STUDY OF COVERING PROPERTIES IN FUZZY TOPOLOGY

by

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DECLARATION

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ABSTRACT

This work is devoted to the study of covering properties both in L-fuzzy topological spaces and in smooth L-fuzzy topological spaces, that is the fuzzy spaces in Šostak's sense, where L is a fuzzy lattice. Based on the satisfactory theory of L-fuzzy compactness build up by Warner, McLean and Kudri, good definitions of feeble compactness and P-closedness are introduced and studied. A unification theory for good L-fuzzy covering axioms is provided.

Following the lines of L-fuzzy compactness, we suggest two kinds of L-fuzzy relative compactness as in general topology, study some of their properties and prove that these notions are good extensions of the corresponding ordinary versions.

We also present L-fuzzy versions of R-compactness, weak compactness and θ-rigidity and discuss some of their properties.

By introducing *'α-Scott continuous functions'*, a *'goodness of extension'* criterion for smooth fuzzy topological properties is established. We propose a good definition of compactness, which we call *'smooth compactness'* in smooth L-fuzzy topological spaces. Smooth compactness turns out to be an extension of L-fuzzy compactness to smooth L-fuzzy topological spaces. We study some properties of smooth compactness and obtain different characterizations. As an extension of the fuzzy Hausdorffness defined by Warner and McLean, *'smooth Hausdorffness'* is introduced in smooth L-fuzzy topological spaces. Good definitions of smooth countable compactness, smooth Lindelöfness and smooth local compactness are introduced and some of their properties studied.

NOMENCLATURE

The following list contains the most frequently used classical symbols in this thesis. Afterwards a list of the most frequently used fuzzy notations will be given.

N	the set of the natural numbers
$\emptyset$	the empty set
I	the closed unit interval [0 , 1]
L	a lattice
$\leq, \nleq$	partial order relation and its negation
$\vee$	join
$\wedge$	meet
'	order reversing involution
0 , 1	the smallest and the largest element of a lattice L
pr(L)	the set of all prime elements of a lattice L
M(L)	the set of all union irreducible (or coprime) elements of a lattice L
$\mathbf{a \ll b}$	a is way below b
$\beta(\alpha)$	the union of all minimal sets relative to α
$\beta^*(\alpha)$	the intersection of $\beta(\alpha)$ and M(L) where $\alpha \in L$
$2^{(\Gamma)}$	the family of all finite subsets of a collection Γ
$\mathbf{x \in X}$	x is an element of X

$x \notin X$	x is not an element of X
$X \setminus A$	the set $\{x : x \in X,\ x \notin A\}$
$\forall$	the quantifier " for each "
$\wp(X)$	the power set of X
χ_A	the characteristic function of A
$\subset, \not\subset$	the relation " is properly contained in " on a power set and its negation
$\subseteq, \not\subseteq$	the relation " is contained in " on a power set and its negation
$(A_i)_{i \in J}$	indexed family of sets
$\bigcup_{i \in J} A_i$	the union of the family $(A_i)_{i \in J}$
$\bigcap_{i \in J} A_i$	the intersection of the family $(A_i)_{i \in J}$
$f: X \to Y$	a function from X to Y
$f(x)$	the image of x under the function f
$f^{-1}(y)$	the inverse image of y under the function f
$f(A)$	the image of A under the function f
$f^{-1}(A)$	the inverse image of A under the function f
$f\vert_A$	the restriction of the function f to the set A
(X,T)	a topological space
$cl(A)$	the closure of the set A
$int(A)$	the interior of the set A

Now we give a list of the most frequently used fuzzy notations.

$\mathbf{L^X}$	the set of all L-fuzzy sets on X
$\emptyset$	the empty fuzzy set on X
$\mathbf{X}$	the full fuzzy set on X
$\mathbf{f'}$	the complement of an L-fuzzy set f
Suppf	the support of an L-fuzzy set f
$\mathbf{pr(L^X)}$	the set of all prime elements of L^X
$\mathbf{M(L^X)}$	the set of all coprime elements of L^X
$\mathbf{x_p}$	an L-fuzzy point
$\mathbf{x_\alpha}$	a coprime element of L^X
$\mathbf{x_p \in f}$	the L-fuzzy point x_p is a member of the L-fuzzy set f
$\vee_{i\in J}\mathbf{f_i}$	the join of the family $(f_i)_{i\in J}$ of L-fuzzy sets
$\wedge_{i\in J}\mathbf{f_i}$	the meet of the family $(f_i)_{i\in J}$ of L-fuzzy sets
$\mathbf{f(g)}$	the image of an L-fuzzy set g under a function f
$\mathbf{f^{-1}(g)}$	the inverse image of an L-fuzzy set g under a function f
$\mathbf{(S_m)_{m\in D}}$	a net in X of term $S_m \in M(L^X)$
$\mathbf{SuppS_m}$	the support of the term S_m of a net $(S_m)_{m\in D}$
$\mathbf{h(S_m)}$	the height of the term S_m of a net $(S_m)_{m\in D}$
$\mathbf{(x^m_{\alpha_m})_{m\in D}}$	an α-net of term $x^m_{\alpha_m}$ where x^m is its support and α_m its height
$\mathbf{(X,\tau)}$	an L-fuzzy topological space
$\mathbf{(X,\tau_Y)}$	a subspace of an L-fuzzy topological space (X,τ)

cl(f)	the closure of the L-fuzzy set f
int(f)	the interior of the L-fuzzy set f
scl(f)	the semi-closure of the L-fuzzy set f
sint(f)	the semi-interior of the L-fuzzy set f
pcl(f)	the pre-closure of the L-fuzzy set f
pint(f)	the pre-interior of the L-fuzzy set f
θ-cl(f)	the θ-closure of the L-fuzzy set f
θ-int(f)	the θ-interior of the L-fuzzy set f
δ-cl(f)	the δ-closure of the L-fuzzy set f
δ-int(f)	the δ-interior of the L-fuzzy set f
ω(T)	the set of all continuous functions from a topological space (X,T) to a fuzzy lattice L with its Scott topology
(X,ω(T))	an induced L-fuzzy topological space
φ(T)	the set of all 'completely Scott continuous' functions from a topological space (X,T) to a fuzzy lattice L with its Scott topology
(X,φ(T))	a completely induced L-fuzzy topological space
$\phi_p(\tau)$	the ordinary topology $\{ f^{-1}(\{t\in L : t\nleq p\}) : f\in\tau \}$ on X where τ is an L-fuzzy topology on X and $p\in pr(L)$
T_s	the ordinary topology with the base $\{ R\subseteq X : R$ is regular open in $(X,T) \}$ where (X,T) is an ordinary topological space

$\mathbf{i_p(f)}$	the set $\{ x\in X : f(x)\not\leq p \}$ where f is an L-fuzzy set on X and $p\in pr(L)$
$\mathbf{i_L(\tau)}$	the ordinary topology with subbase $\{ i_p(f) : p\in pr(L) \text{ and } f\in\tau \}$ where (X,τ) is an L-fuzzy topological space
$\mathbf{G_p}$	the set $\{ x\in X : g(x)\geq p' \}$ where g is an L-fuzzy set on X and $p\in pr(L)$
$\boldsymbol{\beta}$	an α-level filter base
$\mathbf{(X,\Upsilon)}$	a smooth L-fuzzy topological space
$\mathbf{(X,\Upsilon_Y)}$	a smooth subspace of a smooth L-fuzzy topological space (X,Υ)
$\boldsymbol{\Phi}$	the degree of closedness on X
$\mathbf{\Upsilon(g)}$	the degree of openness of an L-fuzzy set g
$\mathbf{\Phi(g)}$	the degree of closedness of an L-fuzzy set g
$\boldsymbol{\omega_T}$	the smooth L-fuzzy topology on X induced by an ordinary topology T
$\mathbf{(X,\omega_T)}$	an induced smooth L-fuzzy topological space
$\mathbf{(X,\Upsilon_p)}$	the ordinary topology on X with the base $\{ \{ f^{-1}(\{t\in L : t\not\leq p\}) : \Upsilon(f)\not\leq p \}$ where Υ is a smooth L-fuzzy topology on X and $p\in pr(L)$; this topology is called a "prime level space" of (X,Υ)

INTRODUCTION

I wish to begin with some examples simply to express the notions of *'fuzziness'* and a *'fuzzy set'*.

Hold an apple in your hand. Is it an apple ? Yes. The object in your hand belongs to the set of apples - all apples anywhere ever. Now take a bite, chew it and swallow it. Is the object in your hand still an apple ? Yes or No? Take another bite. Is the new object still an apple ? Take another bite, and so on down to void.

The apple changes from a thing to nonthing, to nothing. But where does it cross the line from apple to nonapple? When you hold a half of an apple in your hand, the thing in your hand is as much of an apple as it is not. The half apple foils all-or-none descriptions. The half apple is a *fuzzy* apple, the grey between the black and the white. *Fuzziness is greyness*.

Consider a group of people for instance. Let's ask some questions to them. How many of you are male? Raise your hands. Male hands go up and female hands stay down. That gives a set and it is not a fuzzy. How many of you are female? Raise your hands. The reverse happens and again the audience splits into two, black and white sets, males and nonmales or females and nonfemales. Then comes a harder question: How many of you are satisfied with your jobs? The hands bob up and down. A confident few, point their arms straight up or do not raise them at all. Most persons are in between. That defines one fuzzy set, the set of those satisfied with their jobs, the happily employed. Now hands down. How many of you are not satisfied with your jobs? Many of the same hands go up again and bob up and down. This defines another fuzzy set, the unhappily employed, the opposite or the negation of the first fuzzy set.

The job sets differ from the male-female sets. The set of males does not intersect the set of females. No one is both male and female. Every one is either male or female: ***A or not-A***. But most people are both satisfied and not satisfied with their jobs: ***A and not-A***. Few are 100% satisfied or 100% unsatisfied.

The audience example shows the essence of *fuzziness*: fuzzy things resemble fuzzy nonthings. ***A*** resembles ***not-A***. Fuzzy things have vague boundaries with their opposites, with nonthings.

Fuzziness has a formal name in science: ***multivalence***. The opposite of fuzziness is bivalence or two-valuedness; two ways to answer each question : true or false, 1 or 0. Fuzziness means multivalence. It means three or more options, perhaps an infinite spectrum of options instead of just two extremes with no greyness. It means analog instead of binary; infinite shades of grey between black and white.

In 1965 Lotfi Zadeh [105] published a paper called " Fuzzy Sets ". The paper applied the multivalued logic to sets or groups of objects. Zadeh put the label ***"fuzzy"*** on these vague or multivalued sets - sets whose elements belong to it to different degrees, like the set of people satisfied with their jobs. Consider again the apple you hold in your hand and bite. At first what you hold in your hand is 100% an apple. Or 100% of the apple is there. Or your apple belongs 100% to the set of whole apples. As you bite chunks out of the apple, the percentage falls from 100% all the way down to 0% when you have eaten the apple. About half way through the process you hold the half apple or 50% apple. Fit values describe the apple's descent from total presence to total absence; from the bit value 1 down to the bit value 0, where ***'bit'*** and ***'fit'*** stand respectively for ***'binary unit'*** and ***'fuzzy unit'***. In this sense fit values fill in between bit values. The opposite of 1 is 0 and of 0 is 1. The opposites, ***A and not-A***, reflect about the midpoint fit value of 1/2. The bit values 0 and 1 lie the same distance from the midpoint. The same holds true for a fit value and its opposite. The opposite of 3/4 is 1/4, the opposite of 1/3 is 2/3 and so on. This means that the opposite of 1/2 is 1/2, i.e. ***A*** equals ***not-A*** at the midpoint.

As understood from the above examples, a fuzzy set is a sort of generalized 'characteristic function', whose 'degrees of membership' can be more general than ***1*** or ***0***, ***'Yes'*** or ***'No'***, that is, a membership function which describes the gradual transition from membership to non-membership. As a result, fuzzy set theory can be considered as a mathematical model for imprecise concepts.

The notion of a fuzzy set has caused great interest among both pure and applied mathematicians and experts in other areas , who use mathematical ideas and methods in their research . After Zadeh introduced the theory of fuzzy sets in 1965 , extensive work has been done on these sets by many mathematicians , which caused the formation of a new mathematical field called " ***Fuzzy Mathematics*** " . Since then fuzzy sets have been applied to several fields such as Artificial Intelligence , Control Theory , Expert Systems , Pattern Recognition , Economics , Management and so on .

In his classical paper , Zadeh has defined fuzzy sets in terms of function from an ordinary set to the closed unit interval and introduced other basic notions such as fuzzy union , intersection and complement , all of which have now become standard . These notions were explored in 1967 by Goguen [33] who extended the concept of fuzzy set by replacing the closed unit interval by an arbitrary lattice; thus leading to the definition of L-fuzzy sets corresponding to a given lattice L .

General Topology was one of the first branches of pure mathematics to which fuzzy sets have been applied systematically . It was in 1968 , that is, three years after Zadeh 's paper had appeared , that Chang [16] made the first attempt to define the notion of a fuzzy topological space . Chang has also introduced fuzzy image and fuzzy inverse image under a function , which are now standard, and extended a number of properties of functions , such as continuity , to fuzzy topology .

Since the late seventies, the intensity of research on " ***Fuzzy Topology*** ", that is a branch of Fuzzy Mathematics has sharply increased. Different definitions of fuzzy topology and several approaches to fuzzy topology have been pointed out in the literature. After Chang, in 1976 Lowen [49] introduced another definition of fuzzy topology which requires that a fuzzy topology should have one more axiom, namely it includes all the constant fuzzy sets. In [49] and [52], Lowen and Wuyts have given the reasons and advantages of this definition. However, Rodabaugh [78] has pointed out some reasons to justify Chang's definition of fuzzy topology, that is, without demanding that all the constant are open. The third definition of fuzzy topology is due to Hutton [37]. He has defined the so-called 'pointless fuzzy topological spaces' which depend purely upon the lattice structure of the collection of fuzzy sets. In [79], Sarma and Ajmal presented another approach for fuzzy topology which is based on fuzzy nets. In [79, 80, 81] they pointed out that their category is a subcategory of the Chang category and has some advantages.

As Šostak [84] remarked in 1985, so far in all the definitions of fuzzy topology, fuzzy are only sets but the so-called fuzzy topology is always a crisp subfamily of the fuzzy power set of a non-empty classical set which is closed for finite intersections and any union operations. Moreover, fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of classical topological spaces. Therefore, Šostak [84] defined a new kind of fuzzy topology in 1985.

The first object of Šostak's approach is to consider a fuzzy topology to be a fuzzy subset on the fuzzy power set of an ordinary fuzzy set. The second one is to allow fuzzy sets to be open to some degree, and this degree may range from 1 ('completely open fuzzy sets') to 0 ('completely non-open fuzzy sets'). In [84, 85, 86], Šostak has developed the theory of this new kind of fuzzy topological spaces. With the same objective of Šostak's approach, in 1991 Mingsheng [62] introduced the concept of a bifuzzy topology which was based on fuzzy logic and practically the same as Šostak's definition of fuzzy topology. This new definition of fuzzy topological space was later rephrased and studied by several authors [17, 18, 23, 26, 28, 31, 36, 77, 88]. In 1992, Chattopadhyay, Hazra and Samanta [17] have redefined the same concept independently, called such a space " a gradation of openness ", and investigated some of its properties. Also in the same year Ramadan [77] presented the same definition under the name of 'smooth topological spaces'. We shall call this new kind of fuzzy topological spaces as *" smooth fuzzy topological spaces "* and devote the last chapter of the thesis to the study of such spaces.

Many mathematicians have tried to formulate a reasonable definition of fuzzy point and its membership to a fuzzy set. First attempt to define these notions was made by Wong [103] in 1974. But, as Gottwald [35] has shown, some of the results obtained by Wong were not correct and Wong's definition of fuzzy point excluded the so called crisp points that are classical points.

Thus, his definitions of fuzzy point and membership have turned out to be not a good choice. In 1980, Pu and Liu [75] introduced a definition of fuzzy point (that includes crisp points) and the notion of 'quasi-coincident' (not belonging to the complement, where belonging is taken as $\leq$). This, together with their fuzzy membership relation, has provided a fuzzy extension of the classical membership. Pu and Liu have considered these concepts only in the I-fuzzy settings, where I stands for the closed unit interval [0,1]. Taking fuzziness with respect to a fuzzy lattice L, a completely distributive lattice with an order reversing involution, Wang [93] has introduced special fuzzy points called molecules which take only union-irreducible values in L, and replaced membership with the concept of remote neighbourhood (R-neighbourhood), where the relation involved is $\leq$. In 1991, Warner [97], considering a frame L, defined L-fuzzy points locale-wise by frame homomorphisms to the two-point set, and so corresponding bijectively to prime elements. Membership is defined in terms of $\leq$, just as in Wang's molecular theory.

Another big step in the development of fuzzy topology is the invention of the so-called ***" goodness of extension criterion "*** by Lowen [51], which was the result of his recognition of the special place of the fuzzy topology defined by lower semi-continuous functions. We shall refer to such a fuzzy topology as ***"the induced fuzzy topology "***. This goodness criterion has been used as a guide for the fuzzification of classical concepts. In 1990, Warner [96] generalised this criterion to L-fuzzy topological spaces, where L is a continuous frame.

The notion of compactness is one of the most important concepts in general topology. Therefore, the problem of generalisation of the classical compactness to fuzzy topological spaces has been intensively discussed over almost 30 years. Many papers on fuzzy compactness have been published and various kinds of fuzzy compactness [e.g. 16, 30, 39, 49, 51, 59, 92, 98, 108] have been presented and studied. Among these compactnesses, the fuzzy compactness introduced by Warner and McLean [98] and extended to arbitrary fuzzy sets by Kudri [39, 45] possesses more satisfactory properties than others. Based on this fuzzy compactness, a series of works have been done [39-47, 90, 98-100]. It was proved that a compact Hausdorff fuzzy topological space is an induced space[98]. Good extensions of some weaker and stronger forms of compactness (e.g. almost, near, semi and strong) and some other covering properties (e.g. S-closedness, RS-compactness, paracompactness and local compactness) were introduced and studied by Kudri and Warner. As a result, it is sufficient for an adequate compactness theory in fuzzy topology, being defined a good extension [98], defined on arbitrary fuzzy sets [39], proposed other fuzzy covering properties, and with a general Tychonoff theorem [45]. We therefore adopt this fuzzy compactness in fuzzy topological spaces, that is, our work is mainly based on this compactness theory.

Let us outline briefly the contents of the thesis . For the sake of clarity , we divide the thesis in nine chapters . In the first two chapters we shall try to make the reader familiar with the basic notions , constructions and results in this area . In Chapter I we present minimal amount of information on lattice theory and fuzzy sets needed for reading the main body of the work .

Chapter II is devoted to L-fuzzy topological spaces where L is a fuzzy lattice . We present some known assertions concerning open L-fuzzy sets, bases and various forms of fuzzy continuity . We also give the definitions of some special L-fuzzy topological spaces and introduce a good definition of completely Hausdorffness in L-fuzzy topological spaces . We devote much attention to the ***"induced L-fuzzy topological spaces"*** as well as the ***"completely induced L-fuzzy topological spaces"*** . Some useful results in the induced L-fuzzy topological spaces are presented and the goodness of completely Hausdorffness are obtained . To construct completely induced L-fuzzy topological spaces , we introduce ***" completely Scott continuity "*** as a generalisation of the *"completely lower semi-continuity"* proposed by Bhamuk and Mukherjee [4] . Thus we obtain an L-fuzzy topological space from a given ordinary topological space and study some properties of such spaces . Completely Scott continuous functions play an important role for studying completely induced L-fuzzy topological spaces .

In Chapter III , since our work is based on the L- fuzzy compactness introduced by Warner and McLean [98] and extended to arbitrary L-fuzzy sets by

Kudri [39] , we present the basic properties of this compactness and related concepts which will be used in the forthcoming chapters . We also give different formulations of these covering axioms and characterize them in terms of filter bases .

In Chapters IV and V we propose two more covering properties; feeble compactness and P-closedness , in L-fuzzy topological spaces . We prove that they are good extensions of the corresponding notions in general topology . Different characterizations of these covering axioms are obtained and some of their properties studied .

Chapter VI focuses on the unification theory of covering properties in L-fuzzy topological spaces .

In Chapter VII we study relative compactness in L-fuzzy topological spaces . We propose two good definitions of fuzzy relative compactness as in general topology . We obtain different characterizations of them and discuss some of their properties .

Chapter VIII is reserved for the study of some weaker forms of L-fuzzy compactness ; namely R-compactness , weak compactness and θ-rigidity in L-fuzzy topological spaces .

The last chapter of the thesis is dedicated to the study of compactness in smooth L-fuzzy topological spaces. After presenting some basic definitions in such spaces we, introducing " ***α-Scott continuity*** ", build up " ***goodness of extension*** " criterion for smooth L-fuzzy topological spaces. Then we propose a good definition of compactness, which we call " ***smooth compactness*** ", in smooth L-fuzzy topological spaces. Smooth compactness turns out to be an extension of L-fuzzy compactness defined in [39,98] to smooth L-fuzzy topological spaces. We study some properties of smooth compactness and obtain different characterizations. As an extension of fuzzy Hausdorffness defined in [98], smooth Hausdorffness is introduced in smooth L-fuzzy topological spaces. We also propose good definitions of smooth countable compactness, smooth Lindelöfness and smooth local compactness and study some of their properties.

Throughout the thesis, definitions, propositions and theorems adopted from other authors are attributed. All unattributed definitions and results should be understood to be our own contributions.

CHAPTER I

SOME LATTICE THEORY AND L-FUZZY SET THEORY

In this chapter, we recall some basic definitions and results on lattice theory and L-fuzzy set theory which will be used throughout this work. Our aim is to make this work self-contained and readable.

For the sake of clarity, we divide this chapter into three sections :

The first section is devoted to some essential definitions and related properties in lattice theory .

The second section contains the standard definitions and some properties related to L-fuzzy sets .

The third section is reserved for the notions of L-fuzzy point , net and α-net.

1.1. Some Lattice Theory

Definition 1.1.1 (Birkhoff [9]) :

A **directed set** D is a set with a partial order '$\geq$' such that for each pair a , b of elements of D , there exists an element c of D with $c \geq a$ and $c \geq b$.

Definition 1.1.2 (Birkhoff [9]) :

A **lattice** $L = L(\leq, \wedge, \vee)$ is a set L equipped with a partial order '$\leq$', in which every finite subset has a join and meet , where meet and join are denoted by $\wedge$ and $\vee$ respectively.

Definition 1.1.3 (Birkhoff [9]) :

A **complete lattice** is a lattice in which every set has a join and a meet . We denote the largest element of L , $\vee L$, by **1** and the smallest element of L , $\wedge L$, by **0** . We consider 0 as the join of the empty set and 1 as the meet of the empty set .

Definition 1.1.4 (Johnstone [38]) :

A **locale** or a **frame** L is a complete lattice satisfying the infinite distributive law:

$$a \wedge (\vee S) = \vee \{ a \wedge s : s \in S \}$$

for all $a \in L$ and every $S \subseteq L$.

Definition 1.1.5 (Gierz et al. [32]) :

A lattice L is called **completely distributive** if and only if it is complete and the following condition holds :

$$\bigwedge_{i\in I} (\bigvee_{j\in J(i)} a_{i,j}) = \bigvee_{f\in K} (\bigwedge_{i\in I} a_{i,f(i)}),$$

where $\{ a_{i,j} : i\in I , j\in J(i) \} \subseteq L$ and K is the set of all functions $f : I \to \cup J(i)$ such that for every $i\in J$, $f(i) \in J(i)$.

Definition 1.1.6 (Gierz et al. [32]) :

Let L be a complete lattice . We say that a is **way below** b , in symbols **a«b**, if and only if for any directed subset D of L the relation $b \leq \bigvee D$ always implies the existence of $d\in D$ with $a \leq d$.

Definition 1.1.7 (Gierz et al. [32]) :

A lattice L is called a **continuous lattice** if and only if L is complete and satisfies the following :

whenever $a \nleq b$, then there is a $y\in L$ such that $y \ll a$ and $y \nleq b$.

Proposition 1.1.8 :

Every completely distributive lattice is a continuous lattice .

Proof : See pp:59 Corollary 2.5 in Gierz et al. [32] .

Definition 1.1.9 (Birkhoff [9]) :

An **order reversing involution** on a lattice L is a map $x \to x'$ from L to L satisfying the following conditions :

(i) If $a \leq b$ then $b' \leq a'$

(ii) $(a')' = a$

Definition 1.1.10 (Hutton [37]) :

A **fuzzy lattice** is a completely distributive lattice with an order reversing involution.

Proposition 1.1.11 :

Let $(L, ')$ be a complete lattice with an order reversing involution. Then for any family $\{ a_i : i \in J \} \subseteq L$ we have

(i) $(\bigvee_{i \in J} a_i)' = \bigwedge_{i \in J} a_i'$

(ii) $(\bigwedge_{i \in J} a_i)' = \bigvee_{i \in J} a_i'$

Proof : See Theorem 17.1 in [10].

Definition 1.1.12 (Gierz et al. [32]) :

An element p of a lattice L is called **prime** if and only if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$.

The set of all prime elements of a lattice L will be denoted by **pr(L)** .

Definition 1.1.13 (Gierz et al. [32]) :

An element α of a lattice L is called **coprime or union irreducible** if and only if $\alpha \neq 0$ and whenever $a, b \in L$ with $\alpha \leq a \vee b$ then $\alpha \leq a$ or $\alpha \leq b$.

The set of all coprime elements of a lattice L will be denoted by **M(L)**. It is evident that in a lattice with an order reversing involution, we have $p \in pr(L)$ if and only if $p' \in M(L)$.

Proposition 1.1.14 :

In a completely distributive lattice, every element is a join of coprime elements and therefore every element is a meet of prime elements.

Proof : See pp : 355 proposition 2.17 in Wang [94].

Definition 1.1.15 [94 , 108] :

Let L be a complete lattice, $\alpha \in L$ and $\emptyset \neq B \subset L$. B is called a **minimal set relative to α** if and only if the following conditions hold :

(i) $\vee B = \alpha$

(ii) for each subset A of L satisfying $\vee A \geq \alpha$ for each $b \in B$, there exists $a \in A$ such that $b \leq a$.

