the set $\{d_m \mid m \in \mathbb{N}\}$ is $\mathbb{C}[z_0]$-linearly independent in $\mathcal{A}_{2k}^{\text{aff}}$, which will complete the proof. Given any finite subset $I \subset \mathbb{N}$ assume

$$\sum_{m \in I} h_m(z_0) d_m = 0,$$

where $h_m(z_0)$ are polynomials in $\mathbb{C}[z_0]$. We seek to show that $h_m(z_0) = 0$ for each $m \in I$. Let $M \in I$ be the maximal element, and let R be the set of roots for each $h_m(z_0)$. Pick an $n \in \mathbb{N}$ such that $n > M + 1$ and $n \notin R$. Let F be any free $\mathbb{C}\mathfrak{S}_n$-module. For any non-zero $f \in F$ and $(a_1, \ldots, a_k) \in [n]^k$, we have

$$\Psi_{2k,n}^{(F)}(d_m)(f \otimes v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_k}) = \left(T_{n,a_2}^m f\right) \otimes v_{a_2} \otimes v_{a_2} \otimes v_{a_3} \otimes \cdots \otimes v_{a_k}.$$

Since F is free, it will follow that the set $\{\Psi_{2k,n}^{(F)}(d_m) \mid m \in I\}$ is linearly independent in $\mathsf{End}_{\mathfrak{S}_n}(F \otimes V^{\otimes k})$ if the set $\{T_{n,a_2}^m \mid m \in I\}$ is linearly independent in $\mathbb{C}\mathfrak{S}_n$. This follows since $n > M + 1$, and hence T_{n,a_2}^m contains a permutation consisting of a single cycle of size $m+1$, while all permutations in $T^{m'}_{n,a_2}$ must have smaller support whenever $m' < m$. Now consider the equation

$$\Psi^{(F)}_{2k,n}\left(\sum_{m\in I} h_m(z_0)d_m\right) = \sum_{m\in I} h_m(n)\Psi^{(F)}_{2k,n}(d_m) = 0.$$

Since n is not a root of any $h_m(z_0)$, and the set $\{\psi^{(F)}_{n,k}(d_m) \mid m \in I\}$ is linearly independent, we must have that $h_m(z_0) = 0$ for each $m \in I$.

□

4.3 The Heisenberg Category

4.3.1 Definition and Known Results

In this section we recall the definition of the Heisenberg category Heis first defined in [Kho14]. The morphisms in Heis are defined diagrammatically, with the composition given by diagram concatenation. We will present some known results regarding this category which will be helpful for later sections, namely we recall a basis for the morphism spaces, describe certain local relations involving decorations in the diagrammatics, and how the degenerate affine Hecke algebra appears in a certain endomorphism algebra.

The Heisenberg category Heis is a $\mathbb{C}$-linear monoidal category whose objects are generated by the two objects $\uparrow$ and $\downarrow$. We will often use juxtaposition to denote the tensor product of objects, and the monoidal identity object is the empty word $\emptyset$. Hence we view the free monoid $\langle\uparrow,\downarrow\rangle$ as the set of objects in Heis. We require some setup to describe the morphism spaces in Heis. Firstly, we will be working in the planar strip $\mathbb{R}\times[0,1]$ with boundary $\mathbb{B} := \mathbb{R}\times\{1\}\cup\mathbb{R}\times\{0\}$. We call an oriented immersion of the interval $[0,1]$ and circle S^1 a *string* and *loop* respectively. We denote orientations by drawing an arrow on the curve. For any $n, m \geq 0$, we consider the set of points $E(n,m) := [n]\times\{1\}\cup[m]\times\{0\}$ which belong to the boundary $\mathbb{B}$.

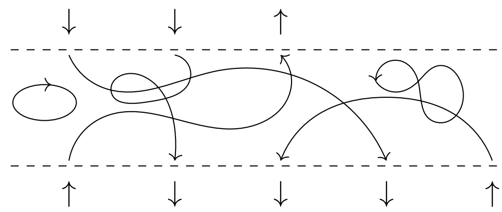
Definition 4.27. For any $n, m \geq 0$ let $\boldsymbol{a} = a_1\cdots a_n$ and $\boldsymbol{b} = b_1\cdots b_m$, for $a_i, b_i \in \{\uparrow,\downarrow\}$, be objects in $\langle\uparrow,\downarrow\rangle$. Colouring the points $(i,1)$ and $(j,0)$ of $E(n,m)$ by the symbols a_i and b_j respectively, we say a set partition of $E(n,m)$ into pairs is an $(\boldsymbol{a},\boldsymbol{b})$-matching if pairs of points in the same row are coloured by opposite arrows, while pairs of points in different rows are coloured by the same arrow.

Definition 4.28. For any $n, m \geq 0$ let $\boldsymbol{a} = a_1\cdots a_n$ and $\boldsymbol{b} = b_1\cdots b_m$, for $a_i, b_i \in \{\uparrow,\downarrow\}$, be objects in $\langle\uparrow,\downarrow\rangle$. We define an $(\boldsymbol{a},\boldsymbol{b})$-diagram to be a finite collection of strings and loops, modulo boundary preserving isotopies, such that the following are upheld:

(D1) The endpoints of the strings induce an $(\boldsymbol{a},\boldsymbol{b})$-matching on $E(n,m)$,

(D2) There are only finitely many points of intersection, and no triple or tangential intersections occur, (D3) The boundary $\mathbb{B}$ does not intersect any loops, and only intersects strings at the endpoints $E(n, m)$.

Example 4.29. Let $\boldsymbol{a} = \downarrow\downarrow\uparrow, \boldsymbol{b} = \uparrow\downarrow\downarrow\downarrow\uparrow \in \langle\uparrow, \downarrow\rangle$, then

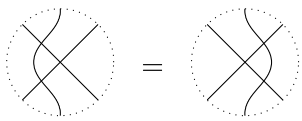


is a $(\boldsymbol{a}, \boldsymbol{b})$-diagram. Isotopic deformation of the interior of $\mathbb{R} \times [0, 1]$ is allowed, and will preserve the relative structure of the ten points of intersection.

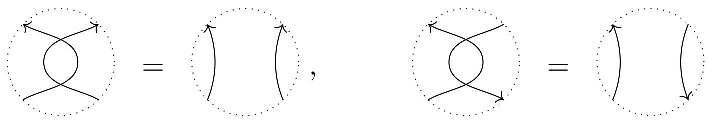
If a loop contains no self-intersections we call it a *bubble*. Bubbles can have *clockwise* or *anti-clockwise* orientation. If the endpoints of a string occur in different rows we call it a *vertical* string, and it has either a *down* or *up* orientation. If the endpoints belong to the same row then we call it an *arc*. Non self-intersecting arcs have either a *clockwise* or *anti-clockwise* orientation. In the above example there are two loops, one of which is a bubble, and four strings, three of which are vertical and one an arc. We call an endpoint of a string a *source* if the arrow of orientation points away from it, and a *target* otherwise.

Definition 4.30. For $n, m \geq 0$, let $\boldsymbol{a} = a_1 \cdots a_n$ and $\boldsymbol{b} = b_1 \cdots b_m$, for $a_i, b_i \in \{\uparrow, \downarrow\}$, be objects in $\langle\uparrow, \downarrow\rangle$. The space of morphisms $\mathsf{Hom}_{\mathsf{Heis}}(\boldsymbol{a}, \boldsymbol{b})$ is the $\mathbb{C}$-vector space generated by the $(\boldsymbol{a}, \boldsymbol{b})$-diagrams modulo the following local relations:

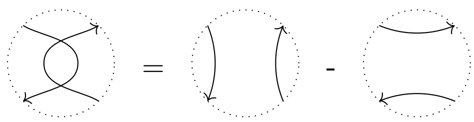
(H1)



(H2)



(H3)



(H4)

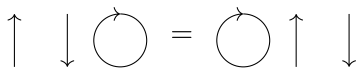
where relation $(H1)$ holds regardless of the orientations.

To apply such a local relation to an $(\boldsymbol{a}, \boldsymbol{b})$-diagram one locates a disk which is isotopic to one of the disks above, then replace such a disk according to the corresponding equation. Note that any of the local relations may be rotated in any way to give an equivalent relation. Relation $(H1)$ tells us that any curve may pass over a crossing, and relations $(H2)$ and $(H3)$ tells us how to pull part orientated curves, where $(H3)$ shows that this can not always be done for free. Relation $(H4)$ tells us that left curls annihilate $(\boldsymbol{a}, \boldsymbol{b})$-diagrams, and that any anti-clockwise bubble may be removed for free.

The composition of morphisms is vertical concatenation of diagrams, and rescaling (and extending $\mathbb{C}$-linearly). We denote composition by juxtaposition of symbols. When $\boldsymbol{a} = \boldsymbol{b}$ we write $\boldsymbol{a}$-diagram instead of $(\boldsymbol{a}, \boldsymbol{a})$-diagram. The morphism space $\mathsf{End}_{\mathsf{Heis}}(\boldsymbol{a})$ is a $\mathbb{C}$-algebra with identity given by the diagram of non-intersecting vertical strings.

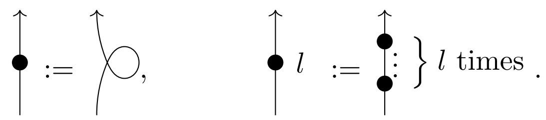
Now for later use, we collect some relations regarding arbitrary $(\boldsymbol{a}, \boldsymbol{b})$-diagrams. The following local relation follows from (H2) and (H3), see also [LS21, (3.5)]:

Lemma 4.31. Clockwise bubbles satisfy the commuting relation

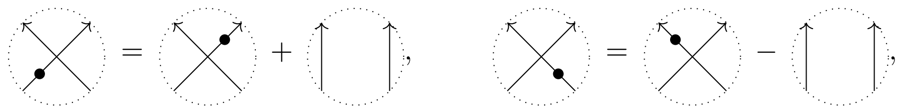


□

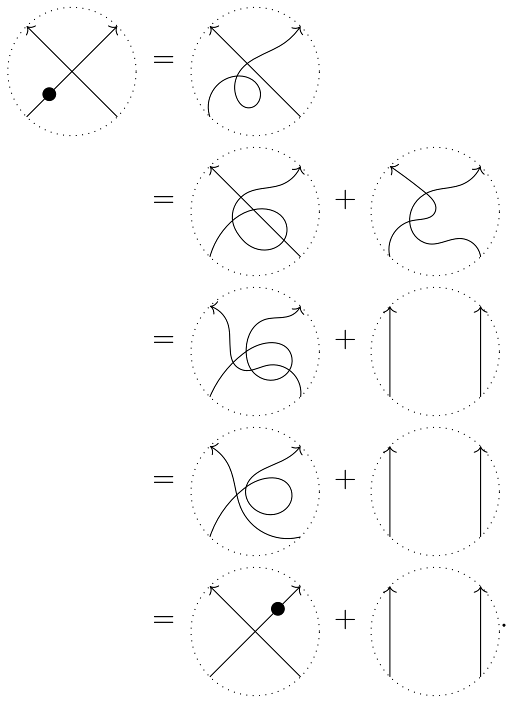
Although left curls annihilate diagrams, right curls do not, and they play an important role in the diagrammatics. We will represent right curls by a decoration, and label such decorations with weights to denote multiplicity:



Lemma 4.32. The following two local relations hold:



Proof. For the first relation we have that



by (H2)

by (H1) and (H2)

by (H2)

The second relation follows in a similar manner.

□

Note that the first relation in the above lemma is comparable to the local relation described in *Equation* (2.6) for the degenerate affine Hecke algebra. In fact the degenerate affine Hecke algebras appear within the endomorphism algebras of certain objects in Heis. For the following result see [Kho14, Proposition 4].

Proposition 4.33. Let $\mathbb{C}[z_0, z_1, \dots]$ be the polynomial $\mathbb{C}$-algebra in countably many commuting variables z_l. For $k \geq 0$ we have an isomorphism of algebras

$$\mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes k}) \cong \mathbb{C}[z_0, z_1, \dots] \otimes \mathcal{H}_k.$$

□

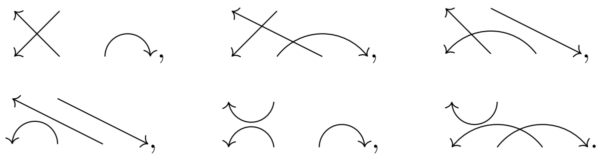
Remark 4.34. Under this isomorphism the variables z_l are acting as clockwise bubbles with l decorations on then. The affine generators and permutations of $\mathcal{H}_k$ corresponding to decorated permutation diagrams as discussed at the end of *Section 2.1.7*, with the exception that the labelling of the vertices $1, 2, \dots, k$ in both rows has been reverse when going from $\mathcal{H}_k$ into $\mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes k})$.

We end this section by recalling a basis for the morphism spaces $\mathsf{Hom}_{\mathsf{Heis}}(\boldsymbol{a}, \boldsymbol{b})$ presented in [Kho14]. We first introduce a few definitions to help us describe this basis in a manner which will lend itself better for later results.

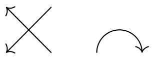
Definition 4.35. For $\boldsymbol{a}, \boldsymbol{b} \in \langle \uparrow, \downarrow \rangle$, we say an $(\boldsymbol{a}, \boldsymbol{b})$-diagram is *simple* if it contains no loops, no self-intersections, and two strings intersect at most once. Let $\mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b})$ denote the set of simple $(\boldsymbol{a}, \boldsymbol{b})$-diagrams, and write $\mathsf{Sim}(\boldsymbol{a})$ for $\mathsf{Sim}(\boldsymbol{a}, \boldsymbol{a})$.

Given words $\boldsymbol{a}, \boldsymbol{b} \in \langle \uparrow, \downarrow \rangle$, let $\boldsymbol{b}^*$ denote the word obtained from $\boldsymbol{b}$ by replacing up arrows with down arrows, and down arrows with up arrows. Let u equal the number of up arrows appearing in $\boldsymbol{a}$ and $\boldsymbol{b}^*$, and d the number of a down arrows. Then by $(D1)$ of *Definition 4.28*, one can deduce that $\mathsf{Hom}_{\mathsf{Heis}}(\boldsymbol{a}, \boldsymbol{b})$ is non-empty if and only if $u = d$. In this situation we have that $|\mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b})| = u!$, since there is one simple $(\boldsymbol{a}, \boldsymbol{b})$-diagram for every $(\boldsymbol{a}, \boldsymbol{b})$-matching. Such a correspondence is given by reading the pairings of endpoints formed from the strings of a simple diagram.

Example 4.36. Consider $\boldsymbol{a} = \uparrow\downarrow$ and $\boldsymbol{b} = \downarrow\uparrow\uparrow\downarrow$. The $6 = 3!$ simple $(\boldsymbol{a}, \boldsymbol{b})$-diagrams are



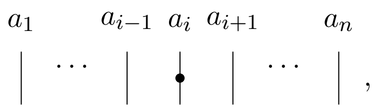
These diagrams are in a one-to-one correspondence with the $(\boldsymbol{a}, \boldsymbol{b})$-matchings of the set of endpoints $E(2,4) = \{(i,1),(j,0) \mid i \in [2], j \in [4]\}$, were we read off the endpoints of the strings. For example we have the correspondence

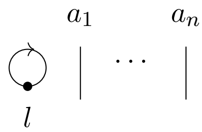


$$\leftrightarrow \quad \Big\{\{(1,1),(2,0)\},\{(2,1),(1,0)\},\{(3,0),(4,0)\}\Big\}.$$

The basis for $\mathsf{Hom}_{\mathsf{Heis}}(\boldsymbol{a}, \boldsymbol{b})$ we describe below is obtained by adding decorations (right curls) and decorated clockwise loops to all the simple $(\boldsymbol{a}, \boldsymbol{b})$-diagrams in a particular manner. We describe this by introducing some basic diagrams and using the composition of diagrams.

Definition 4.37. Let $\boldsymbol{a} = a_1 \cdots a_n \in \langle \uparrow, \downarrow \rangle$ for $a_i \in \{\uparrow, \downarrow\}$. For $i \in [n]$ and $l \in \mathbb{Z}_{\geq 0}$ set

$r_i :=$ 

, and $c_l :=$ 

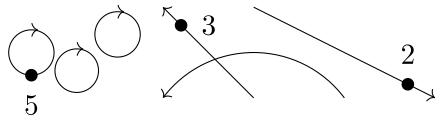
which are $\boldsymbol{a}$-diagrams. The orientations of the strings are taken to match $\boldsymbol{a}$. Although both r_i and c_l depend on $\boldsymbol{a}$, we surpress this fact as it should be clear from context.

Definition 4.38. Given any $\boldsymbol{a} = a_1 \cdots a_n, \boldsymbol{b} = b_1 \cdots b_m \in \langle \uparrow, \downarrow \rangle$ for $a_i, b_j \in \{\uparrow, \downarrow\}$, let $\mathsf{B}(\boldsymbol{a}, \boldsymbol{b})$ be the set of $(\boldsymbol{a}, \boldsymbol{b})$-diagrams of the form

$$c_w^{k_w} \cdots c_1^{k_1} c_0^{k_0} r_1^{s_1} \cdots r_n^{s_n} \alpha r_1^{t_1} \cdots r_m^{t_m}$$

where $\alpha \in \mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b})$, $w, k_l, s_i, t_j \in \mathbb{Z}_{\geq 0}$, and $s_i = t_j = 0$ whenever $(i,1)$ and $(j,0)$ are sources respectively. We write $\mathsf{B}(\boldsymbol{a})$ for $\mathsf{B}(\boldsymbol{a}, \boldsymbol{a})$.

Example 4.39. Given $\boldsymbol{a} = \uparrow\downarrow$ and $\boldsymbol{b} = \downarrow\uparrow\uparrow\downarrow$, an example of an element of $\mathsf{B}(\boldsymbol{a}, \boldsymbol{b})$ is

$c_5 c_0^2 r_1^3 \alpha r_4^2 \quad =$ 

where α is the third simple $(\boldsymbol{a}, \boldsymbol{b})$-diagram in the list given in *Example 4.36*.

The following result is *Proposition 5* of [Kho14].

Theorem 4.40. The set $\mathsf{B}(\boldsymbol{a}, \boldsymbol{b})$ is a basis for $\mathsf{Hom}_{\mathsf{Heis}}(\boldsymbol{a}, \boldsymbol{b})$.

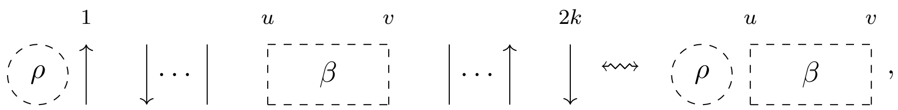
$\square$

Remark 4.41. The description of this basis is analogous to the basis given by regular monomials presented in [N96, Theorem 4.6]. Note that no decorated anti-clockwise bubbles appear. This is due to the fact that any decorated anti-clockwise bubble may be expressed as a linear combination of decorated clockwise bubbles, see for example [Kho14, Proposition 2].

4.3.2 Connections to $\mathcal{A}_{2k}^{\mathrm{aff}}$

In this section we demonstrate a connection between our affine partition algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$ and the Heisenberg category Heis. Namely, we will prove that we have a surjective $\mathbb{C}$-algebra homomorphism $\mathcal{A}_{2k}^{\mathrm{aff}} \to \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$ which sends the generators of $\mathcal{A}_{2k}^{\mathrm{aff}}$ to very natural diagrams. This investigation was inspired from the works of J. Brundan and M. Vargas in [BV21] regarding the *affine partition category* APar which they introduced, and which we will recall in the next section.

It was shown by S. Likeng and A. Savage in [LS21] that there exists a faithful functor from the partition category into the Heisenberg category (see *Theorem* 4.1 and *Theorem* 5.2 in [LS21]). As a result the partition algebra $\mathcal{A}_{2k}$ embeds into $\mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$. To describe this embedding, when drawing a $(\uparrow\downarrow)^{\otimes k}$-diagram, instead of labelling the end-points with arrows, we instead will label the points $(i, 1)$ with i for each $1 \leq i \leq 2k$, since the parity of the label recovers the orientation of the arrow. Also to ease notation we employ the following diagrammatic shorthand for elements of $\mathsf{B}((\uparrow\downarrow)^{\otimes k})$:

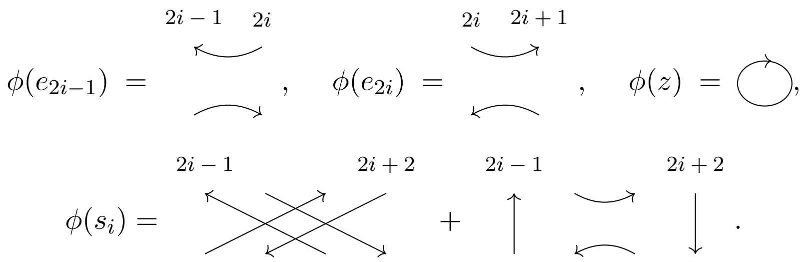


where β is loopless, ρ is a collection of (possibly decorated) clockwise bubbles, and $u, v \in [2k]$. Hence we drop the trivial vertical strings but retain the labels u through v, allowing one to recover the original diagram.

Theorem 4.42 (Theorem 4.1 of [LS21])**.** We have an injective $\mathbb{C}$-algebra homomorphism

$$\phi : \mathcal{A}_{2k} \to \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$$

given on the generators by



□

To go from the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_k$ to the degenerate affine Hecke algebra $\mathcal{H}_k$, one includes the decorations (right curls) into the diagrammatics, which play the role of the affine generators. Thus consider the subalgebra of $\mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$ given by the image of the embedding of the partition algebra $\mathcal{A}_{2k}$ via ϕ, hence the subalgebra $\langle \phi(z), \phi(e_{2i-1}), \phi(e_{2i}), \phi(s_j)\rangle$. If we include decorations into such a subalgebra, i.e. adjoin the generators r_i and c_l of *Definition 4.37*, then we should obtain something akin to an affine counterpart to the partition algebra. Let

$$\widehat{\mathcal{A}}_{2k} := \langle \phi(z), \phi(e_{2i-1}), \phi(e_{2i}), \phi(s_j), r_i, c_l\rangle \subseteq \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k}) \tag{4.8}$$

be this subalgebra. We would hope that $\mathcal{A}_{2k}^{\mathrm{aff}}$ is isomorphic to $\widehat{\mathcal{A}}_{2k}$. Although at this moment we are unable to prove or disprove this, with a significant hurdle being the fact that we lack a basis for $\mathcal{A}_{2k}^{\mathrm{aff}}$. However, what we are able to show in this section is that

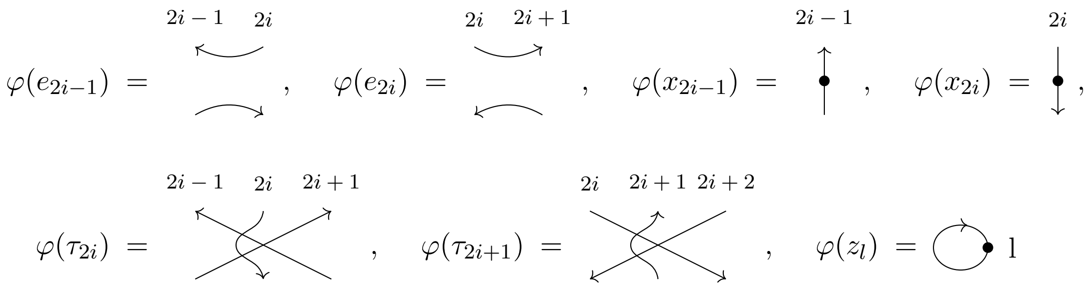
$$\widehat{\mathcal{A}}_{2k} = \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k}),$$

and moreover that this endomorphism algebra is a quotient of $\mathcal{A}_{2k}^{\mathrm{aff}}$.

Proposition 4.43. Let $k \in \mathbb{Z}_{\geq 0}$, then we have a $\mathbb{C}$-algebra homomorphism

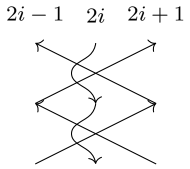
$$\varphi : \mathcal{A}_{2k}^{\mathrm{aff}} \to \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$$

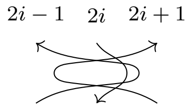
given on the generators by



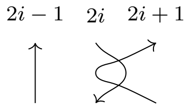
Proof. This will be shown by directly checking that each of the defining relations in *Definition 4.7* is satisfied under the map φ. Most of these are simple to check but lengthy, hence for such relations we do not give full details.

(1)(i):

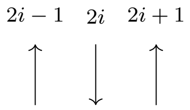
$\varphi(\tau_{2i}^2) =$ 

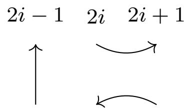
$=$ 

by (H1)

$=$ 

by (H2)

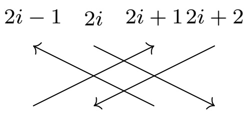
$=$ 

$-$ 

by (H3)

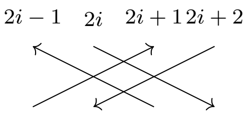
which equals $\varphi(1-e_{2i})$. One can show that relation *(1)(ii)* is upheld in a similar manner.

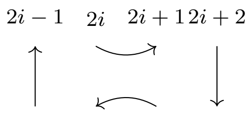
(2): Relation *(2)(i)* is $\tau_{2i+1}\tau_{2j} = \tau_{2j}\tau_{2i+1}$ for all $j \neq i+1$. When $j \neq i$, it is clear to see diagrammatically that this relation is upheld under φ. For case $j = i$, one applies (H1) and then (H2) to see that

$\varphi(\tau_{2i+1}\tau_{2i}) =$ 

$= \varphi(\tau_{2i}\tau_{2i+1}).$

Both relations *(2)(ii)* and *(2)(iii)* can be seen to hold under φ diagrammatically. For relation *(2)(iv)*, we have that

$\varphi(s_i) = \varphi(\tau_{2i+1}\tau_{2i} + e_{2i}) =$ 

$+$ 

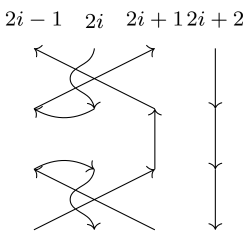
.

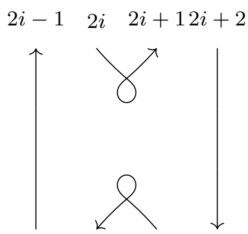
Such elements satisfy the braid relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ by *Theorem 4.42*.

(3): Relation *(3)(i)* is upheld under φ by applying *Lemma 4.31*, and *(3)(ii)* is upheld by (H4). Relations *(3)(iii)* and *(3)(iv)* are upheld by the fact that left curls are annihilated. For relation *(3)(v)*, it is clear that applying (H1) allows one to go from the diagram $\varphi(\tau_{2i}e_{2i-1}e_{2i+1})$ to $\varphi(\tau_{2i+1}e_{2i-1}e_{2i+1})$, and similarly for relation *(3)(vi)*.

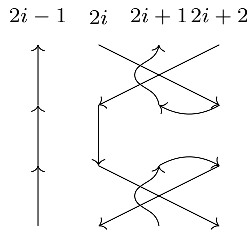
(4): All of these relations follow diagrammatically and from *(3)(iii)* and *(3)(iv)*.

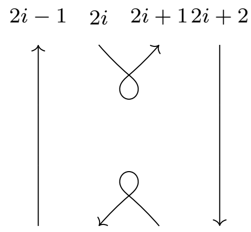
(5): Relation *(5)(i)* is simple to check since the diagrams contain no points of intersection. For *(5)(ii)*, applying (H1) and (H2) we see that

$\varphi(\tau_{2i}e_{2i-1}\tau_{2i}) =$ 

$=$ 

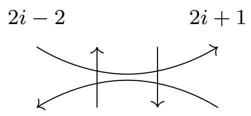
,

$\varphi(\tau_{2i+1}e_{2i+1}\tau_{2i+1}) =$ 

$=$ 

,

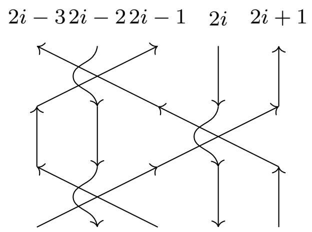
thus $\varphi(\tau_{2i}e_{2i-1}\tau_{2i}) = \varphi(\tau_{2i+1}e_{2i+1}\tau_{2i+1})$. In a similar manner, for relation *(5)(iii)* one can show that

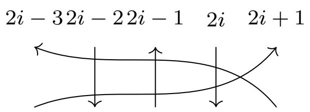
$\varphi(\tau_{2i}e_{2i-2}\tau_{2i}) =$ 

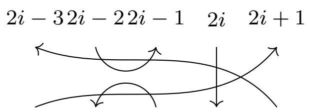
$= \varphi(\tau_{2i-1}e_{2i}\tau_{2i-1})$.

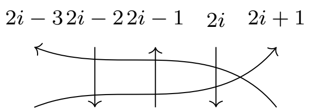
(6): These relations are immediately seen to be upheld diagrammatically.

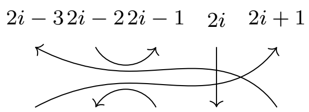
(7)(i): We seek to show $\varphi(\tau_{2i-2}\tau_{2i}\tau_{2i-2}) = \varphi(\tau_{2i}\tau_{2i-2}\tau_{2i}(1 - e_{2i-2}))$. The left hand side gives

$\varphi(\tau_{2i-2}\tau_{2i}\tau_{2i-2}) =$ 

$=$ 

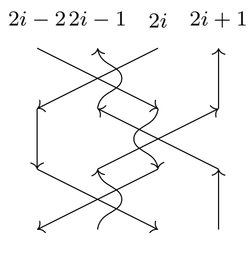
$-$ 

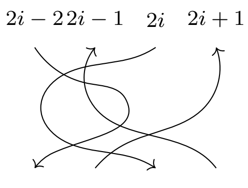
$=$ 

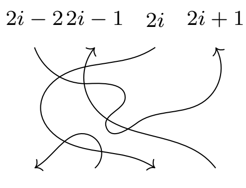
$-$ 

where the second equality follows by applying (H3), and the third equality follows from (H2). By applying (H1) and (H2), one can check that the first term above is $\varphi(\tau_{2i}\tau_{2i-2}\tau_{2i})$ and the second term above is $\varphi(\tau_{2i}\tau_{2i-2}\tau_{2i}e_{2i-2})$, hence *(7)(i)* holds. Relation *(7)(ii)* can be shown in an analogous manner.

(7)(iii): We seek to show that $\varphi(\tau_{2i-1}\tau_{2i}\tau_{2i-1}) = \varphi(\tau_{2i} - e_{2i-2}\tau_{2i} - \tau_{2i}e_{2i-2})$. The left hand side gives

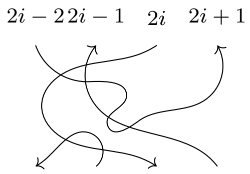
$\varphi(\tau_{2i-1}\tau_{2i}\tau_{2i-1}) =$ 

$=$ 

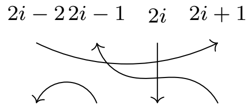
$-$ 

.

By applying (H2) twice and (H1), the second term above straightens out to

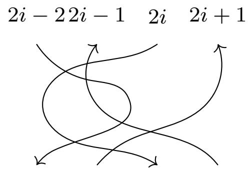


$=$



$= \varphi(\tau_{2i}e_{2i-2})$.

For the first term we get



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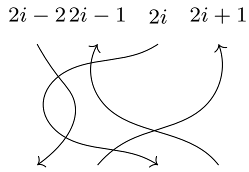

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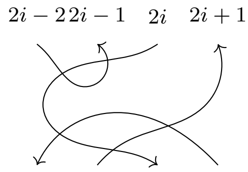

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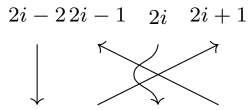

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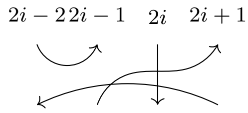


$$= \varphi(\tau_{2i}) + \varphi(e_{2i-2}\tau_{2i})$$

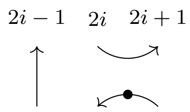
where the first equality follows by applying (H3), and the second equality by (H1) and (H2). Therefore collectively we have show *(7)(iii)*. Relation *(7)(iv)* follows in an analogous manner.

(8)(i): We seek to show that

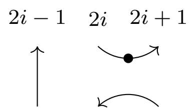
$$\varphi(x_{2i+1}) = \varphi(\tau_{2i}x_{2i-1}\tau_{2i}) + \varphi(e_{2i}e_{i-1}\tau_{2i}) + \varphi(\tau_{2i}e_{2i-1}e_{2i}) - \varphi(\tau_{2i}). \tag{4.9}$$

One can check that

$\varphi(e_{2i}e_{2i-1}\tau_{2i}) =$



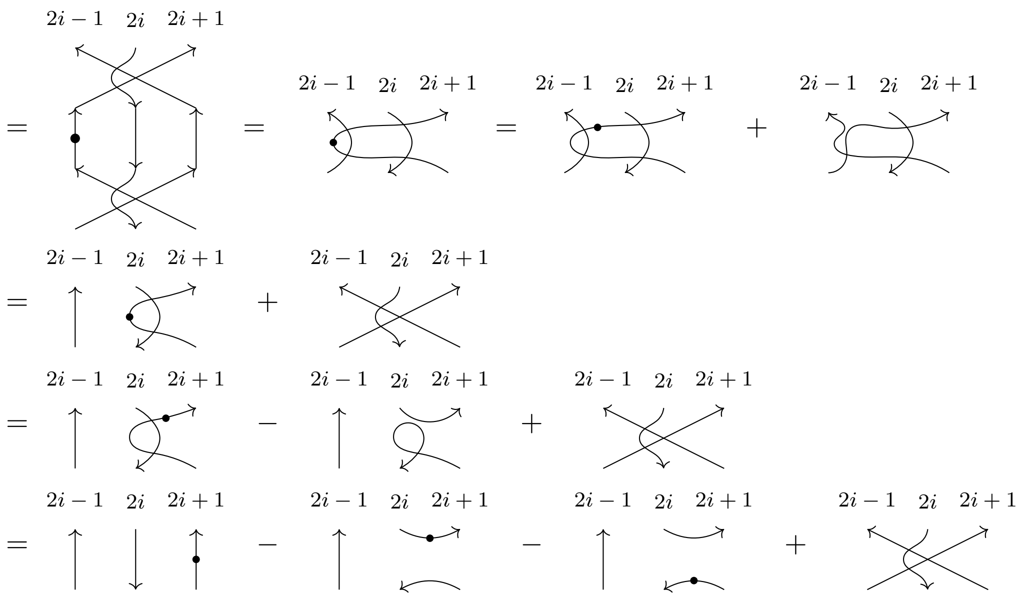
, $\varphi(\tau_{2i}e_{2i-1}e_{2i}) =$



.

By local relations (H1), (H2), and (H3), and applying *Lemma 4.32* (and a 90° clockwise rotation of *Lemma 4.32*), we have

$\varphi(\tau_{2i}x_{2i-1}\tau_{2i}) \;=$



$$= \varphi(x_{2i+1}) - \varphi(\tau_{2i}e_{2i-1}e_{2i}) - \varphi(e_{2i}e_{2i-1}\tau_{2i}) + \varphi(\tau_{2i}).$$

Rearranging yields *Equation* (4.9). The remaining Skein-like relations *(8)(ii)*, *(8)(iii)*, and *(8)(iv)*, following in a similar manner where we employ *Lemma 4.32* to pull the decoration over various oriented crossings.

(9) and *(10)*: These relations are immediately seen to be upheld diagrammatically.