Remark 1.1.16 (Wang [94]) :

The union of minimal sets relative to α is a minimal set relative to α. We shall denote the union of all minimal sets relative to α by $\beta(\alpha)$. We shall denote the set $\beta(\alpha) \cap M(L)$ by $\beta^*(\alpha)$.

Proposition 1.1.17 :

Let L be a complete lattice. Then L is a completely distributive lattice if and only if for every $\alpha \in L$, α has a minimal set and hence $\beta(\alpha)$ exists.

Proof : See pp : 354 Theorem 2.11 in Wang [94] .

Proposition 1.1.18 :

Let L be a completely distributive lattice . If $\alpha \in L\backslash\{0\}$, then $\beta^*(\alpha)$ is a minimal set relative to α . Furthermore, if $\alpha \in M(L)$ then $\beta^*(\alpha)$ is a directed set.

Proof : See pp:68 lemma 4.1 in Zhao [108] .

Example 1.1.19 (Wang [94]) :

Let $L = [0, 1]$. Then, $pr(L) = [0, 1)$, $M(L) = (0, 1]$, $\beta(0) = \{ 0 \}$ and for every $\alpha \in (0, 1]$, $\beta(\alpha) = [0, \alpha)$.

Definition 1.1.20 (Gierz et al. [32]) :

Let L be a complete lattice. A subset U of L is called **Scott open** if and only if it is an upper set and is inaccessible by directed joins, i.e. :

(i) if $a \in U$ and $a \leq b$ then $b \in U$,

(ii) If D is a directed subset of L with $\bigvee D \in U$ then there is a $d \in D$ with $d \in U$.

The collection of all Scott open subsets of L is a topology on L and is called **Scott topology** of L .

Proposition 1.1.21 :

If L is a continuous lattice, then the sets of the form $\{ a \in L : e_o \ll a \}$ form a basis for the Scott topology on L .

Proof : See Remark 3.2 pp : 68 and Proposition 1.10 (ii) pp : 104 in [32].

Proposition 1.1.22 :

The Scott topology of a completely distributive lattice L is generated by the sets of the form $\{ t \in L : t \not\leq p \}$, where $p \in pr(L)$.

Proof : See pp : 104 , proposition 2.1 in Warner and McLean [98] .

1.2. L-fuzzy Sets

In the following, let X be a nonempty set and let $L = L(\leq, \vee, \wedge, ')$ be a fuzzy lattice with a smallest element 0 and a largest element 1 ($0 \neq 1$) and with an order reversing involution '. We consider 0 as the join of the empty set and 1 as the meet of the empty set.

Definition 1.2.1 (Goguen [33]):

An **L-fuzzy set** f on X is a function $f: X \to L$. The set of all L-fuzzy sets on X will be denoted by $\mathbf{L^X}$, that is,

$$L^X = \{ f : f : X \to L \text{ is a function } \}$$

The L-fuzzy sets on X defined by $f(x) = 0$ for every $x \in X$ and $g(x) = 1$ for every $x \in X$ will be denoted by $\emptyset$ and $\mathbf{X}$, respectively. We call them the **empty L-fuzzy set** and the **full L-fuzzy set** respectively.

Definition 1.2.2 (Weiss [101]):

A **crisp set** on X is an ordinary subset of X. In particular, its characteristic function from X to L is an L-fuzzy set.

We shall denote the characteristic function of a subset A of X by χ_A.

Remark 1.2.3 (Goguen [33]):

Since L is a fuzzy lattice, L^X is also a fuzzy lattice with the partial ordering

$\mathbf{f \leq g}$ if and only if $f(x) \leq g(x)$ for all $x \in X$, for $f, g \in L^X$

and the operations of meet and join as :

(i) $(f \wedge g)(x) = f(x) \wedge g(x)$ for every $x \in X$

(ii) $(f \vee g)(x) = f(x) \vee g(x)$ for every $x \in X$

(iii) $(\vee_{i \in J} f_i)(x) = \vee_{i \in J} f_i(x)$ for every $x \in X$ and $\{ f_i : i \in J \} \subset L^X$

(vi) $(\wedge_{i \in J} f_i)(x) = \vee_{i \in J} f_i(x)$ for every $x \in X$ and $\{ f_i : i \in J \} \subset L^X$.

The order reversing involution on L^X is the map $\mathbf{f \to f'}$ from L^X to L^X, where $\mathbf{f'}$ is the L-fuzzy set on X defined by $\mathbf{f'(x) = (f(x))'}$ for all $x \in X$. We shall call f' the **complement** of the L-fuzzy set f.

$\mathbf{f \vee g}$ and $\mathbf{f \wedge g}$ are called the **union** of f and g and the **intersection** of f and g. We shall read $\mathbf{f \leq g}$ as **" f is contained in g "** .

Definition 1.2.4 (Weiss [101]) :

Let f be an L-fuzzy set on X . The subset $\{ x \in X : f(x) > 0 \}$ of X is called the **Support of f** and denoted by **Supp f**, that is, Supp f $= \{ x \in X : f(x) > 0 \}$.

Definition 1.2.5 (Chang [16]) :

Let X and Y be nonempty ordinary sets and let $h : X \to Y$ be a function .

(i) For an L-fuzzy set g on X , the **image** of g under h is the L-fuzzy set on Y defined by $\mathbf{h(g)(y) = \vee \{ g(x) : x \in h^{-1}(y) \}}$ for every $y \in Y$.

(ii) For an L-fuzzy set f on Y , the **inverse image** of f under h is the L-fuzzy set on X defined by $\mathbf{h^{-1}(f)(x) = f(h(x))}$ for every $x \in X$.

Proposition 1.2.6 :

Let X and Y be nonempty ordinary sets, let $h : X \to Y$ be a function and let L be a fuzzy lattice. If $\{ g_i : i \in J \} \subset L^X$, $g, g_1, g_2 \in L^X$ and $\{ f_i : i \in K \} \subset L^Y$, $f, f_1, f_2 \in L^Y$ then we have the following well-known results :

(i) $h^{-1} (\vee_{i \in K} f_i) = \vee_{i \in K} h^{-1} (f_i)$ and $h^{-1} (\wedge_{i \in K} f_i) = \wedge_{i \in K} h^{-1} (f_i)$

(ii) $h (\vee_{i \in J} g_i) = \vee_{i \in J} h(g_i)$ and $h (\wedge_{i \in J} g_i) \leq \wedge_{i \in J} h(g_i)$

(iii) If $f_1 \leq f_2$ then $h^{-1}(f_1) \leq h^{-1} (f_2)$

(iv) If $g_1 \leq g_2$ then $h (g_1) \leq h (g_2)$

(v) $h (h^{-1} (f)) \leq f$. If h is surjective then $h (h^{-1} (f)) = f$.

(vi) $h^{-1} (h (g)) \geq g$. If h is injective then $h^{-1} (h (g)) = g$.

(vii) $h^{-1} (f') = (h^{-1} (f))'$

(viii) If h is surjective then $(h(g))' \leq h(g')$

(ix) If h is injective then $h(g') \leq (h(g))'$ and hence if h is bijective then $h(g') = (h(g))'$.

Proof : See [16, 55, 76, 104]

1.3. L-fuzzy Points and Nets

Let L be a fuzzy lattice and X be a nonempty set. Warner [97] has determined the prime elements of the fuzzy lattice L^X of all L-fuzzy sets on X. We have

$$\mathbf{pr}\,(L^X) = \{\ x_p : x \in X \text{ and } p \in \mathbf{pr}(L)\ \}$$

where for each $x \in X$ and each $p \in \mathrm{pr}(L)$, $x_p : X \to L$ is the L-fuzzy set defined by

$$x_p(y) = \begin{cases} p & \text{if } y = x \\ 1 & \text{otherwise} \end{cases}$$

Definition 1.3.1 (Warner [97]) :

These x_p are called the **L-fuzzy points** of X and an L-fuzzy point x_p is said to be a **member** of an L-fuzzy set f (written $x_p \in f$) if and only if $f(x) \nleq p$.

Remark 1.3.2 (Kudri [45]) :

Since $\mathrm{pr}\,(L^X) = \{\ x_p : x \in X \text{ and } p \in \mathrm{pr}(L)\ \}$, the coprime elements of L^X are the L-fuzzy sets $x_\alpha : X \to L$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

where $x \in X$ and $\alpha \in M(L)$. Thus,

$$M(L^X) = \{\ x_\alpha : x \in X \text{ and } \alpha \in M(L)\ \}.$$

As these x_α are identified with the L-fuzzy points x_p of X, we shall refer to them as the L-fuzzy points.

When $x_\alpha \in M(L^X)$, we shall call x and α the support of x_α ($\mathbf{x = Supp\, x_\alpha}$) and the height of x_α ($\boldsymbol{\alpha = h(x_\alpha)}$), respectively.

By Proposition 1.1.14, we have that every L-fuzzy set on X is a meet of L-fuzzy points in pr(L^X) and hence every L-fuzzy set on X is a join of L-fuzzy points in M(L^X).

Definition 1.3.3 (Zhao [108]) :

Let L be a fuzzy lattice, D be a directed set and let X be a nonempty ordinary set. A **net of L-fuzzy points** (for short a **net**) in X is a map $\mathbf{S : D \to M(L^X)}$. For $m \in D$, we shall denote S(m) by S_m or $x^m_{\alpha_m}$ and the net S by $(S_m)_{m \in D}$ or $(x^m_{\alpha_m})_{m \in D}$.

Since S_m is an L-fuzzy point in M(L^X), we shall denote by $\mathbf{Supp\, S_m}$ and $\mathbf{h(S_m)}$ the support and the height of S_m, respectively.

Definition 1.3.4 (Kudri [45]) :

Let $f \in L^X$ and let $S = (S_m)_{m \in D}$ be a net in X. The net S is called a **net contained in f** if and only if $S_m \leq f$ for each $m \in D$, i.e. $\mathbf{h(S_m) \leq f(\, Supp\, S_m)}$ for each $m \in D$.

Definition 1.3.5 (Zhao [108]) :

Let $\alpha \in M(L)$. A net $(S_m)_{m\in D}$ is called an **α-net** if and only if for each $\gamma \in \beta^*(\alpha)$, the net $h(S) = (h(S_m))_{m\in D}$ is eventually greater than γ , i.e. for each $\gamma \in \beta^*(\alpha)$, there is $m_o \in D$ such that $h(S_m) \geq \gamma$ whenever $m \geq m_o$, where $h(S_m)$ is the height of the L-fuzzy point $S_m \in M(L^X)$.

If $h(S_m) = \alpha$ for all $m \in D$, then we shall say that $(S_m)_{m\in D}$ is a **constant α-net**.

CHAPTER II

L-FUZZY TOPOLOGICAL SPACES

Different definitions of fuzzy topology have appeared in the literature since Chang [16] introduced the concept in 1968. In 1976, Lowen [49] has redefined the concept of fuzzy topology in a somewhat different way. Lowen's definition requires that a fuzzy topology should have one more axiom, namely it includes the constant fuzzy sets. In 1985, Šostak [84] has defined a new kind of fuzzy topology which we shall call 'smooth fuzzy topology'. Here we adopt Chang's definition of fuzzy topology and consider Lowen's definition as a special case. Moreover, we shall devote the last chapter of the thesis to smooth fuzzy topological spaces.

This chapter is divided into four sections:

In section 1, we present the basic notions and results of L-fuzzy topology that will be used throughout the thesis.

Section 2 contains some special L-fuzzy topological spaces.

The third section is devoted to induced L-fuzzy topological spaces and some related properties. We prove that complete Hausdorffness in L-fuzzy topological spaces is a good extension.

In the fourth section, we introduce 'completely Scott continuous functions' from a topological space to a fuzzy lattice with its Scott topology and study some of their properties. We prove that the set of completely Scott continuous functions is an L-fuzzy topology that will be called 'completely induced L-fuzzy topology'. We also get a functor between the category of semi-regular topological spaces and the category of L-fuzzy topological spaces.

2.1. Some Basic Definitions and Results

In the following, let X be a nonempty ordinary set and let $L = L(\leq, \vee, \wedge, ')$ be a fuzzy lattice with a smallest element 0 and a largest element 1 ($0 \neq 1$) and with an order reversing involution $'$.

Definition 2.1.1 (Chang [16]) :

An **L-fuzzy topology** on X is a subset τ of L^X satisfying the following properties :

(i) the L-fuzzy sets $\varnothing$ and X belong to τ .

(ii) if $f, g \in \tau$ then $f \wedge g \in \tau$.

(iii) if $\{ f_i : i \in J \} \subseteq \tau$ then $\vee f_i \in \tau$.

The pair (X, τ) is called an **L-fuzzy topological space** (for short **L-fts**) . An L-fuzzy set f in an L-fuzzy topological space (X, τ) is called **open** or **τ -open** if and only if $f \in \tau$. We say that $f \in L^X$ is **closed** or **τ-closed** in the L-fts (X, τ) if and only if $f' \in \tau$. We say that $f \in L^X$ is **clopen** if and only if it is both open and closed .

Definition 2.1.2 (Gantner et al. [30]) :

Let (X, τ) be an L-fts and $A \subset X$. The family $\tau_A = \{ f|_A : f \in \tau \}$ is an L-fuzzy topology on A , where $f|_A$ is the restriction of $f \in L^X$ on A . We say that (A, τ_A) is an **L-fuzzy subspace** of (X, τ) .

Definition 2.1.3 (Wong [103]) :

Let (X, τ) be an L-fts. A subfamily $\mathscr{B}$ of τ is said to be a **base** for τ if and only if for each $f \in \tau$, there is a subfamily ϱ of $\mathscr{B}$ such that $f = \bigvee_{g \in \varrho} g$.

Definition 2.1.4 (Wong [103]) :

Let (X, τ) be an L-fts. A subfamily $\mathscr{S}$ of τ is said to be a **subbase** for τ if and only if the family of all finite intersection of members of $\mathscr{S}$ forms a base for τ.

Lemma 2.1.5 :

Let (X, τ) be an L-fts and $f \in L^X$. Then $f \in \tau$ if and only if for every $x_p \in pr(L^X)$ with $x_p \in f$ (i.e. $f(x) \nleq p$), there is $g \in \tau$ such that $x_p \in g \leq f$.

Proof : See Proposition 3.1.4 in Kudri [45] pp : 43 .

Lemma 2.1.6 :

Let (X, τ) be an L-fts and κ be a nonempty family of L-fuzzy sets. Then an L-fuzzy set f is a union of elements of κ if and only if for all $x_p \in pr(L^X)$ with $x_p \in f$, there is $g \in \kappa$ such that $x_p \in g \leq f$.

Proof : See Lemma 3.2.6 in Kudri [45], pp : 53 .

Definition 2.1.7 (Pu and Liu [75]) :

Let (X, τ) be an L-fts and $f \in L^X$. The **interior of f, int(f) and the closure of f, cl(f),** are defined as follows :

$$\textbf{int}(f) = \bigvee \{ g \in \tau : g \leq f \}$$

$$\textbf{cl}(f) = \bigwedge \{ g \in L^X : g \geq f \text{ and } g' \in \tau \}$$

Proposition 2.1.8 :

Let (X, τ) be an L-fts and $f, g \in L^X$. Then we have the following :

(i) $int(f)$ is the largest open L-fuzzy set contained in f and $int(int(f)) = f$.

(ii) $cl(f)$ is the smallest closed L-fuzzy set containing f and $cl(cl(f)) = f$.

(iii) if $f \leq g$ then $int(f) \leq int(g)$ and $cl(f) \leq cl(g)$.

(iv) $(cl(f))' = int(f')$ and $(int(f))' = cl(f')$.

(v) for a family $(f_i)_{i \in J}$ of L-fuzzy sets we have :

$$\vee_{i \in J} cl(f_i) \leq cl(\vee_{i \in J} f_i) \quad \text{and} \quad \vee_{i \in J} int(f_i) \leq int(\vee_{i \in J} f_i)$$

If J is finite then $\vee_{i \in J} cl(f_i) = cl(\vee_{i \in J} f_i)$.

Proof : See [3 , 75].

Definition 2.1.9 :

Let (X, τ) be an L-fts and let $f \in L^X$. The L-fuzzy set f is called :

(i) pre-open [82] iff $f \leq int(cl(f))$.

(ii) pre-closed [82] iff $cl(int(f)) \leq f$.

(iii) feebly open or α-open [8] iff $f \leq int(cl(int(f)))$.

(iv) feebly closed or α-closed [8] iff $cl(int(cl(f))) \leq f$.

(v) regularly open [3] iff $f = int(cl(f))$.

(vi) regularly closed [3] iff $f = cl(int(f))$.

(vii) semi open [3] iff there exists $g \in \tau$ such that $g \leq f \leq cl(g)$.

(viii) semi closed [3] iff there exists a closed L-fuzzy set g such that $int(g) \leq f \leq g$.

Remark 2.1.10 (Azad [3]) :

(i) The closure of every open L-fuzzy set is regularly closed .

(ii) The interior of every closed L-fuzzy set is regularly open.

(iii) For every L-fuzzy set f, int(cl(f)) is regularly open and cl(int(f)) is regularly closed .

Definition 2.1.11 [82 , 22] :

Let (X, τ) be an L-fts and $f \in L^X$. The **pre-interior of f , pint (f)** , the **pre-closure of f , pcl(f)** , and the **semi-interior of f , sint(f)** , the **semi-closure of f , scl(f)** , are defined as follows :

pint (f) $= \bigvee \{ g \in L^X : g$ is pre-open and $g \leq f \}$

pcl (f) $= \bigwedge \{ h \in L^X : h \geq f$ and h is pre-closed $\}$

sint (f) $= \bigvee \{ g \in L^X : g$ is semi-open and $g \leq f \}$

scl (f) $= \bigwedge \{ h \in L^X : h \geq f$ and h is semi-closed $\}$

From the definitions it follows that $f \leq \mathrm{pcl}(f) \leq \mathrm{cl}(f)$, $\mathrm{int}(f) \leq \mathrm{pint}(f) \leq f$ and $f \leq \mathrm{scl}(f) \leq \mathrm{cl}(f)$, $\mathrm{int}(f) \leq \mathrm{sint}(f) \leq f$ for every $f \in L^X$. It is also easy to see that f is pre-closed (pre-open) iff $\mathrm{pcl}(f) = f$ ($\mathrm{pint}(f) = f$) and f is semi-closed (semi-open) iff $\mathrm{scl}(f) = f$ ($\mathrm{sint}(f) = f$) .

Definition 2.1.12 :

Let (X, τ) be an L-fts and let $f \in L^X$. The L-fuzzy set f is called :

(i) θ-open [2, 66, 74, 107] if and only if for all $x_p \in \mathrm{pr}(L^X)$ with $x_p \in f$ (i.e. $f(x) \not\leq p$) , there is an open L-fuzzy set g such that $\mathrm{cl}(g) \leq f$ and $x_p \in g$.

(ii) θ-closed [2, 66, 74, 107] if and only if for all $x_p \in pr(L^X)$ with $x_p \in f$, there is a closed L-fuzzy set g such that $f \leq int(g)$ and $x_p \in g$.

(iii) δ-open [1, 66, 83, 74, 107] if and only if for all $x_p \in pr(L^X)$ with $x_p \in f$, there is a regularly open L-fuzzy set g such that $g \leq f$ and $x_p \in g$.

(iv) δ-closed [1, 66, 83, 74, 107] if and only if for all $x_p \in pr(L^X)$ with $x_p \in f$, there is a regularly closed L-fuzzy set g such that $f \leq g$ and $x_p \in g$.

From Lemma 2.1.5, Remark 2.1.10 (iii) and the definitions, it is evident that every θ-open L-fuzzy set is δ-open and every δ-open L-fuzzy set is open.

Definition 2.1.13 [1, 2, 66, 83, 74, 107] :

Let (X, τ) be an L-fts and $f \in L^X$. The **θ-interior of f, θ-int f,** the **θ-closure of f, θ-cl(f),** and the **δ-interior of f, δ-int(f),** the **δ-closure of f, δ-cl(f),** are defined as follows :

$$\theta\text{-}\mathbf{int}\,(f) = \bigvee \{ h \in L^X : h \in \tau \text{ and } cl(h) \leq f \}$$

$$\theta\text{-}\mathbf{cl}\,(f) = \bigwedge \{ g \in L^X : g \text{ is closed and } f \leq int(g) \}$$

$$\delta\text{-}\mathbf{int}\,(f) = \bigvee \{ h \in L^X : h \text{ is } \delta\text{-open and } h \leq f \}$$

$$\delta\text{-}\mathbf{cl}\,(f) = \bigwedge \{ g \in L^X : g \text{ is } \delta\text{- closed and } f \leq g \}$$

From the definitions it is easy to see the following :

$f \leq cl(f) \leq \delta\text{-}cl(f) \leq \theta\text{-}cl(f)$ and $\theta\text{-}int(f) \leq \delta\text{-}int(f) \leq int(f) \leq f$.

Proposition 2.1.14 :

Let (X, τ) be an L-fts. For any open L-fuzzy set, we have $cl(f) = \theta\text{-}cl(f) = \delta\text{-}cl(f)$.

Proof : See [66, 74, 107].

Proposition 2.1.15 :

Let (X, τ) be an L-fts and $f \in L^X$.

(i) f is θ-closed (θ-open) if and only if $\theta\text{-cl}(f) = f$ ($\theta\text{-int}(f) = f$).

(ii) f is δ-closed (δ-open) if and only if $\delta\text{-cl}(f) = f$ ($\delta\text{-int}(f) = f$).

Proof : See [1, 2].

Definition 2.1.16 :

Let (X, τ) be an L-fts, let x_α be an L-fuzzy point in $M(L^X)$ and let $S = (S_m)_{m \in D}$ be a net. The L-fuzzy point x_α is called a :

(i) limit point [45] of S (or S converges to x_α) iff for each closed L-fuzzy set f with $f(x) \not\geq \alpha$, there exists $m_o \in D$ such that $S_m \not\leq f$ whenever $m \geq m_o$, i.e., $h(S_m) \not\leq f(\text{Supp } S_m)$ whenever $m \geq m_o$, where $\text{Supp } S_m$ and $h(S_m)$ are the support and the height of S_m, respectively.

(ii) cluster point [45] of S iff for each closed L-fuzzy set f with $f(x) \not\geq \alpha$ and for all $n \in D$ there is $m \in D$ such that $m \geq n$ and $S_m \not\leq f$, i.e., $h(S_m) \not\leq f(\text{Supp}S_m)$.

(iii) feebly cluster point iff for each feebly closed L-fuzzy set f with $f(x) \not\geq \alpha$ and for all $n \in D$ there is $m \in D$ such that $m \geq n$ and $S_m \not\leq f$.

(iv) γ-cluster point iff for each pair f, g of closed L-fuzzy sets with $\text{int}(f) \geq g$, $f(x) \not\geq \alpha$ and for all $n \in D$ there is $m \in D$ such that $m \geq n$ and $S_m \not\leq g$.

(v) β-cluster point iff for each pair f, g of closed L-fuzzy sets with $\text{int}(f) \geq g$, $f(x) \not\geq \alpha$ and for all $n \in D$ there is $m \in D$ such that $m \geq n$ and $S_m \not\leq \text{int}(g)$, i.e., $h(S_m) \not\leq (\text{int}(g))(\text{Supp}S_m)$.

(vi) pre-θ^*-cluster point iff for each pre-closed L-fuzzy set f with $f(x) \not\geq \alpha$ and for all $n \in D$ there is $m \in D$ such that $m \geq n$ and $S_m \not\leq \text{pint}(f)$.

Definition 2.1.17 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces . A function $f : (X,\tau) \to (Y,\tau^*)$ is called :

(i) continuous [16] iff $f^{-1}(g) \in \tau$ for every $g \in \tau^*$.

(ii) pre-continuous [70] iff $f^{-1}(g)$ is pre-open for every $g \in \tau^*$.

(iii) semi-continuous [3] iff $f^{-1}(g)$ is semi-open for every $g \in \tau^*$.

(iv) feebly continuous or α-continuous [91] iff $f^{-1}(g)$ is feebly open for every $g \in \tau^*$.

(v) almost continuous [3] iff $f^{-1}(g) \in \tau$ for every regularly open L-fuzzy set g in (Y,τ^*) , i.e. $f^{-1}(g)$ is closed for every regularly closed L-fuzzy set g in (Y,τ^*).

(vi) weakly continuous [3] iff $f^{-1}(g) \leq \text{int}(f^{-1}(\text{cl}(g)))$ for all $g \in \tau^*$.

(vii) θ-continuous [66] iff $\theta\text{-cl}(f^{-1}(g)) \leq f^{-1}(\theta\text{-cl}(g))$ for all $g \in L^Y$.

(viii) δ-continuous [1] iff $f^{-1}(g)$ is δ-open in (X,τ) for every regularly open L-fuzzy set g in (Y,τ^*).

(ix) open [103] iff $f(g) \in \tau^*$ for every $g \in \tau$.

(x) almost open [69] iff $f(g) \in \tau^*$ for every regularly open L-fuzzy set g in (X,τ).

(xi) irresolute [19] iff $f^{-1}(g)$ is semi-open in (X,τ) for every semi-open L-fuzzy set g in (Y,τ^*).

(xii) pre-irresolute [72] iff $f^{-1}(g)$ is pre-open in (X,τ) for every pre-open open L-fuzzy set g in (Y,τ^*).

(xiii) weakly pre-irresolute [72] iff $f^{-1}(g) \leq \text{pint}(f^{-1}(\text{pcl}(g)))$ for every pre-open L-fuzzy set g in (Y,τ^*) .

(xiv) feebly irresolute or α-irresolute [91] iff $f^{-1}(g)$ is feebly open in (X,τ) for every feebly open L-fuzzy set g in (Y,τ^*).

Definition 2.1.18 (Thakur and Saraf [91]) :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let τ_η be the L-fuzzy topology on X which has the set of all feebly open L-fuzzy sets of (X,τ) as a subbase. A function $f:(X,\tau)\to(Y,\tau^*)$ is called **η-continuous** if and only if $f:(X,\tau_\eta)\to(Y,\tau^*)$ is continuous and $f:(X,\tau)\to(Y,\tau^*)$ is sid to be **η'-continuous** if and only if $f:(X,\tau_\eta)\to(Y,\tau_\eta{}^*)$ is continuous.