□

Note that restricting φ of *Theorem 4.42* to the partition algebra $\mathcal{A}_{2k}$ gives the morphisms ϕ of *Proposition 4.43*. The remainder of this section seeks to show that the algebra homomorphism φ is surjective. Firstly, from *Theorem 4.40* we know that $\mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$ has a basis given by

$$c_w^{k_w}\dots c_1^{k_1}c_0^{k_0}r_1^{s_1}r_3^{s_3}\dots r_{2k-1}^{s_{2k-1}}\alpha r_2^{t_2}r_4^{t_4}\dots r_{2k}^{t_{2k}}$$

where $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^k)$. Since $\varphi(z_l) = c_l$ and $\varphi(x_i) = r_i$, to prove that φ is surjective it is enough to show that $\mathsf{Sim}((\uparrow\downarrow)^k) \subset \mathsf{Im}(\varphi)$. We will prove this by showing that $\mathsf{Sim}((\uparrow\downarrow)^k) \subset \langle\varphi(e_i), \varphi(\tau_j)\rangle_{i,j} \subset \mathsf{Im}(\varphi)$. We say that a simple diagram is *planar* if no intersections occur among its strings, for example the diagrams $\varphi(e_i)$ are all planar for each $i \in [2k-1]$. The total number of planar diagrams in $\mathsf{Sim}((\uparrow\downarrow)^k)$ is C_{2k}, the $2k$-th Catalan number. These diagrams are precisely oriented versions of the Temperley-Lieb diagrams. The Jones normal form gives a way of writing the Temperley-Lieb diagrams as a product of generators (see [J83], and also [Kau90, Theorem 4.3 and Figure 16]) which does not involve bubbles, and so may be applied here for the elements $\varphi(e_i)$ to show that any planar diagram belongs to $\langle\varphi(e_i)\rangle_i$ and hence to $\mathsf{Im}(\varphi)$. Thus we have the following:

Lemma 4.44. The planar diagrams of $\mathsf{Sim}((\uparrow\downarrow)^{\otimes k})$ belong to $\mathsf{Im}(\varphi)$.

$\square$

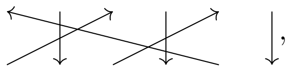
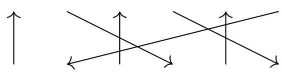
We now define types of simple $(\uparrow\downarrow)^{\otimes k}$-diagrams which are naturally induced from permutations. Recall that any simple $(\uparrow\downarrow)^{\otimes k}$-diagram is determined by the associated $(\uparrow\downarrow)^{\otimes k}$-matching on the set of endpoints $E(2k, 2k)$.

Definition 4.45. Let $\pi \in \mathfrak{S}_k$. Then we define the following simple $(\uparrow\downarrow)^k$-diagrams:

(i) $\pi^\uparrow$ by pairings $\{(2i-1, 0), (2\pi(i)-1, 1)\}$ and $\{(2i, 0), (2i, 1)\}$ of endpoints $E(2k, 2k)$, for each $1 \leq i \leq k$.

(ii) $\pi^\downarrow$ by pairings $\{(2i-1, 0), (2i-1, 1)\}$ and $\{(2\pi(i), 0), (2i, 1)\}$ of endpoints $E(2k, 2k)$, for each $1 \leq i \leq k$.

Example 4.46. For $k = 3$ and $\pi = (1, 2, 3) \in \mathfrak{S}_3$, we have

$\pi^\uparrow =$  , $\quad \pi^\downarrow =$ 

For any $\pi \in \mathfrak{S}_k$, it is shown in [Stem97] that we have a reduced expression of the form

$$\pi = (s_{m_1} s_{m_1+1} \cdots s_{n_1})(s_{m_2} s_{m_2+1} \cdots s_{n_2}) \cdots (s_{m_l} s_{m_l+1} \cdots s_{n_l}),$$

where $k > n_1 > n_2 > \cdots > n_l$ and $n_i \geq m_i$. Noting that $s_i^\uparrow = \varphi(\tau_{2i})$, consider

$$\alpha^\uparrow(\pi) := (s^\uparrow_{m_1} s^\uparrow_{m_1+1} \cdots s^\uparrow_{n_1})(s^\uparrow_{m_2} s^\uparrow_{m_2+1} \cdots s^\uparrow_{n_2}) \cdots (s^\uparrow_{m_l} s^\uparrow_{m_l+1} \cdots s^\uparrow_{n_l}) \in \mathsf{Im}(\varphi).$$

Strings in $\alpha^\uparrow(\pi)$ may intersect one another more than once, but we can resolve such double crossings by pulling strings apart via the local relations (H2). The descending condition on the indices in this reduced expression means we will never need to employ (H3) to pull strings apart, and thus we must have that $\alpha^\uparrow(\pi) = \pi^\uparrow$. Hence $\pi^\uparrow$ belongs to $\mathsf{Im}(\varphi)$. It is easy to see that the image of φ is invariant under taking $180°$ rotation (clockwise or anti-clockwise), and one can note that rotating $\pi^\uparrow$ by $180°$ yields $(\rho\pi\rho^{-1})^\downarrow$ where ρ is the product of transposition $(i, k-i+1)$ for each $i \in [k]$. Thus we also have that $\pi^\downarrow \in \mathsf{Im}(\varphi)$ for all $\pi \in \mathfrak{S}_k$. Hence we have the following:

Lemma 4.47. For any $\pi \in \mathfrak{S}_k$ we have that $\pi^\downarrow$ and $\pi^\uparrow$ belong to $\mathsf{Im}(\varphi)$.

$\square$

To aid upcoming proofs we define a collection of diagrams which loosen the conditions on simple diagrams.

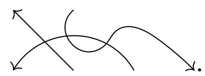
Definition 4.48. Given any $\boldsymbol{a}, \boldsymbol{b} \in \langle\uparrow, \downarrow\rangle$, we call an $(\boldsymbol{a}, \boldsymbol{b})$-diagram *semisimple* if the following hold:

(1) It contains no loops or self intersections.

(2) No top arc intersects a bottom arc.

Let $\mathsf{SSim}(\boldsymbol{a}, \boldsymbol{b})$ denote the set of semisimple $(\boldsymbol{a}, \boldsymbol{b})$-diagrams. We will write $\mathsf{SSim}(\boldsymbol{a})$ instead of $\mathsf{SSim}(\boldsymbol{a}, \boldsymbol{a})$. We have that $\mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b}) \subset \mathsf{SSim}(\boldsymbol{a}, \boldsymbol{b})$.

Example 4.49. Consider $\boldsymbol{a} = \uparrow\downarrow$ and $\boldsymbol{b} = \downarrow\uparrow\uparrow\downarrow$ as was given in *Example 4.36*. The 6 simple $(\boldsymbol{a}, \boldsymbol{b})$-diagrams displayed in that example are all semisimple too. An example of a semisimple $(\boldsymbol{a}, \boldsymbol{b})$-diagram which is not simple would be



The down string and the arc intersect twice, making it non-simple, but such intersections are allowed for semisimple diagrams.

As will be shown in *Proposition 4.52*, any semisimple diagram will decompose into a linear combination of simple diagrams (hence the naming convention).

Any diagram $\alpha \in \mathsf{SSim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$ contains precisely $k+l$ strings, and the endpoints of these strings induce an $((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$-matching of the endpoints $E(2k, 2l)$. We let $\overline{\alpha}$ denote the unique simple diagram corresponding to such a matching (recalling the discussion after *Definition 4.35*).

Lemma 4.50. Given any simple diagram $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$, there exists $\pi \in \mathfrak{S}_k$, $\sigma \in \mathfrak{S}_l$, and a planar diagram $\beta \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$ such that $\pi^\uparrow \beta \sigma^\downarrow$ is semisimple and

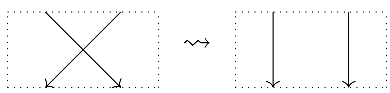
$$\alpha = \overline{\pi^\uparrow \beta \sigma^\downarrow}.$$

Proof. Given any simple diagram $\gamma \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$ let $(2i, 0), (2j, 0) \in E(2k, 2l)$ (respectively $(2i-1, 1), (2i-j, 1)$) be two $\downarrow$ (respectively $\uparrow$) endpoints in the bottom row (respectively top row) of γ. Let γ' be the simple diagram obtained from γ by permuting these two endpoints around. It can be seen that $\gamma(i,j)^\downarrow$ (respectively $(i,j)^\uparrow\gamma$) is semisimple as long as the permutation doesn't swap the orianetation of an arc around, since that is the only way a self intersection can occur. In this case, one can see that

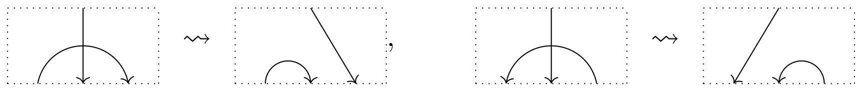
$$\gamma' = \overline{\gamma(i,j)^\downarrow} \quad \left(\text{respectively} = \overline{(i,j)^\uparrow\gamma}\right).$$

Hence to prove this lemma it is enough to show that we can reach a planar diagram β from α by repeatably permuting the endpoints in the bottom row coloured by $\downarrow$, and top row coloured by $\uparrow$, in such a way that the orientations of arcs are preserved. We focus on the bottom row, where the top row will follow in the same manner by a $180°$ rotation of the diagrammatics. Starting with α we remove intersections one at a time by employing a suitable permutation of endpoints. There are a few cases to consider, and in each such case the endpoints of the strings in the following diagrams will be arbitrary:

(Case 1): Crossing of two down strings:

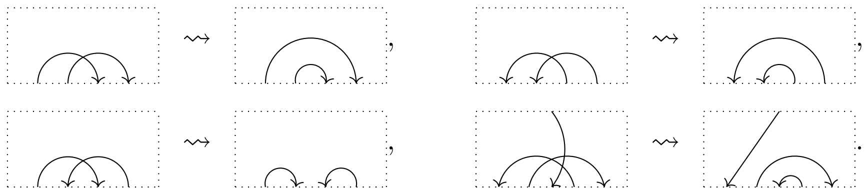


(Case 2): Crossing of a down string with an clockwise/anti-clockwise arc:



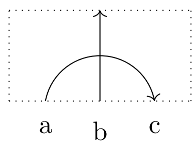
Note in either situation the orientation of the arc is preserved by the permutation.

(Case 3): Crossing of two arcs: There are four cases based on the orientations of the two arcs given by

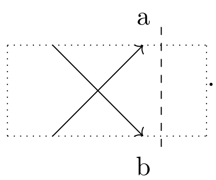


noting that in the last case such a down string must exist. Again, the orientations of the arcs are preserved under the permutation of endpoints in all four of the above situations.

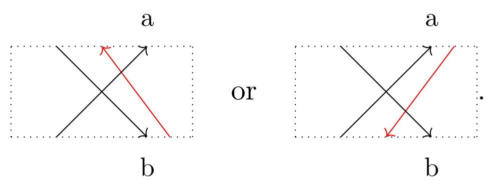
In all three of the cases above, it can be seen that the new simple diagram we obtain after the permutation of endpoints has strictly less number of intersections. We claim that applying the moves above on the bottom row, and their $180°$ counterparts on the top row, until all such intersections are removed will yield a planar diagram. For contradiction, suppose this is not the case. Thus even after removing all such types of intersections, the diagram still contains some other type of intersection. The other such intersections are either between an up string and arc on the bottom row, a down string and arc on the top row, or an up string and a down string. The former two are $180°$ counterparts to one another, hence we only need to consider one such type. Firstly, if an up string intersects a clockwise arc on the bottom we have



Note that the parity of the number of endpoints on the bottom row strictly between a and b must be different to the partity of endpoints strictly between b and c. Thus one can deduce that such an endpoint must be a target to a string which intersects the arc, and such an intersection would be accounted for by Case 2 or 3, hence a contradiction. The same argument can be used to show that the case of an up string intersecting an anti-clockwise arc on the bottom is also impossible. Note all intersections involving arcs have now been accounted for. Lastly assume an up string intersects a down string. We have two cases, one of which is



The dashed vertical line is simply an aid for arguments to come, and has been drawn so that the endpoints a and b are the closest endpoints to its left. The other case is given by rotating the above by 180° and will follow analgously. The parity of the number of endpoints to the right of a is odd, while the parity of the number of endpoints to the right of b is even. This implies that there exists a string s such that one of its endpoints belongs to the right of the dashed line, while the other belongs to the left. Moreover, since the right-most endpoint on the top and bottom row are coloured by $\downarrow$, we can say that the endpoint of s which is to the right of the dashed line is a source while the endpoint to the left is a target. So s must intersect one of the above strings, and must be a vertical string since all intersections with arcs are accounted for. Hence, colouring the string s in red we have



Note that s may intersect both of the other strings and not just one, but it is always forced to intersect the string depicted. As such each situation exhibits an intersection accounted for in Cases 1 (or its 180° counterpart), giving the desired contradiction. Thus removing all intersections of the types presented in Cases 1 to 3 (and their 180° rotated counterparts) will result in a planar diagram, completing the proof.

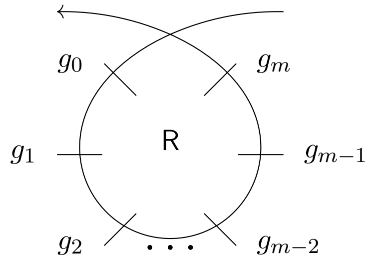
□

Let R be an open subspace of $\mathbb{R} \times [0,1]$ and let α be an $(\boldsymbol{a},\boldsymbol{b})$-diagram. Examining α locally in R will give a configuration of curve segments, and we refer to such as a *region* of α. Within a given region we treat distinct curve segments as different curves, even if in α itself the two segments belong to the same curve. In particular, if in R two distinct curve segments intersect one another, and in α these two segments belong to the same curve, we will not call such an intersection a self-intersection in R, but it is a self-intersection in α.

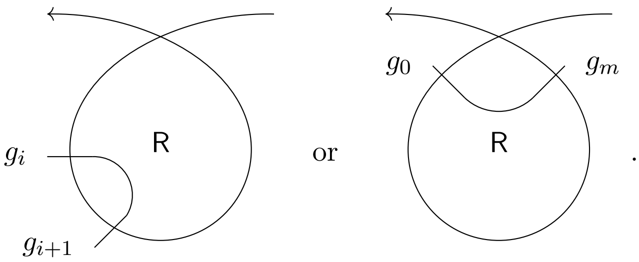
Recall that the local relation (H4) tells us that if a left curl appears in a diagram we may annihilate such a diagram. This relation asks that the region enclosed in the curl is absent of any other strings. The following result shows that even if such a region is non-empty, as long as its contains no loops or self-intersections, we can annihilate the diagram.

Lemma 4.51. Let α be an $(\boldsymbol{a},\boldsymbol{b})$-diagram containing a left curl where the region bounded by the curl contains no loops or self-intersecting curve segments, then $\alpha = 0$.

Proof. By assumption α contains a configuration of the form

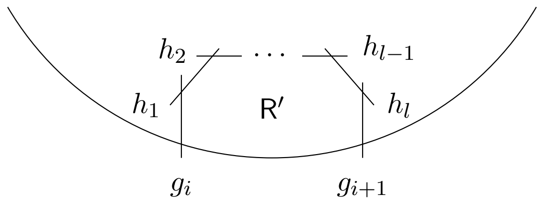


where we let R denote the interior region bounded by the curl, which contains no loops or self-intersecting curve segments, and $g_0, \ldots, g_m$ account for all the intersections which occur on the curl. Note we have only drawn the segments of the g_i's which realise the intersection on the curl. We prove the result by induction on the number of intersections occurring in R. Assume that no intersections occur in R, hence R gives a planar configuration of strings. One can deduce that there exists neighbours $g_{i \bmod (m+1)}$ and $g_{(i+1)\mathrm{mod}(m+1)}$ such that either

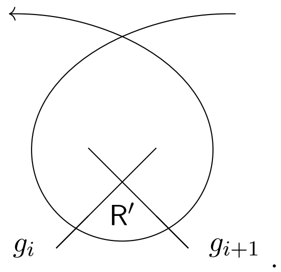


In the former situation, since we are dealing with a left curl, one can check that regardless of the orientation of the depicted string in R, it may be pulled outside the curl by (H2). For the latter situation we may employ (H1) to pull the string out of the curl over the crossing at the top. Continually pulling out such strings one at a time will result in making R empty, and then applying (H4) gives $\alpha = 0$.

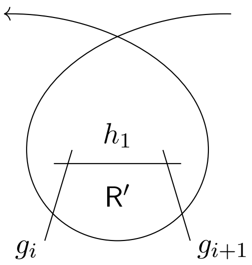
Now suppose that the result holds whenever R contains n or less intersections for some $n \geq 0$, and assume that R contains $n+1$ intersections. It is clear that there must exist an empty region R' in R bounded by the curl and various segments. Diagrammatically we have



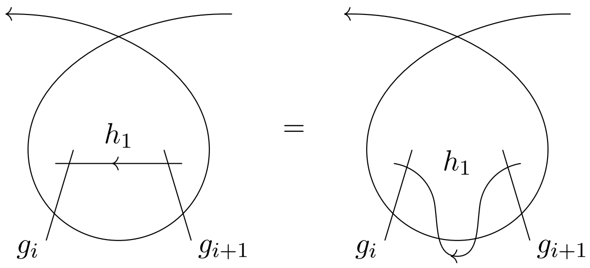
where g_i, g_{i+1}, and the (possibly empty) set of curve segments $H = \{h_1, \ldots, h_l\}$ make up the remainder of the boundary of R'. Note such curve segments may not be pairwise distinct in R. In the case when H is empty, we simply have the situation



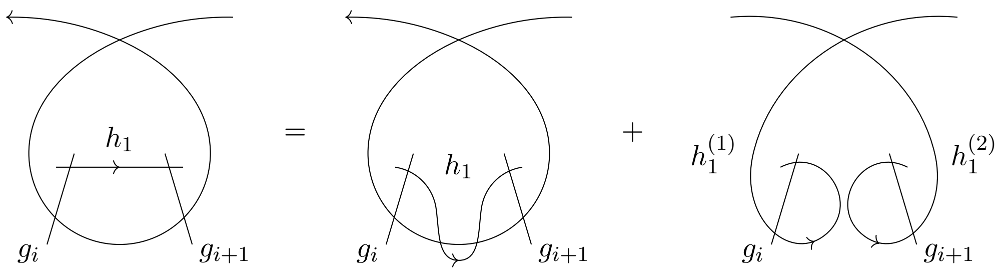
Since R' is empty we may pull this crossing out of the curl by (H1), which will decrease the number of intersection in R and thus by induction $\alpha = 0$. Hence we may assume that H is non-empty. The general case $H = \{h_1, \ldots, h_l\}$ is solved by focusing on h_1, and in fact solving the case $H = \{h_1\}$ is sufficient to understand the general case, hence we only prove this case. So we are working with the sitaution



There are two cases to consider based on the orientation of h_1. For the first case we have



by (H2). Then we may pull the crossing between either g_i and h_1, or g_{i+1} and h_1 out of the curl by (H1), which will decrease the number of intersections in R by one and so $\alpha = 0$ by induction. With the opposite orientation on h_1 we have



by (H3). Here denote the first diagram on the right of the above equation by α_1 and the second by α_2. For α_1, as was done in the previous case we may pull one of the crossings outside of the curl, and thus decrease the number of intersections in R by one, and hence $\alpha_1 = 0$ by induction. For α_2 the curve containing h_1 and the original left curl have been turned into the two new curves $h_1^{(1)}$ and $h_1^{(2)}$. Note the original left curl is no longer present, but regardless of how the original curve containing h_1 intersected the curl, at least one of the new curves $h_1^{(1)}$ and $h_1^{(2)}$ must form a new, smaller, left curl. The region bounded by this new curl is a subregion of R containing strictly less number of intersections. Hence by induction $\alpha_2 = 0$, and so collectively $\alpha = \alpha_1 + \alpha_2 = 0$ completing the proof by induction. Note the general case for $H = \{h_1, \dots, h_l\}$ is tackled in the exact same manner by pulling h_1 out of the curl, the diagrammatics are just more cluttered, but the remaining segments $h_2, \dots, h_l$ do not interfer with the above argumenets.

□

Let $\boldsymbol{a} = a_1 \cdots a_k$ and $\boldsymbol{b} = b_1 \cdots b_l$ for $a_i, b_i \in \{\uparrow, \downarrow\}$, and consider the map

$$\mathsf{deg} : \mathsf{SSim}(\boldsymbol{a}, \boldsymbol{b}) \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

given by $\mathsf{deg}(\alpha) = (A(\alpha), C(\alpha))$ where $A(\alpha)$ is the number of arcs in α, and $C(\alpha)$ is the number of clockwise arcs in α. We call the tuple $(A(\alpha), C(\alpha))$ the *degrees* of α. We order the image of deg by using the lexicographical ordering $<$ on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Note that for any $\alpha \in \mathsf{SSim}(\boldsymbol{a}, \boldsymbol{b})$ we have $\mathsf{deg}(\overline{\alpha}) = \mathsf{deg}(\alpha)$.

Proposition 4.52. Let $\alpha \in \mathsf{SSim}(\boldsymbol{a}, \boldsymbol{b})$. Then

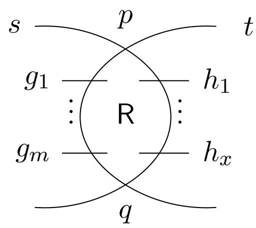
$$\alpha \;=\; \overline{\alpha} \;+ \sum_{\substack{\beta \in \mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b}) \\ \mathsf{deg}(\beta) > \mathsf{deg}(\alpha)}} c_\beta \beta$$

where $c_\beta \in \mathbb{Z}$ for each $\beta \in \mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b})$ such that $\mathsf{deg}(\beta) > \mathsf{deg}(\alpha)$.

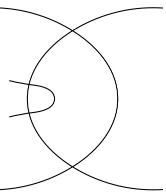
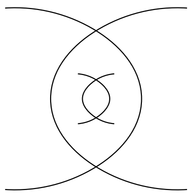
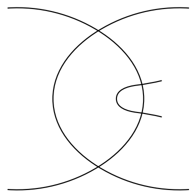
Proof. Given two distinct strings s and t in α, let n be the number of intersections occurring between the two strings. If n is even set $\mu(\{s,t\}) = n$, while if n is odd set $\mu(\{s,t\}) = n - 1$. Note $\mu(\{s,t\})$ is always even. We let

$$\eta(\alpha) = \sum_{s,t} \mu(\{s,t\}),$$

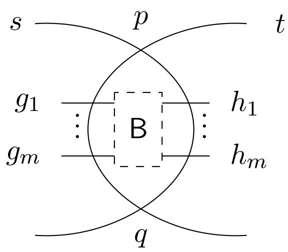
where the sum runs over all unordered pairs of distinct strings of α. Informally, $\eta(\alpha)$ is the number of intersections of α which prevent it from being simple, in particular $\eta(\alpha) = 0$ if and only if $\alpha \in \mathsf{Sim}(\boldsymbol{a}, \boldsymbol{b})$. We will prove this proposition by induction on $\eta(\alpha)$, where the base case of $\eta(\alpha) = 0$ follows immediately since $\alpha = \overline{\alpha}$. Assume the result holds for all $\alpha \in \mathsf{SSim}(\boldsymbol{a}, \boldsymbol{b})$ such that $\eta(\alpha) < n$ for some $n > 0$. Now let α be such that $\eta(\alpha) = n$. Pick two strings s and t in α such that $\mu(\{s,t\}) \geq 2$. Order the points of intersections between s and t according to when they appear as one travels from the source of s to its target. Under this ordering pick two neighbouring points of intersection p and q. Then diagrammatically we have a configuration of strings



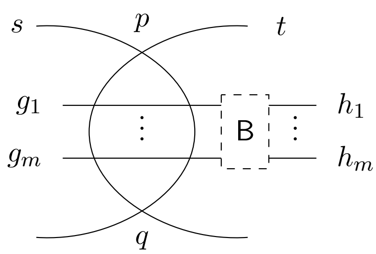
where R is the interior region bounded by the curve segments of s and t between the points of intersection p and q, and the two (possible empty) sets of string segments $G = \{g_1, \ldots, g_m\}$ and $H = \{h_1, \ldots, h_x\}$ account for all the intersections of the boundary of R through t and s respectively. We may assume that we are not in one of the following three situations:

(i) :  (ii) :  or (iii) : 

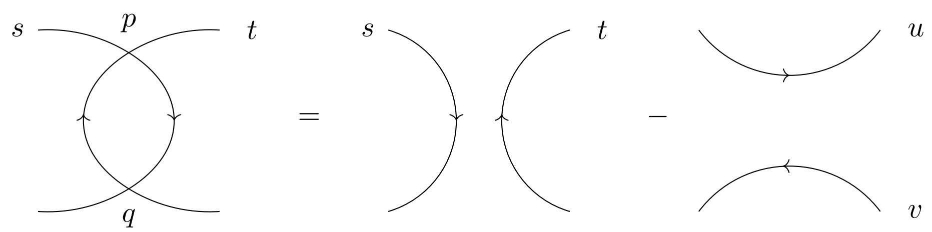
since otherwise we may pick the more nested pair of intersections to work with instead. Since situations (i) and (iii) are not present, any string segment g_i must connect to a h_j (rather than another segment in G). Hence $m = x$ and R realises a pairing of the string segments G with H. Diagrammatically we have



where B is some permutation connecting segments in G with those in H. Moreover, since situation (ii) is not present, this means that no string segments in B can intersect more that once. In other words B is built out of crossings, and so we may pull all of B outside of the region R one crossing at a time by (H1), and thus obtain



Lastly we may pull these horizontal strings out of R through the top or bottom crossing by (H1). Hence we have emptied R by employing only local relation (H1), and so the value $\eta(\alpha)$ has remained the same. Now there are four different cases depending on the orientations of the strings s and t. In three of these cases, since R is empty, we may pull the strings s and t apart by applying (H2), and thus remove the two intersections p and q. This decreases $\eta(\alpha)$ by two, and so the result follows by induction. The last case is given with orientations as follows



where we have applied (H3). Let the two diagrams on the right hand side of the above equation be denoted by α_1 and α_2 respectively, hence $\alpha = \alpha_1 - \alpha_2$. It is clear that $\overline{\alpha_1} = \overline{\alpha}$ and $\mathsf{deg}(\alpha_1) = \mathsf{deg}(\alpha)$. Moreover we have that $\eta(\alpha_1) = \eta(\alpha) - 2$, and so by induction

$$\alpha_1 = \overline{\alpha} + \sum_{\substack{\beta \in \mathsf{Sim}(\boldsymbol{a},\boldsymbol{b}) \\ \mathsf{deg}(\beta) > \mathsf{deg}(\alpha)}} d_\beta \beta \tag{4.10}$$

where $d_\beta \in \mathbb{Z}$. As for α_2, the original strings s and t have been replaced by u and v. Although the points of intersection p and q have been removed, in general we cannot apply the inductive step for α_2 as it may not be semisimple, since the new strings u and v may contain self-intersections. This occurs precisely when there are more intersections between the strings s and t than just p and q. So we break this situation into two cases:

(Case 1) Assume that p and q are the only intersections between the strings s and t in α, and so α_2 is semisimple. Thus by induction we have

$$\alpha_2 = \overline{\alpha_2} + \sum_{\substack{\beta \in \mathsf{Sim}(\boldsymbol{a},\boldsymbol{b}) \\ \mathsf{deg}(\beta) > \mathsf{deg}(\alpha_2)}} f_\beta \beta \tag{4.11}$$

where $f_\beta \in \mathbb{Z}$. We seek to show that $\mathsf{deg}(\alpha_2) > \mathsf{deg}(\alpha)$, and then subtracking *Equation* (4.11) away from *Equation* (4.10) will prove this case. One can show this by comparing the string types of the sets $\{s, t\}$ and $\{u, v\}$. We have the following to consider:

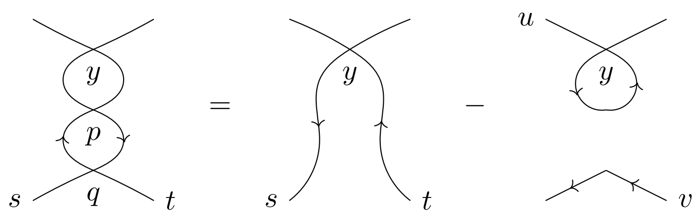
(1) The set $\{s, t\}$ contains a down and up string.

(2) The set $\{s, t\}$ contains a vertical string and clockwise arc.

(3) The set $\{s, t\}$ contains two arcs on the same row, but not both anti-clockwise.

Note $\{s, t\}$ cannot contain a top and bottom arc since α is semisimple. The remaining cases which have been left out are due to the fact they can never realise the orientated double crossing of the strings s and t which we are considering. For (1) it is easy to see that $\{u, v\}$ consists of two arcs. For (2) one can deduce that $\{u, v\}$ contains a vertical string and anti-clockwise arc. For (3), when $\{s, t\}$ consists of two clockwise arcs one can check that $\{u, v\}$ consists of a clockwsie arc and an anti-clockwise arc. When $\{s, t\}$ contains a clockwise and anti-clockwise arc, one can check that $\{u, v\}$ consists of two anti-clockwise arcs. For all these cases we have $\mathsf{deg}(\alpha_2) > \mathsf{deg}(\alpha)$, completing Case 1.

(Case 2) Assume now that there is at least one more point of intersection between the strings s and t beside p or q. In the ordering of intersections discussed previously, pick a neighbouring point which either preceds p or proceds q, say y. Without loss of generality assume y preceds p. Then diagrammatically the equation $\alpha = \alpha_1 - \alpha_2$ is given by

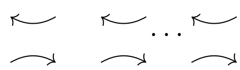


by (H3). In α_2 the interior region bounded by the left curl cannot contain loops or string segments with self-intersections since α is semisimple. Hence by *Lemma 4.51* $\alpha_2 = 0$, and so $\alpha = \alpha_1$ and thus the result follows by *Equation* (4.10).

□

Theorem 4.53. The homomorphism $\varphi : \mathcal{A}^{\mathrm{aff}}_{2k} \to \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$ of *Proposition 4.43* is surjective.

Proof. As discussed previously, this will follow by showing that $\alpha \in \langle \varphi(e_i), \varphi(\tau_j)\rangle_{i,j}$ for all $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^{\otimes k})$. We prove this by downwards induction on $\mathsf{deg}(\alpha)$. It's easy to see that the maximum degree is $\mathsf{deg}(\alpha) = (2k, 2k)$. By considering what endpoints can be targets and sources of clockwise arcs, one can deduce the only element $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^{\otimes k})$ satisfying $\mathsf{deg}(\alpha) = (2k, 2k)$ is given by



$$= \varphi\left(\prod_{i\in[k]} e_{2i-1}\right)$$

This completes the base case. Now, pick α such that $\mathsf{deg}(\alpha) = (x, y) < (2k, 2k)$ and assume that $\gamma \in \langle \varphi(e_i), \varphi(\tau_j)\rangle_{i,j}$ for any simple diagram $\gamma \in \mathsf{Sim}((\uparrow\downarrow)^{\otimes k})$ such that $\mathsf{deg}(\gamma) > (x, y)$. By *Lemma 4.50* there exists $\pi, \sigma \in \mathfrak{S}_k$ and a planar diagram $\beta \in \mathsf{Sim}((\uparrow\downarrow)^{\otimes k})$ such that $\pi^{\uparrow}\beta\sigma^{\downarrow}$ is semisimple and $\alpha = \overline{\pi^{\uparrow}\beta\sigma^{\downarrow}}$, in particular $\mathsf{deg}(\alpha) = \mathsf{deg}(\pi^{\uparrow}\beta\sigma^{\downarrow})$. Hence by *Proposition 4.52* we have that

$$\pi^{\uparrow}\beta\sigma^{\downarrow} = \alpha + \sum_{\substack{\gamma\in\mathsf{Sim}((\uparrow\downarrow)^{k})\\ \mathsf{deg}(\gamma)>\mathsf{deg}(\alpha)}} c_\gamma\gamma, \tag{4.12}$$

where $c_\gamma \in \mathbb{Z}$. By induction all the simple terms in the above summation belong to $\mathsf{Im}(\varphi)$. Also from *Lemma 4.44* and *Lemma 4.47* we know that $\pi^\uparrow \beta \sigma^\downarrow \in \mathsf{Im}(\varphi)$, thus rearranging the above equation shows that $\alpha \in \mathsf{Im}(\varphi)$, completing the proof by induction.

$\square$

Remark 4.54. *Equation* (4.12) is the key to *Theorem 4.53*, and follows from *Proposition 4.52*. This proposition applies to all semisimple diagrams which are much more general than those appearing here. Ideally, one would like to prove that *Equation* (4.12) holds for $\pi^\uparrow \beta \sigma^\downarrow$ by some inductive argument without needing to show it for all semisimple diagrams. However it is a very delicate task to check which properties are preserved by an inductive process. So we ended up using this more general approach instead, even though many of the cases considered in proving *Proposition 4.52* probably will not occur in this case.

By *Theorem 4.53*, the image of any generating set of $\mathcal{A}_{2k}^{\text{aff}}$ provides a generating set for the algebra $\mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$. Hence by *Corollary 4.13* we see that the set

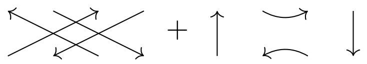
$$\{\varphi(e_i), \varphi(s_j), \varphi(x_m) = r_m, \varphi(z_l) = c_l \mid i \in [2k-1], j \in [k-1], m \in [2k], l \in \mathbb{Z}_{\geq 0}\}$$

gives a generating set of $\mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$. Comparing this to *Equation* (4.8), we see that $\widehat{\mathcal{A}}_{2k} = \mathsf{End}_{\mathsf{Heis}}((\uparrow\downarrow)^{\otimes k})$ as previously claimed.

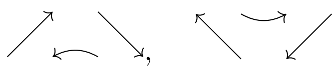
4.3.3 The Affine Partition Category of Brundan and Vargas

In this last section of the chapter we relate our affine partition algebra $\mathcal{A}_{2k}^{\text{aff}}$ to the work of J. Brundan and M. Vargas in [BV21]. We start by recalling the definition of their affine partition category APar as a subcategory of Heis generated by a single object and certain morphisms, and of their affine partition algebra AP_k, which is an endomorphism algebra within APar.

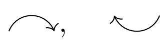
Definition 4.55. [BV21, Definition 4.6 and Equation 4.47] The *affine partition category* APar is the monoidal subcategory of Heis generated by the object $\uparrow\downarrow$ and the following morphisms:



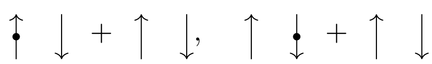
(4.13)



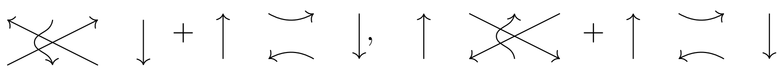
(4.14)



(4.15)



(4.16)



(4.17)

The *affine partition algebra* is defined to be $\mathsf{AP}_k := \mathsf{End}_{\mathsf{APar}}((\uparrow\downarrow)^k)$.

We can generalise the arguments in the proof of *Theorem 4.53* to show the following result.

Theorem 4.56. The category APar is the full monoidal subcategory of Heis generated by the object $\uparrow\downarrow$.

Proof. We need to show that

$$\mathsf{Hom}_{\mathsf{APar}}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l) = \mathsf{Hom}_{\mathsf{Heis}}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l).$$

Using *Theorem 4.40*, we need to show that any element of the form

$$c_w^{k_w} \dots c_1^{k_1} c_0^{k_0} r_1^{s_1} r_3^{s_3} \dots r_{2k-1}^{s_{2k-1}} \alpha r_2^{t_2} r_4^{t_4} \dots r_{2l}^{t_{2l}},$$

where $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$, can be written in terms of the generating morphisms in APar. The morphisms r_i can be obtained by tensoring the generators (4.16) with the appropriate identity morphisms on the left and right (and subtracting the identity). Moreover, the morphisms c_i can be obtained by concatenating r_1^i with the generators (4.15) on top and bottom. Thus, it remains to show that any $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$ can be written in terms of the generating morphism in APar. A generalisation of Jones' normal form shows that any planar $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$ can be written in terms of the generators (4.14) and (4.15) (see for example [RSA14, Proof of Lemma 2.1] for an explicit construction). Now *Lemma 4.50* allows us to write any $\alpha \in \mathsf{Sim}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$ as $\alpha = \overline{\pi^\uparrow \beta \sigma^\downarrow}$ where $\pi \in \mathfrak{S}_k, \sigma \in \mathfrak{S}_l$ and β is planar. Note that $s_i^\uparrow$ and $s_i^\downarrow$ can be written using the generators (4.17) and the composition of the generators (4.14) (and tensoring with the appropriate identity morphism on the left and right). So using the discussion following *Example 4.46* we know that $\pi^\uparrow$ and $\sigma^\downarrow$ belong to $\mathsf{End}_{\mathsf{APar}}((\uparrow\downarrow)^{\otimes k})$ and $\mathsf{End}_{\mathsf{APar}}((\uparrow\downarrow)^{\otimes l})$ respectively. Now we can follow exactly the same proof as for *Theorem 4.53* noting that in this case the maximum degree is $(k+l, k+l)$ and the only simple diagram with that degree is the one containing k consecutive arcs at the top and l consecutive arcs at the bottom, which is planar. The rest of the proof can be followed verbatim simply replacing $\mathsf{Im}(\varphi)$ by $\mathsf{Hom}_{\mathsf{APar}}((\uparrow\downarrow)^k, (\uparrow\downarrow)^l)$. □

We immediately obtain the following consequences.