Proposition 2.1.19 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be a function.

(i) f is continuous iff $cl(f^{-1}(g)) \leq f^{-1}(cl(g))$ for every $g\in L^X$.

(ii) f is almost continuous iff $cl(f^{-1}(int(g))) \leq f^{-1}(cl(int(g)))$ for every $g\in L^X$.

(iii) f is almost continuous iff $f^{-1}(int(g)) \leq int(f^{-1}(g))$ for every closed L-fuzzy g in (Y,τ^*).

(iv) f is almost open iff $f(int(g)) \leq int(f(g))$ for every semi-closed L-fuzzy set g in (X,τ).

(v) If f is almost continuous and almost open then the inverse image of any regularly open L-fuzzy set in (Y,τ^*) is regularly open L-fuzzy set in (X,τ), the inverse image of any regularly closed L-fuzzy set in (Y,τ^*) is regularly closed L-fuzzy set in (X,τ).

Proof : **(i)** See theorem 1.1 in [76]. **(ii)** See theorem 3.4 in [65].

(iii) See proposition 2.3 in [20]. **(iv)** see lemma 4.1 in [67].

(v) see theorem 3.5 in [65].

Remark 2.1.20 :

(i) Every almost continuous almost open function is δ-continuous but the converse need not be true in general [1].

(ii) Every almost continuous function is θ-continuous but the converse need not be true in general [64].

2.2. Some Special L-fuzzy Topological Spaces

Definition 2.2.1 (Pu and Liu [76]) :

An L-fuzzy topological space (X,τ) is **fully stratified** if and only if each constant L-fuzzy set on X is open .

Definition 2.2.2 (Mashour et al. [57]) :

An L-fuzzy topological space (X, τ) is said to be **extremally disconnected** if and only if $cl(f) \in \tau$ for every $f \in \tau$.

Definition 2.2.3 (Warner and McLean [98]) :

An L-fts (X,τ) is **Hausdorff** if and only if for every $p,q \in pr(L)$ and every pair x , y of distinct elements of X , there exist $f, g \in \tau$ with $f(x) \not\leq p$, $g(y) \not\leq q$ and $(\forall z \in X)$ $f(z) = 0$ or $g(z) = 0$, i.e. there exist $f, g \in \tau$ such that $x_p \in f$, $y_q \in \tau$ and $(\forall z \in X)$ $f(z) = 0$ or $g(z) = 0$.

Recall that a topological space (X,T) is called **completely Hausdorff** (or a **Urysohn space**) if and only if for every distinct points x , y of X , there are open sets U and V such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$ (Steen and Seebach [89]) .

We now define this notion in the L-fuzzy setting as follows :

Definition 2.2.4 :

An L-fts (X,τ) is said to be **completely Hausdorff** (or a Urysohn space) if and only if for every distinct points x, y of X and every $p, q \in pr(L)$, there exist open L-fuzzy sets f and g such that $x_p \in f$, $y_q \in g$ and $(\forall z \in X)$ $cl(f)(z)=0$ or $cl(g)(z) = 0$.

It is obvious that every completely Hausdorff L-fts is Hausdorff.

Definition 2.2.5 (Kudri [39]) **:**

An L-fts (X,τ) is said to be **regular** if and only if for every $p \in pr(L)$, for each $x \in X$ and each closed L-fuzzy set f such that there is $y \in X$ with $y_p \notin f$ and $f(x) = 0$, there exist open L-fuzzy sets u, v such that $x_p \in u$, for every $y_p \notin f'$, $y_p \in v$ and $(\forall z \in X)$ $u(z) = 0$ or $v(z) = 0$.

2.3. Induced L-fuzzy Topological Spaces

It is a well known fact that the family of all lower semi-continuous functions from a given topological space (X,T) to the closed unit interval I = [0, 1] with its usual ordinary topology forms an I-fuzzy topology on X. This special type of I-fuzzy topology was first mentioned by Wong [102] who called it 'semi-continuous fuzzy topology'. Since then other names have appeared in the literature; Lowen [49] called 'topologically generated' which was adopted by Srivastava et al. [87], Warner and McLean [98] and others, Weiss [101] referred to it as ' induced fuzzy topology ' which was also used by Pu and Liu [75], Martin [56] and others. We shall use the term ' induced fuzzy topology '. The induced I-fuzzy topology was used by Lowen [49,51] to establish so called 'goodness criterion ' for fuzzification of classical concepts in general topology. This is considered to be a big step in the development of fuzzy topology.

For a continuous frame L, Warner [96] has proved that the family of all Scott continuous functions from a topological space (X,T) to L with its Scott topology constitute an L-fuzzy topology as a generalisation of the I-fuzzy topology of lower semi-continuous functions from (X,T) to I. Thus, Warner has established 'goodness criterion' for L-fuzzy topological properties.

Definition 2.3.1 (Gierz et al. [32]) :

Let (X,T) be a topological space and let L be fuzzy lattice . A function f from (X,T) to L with its Scott topology is called **Scott continuous (or continuous)** if and only if the inverse image of every Scott open set is open in (X,T) .

By Proposition 1.1.22, $f:(X,T)\to L$ is Scott continuous iff $f^{-1}(\{ t\in L : t\nleq p \})\in T$ for every $p\in pr(L)$.

In the case L=I , we have $f:(X,T)\to L$ is Scott continuous iff $f^{-1}((p, 1])\in T$ for every $p\in pr(I) = [0, 1)$ iff f is **lower semi-continuous** (Bourbaki [11]).

Proposition 2.3.2 :

Let (X,T) be a topological space . The set of all Scott continuous functions from (X,T) to L with its Scott topology forms an L-fuzzy topology on X , which will be denoted by **ω(T)** , i.e.

$$\boldsymbol{\omega(T) = \{ f\in L^X \ ; \ f:(X,T)\to L \text{ is Scott continuous } \}.}$$

When L = I , then ω(T) is the set of lower semi-continuous functions from (X,T) to I (Lowen [51]) .

Proof : See pp : 88 Corollary 3.2 in Warner [96] .

Definition 2.3.3 :

For a given topological space (X,T) , the L-fuzzy topology ω(T) of Scott continuous functions is called the **induced L-fuzzy topology** , the pair (X,ω(T)) is called the **induced L-fuzzy topological space** . An L-fts (X,τ) is an induced

L-fuzzy topological space if and only if there exists an ordinary topological space (X,T) such that $\omega(T) = \tau$.

Remark 2.3.4 (Lowen [51]) :

Let **TOP** and **FT** be respectively the category of topological spaces with continuous functions between them and the category of L-fuzzy topological spaces with continuous functions between them . The map $\omega : \text{TOP} \rightarrow \text{FT}$ defined by $\omega((X,T)) = (X,\omega(T))$ for every $(X,T) \in \text{TOP}$, where $\omega(T)$ is the induced L-fuzzy topology , is a functor between the categories TOP and FT .

Remark 2.3.5 :

Let (X,T) be a topological space. Lowen [49] has called an I-fuzzy topological property P_f a good extension of a topological property P if and only if : (X,T) has P if and only if $(X,\omega(T))$ has P_f , where $\omega(T)$ is the I-fuzzy topology of lower semi-continuous functions .

Warner [96] has extended this definition of goodness to L-fuzzy topological properties , where L is a continuous frame , as follows :

Definition 2.3.6 :

Let (X,T) be a topological space . An L-fuzzy topological property P_f is a **' good extension '** of a topological property P if and only if :

the topological space (X,T) has P if and only if the induced L-fuzzy topological space $(X,\omega(T))$ has P_f, where $\omega(T)$ is the L-fuzzy topology of Scott continuous functions from (X,T) to L with its Scott topology.

For the induced L-fuzzy topological spaces, Warner [97] has provided a base which turned out to be a powerful tool to obtain the goodness theorems as well as some other results in such spaces.

Proposition 2.3.7 :

Let (X,T) be a topological space and let L be a fuzzy lattice. The family $\mathbf{\mathit{6} = \{ Z^{(\delta)} : Z \in T \text{ and } \delta \in L \}}$ (where $Z^{(\delta)}(x) = \delta$ if $x \in Z$, $Z^{(\delta)}(x) = 0$ otherwise) is a base for the induced L-fuzzy topological space $(X,\omega(T))$.

Proof : See the lemma in Warner [97], pp : 342 and proposition 5.4 in Warner [96], pp : 90.

Proposition 2.3.8 :

Let (X,T) be a topological space and let $f \in L^X$, $A \subset X$.

(i) The L-fuzzy set f is open in $(X,\omega(T))$ iff $f^{-1}(\{ t \in L : t \nleq p \}) \in T$ for every $p \in pr(L)$.

(ii) The L-fuzzy set f is closed in $(X,\omega(T))$ iff $f^{-1}(\{ t \in L : t \geq a \}) \in T$ for every $a \in L$.

(iii) A is open in (X,T) iff the characteristic function χ_A is open in $(X,\omega(T))$.

(iv) A is closed in (X,T) iff the characteristic function χ_A is closed in $(X,\omega(T))$.

(v) A is pre- open in (X,T) iff the characteristic function χ_A is pre-open in $(X,\omega(T))$.

Proof: See proposition 3.2.9 and proposition 3.2.10 in Kudri [45], pp : 49, 50 .

Proposition 2.3.9 :

Let (X,T) be a topological space, let f be an L-fuzzy set in the induced L-fts $(X,\omega(T))$ and $p \in pr(L)$. Then we have the following :

(i) $(cl(f))^{-1}(\{t \in L : t \nleq p\}) \subseteq cl(f^{-1}(\{t \in L : t \nleq p\}))$

(ii) $(int(f))^{-1}(\{t \in L : t \nleq p\}) \subseteq int(f^{-1}(\{t \in L : t \nleq p\}))$

Proof : See lemma 3.2.12 in Kudri [45], pp : 50 .

Proposition 2.3.10 :

Let (X,T) be a topological space and $A \subseteq X$. Considering the induced L-fts $(X,\omega(T))$

and $f(x) = \begin{cases} e \in L & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$, we have the following :

$cl(f)(x) = \begin{cases} e \in L & \text{if } x \in cl(A) \\ 0 & \text{otherwise} \end{cases}$ and $int(f)(x) = \begin{cases} e & \text{if } x \in int(A) \\ 0 & \text{otherwise} \end{cases}$.

Proof : See proposition 3.2.13 in Kudri [45], pp : 51 .

Theorem 2.3.11 (The goodness of Hausdorfness) :

Let (X,T) be a topological space. Then (X,T) is Hausdorff if and only if the induced L-fts $(X,\omega(T))$ is Hausdorff.

Proof : See Proposition 3.1 in Warner and McLean [98].

The next theorem shows that the complete Hausdorfness in L-fuzzy topological spaces (Definition 2.2.4) is a good extension of the complete Hausdorfness in general topology.

Theorem 2.3.12 :

Let (X,T) be a topological space. Then (X,T) is completely Hausdorff if and only if the induced L-fts $(X,\omega(T))$ is completely Hausdorff.

Proof :

Necessity: Let $x, y \in X$ ($x \neq y$) and $p, q \in pr(L)$. From the complete Hausdorffness of (X,T), there exist $U,V \in \tau$ such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \varnothing$. Then, $\chi_U, \chi_V \in \omega(T)$ and $\chi_U(x) \nleq p$, $\chi_V(y) \nleq q$. We also have ($\forall z \in X$) $cl(\chi_U)(z) = \chi_{cl(U)}(z) = 0$ or $cl(\chi_V)(z) = \chi_{cl(V)}(z) = 0$ because $cl(U) \cap cl(V) = \varnothing$. Hence, $(X,\omega(T))$ is completely Hausdorff.

Sufficiency: Let $x, y \in X$ ($x \neq y$) and $p, q \in pr(L)$. Since $(X,\omega(T))$ is complete Hausdorff, by Proposition 2.3.7, there exist basic open L-fuzzy sets

$$f(z) = \begin{cases} \gamma \in L, & \text{if } z \in U \in T \\ 0 & \text{otherwise} \end{cases} \qquad g(z) = \begin{cases} \beta \in L, & \text{if } z \in V \in T \\ 0 & \text{otherwise} \end{cases}$$

such that $x_p \in f$, $y_p \in g$ and ($\forall z \in X$) $cl(f)(z) = 0$ or $cl(g)(z) = 0$. Then, $x \in U \in T$

and $y \in V \in T$. In addition, by Proposition 2.3.10, we have $cl(U) \cap cl(V) = \emptyset$. Hence, (X,T) is completely Hausdorff.

Theorem 2.3.13 (The goodness of regularity) :

Let (X,T) be a topological space. Then (X,T) is regular if and only if the induced L-fts $(X, \omega(T))$ is regular.

Proof : See Theorem 3.4.9 in Kudri [45] .

2.4. Completely Induced L-fuzzy Topological Spaces

In this section, we first introduce a new class of functions from a topological space (X,T) to a fuzzy lattice L with its Scott topology, called completely Scott continuous functions as a generalisation of the completely lower semi-continuous functions from (X,T) to the closed unit interval I introduced by Bhamuk and Mukherjee [4]. Then we study some of their properties and characterizations. We prove that the set of all completely Scott continuous functions from (X,T) to L is an L-fuzzy topology on X which is a generalisation of the I-fuzzy topology of the completely lower semi-continuous functions presented in [5]. Thus we obtain an L-fuzzy topological space from a given ordinary topological space which will be called completely induced L-fuzzy topological space. Completely Scott continuous functions turn out to be the natural tool for studying completely induced L-fuzzy topological spaces.

Bhaumik and Mukherjee [4] have stated the concept of 'complete lower semi-continuity' as follows:

Definition 2.4.1 (Bhaumik and Mukherjee [4]) **:**

Let (X,T) be a topological space. A function $f:(X,T) \to I$ is said to be **completely lower semi-continuous at $a \in X$** if and only if for every $\varepsilon > 0$, there is a regular open neighbourhood U of a in (X,T) such that $f(x) > f(a)-\varepsilon$ for every $x \in U$. f is called **completely lower semi-continuous** on X if and only if f is completely lower semi-continuous at every point of X.

This definition can be characterized as follows:

$f:(X,T) \to I$ is completely lower semi-continuous at $a \in X$ iff for every $\alpha \in [0,1)$ with $f(a) > \alpha$, there is a regular open neighbourhood U of a in (X,T) such that $f(x) > \alpha$ for every $x \in U$.

It is evident that every completely lower semi-continuous function is lower semi-continuous.

In [5], Bhaumik and Mukherjee have proved that the set of all completely lower semi- continuous functions from a topological space (X,T) to I is an I-fuzzy topology, called a completely induced I-fuzzy topology. In [5, 6, 7], they studied various properties of completely induced I-fuzzy topological spaces.

Considering a fuzzy lattice L with its Scott topology instead of I, we now introduce complete Scott continuity as a generalisation of complete lower semi-continuity.

Definition 2.4.2 :

Let (X,T) be a topological space and $a \in X$. A function $f:(X,T) \to L$, where L has its Scott topology, is said to be **completely Scott continuous at $a \in X$** if and only if for every $p \in pr(L)$ with $f(a) \nleq p$, there is a regular open neighbourhood U of a in (X,T) such that $f(x) \nleq p$ for every $x \in U$, i.e. $U \subset f^{-1}(\{t \in L : t \nleq p\})$. f is called **completely Scott continuous on X** if and only if f is completely Scott continuous at every point of X.

It is clear that every completely Scott continuous function is Scott continuous. When $L = I$, the definition becomes :

$f : (X,T) \to I$ is completely Scott continuous at $a \in X$ iff for every $p \in pr(I)=[0,1)$ with $f(a) > p$, there is a regular open neighbourhood U of a in (X,T) such that $f(x) > p$ for every $x \in U$, i.e. f is completely lower semi-continuous. That is , complete Scott continuity coincides with complete lower semi-continuity in the case of $L = I$.

Proposition 2.4.3 :

Let (X,T) be a topological space. $f : (X,T) \to L$ is completely Scott continuous if and only if for every $p \in pr(L)$, $f^{1}(\{t \in L : t \nleq p\})$ can be expressed as a union of some regular open sets in (X,T).

Proof :

__Necessity :__ Let $p \in pr(L)$ and $x \in f^{-1}(\{t \in L : t \nleq p\})$. Then $f(x) \nleq p$. Since f is completely Scott continuous at x, there exists a regular open set O_x in (X,T) such that $x \in O_x$ and $O_x \subset f^{-1}(\{t \in L : t \nleq p\})$. Hence, $f^{-1}(\{t \in L : t \nleq p\}) = \bigcup O_x$, O_x is regular open.

__Sufficiency__ : Let $a \in X$ and $p \in pr(L)$ with $f(a) \nleq p$. Then $a \in f^{-1}(\{t \in L : t \nleq p\})$. By the hypothesis, there is a regular open set O in (X,T) such that $a \in O$ and

$O \subset f^{-1}(\{t \in L : t \nleq p\})$. This means that f is completely Scott continuous at $a \in X$.

Recall that in an ordinary topological space (X,T), the family of all regular open sets forms a base for a smaller topology $\mathbf{T_s}$ on X, called the **semi-regularization** of T. (X,T) is said to be **semi-regular** if and only if $T = T_s$ [63].

Corollary 2.4.4 :

Let (X,T) be a topological space and T_s be its semi-regularization topology. Then $f:(X,T) \to L$ is completely Scott continuous if and only if for every $p \in pr(L)$, $f^{-1}(\{t \in L : t \nleq p\}) \in T_s$ if and only if $f:(X,T_s) \to L$ is Scott continuous.

Proof :

This follows from the previous proposition and the definition of semi-regularization topology.

Lemma 2.4.5 :

The characteristic function of every regular open set is completely Scott continuous.

Proof :

Let (X,T) be a topological space, let A be a regular open set in (X,T) and $a \in X$, $p \in pr(L)$ with $\chi_A(a) \nleq p$. Then $a \in A$ and A is regular open neighbourhood of a. We also have $\chi_A(x) \nleq p$ for every $x \in A$. Hence, χ_A is completely Scott continuous at $a \in X$.

Lemma 2.4.6 :

Let (X,T) be a topological space. If $f, g : (X,T) \to L$ are completely Scott continuous functions then $f \wedge g : (X,T) \to L$ is completely Scott continuous as well.

Proof :

Let $a \in X$ and $p \in pr(L)$ with $(f \wedge g)(a) \not\leq p$. Then $f(a) \not\leq p$ and $g(a) \not\leq p$. Since f and g are completely Scott continuous at a, there are regular open neighbourhoods U and V of a such that $f(x) \not\leq p$ for all $x \in U$ and $g(x) \not\leq p$ for all $x \in V$. Let $W = U \cap V$. Then W is a regular open neighbourhood of a. Since p is prime, we have $(f \wedge g)(x) \not\leq p$ for all $x \in W$. Hence, $f \wedge g$ is completely Scott continuous at $a \in X$.

Lemma 2.4.7 :

If $(f_i)_{i \in J}$ is a family of completely Scott continuous functions from a topological space (X,T) to L, then $f = \bigvee_{i \in J} f_i$ is completely Scott continuous as well.

Proof :

Let $a \in X$ and $p \in pr(L)$ with $f(a) = \bigvee_{i \in J} f_i(a) \not\leq p$. Then, there is $i \in J$ such that $f_i(a) \not\leq p$. Since f_i is completely Scott continuous at a, there is regular open neighbourhood U of a such that $f_i(x) \not\leq p$ for all $x \in U$. Hence, $f(x) = \bigvee_{i \in J} f_i(x) \not\leq p$ for all $x \in U$. Thus, f is completely Scott continuous at $a \in X$.

Theorem 2.4.8 :

For a topological space (X,T), the collection

$$\varphi(T) = \{\ f \in L^X;\ f : (X,T) \to L \text{ is completely Scott continuous } \}$$

is an L-fuzzy topology on X.

Proof : This follows immediately from Lemma 2.4.5, 2.4.6, 2.4.7.

When $L = I$, then $\varphi(T)$ is the set of completely lower semi-continuous functions from (X,T) to I (Bhamuk and Mukherjee [5]).

Definition 2.4.9 :

The L-fuzzy topology **$\varphi(T)$** obtained in Theorem 2.4.8 is called **completely induced L-fuzzy topology** and the L-fts **(X, $\varphi(T)$)** is called **completely induced L-fuzzy topological space .**

Remark 2.4.10 :

Since every completely Scott continuous function from a topological space (X,T) to a fuzzy lattice L is Scott continuous, we have $\varphi(T) \subseteq \omega(T)$, where $\omega(T)$ is the induced L-fuzzy topology of Scott continuous functions from (X,T) to L (Definition 2.3.3).

Now we provide a base for the completely induced L-fuzzy topological spaces.

Proposition 2.4.11 :

For a topological space (X,T) and a fuzzy lattice L, the collection

$\Delta = \{ R^{(\delta)} : R$ is regular open in (X,T), $\delta \in L \}$

(where $R^{(\delta)} : X \to L$ is defined by $R^{(\delta)}(x) = \delta$ if $x \in R$ and $R^{(\delta)}(x) = 0$ otherwise)

forms a base for $\varphi(T)$.

Proof :

Let $f \in \varphi(T)$ and $x_p \in pr(L^X)$ with $f(x) \not\leq p$. Then, f is completely Scott continuous and $f(x) \not\leq p$. Since L is a continuous lattice (see Definition 1.1.7), there exist $\eta \in L$ such that $\eta \ll f(x)$ and $\eta \not\leq p$. Take $\delta \in L$ with $\eta \ll \delta \ll f(x)$. Hence, $f(x) \in \{t \in L : \delta \ll t\}$. Since $\{ t \in L : \delta \ll t \}$ is Scott open (see Proposition 1.1.21), by Proposition 1.1.22, there is a $q \in pr(L)$ such that $f(x) \in \{t \in L : t \not\leq q\} \subset \{t \in L : \delta \ll t\}$. Then $f(x) \not\leq q$. From the completely Scott continuity of f, there exists a regular open set R in (X,T) such that $x \in R$ and $f(z) \not\leq q$ for every $z \in R$. Thus, $\delta \ll f(z)$ for every $z \in R$ and hence $\delta \leq f(z)$ for every $z \in R$. Moreover, $\delta \not\leq p$ because $\delta \gg \eta \not\leq p$. So, we have $x_p \in R^{(\delta)}$, $R^{(\delta)} \in \Delta$ and $R^{(\delta)} \leq f$. Thus, by Lemma 2.1.6, f is a union of elements of Δ. Consequently, Δ is a base for $\varphi(T)$.

Theorem 2.4.12 :

Let (X,T) and (Y,T^*) be two topological spaces and T_s, T_s^* be their semi-regularization topologies respectively. Let $f : X \to Y$ be a function. $f : (X,T_s) \to (Y,T_s^*)$ is continuous iff $f : (X,\varphi(T)) \to (Y,\varphi(T^*))$ is continuous, where $\varphi(T)$ and $\varphi(T^*)$ are the completely induced L-fuzzy topologies.

Proof :

Suppose that $f : (X,T_s) \to (Y,T_s^*)$ is continuous. Take $g \in \varphi(T^*)$. We are going to prove that $f^{-1}(g) \in \varphi(T)$, i.e. $f^{-1}(g) : (X,T) \to L$ is completely Scott continuous.

Let $p\in pr(L)$ and $a\in X$ with $f^{-1}(g)(a)\nleq p$. Then, $g(f(a))\nleq p$. Since $g:(Y,T^*)\to L$ is completely Scott continuous at $f(a)\in Y$, there exits a regular open set A in (Y,T^*) such that $f(a)\in A$ and $g(y)\nleq p$ for all $y\in A$. Since A is regular open in (Y,T^*), $A\in T_s^*$ and hence $f^{-1}(A)\in T_s$ because $f:(X,T_s)\to(Y,T_s^*)$ is continuous. Now we have $a\in f^{-1}(A)\in T_s$ which implies that there is a regular open set B in (X,T) such that $a\in B\subset f^{-1}(A)$. Thus, $f^{-1}(g)(x)=g(f(x))\nleq p$ for every $x\in B$. This means that $f^1(g)$ is completely Scott continuous. Consequently, $f:(X,\varphi(T))\to(Y,\varphi(T^*))$ is continuous.

Now suppose that $f:(X,\varphi(T))\to(Y,\varphi(T^*))$ is continuous. Let A be a basic open set in (Y,T_s^*). Then, A is regular open in (Y,T^*) and hence $\chi_A\in\varphi(T^*)$. By the hypothesis, $f^{-1}(\chi_A)=\chi_{f^{-1}(A)}\in\varphi(T)$. We shall show that $f^{-1}(A)\in T_s$. Let $p\in pr(L)$ and $x\in f^{-1}(A)$. Then $\chi_{f^{-1}(A)}(x)\nleq p$. Since $\chi_{f^{-1}(A)}\in\varphi(T)$, there exists a regular open set O_x in (X,T) such that $x\in O_x$ and $O_x\subset\chi^{-1}_{f^{-1}(A)}(\{t\in L:t\nleq p\})=f^{-1}(A)$. Thus, we have that for each $x\in f^{-1}(A)$, there exists a regular open set O_x in (X,T) such that $x\in O_x\subset f^{-1}(A)$. This means that $f^{-1}(A)\in T_s$. Consequently, $f:(X,T_s)\to(Y,T_s^*)$ is continuous.

Corollary 2.4.13 :

Let (X,T) and (Y,T^*) be semi-regular topological spaces, i.e. $T=T_s$ and $T^*=T_s^*$. Then, $f:(X,T)\to(Y,T^*)$ is continuous iff $f:(X,\varphi(T))\to(Y,\varphi(T^*))$ is continuous.

Proof : This follows immediately from the previous theorem.

Remark 2.4.14 :

Let **SRT** and **FT** be respectively the category of semi-regular topological spaces with continuous functions between them and the category of L-fuzzy topological spaces with continuous functions between them . Define φ : SRT $\to$ FT by $\varphi((X,T)) = (X,\varphi(T))$ for every $(X,T)\in SRT$, where $\varphi(T)$ is the completely induced L-fuzzy topology . Corollary 2.4.13 ensures that if $f:(X,T)\to(Y,T^*)$ is a morphism in SRT , then $\varphi(f)=f:\varphi((X,T))=(X,\varphi(T))\to(Y,\varphi(T^*))=\varphi((Y,T^*))$ is a morphism in FT . Thus , we get the functor , φ , from SRT into FT .