Corollary 4.57. The map φ gives a surjective homomorphism for $\mathcal{A}_{2k}^{\mathrm{aff}}$ to AP_k.

□

Corollary 4.58. The set $\mathsf{B}((\uparrow\downarrow)^k)$ gives a basis for AP_k.

□

We do not know whether the map φ is an isomorphism. If it were, then we would also have a presentation for AP_k. Although not included within this thesis, we have constructed a basis for the small algebras $\mathcal{A}_2^{\mathrm{aff}}$ and $\mathcal{A}_4^{\mathrm{aff}}$, and from such we have confirmed that $\mathcal{A}_2^{\mathrm{aff}} \cong \mathsf{AP}_1$ and $\mathcal{A}_4^{\mathrm{aff}} \cong \mathsf{AP}_2$. We suspect that the map φ realises an isomorphism between $\mathcal{A}_{2k}^{\mathrm{aff}}$ and AP_k in general.

5 Orbit Affine Partition Algebra

This chapter is broken into two sections. The first section focuses on generalising the results summarised in *Section 2.1.2* and *Section 2.1.3* regarding the centers of group algebras of symmetric groups, to closely related centralizer algebras. Namely, we provide a class sum basis to certain centralizer algebras of the group algebras of the symmetric groups, prove a polynomial property for the corresponding structure constants, and from such define new $\mathbb{C}[z]$-algebras called *marked cycle shape* algebras (see *Definition 5.33*), which generalise the Farahat and Higman algebra Z given in *Definition 2.10*. We then go on to prove various results regarding these new algebras, with one such result establishing an isomorphism between marked cycle shape algebras and the endomorphism algebras $\mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes k})$ discussed in the previous chapter.

The second section of this chapter uses the marked cycle shape algebras to analysis certain subalgebras of the endomorphism algebras $\mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$ (the codomain of the representation $\Psi^{(M)}_{2k,n}$ given in *Theorem 4.24*). As a result, we are able to construct a new algebra $\mathcal{Q}^{\text{aff}}_{2k}$ which may also be thought of as an affine version of the partition algebra. We prove some basic properties of this algebra, and end the chapter by constructing an algebra homomorphism $\mathcal{A}^{\text{aff}}_{2k} \to \mathcal{Q}^{\text{aff}}_{2k}$. We suspect that $\mathcal{A}^{\text{aff}}_{2k} \cong \mathcal{Q}^{\text{aff}}_{2k}$ as $\mathbb{C}$-algebras, although this appears difficult to prove at this stage.

5.1 Marked Cycle Shape Algebras

5.1.1 Conjugacy Classes of $\mathfrak{S}_{\mathbb{N}}^{\times r}$ and $\mathfrak{S}_n^{\times r}$

Throughout this chapter we let X denote a finite subset of $\mathbb{N}$. For any $r \geq 1$ and finite group G, let $G^{\times r}$ denote the group of r direct products of G. Recall that for any permutation $\pi \in \mathfrak{S}_{\mathbb{N}}$ the support $\mathsf{Sup}(\pi)$ is all elements of $\mathbb{N}$ on which π acts non-trivially, and we have set $||\pi|| = |\mathsf{Sup}(\pi)|$.

Definition 5.1. Let $\boldsymbol{\pi} = (\pi_i)_{i=1}^r \in \mathfrak{S}_{\mathbb{N}}^{\times r}$. Then we define the following:

(1) $\mathsf{Sup}(\boldsymbol{\pi}) := \mathsf{Sup}(\pi_1) \cup \cdots \cup \mathsf{Sup}(\pi_r)$ and $||\boldsymbol{\pi}|| := |\mathsf{Sup}(\boldsymbol{\pi})|$.

(2) $\mathsf{Sup}_X(\boldsymbol{\pi}) := \mathsf{Sup}(\boldsymbol{\pi}) \cap X$ and $||\boldsymbol{\pi}||_X := |\mathsf{Sup}_X(\boldsymbol{\pi})|$.

(3) $\mathsf{Sup}^X(\boldsymbol{\pi}) := \mathsf{Sup}(\boldsymbol{\pi}) \backslash X$ and $||\boldsymbol{\pi}||^X := |\mathsf{Sup}^X(\boldsymbol{\pi})|$.

In particular we have the disjoint union $\mathsf{Sup}(\boldsymbol{\pi}) = \mathsf{Sup}_X(\boldsymbol{\pi}) \sqcup \mathsf{Sup}^X(\boldsymbol{\pi})$.

We let $\mathsf{Stab}(X) := \{\pi \in \mathfrak{S}_{\mathbb{N}} \mid \pi(x) = x \text{ for all } x \in X\}$ be the stabilizer subgroup of X in $\mathfrak{S}_{\mathbb{N}}$. The group $\mathsf{Stab}(X)$ acts on $\mathfrak{S}_{\mathbb{N}}^{\times r}$ by component-wise conjugation. We refer to the orbits of this action as the X-*conjugacy classes of* $\mathfrak{S}_{\mathbb{N}}^{\times r}$. For any $\boldsymbol{\pi} = (\pi_i)_{i=1}^r \in \mathfrak{S}_{\mathbb{N}}^{\times r}$ we denote the X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}^{\times r}$ containing $\boldsymbol{\pi}$ by

$$\mathsf{CL}[X](\boldsymbol{\pi}) := \{(\sigma_i)_{i=1}^r \in \mathfrak{S}_{\mathbb{N}}^{\times r} \mid \sigma_i = \tau\pi_i\tau^{-1} \text{ for all } i \in [r], \text{ and some } \tau \in \mathsf{Stab}(X)\}.$$

Let C be an X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}^{\times r}$ and let $\boldsymbol{\pi} = (\pi_i)_{i=1}^r, \boldsymbol{\sigma} = (\sigma_i)_{i=1}^r \in \mathsf{C}$. By considering how conjugation acts on the cycle structure of permutations given in *Lemma 2.3*, one can see that for each $i \in [r]$ the permutations π_i and σ_i must have the same cycle structure, and the relative positions of the elements of X within their cycles must also match. Hence $||\boldsymbol{\pi}|| = ||\boldsymbol{\sigma}||$, $||\boldsymbol{\pi}||_X = ||\boldsymbol{\sigma}||_X$, and $||\boldsymbol{\pi}||^X = ||\boldsymbol{\sigma}||^X$, and so it makes sense to define $||\mathsf{C}|| := ||\boldsymbol{\pi}||$, $||\mathsf{C}||_X := ||\boldsymbol{\pi}||_X$, and $||\mathsf{C}||^X := ||\boldsymbol{\pi}||^X$ for any $\boldsymbol{\pi} \in \mathsf{C}$.

Example 5.2. Let $r = 2$, $X = \{1, 2\}$, and consider $\boldsymbol{\pi} = ((1, 3)(2, 4), (3, 4, 5)) \in \mathfrak{S}_{\mathbb{N}}^{\times 2}$. Then the X-conjugacy class containing $\boldsymbol{\pi}$ is given by

$$\mathsf{CL}[X](\boldsymbol{\pi}) = \{((1, a)(2, b), (a, b, c)) \mid (a, b, c) \in (\mathbb{N}\backslash X)^{!3}\}$$

where $(\mathbb{N}\backslash X)^{!3}$ is the subset of $(\mathbb{N}\backslash X) \times (\mathbb{N}\backslash X) \times (\mathbb{N}\backslash X)$ of all tuples with pairwise distinct entries. For any $\boldsymbol{\sigma} \in \mathsf{CL}[X](\boldsymbol{\pi})$, we have $||\boldsymbol{\sigma}|| = 5$, $||\boldsymbol{\sigma}||_X = 2$, and $||\boldsymbol{\sigma}||^X = 3$.

Let $n \in \mathbb{N}$ and $X \subseteq [n]$, then we set $\mathsf{Stab}_n(X) := \mathsf{Stab}(X) \cap \mathfrak{S}_n$. Similar to the above situation, this subgroup acts on $\mathfrak{S}_n^{\times r}$ by component-wise conjugation. We refer to the orbits of this action as the X-*conjugacy classes of* $\mathfrak{S}_n^{\times r}$. For any $\boldsymbol{\pi} = (\pi_i)_{i=1}^r \in \mathfrak{S}_n^{\times r}$, we denote the X-conjugacy class of $\mathfrak{S}_n^{\times r}$ containing $\boldsymbol{\pi}$ by

$$\mathsf{CL}_n[X](\boldsymbol{\pi}) := \{(\sigma_i)_{i=1}^r \in \mathfrak{S}_n^{\times r} \mid \sigma_i = \tau\pi_i\tau^{-1} \text{ for all } i \in [r], \text{ and some } \tau \in \mathsf{Stab}_n(X)\}.$$

Again we can set $||\mathsf{C}|| := ||\boldsymbol{\pi}||$, $||\mathsf{C}||_X := ||\boldsymbol{\pi}||_X$, and $||\mathsf{C}||^X := ||\boldsymbol{\pi}||^X$ for any X-conjugacy class C of $\mathfrak{S}_n^{\times r}$ and any $\boldsymbol{\pi} \in \mathsf{C}$.

Given any X-conjugacy class C of $\mathfrak{S}_{\mathbb{N}}^{\times r}$ we will let $\mathsf{C}_n := \mathsf{C} \cap \mathfrak{S}_n^{\times r}$.

Example 5.3. Continuing from *Example 5.2* let $\mathsf{C} = \mathsf{CL}[X](\boldsymbol{\pi})$. Note that $X \subseteq [n]$ for all $n \geq 2$. For these values one can deduce that

$$\mathsf{C}_n = \begin{cases} \{((1, a)(2, b), (a, b, c)) \mid (a, b, c) \in ([n]\backslash X)^{!3}\}, & n \geq 5 \\ \emptyset, & 2 \leq n \leq 4 \end{cases}$$

where $([n]\backslash X)^{!3}$ is the subset of $([n]\backslash X) \times ([n]\backslash X) \times ([n]\backslash X)$ of all tuples with pairwise distinct entries. When $n \geq 5$ the set C_n is an X-conjugacy class of $\mathfrak{S}_n^{\times 2}$.

As the above example suggests, when C is an X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}^{\times r}$, $n \in \mathbb{N}$, and $X \subseteq [n]$, then the set C_n is either empty or an X-conjugacy class of $\mathfrak{S}_n^{\times r}$. The following proposition proves this and gives a criteria for when C_n is non-empty.

Proposition 5.4. Let C be an X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}^{\times r}$, $n \in \mathbb{N}$, and $X \subseteq [n]$. Then C_n is non-empty if and only if

$$n \geq ||\mathsf{C}||^X + |X|. \tag{5.1}$$

In this case C_n is an X-conjugacy class of $\mathfrak{S}_n^{\times r}$, and all such X-conjugacy classes of $\mathfrak{S}_n^{\times r}$ appear uniquely in this manner.

Proof. By assumption $|X| \leq n$. Pick an arbitrary element $\boldsymbol{\pi} = (\pi_i)_{i=1}^r \in \mathsf{C}$. All elements of C are of the form $(\tau\pi_i\tau^{-1})_{i=1}^r$ for some $\tau \in \mathsf{Stab}(X)$. The set $\mathsf{Sup}^X(\tau\boldsymbol{\pi}\tau^{-1})$ is all the elements of $\mathbb{N}\backslash X$ for which at least one $\tau\pi_i\tau^{-1}$ acts non-trivially on. As such $\tau\boldsymbol{\pi}\tau^{-1} \in \mathsf{C}_n$ if and only if $\mathsf{Sup}^X(\tau\boldsymbol{\pi}\tau^{-1}) \subset [n]\backslash X$. For this to be the case we must have that

$$|\mathsf{Sup}^X(\tau\boldsymbol{\pi}\tau^{-1})| = ||\mathsf{C}||^X \leq |[n]\backslash X| = n - |X|.$$

Rearranging gives *Equation* (5.1). This tells us that $\mathsf{C}_n = \emptyset$ whenever $n < ||\mathsf{C}||^X + |X|$. Now assume *Equation* (5.1) holds. We have $\mathsf{Sup}^X(\tau\boldsymbol{\pi}\tau^{-1}) = \tau\left(\mathsf{Sup}^X(\boldsymbol{\pi})\right)$, the image of the set $\mathsf{Sup}^X(\boldsymbol{\pi})$ under τ. Let $t : \mathsf{Sup}^X(\boldsymbol{\pi}) \to [n]\backslash X$ be an injective map, which must exist since *Equation* (5.1) holds. Then fix a permutation $\tau_t \in \mathsf{Stab}(X)$ such that $\tau_t(i) = t(i)$ for all $i \in \mathsf{Sup}^X(\boldsymbol{\pi})$. Hence $\mathsf{Sup}^X(\tau_t\boldsymbol{\pi}\tau_t^{-1}) \subset [n]\backslash X$ and so, from our above discussion, the set C_n is non-empty. We now want to prove that such a set C_n is an X-conjugacy class of $\mathfrak{S}_n^{\times r}$. We have shown that $\boldsymbol{\pi}_n := \tau_t\boldsymbol{\pi}\tau_t^{-1} \in \mathsf{C}_n$ and hence, since $\mathsf{C}_n = \mathsf{C} \cap \mathfrak{S}_n^{\times r}$, any element of C_n is of the form $\tau\boldsymbol{\pi}_n\tau^{-1}$ for $\tau \in \mathsf{Stab}(X)$ such that $\mathsf{Sup}^X(\tau\boldsymbol{\pi}_n\tau^{-1}) \subset [n]\backslash X$. This differs to $\mathsf{CL}_n[X](\boldsymbol{\pi}_n)$ only in the fact that τ can be taken from $\mathsf{Stab}(X)$ as opposed to $\mathsf{Stab}_n(X)$. Hence we need to show that whenever $\tau\boldsymbol{\pi}_n\tau^{-1} \in \mathsf{C}_n$ for some $\tau \in \mathsf{Stab}(X)$, then there exists a $\tau_n \in \mathsf{Stab}_n(X)$ such that $\tau\boldsymbol{\pi}_n\tau^{-1} = \tau_n\boldsymbol{\pi}_n\tau_n^{-1}$. Given such a τ, we must have that $\mathsf{Sup}^X(\tau\boldsymbol{\pi}_n\tau^{-1}) = \tau\left(\mathsf{Sup}^X(\boldsymbol{\pi}_n)\right) \subset [n]\backslash X$. We also have that $\mathsf{Sup}^X(\boldsymbol{\pi}_n) \subset [n]\backslash X$, hence let τ_n be any permutation of $[n]\backslash X$ with the property that $\tau_n(i) := \tau(i)$ for all $i \in \mathsf{Sup}^X(\boldsymbol{\pi}_n)$. From how conjugation acts on the cycle structure of permutations given in *Lemma 2.3*, it is clear that $\tau\boldsymbol{\pi}_n\tau^{-1} = \tau_n\boldsymbol{\pi}_n\tau_n^{-1}$. Hence, for any $\boldsymbol{\pi} \in \mathfrak{S}_n^{\times r}$, we have that

$$(\mathsf{CL}[X](\boldsymbol{\pi}))_n = \mathsf{CL}_n[X](\boldsymbol{\pi}).$$

Also, since orbits intersect trivially, all such X-conjugacy classes of $\mathfrak{S}_n^{\times r}$ appear as the intersection of a unique X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}^{\times r}$ with $\mathfrak{S}_n^{\times r}$.

□

We now present a result describing the size of the set C_n when C is an X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}^{\times r}$, $n \in \mathbb{N}$, and $X \subseteq [n]$. In particular we show that the size of such a set is polynomial in n.

Proposition 5.5. Let $n \in \mathbb{N}$ and $X \subseteq [n]$. For any X-conjugacy class C of $\mathfrak{S}_{\mathbb{N}}^{\times r}$,

$$|\mathsf{C}_n| = \frac{1}{b(\mathsf{C})}(n - |X|)(n - |X| - 1)\cdots(n - |X| - ||\mathsf{C}||^X + 1)$$

where $b(\mathsf{C}) \in \mathbb{N}$ is a constant depending only on the class C and not on n.

Proof. Assume that $n \geq ||\mathsf{C}||^X + |X|$, and so by *Proposition 5.4* we have that C_n is an X-conjugacy class of $\mathfrak{S}_n^{\times r}$. Pick $\boldsymbol{\pi} = (\pi_i)_{i=1}^r \in \mathsf{C}_n$, then C_n is the orbit of $\boldsymbol{\pi}$ in $\mathfrak{S}_n^{\times r}$ under the action of $\mathsf{Stab}_n(X)$ by component-wise conjugation. Thus by the *Orbit-Stabilizer Theorem* we have

$$|\mathsf{C}_n| = \frac{|\mathsf{Stab}_n(X)|}{|\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi})|}$$

where $\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi}) = \{\tau \in \mathsf{Stab}_n(X) \mid \tau\boldsymbol{\pi}\tau^{-1} = \boldsymbol{\pi}\}$ is the subgroup of $\mathsf{Stab}_n(X)$ whose elements fix $\boldsymbol{\pi}$ under component-wise conjugation. Consider $\mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi}))$, the subgroup of $\mathfrak{S}_n$ consisting of all permutations which act trivially on $X \cup \mathsf{Sup}^X(\boldsymbol{\pi})$. It is easy to see that $\mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi})) \subset \mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi})$. Now let $\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))$ denote the subgroup of $\mathfrak{S}_n$ consisting of the permutations of $\mathsf{Sup}^X(\boldsymbol{\pi})$. Naturally $\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi})) \subset \mathsf{Stab}_n(X)$, and hence the group

$$\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi}) = \{\tau \in \mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi})) \mid \tau\boldsymbol{\pi}\tau^{-1} = \boldsymbol{\pi}\}$$

is also a subgroup of $\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi})$.

Claim: We have a group isomorphism

$$\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi}) \cong \mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi})) \times \mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi}).$$

Note that the two subgroups $\mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi}))$ and $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi})$ commute and have trivial intersection. Hence to prove this claim we only need to show that any permutation $\tau \in \mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi})$ can be expressed as a product $\tau = \tau_1\tau_2$ for some $\tau_1 \in \mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi}))$ and $\tau_2 \in \mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi})$. Let $\tau \in \mathsf{Stab}_{\mathsf{Stab}_n(X)}(\boldsymbol{\pi})$, then by definition we have $\tau\pi_i\tau^{-1} = \pi_i$ for each $i \in [r]$. We now seek to show that the sets $[n]\backslash\mathsf{Sup}(\boldsymbol{\pi})$ and $\mathsf{Sup}(\boldsymbol{\pi})$ are invariant under the action of τ. Suppose for contradiction this is not the case, hence there exists an $i \in [n]\backslash\mathsf{Sup}(\boldsymbol{\pi})$ such that $\tau(i) = j \in \mathsf{Sup}(\boldsymbol{\pi})$. Then for each $i \in [r]$,

$$\pi_i(j) = (\tau\pi_i\tau^{-1})(j) = (\tau\pi_i)(i) = \tau(i) = j.$$

Thus π_i fixes j for each $i \in [r]$, but this gives the desired contradiction since $j \in \mathsf{Sup}(\boldsymbol{\pi})$. So since the sets $[n]\backslash\mathsf{Sup}(\boldsymbol{\pi})$ and $\mathsf{Sup}(\boldsymbol{\pi})$ are invariant under the action of τ, we must have a decomposition $\tau = \tau_1\tau_2$ where τ_1 is a permutation of $[n]\backslash\mathsf{Sup}(\boldsymbol{\pi})$ and τ_2 is a permutation of $\mathsf{Sup}(\boldsymbol{\pi})$. Naturally τ_1 and τ_2 commute, and since τ fixes X, both τ_1 and τ_2 must also fix X. As such τ_1 is an element of $\mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi}))$ as desired, and $\tau_2 \in \mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))$. Lastly note that for each $i \in [r]$,

$$\pi_i = \tau\pi_i\tau^{-1} = \tau_2\tau_1\pi_i\tau_1^{-1}\tau_2^{-1} = \tau_2\pi_i\tau_2^{-1}$$

since τ_1 commutes with both τ_2 and π_i. Thus $\tau_2\pi_i\tau_2^{-1} = \pi_i$ for each $i \in [r]$, and so τ_2 actually belongs to $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi})$. Hence the claim holds.

Now returning to the size of C_n, recall that $|\mathsf{Sup}^X(\boldsymbol{\pi})| = ||\mathsf{C}||^X$. Also observe that the size of the group $\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))$ depends only on the class C, and in particular it is independent of n. This implies the same for the size of $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi})$, and so we denote $b(\mathsf{C}) := |\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi})|$. Hence we have that

$$\begin{aligned}
|\mathsf{C}_n| &= \frac{|\mathsf{Stab}_n(X)|}{|\mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\boldsymbol{\pi}))||\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\boldsymbol{\pi}))}(\boldsymbol{\pi})|} \\
&= \frac{(n-|X|)!}{(n-|X|-||\mathsf{C}||^X)!b(\mathsf{C})} \\
&= \frac{1}{b(\mathsf{C})}(n-|X|)(n-|X|-1)\cdots(n-|X|-||\mathsf{C}||^X+1).
\end{aligned}$$

We assumed that $n \geq ||\mathsf{C}||^X + |X|$ so that C_n is an X-conjugacy class of $\mathfrak{S}_n^{\times r}$. By *Proposition 5.4* the set C_n is empty when $|X| \leq n < ||\mathsf{C}||^X + |X|$, and such values of n are precisely those which give zero in the above formula for $|\mathsf{C}_n|$. Hence this formula holds for all $n \geq |X|$, completing the proof.

□

5.1.2 Marked Cycle Shapes

In this section we focus on the X-conjugacy classes of $\mathfrak{S}_{\mathbb{N}}$ and $\mathfrak{S}_n$, hence specialising to the case of $r = 1$ when comparing to the previous section. We will provide a natural indexing set of such classes and set up a variety of notation which will be employed throughout this chapter.

Let $\pi \in \mathfrak{S}_{\mathbb{N}}$ and recall that the X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}$ containing π is given by

$$\mathsf{CL}[X](\pi) = \{\sigma \in \mathfrak{S}_{\mathbb{N}} \mid \sigma = \tau\pi\tau^{-1} \text{ for some } \tau \in \mathsf{Stab}(X)\}.$$

Considering how conjugation acts on the cycles of a permutation, as given in *Lemma 2.3*, we have that $\sigma \in \mathsf{CL}[X](\pi)$ if and only if σ has the same cycle structure as π and the relative positions of the elements of X within the cycles are the same. We will encode such information into the following monoid: Let $\mathfrak{S}(X)$ denote the group of permutations of the set X, and let $\mathcal{U}(X)$ denote the free commutative monoid generated by the set $\{u_x \mid x \in X\}$. We have a natural (left) monoid action

$$\varphi : \mathfrak{S}(X) \to \mathsf{End}(\mathcal{U}(X))$$

given by $\varphi(\pi)(u_x) := u_{\pi(x)}$ for all $\pi \in \mathfrak{S}(X)$ and $x \in X$, and where $\mathsf{End}(\mathcal{U}(X))$ is the monoid of all monoid endomorphisms $\mathcal{U}(X) \to \mathcal{U}(X)$. Lastly, recall that $\mathcal{C}$ denotes the free commutative monoid generated by the infinite set $\{c_i \mid i \in \mathbb{N}\}$.

Definition 5.6. We define the *X-marked cycle shape monoid* to be

$$\mathcal{C}[X] := (\mathfrak{S}(X) \ltimes_{\varphi} \mathcal{U}(X)) \times \mathcal{C},$$

where $\ltimes_{\varphi}$ denotes the semidirect product with respect to the action φ.

As such the underlying set of $\mathcal{C}[X]$ is the cartesian product $\mathfrak{S}(X) \times \mathcal{U}(X) \times \mathcal{C}$, and the monoid product is given by

$$(\pi, p, c)(\pi', p', c') = (\pi\pi', \varphi(\pi')(p)p', cc')$$

for all $(\pi, p, c)(\pi', p', c') \in \mathfrak{S}(X) \times \mathcal{U}(X) \times \mathcal{C}$. We abuse notation a little and just write $c = (1, 1, c)$, $p = (1, p, 1)$, and $\pi = (\pi, 1, 1)$ as elements in $\mathcal{C}[X]$. Now for any set A let $\mathbb{Z}_{\geq 0}^A$ denote the set of all functions $\boldsymbol{f} : A \to \mathbb{Z}_{\geq 0}$ with finite support, that is the set of elements $a \in A$ such that $f_a := \boldsymbol{f}(a) \neq 0$ is finite. Such a set may be regarded as a commutative monoid under point-wise addition. Then for any $\boldsymbol{d} \in \mathbb{Z}_{\geq 0}^X$ we let $u^{\boldsymbol{d}}$ denote the element of $\mathcal{U}(X)$ given by the product of terms $u_x^{d_x}$ for each $x \in X$. Similarly, for any $\boldsymbol{l} \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}$ we let $c^{\boldsymbol{l}}$ denote the element of $\mathcal{C}$ given by the product of terms $c_i^{l_i}$ for each $i \in \mathbb{N}$, which is well-defined since $\boldsymbol{l}$ has finite support. Then as sets we have

$$\mathcal{C}[X] = \{\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}} \mid \pi \in \mathfrak{S}(X),\ \boldsymbol{d} \in \mathbb{Z}_{\geq 0}^{X},\ \boldsymbol{l} \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}\}.$$

Moreover, from this perspective the product is given by

$$(\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}})(\sigma u^{\boldsymbol{e}} c^{\boldsymbol{k}}) = \pi\sigma u^{\sigma \circ \boldsymbol{d} + \boldsymbol{e}} c^{\boldsymbol{l}+\boldsymbol{k}} \tag{5.2}$$

where $\sigma \circ \boldsymbol{d} \in \mathbb{Z}_{\geq 0}^{X}$ is defined by $(\sigma \circ \boldsymbol{d})(i) = \boldsymbol{d}(\sigma^{-1}(i))$. In particular, the operator $\circ$ realises the set $\mathbb{Z}_{\geq 0}^{X}$ as a (left) $\mathfrak{S}(X)$-action set.

The monoid $\mathcal{C}[X]$ provides an indexing set for the X-conjugacy classes of $\mathfrak{S}_{\mathbb{N}}$. The submonoid $\mathcal{C}$ encodes the cycle structure of the cycles containing no elements of X, while the submonoid $\mathfrak{S}(X) \ltimes_{\varphi} \mathcal{U}(X)$ encodes the cycle structure of the cycles containing elements of X and the relative positions of such elements in such cycles. Instead of proving this connection now, we will show that we may associate to any element of $\mathcal{C}[X]$ an object called an *X-marked cycle shape*, and then the correspondence between $\mathcal{C}[X]$ and X-conjugacy classes of $\mathfrak{S}_{\mathbb{N}}$ will be immediate.

Given a finite set A, a cycle with entries belonging to A is a tuple $(a_i)_{i=1}^{m} \in A^{\times m}$, for some $m \in \mathbb{N}$, where we only care about the order of the coordinates up to cyclic shifts.

Definition 5.7. Let $*$ be a formal symbol. We define an *X-marked cycle shape* to be a finite collection of cycles with entries belonging to $X \cup \{*\}$ with the following properties:

(1) The multiset of entries among the cycles equals $X \cup \{*^m\}$ for some $m \in \mathbb{Z}_{\geq 0}$, in particular each element of X appears precisely once.

(2) Cycles containing only $*$ must be of length at least two.

We write an X-marked cycle shape as a formal product of its cycles by juxtaposition, where the order of the cycles is immaterial.

Example 5.8. Let $X = \{1, 3, 5, 7, 9, 11\}$. An example of an X-marked cycle shape is

$$\lambda = (3, 11)(9)(1, *, *, 7, *)(5, *, *)(*, *)(*, *, *).$$

The multiset of entries among the cycles of λ is $X \cup \{*^{10}\}$. Any of the six cycles above may be rearranged in any order to give an alternative expression for λ.

Naturally the group $\mathfrak{S}(X)$ may be regarded as the subset of all X-marked cycle shapes whose multiset of entries among the cycles is precisely X. At the other end of the spectrum, the cycle shape monoid $\mathcal{C}$ of *Section 2.1.2* may be regarded as the subset of all X-marked cycle shapes where each element of X belongs to a cycle of length one. Thus the set of X-marked cycle shapes interpolates between permutations of X and cycle shapes.

By consulting *Lemma 2.3*, it is obvious that the set of X-marked cycles shapes index the X-conjugacy classes of $\mathfrak{S}_{\mathbb{N}}$. Explicitly, given an X-marked cycle shape λ, the corresponding X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}$ is the set of permutations obtained from λ by replacing the symbols $*$ with distinct elements from the set $\mathbb{N}\backslash X$. For any X-marked cycle shape λ, we let $\mathsf{CL}[X](\lambda)$ denote the corresponding X-conjugacy class of $\mathfrak{S}_{\mathbb{N}}$.

Example 5.9. Let X and λ be as in *Example 5.8*, then

$$\mathsf{CL}[X](\lambda) = \left\{(3,11)(1,a_1,a_2,7,a_3)(5,a_4,a_5)(a_6,a_7)(a_8,a_9,a_{10}) \;\middle|\; (a_i)_{i=1}^{10} \in (\mathbb{N}\backslash X)^{!10}\right\}$$

where $(\mathbb{N}\backslash X)^{!10}$ is the subset of the ten-fold cartesian product of $\mathbb{N}\backslash X$ of tuples with pairwise distinct entries.

The X-marked cycle shapes are in a natural one-to-one correspondence with the elements of $\mathcal{C}[X]$ (justifying the name of such a monoid): Consider the map from $\mathcal{C}[X]$ to the set of all X-marked cycle shapes given by sending $\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}$ to the X-marked cycle shape consisting of, for each $i \in \mathbb{N}$, $\boldsymbol{l}(i)$ many cycles of length $i+1$ containing only the symbol $*$, and where the other cycles are constructed from those of π by adding $\boldsymbol{d}(x)$ symbols $*$ after the entry x, for each $x \in X$ (see below for an example). It is easy to show that such a map is a bijection since there is a obvious inverse to consider. With this bijection in mind, we identify $\mathcal{C}[X]$ as the set of X-marked cycle shapes. Hence $\mathcal{C}[X]$ indexes the X-conjugacy classes of $\mathfrak{S}_{\mathbb{N}}$ as previously claimed.

Example 5.10. Let X and λ be as in *Example 5.8*, then we have the identification

$$(1,7)(3,11)(5)(9)u_1^2 u_5^2 u_7^1 c_1^1 c_2^1 = (3,11)(9)(1,*,*,7,*)(5,*,*)(*,*)(*,*,*),$$

where we have added colours to aid in demonstrating the correspondence.

Given any element λ of $\mathcal{C}[X]$, we will freely move between viewing it as an X-marked cycle shape or an expression of the form $\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}$ with $\pi \in \mathfrak{S}(X)$, $\boldsymbol{d} \in \mathbb{Z}_{\geq 0}^{X}$, and $\boldsymbol{l} \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}$. As mentioned, a permutation σ belongs to $\mathsf{CL}[X](\lambda)$ for some X-marked cycle shape λ if and only if σ may be obtained from λ by replacing the symbols $*$ with distinct elements from $\mathbb{N}\backslash X$. When expressing $\lambda = \pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}$, then from the above discussion, we have the following equivalent criteria for when σ belongs to the X-conjugacy class $\mathsf{CL}[X](\lambda)$ given in terms of π, $\boldsymbol{d}$, and $\boldsymbol{l}$.

Lemma 5.11. We have $\sigma \in \mathsf{CL}[X](\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}})$ if and only if the following hold:

(i) the number of cycles of σ of length $i+1$ containing no elements of X is $\boldsymbol{l}(i)$,

(ii) $\sigma^{\boldsymbol{d}(x)+1}(x) = \pi(x)$ for each $x \in X$, and $\sigma^m(x) \notin X$ for any $1 \leq m \leq \boldsymbol{d}(x)$.

□

Now recall from the previous section the quantities $||\mathsf{C}||$, $||\mathsf{C}||^X$, and $||\mathsf{C}||_X$, for any X-conjugacy class C of $\mathfrak{S}_{\mathbb{N}}$. Then for $\lambda \in \mathcal{C}[X]$, we write

$$||\lambda|| := ||\mathsf{CL}[X](\lambda)||, \quad ||\lambda||^X := ||\mathsf{CL}[X](\lambda)||^X, \quad \text{and} \quad ||\lambda||_X := ||\mathsf{CL}[X](\lambda)||_X.$$

From the view point of an X-marked cycle shape, the quantity $||\lambda||$ counts the number of entries among the cycles of length at least two, $||\lambda||^X$ is the number of $*$ symbols appearing among the cycles, while $||\lambda||_X$ is the number of elements from X which appear in cycles of length at least two. With $\lambda = \pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}$, one can deduce

$$\begin{aligned}
||\lambda|| &= |\mathsf{Sup}(\pi)| + \sum_{x\in X} \boldsymbol{d}(x) + \sum_{i\in\mathbb{N}} (i+1)\boldsymbol{l}(i),\\
||\lambda||^X &= \sum_{x\in X} \boldsymbol{d}(x) + \sum_{i\in\mathbb{N}} (i+1)\boldsymbol{l}(i),\\
||\lambda||_X &= |\mathsf{Sup}(\pi)|.
\end{aligned}$$

We now describe a grading on $\mathcal{C}[X]$, and to do so we introduce a monoid to grade by. Consider the set $\mathbb{Z}_{\geq|X|}$ of non-negative integers greater than or equal to the cardinality of X. We equip $\mathbb{Z}_{\geq|X|}$ with the binary operator $+_X$ defined by

$$a +_X b := a + b - |X|.$$

Then one can check that $\mathbb{Z}_{\geq|X|}$ is a monoid under $+_X$ with identity $|X|$. As monoids $(\mathbb{Z}_{\geq 0}, +)$ and $(\mathbb{Z}_{\geq|X|}, +_X)$ are isomorphic, with isomorphism given by $n \mapsto n + |X|$ for all $n \in \mathbb{Z}_{\geq 0}$. For our purposes, the monoid $\mathbb{Z}_{\geq|X|}$ will be more convenient to work with.

Definition 5.12. We define the map $\mathsf{deg} : \mathcal{C}[X] \to \mathbb{Z}_{\geq|X|}$ by

$$\mathsf{deg}(\lambda) = ||\lambda||^X + |X|$$

for any $\lambda \in \mathcal{C}[X]$. We refer to $\mathsf{deg}(\lambda)$ as the *degree* of λ.

Note we have defined this degree function with the inequality of *Proposition 5.4* in mind. By considering the description of the product of $\mathcal{C}[X]$ given in *Equation* (5.2), it is easy to check that deg is a homomorphism of monoids. As such deg realises a grading on $\mathcal{C}[X]$. For any $n \in \mathbb{Z}_{\geq 0}$ define $\mathcal{C}_n[X] = \{\lambda \in \mathcal{C}[X] \mid \mathsf{deg}(\lambda) = n\}$ to be the n-th graded component of $\mathcal{C}[X]$. We have that $\mathcal{C}_n[X]$ is non-empty if and only if $n \geq |X|$. We have the disjoint union

$$\mathcal{C}[X] = \bigsqcup_{|X|\leq n} \mathcal{C}_n[X].$$

For any $n, m \in \mathbb{Z}_{\geq|X|}$, $\lambda \in \mathcal{C}_n[X]$, and $\mu \in \mathcal{C}_m[X]$, we have that $\lambda\mu \in \mathcal{C}_{n+m-|X|}[X]$. In particular, the graded component $\mathcal{C}_{|X|}[X]$ is a submonoid, and in fact this is precisely the submonoid $\mathfrak{S}(X)$. For any $n \in \mathbb{Z}_{\geq 0}$ we let $\mathcal{C}_{\leq n}[X] = \{\lambda \in \mathcal{C}[X] \mid \mathsf{deg}(\lambda) \leq n\}$. Again $\mathcal{C}_{\leq n}[X]$ is non-empty if and only if $n \geq |X|$. In this case we have the disjoint union

$$\mathcal{C}_{\leq n}[X] = \bigsqcup_{|X|\leq m\leq n} \mathcal{C}_m[X].$$

By *Proposition 5.4*, for any $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$, we have that $\mathcal{C}_{\leq n}[X]$ indexes the X-conjugacy classes of $\mathfrak{S}_n$. Given such an n, for any $\lambda \in \mathcal{C}[X]$ we set $\mathsf{CL}_n[X](\lambda) := \mathsf{CL}[X](\lambda) \cap \mathfrak{S}_n$. Hence $\mathsf{CL}_n[X](\lambda) = \emptyset$ whenever $\mathsf{deg}(\lambda) > n$, and otherwise is the X-conjugacy class of $\mathfrak{S}_n$ containing all the permutations of $\mathfrak{S}_n$ one can obtain from the X-marked cycle shape λ by replacing the symbols $*$ with distinct elements from $[n]\backslash X$.