Proposition 2.4.15 :

For a topological space (X,T) , we have $\varphi(T)=\varphi(T_s)$, where T_s is the semi-regularization of T .

Proof :

Since $T_s \subseteq T$, we have $\varphi(T_s)\subseteq\varphi(T)$. Now take $f\in\varphi(T)$ and $x_p\in pr(L^X)$ with $x_p\in f$. Then , by the previous proposition , there is an $R^{(\delta)}\in\Delta$ such that $x_p\in R^{(\delta)}\leq f$. Since R is regular open set in (X,T) , $R\in T_s$ and hence $R^{(\delta)}\in\varphi(T_s)$. Thus , by Proposition 2.1.6 , f is a union of some elements of $\varphi(T_s)$ and therefore $f\in\varphi(T_s)$. Hence , $\varphi(T)\subseteq\varphi(T_s)$. Consequently , $\varphi(T)=\varphi(T_s)$.

Corollary 2.4.16 :

For a topological space (X,T) , we have $\varphi(T)=\omega(T_s)$.

Proof : This follows directly from Corollary 2.4.4 .

Corollary 2.4.17 :

Let (X,T) be a topological space. If (X,T) is semi-regular, then $\varphi(T) = \omega(T)$.

Proof : This follows from the definition of semi-regularity and Corollary 2.4.16.

Remark 2.4.18 :

From the previous corollary, we see that the restriction of the functor ω on the category of semi-regular topological space SRT is equal to the functor φ, i.e., $\omega|_{SRT} = \varphi$.

Theorem 2.4.19 :

Let (X,T) be a topological space. (X,T) is Hausdorff if and only if the completely induced L-fts $(X, \varphi(T))$ is Hausdorff.

Proof :

In general topology, we have that (X,T) is Hausdorff iff (X,T_s) is Hausdorff [63]. Then, by the goodness of Hausdorffness and by Corollary 2.4.16, we get (X,T) is Hausdorff iff (X,T_s) is Hausdorff iff $(X,\omega(T_s))$ is Hausdorff iff $(X,\varphi(T))$ is Hausdorff.

Theorem 2.4.20 :

Let (X,T) be a topological space. Then (X,T) is completely Hausdorff if and only if the completely induced L-fts $(X,\varphi(T))$ is fuzzy completely Hausdorff.

Proof:

Necessity : Let x , $y \in X$ ($x \neq y$) and p , $q \in pr(L)$. From the complete Hausdorffness of (X,T) , there exist $U,V \in \tau$ such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$. Let $A = int(cl(U))$ and $B = int(cl(V))$. Then , $\chi_A , \chi_B \in \varphi(T)$ because A and B are regular open sets in (X,T) . We also have $\chi_A(x) \nleq p$, $\chi_B(y) \nleq q$ and ($\forall z \in X$) $cl(\chi_A)(z) = \chi_{cl(A)}(z) = 0$ or $cl(\chi_B)(z) = \chi_{cl(B)}(z) = 0$ because $cl(U) \cap cl(V) = \emptyset$. In fact , $cl(A) = cl(U)$ and $cl(B) = cl(V)$. Hence , $cl(A) \cap cl(B) = \emptyset$ and therefore ($\forall z \in X$) $\chi_{cl(A)}(z) = 0$ or $\chi_{cl(B)}(z) = 0$. Consequently, $(X, \varphi(T))$ is completely Hausdorff .

Sufficiency : Let x , $y \in X$ ($x \neq y$) and p , $q \in pr(L)$. From the complete Hausdorffness of $(X, \varphi(T))$, there exist basic open L-fuzzy sets f, g defined by respectively $f(z) = \gamma$ if $z \in U$, $f(z) = 0$ otherwise and $g(z) = \delta$ if $z \in V$, $g(z) = 0$ otherwise , where U and V are regular open sets in (X,T) and γ , $\delta \in L$, such that $x_p \in f$, $y_p \in g$ and ($\forall z \in X$) $cl(f)(z) = 0$ or $cl(g)(z) = 0$. Hence , we have $x \in U \in T$, $y \in V \in T$ and $cl(U) \cap cl(V) = \emptyset$. Thus , (X,T) is completely Hausdorff.

Recall that a topological space (X,T) is said to be **almost regular** if and only if for each non-empty regular closed subset C of X and each point $x \in X \setminus C$, there exist disjoint open sets U and V such that $x \in U$ and $C \subset V$ [63] .

Theorem 2.4.21 :

Let (X,T) be a topological space . (X,T) is almost regular if and only if the completely induced L-fts $(X, \varphi(T))$ is fuzzy regular .

Proof :

In general topology, we know that (X,T) is almost regular iff its semi-regularization topological space (X,T_s) is regular [63]. Then, by the goodness of regularity and by Corollary 2.4.16, we have that (X,T) is almost regular iff (X,T_s) is regular iff $(X,\omega(T_s))$ is regular iff $(X,\varphi(T))$ is regular.

Corollary 2.4.22 :

Let (X,T) be a topological space. Then (X,T_s) is regular iff the completely induced L-fts $(X,\varphi(T))$ is regular.

Proof : This follows from the goodness of regularity and Corollary 2.4.16.

Corollary 2.4.23 :

Let (X,T) be a semi-regular topological space. Then, (X,T) is regular iff the completely induced L-fts $(X,\varphi(T))$ is regular.

Proof : This follows immediately from the previous corollary.

CHAPTER III

COMPACTNESS IN L-FUZZY TOPOLOGICAL SPACES

Compactness is one of the most important notions in general topology. Hence, the problem of generalization of the classical compactness to fuzzy topological spaces has been intensively discussed over the past 28 years.

The concept of compactness in I-fuzzy topological spaces was first introduced by Chang [16] who simply expressed in fuzzy terms the classical open covering definition of general topology. This compactness turned out to be unsatisfactory as basis for a theory, not least because it is not a good extension of classical compactness (Lowen [49]). Also, Goguen [34] showed that Tychonoff theorem does not hold for infinite products.

In [51], Lowen addressed these problems, producing seven suggested versions of compactness in I-fuzzy topological spaces five of which he proved to be good extensions. Lowen fuzzy compactness (Definition VII in [51]) has given rise to fruitful work, for instance Lowen [50] has proved that a compact Hausdorff fuzzy space is topologically generated (i.e. an induced space). As Warner [100] pointed out, fuzzy compactness in I-fuzzy topological spaces seems to depend on some arithmetic properties of real numbers in the unit interval.

Of more interest are the definitions which depend only on the unit interval's lattice theoretic properties . In particular Lowen's strong compactness (Definition IV in [51]) was based on that of Gantner et al. [30] who produced a compactness theory with respect to a fuzzy lattice L . Different degrees of compactness , called α-compactness , were introduced in [30] and Tychonoff theorem for α-compactness was proved for arbitrary products and some restricted values of α in L .

Originally the fuzzy compactnesses mentioned above were defined only for the whole fuzzy topological space rather than arbitrary fuzzy subsets . Chadwick [14] has defined Lowen fuzzy compactness [51] for arbitrary I-fuzzy subsets and studied some of its properties . Meng [59] has pointed out that , in a work in Chinese , Wang generalized Lowen fuzzy compactness to L-fuzzy topological spaces . In [61] , Meng has obtained some more characterizations for Lowen's fuzzy compactness in L-fuzzy topological spaces .

On the other hand , Wang [92] has introduced a new theory based on fuzzy nets of Pu and Liu [75] in I-fuzzy topological spaces and called it nice compactness, written N-compactness . This compactness is defined for arbitrary I-fuzzy sets and has desirable properties ; namely it is a good extension of the classical compactness and the general Tychonoff theorem holds . However , as Chadwick remarked in [14] , it is possible to have fuzzy sets which are never N-compact, even if the fuzzy topology has only finite numbers of open fuzzy sets . In [108], Zhao has defined N-compactness in L-fuzzy topological spaces. This has the same properties of Wang's theory , as well as being generalisation of it and giving the geometric formulations of N-compactness in terms of remote neighbourhoods . Kudri [45] has proved that the N-compactness in L-fuzzy topological spaces is a good extension . For the relations between the compactnesses in L-fuzzy topological spaces which are good extensions , we refer to Kudri [45] . Kudri has also remarked that Zhao's definition of N-compactness for L-fuzzy topological spaces has the same drawback of Wang's theory.

Meanwhile Warner and McLean [98] have suggested a generalisation of Lowen's strong compactness [51] to L-fuzzy topological spaces . In the same work , they proved that it is a good extension and also that a compact Hausdorff L-fuzzy topological space is an induced L-fts . Kudri [45] has shown that this compactness is implied by Zhao's N-compactness and in a Hausdorff L-fts , these two compactnesses coincide . In [39 , 45] , Kudri has extended this compactness to arbitrary L-fuzzy sets and obtained the satisfactory properties of N-compactness rectifying the drawback mentioned above . Good extensions of weaker and stronger forms of compactness (e.g. almost , nearly , semi-compactness and strong compactness) were introduced and studied by Kudri and Warner in [39, 43 , 47]. They also suggested good extensions of some other covering properties (e.g. S-closedness, RS-compactness, S^*-closedness, paracompactness, local compactness) in [40, 41, 44, 47] . Consequently , this fuzzy compactness is sufficient for an adequate compactness theory in L-fuzzy topology, being a good extension, defined on arbitrary fuzzy sets, given other fuzzy covering axioms, with a general Tychonoff product theorem and a compact Hausdorff L-fts is an induced L-fts. We therefore adopt this fuzzy compactness in L-fuzzy topological spaces.

In this chapter, since our work is based on this fuzzy compactness, we present the basic properties of fuzzy compactness and related concepts which will be used in the forthcoming chapters. We give a different description of these concepts and characterize them in terms of filter bases.

For the sake of clarity, this chapter is divided into three sections:

Section 1 contains the definition of fuzzy compactness and some important properties.

Section 2 consists of the definitions of all the other existing covering axioms introduced by Kudri and Warner and some of their properties which will be needed in the sequel.

The third section is reserved for some more characterizations of fuzzy covering axioms, obtained by us.

3.1. Compactness in L-fuzzy Topological Spaces

In the following L will be a fuzzy lattice unless otherwise stated.

Definition 3.1.1 :

Let (X,τ) be an L-fts and let $g \in L^X$, $r \in L$.

(i) A collection $\mathcal{G} = (f_i)_{i\in J}$ of L-fuzzy sets is called an **r-level cover** of the L-fuzzy set g if and only if $(\bigvee_{i\in J} f_i)(x) \not\leq r$ for all $x \in X$ with $g(x) \geq r'$, i.e. $(\bigvee_{i\in J} f_i)(x) \not\leq r$ for all $x \in X$ with $x_r \notin g'$.

If for every $i \in J$, f_i is open then $\mathcal{G}$ is called an **r-level open cover of g**.

If g is the whole space X, then $\mathcal{G}$ is called an **r-level cover of X**. Then, $\mathcal{G}$ is an r-level cover of X if and only if $(\bigvee_{i\in J} f_i)(x) \not\leq r$ for all $x \in X$.

(ii) An r-level cover $\mathcal{G} = (f_i)_{i\in J}$ of g is said to have a **finite r-level subcover** if there exists a finite subset F of J such that $(\bigvee_{i\in F} f_i)(x) \not\leq r$ for all $x \in X$ with $g(x) \geq r'$.

(iii) An r-level cover $\mathcal{G} = (f_i)_{i\in J}$ of g is said to have a **finite r-level proximate subcover** if there exists a finite subset F of J such that $(\bigvee_{i\in F} cl(f_i))(x) \not\leq r$ for all $x \in X$ with $g(x) \geq r'$.

Definition 3.1.2 (Warner and McLean [98]) **:**

An L-fts (X,τ) is said to be **compact** if and only if every p-level open cover of X, where $p \in pr(L)$, has a finite p-level subcover, i.e.,

for every prime element p of L and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x \in X$, there is a finite subset F of J such that $(\bigvee_{i\in F} f_i)(x) \not\leq p$ for all $x \in X$.

When $L = I$ this is simply the strong compactness introduced by Lowen [51] since in I all elements apart from 1 are prime.

Kudri has extended this definition to arbitrary L-fuzzy subsets as follows:

Definition 3.1.3 (Kudri [39]) **:**

Let (X,τ) be an L-fts. An L-fuzzy subset $g \in L^X$ is said to be compact if and only if every p-level open cover of g, where $p \in pr(L)$, has a finite p-level subcover, i.e.,

for every prime element p of L and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$ (i.e. $x_p \notin g'$), there is a finite subset F of J such that $(\bigvee_{i\in F} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

It is clear that when $g = X$ Definition 3.1.3 reduces to Definition 3.1.2.

Theorem 3.1.4 (The goodness of compactness) :

Let (X,T) be a topological space. Then (X,T) is compact if and only if the induced L-fts $(X,\omega(T))$ is compact.

Proof : See Proposition 4.4 in [98], pp : 108.

Proposition 3.1.5 :

(i) If (X,τ) is an L-fts where τ is finite, then (X,τ) is compact.

(ii) If (X,τ) is an L-fts where X is finite, then (X,τ) is compact.

Proof: (i) See Proposition 4.1.7 in [45], pp:74 (i) Proposition 4.6 in [98], pp:108.

Theorem 3.1.6 :

A fully stratified compact Hausdorff L-fts is an induced L-fts, i.e., If (X,τ) is a fully stratified compact Hausdorff L-fts, then there exists an ordinary topological space (X,T) such that $\tau = \omega(T)$.

Proof: See Theorem 5.1 in [98], pp : 109.

Theorem 3.1.7 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is compact if and only if every constant α-net contained in g has a cluster point, with height α, contained in g, for each $\alpha \in M(L)$.

Proof: See Theorem 4.4.2 in [45], pp : 92.

Corollary 3.1.8 :

An L-fuzzy topological space is compact if and only if every constant α-net has a cluster point with height α.

Proof: This follows immediately from the previous theorem.

Proposition 3.1.9 :

Let (X,τ) be a Hausdorff L-fts and let $A \subset X$. If χ_A is compact then χ_A is closed.

Proof: See Proposition 4.1.16 in [45], pp : 81.

Proposition 3.1.10 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau) \to (Y,\tau^*)$ be a continuous function such that $f^{-1}(y)$ is finite for every $y \in Y$. If g is a compact L-fuzzy set in (X,τ), then $f(g)$ is a compact L-fuzzy set in (Y,τ^*).

Proof: See Proposition 4.1.14 in [45], pp : 79 .

Proposition 3.1.11 :

(i) Finite union of compact L-fuzzy sets is compact as well .

(ii) Every closed L-fuzzy set contained in a compact L-fuzzy set is compact, hence every closed L-fuzzy set in a compact L-fts is compact .

Proof: See Propositions 4.1.10, 11 , 12 in [45], pp : 76, 77, 78 .

Theorem 3.1.12 (Alexander's Subbase Theorem) :

Let (X,τ) be an L-fts, $g \in L^X$ and let $\mathcal{S}$ be a subbase for τ. The L-fuzzy set g is compact if and only if every p-level cover consisting of subbasic open L-fuzzy sets has a p-level subcover, where $p \in pr(L)$.

Proof: See Theorem 4.2.1 in [45], pp : 85 .

Theorem 3.1.13 (Tychonoff Product Theorem) :

The L-fuzzy product space (X,τ) of the indexed family $\{ (X_\lambda,\tau_\lambda) \}_{\lambda \in J}$ of L-fuzzy topological spaces is compact if and only if for each $\lambda \in J$ (X_λ,τ_λ) is compact.

Proof: See Theorem 4.2.3 in [45], pp : 88 .

3.2. Other Existing Covering Properties

Let (X,τ) be an L-fts and let $i_p(f) = \{ x \in X : f(x) \nleq p \}$ where $p \in pr(L)$ and $f \in L^X$. Then the collection $\{ i_p(f) : p \in pr(L) \text{ and } f \in \tau \} \cup \{X\}$ is a subbase for some ordinary topology, $\mathbf{i_L(\tau)}$, on X.

Definition 3.2.1 (Meng [60]) :

An L-fts (X,τ) is said to be **ultra compact** if and only if the ordinary topological space $(X, i_L(\tau))$ is compact.

Kudri [45] has proved that ultra compactness is a good extension of the ordinary compactness in general topology.

Definition 3.2.2 (Kudri [42]) :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is said to be

(i) countably compact if and only if every countable p-level open cover of g, where $p \in pr(L)$, has a finite p-level subcover.

(ii) Lindelöf if and only if every p-level open cover of g, where $p \in pr(L)$, has a countable p-level subcover.

Kudri has proved that these concepts are good extensions of the corresponding properties in general topology and studied their properties.

Definition 3.2.3 (Kudri and Warner [40]) :

Let (X,τ) be an L-fts. An L-fuzzy set k is called **very compact** if and only if for some $e\in L$ and $A\subseteq X$, it is of the form ($k(x) = e$ if $x\in A$ and $k(x) = 0$ otherwise) and χ_A is compact in (X,τ).

It is evident that every very compact L-fuzzy set is compact.

Definition 3.2.4 (Kudri and Warner [46]) :

An L-fts (X,τ) is called **locally compact** if and only if for every $x_p\in pr(L^X)$ there exist a very compact L-fuzzy set k and $f\in\tau$ such that $x_p\in f\leq k$.

Kudri and Warner have studied various properties of the local compactness and proved that it is a good extension.

Definition 3.2.5 :

Let (X,τ) be an L-fts and $g\in L^X$. The L-fuzzy set g is called :

(i) almost compact [43] if and only if every p-level open cover of g, where $p\in pr(L)$, has a finite p-level proximate subcover. If g is the whole space X, then the L-fts (X,τ) is called almost compact.

(ii) nearly compact [43] if and only if for every p-level open cover $(f_i)_{i\in J}$ of g, where $p\in pr(L)$, there is a finite subset F of J such that $(\vee_{i\in F} int(cl(f_i)))(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$, i.e. every p-level regularly open cover of g has a finite p-level cover. If g is the whole space X, then the L-fts (X,τ) is called nearly compact.

(iii) semi-compact [47] if and only if every p-level semi-open cover of g, where $p \in pr(L)$, has a finite p-level subcover. If g is the whole space X, then the L-fts (X,τ) is called semi compact.

(iv) S-closed [44] if and only if every p-level semi-open cover of g, where $p \in pr(L)$, has a finite p-level proximate subcover. If g is the whole space X, then the L-fts (X,τ) is called S-closed.

(v) S^*-closed [47] if and only if for every p-level semi-open cover $(f_i)_{i \in J}$ of g, where $p \in pr(L)$, there is a finite subset F of J such that $(\vee_{i \in F} scl(f_i))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. If g is the whole space X, then the L-fts (X,τ) is called S^*-closed.

(vi) RS- compact [45] if and only if for every p-level semi-open cover $(f_i)_{i \in J}$ of g, where $p \in pr(L)$, there is a finite subset F of J such that $(\vee_{i \in F} int(cl(f_i)))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, i.e. for every p-level regularly semi-open cover of g, there is a finite subset F of J such that $(\vee_{i \in F} int(f_i))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. If g is the whole space X, then the L-fts (X,τ) is called RS-compact.

(vii) strong compact [44] if and only if every p-level pre-open cover of g, where $p \in pr(L)$, has a finite p-level subcover. If g is the whole space X, then the L-fts (X,τ) is called pre-compact.

Kudri and Warner have proved that all the above definitions are good extensions of the corresponding notions in general topology. They also obtained several characterizations of these covering properties and studied their properties. For more details about all the concepts presented in this section we refer to [39-47].

Proposition 3.2.6 :

Let (X,T) be a topological space. Then (X,T) is nearly compact if and only if the completely induced L-fts $(X,\varphi(T))$ is compact.

Proof :

From the goodness of compactness (Theorem 3.1.4) and the fact that (X,T) is nearly compact iff its semi-regularization topological space (X,T_s) is compact [63], we have that (X,T) is nearly compact iff the induced L-fts $(X,\omega(T_s))$ is compact. Then, by Corollary 2.4.16, (X,T) is nearly compact iff the completely induced L-fts $(X,\varphi(T))$ is compact.

Corollary 3.2.7 :

Let (X,T) be a semi-regular topological space. Then (X,T) is compact if and only if the completely induced L-fts $(X,\varphi(T))$ is compact.

Proof : This follows immediately from the goodness of compactness and Corollary 2.4.17.

3.3. Some More Results On Existing Covering Properties

In this section we present some more characterizations of the covering properties in L-fuzzy topological spaces.

Lemma 3.3.1 :

Let (X,τ) be an L-fts and let $p \in pr(L)$. Then the family

$\phi_p(\tau) = \{ f_i^{-1}(\{t \in L : t \nleq p\}) : f_i \in \tau \}$ is an ordinary topology on X.

Proof : This is straightforward and therefore omitted.

The next theorem shows that compactness in an L-fuzzy topological space (X,τ) is characterized by compactness in the ordinary topological spaces $(X,\phi_p(\tau))$, where $p \in pr(L)$.

Theorem 3.3.2 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is compact if and only if for every $p \in pr(L)$, $G_p = \{ x \in X : g(x) \geq p' \}$ is compact in the ordinary topological space $(X, \phi_p(\tau))$.

Proof :

<u>Necessity</u> : Let $p \in pr(L)$ and let $(A_i)_{i \in J}$ be an open covering of G_p, where $A_i = f_i^{-1}(\{t \in L : t \nleq p\})$ and $f_i \in \tau$ for each $i \in J$. Then, $G_p \subseteq \bigcup_{i \in J} f_i^{-1}(\{t \in L : t \nleq p\})$, i.e.,

$(\bigvee_{i\in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Due to the compactness of g, there is a finite subset F of J such that $(\bigvee_{i\in F} f_i)(x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$, i.e., for all $x \in G_p$. Hence, $G_p \subseteq \bigcup_{i\in F} A_i$ and so G_p is compact in $(X, \phi_p(\tau))$.

Sufficiency : Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level open cover of g. Then, $(\bigvee_{i\in J} f_i)(x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Then, $G_p \subseteq \bigcup_{i\in J} f_i^{-1}(\{t\in L: t \nleq p\})$ and $f_i^{-1}(\{t\in L: t\nleq p\}) \in \phi_p(\tau)$ for each $i \in J$. By the compactness of G_p in $(X, \phi_p(\tau))$, there is a finite subset F of J such that $G_p \subseteq \bigcup_{i\in F} f_i^{-1}(\{t\in L: t\nleq p\})$ which implies that $(\bigvee_{i\in F} f_i)(x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Hence, g is compact in (X,τ).

Corollary 3.3.3 :

An L-fuzzy topological space (X,τ) is compact if and only if for every $p\in pr(L)$ the ordinary topological spaces $(X,\phi_p(\tau))$ is compact.

Proof : This follows directly from the previous theorem.

The next theorem provides a different description for compactness in L-fuzzy topological spaces.

Theorem 3.3.4 :

Let (X,τ) be an L-fts and $g\in L^X$. The L-fuzzy set g is compact if and only if for every $p\in pr(L)$ and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $(\bigvee_{i\in J} f_i \vee g')(x) \nleq p$ for all $x\in X$, there is a finite subset F of J such that $(\bigvee_{i\in F} f_i \vee g')(x) \nleq p$ for all $x \in X$.

Proof :

__Necessity__ : Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of open L-fuzzy sets with $(\bigvee_{i \in J} f_i \vee g')(x) \not\leq p$ for all $x \in X$. Then, $(\bigvee_{i \in J} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. Since g is compact, there is a finite subset F of J such that $(\bigvee_{i \in F} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Take an arbitrary $x \in X$. If $g'(x) \leq p$ then $g'(x) \vee (\bigvee_{i \in F} f_i)(x) = (\bigvee_{i \in F} f_i \vee g')(x) \not\leq p$ because $(\bigvee_{i \in F} f_i)(x) \not\leq p$.

If $g'(x) \not\leq p$ then we have $g'(x) \vee (\bigvee_{i \in F} f_i)(x) = (\bigvee_{i \in F} f_i \vee g')(x) \not\leq p$.

Thus, we have $(\bigvee_{i \in F} f_i \vee g')(x) \not\leq p$ for all $x \in X$.

__Sufficiency__ : Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a p-level open cover of g. Then, $(\bigvee_{i \in J} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, $(\bigvee_{i \in J} f_i \vee g')(x) \not\leq p$ for all $x \in X$. From the hypothesis, there is a finite subset F of J such that $(\bigvee_{i \in F} f_i \vee g')(x) \not\leq p$ for all $x \in X$. Then, $(\bigvee_{i \in F} f_i)(x) \not\leq p$ for all $x \in X$ with $g'(x) \leq p$. Hence, g is compact.

Similar descriptions are valid for the other covering properties given in Definition3.2.5 . For instance, the L-fuzzy set g is almost compact if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of open L-fuzzy sets with $(\bigvee_{i \in J} f_i \vee g')(x) \not\leq p$ for all $x \in X$, there is a finite subset F of J such that $(\bigvee_{i \in F} cl(f_i) \vee g')(x) \not\leq p$ for all $x \in X$. The proof is similar to the proof of Theorem 3.3.4 .

Now we shall characterize the covering properties in terms of filter bases. Firstly, we present the definitions of filter base and cluster point .

Definition 3.3.5 :

Let $\alpha \in M(L)$ and $g \in L^X$. A collection β of L-fuzzy sets is said to form an **α-level filter base** in the L-fuzzy set g if and only if for any finite subcollection $\{ f_1 , f_2 , \dots , f_n \}$ of β , there exists $x \in X$ with $g(x) \geq \alpha$ such that $(\wedge^n_{i=1} f_i)(x) \geq \alpha$. When g is the whole space X , then β is an α-level filter base if and only if for any finite subcollection $\{ f_1 , f_2 , \dots , f_n \}$ of β , there exists $x \in X$ such that $(\wedge^n_{i=1} f_i)(x) \geq \alpha$. If every member of β is open , then β is called **α-level open filter base** .

Definition 3.3.6 :

Let (X,τ) be an L-fts and let β be an α-level filter base , where $\alpha \in M(L)$. A fuzzy point $x_r \in M(L^X)$ is called a **cluster point of β** if and only if $(\wedge_{f \in \beta} cl(f)) (x) \geq r$.