We end this section by describing generating functions whose coefficients record the size of $\mathcal{C}_n[X]$ and $\mathcal{C}_{\leq n}[X]$. As a corollary we obtain a formula for the size of $\mathcal{C}_{\leq n}[X]$ given in terms of the number of partitions of integers less than or equal to n. Given a generating function $F(t)$ in a formal variable t, we let $[t^n]F(t)$ denote the coefficient of the n-th degree term t^n.

Proposition 5.13. Let t be a formal variable. As generating functions we have

$$\sum_{n=0}^{\infty} |\mathcal{C}_n[X]| t^n = \frac{|X|! t^{|X|}}{(1-t)^{|X|}} \prod_{n=2}^{\infty} \frac{1}{1-t^n}, \tag{5.3}$$

and

$$\sum_{n=0}^{\infty} |\mathcal{C}_{\leq n}[X]| t^n = \frac{|X|! t^{|X|}}{(1-t)^{|X|}} \prod_{n=1}^{\infty} \frac{1}{1-t^n}. \tag{5.4}$$

Proof. We begin by showing *Equation* (5.3). For any $\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}} \in \mathcal{C}[X]$ we have that

$$\mathsf{deg}(\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}) = |X| + \sum_{x \in X} \boldsymbol{d}(x) + \sum_{i \in \mathbb{N}} (i+1) \boldsymbol{l}(i).$$

Now for any $m \in \mathbb{Z}_{\geq 0}$ define the subsets of $\mathcal{C}[X]$ given by

$$\mathfrak{S}\mathcal{U}^{(m)} := \left\{ \pi u^{\boldsymbol{d}} \;\middle|\; \pi \in \mathfrak{S}(X),\ \boldsymbol{d} \in \mathbb{Z}_{\geq 0}^{X},\ \sum_{x \in X} \boldsymbol{d}(x) = m \right\},$$

and

$$\mathcal{C}^{(m)} := \left\{ c^{\boldsymbol{l}} \;\middle|\; \boldsymbol{l} \in \mathbb{Z}_{\geq 0}^{\mathbb{N}},\ \sum_{i \in \mathbb{N}} (i+1) \boldsymbol{l}(i) = m \right\}.$$

For any $a, b \in \mathbb{Z}_{\geq 0}$, given $\pi u^{\boldsymbol{d}} \in \mathfrak{S}\mathcal{U}^{(a)}$ and $c^{\boldsymbol{l}} \in \mathcal{C}^{(b)}$, we have $\mathsf{deg}(\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}) = a + b + |X|$, and all elements in $\mathcal{C}[X]$ of degree $a + b + |X|$ appear uniquely in such a manner. Hence for any $n \in \mathbb{Z}_{\geq 0}$ we have that

$$|\mathcal{C}_{n+|X|}[X]| = \sum_{\substack{a,b \geq 0 \\ a+b=n}} |\mathfrak{S}\mathcal{U}^{(a)}| |\mathcal{C}^{(b)}|.$$

Therefore, if $F(t)$ is the generating function such that $[t^n]F(t) = |\mathfrak{S}\mathcal{U}^{(n)}|$, and $G(t)$ the generating function such that $[t^n]G(t) = |\mathcal{C}^{(n)}|$, then

$$[t^n](t^{|X|} F(t) G(t)) = |\mathcal{C}_n[X]|. \tag{5.5}$$

The elements of $\mathcal{C}^{(n)}$ are the X-marked cycle shapes where the elements of X belong to cycles of length one, and the remaining cycles contain n symbols $*$ in total. The cycles containing only the symbols $*$ must be of length at least two. Thus one can deduce that

$$G(t) = \sum_{n=0}^{\infty} |\mathcal{C}^{(n)}| t^n = \prod_{n=2}^{\infty} \frac{1}{1-t^n},$$

since the factor $(1-t^n)^{-1}$ accounts for the number of cycles of length n containing only the symbols $*$ present in such an X-marked cycle shape. As for the set $\mathfrak{S}\mathcal{U}^{(n)}$, it is clear that its cardinality is $|X|!$ times the number of maps $\boldsymbol{d} : \mathbb{Z}_{\geq 0}^X \to \mathbb{Z}_{\geq 0}$ such that the sum of $\boldsymbol{d}(x)$ for each $x \in X$ is precisely n. The number of such maps is precisely the coefficient of t^n in the generating function $(1-t)^{-|X|}$, since each factor $(1-t)^{-1}$ accounts for the choice of the image of a given element of X. As such

$$F(t) = \sum_{n=0}^{\infty} |\mathfrak{S}\mathcal{U}^{(n)}| t^n = |X|! \left(\frac{1}{1-t}\right)^{|X|}.$$

Hence from *Equation* (5.5) we have that

$$\sum_{n=0}^{\infty} |\mathcal{C}_n[X]| t^n = t^{|X|} |X|! \left(\frac{1}{1-t}\right)^{|X|} \prod_{n=2}^{\infty} \frac{1}{1-t^n},$$

which is precisely *Equation* (5.3). As for *Equation* (5.4), this follows from *Equation* (5.3) by noting that for any generating function $A(t)$, the new generating function $(1-t)^{-1}A(t)$ records the partial sums of the coefficients of $A(t)$, that is to say

$$[t^n]\left((1-t)^{-1}A(t)\right) = \sum_{i=0}^{n} [t^n] A(t).$$

$\square$

Recall that Λ_n is the set of all partitions of n. Then the above proposition allows us to give a formula for the size of $\mathcal{C}_{\leq n}[X]$ in terms of the sizes of the sets Λ_m for $m \leq n$.

Corollary 5.14. The cardinality of $\mathcal{C}_{\leq n}[X]$ is given by

$$|\mathcal{C}_{\leq n}[X]| = \sum_{\substack{a \geq |X|, b \geq 0 \\ a+b=n}} |X|! \binom{a-1}{a-|X|} |\Lambda_b|.$$

Proof. It is well-known that

$$\sum_{n=0}^{\infty} |\Lambda_n| t^n = \prod_{n=1}^{\infty} \frac{1}{1-t^n}.$$

Also it is known that

$$\left(\frac{1}{1-t}\right)^{|X|} = \sum_{n=0}^{\infty} \binom{|X|+n-1}{n} t^n,$$

and so

$$\left(\frac{t}{1-t}\right)^{|X|} = \sum_{n=0}^{\infty} \binom{|X|+n-1}{n} t^{n+|X|} = \sum_{n=|X|}^{\infty} \binom{n-1}{n-|X|} t^n.$$

Then the result follows from *Equation* (5.4) of *Proposition 5.13*. Note we are using the generalised binomial coefficients here when X is empty.

$\square$

5.1.3 Centralizer Algebras $Z_n(X)$

Assume throughout this section that $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$. In this section we introduce the X-centralizer algebras $Z_n(X)$, which are subalgebras of the group algebras of the symmetric group. When X is empty, these algebras are precisely the centers $Z_n = Z(\mathbb{C}\mathfrak{S}_n)$. We give a basis of the X-centralizer algebras by class sums, and show that the associated structure constants are polynomial in n. We end the section by establishing various notation and proving some technical results for later proofs.

Recall that $\mathsf{Stab}_n(X)$ is the subgroup of $\mathfrak{S}_n$ consisting of all permutations which act trivially on all elements belong to the set X. Then we define the *X-centralizer algebra* as

$$Z_n(X) := \{z \in \mathbb{C}\mathfrak{S}_n \mid \tau z = z\tau \text{ for all } \tau \in \mathsf{Stab}_n(X)\}, \tag{5.6}$$

which is a subalgebra of $\mathbb{C}\mathfrak{S}_n$. When $X = \emptyset$ we have that $\mathsf{Stab}_n(\emptyset) = \mathfrak{S}_n$, and so the X-centralizer algebra is precisely the center $Z_n(\emptyset) = Z_n$. Analogous to the center, we have a natural basis of $Z_n(X)$ given by X-conjugacy class sums of $\mathbb{C}\mathfrak{S}_n$. Recall that the set $\mathcal{C}_{\leq n}[X]$ of X-marked cycle shapes of degree less than or equal to n gives an indexing set for the X-conjugacy classes of $\mathfrak{S}_n$.

Definition 5.15. For $\lambda \in \mathcal{C}_{\leq n}[X]$, we define the X-conjugacy class sum as

$$K_n(\lambda) := \sum_{\pi \in \mathsf{CL}_n[X](\lambda)} \pi.$$

Since X-conjugacy classes of $\mathfrak{S}_n$ are precisely the orbits associated to the action of $\mathsf{Stab}_n(X)$ on $\mathfrak{S}_n$ by conjugation, the proceeding result follows in a completely analogous manner to *Proposition 2.8* of *Section 2.1.3*.

Proposition 5.16. The set of class sums $\{K_n(\lambda) \mid \lambda \in \mathcal{C}_{\leq n}[X]\}$ forms a basis of $Z_n(X)$.

$\square$

As was the case for the class sums of the centers, we prove that the structure constants associated to the basis $\{K_n(\lambda) \mid \lambda \in \mathcal{C}_{\leq n}[X]\}$ are polynomial in n.

Theorem 5.17. Let z be a formal variable, $n \in \mathbb{Z}_{\geq 0}$, and $X \subseteq [n]$. For each $\lambda, \mu, \tau \in \mathcal{C}_{\leq n}[X]$ there exists a unique polynomial $f^{\tau}_{\lambda,\mu}(z)$ such that in the centralizer algebra $Z_n(X)$ we have

$$K_n(\lambda)K_n(\mu) = \sum_{\tau \in \mathcal{C}_{\leq n}[X]} f^{\tau}_{\lambda,\mu}(n) K_n(\tau).$$

We refer to the polynomials $f^{\tau}_{\lambda,\mu}(z)$ as the *structure polynomials*.

Proof. Fix $\lambda, \mu, \tau \in \mathcal{C}_{\leq n}[X]$. Consider the set of pairs

$$A = \{(\pi_1, \pi_2) \in \mathsf{CL}[X](\lambda) \times \mathsf{CL}[X](\mu) \mid \pi_1\pi_2 \in \mathsf{CL}[X](\tau)\} \subset \mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}}.$$

When A is empty, then we may set $f^{\tau}_{\lambda,\mu}(z) := 0$. Assume that A is non-empty and let $(\pi_1, \pi_2) \in A$. Then for any $\sigma \in \mathsf{Stab}(X)$ we have that $(\sigma\pi_1\sigma^{-1})(\sigma\pi_2\sigma^{-1}) = \sigma(\pi_1\pi_2)\sigma^{-1}$

which belongs to $\mathsf{CL}[X](\tau)$ since $\pi_1\pi_2 \in \mathsf{CL}[X](\tau)$. Therefore $(\sigma\pi_1\sigma^{-1}, \sigma\pi_2\sigma^{-1})$ belongs to A for any $\sigma \in \mathsf{Stab}(X)$. Thus for some indexing set I, we have the disjoint union

$$A = \bigsqcup_{i \in I} \mathsf{C}^{(i)}$$

where $\mathsf{C}^{(i)}$ is an X-conjugacy class of $\mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}}$ for each $i \in I$. For any $(\pi_1^{(i)}, \pi_2^{(i)}) \in \mathsf{C}^{(i)}$,

$$||\mathsf{C}^{(i)}||^X = |\mathsf{Sup}^X(\pi_1^{(i)}) \cup \mathsf{Sup}^X(\pi_2^{(i)})| \leq |\mathsf{Sup}^X(\pi_1^{(i)})| + |\mathsf{Sup}^X(\pi_2^{(i)})| = ||\lambda||^X + ||\mu||^X.$$

Thus for any $n \geq ||\lambda||^X + ||\mu||^X + |X|$ such that $X \subseteq [n]$, by *Proposition 5.4* we have that $\mathsf{C}_n^{(i)}$ is an X-conjugacy class of $\mathfrak{S}_n \times \mathfrak{S}_n$ for each $i \in I$. This implies that the indexing set I is finite. Also by *Proposition 5.5* we have that

$$|A \cap (\mathfrak{S}_n \times \mathfrak{S}_n)| = \sum_{i \in I} \frac{1}{b(\mathsf{C}^{(i)})}(n - |X|)(n - |X| - 1) \cdots (n - |X| - ||\mathsf{C}^{(i)}||^X + 1),$$

where $b(\mathsf{C}^{(i)})$ are constants independent of n. By definition of A, the multiplicity of $K_n(\tau)$ in the product $K_n(\lambda)K_n(\mu)$ is $|A \cap (\mathfrak{S}_n \times \mathfrak{S}_n)|$ divided by $|\mathsf{CL}_n[X](\tau)|$. Again by *Proposition 5.5* we have that

$$|\mathsf{CL}_n[X](\tau)| = \frac{1}{b(\tau)}(n - |X|)(n - |X| - 1) \cdots (n - |X| - ||\tau||^X + 1),$$

where $b(\tau) = b(\mathsf{CL}[X](\tau))$ is a constant independent of n. Now for any $(\pi_1^{(i)}, \pi_2^{(i)}) \in \mathsf{C}^{(i)}$,

$$||\tau||^X = |\mathsf{Sup}^X(\pi_1^{(i)}\pi_2^{(i)})| \leq |\mathsf{Sup}^X(\pi_1^{(i)}) \cup \mathsf{Sup}^X(\pi_2^{(i)})| = ||\mathsf{C}^{(i)}||.$$

Thus we have that $|A \cap (\mathfrak{S}_n \times \mathfrak{S}_n)|$ divided by $|\mathsf{CL}_n[X](\tau)|$ is given by

$$b(\tau) \sum_{i \in I} \frac{1}{b(\mathsf{C}^{(i)})}(n - |X| - ||\tau||^X)(n - |X| - ||\tau||^X - 1) \cdots (n - |X| - ||\mathsf{C}^{(i)}||^X + 1).$$

Hence letting $f_{\lambda,\mu}^{\tau}(z)$ be the polynomial obtained from the above expression by replacing n with z gives the desired structure polynomial.

□

We now want to compare different centralizer algebras and their corresponding structure polynomials. Most of this is fairly obvious, but it is convenient to explicitly state such results for later use.

Definition 5.18. For any $\sigma \in \mathfrak{S}_{\mathbb{N}}$ and $\lambda \in \mathcal{C}[X]$, let λ^σ be the element of $\mathcal{C}[\sigma(X)]$ obtained from λ by replacing the elements of X in the cycles of λ with their corresponding images under σ.

Example 5.19. Let $X = \{1, 3, 5\}$ and $\lambda = (1, *, 5, *)(3, *, *)(*, *) \in \mathcal{C}[X]$. Given the permutation $\sigma = (1, 4, 9)(2, 5)(7, 8)$, we have the $\sigma(X)$-marked cycle shape

$$\lambda^\sigma = (\sigma(1), *, \sigma(5), *)(\sigma(3), *, *)(*, *) = (4, *, 2, *)(3, *, *)(*, *).$$

For any $\sigma \in \mathfrak{S}_{\mathbb{N}}$, consider the map $(-)^\sigma : \mathcal{C}[X] \to \mathcal{C}[\sigma(X)]$. Given $\lambda = \pi u^{\boldsymbol{d}} c^{\boldsymbol{l}} \in \mathcal{C}[X]$, then one can deduce that

$$\lambda^\sigma = \sigma\pi\sigma^{-1} u^{\sigma\circ\boldsymbol{d}} c^{\boldsymbol{l}}$$

where $\sigma \circ \boldsymbol{d} \in \mathbb{Z}_{\geq 0}^{\sigma(X)}$ is given by $(\sigma \circ \boldsymbol{d})(y) = \boldsymbol{d}(\sigma^{-1}(y))$ for each $y \in \sigma(X)$. One can show from *Equation* (5.2) that $(-)^\sigma$ is an isomorphism of monoids. Also by *Lemma 2.3* it is clear that $\mathsf{CL}[\sigma(X)](\lambda^\sigma) = \sigma\mathsf{CL}[X](\lambda)\sigma^{-1}$ for any $\lambda \in \mathcal{C}[X]$ and $\sigma \in \mathfrak{S}_{\mathbb{N}}$. Hence for any $\sigma \in \mathfrak{S}_{\mathbb{N}}$, picking $n \in \mathbb{Z}_{\geq 0}$ such that $X, \sigma(X) \subseteq [n]$, we may regard $(-)^\sigma$ as a map $Z_n(X) \to Z_n(\sigma(X))$ by the $\mathbb{C}$-linear extension of $K_n(\lambda)^\sigma := K_n(\lambda^\sigma) = \sigma K_n(\lambda)\sigma^{-1}$. It is simple to check that $(-)^\sigma$ realised an isomorphism of $\mathbb{C}$-algebras $Z_n(X) \cong Z_n(\sigma(X))$. Also, it is simple to check that for any $\pi \in \mathfrak{S}_n$ and $K \in Z_n(X)$ we have

$$[\pi]K = [\pi^\sigma]K^\sigma. \tag{5.7}$$

That is the coefficient of π in K is the same as that of $\sigma\pi\sigma^{-1}$ in $\sigma K\sigma^{-1}$. As such we immediately obtain the following results.

Lemma 5.20. For $\lambda, \mu, \tau \in \mathcal{C}[X]$ and $\sigma \in \mathfrak{S}_{\mathbb{N}}$, we have an equality $f_{\lambda,\mu}^{\tau}(z) = f_{\lambda^\sigma,\mu^\sigma}^{\tau^\sigma}(z)$ of structure polynomials.

□

Let $X \subseteq Y \subset \mathbb{N}$ with both such subsets finite. Let $n \in \mathbb{Z}_{\geq 0}$ be such that $X, Y \subseteq [n]$. Then $\mathsf{Stab}_n(Y) \subseteq \mathsf{Stab}_n(X)$, and hence from *Equation* (5.6) one can see that $Z_n(X) \subseteq Z_n(Y)$. We now describe this embedding in terms of the class sum basis.

Definition 5.21. Given finite subsets $X \subseteq Y \subset \mathbb{N}$ and $\lambda \in \mathcal{C}[X]$, we say that $\mu \in \mathcal{C}[Y]$ is a Y-*filling of* λ if it can be obtained from λ by adding to it the elements of $Y \backslash X$ by either replacing symbols $*$ or adding cycles of length one. We write $\mathsf{Fill}_X^Y(\lambda)$ to denote the set of all Y-fillings of λ.

Example 5.22. Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3, 4, 5\}$. Consider the X-marked cycle shape $\lambda = (1, *, 2)(3, *)$, then the Y-fillings of λ are given by

$$\begin{aligned}\mathsf{Fill}_X^Y(\lambda) = \big\{&(1,*,2)(3,*)(4)(5),\ (1,4,2)(3,*)(5),\ (1,*,2)(3,4)(5),\\ &(1,5,2)(3,*)(4),\ (1,*,2)(3,5)(4),\ (1,4,2)(3,5),\ (1,5,2)(3,4)\big\}.\end{aligned}$$

Given $\lambda \in \mathcal{C}[X]$ and a finite set $Y \subset \mathbb{N}$ containing X, it is clear that

$$\{\mathsf{CL}[Y](\mu) \mid \mu \in \mathsf{Fill}_X^Y(\lambda)\}$$

gives a set partition of the X-conjugacy class $\mathsf{CL}[X](\lambda)$. In particular, each $\mathsf{CL}[Y](\mu)$ is refining the elements of $\mathsf{CL}[X](\lambda)$ by also encoding the relative positions of the elements of $Y \backslash X$ among the cycles of the permutations. Thus, given any $n \in \mathbb{Z}_{\geq 0}$ such that $Y \subseteq [n]$, we have that $\{\mathsf{CL}_n[Y](\mu) \mid \mu \in \mathsf{Fill}_X^Y(\lambda)\}$ gives a set partition $\mathsf{CL}_n[X](\lambda)$, and hence we have the equality

$$K_n(\lambda) = \sum_{\mu \in \mathsf{Fill}_X^Y(\lambda)} K_n(\mu). \tag{5.8}$$

Hence the inclusion $Z_n(X) \subseteq Z_n(Y)$ may be described by the embedding given on the class sum basis by $K_n(\lambda) \mapsto \sum_{\mu \in \mathsf{Fill}_X^Y(\lambda)} K_n(\mu)$.

We end this section by proving a few technical results which will play roles in later proofs. The last of such will prove that another quantity which arises when dealing with centralizer algebras is polynomial in n.

Let G be a group and H a subgroup of G. Recall that a left transversal of H in G is a set T containing exactly one element from each left coset of H in G. In particular we have the disjoint union

$$G = \bigsqcup_{t \in T} tH.$$

Given finite sets $X \subseteq Y \subset \mathbb{N}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $Y \subseteq [n]$, we have $\mathsf{Stab}_n(Y) \subseteq \mathsf{Stab}_n(X)$. Let $\mathcal{T}_n$ be a left transversal of $\mathsf{Stab}_n(Y)$ in $\mathsf{Stab}_n(X)$. Then, as we show below, the elements of $\mathcal{T}_n$ are precisely categorised by their action on $Y \backslash X$.

Lemma 5.23. Given any injective map $\phi : Y \backslash X \to [n] \backslash X$, there exists one and only one element $\tau \in \mathcal{T}_n$ such that $\tau(i) = \phi(i)$ for each $i \in Y \backslash X$.

Proof. Fix some injective map $\phi : Y \backslash X \to [n] \backslash X$. Since elements of $\mathsf{Stab}_n(X)$ are all those in $\mathfrak{S}_n$ which fix X element-wise, there clearly exists a permutation $\pi \in \mathsf{Stab}_n(X)$ such that $\pi(i) = \phi(i)$ for each $i \in Y \backslash X$. Now suppose $\tau \in \mathcal{T}_n$ is such that $\pi \in \tau \mathsf{Stab}_n(Y)$, hence $\pi = \tau\sigma$ for some $\sigma \in \mathsf{Stab}_n(Y)$. Thus for each $i \in Y \backslash X$, since σ acts trivially on Y, we have that $\phi(i) = \pi(i) = (\tau\sigma)(i) = \tau(\sigma(i)) = \tau(i)$. Therefore we have show that there exists such an element $\tau \in \mathcal{T}_n$ with the desired property. Consider the map $I : \mathcal{T}_n \to \{\phi : Y \backslash X \to [n] \backslash X \mid \phi \text{ is injective}\}$ sending τ to its restriction on $Y \backslash X$. We have just shown that I is surjective. However, note that

$$|\mathcal{T}_n| = \frac{|\mathsf{Stab}_n(X)|}{|\mathsf{Stab}_n(Y)|} = \frac{(n - |X|)!}{(n - |Y|)!} = (n - |X|)(n - |X| - 1) \cdots (n - |X| - |Y \backslash X| + 1).$$

As such $\mathcal{T}_n$ has the same cardinality as the set $\{\phi : Y \backslash X \to [n] \backslash X \mid \phi \text{ is injective}\}$, and so I must be injective also, and thus bijective, which completes the proof.

□

Definition 5.24. Given finite sets $X \subseteq Y \subset \mathbb{N}$ and $\mu \in \mathcal{C}[Y]$, let $\mu \downarrow_X$ denote the element of $\mathcal{C}[X]$ obtained from μ by replacing the elements of $Y \backslash X$ with the symbols $*$, or when such an element belongs to a cycle of length one, remove such a cycle.

Example 5.25. Let $X = \{1, 2, 3\}$, $Y = \{1, 2, 3, 4, 5, 6\}$, and consider the Y-marked cycle shape given by $\mu = (2, 5)(4)(3, *, 6, *)(1, *)(*, *)$. Then

$$\mu \downarrow_X = (2, *)(3, *, *, *)(1, *)(*, *),$$

hence the cycle (4) was removed, and the entries 5 and 6 where replaced with $*$.

Remark 5.26. Comparing to *Definition 5.21*, it is easy to see that given finite sets $X \subseteq Y \subset \mathbb{N}$ and any $\lambda \in \mathcal{C}[X]$, we have $\mathsf{Fill}_X^Y(\lambda) = \{\mu \in \mathcal{C}[Y] \mid \mu \downarrow_X = \lambda\}$.

Lemma 5.27. Let $X \subseteq Y \subset \mathbb{N}$ be finite sets, $\mu \in \mathcal{C}[Y]$, and $n \in \mathbb{Z}_{\geq 0}$ such that $Y \subseteq [n]$. Let $\mathcal{T}_n$ be a left transversal of $\mathsf{Stab}_n(Y)$ in $\mathsf{Stab}_n(X)$, then as sets we have

$$\mathsf{CL}_n[X](\mu\downarrow_X) = \bigcup_{\tau\in\mathcal{T}_n} \mathsf{CL}_n[Y](\mu^\tau).$$

Proof. It is clear that $\mathsf{CL}_n[Y](\mu) \subseteq \mathsf{CL}_n[X](\mu\downarrow_X)$, thus for $\pi \in \mathsf{CL}[Y](\mu)$ we have

$$\mathsf{CL}_n[Y](\mu) = \{h\pi h^{-1} \mid h \in \mathsf{Stab}_n(Y)\} \text{ and } \mathsf{CL}_n[X](\mu\downarrow_X) = \{g\pi g^{-1} \mid g \in \mathsf{Stab}_n(X)\}.$$

Since $\mathsf{Stab}_n(X)$ is the disjoint union of $\tau\mathsf{Stab}_n(Y)$ as τ runs over $\mathcal{T}_n$, we have that

$$\begin{aligned}
\mathsf{CL}_n[X](\mu\downarrow_X) &= \{(\tau h)\pi(\tau h)^{-1} \mid (\tau,h) \in \mathcal{T}_n \times \mathsf{Stab}_n(Y)\} \\
&= \{\tau(h\pi h^{-1})\tau^{-1} \mid (\tau,h) \in \mathcal{T}_n \times \mathsf{Stab}_n(Y)\} \\
&= \bigcup_{\tau\in\mathcal{T}_n} \tau\{h\pi h^{-1} \mid h \in \mathsf{Stab}_n(Y)\}\tau^{-1} \\
&= \bigcup_{\tau\in\mathcal{T}_n} \tau\mathsf{CL}_n[Y](\mu)\tau^{-1} = \bigcup_{\tau\in\mathcal{T}_n} \mathsf{CL}_n[Y](\mu^\tau).
\end{aligned}$$

□

Proposition 5.28. Let $X \subseteq Y \subset \mathbb{N}$ be finite sets and $\mu \in \mathcal{C}[Y]$. For all $n \in \mathbb{Z}_{\geq 0}$ where $Y \subseteq [n]$, let $\mathcal{T}_n$ be any left transversal of $\mathsf{Stab}_n(Y)$ in $\mathsf{Stab}_n(X)$, then there exists a polynomial $f^\mu_X(z)$ such that

$$K := \sum_{\tau\in\mathcal{T}_n} K_n(\mu^\tau) = f^\mu_X(n)K_n(\mu\downarrow_X).$$

We call the polynomials $f^\mu_X(z)$ the *Transversal Polynomials.*

Proof. For any $\pi \in \mathfrak{S}_n$ and $A \in \mathbb{C}\mathfrak{S}_n$, let $[\pi]A \in \mathbb{C}$ denote the coefficient of π in A, and write $\pi \in A$ whenever $[\pi]A \neq 0$. From *Lemma 5.27* we know that $\pi \in K$ if and only if $\pi \in K_n(\mu\downarrow_X)$. We first prove that for any two $\pi_1, \pi_2 \in K$ we have $[\pi_1]K = [\pi_2]K$. Given any $\tau \in \mathcal{T}_n$, since $K_n(\mu^\tau)$ is a class sum we have that $[\pi]K_n(\mu^\tau) \in \{0,1\}$ for any $\pi \in \mathfrak{S}_n$. Now let $\mathcal{T}_n(\pi)$ be the subset of $\mathcal{T}_n$ consisting of all τ such that $\pi \in K_n(\mu^\tau)$, then we have that

$$[\pi]K = \sum_{\tau\in\mathcal{T}_n} [\pi]K_n(\mu^\tau) = |\mathcal{T}_n(\pi)|.$$

So we want to show $|\mathcal{T}_n(\pi_1)| = |\mathcal{T}_n(\pi_2)|$. For any $h \in \mathfrak{S}_n$, let $L_h : \mathcal{T}_n \to \mathcal{T}_n$ be given by sending τ to the unique element $L_h(\tau)$ of $\mathcal{T}_n$ such that $h\tau \in L_h(\tau)\mathsf{Stab}_n(Y)$. It is easy to check that L_h is bijective. Now since $\pi_1, \pi_2 \in K_n(\mu\downarrow_X)$ there exists $h \in \mathsf{Stab}_n(X)$ such that $\pi_2 = h\pi_1 h^{-1}$. Also for $\tau \in \mathcal{T}_n(\pi_1)$ we have $\pi_1 \in K_n(\mu^\tau)$, and so there exists a unique $g \in K_n(\mu)$ such that $\pi_1 = \tau g\tau^{-1}$. Thus we have

$$\pi_2 = (h\tau)g(h\tau)^{-1} = L_h(\tau)\omega g\omega^{-1}L_h(\tau)^{-1} = L_h(\tau)fL_h(\tau)^{-1},$$

where $\omega \in \mathsf{Stab}_n(Y)$ and $f = \omega g \omega^{-1} \in K_n(\mu)$. So $\pi_2 = L_h(\tau) f L_h(\tau)^{-1}$, which implies that $L_h(\tau) \in \mathcal{T}_n(\pi_2)$. Hence restricting L_h to $\mathcal{T}_n(\pi_1)$ gives a mapping $\mathcal{T}_n(\pi_1) \to \mathcal{T}_n(\pi_2)$. Analgously one can show that restricting $L_{h^{-1}}$ to $\mathcal{T}_n(\pi_2)$ gives a mapping $\mathcal{T}_n(\pi_2) \to \mathcal{T}_n(\pi_1)$, and naturally this map is an inverse to L_h. Therefore $|\mathcal{T}_n(\pi_1)| = |\mathcal{T}_n(\pi_2)|$. So we have currently shown that $K = f_X^\mu(n) K_n(\mu\downarrow_X)$ for some constant $f_X^\mu(n) \in \mathbb{N}$, and now need to show that such a constant is polynomial in n. Since $|\mathsf{CL}_n[Y](\mu)| = |\mathsf{CL}_n[Y](\mu^\tau)|$ for any $\tau \in \mathcal{T}_n$, then by definition of K we have that

$$f_X^\mu(n) = \frac{|\mathcal{T}_n||\mathsf{CL}_n[Y](\mu)|}{|\mathsf{CL}_n[X](\mu\downarrow_X)|}. \tag{5.9}$$

Let $\pi \in \mathsf{CL}_n[Y](\mu) \subseteq \mathsf{CL}_n[X](\mu\downarrow_X)$, then by the orbit-stabilizer theorem we have

$$|\mathsf{CL}_n[Y](\mu)| = \frac{|\mathsf{Stab}_n(Y)|}{|\mathsf{Stab}_{\mathsf{Stab}_n(Y)}(\pi)|}, \quad \text{and} \quad |\mathsf{CL}_n[X](\mu\downarrow_X)| = \frac{|\mathsf{Stab}_n(X)|}{|\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\pi)|}.$$

As a special case of the claim proved in the proof of *Proposition 5.5*, one can deduce the following isomorphisms of groups:

$$\begin{aligned}
\mathsf{Stab}_{\mathsf{Stab}_n(Y)}(\pi) &\cong \mathsf{Stab}_n(Y \cup \mathsf{Sup}^Y(\pi)) \times \mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^Y(\pi))}(\pi), \\
\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\pi) &\cong \mathsf{Stab}_n(X \cup \mathsf{Sup}^X(\pi)) \times \mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\pi))}(\pi).
\end{aligned}$$

Since the size of the sets $\mathsf{Sup}^Y(\pi)$ and $\mathsf{Sup}^X(\pi)$ are independent in n, so are the sizes of the groups $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^Y(\pi))}(\pi)$ and $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\pi))}(\pi)$. Hence we set

$$\begin{aligned}
b(\mu) &:= |\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^Y(\pi))}(\pi)|, \\
b(\mu\downarrow_X) &:= |\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\pi))}(\pi)|.
\end{aligned}$$

Also note that $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^Y(\pi))}(\pi)$ is a subgroup of $\mathsf{Stab}_{\mathfrak{S}(\mathsf{Sup}^X(\pi))}(\pi)$, and hence $b(\mu)$ divides $b(\mu\downarrow_X)$. We set $c(\mu) := b(\mu\downarrow_X)/b(\mu) \in \mathbb{N}$. Then from *Equation* (5.9) we have

$$\begin{aligned}
f_X^\mu(n) &= |\mathcal{T}_n||\mathsf{CL}_n[Y](\mu)|\frac{1}{|\mathsf{CL}_n[X](\mu\downarrow_X)|} \\
&= \frac{|\mathsf{Stab}_n(X)|}{|\mathsf{Stab}_n(Y)|}\frac{|\mathsf{Stab}_n(Y)|}{|\mathsf{Stab}_{\mathsf{Stab}_n(Y)}(\pi)|}\frac{|\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\pi)|}{|\mathsf{Stab}_n(X)|} \\
&= \frac{|\mathsf{Stab}_{\mathsf{Stab}_n(X)}(\pi)|}{|\mathsf{Stab}_{\mathsf{Stab}_n(Y)}(\pi)|} \\
&= \frac{b(\mu\downarrow_X)}{b(\mu)}\frac{(n-|\mathsf{Sup}^X(\pi)|)!}{(n-|\mathsf{Sup}^Y(\pi)|)!} \\
&= c(\mu)(n-|\mathsf{Sup}^X(\pi)|)(n-|\mathsf{Sup}^X(\pi)|-1)\cdots(n-|\mathsf{Sup}^X(\pi)|-|\mathsf{Sup}^Y(\pi)|+1).
\end{aligned}$$

Hence replacing the occurences of n in the above expression with the variable z give the desired polynomial $f_Y^\mu(z)$.

□

Lemma 5.29. Let $X \subseteq Y \subset \mathbb{N}$ be finite sets and $\mu \in \mathcal{C}[Y]$. For all $n \in \mathbb{Z}_{\geq 0}$ where $Y \subseteq [n]$, and any $\sigma \in \mathfrak{S}_n$, we have the equality $f_X^{\mu}(z) = f_{\sigma(X)}^{\mu^\sigma}(z)$ of transversal polynomials.

Proof. As mentioned previously, the linear extension of $(-)^\sigma$ realises an isomorphism of centralizer algebras $Z_n(Y) \cong Z_n(\sigma(Y))$. Let $\mathcal{T}_n$ be a left transversal of $\mathsf{Stab}_n(Y)$ in $\mathsf{Stab}_n(X)$, then one can check that $\mathcal{T}_n^\sigma := \sigma\mathcal{T}_n\sigma^{-1}$ gives a left transversal of $\mathsf{Stab}_n(\sigma(Y))$ in $\mathsf{Stab}_n(\sigma(X))$. Then by *Proposition 5.28*, under the isomorphism $(-)^\sigma$ we have that

$$\begin{aligned} f_X^{\mu}(n)K_n(\mu\downarrow_X) = \sum_{\tau\in\mathcal{T}_n} K_n(\mu^\tau) \mapsto \sum_{\tau\in\mathcal{T}_n} K_n(\mu^\tau)^\sigma &= \sum_{\tau\in\mathcal{T}_n} K_n(\mu^{\tau\sigma}) \\ &= \sum_{\tau\in\mathcal{T}_n^\sigma} K_n(\mu^{\sigma\tau}) = \sum_{\tau\in\mathcal{T}_n^\sigma} K_n((\mu^\sigma)^\tau) \\ &= f_{\sigma(X)}^{\mu^\sigma}(n)K_n((\mu^\sigma)\downarrow_{\sigma(X)}) \\ &= f_{\sigma(X)}^{\mu^\sigma}(n)K_n(\mu\downarrow_X)^\sigma \end{aligned}$$

Hence by *Equation* (5.7) these polynomials agree on infinitely many natural numbers, and so must be equal.

$\square$

5.1.4 Dimension Formula

By *Proposition 5.16* we know that $\mathsf{dim}(Z_n(X)) = |\mathcal{C}_{\leq n}[X]|$, and from *Corollary 5.14* given at the end of *Section 5.1.2*, we have a formula to calculate such a quantity. In this section we present an alterative expression for the dimension of $Z_n(X)$ which comes from the representation theory of the symmetric groups. We will first prove a few lemmas to help with such a result.

Consider the algebra $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_n$ with component-wise addition and multiplication. The group algebra $\mathbb{C}\mathfrak{S}_n$ may be regarded as a left $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_n$-module by the extension of the action $(\pi_1 \otimes \pi_2)(\sigma) = \pi_1\sigma\pi_2^{-1}$, for all $\pi_1, \pi_2, \sigma \in \mathfrak{S}_n$.

Lemma 5.30. For any $n \in \mathbb{Z}_{\geq 0}$ such that $X \subseteq [n]$, we have an isomorphism of algebras

$$Z_n(X) \cong \mathsf{End}_{\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathsf{Stab}_n(X)}(\mathbb{C}\mathfrak{S}_n),$$

with the latter being the algebra of $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathsf{Stab}_n(X)$-module endomorphism of $\mathbb{C}\mathfrak{S}_n$.

Proof. Consider the map

$$\phi : Z_n(X) \to \mathrm{End}_{\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathsf{Stab}_n(X)}(\mathbb{C}\mathfrak{S}_n)$$

given by $\phi(z)(x) = xz$ for any $z \in Z_n(X)$ and $x \in \mathbb{C}\mathfrak{S}_n$. We will show that ϕ is an isomorphism of algebras. Let us first show that ϕ is well-defined, that is for any $z \in Z_n(X)$ we show that $\phi(z) : \mathbb{C}\mathfrak{S}_n \to \mathbb{C}\mathfrak{S}_n$ is indeed a $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathsf{Stab}_n(X)$-module homomorphism. Since $\phi(z)$ is given by right multiplying by the element z, it is clear to see that $\phi(z)$ is linear. Also, given any $\pi_1 \otimes \pi_2 \in \mathfrak{S}_n \times \mathsf{Stab}_n(X)$ and $x \in \mathbb{C}\mathfrak{S}_n$,

$$\phi(z)(\pi_1 x\pi_2^{-1}) = \pi_1 x\pi_2^{-1} z = \pi_1 xz\pi_2^{-1} = \pi_1\phi(z)(x)\pi_2^{-1} = (\pi_1 \otimes \pi_2)(\phi(z)(x)),$$

where we used the fact that $Z_n(X)$ commutes with $\mathsf{Stab}_n(X)$. Hence $\phi(z)$ is a module homomorphism. Now it is clear that ϕ is an algebra homomorphism since its given by right multiplication. So we are left with showing that ϕ is both injective and surjective. For injectivity, assume $z \in \mathrm{Ker}(\phi)$, hence $xz = 0$ for all $x \in \mathbb{C}\mathfrak{S}_n$, but picking $x = 1$ yields $z = 0$, showing that $\mathrm{Ker}(\phi) = \{0\}$. For surjectivity, let $f \in \mathrm{End}_{\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathsf{Stab}_n(X)}(\mathbb{C}\mathfrak{S}_n)$, then for any $\pi \in \mathsf{Stab}_n(X)$, we have that $\pi f(1)\pi^{-1} = f(\pi\pi^{-1}) = f(1)$. Therefore $f(1) \in Z_n(X)$. Moreover, for any $x \in \mathbb{C}\mathfrak{S}_n$ we have that $\phi(f(1))(x) = xf(1) = f(x)$. Thus $\phi(f(1)) = f$, and hence ϕ is also surjective.

$\square$

For M and N two left $\mathbb{C}\mathfrak{S}_n$-modules, then $M \otimes N$ is a left $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_n$-module given by component-wise action.

Lemma 5.31. As left $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_n$-module we have the isomorphism

$$\mathbb{C}\mathfrak{S}_n \cong \bigoplus_{\lambda \in \Lambda_n} \mathsf{S}^\lambda \otimes \mathsf{S}^\lambda.$$

Proof. By the Artin-Wedderburn Theorem we have that

$$\mathbb{C}\mathfrak{S}_n \cong \bigoplus_{\lambda \in \Lambda_n} M_{\mathsf{dim}(\lambda)}(\mathbb{C}) \tag{5.10}$$

as algebras. We identify $\mathbb{C}\mathfrak{S}_n$ as the direct product of matrix algebras described in this isomorphism, and let $\{E^\lambda_{ij} \mid \lambda \in \Lambda_n,\ i, j, \in [\mathsf{dim}(\lambda)]\}$ be the basis of $\mathbb{C}\mathfrak{S}_n$ given by the standard matrix units. From this identification, the simple $\mathbb{C}\mathfrak{S}_n$-module S^λ corresponds to the column space associated to the block $M_{\mathsf{dim}(\lambda)}(\mathbb{C})$. From this let $\{e^\lambda_i \mid i \in [\mathsf{dim}(\lambda)]\}$ be the standard basis of S^λ as a column space. Hence $\mathbb{C}\mathfrak{S}_n$ acts on S^λ by the linear extension of $E^\lambda_{ij} e^\mu_k = \delta_{jk}\delta_{\lambda\mu} e^\lambda_i$, where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise. Now let

$$\varphi : \mathbb{C}\mathfrak{S}_n \to \bigoplus_{\lambda \in \Lambda_n} \mathsf{S}^\lambda \otimes \mathsf{S}^\lambda$$

be the map defined by $E^\lambda_{ij} \mapsto e^\lambda_i \otimes e^\lambda_j$, extended linearly across $\mathbb{C}\mathfrak{S}_n$. It is immediate that φ is both linear and bijective, so what remains is to show that it preserves the $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_n$-module structure. We show this on the standard matrix units of $\mathbb{C}\mathfrak{S}_n$, where the result will then follow from linearity of φ. Note that, since irreducible representations of finite group algebras over $\mathbb{C}$ are unitary, the linear extension of the inverse $(-)^{-1}$ of $\mathbb{C}\mathfrak{S}_n$ corresponds to conjugate transpose $(-)^*$ under the isomorphism of *Equation* (5.10). Then we have

$$\begin{aligned}
(E^\lambda_{ij} \otimes E^\mu_{lk})\varphi(E^\tau_{ab}) &= (E^\lambda_{ij} \otimes E^\mu_{lk})(e^\tau_a \otimes e^\tau_b) \\
&= (E^\lambda_{ij} e^\tau_a) \otimes (E^\mu_{lk} e^\tau_b) \\
&= \delta_{\lambda\tau}\delta_{\mu\tau}\delta_{ja}\delta_{kb}(e^\tau_i \otimes e^\tau_l)
\end{aligned}$$

and

$$\begin{aligned}
\varphi((E^{\lambda}_{ij} \otimes E^{\mu}_{lk})E^{\tau}_{ab}) &= \varphi(E^{\lambda}_{ij}E^{\tau}_{ab}(E^{\mu}_{lk})^*) \\
&= \varphi(E^{\lambda}_{ij}E^{\tau}_{ab}E^{\mu}_{kl}) \\
&= \delta_{\lambda\tau}\delta_{\mu\tau}\delta_{ja}\delta_{kb}\varphi(E^{\tau}_{il}) \\
&= \delta_{\lambda\tau}\delta_{\mu\tau}\delta_{ja}\delta_{kb}(e^{\tau}_{i} \otimes e^{\tau}_{l}).
\end{aligned}$$

So $(E^{\lambda}_{ij} \otimes E^{\mu}_{lk})\varphi(E^{\tau}_{ab}) = \varphi((E^{\lambda}_{ij} \otimes E^{\mu}_{lk})E^{\tau}_{ab})$, thus φ is an isomorphism.

□

Recall *Young's lattice* $\widehat{\mathsf{S}}$ given in *Definition 2.16*. For any $a \geq b$, and partitions $\mu \in \Lambda_b$ and $\lambda \in \Lambda_a$, let $\mathsf{Path}(\mu, \lambda)$ denote the set of paths in $\widehat{\mathsf{S}}$ starting at μ and ending at λ. Such a set of paths is non-empty if and only if $\mu \subseteq \lambda$.

Proposition 5.32. Given $n \in \mathbb{Z}_{\geq 0}$ such that $X \subseteq [n]$, set $m := |X|$. Then we have

$$\mathsf{dim}_{\mathbb{C}}(Z_n(X)) = \sum_{\mu \in \Lambda_{n-m}} \sum_{\lambda \in \Lambda_n} |\mathsf{Path}(\mu, \lambda)|^2.$$

Proof. Note that for any subsets $X, Y \subseteq [n]$ such that $|X| = |Y|$, we have an isomorphism of algebras $Z_n(X) \cong Z_n(Y)$. Therefore to prove this result, it suffices to prove it for $X = \{n-m, n-m+1, \ldots, n\}$. Now by *Lemma 5.31* we have

$$\mathbb{C}\mathfrak{S}_n \cong \bigoplus_{\lambda \in \Lambda_n} \mathsf{S}^{\lambda} \otimes \mathsf{S}^{\lambda},$$

as left $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_n$-modules. Restricting the action to $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathsf{Stab}_n(X) \cong \mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_{n-m}$ gives the following decomposition,

$$\begin{aligned}
\mathsf{Res}_{\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_{n-m}}(\mathbb{C}\mathfrak{S}_n) &\cong \bigoplus_{\lambda \in \Lambda_n} \mathsf{S}^{\lambda} \otimes \mathsf{Res}_{\mathbb{C}\mathfrak{S}_{n-m}}(\mathsf{S}^{\lambda}) \\
&\cong \bigoplus_{\lambda \in \Lambda_n} \mathsf{S}^{\lambda} \otimes \left(\bigoplus_{\mu \in \Lambda_{n-m}} |\mathsf{Path}(\mu, \lambda)| \mathsf{S}^{\mu} \right) \\
&\cong \bigoplus_{\lambda \in \Lambda_n} \bigoplus_{\mu \in \Lambda_{n-m}} |\mathsf{Path}(\mu, \lambda)| (\mathsf{S}^{\lambda} \otimes \mathsf{S}^{\mu})
\end{aligned}$$

where the second isomorphism above follows from item (2) of *Theorem 2.18*. By *Lemma 5.30* we know that

$$Z_n(X) \cong \mathsf{End}_{\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_{n-m}} \left(\bigoplus_{\lambda \in \Lambda_n} \bigoplus_{\mu \in \Lambda_{n-m}} |\mathsf{Path}(\mu, \lambda)| (\mathsf{S}^{\lambda} \otimes \mathsf{S}^{\mu}) \right).$$

The proposition thus follows since $\mathsf{dim}(\mathsf{End}_A(S^{\oplus m})) = m^2$ for any algebra A, simple A-module S, and $m \in \mathbb{Z}_{\geq 0}$.

□

5.1.5 The Marked Cycle Shape Algebra $Z(X)$

In this section we define a new $\mathbb{C}[z]$-algebra $Z(X)$ which we call the X-marked cycle shape algebra, for any finite subset $X \subseteq \mathbb{N}$. This is not the monoid algebra of the X-marked cycle shape monoid $\mathcal{C}[X]$, but rather an algebra which can be interpreted as the X-centralizer algebra $Z_n(X)$ as $n \to \infty$. In particular, in the case when X is empty, the algebra $Z(\emptyset)$ will coincide with the cycle shape algebra Z of H. Farahat and G. Higman presented in *Section 2.1.3*.

Recall the X-centralizer algebra $Z_n(X)$ given by *Equation 5.6*, which has a basis given by class sums $K_n(\lambda)$ for $\lambda \in \mathcal{C}_{\leq n}[X]$, presented in *Definition 5.15*. Also recall that the corresponding structure constants for this class sum basis were shown to be polynomial in n by *Theorem 5.17*.

Definition 5.33. Let $X \subseteq \mathbb{N}$ be finite. We define $Z(X)$ to be the free $\mathbb{C}[z]$-module with basis $\{K(\lambda) \mid \lambda \in \mathcal{C}[X]\}$. Equip $Z(X)$ with the product given by the $\mathbb{C}[z]$-linear extension of

$$K(\lambda)K(\mu) = \sum_{\tau \in \mathcal{C}[X]} f^{\tau}_{\lambda,\mu}(z)K(\tau),$$

where $f^{\tau}_{\lambda,\mu}(z)$ are the structure polynomials given in *Theorem 5.17*. We call $Z(X)$ the *X-marked cycle shape algebra*.

Since $f^{\tau}_{\lambda,\mu}(n)$ is the multiplicity of $K_n(\tau)$ in the product $K_n(\lambda)K_n(\mu)$, it is clear that $f^{\tau}_{\lambda,\mu}(z)$ equals zero whenever $||\tau|| > ||\lambda|| + ||\mu||$. So the product of $Z(X)$ described above is well-defined as only finitely many terms will appear in the product of two elements. We may also view $Z(X)$ as a $\mathbb{C}$-algebra with basis $\{z^n K(\lambda) \mid n \in \mathbb{Z}_{\geq 0},\ \lambda \in \mathcal{C}[X]\}$, and where z is interpreted as a central generator.

As it stands, we need to prove that $Z(X)$ is indeed an $\mathbb{C}[z]$-algebra. Recall that a *distributive ring* is an object satisfying all the axioms of a ring except possibly the associativity of the product, and the existence of a multiplicative identity element. Then by definition $Z(X)$ is a distributive ring. By *Theorem 5.17*, for any $n \in \mathbb{Z}_{\geq 0}$ with $X \subseteq [n]$, we have a surjective homomorphism of distributive rings $\mathsf{pr}_n[X] : Z(X) \to Z_n(X)$ given by

$$\mathsf{pr}_n[X](K(\lambda)) = K_n(\lambda) \ \text{ and } \ \mathsf{pr}_n[X](z) = n.$$

By *Proposition 5.4* we have that $K_n(\lambda) = 0$ if and only if $n < ||\lambda||^X + |X| = \mathsf{deg}(\lambda)$. Therefore, one can deduce that $\mathsf{Ker}(\mathsf{pr}_n[X])$ is the ideal generated by the polynomial $z-n$ and the set $\{K(\lambda) \mid n < \mathsf{deg}(\lambda)\}$. As such, it is easy to show that $\cap_n \mathsf{Ker}(\mathsf{pr}_n[X]) = \{0\}$ as one lets n run over all $n \in \mathbb{Z}_{\geq 0}$ such that $X \subseteq [n]$. We thus obtain the following:

Lemma 5.34. For any $R_1, R_2 \in Z(X)$, we have that $R_1 = R_2$ if and only if

$$\mathsf{pr}_n[X](R_1) = \mathsf{pr}_n[X](R_2)$$

for all $n \in \mathbb{Z}_{\geq 0}$ such that $X \subseteq [n]$.

Proof. The forward implication is immediate, while the reverse implication follows since it implies that $R_1 - R_2$ belongs to $\cap_n \mathsf{Ker}(\mathsf{pr}_n[X]) = \{0\}$.

∎

The above result will be the main tool we use in confirming relations within $Z(X)$.

Example 5.35. Let $X = \{1, 2\}$, and so we only consider $n \geq 2$. Consider the X-marked cycle shapes $\lambda = (1, 2)(*, *)$ and $\mu = (1)(2)(*, *)$. We have that $\mathsf{deg}_2(\lambda) = \mathsf{deg}_2(\mu) = 4$, hence the class sums $K_n(\lambda)$ and $K_n(\mu)$ are non-zero if and only if $n \geq 4$. Let $n \geq 2$ so that $X \subseteq [n]$, then we have that

$$\begin{aligned} K_n(\lambda)K_n(\mu) &= \left(\sum_{\{a,b\}\subseteq[n]\setminus X} (1, 2)(a, b)\right)\left(\sum_{\{c,d\}\subseteq[n]\setminus X} (c, d)\right) \\ &= \sum_{\{a,b\}\subseteq[n]\setminus X}\ \sum_{\{c,d\}\subseteq[n]\setminus X} (1, 2)(a, b)(c, d). \end{aligned}$$

If the two cycles (a, b) and (c, d) are disjoint then the resulting permutation is simply $(a, b)(c, d)$, and there are two such ways to do so. If the two cycles share a single element, then the resulting permutation gives a 3-cycle (a, b, c), and the number of ways to arrive at this 3-cycle from a product of two transpositions is three. Lastly if the two cycles agree then the result is the identity, and there are as many ways to do this as there are two-element subsets of $[n]\setminus[2]$. Thus altogether we have that

$$K_n(\lambda)K_n(\mu) = 2K_n(\tau_1) + 3K_n(\tau_2) + \binom{n-2}{2}K_n(\tau_3)$$

where $\tau_1 = (1, 2)(*, *)(*, *)$, $\tau_2 = (1, 2)(*, *, *)$, and $\tau_3 = (1, 2)$. Note that

$$\binom{n-2}{2} = \frac{1}{2}(n-2)(n-3).$$

Thus $K_n(\lambda)K_n(\mu) = 2K_n(\tau_1)+3K_n(\tau_2)+\frac{1}{2}(n-2)(n-3)K_n(\tau_3)$ for all $n \geq 2$, noting that when $n = 2, 3$ both sides of the equality are zero since $K_n(\tau) = 0$ for all $\tau \in \{\lambda, \mu, \tau_1, \tau_2\}$, and the polynomial in n which is the coefficient of $K_n(\tau_3)$ has both 2 and 3 as roots. Thus by *Lemma 5.34* we have that the relation

$$K(\lambda)K(\mu) = 2K(\tau_1) + 3K(\tau_2) + \frac{1}{2}(z-2)(z-3)K(\tau_3)$$

holds in $Z(X)$, and so $f^{\tau_1}_{\lambda,\mu}(z) = 2$, $f^{\tau_2}_{\lambda,\mu}(z) = 3$, and $f^{\tau_3}_{\lambda,\mu}(z) = \frac{1}{2}(z-2)(z-3)$.

As one can imagine from the above example, if the X-marked cycles shapes λ and μ of $\mathcal{C}[X]$ possess many symbols $*$, then calculating the product $K(\lambda)K(\mu)$ in $Z(X)$ becomes quite difficult. At the moment, carrying out computations analogous to the above is the easiest way to calculate $K(\lambda)K(\mu)$. One could also follow along with the proof of *Theorem 5.17* to compute which non-zero structure polynomials are present in the product $K(\lambda)K(\mu)$, and to calculate such structure polynomials. This would require one to evaluate the constants $b(\mathsf{C})$ for various X-conjugacy class of $\mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}}$. However, it turns out that such an approach is comparable to carring out similar calculations to the above anyway.

We now use *Lemma 5.34* to confirm that $Z(X)$ is a $\mathbb{C}[z]$-algebra.

Proposition 5.36. The distributive ring $Z(X)$ is in fact an $\mathbb{C}[z]$-algebra.

Proof. We only need to confirm that the product is associative and that a multiplicative identity exists. For the identity, consider the element $K(1)$ where $1 \in \mathcal{C}[X]$ is the X-marked cycle shape with no symbols $*$ present, and where the elements of X belong to cycles of length one. Thus $\mathsf{CL}_n[X](1)$ is the singleton containing only the identity permutation, and hence $K_n(1)$ is the identity permutation belonging to $Z_n(X)$ for any $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$. As such, by *Lemma 5.34*, $K(1)$ must be the identity of $Z(X)$. Now for any triple $(R_1, R_2, R_3) \in Z(X) \times Z(X) \times Z(X)$, let

$$[R_1, R_2, R_3] := (R_1R_2)R_3 - R_1(R_2R_3)$$

be the associator. Then $\mathsf{pr}_n[X]([R_1, R_2, R_3]) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$ since $Z_n(X)$ is associative. As such, by *Lemma 5.34*, $[R_1, R_2, R_3] = 0$ showing that $Z(X)$ is also associative.

□

We end this section by describing some basic facts about the algebra $Z(X)$.

Lemma 5.37. We have an injective $\mathbb{C}$-algebra homomorphism $\iota : \mathbb{C}\mathfrak{S}(X) \to Z(X)$ defined on the basis elements by $\iota(\pi) = K(\pi)$ for all $\pi \in \mathfrak{S}(X)$.

Proof. For any $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$, the X-conjugacy class $\mathsf{CL}_n[X](\pi)$ is precisely the singleton $\{\pi\}$ for any $\pi \in \mathfrak{S}(X) \subset \mathcal{C}[X]$. Hence $\mathsf{pr}_n[X](K(\pi)) = K_n(\pi) = \pi$, and so it is clear by *Lemma 5.34* that ι is a homomorphism of $\mathbb{C}$-algebras. Injectivity follows from construction of $Z(X)$.

□

Let $X \subseteq Y \subset \mathbb{N}$ be finite sets such that $|X| = |Y|$. By *Lemma 5.20* $Z(X) \cong Z(Y)$ as $\mathbb{C}[z]$-algebras, where such an isomorphism can be realised as the extension of basis elements by $K(\lambda) \mapsto K(\lambda^\sigma)$ for any $\sigma \in \mathfrak{S}_{\mathbb{N}}$ such that $\sigma(X) = Y$.

Lastly, by *Equation* (5.8) we immediately obtain the following result.

Lemma 5.38. Let $X \subseteq Y \subset \mathbb{N}$ be finite sets. Then $Z(X)$ may be realised as the subalgebra of $Z(Y)$ by the embedding

$$K(\lambda) \mapsto \sum_{\mu \in \mathsf{Fill}_X^Y(\lambda)} K(\mu).$$

□

Thus the cycle shape algebra $Z = Z(\emptyset)$ is a subalgebra of $Z(X)$ for any finite $X \subset \mathbb{N}$.

5.1.6 Structural Properties of $Z(X)$

In this section we will prove that the X-marked cycle shape algebra $Z(X)$ is filtered by a degree function induced from that of $\mathcal{C}[X]$. Moreover, we show that the multiplication of two basis elements of $Z(X)$ produces a unique leading term of highest degree. From this we are able to determine a generating set for $Z(X)$, which is also induced from the natural generators of $\mathcal{C}[X]$. We end the section by proving that $Z(X)$ is isomorphic as $\mathbb{C}$-algebras to the tensor product of the degenerate affine Hecke algebra $\mathcal{H}_{|X|}$ with the polynomial algebra in countably many commuting variables.

Recall the degree function $\mathsf{deg} : \mathcal{C}[X] \to \mathbb{Z}_{\geq |X|}$ given by $\mathsf{deg}(\lambda) = ||\lambda||^X + |X|$ for any $\lambda \in \mathcal{C}[X]$. We extend this function to $Z(X)$ by letting

$$\mathsf{deg}\left(\sum_{\lambda\in\mathcal{C}[X]} f_\lambda(z)K(\lambda)\right) = \mathsf{max}\{\mathsf{deg}(\lambda) \mid f_\lambda(z) \neq 0\}.$$

In particular $\mathsf{deg}(K(\lambda)) = \mathsf{deg}(\lambda)$. We seek to show that $Z(X)$ is filtered by deg. To prove this we will use the following lemma.

Lemma 5.39. For $\lambda, \mu \in \mathcal{C}[X]$, let $g \in \mathsf{CL}[X](\lambda)$ and $h \in \mathsf{CL}[X](\mu)$. Suppose that

$$\mathsf{Sup}^X(g) \cap \mathsf{Sup}^X(h) = \emptyset, \tag{5.11}$$

then we must have that $gh \in \mathsf{CL}[X](\lambda\mu)$.

Proof. Consider the expressions

$$\lambda = \pi u^{\boldsymbol{d}} c^{\boldsymbol{l}} \quad \text{and} \quad \mu = \sigma u^{\boldsymbol{e}} c^{\boldsymbol{k}}$$

for $\pi, \sigma \in \mathfrak{S}(X)$, $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{Z}^X_{\geq 0}$, and $\boldsymbol{l}, \boldsymbol{k} \in \mathbb{Z}^{\mathbb{N}}_{\geq 0}$. By *Equation* (5.2),

$$\lambda\mu = \pi\sigma u^{\sigma\circ\boldsymbol{d}+\boldsymbol{e}} c^{\boldsymbol{l}+\boldsymbol{k}}.$$

Hence the result follows if we show that gh satisfies items (i) and (ii) of *Lemma 5.11* with respect to $\pi\sigma u^{\sigma\circ\boldsymbol{d}+\boldsymbol{e}} c^{\boldsymbol{l}+\boldsymbol{k}}$. For item (i), since *Equation* (5.11) is upheld, it is clear that the number of cycles of gh of length $i+1$ which contain no elements of X is $\boldsymbol{l}(i) + \boldsymbol{k}(i)$, since this is the sum of such cycles of g and h. For item (ii), pick any $x \in X$ and set $y := \sigma(x)$ and $z := \pi(y)$. Since $h \in \mathsf{CL}[X](\sigma u^{\boldsymbol{e}} c^{\boldsymbol{k}})$, item (ii) of *Lemma 5.11* tells us that

$$h : x \mapsto i_1 \mapsto i_2 \mapsto \cdots \mapsto i_{\boldsymbol{e}(x)} \mapsto y$$

where $\{i_1, i_2, \ldots, i_{\boldsymbol{e}(x)}\} \cap X = \emptyset$. Similarly since $g \in \mathsf{CL}[X](\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}})$,

$$g : y \mapsto j_1 \mapsto j_2 \mapsto \cdots \mapsto j_{\boldsymbol{d}(y)} \mapsto z$$

where $\{j_1, j_2, \ldots, j_{\boldsymbol{d}(y)}\} \cap X = \emptyset$. Since *Equation* (5.11) is upheld, we must have that

$$gh : x \mapsto i_1 \mapsto \cdots \mapsto i_{\boldsymbol{e}(x)} \mapsto j_1 \mapsto \cdots j_{\boldsymbol{d}(y)} \mapsto z.$$

Thus $(gh)^{\boldsymbol{d}(y)+\boldsymbol{e}(x)+1}(x) = (\pi\sigma)(x)$ and $(gh)^m(x) \notin X$ for any $1 \leq m \leq \boldsymbol{d}(y) + \boldsymbol{e}(x)$. Note that $\boldsymbol{d}(y) = \boldsymbol{d}(\sigma^{-1}(x)) = (\sigma \circ \boldsymbol{d})(x)$, and hence gh also upholds item (ii) of *Lemma 5.11*.

□

Proposition 5.40. Let $\lambda = \pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}, \mu = \sigma u^{\boldsymbol{e}} c^{\boldsymbol{k}} \in \mathcal{C}[X]$. In $Z(X)$ we have that

$$K(\lambda)K(\mu) = c_{\lambda,\mu}K(\lambda\mu) + \sum_{\substack{\tau\in\mathcal{C}[X] \\ \mathsf{deg}(\tau)<\mathsf{deg}(\lambda\mu)}} f^{\tau}_{\lambda,\mu}(z)K(\tau)$$

where $c_{\lambda,\mu} \in \mathbb{N}$ is a constant given by

$$c_{\lambda,\mu} = \prod_{i=1}^{\infty} \binom{(\boldsymbol{l}+\boldsymbol{k})(i)}{\boldsymbol{l}(i)},$$

Proof. Let $g \in \mathsf{CL}[X](\lambda)$ and $h \in \mathsf{CL}[X](\mu)$, then clearly we have

$$\mathsf{Sup}^X(gh) \subseteq \mathsf{Sup}^X(g) \cup \mathsf{Sup}^X(h).$$

Thus if $gh \in \mathsf{CL}[X](\tau)$ for some $\tau \in \mathcal{C}[X]$, then $||\tau||^X \leq ||\lambda||^X + ||\mu||^X$. Now recall that $\mathsf{deg}(\tau) = ||\tau||^X + |X|$, hence $\mathsf{deg}(\tau) \leq \mathsf{deg}(\lambda) +_X \mathsf{deg}(\mu) = \mathsf{deg}(\lambda\mu)$. Thus we have

$$K(\lambda)K(\mu) = \sum_{\substack{\tau\in\mathcal{C}[X] \\ \mathsf{deg}(\tau)\leq\mathsf{deg}(\lambda\mu)}} f^{\tau}_{\lambda,\mu}(z)K(\tau).$$

Now suppose $gh \in \mathsf{CL}[X](\tau)$ with $\mathsf{deg}(\tau) = \mathsf{deg}(\lambda\mu)$. This implies that $||\tau||^X = ||\lambda\mu||^X$, which means that $\mathsf{Sup}^X(g) \cap \mathsf{Sup}^X(h) = \emptyset$. Hence by *Lemma 5.39* it must be the case that $\tau = \lambda\mu$. Therefore

$$K(\lambda)K(\mu) = f^{\lambda\mu}_{\lambda,\mu}(z)K(\lambda\mu) + \sum_{\substack{\tau\in\mathcal{C}[X] \\ \mathsf{deg}(\tau)<\mathsf{deg}(\lambda\mu)}} f^{\tau}_{\lambda,\mu}(z)K(\tau).$$

So what remains to be shown is that $f^{\lambda\mu}_{\lambda,\mu}(z) = c_{\lambda,\mu}$. Well let $f \in \mathsf{CL}[X](\lambda\mu)$ and consider the set $A_{\lambda,\mu}(f) = \{(g,h) \in \mathsf{CL}[X](\lambda) \times \mathsf{CL}[X](\mu) \mid gh = f\}$. By *Equation* (5.2) we have $\lambda\mu = \pi\sigma u^{\sigma\circ\boldsymbol{d}+\boldsymbol{e}} c^{\boldsymbol{l}+\boldsymbol{k}}$, and so by *Lemma 5.11*, for any $x \in X$, we have that

$$f : x \mapsto i_1 \mapsto \cdots \mapsto i_{(\sigma\circ\boldsymbol{d}+\boldsymbol{e})(x)} \mapsto (\pi\sigma)(x),$$

where $\{i_1, \cdots, i_{(\sigma\circ\boldsymbol{d}+\boldsymbol{e})(x)}\} \cap X = \emptyset$. Any pair $(g,h) \in A_{\lambda,\mu}(f)$ satisfies *Equation* (5.11), hence we have that

$$\begin{aligned} &h : x \mapsto i_1 \mapsto \cdots \mapsto i_{\boldsymbol{e}(x)} \mapsto \sigma(x), \\ &g : \sigma(x) \mapsto i_{\boldsymbol{e}(x)+1} \mapsto \cdots \mapsto i_{(\sigma\circ\boldsymbol{d}+\boldsymbol{e})(x)} \mapsto (\pi\sigma)(x). \end{aligned}$$

As such, if we are to construct a pair of permutations $(g,h) \in \mathsf{CL}[X](\lambda) \times \mathsf{CL}[X](\mu)$ such that $gh = f$, the cycles containing elements of X in g and h are predetermined by f. Hence we are just concerned with the cycles which contain no elements of X. In f there are $(\boldsymbol{l}+\boldsymbol{k})(i)$ number of such cycles of length $i+1$, while g and h containg $\boldsymbol{l}(i)$ and $\boldsymbol{k}(i)$ such cycles respectively. Thus to construct a pair $(g,h) \in A_{\lambda,\mu}(f)$, it is simply a matter of how one distributes the cycles containing no elements of X of f among either g or h. The binomial coefficient

$$\binom{(\boldsymbol{l}+\boldsymbol{k})(i)}{\boldsymbol{l}(i)}$$

counts the number of ways to allocate such cycles of length $i+1$ of f to the permutation g (with the remaining cycles allocated to h). Therefore

$$f_{\lambda,\mu}^{\lambda\mu}(z) = |A_{\lambda,\mu}(f)| = \prod_{i=1}^{\infty} \binom{(\boldsymbol{l}+\boldsymbol{k})(i)}{\boldsymbol{l}(i)}.$$

□

We immediately obtain the following corollary.

Corollary 5.41. The algebra $Z(X)$ is filtered by deg, with filtration

$$\mathbb{C}[z]\mathfrak{S}(X) = Z^{\leq|X|}(X) \subset Z^{\leq|X|+1}(X) \subset Z^{\leq|X|+2}(X) \subset \cdots,$$

where for any $n \geq |X|$, $Z^{\leq n}(X)$ is the $\mathbb{C}[z]$-submonoid generated by $\{K(\lambda) \mid \lambda \in \mathcal{C}_{\leq n}[X]\}$.

□

This filtration is quite compatible with the projections $\mathsf{pr}_n[X]$ down to the centralizer algebras. Namely, for any $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$, by *Proposition 5.4* one can deduce that $\mathsf{Im}(\mathsf{pr}_n[X]) = \mathsf{pr}_n[X](Z^{\leq n}(X))$, the image of the n-th filtered component of $Z(X)$.

Beyond confirming that $Z(X)$ is filtered, *Proposition 5.40* shows that $K(\lambda\mu)$ is the unique basis element of maximal degree which appears in the decomposition of $K(\lambda)K(\mu)$, and more importantly, the coefficient of $K(\lambda\mu)$ in this decomposition belongs to $\mathbb{N}$. As such the associated graded $\mathbb{C}[z]$-algebra $Z^{\mathsf{gr}}(X)$, of $Z(X)$ with respect to deg, may be regarded as a $\mathbb{C}$-algebra. Also one can prove that $Z^{\mathsf{gr}}(X)$ is the twisted semigroup algebra of the monoid $\mathcal{C}[X]$ with twisting $t : \mathcal{C}[X] \times \mathcal{C}[X] \to \mathbb{N}$ given by

$$t(\pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}, \sigma u^{\boldsymbol{e}} c^{\boldsymbol{k}}) = \prod_{i=1}^{\infty} \binom{(\boldsymbol{l}+\boldsymbol{k})(i)}{\boldsymbol{l}(i)}.$$

We now use *Proposition 5.40* to detemine a generating set for $Z(X)$ as a $\mathbb{C}[z]$-algebra, which is induced from the natural generators of the monoid $\mathcal{C}[X]$. Firstly we will write $X = \{x_1, \ldots, x_{|X|}\}$ such that $x_i < x_{i+1}$ for each $i \in [|X|-1]$. Then for any $i \in [|X|-1]$ we let $\tilde{s}_i := (x_i, x_{i+1}) \in \mathcal{C}[X]$ denote the transpositions exchanging x_i and x_{i+1}, and for any $j \in [|X|]$ we let $\tilde{u}_j := u_{x_j} \in \mathcal{C}[X]$. Now consider the set

$$\mathcal{G}_X := \{K(\tilde{s}_i), K(\tilde{u}_j), K(c_m) \mid i \in [|X|-1], j \in [|X|], m \in \mathbb{N}\} \tag{5.12}$$

of elements in $Z(X)$.

Proposition 5.42. The set $\mathcal{G}_X$ generates $Z(X)$ as a $\mathbb{C}[z]$-algebra.

Proof. It suffices to prove, for any $\lambda \in \mathcal{C}[X]$, that $K(\lambda)$ is a $\mathbb{C}[z]$-linear combination of monomials in elements of $\mathcal{G}_X$. We prove this by induction on $\mathsf{deg}(\lambda)$. For the base case assume $\mathsf{deg}(\lambda) = |X|$, thus $\lambda \in \mathfrak{S}(X)$. As mentioned before, the subalgebra of $Z(X)$ generated by $K(\pi)$ for $\pi \in \mathfrak{S}(X)$ is isomorphic to $\mathbb{C}[z]\mathfrak{S}(X)$ via $K(\pi) \mapsto \pi$. Hence this case follows since the set of transpositions $\{\tilde{s}_i \mid i \in [|X|-1]\}$ generate $\mathbb{C}[z]\mathfrak{S}(X)$ as a $\mathbb{C}[z]$-algebra. Now assume that $K(\mu) \in \langle \mathcal{G}_X \rangle$ for all $\mu \in \mathcal{C}[X]$ such that $\mathsf{deg}(\mu) < n$ for some $n > |X|$. We seek to show that $K(\lambda) \in \langle \mathcal{G}_X \rangle$ for all $\lambda \in \mathcal{C}[X]$ such that $\mathsf{deg}(\lambda) = n$. Let $\lambda = \pi u^{\boldsymbol{d}} c^{\boldsymbol{l}}$ be such an element, and let $\pi = \tilde{s}_{i_1} \cdots \tilde{s}_{i_l}$ be an expression for π in terms of transpositions. Then by repeat application of *Proposition 5.40*, the expression

$$K_\lambda := K(\tilde{s}_{i_1}) \cdots K(\tilde{s}_{i_l}) K(\tilde{u}_1)^{\boldsymbol{d}(x_1)} \cdots K(\tilde{u}_{|X|})^{\boldsymbol{d}(x_m)} K(c_1)^{\boldsymbol{l}(i)} K(c_2)^{\boldsymbol{l}(2)} \cdots ,$$

is equal to

$$c_\lambda K(\lambda) + \sum_{\substack{\mu \in \mathcal{C}[X] \\ \mathsf{deg}(\mu) < \mathsf{deg}(\lambda)}} h_\mu(z) K(\mu)$$

where c_λ is some constant belonging to $\mathbb{N}$ and $h_\mu(z)$ belong to $\mathbb{C}[z]$ for all $\mu \in \mathcal{C}[X]$ such that $\mathsf{deg}(\mu) < \mathsf{deg}(\lambda)$. Rearranging yields

$$K(\lambda) = \frac{1}{c_\lambda} \left(K_\lambda - \sum_{\substack{\mu \in \mathcal{C}[X] \\ \mathsf{deg}(\mu) < \mathsf{deg}(\lambda)}} h_\mu(z) K(\mu) \right)$$

which belongs to $\langle \mathcal{G}_X \rangle$ since K_λ does by construction, and the summation does by the inductive hypothesis.