Theorem 3.3.7 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is compact if and only if every α-level filter base in g , where $\alpha \in M(L)$, has a cluster point x_α , with height α , contained in g .

Proof :

<u>Necessity</u> : Suppose that β is an α-level filter base in g with no cluster point, with height α, contained in g , where $\alpha \in M(L)$. Then , for each $x \in X$ with

$g(x) \geq \alpha$, x_α is not a cluster point of β , i.e. there is $f_x \in \beta$ with $cl(f_x)(x) \ngeq \alpha$. Hence, $(cl(f_x))' (x) \nleq \alpha' = p \in pr(L)$. This means that the collection $((cl(f_x))')_{x \in X \text{ with } g(x) \geq \alpha}$ is a p-level open cover of g . From the compactness of g , there are $cl(f_{x_1}), \ldots, cl(f_{x_n})$ such that $(\vee^n_{i=1} (cl(f_{x_i}))')(x) \nleq p$ for all $x \in X$ with $g(x) \geq p' = \alpha$. Hence , $(\wedge^n_{i=1} cl(f_{x_i})) (x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$ which implies that $(\wedge^n_{i=1} f_{x_i}) (x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$. This is a contradiction.

Sufficiency : Suppose that g is not compact . Then , there is a p-level open cover $\mathcal{B}$ of g with no finite p-level subcover , where $p \in pr(L)$. Hence , for each finite subcollection $\{ h_1 , \ldots , h_n \}$ of $\mathcal{B}$, there exists $x \in X$ with $g(x) \geq p'$ such that $(\vee^n_{i=1} h_i)(x) \leq p$, i.e. $(\wedge^n_{i=1} h'_i)(x) \geq p' = \alpha \in M(L)$. Thus , $\beta = \{ h' : h \in \mathcal{B} \}$ forms an α-level filter base in g . By the hypothesis , β has a cluster point $z_\alpha \in M(L^X)$, with height α , contained in g , i.e. $g(z) \geq \alpha$ and $(\wedge_{h \in \mathcal{B}} cl(h'))(z) = (\wedge_{h \in \mathcal{B}} h')(z) \geq \alpha$. Then , $(\vee_{h \in \mathcal{B}} h')(z) \leq p$ which yields a contradiction . This completes the proof.

Corollary 3.3.8 :

An L-fts (X,τ) is compact if and only if every α-level filter base has a cluster point with height α , where $\alpha \in M(L)$.

Proof : This follows easily from the previous theorem .

Theorem 3.3.9 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is almost compact if and only if every α-level open filter base in g , where $\alpha \in M(L)$, has a cluster point x_α , with height α , contained in g .

Proof : This is similar to the proof of Theorem 3.3.7.

Corollary 3.3.10 :

An L-fts (X,τ) is almost compact if and only if every α-level open filter base has a cluster point with height α, where $\alpha \in M(L)$.

Proof : This is an immediate result of the previous theorem.

Definition 3.3.11 :

Let (X,τ) be an L-fts and let β be an α-level filter base, where $\alpha \in M(L)$. A fuzzy point $x_r \in M(L^X)$ is called a **θ-cluster point of β** if and only if $(\wedge_{f \in \beta} \theta\text{-cl}(f))(x) \geq r$.

Theorem 3.3.12 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is almost compact if and only if every α-level filter base in g, where $\alpha \in M(L)$, has a θ-cluster point x_α, with height α, contained in g.

Proof : <u>Necessity</u> : Using the definition of θ-closure (Definition 2.1.13), this is similar to the necessity of Theorem 3.3.7.

<u>Sufficiency</u> : This follows from Theorem 3.3.9 and Proposition 2.1.14.

Corollary 3.3.13 :

An L-fts (X,τ) is almost compact if and only if every α-level filter base has a θ-cluster point with height α, where $\alpha \in M(L)$.

Proof : This is an immediate consequence of the previous theorem.

Proposition 3.3.14 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is almost compact if and only if for every $\alpha \in M(L)$ and every collection $(f_i)_{i\in J}$ of L-fuzzy sets with $(\wedge_{i\in J}\theta\text{-}cl(f_i))(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J such that $(\wedge_{i\in F} f_i)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : This follows easily from Definitions 3.3.5, 3.3.11 and Theorem 3.3.12.

Remark 3.3.15 :

Kudri and Warner [43] have proved that the almost continuous image of every almost compact L-fuzzy set is almost compact. With Remark 2.1.20 (ii) and the next proposition we improve this result.

Proposition 3.3.16 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau) \to (Y,\tau^*)$ be a θ-continuous function (Definition 2.1.17 (vii)) such that $f^{-1}(y)$ is finite for every $y \in Y$. If g is an almost compact L-fuzzy set in (X,τ), then f(g) is almost compact L-fuzzy set in (Y,τ^*).

Proof :

Let $\alpha \in M(L)$ and let $\beta = (g_i)_{i\in J}$ be an α-level filter base in f(g). Then, $\beta^* = (f^{-1}(g_i))_{i\in J}$ is an α-level filter base in g. In fact, take a finite subcollection $\{f^{-1}(g_1), \dots, f^{-1}(g_n)\}$ of β^*. Since β is an α-level filter base in f(g), there exists $y \in Y$ with $f(g)(y) \geq \alpha$ such that $(\wedge^n_{i=1} g_i)(y) \geq \alpha$. Since $f(g)(y) \geq \alpha$ and $f^{-1}(y)$ is finite, there exists $x \in X$ with $f(x) = y$ and $g(x) \geq \alpha$. Then,

$(\wedge^n_{i=1} f^{-1}(g_i))(x) = (\wedge^n_{i=1} g_i)(f(x)) \geq \alpha$.

From the almost compactness of g, by Theorem 3.3.12, β^* has a θ-cluster point z_α, with height α, contained in g.

Claim : u_α is a θ-cluster point of β contained in f(g), where $u = f(z)$. In fact,

$$\alpha \leq (\wedge_{i\in J} \theta\text{-cl}(f^{-1}(g_i)))(z) \quad (\text{because } z_\alpha \text{ is } \theta\text{-cluster point of } \beta^*)$$

$$\leq (\wedge_{i\in J} f^{-1}(\theta\text{-cl}(g_i)))(z) \quad (\text{due to the } \theta\text{-continuity of } f)$$

$$\leq (\wedge_{i\in J} \theta\text{-cl}(g_i))(f(z)) = (\wedge_{i\in J} \theta\text{-cl}(g_i))(u) .$$

Moreover, $f(g)(u) \geq \alpha$ because $g(z) \geq \alpha$ and $u = f(z)$. Hence, by Theorem 3.3.12, f(g) is almost compact.

Corollary 3.3.17 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f: (X,\tau) \to (Y,\tau^*)$ be a θ-continuous surjection. If (X,τ) is almost compact, then so is (Y,τ^*).

Proof : This follows immediately from the previous proposition.

The next proposition shows that near compactness in L-fuzzy topological spaces is characterized by δ-open L-fuzzy sets (Definition 2.1.12 (iii)).

Proposition 3.3.18 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is nearly compact if and only if every p-level δ-open cover of g has a finite α-level subcover, where $p \in pr(L)$.

Proof :

Necessity : Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level δ-open cover of g. Then, for every $x \in X$ with $g(x) \geq p'$, there is $i \in J$ such that $f_i(x) \nleq p$. Since f_i is δ-open, by Definition 2.1.12 (iii), there is a regularly open L-fuzzy set h_i such that $h_i \leq f_i$ and $h_i(x) \nleq p$. Hence, $(h_i)_{i\in J}$ is a p-level regularly open cover of g. Since g is nearly compact, there is a finite subset F of J such that $(\vee_{i\in F} h_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Thus, $(\vee_{i\in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$ because for each $i \in J$, $h_i \leq f_i$.

Sufficiency : Since every regularly open L-fuzzy set is δ-open, this is obvious.

Proposition 3.3.19 :

Let (X,τ) be an L-fts and let $g,h \in L^X$. If g is nearly compact and h is δ-closed then $g \wedge h$ is nearly compact as well.

Proof :

Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level δ-open cover of $g \wedge h$. Then, $(f_i)_{i\in J} \cup \{h'\}$ is p-level δ-open cover of g. Since g is nearly compact, by the previous proposition, there are $f_1, \dots, f_n$ such that $(\vee^n_{i=1} f_i \vee h')(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, $(\vee^n_{i=1} f_i)(x) \nleq p$ for all $x \in X$ with $(g \wedge h)(x) \geq p'$. Thus, by the previous proposition, $g \wedge h$ is nearly compact.

Corollary 3.3.20 :

Let (X,τ) be a nearly compact L-fts. Then each δ-closed L-fuzzy set is nearly compact.

Proof : This follows directly from Proposition 3.3.19.

Remark 3.3.21 :

Kudri and Warner [43] have proved that the almost continuous almost open image of every nearly compact L-fuzzy set is nearly compact. With Remark2.1.20 (i) and the next proposition we improve this result.

Proposition 3.3.22 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be a δ-continuous function (Definition 2.1.17 (viii)) such that $f^{-1}(y)$ is finite for every $y\in Y$. If g is an nearly compact L-fuzzy set in (X,τ), then f(g) is nearly compact L-fuzzy set in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level regularly open cover of g. Then, $(\vee_{i\in J} h_i)(y) \nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. Hence, $(\vee_{i\in J} f^{-1}(h_i))(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$ because $f^{-1}(y)$ is finite for all $y\in Y$. Since f is δ-continuous, for each $i\in J$ $f^{-1}(h_i)$ is δ-open in (X,τ). Thus, $\{f^{-1}(h_i)\}_{i\in J}$ is a p-level δ-open cover of g. Since g is nearly compact, by Proposition 3.3.18, there is finite subset F of J such that $(\vee^{n}_{i=1} f^{-1}(h_i))(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. Hence, $(\vee^{n}_{i=1} h_i)(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. Consequently, f(g) is nearly compact.

Definition 3.3.23 :

Let (X,τ) be an L-fts and let β be an α-level filter base, where $\alpha\in M(L)$. A fuzzy point $x_r\in M(L^X)$ is called a **δ-cluster point of β** if and only if $(\wedge_{f\in\beta}\ \delta\text{-cl}(f))(x)\geq r$.

Theorem 3.3.24 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is nearly compact if and only if every α-level filter base in g, where $\alpha \in M(L)$, has a δ-cluster point x_α, with height α, contained in g.

Proof: By using Proposition 3.3.18, this is similar to the proof of Theorem 3.3.7.

Corollary 3.3.25 :

An L-fts (X,τ) is nearly compact if and only if every α-level filter base has a δ-cluster point with height α, where $\alpha \in M(L)$.

Proof: This is an immediate consequence of the previous theorem.

Proposition 3.3.26 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is nearly compact if and only if for every $\alpha \in M(L)$ and every collection $(f_i)_{i \in J}$ of L-fuzzy sets with $(\wedge_{i \in J} \delta\text{-cl}(f_i))(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J such that $(\wedge_{i \in F} f_i)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof: This follows from Theorem 3.3.24 and Definitions 3.3.5, 3.3.23.

Definition 3.3.27 :

Let (X,τ) be an L-fts and let β be an α-level filter base, where $\alpha \in M(L)$. A fuzzy point $x_r \in M(L^X)$ is called a :

(i) semi-cluster point of β if and only if $(\wedge_{f \in \beta} \text{scl}(f))(x) \geq r$.

(ii) pre-cluster point of β if and only if $(\wedge_{f \in \beta} \text{pcl}(f))(x) \geq r$.

Theorem 3.3.28 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is semi-compact if and only if every α-level filter base in g, where $\alpha \in M(L)$, has a semi-cluster point x_α, with height α, contained in g.

Proof : This is similar to the proof of Theorem 3.3.7 .

Corollary 3.3.29 :

An L-fts (X,τ) is semi-compact if and only if every α-level filter base has a semi-cluster point with height α, where $\alpha \in M(L)$.

Proof : This is an immediate consequence of the previous theorem .

Theorem 3.3.30 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is S^*-closed if and only if every α-level semi-open filter base in g, where $\alpha \in M(L)$, has a semi-cluster point x_α, with height α, contained in g.

Proof : This is similar to the proof of Theorem 3.3.7.

Corollary 3.3.31 :

An L-fts (X,τ) is S^*-closed if and only if every α-level semi-open filter base has a semi-cluster point with height α, where $\alpha \in M(L)$.

Proof : This is an immediate consequence of the previous theorem .

Theorem 3.3.32 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is strong compact if and only if every α-level filter base in g, where $\alpha \in M(L)$, has a pre-cluster point x_α, with height α, contained in g.

Proof : This is similar to the proof of Theorem 3.3.7.

Corollary 3.3.33 :

An L-fts (X,τ) is strong compact if and only if every α-level filter base has a pre-cluster point with height α, where $\alpha \in M(L)$.

Proof : This is an immediate consequence of the previous theorem.

CHAPTER IV

FEEBLE COMPACTNESS IN L-FUZZY TOPOLOGICAL SPACES

In ordinary topology, feebly open (or α-open) sets were introduced and studied by Njastad [71]. A subset A in a topological space (X,T) is called feebly open (or α-open) if and only if $A \subseteq \text{int}(\text{cl}(\text{int}A))$. By using these sets, Maheshwari and Thakur [54] have presented the notion of feeble compactness (or α-compactness) in general topology. A topological space (X,T) is said to be feebly compact if and only if every feebly open cover of X has a finite subcover.

In I-fuzzy topological spaces, feeble compactness has been initiated and studied by Thakur and Saraf [91]. Their definition is based on Chang's compactness [16] which is not a good extension of ordinary compactness.

In this chapter, a good definition of feeble compactness is introduced in L-fuzzy topological spaces. We prove the goodness of the proposed definition, obtain different characterizations and study some of its properties.

This chapter is divided in three sections:

In section 1 we introduce our definition and prove that it is a good extension of the feeble compactness in classical topology.

The second section contains some other characterizations of feeble compactness.

In the third section we study some of its properties.

4.1. Proposed Definition and Its Goodness

Definition 4.1.1 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is said to be **feebly compact** if and only if every p-level feebly open cover of g has a finite p-level subcover, where $p \in pr(L)$.

If g is the whole space, then we say that the L-fts is feebly compact.

Lemma 4.1.2 :

Let (X,T) be a topological space and $A \subseteq X$. Then, A is feebly open in (X,T) if and only if χ_A is feebly open in the induced L-fts $(X,\omega(T))$.

Proof :

A is feebly open in (X,T) iff $A \subseteq int(cl(int(A)))$ iff $\chi_A \leq \chi_{int(cl(int(A)))} = int(cl(int(\chi_A)))$ (the equality is due to Proposition 2.3.10) iff χ_A is feebly open in $(X,\omega(T))$.

Theorem 4.1.3 (The goodness of feeble compactness) :

Let (X,T) be a topological space. Then (X,T) is feebly compact if and only if the induced L-fts $(X,\omega(T))$ is feebly compact.

Proof :

Necessity : Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a p-level feebly open cover of

$(X,\omega(T))$. Then, $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x\in X$. Hence, for each $x\in X$ there is $i\in J$ such that $f_i(x)\not\leq p$, i.e. $x\in f_i^{-1}(\{t\in L : t\not\leq p\})$. So, $X = \bigcup_{i\in J} f_i^{-1}(\{t\in L : t\not\leq p\})$.

We also have that $f_i \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}\, f_i))$ for every $i\in J$ because each f_i is feebly open in $(X,\omega(T))$. Hence, by Proposition 2.3.9, we get

$f_i^{-1}(\{t\in L : t\not\leq p\}) \subseteq (\,\mathrm{int}(\mathrm{cl}(\mathrm{int}\, f_i))\,)^{-1} (\{t\in L : t\not\leq p\}) \subseteq \mathrm{int}\{\mathrm{cl}[\mathrm{int}(f_i^{-1}(\{t\in L : t\not\leq p\}))]\}$

which means that for every $i\in J$, $f_i^{-1}(\{t\in L : t\not\leq p\})$ is feebly open in (X,T). Thus, $\{ f_i^{-1}(\{t\in L : t\not\leq p\}) \}_{i\in J}$ is a feebly open cover of (X,T). Due to the feeble compactness of (X,T), there is a finite subset F of J such that $X = \bigcup_{i\in F} f_i^{-1}(\{t\in L : t\not\leq p\})$, i.e. $(\bigvee_{i\in F} f_i)(x) \not\leq p$ for all $x\in X$. Hence, $(X,\omega(T))$ is feebly compact.

__Sufficiency :__ Let $(A_i)_{i\in J}$ be a feebly open cover of (X,T). Then, by the previous lemma, $(\chi_{A_i})_{i\in J}$ is a family of feebly open L-fuzzy sets in $(X,\omega(T))$ such that $1 = (\bigvee_{i\in J} \chi_{A_i})(x) \not\leq p$ for all $x\in X$ and for all $p\in pr(L)$, i.e. $(\chi_{A_i})_{i\in J}$ is a p-level feebly open cover of $(X,\omega(T))$. Since $(X,\omega(T))$ is feebly compact, there is a finite subset F of J such that $(\bigvee_{i\in F} \chi_{A_i})(x) \not\leq p$ for all $x\in X$. Hence, $(\bigvee_{i\in F} \chi_{A_i})(x) = 1$ for all $x\in X$, i.e. $X = \bigcup_{i\in F} A_i$ and therefore (X,T) is feebly compact.

4.2. Other Characterizations

Theorem 4.2.1 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is feebly compact if and only if for every $\alpha \in M(L)$ and every collection $(h_i)_{i\in J}$ of feebly closed L-fuzzy sets with $(\bigwedge_{i\in J} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there is a finite subset F of J such that $(\bigwedge_{i\in F} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : This follows immediately from Definition 4.1.1.

Theorem 4.2.2 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is feebly compact if and only if every constant α-net contained in g has a feebly cluster point (see Definition 2.1.16 (iii)) $x_\alpha \in M(L^X)$, with height α, contained in g, for each $\alpha \in M(L)$.

Proof : Necessity : Let $\alpha \in M(L)$ and let $(S_m)_{m\in D}$ be a constant α-net contained in g without any feebly cluster point with height α contained in g. Then, for each $x \in X$ with $g(x) \geq \alpha$, x_α is not a feebly cluster point of $(S_m)_{m\in D}$, i.e. there are $n_x \in D$ and a feebly closed L-fuzzy set f_x with $f_x(x) \not\geq \alpha$ and $S_m \leq f_x$ for each $m \geq n_x$.

Let $x^1, \dots, x^k$ be elements of X with $g(x^i) \geq \alpha$ for each $i \in \{1,\dots,k\}$. Then, there are $n_{x_1}, \dots, n_{x_k} \in D$ and feebly closed L-fuzzy sets f_{x_i} with $f_{x_i}(x^i) \not\geq \alpha$ and

$S_m \leq f_{x_i}$ for each $m \geq n_{x_i}$ and for each $i \in \{1,...,k\}$. Since D is a directed set, there is $n_o \in D$ such that $n_o \geq n_{x_i}$ for every $i \in \{1,...,k\}$ and $S_m \leq f_{x_i}$ for $i \in \{1,...,k\}$ and for each $m \geq n_o$.

Now consider the family $\Gamma = (f_x)_{x \in X \text{ with } g(x) \geq \alpha}$.

Then $(\wedge_{f_x \in \Gamma} f_x)(y) \ngeq \alpha$ for all $y \in X$ with $g(y) \geq \alpha$ because $f_y(y) \ngeq \alpha$. We also have that for any finite subfamily $\Lambda = \{ f_{x_1}, \ldots , f_{x_k} \}$ of Γ, there is $y \in X$ with $g(y) \geq \alpha$ and $(\wedge^k_{i=1} f_{x_i})(y) \geq \alpha$ since $S_m \leq \wedge^k_{i=1} f_{x_i}$ for each $m \geq n_o$ because $S_m \leq f_{x_i}$ for each $i \in \{1,...,k\}$ and for each $m \geq n_o$.

Hence, by the previous theorem, g is not feebly compact.

Sufficiency: Suppose that g is not feebly compact. Then, by the previous theorem, there exist $\alpha \in M(L)$ and a collection $\Gamma = (f_i)_{i \in J}$ of feebly closed L-fuzzy sets with $(\wedge_{i \in J} f_i)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, but for any finite subfamily β of Γ there is $x \in X$ with $g(x) \geq \alpha$ and $(\wedge_{i \in F} f_i) (x) \geq \alpha$.

Consider the family of all finite subsets of Γ, $2^{(\Gamma)}$, with the order $\beta_1 \leq \beta_2$ if and only if $\beta_1 \subseteq \beta_2$. Then $2^{(\Gamma)}$ is a directed set.

So, writing x_α as S_β for every $\beta \in 2^{(\Gamma)}$, $(S_\beta)_{\beta \in 2^{(\Gamma)}}$ is a constant α-net contained in g because the height of S_β for all $\beta \in 2^{(\Gamma)}$ is α and $S_\beta \leq g$ for all $\beta \in 2^{(\Gamma)}$, i.e. $g(x) \geq \alpha$.

$(S_\beta)_{\beta \in 2^{(\Gamma)}}$ also satisfies the condition that for each feebly closed L-fuzzy set $f_i \in \beta$ we have $x_\alpha = S_\beta \leq f_i$.

Let $y \in X$ with $g(y) \geq \alpha$. Then, $(\wedge_{i \in J} f_i)(y) \ngeq \alpha$, i.e. there exists $j \in J$ with $f_j(y) \ngeq \alpha$.

Let $\beta_o = \{ f_j \}$. So, for any $\beta \geq \beta_o$, $S_\beta \leq \wedge_{f \in \beta} f_i \leq \wedge_{f \in \beta_o} f_i = f_j$.

Thus, we got a feebly closed L-fuzzy set f_j with $f_j(y) \not\geq \alpha$ and $\beta_o \in 2^{(\Gamma)}$ such that for any $\beta \geq \beta_o$, $S_\beta \leq f_j$, that means $y_\alpha \in M(L^X)$ is not a feebly cluster point of $(S_\beta)_{\beta \in 2^{(\Gamma)}}$ for all $y \in X$ with $g(y) \geq \alpha$. Hence, the constant α-net $(S_\beta)_{\beta \in 2^{(\Gamma)}}$ has no feebly cluster point with height α, contained in g.

Corollary 4.2.3 :

An L-fts (X,τ) is feebly compact if and only if every constant α-net in (X,τ) has a feebly cluster with height α.

Proof : This is an immediate consequence of the previous theorem.

Definition 4.2.4 :

Let (X,τ) be an L-fts and let β be an α-level filter base, where $\alpha \in M(L)$. A fuzzy point $x_r \in M(L^X)$ is called a **feebly cluster point** of β if and only if $(\wedge_{f \in \beta} fcl(f))(x) \geq r$, where $fcl(f) = \wedge \{ g \in L^X : g \geq f$ and g is feebly closed $\}$ [21].

Theorem 4.2.5 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is feebly compact if and only if every α-level filter base in g, where $\alpha \in M(L)$, has a feebly cluster point x_α, with height α, contained in g.

Proof : This is similar to the proof of Theorem 3.3.7.

Corollary 4.2.6 :

An L-fts (X,τ) is feebly compact if and only if every α-level filter base, where $\alpha \in M(L)$, has a feebly cluster point with height α.

Proof : This follows immediately from Theorem 4.2.5 .

Theorem 4.2.7 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is feebly compact if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of feebly open L-fuzzy sets with $(\vee_{i \in J} f_i \vee g')(x) \nleq p$ for all $x \in X$, there is a finite subset F of J such that $(\vee_{i \in F} f_i \vee g')(x) \nleq p$ for all $x \in X$.

Proof : This is similar to the proof of Theorem 3.3.4

4.3. Some Properties

Proposition 4.3.1 :

Let (X,τ) be an L-fts where X is a finite set. Then (X,τ) is feebly compact.

Proof : This follows easily from the definition.

Proposition 4.3.2 :

Let (X,τ) be an L-fts and $g, h \in L^X$. If g and h are feebly compact then $g \vee h$ is feebly compact as well.

Proof :

Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level feebly open cover of $g\vee h$. Then, $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x\in X$ with $(g\vee h)(x) \geq p'$. Since p is prime, we have $(g\vee h)(x)\geq p'$ if and only if $g(x)\geq p'$ or $h(x)\geq p'$. So, by the feebly compactness of g and h, there are finite subsets E, F of J such that $(\vee_{i\in E} f_i)(x) \nleq p$ for all $x\in X$ with $g(x)\geq p'$ and $(\vee_{i\in F} f_i)(x)\nleq p$ for all $x\in X$ with $h(x)\geq p'$. Then, $(\vee_{i\in E\cup F} f_i)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$ or $h(x)\geq p'$, i.e. $(\vee_{i\in E\cup F} f_i)(x)\nleq p$ for all $x\in X$ with $(g\vee h)(x)\geq p'$. Hence, $g\vee h$ is feebly compact.

Proposition 4.3.3 :

Let (X,τ) be an L-fts and $g, h\in L^X$. If g is feebly compact and h is feebly closed, then $g\wedge h$ is feebly compact.

Proof :

Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level feebly open cover of $g \wedge h$. Then, $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. Thus, $\Gamma = (f_i)_{i\in J} \cup \{h'\}$ is a p-level feebly open cover of g, i.e. $(\vee_{k\in\Gamma} k)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. In fact, for each $x \in X$ with $g(x) \geq p'$, if $h(x) \geq p'$ then $(g\wedge h)(x) \geq p'$ which implies that $(\vee_{i\in J} f_i)(x) \nleq p$, thus $(\vee_{k\in\Gamma} k)(x) \nleq p$. If $h(x) \ngeq p'$ then $h'(x) \nleq p$ which implies that $(\vee_{k\in\Gamma} k)(x) \nleq p$. From the feebly compactness of g, there is a finite subfamily Λ of Γ, say $\Lambda = \{ f_1, f_2, \ldots, f_n, h' \}$ with $(\vee_{k\in\Lambda} k)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Then, $(\vee_{i=1}^{n} f_i)(x) \nleq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. In fact, if $(g\wedge h)(x) \geq p'$ then $g(x) \geq p'$ and hence $(\vee_{k\in\Lambda} k)(x) \nleq p$. So, there exists $k \in \Lambda$ such that $k(x) \nleq p$. Moreover, $h(x) \geq p'$ as well, i.e. $h'(x) \leq p$. So, $(\vee_{i=1}^{n} f_i)(x) \nleq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. Hence, $g\wedge h$ is feebly compact.