$\square$

When viewing $Z(X)$ as a $\mathbb{C}$-algebra, naturally it is generated by the variable z and the set $\mathcal{G}_X$. Now let $\mathbb{C}[z_0, z_1, \dots]$ be the polynomial $\mathbb{C}$-algebra in commuting variables z_i for $i \in \mathbb{N}$. We end this section by showing that, as $\mathbb{C}$-algebras, the X-marked cycle shape algebra is isomorphic to $\mathcal{H}_{|X|} \otimes \mathbb{C}[z_0, z_1, \dots]$, where $\mathcal{H}_{|X|}$ is the affine degenerate Hecke algebra discussed in *Section 2.1.7*. For $h \in \mathcal{H}_{|X|}$ and $f \in \mathbb{C}[z_0, z_1, \dots]$, we write $hf := h \otimes f \in \mathcal{H}_{|X|} \otimes \mathbb{C}[z_0, z_1, \dots]$.

Proposition 5.43. We have a $\mathbb{C}$-algebra isomorphism $\varphi : \mathcal{H}_{|X|} \otimes \mathbb{C}[z_0, z_1, \dots] \to Z(X)$ defined on generators by

$$\varphi(s_i) = K(\tilde{s}_i), \quad \varphi(y_j) = K(\tilde{u}_j) + \sum_{l<j} K((x_l, x_j)), \quad \text{and} \quad \varphi(z_m) = \begin{cases} z & m = 0 \\ K(c_m) & m \geq 1 \end{cases},$$

for all $i \in [|X|-1]$, $j \in [|X|]$, and $m \in \mathbb{Z}_{\geq 0}$.

Proof. We begin by showing that φ is a homomorphism of algebras. It suffices to check that φ respects the defining relations of $\mathcal{H}_{|X|}$ given in *Definition 2.23*, and that φ respects the fact that the variable generators z_i commute with the other generators. To show this we will use the projection $\mathsf{pr}_n[X] : Z(X) \to Z_n(X)$ and employ *Lemma 5.34*. We start with checking relations in *Definition 2.23*.

(1): The $\mathbb{C}$-subalgebra of $Z(X)$ generated by the set $\{K(\pi) \mid \pi \in \mathfrak{S}(X) \subset \mathcal{C}[X]\}$ is isomorphic to the group algebra $\mathbb{C}\mathfrak{S}(X)$ by associating $K(\pi)$ with π. Hence it is clear that relations (1) of *Definition 2.23* are respected under φ.

(2)(ii): For any $i \in [|X|-1]$ and $j \in [|X|]$ such that $j \neq i, i+1$, we seek to show that $\varphi(s_i y_j) = \varphi(y_j s_i)$ which is the same as $\varphi(s_i y_j s_i) = \varphi(y_j)$ since (1)(i) of *Definition 2.23* is respected by φ. For any $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$, we have that

$$\begin{aligned}
\mathsf{pr}_n[X](\varphi(s_i y_j s_i)) &= \mathsf{pr}_n[X](\varphi(s_i)\varphi(y_j)\varphi(s_i)) \\
&= \mathsf{pr}_n[X]\left(K(\tilde{s}_i)\left(K(\tilde{u}_j) + \sum_{l<j} K((x_l, x_j))\right) K(\tilde{s}_i)\right) \\
&= K_n(\tilde{s}_i)\left(K_n(\tilde{u}_j) + \sum_{l<j} K_n((x_l, x_j))\right) K_n(\tilde{s}_i) \\
&= (x_i, x_{i+1})\left(\sum_{a \in [n]\backslash X} (a, x_j) + \sum_{l<j} (x_l, x_j)\right)(x_i, x_{i+1}) \\
&= \sum_{a \in [n]\backslash X} (a, x_j) + \sum_{l<j} (x_l, x_j) \\
&= \mathsf{pr}_n[X](\varphi(y_j))
\end{aligned}$$

where the fifth equality follows since $j \neq i, i+1$, and so (x_i, x_{i+1}) commutes with (a, x_j) for any $a \in [n]\backslash X$, and $(x_i, x_{i+1})(\{x_1, \ldots, x_{j-1}\}) = \{x_1, \ldots, x_{j-1}\}$, thus

$$\sum_{l<j} (x_i, x_{i+1})(x_l, x_j)(x_i, x_{i+1}) = \sum_{l<j} ((x_i, x_{i+1})(x_l), x_j) = \sum_{l<j} (x_l, x_j).$$

So $\mathsf{pr}_n[X](\varphi(s_i y_j s_i)) = \mathsf{pr}_n[X](\varphi(y_j))$ for all $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$. Hence applying *Lemma 5.34* tells us that $\varphi(s_i y_j s_i) = \varphi(y_j)$, showing that (2)($ii$) is respected under φ.

(2)(i): We seek to show that $\varphi(y_i y_j) = \varphi(y_j y_i)$ for any $i, j \in [|X|]$. Recall that the set $X = \{x_1, \ldots, x_{|X|}\}$ such that $x_i < x_{i+1}$, and let $[n]\backslash X = \{y_1, \ldots, y_{n-|X|}\}$ be such that $y_i < y_{i+1}$. Then let $\sigma \in \mathfrak{S}_n$ be the unique permutation which rearranges the list $1, 2, \ldots, n$ into $y_1, \ldots, y_{n-|X|}, x_1, \ldots, x_{|X|}$. Then for any $j \in [|X|]$ one can deduce that

$$\mathsf{pr}_n[X](\varphi(y_j)) = \sum_{a \in [n]\backslash X} (a, x_j) + \sum_{l<j} (x_l, x_j) = \sigma Y_{n-|X|+j} \sigma^{-1},$$

where Y_i is the i-th Jucys-Murphy element of $\mathbb{C}\mathfrak{S}_n$ given in *Definition 2.11*. Since the Jucys-Murphy elements pairwise commute, it is easy to see that $\mathsf{pr}_n[X](\varphi(y_j))$ and $\mathsf{pr}_n[X](\varphi(y_i))$ commute for all $i, j \in [|X|]$. Thus $\mathsf{pr}_n[X](\varphi(y_i y_j)) = \mathsf{pr}_n[X](\varphi(y_j y_i))$ for all $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$, hence applying *Lemma 5.34* we see that relation $(2)(i)$ is respected by φ.

$(2)(iii)$: For any $i \in [|X|-1]$ we seek to show that $\varphi(y_{i+1}) = \varphi(s_i y_i s_i + s_i)$. For any $n \in \mathbb{Z}_{\geq 0}$ and $X \subseteq [n]$, we have that

$$\begin{aligned}
\mathsf{pr}_n[X](\varphi(s_i y_i s_i + s_i)) &= \mathsf{pr}_n[X](\varphi(s_i)\varphi(y_i)\varphi(s_i) + \varphi(s_i)) \\
&= \mathsf{pr}_n[X]\left(K(\tilde{s}_i)\left(K(\tilde{u}_i) + \sum_{k<i} K((x_k, x_i))\right) K(\tilde{s}_i) + K(\tilde{s}_i)\right) \\
&= K_n(\tilde{s}_i)\left(K_n(\tilde{u}_i) + \sum_{k<i} K_n((x_k, x_i))\right) K_n(\tilde{s}_i) + K_n(\tilde{s}_i) \\
&= (x_i, x_{i+1})\left(\sum_{a\in[n]\setminus X} (a, x_i) + \sum_{k<i}(x_k, x_i)\right)(x_i, x_{i+1}) + (x_i, x_{i+1}) \\
&= \sum_{a\in[n]\setminus X} (a, x_{i+1}) + \sum_{k<i}(x_k, x_{i+1}) + (x_i, x_{i+1}) \\
&= \sum_{a\in[n]\setminus X} (a, x_{i+1}) + \sum_{k<i+1}(x_k, x_{i+1}) \\
&= \mathsf{pr}_n[X](\varphi(y_{i+1})).
\end{aligned}$$

Hence employing *Lemma 5.34* shows that φ respects relation $(2)(iii)$ of *Definition 2.23*.

We now need to show that φ respects the fact that z_m is central for each $m \in \mathbb{Z}_{\geq 0}$. Note this is immediate for the case $m = 0$. Assume that $m > 1$, then for any $n \geq m+1$ and $X \subseteq [n]$, we have that

$$\mathsf{pr}_n[X](\varphi(z_m)) = \sum (a_1, \ldots, a_{m+1}),$$

where the sum runs over all cycles of length $m+1$ whose entries $a_1, \ldots, a_{m+1}$ belong to $[n]\setminus X$. As such we have that $\mathsf{pr}_n[X](\varphi(z_m))$ belongs to $\mathbb{C}\mathsf{Stab}_n(X)$, and so recalling that $Z_n(X) = \{z \in \mathbb{C}\mathfrak{S}_n \mid \tau z = z\tau \text{ for all } \tau \in \mathsf{Stab}_n(X)\}$, it is easily seen that $\mathsf{pr}_n[X](\varphi(z_m))$ is central in $Z_n(X)$. Thus applying *Lemma 5.34* shows that $\varphi(z_m)$ is central in $Z(X)$. Therefore we have shown that φ is a homomorphism of $\mathbb{C}$-algebras.

We now show that φ is bijective. For surjectivity, by *Proposition 5.42*, we only need to show that the elements of $\mathcal{G}_X$ belong to the image $\mathsf{Im}(\varphi)$. Clearly $K(\tilde{s}_i)$ and $K(c_m)$ belong to $\mathsf{Im}(\varphi)$. Also it is easy to check that, for any $j \in [|X|]$ and $l < j$,

$$K((x_l, x_j)) = K(\tilde{s}_l)\cdots K(\tilde{s}_{j-1})K(\tilde{s}_j)K(\tilde{s}_{j-1})\cdots K(\tilde{s}_l).$$

Hence $K((x_l, x_j))$ belongs to $\mathsf{Im}(\varphi)$ for any $l, j \in [|X|]$. Therefore

$$K(\tilde{u}_j) = \varphi(y_j) - \sum_{l<j} K((x_l, x_j)) \in \mathsf{Im}(\varphi),$$

and so $\mathcal{G}_X \subset \mathsf{Im}(\varphi)$. For injectivity we first set up a little notation. For any $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{|X|}$ and $\boldsymbol{b} \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}$, we let

$$y^{\boldsymbol{a}} := \prod_{i \in [|X|]} y_i^{\boldsymbol{a}(i)} \quad \text{and} \quad z^{\boldsymbol{b}} := \prod_{m \in \mathbb{N}} z_m^{\boldsymbol{b}(m)},$$

which is well-defined since $\boldsymbol{b}$ has finite support. Then by *Theorem 2.25*, the set

$$B := \{z_0^n \pi y^{\boldsymbol{a}} z^{\boldsymbol{b}} \mid n \in \mathbb{Z}_{\geq 0},\ \pi \in \mathfrak{S}_{|X|},\ \boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{|X|},\ \boldsymbol{b} \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}\}$$

forms a basis for $\mathcal{H}_{|X|} \otimes \mathbb{C}[z_0, z_1, \dots]$. Given any $\pi \in \mathfrak{S}_{|X|}$ let $\tilde{\pi}$ denote the permutation in $\mathfrak{S}(X)$ defined by $\tilde{\pi}(x_i) = x_{\pi(i)}$ for all $i \in [|X|]$. Then $\varphi(\pi) = K(\tilde{\pi})$. Similarly, given any $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{|X|}$ let $\tilde{\boldsymbol{a}}$ denote the element of $\mathbb{Z}_{\geq 0}^{X}$ defined by $\tilde{\boldsymbol{a}}(x_i) = \boldsymbol{a}(i)$ for each $i \in [|X|]$. Then, by *Proposition 5.40*, one can deduce that

$$\varphi(z_0^n \pi y^{\boldsymbol{a}} z^{\boldsymbol{b}}) = c z^n K(\tilde{\pi} u^{\tilde{\boldsymbol{a}}} c^{\boldsymbol{b}}) + \sum_{\substack{\mu \in \mathbb{C}[X] \\ \mathsf{deg}(\mu) < \mathsf{deg}(\tilde{\pi} u^{\tilde{\boldsymbol{a}}} c^{\boldsymbol{b}})}} h_\mu(z) K(\mu)$$

where $c \in \mathbb{N}$ and $h_\mu(z) \in \mathbb{C}[z]$ for each $\mu \in \mathbb{C}[X]$ such that $\mathsf{deg}(\mu) < \mathsf{deg}(\tilde{\pi} u^{\tilde{\boldsymbol{a}}} c^{\boldsymbol{b}})$. Thus it is clear that $\varphi(B)$ is $\mathbb{C}$-linearly independent since the leading terms appearing in $\varphi(z_0^n \pi y^{\boldsymbol{a}} z^{\boldsymbol{b}})$ are $\mathbb{C}$-linearly independent.

□

By *Proposition 4.33* we immediately obtain the following corollary.

Corollary 5.44. We have an isomorphism of algebras $Z(X) \cong \mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes |X|})$.

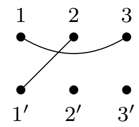
5.2 Orbit Affine Partition Algebra

5.2.1 The Subalgebra $Q_{2k}(M, n)$ of $\mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$

In this section we define a subalgebra $Q_{2k}(M, n)$ of $\mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$ (the codomain of the representation $\Psi_{2k,n}^{(M)}$ given in *Theorem 4.24*) which, in a sense, takes the action of the partition algebra $\mathcal{Q}_{2k}(z)$ on $V^{\otimes k}$ and fuses it with the action of the centraliser algebra $Z(X)$ on M via inflation through $Z_n(X)$ for any $X \subseteq [n]$. This subalgebra will be defined by generators, and we prove a polynomial property regarding the coefficients appearing in the product of any two such generators. As a corollary we obtain a spanning set for $Q_{2k}(M, n)$, which we show specialises to a basis when M is a free $\mathbb{C}\mathfrak{S}_n$-module. This subalgebra, and the polynomial property of coefficients, will be used in the next section to define a new $\mathbb{C}[z]$-algebra.

Recall the definition of an X-marked cycle shape in *Definition 5.7* and the X-marked cycle shape monoid $\mathcal{C}[X]$ in *Definition 5.6*. Naturally such definitions may be generalised to any finite set, not just a finite subset of $\mathbb{N}$. We replace the role of X with the set of blocks for a set partition $\alpha \in \Pi_{2k}$, hence we consider the α-marked cycle shapes and the corresponding α-marked cycle shape monoid $\mathcal{C}[\alpha]$. The grading given by the degree function $\mathsf{deg} : \mathcal{C}[\alpha] \to \mathbb{Z}_{\leq |\alpha|}$ naturally carries over also, with $\mathsf{deg}(\lambda)$ equaling the number of symbols $*$ appearing in λ plus $|\alpha|$, the number of blocks of α, for any $\lambda \in \mathcal{C}[\alpha]$.

Example 5.45. Let $k = 3$, and consider the element of Π_6 given by

$$\alpha = \{\{1,3\},\{2,1'\},\{2'\},\{3'\}\} =$$  .

Label the blocks by $A := \{1,3\}$, $B := \{2,1'\}$, $C := \{2'\}$, and $D := \{3'\}$. An example of a α-marked cycle shape is

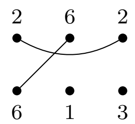
$$\lambda = (A,C)(B)(D)u_A^2 u_B u_C c_1 = (A,*,*,C,*)(B,*)(D)(*,*).$$

We have that $\mathsf{deg}(\lambda) = 6 + |\alpha| = 10$, hence $\lambda \in \mathcal{C}_{\leq 10}[\alpha] \subset \mathcal{C}[\alpha]$.

Let $k,n \geq 0$, and recall from *Definition 2.56* that the set of perfect colourings $\mathsf{PC}_n(\alpha)$ of a partition diagram $\alpha \in \Pi_{2k}$ consists of all pairs of tuples $(\boldsymbol{a},\boldsymbol{b}) \in [n]^k \times [n]^k$ such that for any $i,j \in [k] \cup [k']$ we have $i \sim_\alpha j$ if and only if $(\boldsymbol{a},\boldsymbol{b})(i) = (\boldsymbol{a},\boldsymbol{b})(j)$. We write $(\boldsymbol{a},\boldsymbol{b}) \hookrightarrow \alpha$. Any such perfect colouring assigns a distinct colour from $[n]$ to each block of α. Let $B \in \alpha$ be a block and $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ a perfect colouring, then we let $B_{(\boldsymbol{a},\boldsymbol{b})}$ denote the colour of $[n]$ that $(\boldsymbol{a},\boldsymbol{b})$ has assigned it. Also let $[\boldsymbol{a},\boldsymbol{b}]$ denote the subset of $[n]$ consisting of the entries in $\boldsymbol{a}$ and $\boldsymbol{b}$, that is $[\boldsymbol{a},\boldsymbol{b}] := \{(\boldsymbol{a},\boldsymbol{b})(i) \mid \text{for all } i \in [k] \cup [k']\}$.

Let $\lambda \in \mathcal{C}[\alpha]$ be an α-marked cycle shape. For any perfect colouring $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ we let $\lambda_{(\boldsymbol{a},\boldsymbol{b})}$ denote the $[\boldsymbol{a},\boldsymbol{b}]$-marked cycle shape obtained from λ by replacing each block B with $B_{(\boldsymbol{a},\boldsymbol{b})}$. Clearly we have a monoid isomorphism $(-)_{(\boldsymbol{a},\boldsymbol{b})} : \mathcal{C}[\alpha] \to \mathcal{C}[\boldsymbol{a},\boldsymbol{b}]$ for any perfect colouring $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ given by $\lambda \mapsto \lambda_{(\boldsymbol{a},\boldsymbol{b})}$.

Example 5.46. Let $\alpha \in \Pi_6$ and $\lambda \in \mathcal{C}[\alpha]$ be as in *Example 5.45*. Let $n \geq 10 = \mathsf{deg}(\lambda)$, and consider the tuples $\boldsymbol{a} = (2,6,2)$ and $\boldsymbol{b} = (6,1,3)$ belonging to $[n]^3$, so we are working with the set of colours $[\boldsymbol{a},\boldsymbol{b}] = \{1,2,3,6\}$. We have $(\boldsymbol{a},\boldsymbol{b}) \hookrightarrow \alpha$ with the coloured diagram

$$\alpha_{\boldsymbol{b}}^{\boldsymbol{a}} =$$  .

With the blocks of α labelled as before we have that $A_{(\boldsymbol{a},\boldsymbol{b})} = 2$, $B_{(\boldsymbol{a},\boldsymbol{b})} = 6$, $C_{(\boldsymbol{a},\boldsymbol{b})} = 1$, and $D_{(\boldsymbol{a},\boldsymbol{b})} = 3$. Then $\lambda_{(\boldsymbol{a},\boldsymbol{b})}$ is the $\{1,2,3,6\}$-marked cycle shape given by

$$\lambda_{(\boldsymbol{a},\boldsymbol{b})} = (A_{(\boldsymbol{a},\boldsymbol{b})},*,*,C_{(\boldsymbol{a},\boldsymbol{b})},*)(B_{(\boldsymbol{a},\boldsymbol{b})},*)(D_{(\boldsymbol{a},\boldsymbol{b})})(*,*) = (2,*,*,1,*)(6,*)(3)(*,*).$$

Although we have replaced the role of X with the blocks of a partition diagram α, this does not generalise to the centralizer algebras, that is, it does not make sense to consider the algebra $Z_n(\alpha)$ or elements $K_n(\lambda)$ for some $\lambda \in \mathcal{C}[\alpha]$. However it does make sense to consider the algebra $Z_n([\boldsymbol{a},\boldsymbol{b}])$ and the class sum elements $K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})$ for any $\lambda \in \mathcal{C}[\alpha]$ and perfect colouring $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$.

Recall the set up of *Section 2.2.5* and Schur-Weyl duality in *Theorem 2.58*, in particular V is the n-dimensional vector space with basis $\{v_a \mid a \in [n]\}$ which is viewed as a $\mathbb{C}\mathfrak{S}_n$-module via $\pi(v_a) := v_{\pi(a)}$ for any $\pi \in \mathfrak{S}_n$. Also we have the tensor space

$$V^{\otimes k} = \mathsf{Span}_{\mathbb{C}}\{v_{\boldsymbol{a}} \mid \boldsymbol{a} \in [n]^k\}$$

which is a $\mathbb{C}\mathfrak{S}_n$-module by extending the action of V diagonally. We now seek to extend the action of the orbit basis of $\mathcal{A}_{2k}(n)$ on $V^{\otimes k}$, described in item (2) of *Theorem 2.58*, to an action on $M \otimes V^{\otimes k}$ where M is any $\mathbb{C}\mathfrak{S}_n$-module. From here on let M have a basis $\{m_i \mid i \in I\}$ for some (possibly infinite) indexing set I, and we denote the action of $\pi \in \mathfrak{S}_n$ on any $m \in M$ by concatenation of symbols πm. Consider the $\mathbb{C}$-algebra $\mathsf{End}_{\mathbb{C}}(M \otimes V^{\otimes k})$ of linear endomorphisms $M \otimes V^{\otimes k} \to M \otimes V^{\otimes k}$. For any $f \in \mathsf{End}_{\mathbb{C}}(M)$ and $g \in \mathsf{End}_{\mathbb{C}}(V^{\otimes k})$, we let $f \otimes g \in \mathsf{End}_{\mathbb{C}}(M \otimes V^{\otimes k})$ be given by $(f \otimes g)(m_{a_0} \otimes v_{\boldsymbol{a}}) = (fm_{a_0}) \otimes (gv_{\boldsymbol{a}})$, for any $a_0 \in I$ and $\boldsymbol{a} \in [n]^k$.

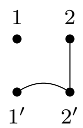
Definition 5.47. For any $\alpha \in \Pi_{2k}$ and α-marked cycle shape $\lambda \in \mathcal{C}[\alpha]$, we let

$$\overline{O}_{M,n}(\lambda, \alpha) := \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}$$

be an endomorphism of $M \otimes V^{\otimes k}$.

Comparing $\overline{O}_{M,n}(\lambda, \alpha)$ to the action of the orbit basis element $O_n(\alpha)$ described by item (2) of *Theorem 2.58*, we have extended it onto the M component by acting by the class sum element $K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})$ in unison with $E^{\boldsymbol{a}}_{\boldsymbol{b}}$ as we run over all perfect colourings in $\mathsf{PC}_n(\alpha)$.

Example 5.48. Let $k = 2$ and consider the partition diagram $\alpha \in \Pi_4$ given by

$\alpha =$ 

.

For $n = 4$ we have $\mathsf{PC}_4(\alpha) = \{((a, b)(b, b)) \mid a, b \in [4],\ a \neq b\}$. Consider the α-marked cycle shape given by $\lambda = (\{2, 1', 2'\}, *)(\{1\})$. Then, for any $\mathbb{C}\mathfrak{S}_4$-module M, the operator $\overline{O}_{M,4}(\lambda, \alpha)$ is given by

$$\begin{aligned}
\overline{O}_{M,4}(\lambda, \alpha) &= \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_4(\alpha)} K_4(\lambda_{(\boldsymbol{a},\boldsymbol{b})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}} \\
&= ((2,3) + (2,4)) \otimes E^{(1,2)}_{(2,2)} + ((3,2) + (3,4)) \otimes E^{(1,3)}_{(3,3)} + ((4,2) + (4,3)) \otimes E^{(1,4)}_{(4,4)} \\
&+ ((1,3) + (1,4)) \otimes E^{(2,1)}_{(1,1)} + ((3,1) + (3,4)) \otimes E^{(2,3)}_{(3,3)} + ((4,1) + (4,3)) \otimes E^{(2,4)}_{(4,4)} \\
&+ ((1,2) + (1,4)) \otimes E^{(3,1)}_{(1,1)} + ((2,1) + (2,4)) \otimes E^{(3,2)}_{(2,2)} + ((4,1) + (4,2)) \otimes E^{(3,4)}_{(4,4)} \\
&+ ((1,2) + (1,3)) \otimes E^{(4,1)}_{(1,1)} + ((2,1) + (2,3)) \otimes E^{(4,2)}_{(2,2)} + ((3,1) + (3,2)) \otimes E^{(4,3)}_{(3,3)}.
\end{aligned}$$

Given $m_{a_0}\otimes v_{a_1}\otimes v_{a_2}\in M\otimes V^{\otimes 2}$ such that $a_1\neq a_2$, then $\overline{O}_{M,4}(\lambda,\alpha)(m_{a_0}\otimes v_{a_1}\otimes v_{a_2})=0$. For a particular non-trivial example, we have that

$$\begin{aligned}\overline{O}_{M,4}(\lambda,\alpha)(m_{a_0}\otimes v_2\otimes v_2)=((2,3)+(2,4))m_{a_0}\otimes v_1\otimes v_2+((2,1)+(2,4))m_{a_0}\otimes v_3\otimes v_2\\+((2,1)+(2,3))m_{a_0}\otimes v_4\otimes v_2\end{aligned}$$

for any $a_0\in I$. We can vary n to obtain an analogous operator on $M\otimes V^{\otimes 2}$ where $\mathsf{dim}(V)=n$. Note when $n=2$ the set $\mathsf{PC}_2(\alpha)$ is non-empty, however since $\mathsf{deg}(\lambda)=3$, one can see that $K_2(\lambda_{(\boldsymbol{a},\boldsymbol{b})})=0$ for any $(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_2(\alpha)$. Hence $\overline{O}_{M,n}(\lambda,\alpha)=0$ for $n=1,2$.

Proposition 5.49. For any $\alpha\in\Pi_{2k}$ and $\lambda\in\mathcal{C}[\alpha]$, then $\overline{O}_{M,n}(\lambda,\alpha)\in\mathsf{End}_{\mathfrak{S}_n}(M\otimes V^{\otimes k})$.

Proof. We need to show that $\overline{O}_{M,n}(\lambda,\alpha)$ commutes with the diagonal action of $\mathbb{C}\mathfrak{S}_n$. We prove this by showing that $\pi\overline{O}_{M,n}(\lambda,\alpha)\pi^{-1}=\overline{O}_{M,n}(\lambda,\alpha)$ for any $\pi\in\mathfrak{S}_n$. Well,

$$\begin{aligned}\pi\overline{O}_{M,n}(\lambda,\alpha)\pi^{-1}&=\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)}\pi K(\lambda_{(\boldsymbol{a},\boldsymbol{b})})\pi^{-1}\otimes\pi E^{\boldsymbol{a}}_{\boldsymbol{b}}\pi^{-1}\\&=\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)}K((\lambda_{(\boldsymbol{a},\boldsymbol{b})})^{\pi})\otimes E^{\pi\boldsymbol{a}}_{\pi\boldsymbol{b}}\\&=\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)}K(\lambda_{(\pi\boldsymbol{a},\pi\boldsymbol{b})})\otimes E^{\pi\boldsymbol{a}}_{\pi\boldsymbol{b}}\\&=\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)}K(\lambda_{(\boldsymbol{a},\boldsymbol{b})})\otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}=\overline{O}_{M,n}(\lambda,\alpha),\end{aligned}$$

where the second and third equalities follows from definitions, and the fourth equality follows since $\mathsf{PC}_n(\alpha)$ is an orbit of the action of $\mathfrak{S}_n$ on $[n]^k\times[n]^k$.

□

Recall the decomposition of the operator $\overline{O}_{M,4}(\lambda,\alpha)$ given in *Example 5.48* into twelve terms. Then one can check the above proposition for this operator by noting that conjugating by any $\pi\in\mathfrak{S}_4$ simply permutes the twelve terms around in some manner.

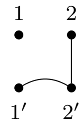
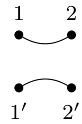
Definition 5.50. For any $n,k\in\mathbb{Z}_{\geq 0}$ and $\mathbb{C}\mathfrak{S}_n$-module M, we define $Q_{2k}(M,n)$ to be the subalgebra of $\mathsf{End}_{\mathfrak{S}_n}(M\otimes V^{\otimes k})$ generated by the operators $\overline{O}_{M,n}(\lambda,\alpha)$ for all $\alpha\in\Pi_{2k}$ and $\lambda\in\mathcal{C}[\alpha]$.

For any partition diagram $\alpha\in\Pi_{2k}$, one can see that $\mathsf{PC}_n(\alpha)\neq\emptyset$ if and only if $n\geq|\alpha|$. Also, by *Proposition 5.4* and *Definition 5.12*, we have for any $\lambda\in\mathcal{C}[\alpha]$ and $(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)$ that $K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})\neq 0$ if and only if $n\geq\deg(\lambda_{(\boldsymbol{a},\boldsymbol{b})})=\mathsf{deg}(\lambda)$. Therefore, the operator $\overline{O}_{M,n}(\lambda,\alpha)\neq 0$ if and only if $n\geq|\alpha|$ and $n\geq\mathsf{deg}(\lambda)$. As such the algebra $Q_{2k}(M,n)$ is finitely generated by

$$Q_{2k}(M,n)=\left\langle\overline{O}_{M,n}(\lambda,\alpha)\mid\alpha\in\Pi_{2k},\ n\geq|\alpha|,\lambda\in\mathcal{C}_{\leq n}[\alpha]\right\rangle.\tag{5.13}$$

We now present an example of multiplying two generators of $Q_{2k}(M,n)$ together. Informally, such a product is comparable to “smashing” together the products of class sum elements of centralizer algebras $Z_n(X)$ with the product of orbit basis elements of the partition algebra $\mathcal{A}_{2k}(n)$. In particular it is worth comparing with *Theorem 5.17* and *Proposition 2.67*. One will note that the computations are quite lengthy even though we picked simple partition diagrams and associated marked cycle shapes.

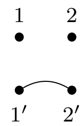
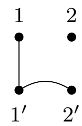
Example 5.51. Let $k = 2$ and consider the partition diagrams $\alpha, \beta \in \Pi_4$ given by

$\alpha =$  and $\beta =$  .

The sets of perfect colourings for α and β are given by

$$\mathsf{PC}_n(\alpha) = \{((a,b),(b,b)) \mid (a,b) \in [n]^{!2}\} \quad \text{and} \quad \mathsf{PC}_n(\beta) = \{((d,d),(c,c)) \mid (c,d) \in [n]^{!2}\},$$

where $[n]^{!m}$ denotes the subset of $[n]^m$ consisting of tuples with pairwise distinct entries. Recall the definition of top-bottom coarsenings given in *Definition 2.64*, then one can see that

$\mathsf{TBC}(\alpha,\beta) = \Big\{ \gamma :=$  , $\delta :=$  $\Big\}$.

One can deduce that the sets of perfect colourings of γ and δ are

$$\mathsf{PC}_n(\gamma) = \{((a,b),(c,c)) \mid (a,b,c) \in [n]^{!3}\} \quad \text{and} \quad \mathsf{PC}_n(\delta) = \{((a,b),(a,a)) \mid (a,b) \in [n]^{!2}\}.$$

Now consider the marked cycle shapes

$$\lambda = (\{2,1',2'\},*)(\{1\}), \quad \text{and} \quad \mu = (\{1,2\},*)(\{1',2'\}),$$

with $\lambda \in \mathcal{C}[\alpha]$ and $\mu \in \mathcal{C}[\beta]$. We now evaluate the product $\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta)$ for arbitrary $n \in \mathbb{Z}_{\geq 0}$ (and arbitrary $\mathbb{C}\mathfrak{S}_n$-module M):

$$\begin{aligned}
\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) &= \left(\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}\right)\left(\sum_{(\boldsymbol{d},\boldsymbol{c})\in\mathsf{PC}_n(\beta)} K_n(\lambda_{(\boldsymbol{d},\boldsymbol{c})}) \otimes E^{\boldsymbol{d}}_{\boldsymbol{c}}\right) \\
&= \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)} \sum_{(\boldsymbol{d},\boldsymbol{c})\in\mathsf{PC}_n(\beta)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})K_n(\lambda_{(\boldsymbol{d},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}E^{\boldsymbol{d}}_{\boldsymbol{c}} \\
&= \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)} \sum_{(\boldsymbol{b},\boldsymbol{c})\in\mathsf{PC}_n(\beta)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})K_n(\lambda_{(\boldsymbol{b},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\substack{a,b,c\in[n] \\ a\neq b, b\neq c}} K_n\left(\lambda_{((a,b)(b,b))}\right) K_n\left(\mu_{((b,b),(c,c))}\right) \otimes E^{(a,b)}_{(c,c)} \\
&= S_1 + S_2,
\end{aligned}$$

where S_1 and S_2 have broken the summation up into the two case $a \neq c$ and $a = c$, thus

$$S_1 := \sum_{(a,b,c)\in[n]^{!3}} K_n\left(\lambda_{((a,b)(b,b))}\right) K_n\left(\mu_{((b,b),(c,c))}\right) \otimes E^{(a,b)}_{(c,c)},$$

$$S_2 := \sum_{(a,b)\in[n]^{!2}} K_n\left(\lambda_{((a,b)(b,b))}\right) K_n\left(\mu_{((b,b),(a,a))}\right) \otimes E^{(a,b)}_{(a,a)}.$$

Evaluating S_1: Given $(a,b,c) \in [n]^{!3}$, we have that

$$\begin{aligned}
K_n\left(\lambda_{((a,b)(b,b))}\right) K_n\left(\mu_{((b,b),(c,c))}\right) &= \left(\sum_{i\in[n]\setminus\{a,b\}} (b,i)\right)\left(\sum_{j\in[n]\setminus\{b,c\}} (b,j)\right) \\
&= \left(\sum_{i\in[n]\setminus\{a,b,c\}} (b,i) + (b,c)\right)\left(\sum_{j\in[n]\setminus\{b,c,a\}} (b,j) + (b,a)\right) \\
&= \sum_{i,j\in[n]\setminus\{a,b,c\}} (b,i)(b,j) + \sum_{j\in[n]\setminus\{a,b,c\}} (b,c)(b,j) + \sum_{i\in[n]\setminus\{a,b,c\}} (b,i)(b,a) + (b,c)(b,a) \\
&= (n-3) + \sum_{\substack{i,j\in[n]\setminus\{a,b,c\}\\ i\neq j}} (b,j,i) + \sum_{j\in[n]\setminus\{a,b,c\}} (b,j,c) + \sum_{i\in[n]\setminus\{a,b,c\}} (b,a,i) + (b,a,c) \\
&= (n-3)K_n\left(1^{(\gamma)}_{((a,b),(c,c))}\right) + K_n\left(\tau^{(1)}_{((a,b),(c,c))}\right) + K_n\left(\tau^{(2)}_{((a,b),(c,c))}\right) \\
&\qquad + K_n\left(\tau^{(3)}_{((a,b),(c,c))}\right) + K_n\left(\tau^{(4)}_{((a,b),(c,c))}\right)
\end{aligned}$$

where $1^{(\gamma)} = (\{1\})(\{2\})(\{1',2'\}) \in \mathcal{C}[\gamma]$ and $\tau^{(2)}, \tau^{(3)}, \tau^{(4)} \in \mathcal{C}[\gamma]$ are given by

$$\begin{aligned}
\tau^{(1)} &= (\{2\},*,*)(\{1\})(\{1',2'\}), & \tau^{(3)} &= (\{2\},\{1\},*)(\{1',2'\}), \\
\tau^{(2)} &= (\{2\},*,\{1',2'\})(\{1\}), & \tau^{(4)} &= (\{2\},\{1\},\{1',2'\}).
\end{aligned}$$

Therefore, recalling the description of $\mathsf{PC}_n(\gamma)$, we have

$$\begin{aligned}
S_1 &= \sum_{(a,b,c)\in[n]^{!3}} (n-3)K_n\left(1^{(\gamma)}_{((a,b),(c,c))}\right) \otimes E^{(a,b)}_{(c,c)} + \sum_{i\in[4]} \sum_{(a,b,c)\in[n]^{!3}} K_n\left(\tau^{(i)}_{((a,b),(c,c))}\right) \otimes E^{(a,b)}_{(c,c)} \\
&= (n-3)\overline{O}_{M,n}(1^{(\gamma)},\gamma) + \sum_{i\in[4]} \overline{O}_{M,n}(\tau^{(i)},\gamma).
\end{aligned}$$

Evaluating S_2: Given $(a,b) \in [n]^{!2}$, we have that

$$\begin{aligned}
K_n\left(\lambda_{((a,b)(b,b))}\right) K_n\left(\mu_{((b,b),(a,a))}\right) &= \left(\sum_{i\in[n]\setminus\{a,b\}} (b,i)\right)\left(\sum_{j\in[n]\setminus\{a,b\}} (b,j)\right) \\
&= (n-2) + \sum_{\substack{i,j\in[n]\setminus\{a,b\}\\ i\neq j}} (b,j,i) \\
&= (n-2)K_n\left(1^{(\delta)}_{((a,b),(b,b))}\right) + K_n\left(\nu_{((a,b),(b,b))}\right)
\end{aligned}$$

where $1^{(\delta)} = (\{1,1',2'\})(\{2\}) \in \mathcal{C}[\delta]$ and $\nu = (\{2\},*,*)(\{1,1',2'\}) \in \mathcal{C}[\delta]$. Hence, recalling the description of $\mathsf{PC}_n(\delta)$, we have that

$$\begin{aligned} S_2 &= (n-2)\sum_{(a,b)\in[n]^{!2}} K_n\left(1^{(\delta)}_{((a,b),(b,b))}\right) + \sum_{(a,b)\in[n]^{!2}} K_n\left(\nu_{((a,b),(b,b))}\right) \\ &= (n-2)\overline{O}_{M,n}(1^{(\delta)},\delta) + \overline{O}_{M,n}(\nu,\delta). \end{aligned}$$

Thus collectively we have that

$$\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) = (n-3)\overline{O}_{M,n}(1^{(\gamma)},\gamma)+\sum_{i\in[4]}\overline{O}_{M,n}(\tau^{(i)},\gamma)+(n-2)\overline{O}_{M,n}(1^{(\delta)},\delta)+\overline{O}_{M,n}(\nu,\delta).$$

In the above example the product of two generators of $Q_4(M,n)$ decomposed into a linear combination of other generators of $Q_4(M,n)$, and the coefficients which appear are polynomial in n. It turns out, and we prove below, that this holds in general. As such the generating set $\{\overline{O}_{M,n}(\lambda,\alpha) \mid \alpha \in \Pi_{2k},\ n \geq |\alpha|, \lambda \in \mathcal{C}_{\leq n}[\alpha]\}$ for $Q_{2k}(M,n)$ is in fact a spanning set. It is worth comparing the proof below to that of *Proposition 2.67*.

Theorem 5.52. Let $n,k \in \mathbb{Z}_{\geq 0}$, M an $\mathbb{C}\mathfrak{S}_n$-module, $\alpha,\beta,\in \Pi_{2k}$, $\lambda \in \mathcal{C}[\alpha]$, and $\mu \in \mathcal{C}[\beta]$. Then in $Q_{2k}(M,n)$ we have that

$$\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) = \sum_{\gamma\in\mathsf{TBC}(\alpha,\beta)}\sum_{\tau\in\mathcal{C}[\gamma]} F^{\tau}_{\lambda,\mu}(n)\overline{O}_{M,n}(\tau,\gamma)$$

with $F^{\tau}_{\lambda,\mu}(z) \in \mathbb{C}[z]$ unique. In particular $F^{\tau}_{\lambda,\mu}(z) = 0$ whenever $\tau \notin \mathsf{TBC}(\alpha,\beta)$.

Proof. We have that

$$\begin{aligned} \overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) &= \left(\sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})\otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}\right)\left(\sum_{(\boldsymbol{d},\boldsymbol{c})\in\mathsf{PC}_n(\beta)} K_n(\mu_{(\boldsymbol{d},\boldsymbol{c})})\otimes E^{\boldsymbol{d}}_{\boldsymbol{c}}\right) \\ &= \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)}\sum_{(\boldsymbol{d},\boldsymbol{c})\in\mathsf{PC}_n(\beta)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})K_n(\mu_{(\boldsymbol{d},\boldsymbol{c})})\otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}E^{\boldsymbol{d}}_{\boldsymbol{c}}. \end{aligned}$$

We have that $E^{\boldsymbol{a}}_{\boldsymbol{b}}E^{\boldsymbol{d}}_{\boldsymbol{c}} = \delta_{\boldsymbol{b},\boldsymbol{d}}E^{\boldsymbol{a}}_{\boldsymbol{c}}$ where $\delta_{\boldsymbol{b},\boldsymbol{d}} = 1$ if $\boldsymbol{b} = \boldsymbol{d}$ and 0 otherwise. Note if the pair (α,β) does not match in the middle, then there is no tuple $\boldsymbol{b} = \boldsymbol{d}$ which perfectly colours both the top row of β and the bottom row of α. Hence in such a case we must have that $\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) = 0$, so we can set $F^{\tau}_{\lambda,\mu}(z) = 0$ whenever (α,β) does not match in the middle. Assume that (α,β) does match in the middle, then continuing from above we have

$$\overline{O}_n(\lambda,\alpha)\overline{O}_n(\mu,\beta) = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathsf{PC}_n(\alpha)}\sum_{(\boldsymbol{b},\boldsymbol{c})\in\mathsf{PC}_n(\beta)} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})K_n(\mu_{(\boldsymbol{b},\boldsymbol{c})})\otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}. \tag{5.14}$$

For any $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ and $(\boldsymbol{b},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)$, recalling *Equation* (5.8) we have that

$$\begin{aligned}
K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})})K_n(\mu_{(\boldsymbol{b},\boldsymbol{c})}) &= \sum_{\kappa \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{a},\boldsymbol{b}]}(\lambda_{(\boldsymbol{a},\boldsymbol{b})})} \sum_{\nu \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{b},\boldsymbol{c}]}(\mu_{(\boldsymbol{b},\boldsymbol{c})})} K_n(\kappa)K_n(\nu) \\
&= \sum_{\kappa \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{a},\boldsymbol{b}]}(\lambda_{(\boldsymbol{a},\boldsymbol{b})})} \sum_{\nu \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{b},\boldsymbol{c}]}(\mu_{(\boldsymbol{b},\boldsymbol{c})})} \left(\sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} f^{\varpi}_{\kappa,\nu}(n) K_n(\varpi) \right) \\
&= \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} \left(\sum_{\kappa \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{a},\boldsymbol{b}]}(\lambda_{(\boldsymbol{a},\boldsymbol{b})})} \sum_{\nu \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{b},\boldsymbol{c}]}(\mu_{(\boldsymbol{b},\boldsymbol{c})})} f^{\varpi}_{\kappa,\nu}(n) \right) K_n(\varpi).
\end{aligned}$$

Hence for any $\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$ define the polynomial

$$h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(z) := \sum_{\kappa \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{a},\boldsymbol{b}]}(\lambda_{(\boldsymbol{a},\boldsymbol{b})})} \sum_{\nu \in \mathsf{Fill}^{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}_{[\boldsymbol{b},\boldsymbol{c}]}(\mu_{(\boldsymbol{b},\boldsymbol{c})})} f^{\varpi}_{\kappa,\nu}(z).$$

Note that given any $\sigma \in \mathfrak{S}([\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}])$, by *Lemma 5.20* one can see that

$$h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(z) = h^{\varpi^{\sigma}}_{\lambda^{\sigma}_{(\boldsymbol{a},\boldsymbol{b})},\mu^{\sigma}_{(\boldsymbol{b},\boldsymbol{c})}}(z). \tag{5.15}$$

From *Equation* (5.14) we have that

$$\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) = \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} \sum_{(\boldsymbol{b},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}. \tag{5.16}$$

Consider any pair $\boldsymbol{a},\boldsymbol{c} \in [n]^k$ such that there exists $\boldsymbol{b} \in [n]^k$ where $(\boldsymbol{a},\boldsymbol{b}) \hookrightarrow \alpha$ and $(\boldsymbol{b},\boldsymbol{c}) \hookrightarrow \beta$. Thus $\boldsymbol{a}$ perfectly colours the top row of α and $\boldsymbol{c}$ perfectly colours the bottom row of β. Let

$$\mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha,\beta) := \{\boldsymbol{b} \in [n]^k \mid (\boldsymbol{a},\boldsymbol{b}) \hookrightarrow \alpha, (\boldsymbol{b},\boldsymbol{c}) \hookrightarrow \beta\}.$$

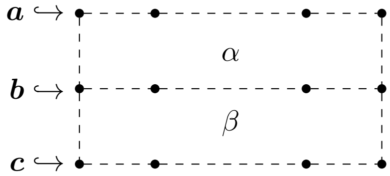
Each block of the coloured diagram $(\alpha \circ \beta)^{\boldsymbol{a}}_{\boldsymbol{c}}$ will have a colour associated to them, but in general these colours may not be distinct. There may be blocks from $\mathsf{Top}(\alpha)$ which share a colour with blocks from $\mathsf{Bot}(\beta)$, in a manner encoded by a partial bijection. As such there exists a unique $\gamma \in \mathsf{TBC}(\alpha,\beta)$ such that $(\boldsymbol{a},\boldsymbol{c})$ perfectly colours γ (where γ is obtained by merging blocks from $\mathsf{Top}(\alpha)$ with those in $\mathsf{Bot}(\beta)$ which share the same colour in $(\alpha \circ \beta)^{\boldsymbol{a}}_{\boldsymbol{c}}$). Hence for any $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ and $(\boldsymbol{b},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)$ we have that $(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)$ and $\boldsymbol{b} \in \mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha,\beta)$ for some unique $\gamma \in \mathsf{TBC}(\alpha,\beta)$. Thus, by *Equation* (5.16), the product $\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta)$ is given by

$$\begin{aligned}
&\sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} \sum_{(\boldsymbol{b},\boldsymbol{c}) \in \mathsf{PC}_n(\beta)} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} = \\
&\qquad \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} \sum_{\boldsymbol{b} \in \mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha,\beta)} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}
\end{aligned} \tag{5.17}$$

We seek to evaluate the right hand side of *Equation* (5.17) by breaking it up into more manageable pieces. Fix $\gamma \in \mathsf{TBC}(\alpha, \beta)$ and a perfect colouring $(\boldsymbol{a}, \boldsymbol{c}) \in \mathsf{PC}_n(\gamma)$. Let

$$W^{\gamma}_{(\boldsymbol{a},\boldsymbol{c})} := \sum_{\boldsymbol{b} \in \mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha,\beta)} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}.$$

Now the set $\mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta)$ is all the tuples $\boldsymbol{b} \in [n]^k$ which perfectly colour the bottom row of α and the top row of β such that $(\boldsymbol{a}, \boldsymbol{b})$ perfectly colours α and $(\boldsymbol{b}, \boldsymbol{c})$ perfectly colours β. Diagrammatically we are working with the situation



The only entries of $\boldsymbol{b}$ that are not predetermined by how $\boldsymbol{a}$ and $\boldsymbol{c}$ have coloured the top and bottom rows are the entries corresponding to middle blocks in the stacked diagram $\alpha \star \beta$, that is the blocks in $\mathsf{Mid}(\alpha \star \beta)$. Now fix $\boldsymbol{b}' \in \mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta)$, and let $\mathcal{T}_{(\boldsymbol{a},\boldsymbol{c})}$ be a set of left transversals of $\mathsf{Stab}_n([\boldsymbol{a}, \boldsymbol{b}', \boldsymbol{c}])$ in $\mathsf{Stab}_n([\boldsymbol{a}, \boldsymbol{c}])$. Note that the set $C := [\boldsymbol{a}, \boldsymbol{b}', \boldsymbol{c}] \backslash [\boldsymbol{a}, \boldsymbol{c}]$ is precisely the entries of $\boldsymbol{b}'$ which perfectly colour the middle blocks of $\alpha \star \beta$. By *Lemma 5.23*, a defining property of $\mathcal{T}_{(\boldsymbol{a},\boldsymbol{c})}$ is the fact that each element $\sigma \in \mathcal{T}_{(\boldsymbol{a},\boldsymbol{c})}$ encodes a unique way of sending the elements of C into the set $[n] \backslash [\boldsymbol{a}, \boldsymbol{c}]$, and every such way of embedding C into $[n] \backslash [\boldsymbol{a}, \boldsymbol{c}]$ is accounted for by a unique $\sigma \in \mathcal{T}_{(\boldsymbol{a},\boldsymbol{b})}$. So an alternative way of representing the set $\mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta)$ is given by

$$\mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha, \beta) := \{\sigma \boldsymbol{b}' \mid \sigma \in \mathcal{T}_{(\boldsymbol{a},\boldsymbol{c})}\}.$$

As such we have

$$\begin{aligned}
W^{\gamma}_{(\boldsymbol{a},\boldsymbol{c})} &= \sum_{\boldsymbol{b} \in \mathsf{C}_n^{(\boldsymbol{a},\boldsymbol{c})}(\alpha,\beta)} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\sigma \in \mathcal{T}_{(\boldsymbol{a},\boldsymbol{c})}} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\sigma\boldsymbol{b}',\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\sigma\boldsymbol{b}')},\mu_{(\sigma\boldsymbol{b}',\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\sigma \in \mathcal{T}_{(\boldsymbol{a},\boldsymbol{c})}} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b}',\boldsymbol{c}]} h^{\varpi^{\sigma}}_{\lambda^{\sigma}_{(\boldsymbol{a},\boldsymbol{b}')},\mu^{\sigma}_{(\boldsymbol{b}',\boldsymbol{c})}}(n) K_n(\varpi^{\sigma}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b}',\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(n) f^{\varpi}_{[\boldsymbol{a},\boldsymbol{c}]}(n) K_n(\varpi \downarrow_{[\boldsymbol{a},\boldsymbol{c}]}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}
\end{aligned}$$

where the third equality follows since $Z_n([\boldsymbol{a}, \boldsymbol{b}', \boldsymbol{c}]) \cong Z_n([\boldsymbol{a}, \sigma\boldsymbol{b}', \boldsymbol{c}])$ via $(-)^{\sigma}$, and where the fourth equality follows from *Proposition 5.28* and *Equation* (5.15). For any $\varpi \in \mathcal{C}[\boldsymbol{a}, \boldsymbol{b}', \boldsymbol{c}]$, since $(\boldsymbol{a}, \boldsymbol{c}) \hookrightarrow \gamma$, there exists a unique $\tau \in \mathcal{C}[\gamma]$ such that $\varpi \downarrow_{[\boldsymbol{a},\boldsymbol{c}]} = \tau_{(\boldsymbol{a},\boldsymbol{c})}$. So for any $\tau \in \mathcal{C}[\gamma]$, define the polynomial

$$F^{\tau_{(\boldsymbol{a},\boldsymbol{c})}}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(z) := \sum_{\substack{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b}',\boldsymbol{c}] \\ \varpi\downarrow_{[\boldsymbol{a},\boldsymbol{c}]} = \tau_{(\boldsymbol{a},\boldsymbol{c})}}} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(z) f^{\varpi}_{[\boldsymbol{a},\boldsymbol{c}]}(z).$$

Then we have that

$$W^{\gamma}_{(\boldsymbol{a},\boldsymbol{c})} = \sum_{\tau \in \mathcal{C}[\gamma]} F^{\tau(\boldsymbol{a},\boldsymbol{c})}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(n) K_n(\tau_{(\boldsymbol{a},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}.$$

Therefore recalling that $\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta)$ is given by the right hand side of *Equation* (5.17), then we have that

$$\begin{aligned}
\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) &= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} \sum_{\boldsymbol{b} \in \mathsf{C}^{(\boldsymbol{a},\boldsymbol{c})}_n(\alpha,\beta)} \sum_{\varpi \in \mathcal{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]} h^{\varpi}_{\lambda_{(\boldsymbol{a},\boldsymbol{b})},\mu_{(\boldsymbol{b},\boldsymbol{c})}}(n) K_n(\varpi) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} W^{\gamma}_{(\boldsymbol{a},\boldsymbol{c})} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} \sum_{\tau \in \mathcal{C}[\gamma]} F^{\tau(\boldsymbol{a},\boldsymbol{c})}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(n) K_n(\tau_{(\boldsymbol{a},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{\tau \in \mathcal{C}[\gamma]} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} F^{\tau(\boldsymbol{a},\boldsymbol{c})}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(n) K_n(\tau_{(\boldsymbol{a},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}}
\end{aligned}$$

By *Equation* (5.15) and *Lemma 5.29* one can deduce that, for any $\sigma \in \mathfrak{S}_n$, we have

$$F^{\tau(\boldsymbol{a},\boldsymbol{c})}_{\lambda_{(\boldsymbol{a},\boldsymbol{b}')},\mu_{(\boldsymbol{b}',\boldsymbol{c})}}(n) = F^{\tau(\sigma\boldsymbol{a},\sigma\boldsymbol{c})}_{\lambda_{(\sigma\boldsymbol{a},\sigma\boldsymbol{b}')},\mu_{(\sigma\boldsymbol{b}',\sigma\boldsymbol{c})}}(n).$$

Hence such polynomials are independent of the particular perfect colourings, and so we can drop such perfect colours as subscripts and just write $F^{\tau}_{\lambda,\mu}(n)$. Thus

$$\begin{aligned}
\overline{O}_{M,n}(\lambda,\alpha)\overline{O}_{M,n}(\mu,\beta) &= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{\tau \in \mathcal{C}[\gamma]} \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} F^{\tau}_{\lambda,\mu}(n) K_n(\tau_{(\boldsymbol{a},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{\tau \in \mathcal{C}[\gamma]} F^{\tau}_{\lambda,\mu}(n) \sum_{(\boldsymbol{a},\boldsymbol{c}) \in \mathsf{PC}_n(\gamma)} K_n(\tau_{(\boldsymbol{a},\boldsymbol{c})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{c}} \\
&= \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{\tau \in \mathcal{C}[\gamma]} F^{\tau}_{\lambda,\mu}(n) \overline{O}_{M,n}(\tau,\gamma).
\end{aligned}$$

□

Corollary 5.53. Let $n, k \in \mathbb{Z}_{\geq 0}$ and M an $\mathbb{C}\mathfrak{S}_n$-module. As a $\mathbb{C}$-algebra we have that

$$Q_{2k}(M,n) = \mathsf{Span}_{\mathbb{C}}\{\overline{O}_{M,n}(\lambda,\alpha) \mid \alpha \in \Pi_{2k},\ n \geq |\alpha|, \lambda \in \mathcal{C}_{\leq n}[\alpha]\}.$$

□

In $Q_{2k}(M,n)$, the $\mathbb{C}$-linear dependencies among the operators $\overline{O}_{M,n}(\lambda,\alpha)$ for $\alpha \in \Pi_{2k}$ with $n \geq |\alpha|$, and $\lambda \in \mathcal{C}_{\leq n}[\alpha]$, depends on the module M. As we show now, no such dependencies are present whenever M is free.

Proposition 5.54. Let F be a free $\mathbb{C}\mathfrak{S}_n$-module, then the set

$$\{\overline{O}_{F,n}(\lambda,\alpha) \mid \alpha \in \Pi_{2k},\ n \geq |\alpha|, \lambda \in \mathcal{C}_{\leq n}[\alpha]\}$$

forms a basis of $Q_{2k}(F,n)$.

Proof. Let us set

$$I := \{(\lambda,\alpha) \mid \alpha \in \Pi_{2k},\ n \geq |\alpha|, \lambda \in \mathcal{C}_{\leq n}[\alpha]\}.$$

By *Corollary 5.53*, we only need to show that $\{\overline{O}_{M,n}(\lambda,\alpha) \mid (\lambda,\alpha) \in I\}$ is linearly independent in $Q_{2k}(F,n)$. Also any free $\mathbb{C}\mathfrak{S}_n$-module is isomorphic to $(\mathbb{C}\mathfrak{S}_n)^{\oplus r}$ viewed as a module by left componentwise multiplication. It will suffice to prove this result for $F = \mathbb{C}\mathfrak{S}_n$. Now assume that

$$\sum_{(\lambda,\alpha)\in I} c_{\lambda,\alpha}\overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha) = 0 \tag{5.18}$$

where $c_{\lambda,\alpha} \in \mathbb{C}$ for each $(\lambda,\alpha) \in I$. We seek to show that $c_{\lambda,\alpha} = 0$ for all $(\lambda,\alpha) \in I$. The space $\mathbb{C}\mathfrak{S}_n \otimes V^{\otimes k}$ has a basis given by $\{\pi \otimes v_{\boldsymbol{a}} \mid \pi \in \mathfrak{S}_n,\ \boldsymbol{a} \in [n]^k\}$. For any vector $w \in \mathbb{C}\mathfrak{S}_n \otimes V^{\otimes k}$ and basis element $\pi \otimes v_{\boldsymbol{a}} \in \mathbb{C}\mathfrak{S}_n \otimes V^{\otimes k}$ we write $\pi \otimes v_{\boldsymbol{a}} \in w$ whenever the basis element $\pi \otimes v_{\boldsymbol{a}}$ appears with non-zero coefficient in the decomposition of w into said basis elements. Now for any $(\lambda,\alpha) \in I$, let $\boldsymbol{b} \in [n]^k$ be a tuple which perfectly colours the bottom row of α (which exists since $|\alpha| \leq n$). Then the operator $\overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha)$ acts on the basis element $1 \otimes v_{\boldsymbol{b}} \in \mathbb{C}\mathfrak{S}_n \otimes V^{\otimes k}$ by

$$\overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha)(1 \otimes v_{\boldsymbol{b}}) = \sum_{\substack{\boldsymbol{a}\in[n]^k \\ (\boldsymbol{a},\boldsymbol{b})\in \mathsf{PC}_n(\alpha)}} K_n(\lambda_{(\boldsymbol{a},\boldsymbol{b})}) \otimes v_{\boldsymbol{a}},$$

which is non-zero since $n \geq \mathsf{deg}(\lambda)$. Thus $\pi \otimes v_{\boldsymbol{a}} \in \overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha)(1 \otimes v_{\boldsymbol{b}})$ if and only if $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ and $\pi \in \mathsf{CL}_n[\boldsymbol{a},\boldsymbol{b}](\lambda_{(\boldsymbol{a},\boldsymbol{b})})$. Hence assume for $(\lambda,\alpha),(\mu,\beta) \in I$ that we have $\pi \otimes v_{\boldsymbol{a}} \in \overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha)(1 \otimes v_{\boldsymbol{b}})$ and $\pi \otimes v_{\boldsymbol{a}} \in \overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\mu,\beta)(1 \otimes v_{\boldsymbol{b}})$. Then $(\boldsymbol{a},\boldsymbol{b})$ belongs to $\mathsf{PC}_n(\alpha)$ and $\mathsf{PC}_n(\beta)$, which implies that $\alpha = \beta$. Moreover we must have that π belongs to $\mathsf{CL}_n[\boldsymbol{a},\boldsymbol{b}](\lambda_{(\boldsymbol{a},\boldsymbol{b})})$ and $\mathsf{CL}_n[\boldsymbol{a},\boldsymbol{b}](\mu_{(\boldsymbol{a},\boldsymbol{b})})$. Since both these sets are orbits under the action of $\mathsf{Stab}_n([\boldsymbol{a},\boldsymbol{b}])$ on $\mathfrak{S}_n$ by conjugation, they are either disjoint or equal, thus we must have that $\lambda = \mu$. Now by *Equation* (5.18), picking any $(\mu,\beta) \in I$ and $\boldsymbol{b} \in [n]^k$ which perfectly colours the bottom row of β, we have that

$$\sum_{(\lambda,\alpha)\in I\setminus\{(\mu,\beta)\}} c_{\lambda,\alpha}\overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha)(1 \otimes v_{\boldsymbol{b}}) = -c_{\mu,\beta}\overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\mu,\beta)(1 \otimes v_{\boldsymbol{b}}).$$

However, we have shown that for any $\pi \otimes v_{\boldsymbol{a}} \in \overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\mu,\beta)(1 \otimes v_{\boldsymbol{b}})$, it must be the case that $\pi \otimes v_{\boldsymbol{a}} \notin \overline{O}_{\mathbb{C}\mathfrak{S}_n,n}(\lambda,\alpha)(1 \otimes v_{\boldsymbol{b}})$ whenever $(\lambda,\alpha) \neq (\mu,\beta)$. This implies that $c_{\mu,\beta} = 0$, and since (μ,β) was an arbitrary element of I, we have that $c_{\lambda,\alpha} = 0$ for all $(\lambda,\alpha) \in I$. □

Remark 5.55. From above, whenever F is a free $\mathbb{C}\mathfrak{S}_n$-module, then the operators $\overline{O}_{F,n}(\lambda,\alpha)$, for $\alpha \in \Pi_{2k}$ and $\lambda \in \mathcal{C}[\alpha]$, form a basis for the subalgebra $Q_{2k}(F,n)$ of the endomorphism space $\mathsf{End}_{\mathfrak{S}_n}(F \otimes V^{\otimes k})$. Thus *Theorem 5.52* has proved the existence of certain polynomials which have "globalised" the structure constants of such a basis for $Q_{2k}(F,n)$. From this, we will be able to define a new algebra in the next section which "globalises" the algebras $Q_{2k}(F,n)$. It is worth comparing this theory to that presented by P. Martin and D. Woodcock in [MW98]. In such, they defined a new algebra called the *global Schur algebra* which appears to play an analogous role for another endomorphism algebra, in particular [MW98, Proposition 3.3] is comparable to our *Theorem 5.52*, with both proofs being combinatorial in nature. In their setting they have "globalised" the endomorphism algebra $\mathsf{End}_{\mathfrak{S}_n}(E^{\otimes k})$ where E is a countably infinite analog to V.

5.2.2 The Orbit Affine Partition Algebra $\mathcal{Q}_{2k}^{\mathrm{aff}}$

In this section we define an algebra $\mathcal{Q}_{2k}^{\mathrm{aff}}$ which may be interpreted as another affinization of the partition algebra. We will call it the *orbit affine partition algebra*. This algebra will act on the space $M \otimes V^{\otimes k}$ in a manner which generalised the action of $\Psi_{2k,n}$ described in item (2) of *Theorem 2.58*. In fact such an action comes more or less for free from the construction of $\mathcal{Q}_{2k}^{\mathrm{aff}}$. We will prove that the partition algebra $\mathcal{Q}_{2k}(z)$ and certain marked cycle shape algebras are subalgebras of $\mathcal{Q}_{2k}^{\mathrm{aff}}$. We end the section by constructing a $\mathbb{C}$-algebra homomorphism $\mathcal{A}_{2k}^{\mathrm{aff}} \to \mathcal{Q}_{2k}^{\mathrm{aff}}$.

We construct the orbit affine partition algebra $\mathcal{Q}_{2k}^{\mathrm{aff}}$ from the subalgebras $Q_{2k}(F_n, n)$ of $\mathsf{End}_{\mathfrak{S}_n}(F_n \otimes V^{\otimes k})$, where F_n is a free $\mathbb{C}\mathfrak{S}_n$-module. This is done in a completely analogous manner to how the X-marked cycle shape algebra $Z(X)$ was constructed from the centralizer algebras $Z_n(X)$ in *Section 5.1.5*, and how the partition algebra $\mathcal{Q}_{2k}(z)$ was constructed from the endomorphism algebras $\mathsf{End}_{\mathfrak{S}_n}(V^{\otimes k})$ in *Section 2.2.5*.

Definition 5.56. Let $k \in \mathbb{Z}_{\geq 0}$ and z a formal variable. We define $\mathcal{Q}_{2k}^{\mathrm{aff}}$ to be the free $\mathbb{C}[z]$-module with basis given by $\{O(\lambda, \alpha) \mid \alpha \in \Pi_{2k}, \lambda \in \mathcal{C}[\alpha]\}$. We equip $\mathcal{Q}_{2k}^{\mathrm{aff}}$ with the product given by the $\mathbb{C}[z]$-linear extension of

$$O(\lambda, \alpha)O(\mu, \beta) = \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} \sum_{\tau \in \mathcal{C}[\gamma]} F_{\lambda,\mu}^{\tau}(z) O(\tau, \gamma),$$

where $F_{\lambda,\mu}^{\tau}(z)$ are the polynomials of *Theorem 5.52*.

By definition, $\mathcal{Q}_{2k}^{\mathrm{aff}}$ is a distributive ring. By *Theorem 5.52*, for any $n \in \mathbb{Z}_{\geq 0}$ and $\mathbb{C}\mathfrak{S}_n$-module M, we have a surjective homomorphism of distributive rings

$$\theta_{2k,n}^{(M)} : \mathcal{Q}_{2k}^{\mathrm{aff}} \to Q_{2k}(M, n) \subseteq \mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k}),$$

defined on the generators by $z \mapsto n$ and $O(\lambda, \alpha) \mapsto \overline{O}_{M,n}(\lambda, \alpha)$. The map $\theta_{2k,n}^{(M)}$ is the orbit affine partition algebra counterpart to the map $\Psi_{2k,n}^{(M)}$ given in *Theorem 4.24* for the affine partition algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$.

Lemma 5.57. For each $n \in \mathbb{Z}_{\geq 0}$ let F_n be a free $\mathbb{C}\mathfrak{S}_n$-module. Then

$$\bigcap_{n \geq 0} \mathsf{Ker}\left(\theta_{2k,n}^{(F_n)}\right) = \{0\}.$$

Proof. Let I be a finite subset of $\{(\lambda,\alpha) \mid \alpha \in \Pi_{2k}, \lambda \in \mathcal{C}[\alpha]\}$. Assume that

$$K := \sum_{(\lambda,\alpha)\in I} f_{\lambda,\alpha}(z) O(\lambda,\alpha)$$

belongs to $\mathsf{Ker}(\theta^{(F_n)}_{2k,n})$ for each $n \in \mathbb{Z}_{\geq 0}$, where $f_{\lambda,\alpha}(z) \in \mathbb{C}[z]$ for all $(\lambda,\alpha) \in I$. Now by *Proposition 5.54*, for any $N \geq \max\{\deg(\lambda), |\alpha| \mid (\lambda,\alpha) \in I\}$ we have that

$$\left\{\theta^{(F_N)}_{2k,N}(O(\lambda,\alpha)) \mid (\lambda,\alpha) \in I\right\} = \{\overline{O}_{F_N,N}(\lambda,\alpha) \mid (\lambda,\alpha) \in I\}$$

is linearly independent in $Q_{2k}(F_N,N)$. Since $\theta^{(F_N)}_{2k,N}(K) = 0$, we must have for all $(\lambda,\alpha) \in I$ that $f_{\lambda,\alpha}(N) = 0$ for infinitely many natural numbers N, which implies that $f_{\lambda,\alpha}(z) = 0$ for all $(\lambda,\alpha) \in I$. Therefore $K = 0$, and since I and K were arbitrary, the result is shown. □

We may now give an analogous result to both *Lemma 5.34* and *Lemma 2.59*.

Lemma 5.58. Let $R_1, R_2 \in \mathcal{Q}^{\text{aff}}_{2k}$. Then $R_1 = R_2$ if and only if for all $n \in \mathbb{Z}_{\geq 0}$

$$\theta^{(F_n)}_{2k,n}(R_1) = \theta^{(F_n)}_{2k,n}(R_1),$$

where F_n is a free $\mathbb{C}\mathfrak{S}_n$-module.

Proof. The forward implication is immediate, while the reverse implication follows since it implies that $R_1 - R_2$ belongs to $\cap_{n\geq 0}\mathsf{Ker}(\theta^{(F_n)}_{2k,n}) = \{0\}$. □

The above lemma will be the main tool we use to confirm relations within $\mathcal{Q}^{\text{aff}}_{2k}$.

Example 5.59. Continuing from *Example 5.51*, it was shown that in $Q_4(F_n,n)$ we have

$$\overline{O}_{F_n,n}(\lambda,\alpha)\overline{O}_{F_n,n}(\mu,\beta) = (n-3)\overline{O}_{F_n,n}(1^{(\gamma)},\gamma)+\sum_{i\in[4]}\overline{O}_{F_n,n}(\tau^{(i)},\gamma)+(n-2)\overline{O}_{F_n,n}(1^{(\delta)},\delta)+\overline{O}_{F_n,n}(\nu,\delta),$$

for any $n \in \mathbb{Z}_{\geq 0}$, and free $\mathbb{C}\mathfrak{S}_n$-modules F_n. Hence by *Lemma 5.58* we have that

$$O(\lambda,\alpha)O(\mu,\beta) = (z-3)O(1^{(\gamma)},\gamma) + \sum_{i\in[4]} O(\tau^{(i)},\gamma) + (z-2)O(1^{(\delta)},\delta) + O(\nu,\delta)$$

is a relation in $\mathcal{Q}^{\text{aff}}_4$.

Proposition 5.60. The distributive ring $\mathcal{Q}^{\text{aff}}_{2k}$ is a $\mathbb{C}[z]$-algebra.

Proof. We need to show that a multiplicative identity exists, and that the product described in *Definition 5.56* is associative. For any partition diagram $\alpha \in \Pi_{2k}$ we let $1^{(\alpha)} \in \mathcal{C}[\alpha]$ denote the α-marked cycle shape containing no symbols $*$ and where the blocks of α appear in cycles of length one. So for any perfect colouring $(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)$ and $n \geq |\alpha|$, we have that $K_n(1^{(\alpha)}_{(\boldsymbol{a},\boldsymbol{b})}) = 1$. Recall *Definition 2.55*, then one can see that

$$\theta^{(F_n)}_{2k,n}\left(\sum_{S \vdash [k]} O(1^{(I(S))}, I(S))\right)$$

is the identity element in $Q_{2k}(F_n, n)$ for any $n \geq k$ and free $\mathbb{C}\mathfrak{S}_n$-module F_n. Thus by *Lemma 5.58* the argument of $\theta^{(F_n)}_{2k,n}$ above is the identity element of $\mathcal{Q}^{\mathrm{aff}}_{2k}$. Now let $A, B, C \in \mathcal{Q}^{\mathrm{aff}}_{2k}$ and $[A, B, C] := (AB)C - A(BC)$. Then for all $n \in \mathbb{Z}_{\geq 0}$ we have that $\psi^{(F_n)}_{2k,n}([A, B, C]) = 0$ since $Q_{2k}(F_n, n)$ is an associative algebra. Thus by *Lemma 5.58* $[A, B, C] = 0$, showing that the product of $\mathcal{Q}^{\mathrm{aff}}_{2k}$ is also associative.

□

We now show that the partition algebra $\mathcal{Q}_{2k}(z)$ is a subalgebra of $\mathcal{Q}^{\mathrm{aff}}_{2k}$.

Proposition 5.61. We have an injective $\mathbb{C}[z]$-algebra homomorphism $\iota : \mathcal{Q}_{2k}(z) \to \mathcal{Q}^{\mathrm{aff}}_{2k}$ given by the $\mathbb{C}[z]$-linear extension of $\iota(O(\alpha)) = O(1^{(\alpha)}, \alpha)$ for each $\alpha \in \Pi_{2k}$ and where $1^{(\alpha)} \in \mathcal{C}[\alpha]$ is the identity.

Proof. For any $\mathbb{C}\mathfrak{S}_n$-module M, as elements in $Q_{2k}(M, n)$ we have that

$$\overline{O}_{M,n}(1^{(\alpha)}, \alpha) = \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} K_n(1^{(\alpha)}_{(\boldsymbol{a},\boldsymbol{b})}) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}} = \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathsf{PC}_n(\alpha)} 1 \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}},$$

for any $\alpha \in \Pi_{2k}$. Hence it is clear that

$$\overline{O}_{M,n}(1^{(\alpha)}, \alpha)\overline{O}_{M,n}(1^{(\beta)}, \beta) = \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(n)\overline{O}_n(1^{(\gamma)}, \gamma)$$

for any $\alpha, \beta \in \Pi_{2k}$ and where $p^{\gamma}_{\alpha,\beta}(z)$ are the polynomials in *Proposition 2.67*. Hence by *Lemma 5.58* we have in $\mathcal{Q}^{\mathrm{aff}}_{2k}$ that

$$O(1^{(\alpha)}, \alpha)O(1^{(\beta)}, \beta) = \sum_{\gamma \in \mathsf{TBC}(\alpha,\beta)} p^{\gamma}_{\alpha,\beta}(z)O(1^{(\gamma)}, \gamma),$$

which confirms that ι is a homomorphism. By definition $\{O(1^{(\alpha)}, \alpha) \mid \alpha \in \Pi_{2k}\}$ is $\mathbb{C}[z]$-linearly independent in $\mathcal{Q}^{\mathrm{aff}}_{2k}$, hence ι is injective.