Corollary 4.3.4 :

Let (X,τ) be an L-fts. If g is a feebly compact L-fuzzy set, then each feebly closed L-fuzzy set contained in g is feebly compact as well.

Proof : This is an immediate consequence of the previous proposition.

Proposition 4.3.5 :

Let (X,τ) be an L-fts and $g \in L^X$.

(i) If g is strong compact (Definition 3.2.5 (vii)) then g is feebly compact.

(ii) If g is feebly compact then g is compact.

Proof :

(i) Since every feebly open L-fuzzy set is pre-open, this directly follows from the definitions.

(ii) Since every open L-fuzzy set is feebly open, this directly follows from the definitions.

Theorem 4.3.6 :

Let (X,τ) be an L-fts, $g \in L^X$ and let τ_η be the L-fuzzy topology defined in Definition 2.1.18. Then g is feebly compact in (X,τ) if and only if g is compact in (X,τ_η).

Proof :

__Necessity :__ Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of subbasic τ_η-open L-fuzzy sets with $(\vee_{i \in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Then, each f_i is feebly open in (X,τ) and so $(f_i)_{i \in J}$ is a p-level feebly open cover of g. By the feeble compactness of g in (X,τ), there exists a finite subset F of J such that $(\vee_{i \in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, by Theorem 3.1.12, g is compact in (X,τ_η).

__Sufficiency :__ Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of feebly open L-fuzzy sets in (X,τ) with $(\vee_{i \in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Since every feebly open L-fuzzy set in (X,τ) is τ_η-open, by the compactness of g in (X,τ_η), there exists a finite subset F of J such that $(\vee_{i \in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, g is feebly compact in (X,τ).

Corollary 4.3.7 :

An L-fts (X,τ) is feebly compact if and only if the L-fts (X,τ_η) is compact.

Proof : Taking g as the whole space, this easily follows from the previous theorem.

Proposition 4.3.8 :

Let (X,τ) be an L-fts. If g is a feebly compact L-fuzzy set in (X,τ), then for each closed L-fuzzy set h in (X,τ_η), $h\wedge g$ is feebly compact in (X,τ).

Proof :

Let g be a feebly compact L-fuzzy set in (X,τ). Then, by Theorem 4.3.6, g is compact in (X,τ_η). Since h is closed in (X,τ_η), by Proposition 3.1.11 (ii), $h\wedge g$ is compact in (X,τ_η). Hence, again by Theorem 4.3.6, $h\wedge g$ is feebly compact in (X,τ).

Proposition 4.3.9 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be a η-continuous mapping (Definition 2.1.18) with $f^1(y)$ is finite for every $y\in Y$. If $g\in L^X$ is feebly compact in (X,τ) then $f(g)$ is compact in (Y,τ^*).

Proof :

If g is feebly compact in (X,τ), by Theorem 4.3.6, g is compact in (X,τ_η). Since $f:(X,\tau)\to(Y,\tau^*)$ is η-continuous, $f:(X,\tau_\eta)\to(Y,\tau^*)$ is continuous. So, by Proposition 3.1.10, $f(g)$ is compact in (Y,τ^*).

Proposition 4.3.10 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be a feebly continuous mapping (Definition 2.1.17. (iv)) with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is feebly compact in (X,τ), then $f(g)$ is compact in (Y,τ^*).

Proof : Since every feebly continuous mapping is η-continuous, this follows from Proposition 4.3.9 .

Proposition 4.3.11 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be a η'-continuous mapping (Definition 2.1.18) with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is feebly compact in (X,τ), then $f(g)$ is feebly compact in (Y,τ^*).

Proof :

If g is feebly compact in (X,τ), then by Theorem 4.3.6, g is compact in (X,τ_η). Since $f:(X,\tau)\to(Y,\tau^*)$ is η'-continuous, we have that $f:(X,\tau_\eta)\to(Y,\tau_\eta^*)$ is continuous. Hence, by Proposition 3.1.10, $f(g)$ is compact in (Y,τ_η^*). So, by Theorem 4.3.6, $f(g)$ is feebly compact in (Y,τ^*).

Corollary 4.3.12 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$

be a η'-continuous surjection. If (X,τ) is feebly compact then (Y,τ^*) is feebly compact as well.

Proof : This follows from Proposition 4.3.11 and Corollary 4.3.7.

Corollary 4.3.13 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau) \to (Y,\tau^*)$ be a feebly irresolute mapping (Definition 2.1.17 (xiv)) with $f^{-1}(y)$ is finite for every $y \in Y$. If $g \in L^X$ is feebly compact in (X,τ), then $f(g)$ is feebly compact in (Y,τ^*).

Proof : Since every feebly irresolute mapping is η'-continuous, this is an immediate consequence of Proposition 4.3.11.

CHAPTER V

P-CLOSEDNESS IN L-FUZZY TOPOLOGICAL SPACES

We say that a topological space (X,T) is P-closed if and only if every pre-open cover of X has a finite subfamily whose pre-closures cover X.

Considering strong compactness introduced by Nanda [70], Zahran [106] has defined and studied P-closedness in I-fuzzy topological spaces.

In this chapter, along the line of strong compactness (Definition 3.2.5 (vii)), we introduce a good definition of P-closedness in L-fuzzy topological spaces. We define P-closedness for arbitrary L-fuzzy sets, prove its goodness, obtain different characterizations of this notion and study some of their properties.

This chapter contains three sections:

In the first section we present our definition and prove its goodness.

In the second section we obtain some other characterizations of our definition.

The third section focuses on some properties.

5.1. Proposed Definition and Its Goodness

Definition 5.1.1 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be **P-closed** if and only if for every p-level open cover $(f_i)_{i \in J}$ of g, where $p \in pr(L)$, there is a finite subset F of J such that $(\vee_{i \in F} pcl(f_i))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, where pcl stands for pre-closure (Definition 2.1.11).

If the L-fuzzy set is the whole space X, then we say that the L-fts (X,τ) is P-closed.

Lemma 5.1.2 :

Let (X,T) be a topological space. If f is a pre-open L-fuzzy set in the induced L-fts $(X,\omega(T))$ then f is pre-continuous (Definition 2.1.17 (ii)) as a function from (X,T) to L with its Scott topology.

Proof :

Let f be a pre-open L-fuzzy set in $(X,\omega(T))$. We shall prove that $f : (X,T) \to L$ is pre-continuous, i.e. the inverse image of every Scott open subset of L is pre-open in (X,T). Since any union of pre-open sets is pre-open, by Proposition 1.1.22, it is sufficient to prove that for every $p \in pr(L)$, $f^{-1}(\{t \in L : t \nleq p\})$ is pre-open in (X,T). Because f is pre-open in $(X,\omega(T))$, we have $f \leq int(cl(f))$ and hence for all $p \in pr(L)$, we get

$f^{1}(\{t\in L: t\nleq p\}) \subseteq (\mathrm{int}(\mathrm{cl}(f)))^{-1}(\{t\in L: t\nleq p\}) \subseteq \mathrm{int}(\mathrm{cl}(f^{-1}(\{t\in L: t\nleq p\})))$,

where the last inclusion is due to Proposition 2.3.9 . Consequently , for all $p\in pr(L)$, $f^{1}(\{t\in L: t\nleq p\})$ is pre-open in (X,T) and therefore $f:(X,T)\to L$ is pre-continuous .

Lemma 5.1.3 :

Let (X,T) be a topological space . Then every pre-open L-fuzzy set in the induced L-fts $(X,\omega(T))$ is a union of elements of the collection $\mathcal{G} = (f_i)_{i\in J}$, where

$$f_i(x) = \begin{cases} e_i \in L & \text{if } x\in A_i \subseteq X \\ & \\ 0 & \text{otherwise} \end{cases} \quad , A_i \text{ is pre-open in } (X,T) .$$

Proof : Let $x_p\in pr(L^X)$ and let g be a pre-open L-fuzzy set in $(X,\omega(T))$ with $x_p\in g$. By Lemma 2.1.6 , it is sufficient to prove that there is $f_i\in\mathcal{G}$ such that $x_p\in f_i\leq g$. Since $x_p\in g$, we have $g(x)\nleq p$. Hence, there is $b\in L$ such that $b \ll g(x)$ and $b\nleq p$ because L is a continuous lattice (Definition 1.1.7) .

Take $e_0\in L$ with $b \ll e_0 \ll g(x)$. Then , $g(x)\in H = \{ q\in L : e_0 \ll q \}$ and by Proposition 1.1.21 , H is Scott open in L . Since g is pre-open in $(X,\omega(T))$, by Lemma 5.1.2 , $g:(X,T)\to L$ is pre-continuous . So, there is a pre-open set A_0 in (X,T) such that $x\in A_0$ and $g(A_0)\subseteq H$. Thus, $g(x)\geq e_0$ for all $x\in A_0$ and $e_0\nleq p$. Hence , $x_p\in f_0$ and $f_0\leq g$, where $f_0(x)=e_0$ if $x\in A_0$ and $f_0(x)=0$ otherwise .

Lemma 5.1.4 :

Let (X,T) be a topological space and $A\subseteq X$. Considering the induced L-fts $(X,\omega(T))$

and $f(x) = \begin{cases} e \in L & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$, we have $pcl(f) = \begin{cases} e & \text{if } x \in pcl(A) \\ 0 & \text{otherwise} \end{cases}$.

Proof :

Let $g(x) = \begin{cases} e & \text{if } x \in pcl(A) \\ 0 & \text{otherwise} \end{cases}$. We shall prove that $pcl(f) = g$.

By Proposition 2.3.10, we have

$$cl(int(g))(x) = \begin{cases} e & \text{if } x \in cl(int(pcl(A))) \\ 0 & \text{otherwise} \end{cases}$$

Since $pcl(A)$ is pre-closed in (X,T), we have $cl(int(pcl(A))) \subseteq pcl(A)$ and hence $cl(int(g)) \leq g$, i.e. g is pre-closed in $(X,\omega(T))$. On the other hand, we have $f \leq g$ and hence $f \leq pcl(f) \leq pcl(g) = g$. Thus, $(pcl(f)) (x) = 0$ for all $x \notin pcl(A)$ and $(pcl(f)) (x) = e$ for all $x \in A$.

From $pcl(f) \leq g$ we get $(pcl(f))^{-1} (\{t \in L : t \neq e\}) \subseteq g^{-1} (\{t \in L : t \neq e\}) = (pcl(A))'$. Hence, $pcl(f)(x) = e$ for all $x \in pcl(A)$ and $pcl(f)(x) = 0$ for all $x \notin pcl(A)$. Consequently, $pcl(f) = g$. This completes the proof.

Theorem 5.1 5 (The goodness of P-closedness) :

Let (X,T) be a topological space. Then (X,T) is P-closed if and only if the induced L-fts $(X,\omega(T))$ is P-closed.

Proof :

<u>**Necessity**</u>: Let $p \in pr(L)$ and $\Gamma = (f_i)_{i \in J}$ be a family of basic pre-open L-fuzzy sets

in $(X,\omega(T))$ with $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x\in X$. Thus, by Lemma 5.1.3, for each $i\in J$

$$f_i(x) = \begin{cases} e_i \in L & \text{if } x\in A_i \subseteq X \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } A_i \text{ is pre-open in } (X,T).$$

Since $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x\in X$, for each $x\in X$ there is $i\in J$ such that $f_i(x) \nleq p$, i.e. $e_i \nleq p$. Let $\Lambda = \{ A_i : \text{there is } i\in J \text{ with } e_i \nleq p \text{ and } f_i \in \Gamma \}$. Then, Λ is a family of pre-open sets in (X,T) covering X. From the P-closedness of (X,T), there is a finite subfamily $\Lambda^* = \{ A_1, A_2, \ldots, A_n\}$ of Λ such that $X = \cup_{i=1}^{n} pcl(A_i)$. Since, by Lemma 5.1.4,

$$pcl(f_i)(x) = \begin{cases} e_i \in L & \text{if } x\in pcl(A_i) \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

we have $(\vee_{i=1} pc(f_i))(x) \nleq p$ for all $x\in X$. Hence, $(X,\omega(T))$ is P-closed.

<u>**Sufficiency**</u>: Let $(A_i)_{i\in J}$ be a pre-open cover of (X,T). Then, by Proposition 2.3.8, $(\chi_{A_i})_{i\in J}$ is a family of pre-open L-fuzzy sets in $(X,\omega(T))$ such that $1 = (\vee_{i\in J}\chi_{A_i})(x) \nleq p$ for all $x\in X$ and for all $p\in pr(L)$. Since $(X,\omega(T))$ is P-closed, there is a finite subset F of J such that $(\vee_{i\in F} pcl(\chi_{A_i}))(x) = (\vee_{i\in F}\chi_{pcl(A_i)})(x) \nleq p$ for all $x\in X$. Hence, $X = \cup_{i\in F} pcl(A_i)$ and thus (X,T) is P-closed.

5.2. Other Characterizations

Theorem 5.2.1 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is P-closed if and only if for every $\alpha \in M(L)$ and every collection $(h_i)_{i\in J}$ of pre-closed L-fuzzy sets with $(\wedge_{i\in J} h_i)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there is a finite subset F of J such that $(\wedge_{i\in F}\, \mathrm{pint}(h_i))(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : This follows immediately from Definition 5.1.1 .

Theorem 5.2.2 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is P-closed if and only if every constant α-net contained in g has a pre-θ^*-cluster point (Definition 2.1.16 (vi)) $x_\alpha \in M(L^X)$, with height α, contained in g, for each $\alpha \in M(L)$.

Proof : This is similar to the proof of Theorem 4.2.2 .

Corollary 5.2.3 :

An L-fts (X,τ) is P-closed if and only if every constant α-net in (X,τ) has a pre-θ^*-cluster with height α .

Proof : This follows immediately from the previous theorem .

Theorem 5.2.4 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is P-closed if and only if every α-level pre-open filter base in g has a pre-cluster point (Definition3.3.27) $x_\alpha \in M(L^X)$, with height α, contained in g.

Proof : This is similar to the proof of Theorem 3.3.7.

Corollary 5.2.5 :

An L-fts (X,τ) is P-closed if and only if every α-level pre-open filter base has a pre-cluster point $x_\alpha \in M(L^X)$ with height α.

Proof : This is an immediate consequence of Theorem 5.2.4.

Theorem 5.2.6 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is P-closed if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i\in J}$ of pre-open L-fuzzy sets with $(\bigvee_{i\in J} f_i \vee g')(x) \nleq p$ for all $x \in X$, there is a finite subset F of J such that $(\bigvee_{i\in F} pcl(f_i) \vee g')(x) \nleq p$ for all $x \in X$.

Proof : This is similar to the proof of Theorem 3.3.4.

5.3. Some Properties

Proposition 5.3.1 :

Let (X,τ) be an L-fts and $g, h \in L^X$. If g and h are P-closed then $g \vee h$ is P-closed as well.

Proof : This is similar to the proof of Proposition 4.3.2 .

Proposition 5.3.2 :

Let (X,τ) be an L-fts and $g, h \in L^X$. If g is P-closed and h is pre-clopen, then $g \wedge h$ is P-closed.

Proof :

Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level pre-open cover of $g \wedge h$. Then, $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. Thus, $\Gamma = (f_i)_{i\in J} \cup \{h'\}$ is a family of pre-open L-fuzzy sets with $(\vee_{k\in\Gamma} k)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. In fact, for each $x \in X$ with $g(x) \geq p'$, if $h(x) \geq p'$ then $(g\wedge h)(x) \geq p'$ which implies that $(\vee_{i\in J} f_i)(x) \nleq p$, thus $(\vee_{k\in\Gamma} k)(x) \nleq p$. If $h(x) \ngeq p'$ then $h'(x) \nleq p$ which implies $(\vee_{k\in\Gamma} k)(x) \nleq p$. From the P-closedness of g, there is a finite subfamily Λ of Γ, say $\Lambda = \{ f_1, f_2, \dots, f_n, h' \}$ with $(\vee_{k\in\Lambda} pcl(k))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Then, $(\vee_{i=1}^{n} pcl(f_i))(x) \nleq p$ for all $x \in X$ with $(g\wedge h)(x) \geq p'$. In fact, if $(g\wedge h)(x) \geq p'$ then $g(x) \geq p'$, hence $(\vee_{k\in\Lambda} pcl(k))(x) \nleq p$. So, there exists $k \in \Lambda$ such that $(pcl(k))(x) \nleq p$. Moreover, $h(x) \geq p'$ as well, i.e. $h'(x) \leq p$. Since h is pre-open, h' is pre-closed, i.e. $h' = pcl(h')$. So, $h'(x) \leq p$ implies that $pcl(h')(x) \leq p'$. Consequently, $(\bigvee_{i=1}^{n} pcl(f_i))(x) \leq p$ for all $x \in X$ with $(g \wedge h)(x) \geq p'$. Hence, $g \wedge h$ is P-closed.

Corollary 5.3.3 :

Let (X,τ) be a P-closed L-fts. Then each pre-clopen L-fuzzy set in (X,τ) is P-closed.

Proof : This is an immediate consequence of the previous proposition.

Proposition 5.3.4 :

Let (X,τ) be an L-fts and $g \in L^X$.

(i) If g is strong compact (Definition 3.2.5 (vii)) then g is P-closed.

(ii) If g is P-closed then g is almost compact (Definition 3.2.5 (i)).

Proof :

(i) Since we have $f \leq pcl(f)$ for every $f \in L^X$, this directly follows from the definitions.

(ii) Since we have $pcl(f) \leq cl(f)$ for every $f \in L^X$ and every open L-fuzzy set is pre-open, this directly follows from the definitions.

Proposition 5.3.5 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and $f : (X,\tau) \to (Y,\tau^*)$ be a pre-irresolute mapping (Definition 2.1.17 (xii)) with $f^{-1}(y)$ is finite for every $y \in Y$. If $g \in L^X$ is P-closed in (X,τ), then $f(g)$ is P-closed in (Y,τ^*) as well.

Proof :

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level pre-open cover of $f(g)$. Then, $(\vee_{i\in J}f_i)(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. Because f is pre-irresolute, $f^{-1}(f_i)$ is pre-open in (X,τ) for every $i\in J$. We also have $(\vee_{i\in J}f^{-1}(f_i))(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. Hence, $(f^{-1}(f_i))_{i\in J}$ is a p-level pre-open cover of g. By the P-closedness of g, there exists a finite subset F of J such that $[\vee_{i\in F}pcl(f^{-1}(f_i))](x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. We are going to show that $(\vee_{i\in F}pcl(f_i))(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

Since $f^{-1}(y)$ is finite for every $y\in Y$, $f(g)(y)\geq p'$ implies that there is $x\in X$ with $g(x)\geq p'$ and $f(x)=y$. Thus, we have that

$(\vee_{i\in F}pcl(f_i))(y) = (\vee_{i\in F}pcl(f_i))(f(x)) = [\vee_{i\in F}f^{-1}(pcl(f_i))](x) = \{\vee_{i\in F}pcl[f^{-1}(pcl(f_i))]\}(x) \geq [\vee_{i\in F}pcl(f^{-1}(f_i))](x)\nleq p$.

The third equality is due to fact that f is pre-irresolute if and only if $f^{-1}(h)$ is pre-closed in (X,τ) for every pre-closed h in (Y,τ^*). Finally, we have that $(\vee_{i\in F}pcl(f_i))(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. Hence, $f(g)$ is P-closed.

Proposition 5.3.6 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and $f:(X,\tau)\to(Y,\tau^*)$ be a weakly pre-irresolute mapping (Definition 2.1.17 (xiii)) with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is strong compact in (X,τ), then $f(g)$ is P-closed in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a p-level pre-open cover of $f(g)$. Then,

$(\vee_{i\in J} f_i)(y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$. Since f is weakly pre-irresolute, we have that for every $i \in J$, $f^{-1}(f_i) \leq \mathrm{pint}[\,f^{-1}(\mathrm{pcl}(f_i))]$. Then, $\{\,\mathrm{pint}\,[\,f^{-1}(\mathrm{pcl}(f_i))\,]\,\}_{i\in J}$ is a family of pre-open L-fuzzy sets in (X,τ) such that $(\,\vee_{i\in J}\,\mathrm{pint}\,[\,f^{-1}(\mathrm{pcl}(f_i))\,]\,)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, i.e. $(\,\mathrm{pint}\,[\,f^{-1}(\mathrm{pcl}(f_i))\,]\,)_{i\in J}$ is a p-level pre-open cover of g. By the strong compactness of g, there is a finite subset F of J such that $(\,\vee_{i\in F}\,\mathrm{pint}\,[\,f^{-1}(\mathrm{pcl}(f_i))\,])(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

We are going to show that $(\,\vee_{i\in F}\,\mathrm{pcl}(f_i)\,)(y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$. Take $y \in Y$ with $f(g)(y) \geq p'$. Because $f^{-1}(y)$ is finite, there exists $x \in X$ with $g(x) \geq p'$ and $f(x) = y$. Then, we have

$(\vee_{i\in F}\mathrm{pcl}(f_i))(y) = (\vee_{i\in F}\mathrm{pcl}(f_i))(f(x)) = [\,\vee_{i\in F} f^{-1}(\mathrm{pcl}(f_i))\,](x) \geq \{\,\vee_{i\in F}\,\mathrm{pint}[f^{-1}(\mathrm{pcl}(f_i))]\,\}(x) \nleq p$.

Hence, $f(g)$ is P-closed.

CHAPTER VI

UNIFICATION THEORY

Mukherjee and Malakar [68] have initiated a unified theory for Chang types of fuzzy covering properties; namely Chang compactness [16], near compactness [27], almost compactness [24] in I-fuzzy topological spaces.

In a similar manner, we attempt to unify several good forms of L-fuzzy covering axioms; namely compactness, strong compactness, feeble compactness, semi-compactness, almost, near compactness and P-closedness etc. studied in the previous chapters.

Taking Ω as the collection of some L-fuzzy sets on a non-empty set X satisfying only the condition $0, 1 \in \Omega$ and with the operator Γ on L^X satisfying $\Gamma(1) = 1$, we obtain the definitions of the concepts mentioned above by different interpretations of the family Ω and the operator Γ in different particular L-fuzzy settings. Some of the known results on the good forms of fuzzy covering axioms are thus obtained as particular cases of the results of the unified theory presented in this chapter.

This chapter is divided into two sections :

In the first section we present some basic definitions to establish the unification theory . We also introduce Ω-compactness of a fuzzy system and obtain its several characterizations .

The second section is reserved for the conclusion .

6.1 Some Definitions and Ω-compactness

Definition 6.1.1 :

Let X be a non-empty set and $\Omega \subseteq L^X$ with $0, 1 \in \Omega$. Let $\Gamma : L^X \to L^X$ be an operator satisfying the property $\Gamma(1) = 1$. Then, the triple (X, Ω, Γ) will be called a **fuzzy system**. The members of Ω will be called **Ω-fuzzy sets**.

Definition 6.1.2 :

Let (X, Ω, Γ) be a fuzzy system and $g \in L^X$. The fuzzy set g is called an **Ω'-fuzzy set** if and only if g' is an Ω-fuzzy set, i.e. $g' \in \Omega$.

Example 6.1.3 :

Let (X, τ) be an L-fuzzy topological space.

(i) Let $\Gamma_1 : L^X \to L^X$ be the identity operator, i.e. $\Gamma_1(f) = f$ for every $f \in L^X$. Then, (X, τ, Γ_1) is a fuzzy system.

(ii) Let $\Gamma_2 : L^X \to L^X$ be the closure operator, i.e. $\Gamma_2(f) = cl(f)$ for every $f \in L^X$. Then, (X, τ, Γ_2) is a fuzzy system as well.

Definition 6.1.4 :

Let (X, Ω, Γ) be a fuzzy system. We define an operator $\Gamma^* : L^X \to L^X$ as follows:
$\Gamma^*(f) = (\Gamma(f'))'$ for every $f \in L^X$.

Definition 6.1.5 :

Let (X, Ω, Γ) be a fuzzy system, let $S = (x^{m}_{\alpha_m})_{m\in D}$ be a net on X and $x_\alpha \in M(L^X)$. The fuzzy point x_α is called an **Ω-cluster point** of S if and only if for every Ω'-fuzzy set f with $f(x) \not\geq \alpha$ and for every $j\in D$, there is $m\in D$ with $m\geq j$ such that $(\Gamma^*(f))(x^m) \not\geq \alpha_m$.

Definition 6.1.6 :

A fuzzy system (X, Ω, Γ) is said to be **Ω-compact** if and only if for every $p\in pr(L)$ and every collection $(f_i)_{i\in J}$ of Ω-fuzzy sets with $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x\in X$, there is a finite subset F of J such that $(\bigvee_{i\in F} \Gamma(f_i))(x) \not\leq p$ for all $x\in X$.

Theorem 6.1.7 :

A fuzzy system (X, Ω, Γ) is Ω-compact if and only if for every $\alpha\in M(L)$ and every collection $(h_i)_{i\in J}$ of Ω'-fuzzy sets with $(\bigwedge_{i\in J} h_i)(x) \not\geq \alpha$ for all $x\in X$, there is a finite subset F of J such that $(\bigwedge_{i\in F}\Gamma^*(h_i))(x) \not\geq \alpha$ for all $x\in X$.

Proof : This follows easily from the definition of Ω-compactness and the definitions of the operators Γ and Γ^*.

Theorem 6.1.8 :

A fuzzy system (X, Ω, Γ) is Ω-compact if and only if for every $\alpha\in M(L)$ and every family $(g_i)_{i\in J}$ of Ω'-fuzzy sets with the property for any finite subset F of J there is $x\in X$ with $(\bigwedge_{i\in F} \Gamma^*(g_i))(x) \geq \alpha$, we have that there exists $z\in X$ with $(\bigwedge_{i\in J} g_i)(z) \geq \alpha$.

Proof :

<u>**Necessity**</u> : Let $\alpha \in M(L)$ and let $(g_i)_{i\in J}$ be a family of Ω'-fuzzy sets satisfying the property given in the theorem. Suppose that $(\wedge_{i\in J} g_i)(x) \not\geq \alpha$ for all $x\in X$. Then, $(\vee_{i\in J} g'_i)(x) \not\leq p$ for all $x\in X$, where $p = \alpha' \in pr(L)$. So, by the Ω-compactness of the system, there is a finite subset F of J such that $(\vee_{i\in F}\Gamma(g'_i))(x) \not\leq p$ for all $x\in X$, i.e. $[\vee_{i\in F}(\Gamma^*(g_i))'](x) \not\leq p$ for all $x\in X$. Hence, $(\wedge_{i\in F}\Gamma^*(g_i))(x) \not\leq p$ for all $x\in X$ which yields a contradiction with the hypothesis.

<u>**Sufficiency**</u> : Suppose that (X, Ω, Γ) is not Ω-compact. Then, there is $p\in pr(L)$ and a family $(f_i)_{i\in J}$ of Ω-fuzzy sets with $(\vee_{i\in J} f_i)(x) \not\leq p$ for all $x\in X$ and for any finite subset F of J, there is $y\in X$ with $(\vee_{i\in F}\Gamma(f_i))(y) \leq p$. Thus, $(f'_i)_{i\in J}$ is a family of Ω'-fuzzy sets such that $(\wedge_{i\in J} f'_i)(x) \not\geq \alpha$ for all $x\in X$, and for any finite subset F of J, there is $y\in X$ with $(\wedge_{i\in F}\Gamma^*(f'_i))(x) = (\wedge_{i\in F}(\Gamma(f_i))')(y) \geq \alpha$, where $\alpha = p' \in M(L)$. So, by the hypothesis, we have that there exists $z\in X$ with $(\wedge_{i\in J} f'_i)(z) \geq \alpha$ which is a contradiction. This completes the proof.

Theorem 6.1.9 :

A fuzzy system (X, Ω, Γ) is Ω-compact if and only if every constant α-net on X has an Ω-cluster point $x_\alpha \in M(L^X)$ with height α, for each $\alpha \in M(L)$.

Proof :

<u>**Necessity**</u> : Let $\alpha \in M(L)$ and let $(S_m)_{m\in D}$ be a constant α-net on X without any Ω-cluster point with height α. Then, for each $x\in X$, x_α is not an Ω-cluster

point of $(S_m)_{m\in D}$, i.e. there are $n_x\in D$ and an Ω'-fuzzy set f_x with $f_x(x)\not\geq \alpha$ and $S_m\leq \Gamma^*(f_x)$ for each $m\geq n_x$.

Let $x^1,\dots,x^k$ be elements of X. Then, there are $n_{x_1},\dots,n_{x_k}\in D$ and Ω'-fuzzy sets f_{x_i} with $f_{x_i}(x^i)\not\geq\alpha$ and $S_m\leq\Gamma^*(f_{x_i})$ for each $m\geq n_{x_i}$ and for each $i\in\{1,\dots,k\}$.

Since D is a directed set, there is $n_0\in D$ such that $n_0\geq n_{x_i}$ for every $i\in\{1,\dots,k\}$ and $S_m\leq\Gamma^*(f_{x_i})$ for $i\in\{1,\dots,k\}$ and for each $m\geq n_0$.

Now consider the family $\Phi=(f_x)_{x\in X}$.

Then $(\bigwedge_{f_x\in\Phi} f_x)(y)\not\geq\alpha$ for all $y\in X$ because $f_y(y)\not\geq\alpha$. We also have that for any finite subfamily $\Psi=\{f_{x_1},\dots,f_{x_k}\}$ of Φ, there is $y\in X$ with $(\bigwedge_{i=1}^k\Gamma^*(f_{x_i}))(y)\geq\alpha$ since $S_m\leq\bigwedge_{i=1}^k\Gamma^*(f_{x_i})$ for each $m\geq n_0$ because $S_m\leq f_{x_i}$ for each $i\in\{1,\dots,k\}$ and for each $m\geq n_0$.

Hence, by Theorem 6.1.7, the fuzzy system (X,Ω,Γ) is not Ω-compact.

Sufficiency : Suppose that (X,Ω,Γ) is not Ω-compact. Then, by Theorem6.1.7, there exist $\alpha\in M(L)$ and a collection $\Phi=(f_i)_{i\in J}$ of Ω'-fuzzy sets with $(\bigwedge_{i\in J}f_i)(x)\not\geq\alpha$ for all $x\in X$, but for any finite subfamily β of Φ, there is $x\in X$ with $(\bigwedge_{f_i\in\beta}f_i)(x)\geq\alpha$.

Consider the family of all finite subsets of Φ, $2^{(\Phi)}$, with the order $\beta_1\leq\beta_2$ if and only if $\beta_1\subseteq\beta_2$. Then $2^{(\Phi)}$ is a directed set.

So, writing x_α as S_β for every $\beta\in 2^{(\Phi)}$, $(S_\beta)_{\beta\in 2^{(\Phi)}}$ is a constant α-net on X because the height of S_β for all $\beta\in 2^{(\Phi)}$ is α.

$(S_\beta)_{\beta\in 2^{(\Phi)}}$ also satisfies the condition that for each Ω'-fuzzy set $f_i\in\beta$ we have $x_\alpha=S_\beta\leq\Gamma^*(f_i)$.

Let $y\in X$. Then, $(\bigwedge_{i\in J}f_i)(y)\not\geq\alpha$, i.e. there exists $j\in J$ with $f_j(y)\not\geq\alpha$.

Let $\beta_o = \{ f_j \}$. So, for any $\beta \geq \beta_o$, $S_\beta \leq \wedge_{f_i \in \beta} (\Gamma^*(f_i)) \leq \wedge_{f \in \beta_o} (\Gamma^*(f_i)) = \Gamma^*(f_j)$ Thus, we got an Ω'-fuzzy set f_j with $f_j(y) \ngeq \alpha$ and $\beta_o \in 2^{(\Phi)}$ such that for any $\beta \geq \beta_o$, $S_\beta \leq \Gamma^*(f_j)$. Hence, $y_\alpha \in M(L^X)$ is not an Ω-cluster point of $(S_\beta)_{\beta \in 2^{(\Phi)}}$ for all $y \in X$. Consequently, the constant α-net $(S_\beta)_{\beta \in 2^{(\Phi)}}$ has no Ω-cluster point with height α.

6.2. Conclusion

From the definitions and the theorems in the previous section we see that different interpretations of the family Ω and the operator Γ in different settings give rise to the definitions and characterizations of different types of L-fuzzy covering axioms presented in the previous chapters . Some such cases are illustrated as follows :

1 - Let Ω be the L-fuzzy topology of an L-fts (X, τ).

(i) If Γ stands for the identity operator, then the Ω-compactness becomes the **compactness** in L-fuzzy topological spaces (Definition 3.1.2 , 3.1.3).

(ii) If Γ is the fuzzy closure operator , then the Ω-compactness becomes the **almost compactness** in L-fuzzy topological spaces (Definition 3.2.5 (i)).

(iii) If Γ is taken to represent the fuzzy interior-closure operator , then the Ω-compactness becomes the **near compactness** in L-fuzzy topological spaces (Definition 3.2.5 (ii)).

2 - Let Ω denote the class of all semi-open L-fuzzy sets of an L-fts (X, τ).

(i) If Γ is the fuzzy closure operator , then the Ω-compactness becomes the **S-closedness** in L-fuzzy topological spaces (Definition 3.2.5 (iv)).

(ii) If Γ is the fuzzy interior-closure operator , then the Ω-compactness becomes the **RS-compactness** in L-fuzzy topological spaces (Definition 3.2.5 (vi)).

(iii) If Γ is the identity operator, then the Ω-compactness coincides with the **semi-compactness** in L-fuzzy topological spaces (Definition 3.2.5 (iii)).

(iv) When Γ is taken to represent the semi-closure operator, the Ω-compactness becomes the **S^*-closedness** in L-fuzzy topological spaces (Definition 3.2.5 (v)).

3 - Let Ω denote the class all pre-open L-fuzzy sets of an L-fts (X, τ).

(i) If Γ is the identity operator, then the Ω-compactness coincides the **strong compactness** in L-fuzzy topological spaces (Definition 3.2.5 (vii)).

(ii) If Γ is taken to represent the pre-closure operator, then the Ω-compactness becomes the **P-closedness** in L-fuzzy topological spaces (Definition 5.1.1).

4 - Let Ω denote the class of all feebly open L-fuzzy sets of an L-fuzzy topological space (X, τ) and let Γ be the identity operator. It then follows that the Ω-compactness turns out to be the **feeble compactness** in L-fuzzy topological spaces (Definition 4.1.1).

CHAPTER VII

RELATIVE COMPACTNESS IN L-FUZZY TOPOLOGICAL SPACES

In general topology, the term " relatively compact " is used in more than one sense. Since we deal with two different senses, we shall use the terms "relatively compact " and " strong relatively compact ". If (X,T) is a topological space and $A \subset X$ then we say that A is relatively compact if and only if every net in A has a cluster point in X [15]. This is also characterized as - A is relatively compact if and only if every open covering of X has a finite subfamily which covers A. We say that A is strongly relatively compact if and only if A is contained in a compact set [11].

In this chapter, we present two kinds of L-fuzzy relative compactness - L-fuzzy relative compactness and L-fuzzy strong relative compactness - as in general topology. We prove that they are good extension of the corresponding formulations in ordinary topology and study some of their properties in the fuzzy setting.

Concerned with one of the definitions of relative compactness in general topology- that one we called here relative compactness- Chadwick [15] proposed a different fuzzyfication of this concept and restricted his work to [0,1]-fuzzy topological spaces.

The structure of this chapter is as follows :

In the first section we present our proposed definitions .

In Section 2 we give different characterizations of the proposed concepts .

Section 3 contains the goodness theorems .

The last section is reserved for some basic properties .

7.1. Proposed Definitions

Definition 7.1.1 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be **relatively compact** if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of open L-fuzzy sets with $(\bigvee_{i \in J} f_i)(x) \nleq p$ for all $x \in X$, there is a finite subset F of J such that $(\bigvee_{i \in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

It is clear that every compact L-fuzzy set is relatively compact.

Definition 7.1.2 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be **strongly relatively compact** if and only if g is contained in a very compact L-fuzzy set (Definition 3.2.3), i.e. there exists a very compact L-fuzzy set k with $g \leq k$.

Obviously, every very compact L-fuzzy set is strongly relatively compact and every strongly relatively compact L-fuzzy set is relatively compact.

7.2. Other Characterizations

Theorem 7.2.1 :

Let (X,τ) be an L-fts and $g\in L^X$. The L-fuzzy set g is relatively compact if and only if for every $\alpha\in M(L)$ and every collection $(h_i)_{i\in J}$ of closed L-fuzzy sets with $(\wedge_{i\in J}h_i)(x)\not\geq\alpha$ for all $x\in X$, there is a finite subset F of J such that $(\wedge_{i\in F}h_i)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$.

Proof : This follows directly from Definition 7.1.1.

Theorem 7.2.2 :

Let (X,τ) be an L-fts, $g\in L^X$ and $\phi_p(\tau)$ be the ordinary topology defined in Lemma 3.3.1 . The L-fuzzy set g is relatively compact if and only if for every $p\in pr(L)$, $G_p = \{ x\in X : g(x)\geq p' \}$ is relatively compact in the ordinary topological space $(X, \phi_p(\tau))$.

Proof :

<u>Necessity</u> : Let $p\in pr(L)$ and let $(A_i)_{i\in J}$ be an open covering of $(X, \phi_p(\tau))$, where $A_i= f_i^{-1}(\{t\in L:t\not\leq p\})$ and $f_i\in\tau$ for each $i\in J$. Then , $X = \cup_{i\in J} f_i^{-1}(\{t\in L:t\not\leq p\})$, i.e. $(\vee_{i\in J}f_i)(x)\not\leq p$ for all $x\in X$. Due to the relative compactness of g , there is a finite subset F of J such that $(\vee_{i\in F}f_i)(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$, i.e. for all $x\in G_p$. Hence , $G_p \subseteq \cup_{i\in F}A_i$ and so G_p is relatively compact in $(X, \phi_p(\tau))$.

Sufficiency : Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be collection of open L-fuzzy sets with $(\vee_{i\in J}f_i)(x) \not\leq p$ for all $x\in X$. Then, $X=\cup_{i\in J}f_i^{-1}(\{t\in L:t\not\leq p\})$ and $f_i^{-1}(\{t\in L:t\not\leq p\})\in\phi_p(\tau)$ for each $i\in J$. By the relative compactness of G_p in $(X, \phi_p(\tau))$, there is a finite subset F of J such that $G_p \subseteq \cup_{i\in F}f_i^{-1}(\{t\in L:t\not\leq p\})$ which implies that $(\vee_{i\in F}f_i)(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Hence, g is relatively compact in (X,τ).

Theorem 7.2.3 :

Let (X,τ) be an L-fts and let $g\in L^X$. The L-fuzzy set g is relatively compact if and only if for every $p\in pr(L)$ and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $(\vee_{i\in J}f_i)(x)\not\leq p$ for all $x\in X$, there is a finite subset F of J such that $(\vee_{i\in F}f_i\vee g')(x) \not\leq p$ for all $x\in X$.

Proof : This is similar to the proof of Theorem 3.3.4 .

Theorem 7.2.4 :

Let (X,τ) be an L-fts and let $g\in L^X$. The L-fuzzy set g is relatively compact if and only if every constant α-net contained in g has a cluster point in X with height α, for each $\alpha\in M(L)$.

Proof :

Necessity : Let $\alpha\in M(L)$ and let $S=(x_\alpha^m)_{m\in D}$ be a constant α-net contained in g , i.e. $g(x^m)\geq \alpha$ for all $m\in D$. Let $G=\{ z\in X : g(z)\geq\alpha \}$ and let $p=\alpha'$.

Since g is relatively compact , by Theorem 7.2.2 , G is relatively compact in the ordinary topological space $(X, \phi_p(\tau))$. Moreover, $x^m \in G$ for all $m \in D$. Thus, $(x^m)_{m \in D}$ is a net in G. Since G is relatively compact in $(X, \phi_p(\tau))$, the net $(x^m)_{m \in D}$ has a cluster point in X, say y.

We are going to show that y_α is cluster point of the net S.

Let f be a closed L-fuzzy set with $f(x) \not\geq \alpha$ and let $j \in D$. Put $F = \{ z \in X : f(z) \geq \alpha \}$. Then, F is a closed set in $(X, \phi_p(\tau))$ and $y \notin F$. Since y is a cluster point of $(x^m)_{m \in D}$, there is $m \in D$ with $m \geq j$ such that $x^m \notin F$, i.e. $f(x^m) \not\geq \alpha$. Hence, y_α is a cluster point of the net S.

Sufficiency : Suppose that g is not relatively compact. Then, by Theorem 7.2.2, there is a prime element p such that $G_p = \{ x \in X : g(x) \geq p' \}$ is not relatively compact in the ordinary topological space $(X, \phi_p(\tau))$. Hence, there is a net $(x^m)_{m \in D}$ in G_p with no cluster point in X. Let $\alpha = p'$. Then, the constant α-net $S=(x_\alpha{}^m)_{m \in D}$ has no cluster point with height α. This completes the proof.

Theorem 7.2.5 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is relatively compact if and only if every α-level filter base in g has a cluster point in X with height α, for each $\alpha \in M(L)$.

Proof : This is similar to the proof of Theorem 3.3.7.

7.3. The Goodness Theorems

Lemma 7.3.1 :

Let (X,T) be a topological space. For every $p \in pr(L)$, $\phi_p(\omega(T)) = T$.

Proof :

Let $p \in pr(L)$. Take $A \in T$. Then, by Proposition 2.3.8 (iii), $\chi_A \in \omega(T)$. Hence, $\chi_A^{-1}(\{t \in L : t \nleq p\}) = A \in \phi_p(\omega(T))$ and so $T \subseteq \phi_p(\omega(T))$.

Now take $f^{-1}(\{t \in L : t \nleq p\}) \in \phi_p(\omega(T))$, where $f \in \omega(T)$. Since $f \in \omega(T)$, by Proposition 2.3.8 (i), $f^{-1}(\{t \in L : t \nleq p\}) \in T$. Thus, $\phi_p(\omega(T)) \subseteq T$. Consequently, we get the desired equality.

Theorem 7.3.2 (The goodness of relative compactness) :

Let (X,T) be a topological space and $A \subset X$. A is relatively compact in (X,T) if and only if χ_A is relatively compact in (X, $\omega(T)$).

Proof : This easily follows from Lemma 7.3.1 and Theorem 7.2.2.

Theorem 7.3.3 (The goodness of strong relative compactness) :

Let (X,T) be a topological space and $A \subset X$. A is strongly relatively compact in (X,T) if and only if χ_A is strongly relatively compact in (X, $\omega(T)$).

Proof :

Suppose that A is strongly relatively compact in (X,T). Then, there is a compact subset K of X with $A \subseteq K$. Hence, $\chi_A \leq \chi_K$ and χ_K is compact in $(X,\omega(T))$ because of the goodness of compactness (Theorem 3.1.4). Thus, χ_K is also very compact in $(X,\omega(T))$ and therefore χ_A is strongly relatively compact.

Now suppose that χ_A is strongly relatively compact in $(X,\omega(T))$. Then, there is a very compact L-fuzzy set k in $(X,\omega(T))$ with $\chi_A \leq k$, where

$$k(x) = \begin{cases} e \in L & \text{if } x \in K \subseteq X \\ 0 & \text{otherwise} \end{cases} \qquad \text{and } \chi_K \text{ is compact in } (X,\omega(T)).$$

Since $\chi_A \leq k$, we have $A \subseteq K$. Because χ_K is compact in $(X,\omega(T))$, by the goodness of compactness (Theorem 3.1.4), K is compact in (X,T). This completes the proof.

7.4. Some Properties

Proposition 7.4.1 :

Let (X,τ) be an L-fts and $f, g, h \in L^X$.

(i) If g and h are relatively compact, then $g \vee h$ is relatively compact.

(ii) If g is relatively compact and $f \le g$, then f is relatively compact as well.

Proof :

(i) Let $p \in pr(L)$ and let g, h be relatively compact. Then, by Theorem 7.2.2, $G_p = \{ x \in X : g(x) \ge p' \}$ and $H_p = \{ x \in X : h(x) \ge p' \}$ are relatively compact in the ordinary topological space $(X, \phi_p(\tau))$. Hence, $G_p \cup H_p$ is relatively compact in $(X,\phi_p(\tau))$. On the other hand, we have

$K_p = \{ x \in X : (g \vee h)(x) \ge p' \} = G_p \cup H_p$ because p is prime. Thus, K_p is relatively compact in $(X, \phi_p(\tau))$ and hence, by Theorem 7.2.2, $g \vee h$ is relatively compact in (X,τ).

(ii) This is very similar to the proof of (i) and therefore omitted.

Lemma 7.4.2 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f : X \to Y$ be a function. If $f:(X,\tau) \to (Y,\tau^*)$ is continuous then for every $p \in pr(L)$, $f:(X,\phi_p(\tau)) \to (Y,\phi_p(\tau^*))$ is continuous.

Proof :

Suppose that $f:(X,\tau)\to(Y,\tau^*)$ is continuous and $p\in pr(L)$. Take $f^1(\{t\in L:t\nleq p\})\in\phi_p(\tau^*)$, where $f\in\tau^*$. Then $f^1(g)\in\tau$ because $f:(X,\tau)\to(Y,\tau^*)$ is continuous. Moreover, we have that

$f^1(g^{-1}(\{t\in L:t\nleq p\}))=\{x\in X:g(f(x))\nleq p\}=\{x\in X: f^1(g)(x)\nleq p\}=(f^1(g))^{-1}(\{t\in L:t\nleq p\})$

Since $f^1(g)\in\tau$, $f^1(g^{-1}(\{t\in L:t\nleq p\}))=(f^1(g))^{-1}(\{t\in L:t\nleq p\})\in\phi_p(\tau)$ and therefore $f:(X,\phi_p(\tau))\to(Y,\phi_p(\tau^*))$ is continuous.

Proposition 7.4.3 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be a continuous mapping such that $f^1(y)$ is finite for every $y\in Y$. If $g\in L^X$ is relatively compact in (X,τ) then $f(g)$ is relatively compact in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and g be a relatively compact L-fuzzy set in (X,τ). Then, by Theorem 7.2.2, $G_p=\{x\in X: g(x)\geq p'\}$ is relatively compact in $(X,\phi_p(\tau))$. On the other hand, we have that $f(G)_p=\{y\in Y: f(g)(y)\geq p'\}=f(G_p)$ because p is prime and for every $y\in Y$ $f^1(y)$ is finite. Since, by Lemma 7.4.2, $f:(X,\phi_p(\tau))\to(Y,\phi_p(\tau^*))$ is continuous, $f(G_p)$ is relatively compact in $(Y,\phi_p(\tau^*))$. So, by Theorem 7.2.2, $f(g)$ is relatively compact in (Y,τ^*).

Proposition 7.4.4 :

Let (X,τ) be an L-fts and $f,g,h\in L^X$.

(i) If g and h are strongly relatively compact then $g \vee h$ is strongly relatively compact .

(ii) If g is strongly relatively compact and $f \leq g$ then f is strongly relatively compact as well .

Proof :

(i) Suppose that g and h are strongly relatively compact . Then , there are very compact L-fuzzy sets k_1 and k_2 with $g \leq k_1$ and $h \leq k_2$, where

$$k_1(x) = \begin{cases} e_1 \in L & \text{if } x \in K_1 \subseteq X \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad k_2(x) = \begin{cases} e_2 \in L & \text{if } x \in K_2 \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

such that χ_K and χ_K are compact.

$$\text{Let} \quad k(x) = \begin{cases} e_1 \vee e_2 & \text{if } x \in K_1 \vee K_2 \\ 0 & \text{otherwise} \end{cases} \quad . \text{ Then } g \vee h \leq k.$$

Since $\chi_{K \cup K} = \chi_K \vee \chi_K$ is compact , k is a very compact L-fuzzy set and therefore $g \vee h$ is strongly relatively compact .

(ii) This follows easily from the definition .

Proposition 7.4.5 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f : (X,\tau) \to (Y,\tau^*)$ be a continuous mapping and g is strongly relatively compact L-fuzzy set in (X,τ) then $f(g)$ is strongly relatively compact in (Y,τ^*) .

Proof :

Let $f : (X,\tau) \to (Y,\tau^*)$ be a continuous mapping and g be a strongly relatively compact L-fuzzy set in (X,τ). Then, there is a very compact L-fuzzy set k with $g \leq k$, where

$$k(x) = \begin{cases} e \in L & \text{if } x \in K \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

and χ_K is compact in (X,τ).

Hence, $f(g) \leq f(k)$ and

$$f(k)(y) = \begin{cases} e & \text{if } y \in f(K) \subseteq Y \\ 0 & \text{otherwise} \end{cases}$$

Since f is continuous and χ_K is compact in (X,τ), by Proposition 3.1.10, $f(\chi_K) = \chi_{f(K)}$ is compact in (Y,τ^*) and therefore $f(k)$ is very compact in (Y,τ^*). This completes the proof.

Proposition 7.4.6 :

Let (X,τ) be a Hausdorff L-fts and let $g \in L^X$. If g is strongly relatively compact then $cl(\chi_{Supp\, g})$ is compact.

Proof :

Suppose that g is a strongly relatively compact L-fuzzy set. Then, there is a very compact L-fuzzy set k with $g \leq k$, where

$$k(x) = \begin{cases} e \in L & \text{if } x \in K \subseteq X \\ 0 & \text{otherwise} \end{cases}$$ and χ_K is compact in (X,τ).

Thus, we have $\text{Supp}\, g \subseteq \text{Supp}\, k = K$ and hence $cl(\chi_{\text{Supp}\, g}) \leq cl(\chi_K)$. Since χ_K is compact and (X,τ) is Hausdorff, by Proposition 3.1.9, χ_K is closed. So, we have $cl(\chi_{\text{Supp}\, g}) \leq \chi_K$. Hence, by Proposition 3.1.11 (ii), $cl(\chi_{\text{Supp}\, g})$ is compact.

Remark 7.4.7 :

In an L-fts, we can have a relatively compact L-fuzzy set which is not strongly relatively compact.

Consider $X = [0, 1] = L$ and τ the fuzzy topology with the subbase

$\delta = \{ 0 , 1 \} \cup \{ f_x^t : x \in X \text{ and } t < 1/2 \}$ where $f_x^t : X \to L$ is defined by

$$f_x^t(y) = \begin{cases} 1/2 & \text{if } y=x \\ t & \text{if } y \neq x \end{cases}$$

Define the fuzzy set $g : X \to L$ by $g(z) = \begin{cases} 1/2 & \text{if } z=1/4 \\ 1/4 & \text{if } z \neq 1/4 \end{cases}$

We are going to show that g is relatively compact but not strongly relatively compact.

Let β be a family of subbasic open fuzzy sets and let $p \in [0, 1) = pr(L)$.

(i) If $p<1/2$ and $(\vee_{f \in \beta} f) (y) > p$ for all $y \in X$, then there are $f_1 , f_2 , \ldots , f_n \in \beta$ such that $(\vee_{i=1}^n f_i) (y) > p$ for all $y \in X$ with $g(y) \geq p' > 1/2$.

(ii) If $p \geq 1/2$ and $(\bigvee_{f\in\beta} f)(y) > p$ for all $y \in X$, then $1 \in \beta$. Let $\beta^* = \{ 1 \}$. Then, we have that $(\bigvee_{f\in\beta^*} f)(y) > p$ for all $y \in X$ with $g(y) \geq p'$.

Consequently, g is relatively compact in (X, τ).

But g is not strongly relatively compact. Because, there is no very compact fuzzy set containing g. In fact, suppose that k is a very compact fuzzy set in (X, τ) with $g \leq k$. Then, for all $z \in X$ $k(z) \equiv s \geq 1/2$. Since (X, τ) is not compact, $k(z) \equiv s$ is not very compact.