□

We now show that certain marked cycle shape algebras are subalgebras of $\mathcal{Q}^{\mathrm{aff}}_{2k}$. Let $k \in \mathbb{Z}_{\geq 0}$ and $m \in [k]$. Take any set partition $S_m = \{B_1, \ldots, B_m\}$ of $[k]$ consisting of m blocks. By ordering the blocks of S_m according to minimal elements, suppose that $B_i < B_{i+1}$ for each $i \in [m-1]$. Recalling *Definition 2.55*,

$$I(S_m) = \{C_i := B_i \cup B'_i \mid i \in [m]\}$$

is a set partition of $[k] \cup [k']$ also containing precisely m blocks. Given any $[m]$-marked cycle shape $\lambda \in \mathcal{C}[[m]]$ let $\lambda^{(S_m)}$ denote the $I(S_m)$-marked cycle shape in $\mathcal{C}[I(S_m)]$ obtained from λ by replacing colour i with block C_i for each $i \in [m]$. Naturally the map $(-)^{(S_m)}$ gives a monoid isomorphism $\mathcal{C}[[m]] \cong \mathcal{C}[I(S_m)]$. It also induces an embedding of $Z([m])$ into $\mathcal{Q}_{2k}^{\mathrm{aff}}$, as we now show.

Proposition 5.62. Let $k \in \mathbb{Z}_{\geq 0}$ and $m \in [k]$. We have an injective $\mathbb{C}[z]$-algebra homomorphism $\iota : Z([m]) \to \mathcal{Q}_{2k}^{\mathrm{aff}}$ given by the $\mathbb{C}[z]$-linear extension of

$$\iota(K(\lambda)) = O(\lambda^{(S_m)}, I(S_m))$$

for any $\lambda \in \mathcal{C}[[m]]$.

Proof. For any $n \in \mathbb{Z}_{\geq 0}$ and $\mathbb{C}\mathfrak{S}_n$-module M, we have that

$$\overline{O}_{M,n}(\lambda^{(S_m)}, I(S_m)) = \sum_{(\boldsymbol{a},\boldsymbol{a}) \in \mathsf{PC}_n(I(S_m))} K_n\left(\lambda^{(S_m)}_{(\boldsymbol{a},\boldsymbol{a})}\right) \otimes E_{\boldsymbol{a}}^{\boldsymbol{a}},$$

noting that $\boldsymbol{a} \in [n]^k$ perfectly colours the bottom row of $I(S_m)$ if and only if it perfectly colours the top row. Let $\lambda, \mu \in \mathcal{C}[[m]]$, then $\overline{O}_{M,n}(\lambda^{(S_m)}, I(S_m))\overline{O}_{M,n}(\mu^{(S_m)}, I(S_m))$ equals

$$\begin{aligned}
&\left(\sum_{(\boldsymbol{a},\boldsymbol{a}) \in \mathsf{PC}_n(I(S_m))} K_n\left(\lambda^{(S_m)}_{(\boldsymbol{a},\boldsymbol{a})}\right) \otimes E_{\boldsymbol{a}}^{\boldsymbol{a}}\right)\left(\sum_{(\boldsymbol{b},\boldsymbol{b}) \in \mathsf{PC}_n(I(S_m))} K_n\left(\mu^{(S_m)}_{(\boldsymbol{b},\boldsymbol{b})}\right) \otimes E_{\boldsymbol{b}}^{\boldsymbol{b}}\right) \\
&= \sum_{(\boldsymbol{a},\boldsymbol{a}) \in \mathsf{PC}_n(I(S_m))} \sum_{(\boldsymbol{b},\boldsymbol{b}) \in \mathsf{PC}_n(I(S_m))} K_n\left(\lambda^{(S_m)}_{(\boldsymbol{a},\boldsymbol{a})}\right) K_n\left(\mu^{(S_m)}_{(\boldsymbol{b},\boldsymbol{b})}\right) \otimes E_{\boldsymbol{a}}^{\boldsymbol{a}} E_{\boldsymbol{b}}^{\boldsymbol{b}} \\
&= \sum_{(\boldsymbol{a},\boldsymbol{a}) \in \mathsf{PC}_n(I(S_m))} K_n\left(\lambda^{(S_m)}_{(\boldsymbol{a},\boldsymbol{a})}\right) K_n\left(\mu^{(S_m)}_{(\boldsymbol{a},\boldsymbol{a})}\right) \otimes E_{\boldsymbol{a}}^{\boldsymbol{a}} \\
&= \sum_{(\boldsymbol{a},\boldsymbol{a}) \in \mathsf{PC}_n(I(S_m))} \left(\sum_{\tau \in \mathcal{C}[[m]]} f_{\lambda,\mu}^{\tau}(n) K_n\left(\tau^{(S_m)}_{(\boldsymbol{a},\boldsymbol{a})}\right)\right) \otimes E_{\boldsymbol{a}}^{\boldsymbol{a}} \\
&= \sum_{\tau \in \mathcal{C}[[m]]} f_{\lambda,\mu}^{\tau}(n) \overline{O}_{M,n}(\tau^{(S_m)}, I(S_m)),
\end{aligned}$$

where $f_{\lambda,\mu}^{\tau}(z)$ are the structure polynomials in *Theorem 5.17*. Hence employing *Lemma 5.58* we have that

$$O(\lambda^{(S_m)}, I(S_m))O(\mu^{(S_m)}, I(S_m)) = \sum_{\tau \in \mathcal{C}[m]} f_{\lambda,\mu}^{\tau}(z) O(\tau^{(S_m)}, I(S_m)),$$

which confirms that ι is a homomorphism of $\mathbb{C}[z]$-algebras. Injectivity follows since by definition the set $\{O(\lambda^{(S_m)}, I(S_m)) \mid \lambda \in \mathcal{C}[m]\}$ is $\mathbb{C}[z]$-linearly independent in $\mathcal{Q}_{2k}^{\mathrm{aff}}$. $\square$

We may view $\mathcal{Q}_{2k}^{\mathrm{aff}}$ as a $\mathbb{C}$-algebra with basis $\{z^n O(\lambda, \alpha) \mid \alpha \in \Pi_{2k}, \lambda \in \mathcal{C}[\alpha], n \in \mathbb{Z}_{\geq 0}\}$. From this perspective z is playing the role of a central generator.

Remark 5.63. By *Proposition 5.43* and *Corollary 5.44* we know that

$$Z([m]) \cong \mathbb{C}[z_0, z_1, \dots] \otimes \mathcal{H}_m \cong \mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes m}),$$

As such for any $m \leq k$, *Proposition 5.62* tells us that $\mathsf{End}_{\mathsf{Heis}}(\uparrow^{\otimes m})$, and in particular $\mathcal{H}_m$, are subalgebras of the orbit affine partition algebra $\mathcal{Q}_{2k}^{\mathrm{aff}}$.

We now wish to connect the algebra $\mathcal{Q}_{2k}^{\mathrm{aff}}$ to $\mathcal{A}_{2k}^{\mathrm{aff}}$ by a homomorphism. To do so, it will be helpful to give a more minimal generating set for $\mathcal{A}_{2k}^{\mathrm{aff}}$.

Lemma 5.64. The affine partition algebra $\mathcal{A}_{2k}^{\mathrm{aff}}$ is generated by the set

$$\mathsf{G}_{2k} := \{e_a, s_j, x_1, x_2, \tau_2, z_l \mid a \in [2k-1], j \in [k-1], l \in \mathbb{Z}_{\geq 0}\}.$$

Proof. From the definition of $\mathcal{A}_{2k}^{\mathrm{aff}}$, is it clear that we only need to show that, for each $i \in [2k]$ and $2 \leq j \leq 2k-1$, the elements x_i and τ_j belong to $\langle \mathsf{G}_{2k} \rangle$, the subalgebra of $\mathcal{A}_{2k}^{\mathrm{aff}}$ generated by the elements of G_{2k}. We first prove that x_{2i-1} and τ_{2j} belong to $\langle \mathsf{G}_{2k} \rangle$ for any $i \in [k]$ and $j \in [k-1]$, by the repeat application of two steps:

(Step 1): Assume for some $i \in [k-1]$ that x_{2i-1} and τ_{2i} belong to $\langle \mathsf{G}_{2k} \rangle$. Then from (i) of *Lemma 4.12* we must have that x_{2i+1} belongs to $\langle \mathsf{G}_{2k} \rangle$.

(Step 2): Assume for some $2 \leq i \leq k$ that x_{2i-1} and τ_{2i-2} belong to $\langle \mathsf{G}_{2k} \rangle$. Then (i) of *Lemma 4.17* may be expressed as

$$\begin{aligned}\tau_{2i} = s_{i-1}s_i\tau_{2i-2}s_is_{i-1} + e_{2i-2}x_{2i-1}s_ie_{2i-2}s_i + s_ie_{2i-2}x_{2i-1}s_ie_{2i-2} \\ - e_{2i-2}x_{2i-1}s_{i-1}e_{2i}e_{2i-1}e_{2i-2} - s_ie_{2i-2}e_{2i-1}e_{2i}s_{i-1}x_{2i-1}e_{2i-2}s_i,\end{aligned}$$

where we employed the relations $e_{2i-2}x_{2i-2} = e_{2i-2}x_{2i-1}$ and $x_{2i-2}e_{2i-2} = x_{2i-1}e_{2i-2}$ given by (9)(i) and (9)(ii) of *Definition 4.7*. Hence we see that τ_{2i} must belong to $\langle \mathsf{G}_{2k} \rangle$.

We already know that x_1 and τ_2 belong to $\langle \mathsf{G}_{2k} \rangle$, hence alternating applications of (Step 1) and then (Step 2) show that x_{2i-1} and τ_{2j} belong to G_{2k} for any $i \in [k]$ and $j \in [k-1]$. A similar argument can be given for the pairs x_{2i} and τ_{2i+1}, noting that x_2 and $\tau_3 = \tau_2 s_1$ both belong to G_{2k}.

□

Let $n \in \mathbb{Z}_{\geq 0}$ and M be an $\mathbb{C}\mathfrak{S}_n$-module with basis $\{m_i \mid i \in I\}$ for I some indexing set. To ease reference checking in the next theorem, we recall the actions of the generators x_1, x_2, and τ_2 on the tensor space $M \otimes V^{\otimes k}$ given by the map $\Psi_{2k,n}^{(M)}$ of *Theorem 4.24*. Let $\boldsymbol{b} = (\boldsymbol{b}(1), \dots, \boldsymbol{b}(k)) \in [n]^k$, $b_0 \in I$, and $i \in [2k]$, then we have the following:

$$\Psi_{2k,n}^{(M)}(x_1)(m_{b_0} \otimes v_{\boldsymbol{b}}) = \sum_{a \in [n] \setminus \{\boldsymbol{b}(1)\}} (a, \boldsymbol{b}(1)) m_{b_0} \otimes v_{\boldsymbol{b}},$$

$$\Psi_{2k,n}^{(M)}(x_2)(m_{b_0} \otimes v_{\boldsymbol{b}}) = \sum_{a \in [n] \setminus \{\boldsymbol{b}(1)\}} (a, \boldsymbol{b}(1)) m_{b_0} \otimes v_a \otimes v_{\boldsymbol{b}(2)} \otimes \cdots \otimes v_{\boldsymbol{b}(k)},$$

$$\Psi_{2k,n}^{(M)}(\tau_2)(m_{b_0} \otimes v_{\boldsymbol{b}}) = (1 - \delta_{\boldsymbol{b}(1),\boldsymbol{b}(2)})(\boldsymbol{b}(1), \boldsymbol{b}(2)) m_{b_0} \otimes v_{\boldsymbol{b}},$$

where $\delta_{a,b}$ is the kronecker delta.

Theorem 5.65. There exists a $\mathbb{C}$-algebra homomorphism $Q : \mathcal{A}_{2k}^{\mathrm{aff}} \to \mathcal{Q}_{2k}^{\mathrm{aff}}$.

Proof. For each of the generators $g \in \mathsf{G}_{2k}$ we construct an element $Q(g) \in \mathcal{Q}_{2k}^{\mathrm{aff}}$ such that

$$\theta_{2k,n}^{(M)}(Q(g)) = \Psi_{2k,n}^{(M)}(g) \in \mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$$

for every $n \in \mathbb{Z}_{\geq 0}$ and every $\mathbb{C}\mathfrak{S}_n$-module M. As $\Psi_{2k,n}^{(M)}$ is a $\mathbb{C}$-algebra homomorphism, we have that all the relations between the generators $g \in \mathsf{G}_{2k}$ are also satisfied by the corresponding elements $\theta_{2k,n}^{(M)}(Q(g)) = \Psi_{2k,n}^{(M)}(g)$. As these hold for any n and any M, we can then use *Lemma 5.58* to deduce that these relations also hold between the corresponding elements $Q(g)$. So this will prove the theorem. Before we construct such elements we introduce a little notation. For any $\alpha \in \Pi_{2k}$, let

$$\Lambda = \sum_{\lambda \in I} f_\lambda(z)\lambda$$

be a formal $\mathbb{C}[z]$-linear combination of elements in $\mathcal{C}[\alpha]$, for some finite set $I \subset \mathcal{C}[\alpha]$. Then in $\mathcal{Q}_{2k}^{\mathrm{aff}}$ we will define

$$O(\Lambda, \alpha) := \sum_{\lambda \in I} f_\lambda(z) O(\lambda, \alpha).$$

We also let $\overline{O}_{M,n}(\Lambda, \alpha) := \theta_{2k,n}^{(M)}(O(\Lambda, \alpha))$ for any $n \in \mathbb{Z}_{\geq 0}$ and $\mathbb{C}\mathfrak{S}_n$-module M. We may now construct elements $Q(g)$ for each family of generators $g \in \mathsf{G}_{2k}$:

Constructing $Q(s_j)$ and $Q(e_a)$: By *Proposition 5.61*, it is clear that

$$Q(e_a) = \sum_{\substack{\alpha \in \Pi_{2k} \\ e_a \preceq \alpha}} O(1^{(\alpha)}, \alpha), \quad \text{and} \quad Q(s_j) = \sum_{\substack{\alpha \in \Pi_{2k} \\ s_j \preceq \alpha}} O(1^{(\alpha)}, \alpha)$$

will satisfy the desired property.

Constructing $Q(x_1)$: Recall the notation established in *Definition 2.55*, let S be a set partition of $[k]$ and $I(S)$ the corresponding set partition of $[k] \cup [k']$. Note any perfect colouring in $\mathsf{PC}_n(I(S))$ is of the form $(\boldsymbol{b}, \boldsymbol{b})$ where $\boldsymbol{b}(i) = \boldsymbol{b}(j)$ if and only if i and j belong to the same block of S. Let $1_{I(S)}$ denote the block of $I(S)$ containing 1. Consider the formal $\mathbb{C}[z]$-linear combination of elements in $\mathcal{C}[I(S)]$ given by

$$U_{1_{I(S)}} := u_{1_{I(S)}} + \sum_{\substack{B \in I(S) \\ B \neq 1_{I(S)}}} (B, 1_{I(S)}).$$

Then we have that

$$
\begin{aligned}
\theta_{2k,n}^{(M)}(O(U_{1_{I(S)}}, I(S))) &= \sum_{(\boldsymbol{b},\boldsymbol{b})\in\mathsf{PC}_n(I(S))} \left(K_n((u_{1_{I(S)}})_{(\boldsymbol{b},\boldsymbol{b})}) + \sum_{\substack{B\in I(S)\\ B\neq 1_{I(S)}}} K_n((B, 1_{I(S)})_{(\boldsymbol{b},\boldsymbol{b})}) \right) \otimes E_{\boldsymbol{b}}^{\boldsymbol{b}} \\
&= \sum_{(\boldsymbol{b},\boldsymbol{b})\in\mathsf{PC}_n(I(S))} \left(K_n(u_{\boldsymbol{b}(1)}) + \sum_{\substack{a\in[\boldsymbol{b}]\\ a\neq \boldsymbol{b}(1)}} K_n((a, \boldsymbol{b}(1))) \right) \otimes E_{\boldsymbol{b}}^{\boldsymbol{b}} \\
&= \sum_{(\boldsymbol{b},\boldsymbol{b})\in\mathsf{PC}_n(I(S))} \left(\sum_{a\in[n]\setminus[\boldsymbol{b}]} (a, \boldsymbol{b}(1)) + \sum_{\substack{a\in[\boldsymbol{b}]\\ a\neq \boldsymbol{b}(1)}} (a, \boldsymbol{b}(1)) \right) \otimes E_{\boldsymbol{b}}^{\boldsymbol{b}} \\
&= \sum_{(\boldsymbol{b},\boldsymbol{b})\in\mathsf{PC}_n(I(S))} \left(\sum_{a\in[n]\setminus\{\boldsymbol{b}(1)\}} (a, \boldsymbol{b}(1)) \right) \otimes E_{\boldsymbol{b}}^{\boldsymbol{b}}.
\end{aligned}
$$

Hence given any $b_0 \in I$ and $\boldsymbol{b} \in [n]^k$ which perfectly colours the top and bottom rows of $I(S)$, we have that $\theta_{2k,n}^{(M)}(O(U_{1_S}, I(S)))$ acts on $m_{b_0} \otimes v_{\boldsymbol{b}}$ in the same manner as $\Psi_{2k,n}^{(M)}(x_1)$, and acts on $m_{b_0} \otimes v_{\boldsymbol{c}}$ by 0 whenever $(\boldsymbol{c}, \boldsymbol{c})$ is not a perfect colouring of $I(S)$. Hence the element $Q(x_1)$ we are looking for is

$$
Q(x_1) := \sum_{S\vdash[k]} O(U_{1_{I(S)}}, I(S)),
$$

where the sum runs over all set partitions S of $[k]$.

Constructing $Q(x_2)$: Let S be a set partition of $[k]$, and let $I^{(1)}(S)$ denote the set partition of $[k] \cup [k']$ we obtain from $I(S)$ by removing 1 from its block and letting it be in its own block $\{1\}$. Let $1'_{I^{(1)}(S)}$ denote the block of $I^{(1)}(S)$ containing $\{1'\}$. Consider the formal $\mathbb{C}[z]$-linear combintaion of elements in $\mathcal{C}[I^{(1)}(S)]$ given by

$$
(1, 1')_{I^{(1)}(S)} := (\{1\}, 1'_{I^{(1)}(S)}) + \sum_{\substack{B\in I^{(1)}(S)\\ B\neq 1'_{I^{(1)}(S)}, \{1\}}} (B, 1'_{I^{(1)}(S)}).
$$

Note that any perfect colourings of $I^{(1)}(S)$ is of the form $(\boldsymbol{b}^{(a)}, \boldsymbol{b})$ where $\boldsymbol{b}$ perfectly colours the top and bottom rows of $I(S)$, $\boldsymbol{b}^{(a)}(j) = \boldsymbol{b}(j)$ for all $j \in [k]\setminus\{1\}$, and $\boldsymbol{b}^{(a)}(1) = a$ for some $a \in [n]\backslash[\boldsymbol{b}]$. Then in $Q_{2k}(M,n)$ we have that $\theta^{(M)}_{2k,n}(O((1,1')_{I^{(1)}(S)}, I^{(1)}(S)))$ equals

$$\sum_{(\boldsymbol{a},\boldsymbol{b})\in \mathsf{PC}_n(I^{(1)}(S))} \left(K_n((\{1\}, 1'_{I^{(1)}(S)})_{(\boldsymbol{a},\boldsymbol{b})}) + \sum_{\substack{B\in I^{(1)}(S)\\ B\neq 1'_{I^{(1)}(S)},\{1\}}} K_n((B, 1'_{I^{(1)}(S)})_{(\boldsymbol{a},\boldsymbol{b})}) \right) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}$$

$$= \sum_{(\boldsymbol{a},\boldsymbol{b})\in \mathsf{PC}_n(I^{(1)}(S))} \left((\boldsymbol{a}(1), \boldsymbol{b}(1)) + \sum_{\substack{b\in[\boldsymbol{b}]\\ b\neq \boldsymbol{b}(1)}} (b, \boldsymbol{b}(1)) \right) \otimes E^{\boldsymbol{a}}_{\boldsymbol{b}}$$

$$= \sum_{(\boldsymbol{b},\boldsymbol{b})\in \mathsf{PC}_n(I(S))} \left(\sum_{a\in[n]\backslash[\boldsymbol{b}]} (a, \boldsymbol{b}(1)) \otimes E^{\boldsymbol{b}^{(a)}}_{\boldsymbol{b}} + \sum_{\substack{b\in[\boldsymbol{b}]\\ b\neq \boldsymbol{b}(1)}} (b, \boldsymbol{b}(1)) \otimes E^{\boldsymbol{b}^{(a)}}_{\boldsymbol{b}} \right)$$

$$= \sum_{(\boldsymbol{b},\boldsymbol{b})\in \mathsf{PC}_n(I(S))} \sum_{\substack{a\in[n]\\ a\neq \boldsymbol{b}(1)}} (a, \boldsymbol{b}(1)) \otimes E^{\boldsymbol{b}^{(a)}}_{\boldsymbol{b}},$$

Therefore, given any $b_0 \in I$ and $\boldsymbol{b} \in [n]^k$ such that $(\boldsymbol{b},\boldsymbol{b})$ perfectly colors $I(S)$, we have that $\theta^{(M)}_{2k,n}(O((1,1')_{I^{(1)}(S)}, I^{(1)}(S)))$ acts on $m_{b_0} \otimes v_{\boldsymbol{b}}$ in the same manner as $\Psi^{(M)}_{2k,n}(x_2)$, and acts on $m_{b_0} \otimes v_{\boldsymbol{c}}$ by 0 whenever $(\boldsymbol{c},\boldsymbol{c})$ is not a perfect colouring of $I(S)$. Hence the element $Q(x_2)$ we are looking for is

$$Q(x_2) = \sum_{S\vdash[k]} O((1,1')_{I^{(1)}(S)}, I^{(1)}(S)),$$

where the sum runs over all set partitions S of $[k]$.

Constructing $Q(\tau_2)$: Let S be a set partition of $[k]$ such that $1 \not\sim_S 2$, that is 1 and 2 belong to distinct blocks of S. Let $1_{I(S)}$ and $2_{I(S)}$ denote the distinct blocks of S containing 1 and 2 respectively. Consider the element $(1_{I(S)}, 2_{I(S)})$ in $\mathcal{C}[I(S)]$, then

$$\begin{aligned}\theta^{(M)}_{2k,n}(O((1_{I(S)}, 2_{I(S)}), I(S))) &= \sum_{(\boldsymbol{b},\boldsymbol{b})\in \mathsf{PC}_n(I(S))} K_n((1_{I(S)}, 2_{I(S)})_{(\boldsymbol{b},\boldsymbol{b})}) \otimes E^{\boldsymbol{b}}_{\boldsymbol{b}} \\ &= \sum_{(\boldsymbol{b},\boldsymbol{b})\in \mathsf{PC}_n(I(S))} K_n((\boldsymbol{b}(1), \boldsymbol{b}(2))) \otimes E^{\boldsymbol{b}}_{\boldsymbol{b}} \\ &= \sum_{(\boldsymbol{b},\boldsymbol{b})\in \mathsf{PC}_n(I(S))} (\boldsymbol{b}(1), \boldsymbol{b}(2)) \otimes E^{\boldsymbol{b}}_{\boldsymbol{b}}.\end{aligned}$$

Therefore, given any $b_0 \in I$ and $\boldsymbol{b} \in [n]^k$ such that $(\boldsymbol{b},\boldsymbol{b})$ perfectly colours $I(S)$, we have that $\theta^{(M)}_{2k,n}(O((1_{I(S)}, 2_{I(S)}), I(S)))$ acts on $m_{b_0} \otimes v_{\boldsymbol{b}}$ in the same manner as $\Psi^{(M)}_{2k,n}(\tau_2)$, and acts on $m_{b_0} \otimes v_{\boldsymbol{c}}$ by 0 whenever $(\boldsymbol{c},\boldsymbol{c})$ is not a perfect colouring of $I(S)$. Hence the element $Q(\tau_2)$ we are looking for is

$$Q(\tau_2) = \sum_{\substack{S \vdash [k] \\ 1 \not\sim_S 2}} O((1_{I(S)}, 2_{I(S)}), I(S)),$$

where the sum runs over all set partitions S of $[k]$ such that $1 \not\sim_S 2$.

Constructing $Q(z_l)$: Let $b_0 \in I$ and $\boldsymbol{b} \in [n]^k$, then recall that

$$\Psi^{(M)}_{2k,n}(z_l)(m_{b_0} \otimes v_{\boldsymbol{b}}) = (Z_{n,l} m_{b_0}) \otimes v_{\boldsymbol{b}},$$

where $Z_{n,l}$ belongs to the center $Z_n = Z_n(\emptyset)$ of $\mathbb{C}\mathfrak{S}_n$ and is given by

$$Z_{n,l} = \sum_{b \in [n]} T^l_{n,b}, \quad \text{where} \quad T_{n,b} = \sum_{a \in [n] \setminus \{b\}} (a, b).$$

To construct $Q(z_l)$ we first want to show that there exists an element in the $\emptyset$-marked cycle shape algebra $Z(\emptyset)$ such that its projection into the center $Z_n(\emptyset)$ under $\mathsf{pr}_n[\emptyset]$ equals $Z_{n,l}$ for all $n \in \mathbb{Z}_{\geq 0}$. We introduce the following notation: Let $X \subseteq [n]$ and $r \geq 1$, and consider any r-tuple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathcal{C}[X]^{\times r}$ of X-marked cycle shapes and any $\mu \in \mathcal{C}[X]$. Then we define

$$f^{\mu}_{\boldsymbol{\lambda}}(z) := [K(\mu)](K(\lambda_1) \cdots K(\lambda_r)),$$

the polynomial in $\mathbb{C}[z]$ which appears as the coefficient of the term $K(\mu)$ in the product $K(\lambda_1) \cdots K(\lambda_r)$ in $Z(X)$. Now for any $b \in [n]$, one can see that $T_{n,b} = K_n((*, b))$ which belongs to $Z_n(\{b\})$. Hence we have that

$$\sum_{b \in [n]} T^l_{n,b} = \sum_{b \in [n]} K_n((*, b))^l = \sum_{\sigma \in \mathcal{T}} \sigma K_n((*, 1))^l \sigma^{-1}$$

where $\mathcal{T} = \{(1, 2), (1, 3), \dots, (1, n)\}$ a set of left transversals of $\mathsf{Stab}_n(\{1\})$ within the group $\mathsf{Stab}_n(\emptyset) = \mathfrak{S}_n$. Now set $(*, 1)^{(l)} := ((*, 1), \dots, (*, 1)) \in \mathcal{C}[\{1\}]^{\times l}$. Then in the centralizer algebra $Z_n(\{1\})$ we have that

$$K_n((1, *))^l = \sum_{\mu \in \mathcal{C}[\{1\}]} f^{\mu}_{(*,1)^{(l)}}(n) K_n(\mu).$$

Therefore the elements $Z_{n,l}$ can be expressed as

$$\begin{aligned}
\sum_{b \in [n]} T^l_{n,b} &= \sum_{\sigma \in \mathcal{T}} \sigma \left(\sum_{\mu \in \mathcal{C}[\{1\}]} f^{\mu}_{(*,1)^{(l)}}(n) K_n(\mu) \right) \sigma^{-1} = \sum_{\sigma \in \mathcal{T}} \sum_{\mu \in \mathcal{C}[\{1\}]} f^{\mu}_{(*,1)^{(l)}}(n) K_n(\mu^{\sigma}) \\
&= \sum_{\mu \in \mathcal{C}[\{1\}]} f^{\mu}_{(*,1)^{(l)}}(n) f^{\mu}_{\emptyset}(n) K_n(\mu \downarrow_{\emptyset}) = \sum_{\lambda \in \mathcal{C}[\emptyset]} \left(\sum_{\substack{\mu \in \mathcal{C}[\{1\}] \\ \mu \downarrow_{\emptyset} = \lambda}} f^{\mu}_{(*,1)^{(l)}}(n) f^{\mu}_{\emptyset}(n) \right) K_n(\lambda)
\end{aligned}$$

where $f^{\mu}_{\emptyset}(z)$ are the polynomials given in *Proposition 5.28*. Thus set

$$h^{\lambda}_{l}(z) := \sum_{\substack{\mu \in \mathcal{C}[\{1\}] \\ \mu\downarrow_{\emptyset}=\lambda}} f^{\mu}_{(*,1)^{(l)}}(z) f^{\mu}_{\emptyset}(z). \tag{5.19}$$

Then the element

$$\sum_{\lambda \in \mathcal{C}[\emptyset]} h^{\lambda}_{l}(z) K(\lambda)$$

of $Z(\emptyset)$ projects down to $Z_{n,l}$ under the morphism $\mathsf{pr}_n[\emptyset]$ for all $n \in \mathbb{Z}_{\geq 0}$. Lastly, for any $l \in \mathbb{Z}_{\geq 0}$ and set partition S of $[k]$, consider the formal $\mathbb{C}[z]$-linear combination of elements of $\mathcal{C}[I(S)]$ given by

$$w_l := \sum_{\lambda \in \mathcal{C}[\emptyset]} h^{\lambda}_{l}(z) \left(\sum_{\mu \in \mathsf{Fill}^{I(S)}_{\emptyset}(\lambda)} \mu \right),$$

Then for any $n \in \mathbb{Z}_{\geq 0}$ we have that

$$\begin{aligned}
\theta^{(M)}_{2k,n}(O(w_l, I(S))) &= \sum_{(\boldsymbol{b},\boldsymbol{b}) \in \mathsf{PC}_n(I(S))} \left(\sum_{\lambda \in \mathcal{C}[\emptyset]} h^{\lambda}_{l}(n) \sum_{\mu \in \mathsf{Fill}^{I(S)}_{\emptyset}(\lambda)} K_n(\mu_{(\boldsymbol{b},\boldsymbol{b})}) \right) \otimes E^{\boldsymbol{b}}_{\boldsymbol{b}}, \\
&= \sum_{(\boldsymbol{b},\boldsymbol{b}) \in \mathsf{PC}_n(I(S))} \left(\sum_{\lambda \in \mathcal{C}[\emptyset]} h^{\lambda}_{l}(n) K_n(\lambda) \right) \otimes E^{\boldsymbol{b}}_{\boldsymbol{b}}, \\
&= \sum_{(\boldsymbol{b},\boldsymbol{b}) \in \mathsf{PC}_n(I(S))} Z_{n,l} \otimes E^{\boldsymbol{b}}_{\boldsymbol{b}},
\end{aligned}$$

where for the second equality we employed *Equation* (5.8). Therefore, given any $b_0 \in I$ and $\boldsymbol{b} \in [n]^k$ such that $(\boldsymbol{b}, \boldsymbol{b})$ perfectly colours $I(S)$, we have that $\theta^{(M)}_{2k,n}(O(w_l, I(S)))$ acts on $m_{b_0} \otimes v_{\boldsymbol{b}}$ in the same manner as $\Psi^{(M)}_{2k,n}(z_l)$, and acts on $m_{b_0} \otimes v_{\boldsymbol{c}}$ by 0 whenever $(\boldsymbol{c}, \boldsymbol{c})$ is not a perfect colouring of $I(S)$. Hence the element $Q(z_l)$ we are looking for is

$$Q(z_l) = \sum_{S \vdash [k]} O_n(w_l, I(S)),$$

where the sum runs over all set partitions S of $[k]$.

□

Corollary 5.66. For any $k, n \in \mathbb{Z}_{\geq 0}$ and $\mathbb{C}\mathfrak{S}_n$-module M, the image of $\Psi^{(M)}_{2k,n} : \mathcal{A}^{\mathrm{aff}}_{2k} \to \mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$ (given in *Theorem 4.24*) belongs to $Q_{2k}(M, n)$.

□

Remark 5.67. Both the algebra $Q_{2k}(M,n)$ and $\mathcal{Q}^{\text{aff}}_{2k}$ came about from investigating the image of $\Psi^{(M)}_{2k,n}$ in *Theorem 4.24*. We suspect that we have an isomorphism of $\mathbb{C}$-algebras $\mathcal{A}^{\text{aff}}_{2k} \cong \mathcal{Q}^{\text{aff}}_{2k}$, which in turn would give us $\mathsf{Im}(\Psi^{(M)}_{2k,n}) = Q_{2k}(M,n)$. Although not included within this thesis, we believe we have a proof of the surjectivity of the homomorphism $Q : \mathcal{A}^{\text{aff}}_{2k} \to \mathcal{Q}^{\text{aff}}_{2k}$, but injectivity will probably require knowing a basis for $\mathcal{A}^{\text{aff}}_{2k}$ which we do not yet have.

Remark 5.68. The algebra $\mathcal{Q}^{\text{aff}}_{2k}$ is interesting in its own right as it allows for greater analysis of the image $\mathsf{Im}(\Psi^{(M)}_{2k,n}) \subseteq Q_{2k}(M,n)$. In particular, using the formula presented in *Corollary 5.14*, one can compute an upper bound for the dimension of the image $\mathsf{Im}(\Psi^{(M)}_{2k,n})$, at least for small n. Also the algebra $\mathcal{Q}^{\text{aff}}_{2k}$, if isomorphic to $\mathcal{A}^{\text{aff}}_{2k}$, provides a lot of non-trivial structure to investigate. For example we have shown that the degenerate affine Hecke algebra $\mathcal{H}_k$, and any X-marked cycle shape algebra $Z(X)$ (for $|X| \leq k$) are subalgebras of $\mathcal{Q}^{\text{aff}}_{2k}$, which has not been shown for $\mathcal{A}^{\text{aff}}_{2k}$. Also, we know a basis of $\mathcal{Q}^{\text{aff}}_{2k}$ which projects down to $\mathsf{End}_{\mathfrak{S}_n}(M \otimes V^{\otimes k})$ in a very natural manner, and the non-vanishing basis elements provide a spanning set (and in fact a basis whenever M is free). This gives us clues as to what structure a yet to be defined *cyclotomic quotient* of the affine partition algebra should have, at least vaguely. A long term hope would be that some of the theory produced by J. Brundan and A. Kleshchev in [BK08] would have analogs in the setting of the affine partition algebra, most notably having some analogs to higher Schur-Weyl dualities between the group algebra of the symmetric group and the partition algebra. In the classical case, the module M is specialised to certain weight modules of $\mathfrak{gl}_n$. It is not obvious what would make an appropriate choice of modules in the setting of the affine partition algebra, but we have done some basic analysis of the algebra $Q_{2k}(M,n)$ when $M = \mathsf{S}^\lambda$ is a Specht module for some $\lambda \in \Lambda_n$, and n is small. In the cases we have investigated, we have not been able to rule out the surjectivity of $\Psi^{(\mathsf{S}^\lambda)}_{2k,n}$. We suspect that $\Psi^{(\mathsf{S}^\lambda)}_{2k,n}$ should be surjective for some Specht modules, but not all.

Remark 5.69. We treat $\mathcal{A}^{\text{aff}}_{2k}$ as our primary definition for an affine partition algebra, since it is much nicer to work with compared to $\mathcal{Q}^{\text{aff}}_{2k}$. In particular, we focused on proving the affinization properties *1* to *5* for $\mathcal{A}^{\text{aff}}_{2k}$ but not for $\mathcal{Q}^{\text{aff}}_{2k}$. However, one can confirm some of the affinization properties for $\mathcal{Q}^{\text{aff}}_{2k}$ by using the homomorphism Q in *Theorem 5.65*. If it is true that $\mathcal{A}^{\text{aff}}_{2k} \cong \mathcal{Q}^{\text{aff}}_{2k}$, then the algebra $\mathcal{Q}^{\text{aff}}_{2k}$ would be the orbit description of $\mathcal{A}^{\text{aff}}_{2k}$, much in the same way the $\mathcal{Q}_{2k}(z)$ is an orbit description of $\mathcal{A}_{2k}(z)$ as present in *Section 2.2.5*.

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