CHAPTER VIII

R-COMPACTNESS, WEAK COMPACTNESS AND θ-RIGIDITY

In ordinary topology, R-compactness and weak compactness were introduced and studied by Cammarato and Noiri [12,13] and θ-rigidity was presented by Dickman and Porter [25]. An open cover $(A_i)_{i\in J}$ of a topological space (X,T) is said to be regular if and only if for each $i\in J$, there is a non-empty regular closed set B_i of X such that $B_i\subset A_i$ and $X=\bigcup_{i\in J}\text{int}(B_i)$. A topological space (X,T) is said to be R-compact (weak compact) if and only if for every regular cover $(A_i)_{i\in J}$ of X, there is a finite subset F of J such that $X=\bigcup_{i\in F}A_i$ ($X=\bigcup_{i\in F}\text{cl}(A_i)$). A subset G of a topological space (X,T) is said to be θ-rigid if and only if for every open cover $(A_i)_{i\in J}$ of G, there is a finite subset F of J with $G\subseteq\text{int}(\text{cl}(\bigcup_{i\in F}A_i))$.

Based on Chang's fuzzy compactess, weak compactness and θ-rigidity were introduced and studied in [0,1]-fuzzy topological spaces by Çoker and Eş [20] and Mukherjee and Ghosh [67], respectively. As far as we know R-compactness has not been initiated in the fuzzy setting so far.

In this chapter, we introduce R-compactness, weak compactness and θ-rigidity in L-fuzzy topological spaces. We define these weak forms of L-fuzzy compactness for arbitrary L-fuzzy sets and study some of their properties.

This chapter consists of three sections :

In the first section we present the proposed definitions .

The second section is reserved for the other characterizations of these weak forms of the compactness .

And lastly , the third section is devoted to some properties .

8.1. Proposed Definitions

Definition 8.1.1 :

Let (X,τ) be an L-fts, $g \in L^X$ and $r \in L$. An r-level open cover $(f_i)_{i\in J}$ of g is called an **r-level regular cover** of g if and only if for every $i \in J$ there is a non-empty regular closed L-fuzzy set h_i such that $h_i \leq f_i$ and $(\vee_{i\in J} int(h_i))(x) \nleq r$ for all $x \in X$ with $g(x) \geq r'$.

Definition 8.1.2 :

Let (X,τ) be an L-fts, $g \in L^X$ and $r \in L$. Non-empty open L-fuzzy sets f and h are called **ordered pair of open L-fuzzy sets**, denoted by **(f , h)**, if $cl(f) \leq h$. A family $\{ (f_i , h_i) : i \in J \}$ of ordered pair of open L-fuzzy sets is called an **ordered pair of r-level open cover** (for short, **r-level OPO cover**) of g if and only if $(\vee_{i\in J} f_i) (x) \nleq r$ for all $x \in X$ with $g(x) \geq r'$.

Definition 8.1.3 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be **R-compact** if and only if every p-level regular cover of g has a finite p-level subcover, where $p \in pr(L)$.

If g is the whole space, then we say that the L-fts (X,τ) is R-compact.

Definition 8.1.4 :

Let (X,τ) be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be **weakly compact** if and only if every p-level regular cover of g has a finite p-level proximate subcover, where $p \in pr(L)$.

If g is the whole space, then we say that the L-fts (X,τ) is weakly compact.

Definition 8.1.5 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is said to be **θ-rigid** if and only if for every p-level cover $(f_i)_{i \in J}$ of g , there is a finite subset F of J such that $(\mathrm{int}(\mathrm{cl}(\vee_{i \in F} f_i)))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, where $p \in pr(L)$.

If g is the whole space, then we say that the L-fts (X,τ) is θ-rigid.

8.2. Other Characterizations

Theorem 8.2.1 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is R-compact if and only if for every p-level OPO cover $\{ (f_i, h_i) : i \in J \}$ of g, where $p \in pr(L)$, there exists a finite subset F of J such that $(\vee_{i \in F} h_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Proof :

__Necessity__ : Let $p \in pr(L)$ and $\{ (f_i, h_i) : i \in J \}$ be a p-level OPO cover of g. Then, for each $i \in J$ $cl(f_i) \leq h_i$ and $(\vee_{i \in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, we have $(\vee_{i \in J} h_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Thus, $(h_i)_{i \in J}$ is a p-level regular cover of g. In fact, for each $i \in J$ $cl(f_i) \leq h_i$ and $cl(f_i)$ is regular closed. Furthermore, we have $[\vee_{i \in J} int (cl(f_i))](x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$ because $f_i \leq int(cl(f_i))$ for ever $i \in J$. So, by the R-compactness of g, there is a finite subset F of J such that $(\vee_{i \in F} h_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

__Sufficiency__ : Let $p \in pr(L)$ and $(h_i)_{i \in J}$ be a p-level regular cover of g. Then, for each $i \in J$, there exists a non-empty regular closed L-fuzzy set f_i such that $f_i \leq h_i$ and $(\vee_{i \in J} int(f_i))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Consider the family $\{ (int(f_i), h_i) : i \in J \}$. This is a p-level OPO cover of g. In fact, for each $i \in J$, $cl(int(f_i)) = f_i \leq h_i$ and $(\vee_{i \in J} int(f_i))(x) \nleq p$ for all $x \in X$

with $g(x) \geq p'$. So , by the hypothesis , there is a finite subset F of J such that $(\bigvee_{i\in F} h_i)(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$. Hence , g is R-compact.

Theorem 8.2.2 :

Let (X,τ) be an L-fts and $g\in L^X$. The L-fuzzy set g is weakly compact if and only if for every p-level OPO cover $\{ (f_i, h_i) : i\in J \}$ of g , where $p\in pr(L)$, there exists a finite subset F of J such that $(\bigvee_{i\in F} cl(h_i))(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$.

Proof : This is very similar to the proof of the previous theorem .

Theorem 8.2.3 :

Let (X,τ) be an L-fts and $g\in L^X$. The L-fuzzy set g is R-compact if and only if for every $\alpha\in M(L)$ and every collection $(f_i)_{i\in J}$ of closed L-fuzzy sets such that for each $i\in J$, there is a regular open L-fuzzy set h_i with $f_i \leq h_i$ and $(\bigwedge_{i\in J} cl(h_i))(x) \not\geq \alpha$ for all $x\in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J such that $(\bigwedge_{i\in F} f_i)(x) \not\geq \alpha$ for all $x\in X$ with $g(x) \geq \alpha$.

Proof :

<u>**Necessity**</u> : Let $(f_i)_{i\in J}$ be a collection of closed L-fuzzy sets satisfying the property in the theorem . Then , $(\bigvee_{i\in J} (cl(h_i))')(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$, where $p = \alpha'$. We also have $(cl(h_i))' \leq f_i'$ for each $i\in J$ because $f_i \leq h_i$ and each f_i is closed . Thus , $(\bigvee_{i\in J} f_i')(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$. Hence , $(f_i')_{i\in J}$ is a p-level open cover of g . On the other hand ,

each h_i is regular closed, $h_i' \le f_i'$ for every $i \in J$ and $(\vee_{i\in J} \operatorname{int}(h_i'))(x) = (\vee_{i\in J} (cl(h_i))')(x) \not\le p$ for all $x \in X$ with $g(x) \ge p'$, i.e. $(f_i')_{i\in J}$ is a p-level regular cover of g. Since g is R-compact, there exists a finite subset F of J such that $(\vee_{i\in F} f_i')(x) \not\le p$ for all $x \in X$ with $g(x) \ge p'$. Hence, $(\wedge_{i\in F} f_i)(x) \not\ge \alpha$ for all $x \in X$ with $g(x) \ge \alpha$.

Sufficiency: Let $(g_i)_{i\in J}$ be a p-level regular cover of g. Then, for each $i \in J$ there is a regular closed L-fuzzy set k_i such that $k_i \le g_i$ and $(\vee_{i\in J} \operatorname{int}(k_i))(x) \not\le p$ for all $x \in X$ with $g(x) \ge p'$. We also have $(\vee_{i\in J} g_i)(x) \not\le p$ for all $x \in X$ with $g(x) \ge p'$. Then, $(\wedge_{i\in J} g_i')(x) \not\ge \alpha$ for all $x \in X$ with $g(x) \ge \alpha$, where $\alpha = p' \in M(L)$. Furthermore, $g_i' \le k_i'$, each k_i is regular open and

$(\wedge_{i\in J} cl(k_i'))(x) = (\wedge_{i\in J} (\operatorname{int}(k_i))')(x) \not\ge \alpha$ for all $x \in X$ with $g(x) \ge \alpha$.

Hence, $(g_i')_{i\in J}$ is a collection of closed L-fuzzy sets satisfying the property in the theorem. So, by the hypothesis, there exists a finite subset F of J such that $(\wedge_{i\in F} g_i')(x) \not\ge \alpha$ for all $x \in X$ with $g(x) \ge \alpha$, i.e. $(\vee_{i\in F} g_i)(x) \not\le p$ for all $x \in X$ with $g(x) \ge p'$. Hence, g is R-compact.

Theorem 8.2.4 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is weakly compact if and only if for every $\alpha \in M(L)$ and every collection $(f_i)_{i\in J}$ of closed L-fuzzy sets such that for each $i \in J$ there is a regular open L-fuzzy set h_i with $f_i \le h_i$ and $(\wedge_{i\in J} cl(h_i))(x) \not\ge \alpha$ for all $x \in X$ with $g(x) \ge \alpha$, there exists a finite subset F of J such that $(\wedge_{i\in F} \operatorname{int}(f_i))(x) \not\ge \alpha$ for all $x \in X$ with $g(x) \ge \alpha$.

Proof: This is very similar to the proof of the previous theorem.

Theorem 8.2.5 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is θ-rigid if and only if for every $\alpha \in M(L)$ and every collection $(f_i)_{i\in J}$ of closed L-fuzzy sets with $(\wedge_{i\in J} f_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J such that $(\, cl(\, int\, (\wedge_{i\in F} f_i)\,)\,)\,(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : Using Proposition 2.1.8 (iv) this follows easily from Definition 8.1.5 .

Theorem 8.2.6 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is R-compact if and only if for every $\alpha \in M(L)$ and every family $\{\, (u_i, v_i) : i \in J \,\}$ of pair of closed L-fuzzy sets u_i, v_i such that $int(u_i) \geq v_i$ and $(\, \wedge_{i\in J} u_i \,)\,(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J such that $(\, \wedge_{i\in F} v_i \,)\,(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : This follows from Theorem 8.2.1 .

Theorem 8.2.7 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is weakly compact if and only if for every $\alpha \in M(L)$ and every family $\{\, (u_i, v_i) : i \in J \,\}$ of pair of closed L-fuzzy sets u_i, v_i such that $int(u_i) \geq v_i$ and $(\, \wedge_{i\in J} u_i \,)\,(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J such that $(\, \wedge_{i\in F} int(v_i)\,)\,(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : This follows from Theorem 8.2.2 .

Theorem 8.2.8 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is R-compact if and only if every constant α-net contained in g has a γ-cluster point (Definition 2.1.16 (iv)) $x_\alpha \in M(L^X)$, with height α, contained in g, i.e. $g(x) \geq \alpha$ for each $\alpha \in M(L)$.

Proof : By using Theorem 8.2.6 , this is similar to the proof of Theorem 4.2.2 .

Theorem 8.2.9 :

Let (X,τ) be an L-fts and $g \in L^X$. The L-fuzzy set g is weakly compact if and only if every constant α-net contained in g has a β-cluster point (Definition 2.1.16 (v)) $x_\alpha \in M(L^X)$, with height α , contained in g , i.e. $g(x) \geq \alpha$ for each $\alpha \in M(L)$.

Proof : By using Theorem 8.2.7 , this is similar to the proof of Theorem 4.2.2 .

8.3. Some Properties

Proposition 8.3.1 :

Let (X, τ) be an L-fts and let $g, h \in L^X$. If g and h are R-compact then $g \vee h$ is R-compact as well.

Proof : This is similar to the proof of Proposition 4.3.2 .

Proposition 8.3.2 :

Let (X, τ) be an L-fts and $g, h \in L^X$. If g is R-compact and h is clopen then $g \wedge h$ is R-compact.

Proof : This is similar to the proof of Proposition 5.3.2 .

Corollary 8.3.3 :

Let (X, τ) be an R-compact L-fts . Then every clopen L-fuzzy set in (X, τ) is R-compact .

Proof : This follows immediately from the previous proposition .

Proposition 8.3.4 :

Let (X, τ) be an L-fuzzy topological space and let $g, h \in L^X$.

(i) If g and h are weakly compact then $g \vee h$ is weakly compact as well.

(ii) If g is weakly compact and h is clopen then $g \wedge h$ is weakly compact.

Proof : These are similar to the proofs of Proposition 4.3.2 and Proposition 5.3.2.

Corollary 8.3.5 :

Let (X, τ) be a weakly compact L-fts. Then every clopen L-fuzzy set in (X, τ) is weakly compact.

Proof : This follows immediately from Proposition 8.3.4 (ii).

Proposition 8.3.6 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau) \to (Y,\tau^*)$ be a continuous mapping with $f^{-1}(y)$ is finite for every $y \in Y$. If $g \in L^X$ is R-compact in (X,τ), then $f(g)$ is R-compact in (Y,τ^*).

Proof :

Let $p \in pr(L)$ and $\{ (g_i, h_i) : i \in J \}$ be a p-level OPO cover of $f(g)$. Since f is continuous and $cl(g_i) \leq h_i$ for every $i \in J$, by Proposition 2.1.19 (i), we have $cl(f^{-1}(g_i)) \leq f^{-1}(cl(g_i)) \leq f^{-1}(h_i)$ for every $i \in J$ and $f^{-1}(g_i)$, $f^{-1}(h_i)$ are open in (X,τ). Furthermore, we have that $(\vee_{i \in J} f^{-1}(g_i)) (x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$ because $(\vee_{i \in J} g_i) (y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$. Thus, $\{ (f^{-1}(g_i) , f^{-1}(h_i)) : i \in J \}$ is a p-level OPO cover of g. From the R-compactness of g, there is a finite subset F of J such that $(\vee_{i \in F} f^{-1}(h_i)) (x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Hence, $(\vee_{i \in F} h_i) (y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$. In fact, if $f(g)(y) = \vee\{ g(x) : x \in f^{-1}(y) \} \geq p'$ then there exists $x \in X$ with $f(x) = y$ and $g(x) \geq p'$ because $f^{-1}(y)$ is finite and $p' \in M(L)$. So, we have

$$(\vee_{i \in F} h_i) (y) = (\vee_{i \in F} h_i) (f(x)) = (\vee_{i \in F} f^{-1}(h_i)) (x) \nleq p .$$

Consequently, $f(g)$ is R-compact in (Y,τ^*).

Proposition 8.3.7 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau)\to(Y,\tau^*)$ be an almost continuous and almost open function with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is R-compact in (X,τ), then $f(g)$ is R-compact in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a p-level regular cover of $f(g)$. Then, for each $i\in J$, there is a non-empty regular closed L-fuzzy set g_i with $g_i \leq f_i$ and $(\bigvee_{i\in J}int(g_i))(y) \not\leq p$ for all $y\in Y$ with $f(g)(y) \geq p'$. Hence, $(\bigvee_{i\in J} f^{-1}(int(g_i)))(x)\not\leq p$ for all $x\in X$ with $g(x) \geq p'$. Since each g_i is also closed, by Remark 2.1.10 (ii), $int(g_i)$ is regular open for each $i\in J$. Since f is almost continuous and almost open, by Proposition 2.1.19 (v), $f^{-1}(int(g_i))$ is regular open in (X,τ) for each $i\in J$. By Proposition 2.1.19 (ii), We also have

$$cl(f^{-1}(int(g_i))) \leq f^{-1}(cl(int(g_i))) = f^{-1}(g_i) \leq f^{-1}(f_i) \quad \text{for every } i\in J.$$

Furthermore, for each $i\in J$ $cl(f^{-1}(int(g_i)))$ is regular closed and therefore

$(\bigvee_{i\in J} int[cl(f^{-1}(int(g_i)))])(x) = (\bigvee_{i\in J} f^{-1}(int(g_i)))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

So, the collection $(f^{-1}(f_i))_{i\in J}$ is a p-level regular cover of g. From the R-compactness of g, there is a finite subset F of J such that

$(\bigvee_{i\in F} f^{-1}(f_i))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Hence, $(\bigvee_{i\in F} f_i)(y) \not\leq p$ for all $y\in Y$ with $f(g)(y) \geq p'$ because $f^{-1}(y)$ is finite for every $y\in Y$. Consequently, $f(g)$ is R-compact in (Y,τ^*).

Lemma 8.3.8 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces, $g \in L^X$ and let $f:(X,\tau) \to (Y,\tau^*)$ be an almost continuous function. If $(f_i)_{i\in J}$ is a p-level regular cover of $f(g)$ in (Y,τ^*), then $\{ f^{1}(int(cl(f_i))) \}_{i\in J}$ is a p-level regular cover of g in (X,τ), where $p \in pr(L)$.

Proof :

If $(f_i)_{i\in J}$ is a p-level regular cover of $f(g)$, then for each $i\in J$ there exists a non-empty regular closed L-fuzzy set g_i such that $g_i \leq f_i \leq int(cl(f_i))$ and $(\bigvee_{i\in J} int(g_i))(y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$. Hence, $(\bigvee_{i\in J} f^{1}(int(g_i)))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$ and $f^{1}(g_i) \leq f^{1}(f_i) \leq f^{1}(int(cl(f_i)))$ for every $i \in J$. Since f is almost continuous and g_i is also closed L-fuzzy set, by Proposition 2.1.19 (iii), $f^{1}(int(g_i)) \leq int(f^{1}(g_i))$ and $f^{1}(g_i)$ must be closed. So, we have $cl(int(f^{1}(g_i)) \leq cl(f^{1}(g_i)) = f^{1}(g_i) \leq f^{1}(int(cl(f_i)))$. By Remark 2.1.10 (iii), $cl(int(f^{1}(g_i))$ is regular closed for each $i \in J$. Hence, we have $\bigvee_{i\in J} f^{1}(int(g_i)) \leq \bigvee_{i\in J} int(f^{1}(g_i)) \leq \bigvee_{i\in J} int(cl(int(f^{1}(g_i))))$. So, $[\bigvee_{i\in J} int(cl(int(f^{1}(g_i))))](x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. We also have $(\bigvee_{i\in J} f^{1}(int(cl(f_i))))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Consequently, $\{f^{1}(int(cl(f_i)))\}_{i\in J}$ is a p-level regular cover of g.

Proposition 8.3.9 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau) \to (Y,\tau^*)$ be an almost continuous function with $f^{1}(y)$ is finite for every $y \in Y$. If $g \in L^X$ is weakly compact in (X,τ), then $f(g)$ is weakly compact in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a p-level regular cover of $f(g)$. Since f is almost continuous, by the previous lemma, $\{ f^{-1}(int(cl(f_i))) \}_{i\in J}$ is a p-level regular cover of g in (X,τ). From the weak compactness of g, there is a finite subset F of J such that $(\bigvee_{i\in F} cl[f^{-1}(int(cl(f_i)))])(x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Then, $(\bigvee_{i\in F} cl(f_i))(y) \nleq p$ for all $y\in Y$ with $f(g)(y) \geq p'$.

In fact, if $f(g)(y) = \bigvee\{ g(x) : x\in f^{-1}(y) \} \geq p'$ then there exists $x\in X$ with $f(x)=y$ and $g(x)\geq p'$ because $f^{-1}(y)$ is finite and $p'\in M(L)$. So,

$(\bigvee_{i\in F} cl(f_i))(y) = (\bigvee_{i\in F} cl(f_i))(f(x)) = (\bigvee_{i\in F} f^{-1}(cl(f_i)))(x)$ and

$(\bigvee_{i\in F} cl[f^{-1}(int(cl(f_i)))])(x) \leq (\bigvee_{i\in F} f^{-1}(cl(int(cl(f_i)))))(x) = (\bigvee_{i\in F} f^{-1}(cl(f_i)))(x)$,

where the inequality is due to the almost continuity of f and the last equality is due to the fact that the closure of each open L-fuzzy set is regular closed. This completes the proof.

Proposition 8.3.10 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f:(X,\tau) \rightarrow (Y,\tau^*)$ be a weakly continuous function with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is compact in (X,τ), then $f(g)$ is weakly compact in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a p-level regular cover of $f(g)$. Then, for each $i\in J$ there exists a non-empty regular closed L-fuzzy set g_i such that $g_i \leq f_i$ and $(\bigvee_{i\in J} int(g_i))(y) \nleq p$ for all $y\in Y$ with $f(g)(y) \geq p'$. Hence, we have

$(\bigvee_{i\in J} f^{-1}(int(g_i)))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Since f is weakly continuous and g_i is regular closed, we also have $f^{-1}(int(g_i)) \leq int[f^{-1}(cl(int(g_i)))] = int(f^{-1}(g_i))$ for every $i\in J$. Thus, $(\bigvee_{i\in J} int(f^{-1}(g_i)))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$, i.e. $\{ int(f^{-1}(g_i)) \}_{i\in J}$ is a p-level open cover of g. From the compactness of g, there are $g_1, g_2, \ldots, g_n$ such that $(\bigvee_{i=1}^{n} int(f^{-1}(g_i)))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Then,
$p \not\geq f (\bigvee_{i=1}^{n} int(f^{-1}(g_i)))(y) = [\bigvee_{i=1}^{n} f(int(f^{-1}(g_i)))](y) \leq [\bigvee_{i=1}^{n} f(f^{-1}(g_i))](y) \leq (\bigvee_{i=1}^{n} g_i)(y) \leq (\bigvee_{i=1}^{n} cl(f_i))(y)$.

Hence, $(\bigvee_{i=1}^{n} cl(f_i))(y) \not\leq p$ for all $y\in Y$ with $f(g)(y) \geq p'$. Consequently, $f(g)$ is weakly compact in (Y,τ^*).

Proposition 8.3.11 :

Let (X,τ) and (Y,τ^*) be L-fuzzy topological spaces and let $f: (X,\tau) \to (Y,\tau^*)$ be an almost open and almost continuous function with $f^{-1}(y)$ is finite for every $y\in Y$. If $g\in L^X$ is θ-rigid in (X,τ), then $f(g)$ is θ-rigid in (Y,τ^*).

Proof :

Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a collection of open L-fuzzy sets in (Y,τ^*) with $(\bigvee_{i\in J} f_i)(y) \not\leq p$ for all $y\in Y$ with $f(g)(y) \geq p'$. Then,
$(\bigvee_{i\in J} int(cl(f_i)))(y) \not\leq p$ for all $y\in Y$ with $f(g)(y) \geq p'$ because $f_i \leq int(cl(f_i))$ for every $i\in J$. Hence, $(\bigvee_{i\in J} f^{-1}(int(cl(f_i))))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$. From the almost continuity of f, $\{ f^{-1}(int(cl(f_i))) \}_{i\in J}$ is a family of open L-fuzzy sets in (X,τ). So, by the θ-rigidity of g, there is a finite subset F of J such that
$int(cl(\bigvee_{i\in F} f^{-1}(int(cl(f_i)))))(x) \not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Then,

$p \not\geq f[\, int(\, cl(\, \bigvee_{i\in F} f^{-1}(int(cl(f_i)))\,)\,)\,](y) \;\leq\; int[\, f(\, cl(\, \bigvee_{i\in F} f^{-1}(int(cl(f_i)))\,)\,](y) \;=$

$int[\, f(\, \bigvee_{i\in F} cl(\, f^{-1}(int(cl(f_i)))\,)\,](y) = int[\, \bigvee_{i\in F} f(cl(\, f^{-1}(int(cl(f_i)))))\,](y) \leq$

$int[\, \bigvee_{i\in F} f(\, f^{-1}(\, cl(int(cl(f_i)))))\,](y) \;\leq\; int[\, \bigvee_{i\in F} cl(int(cl(f_i)))\,](y) \leq$

$int[\, \bigvee_{i\in F} cl(f_i)\,](y) \;=\; int(cl(\, \bigvee_{i\in F} f_i\,))(y)$

where the second inequality is due to the almost openness of f and the third inequality is due to the almost continuity of f. The first and the last equality follows from Proposition 2.1.8 (v). Hence,

$int(cl(\, \bigvee_{i\in F} f_i\,))(y) \not\leq p$ for all $y\in Y$ with $f(g)(y) \geq p'$.

In fact, if $f(g)(y) = \bigvee\{\, g(x) : x\in f^{-1}(y)\,\} \geq p'$ then there exists $x\in X$ with $f(x)=y$ and $g(x)\geq p'$ because $f^{-1}(y)$ is finite and $p'\in M(L)$. Then,

$int(\, cl(\, \bigvee_{i\in F} f^{-1}(int(cl(f_i)))\,)\,)(x) \not\leq p$ and therefore $int(cl(\, \bigvee_{i\in F} f_i\,))(y) \not\leq p$.

Consequently, $f(g)$ is θ-rigid in (Y,τ^*).

Proposition 8.3.12 :

Let (X, τ) be an L-fts and $g\in L^X$. Then the following implications hold :

g is nearly compact (Definition 3.2.5 (ii)) $\overset{(i)}{\Rightarrow}$ g is θ-rigid $\overset{(ii)}{\Rightarrow}$ g is almost compact (Definition 3.2.5 (i)) $\overset{(iii)}{\Rightarrow}$ g is R-compact $\overset{(iv)}{\Rightarrow}$ g is weakly compact.

Proof :

(i) Let $p\in pr(L)$ and $(\, f_i\,)_{i\in J}$ be a p-level open cover of g. Then, by the near compactness of g, there is a finite subset F of J such that $(\bigvee_{i\in F} int(cl(f_i))\,)(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$. We also have

$\bigvee_{i\in F} \mathrm{int}(\mathrm{cl}(f_i)) \leq \mathrm{int}(\bigvee_{i\in F} \mathrm{cl}(f_i)) = \mathrm{int}(\mathrm{cl}(\bigvee_{i\in F} f_i))$,

where the inequality and equality follow from Proposition 2.1.8 (v) . So , we get

$(\mathrm{int}(\mathrm{cl}(\bigvee_{i\in F} f_i))) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Hence, g is θ-rigid.

(ii) Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a p-level open cover of g . Then , by the θ-rigidity of g , there is a finite subset F of J such that

$(\mathrm{int}(\mathrm{cl}(\bigvee_{i\in F} f_i))) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$.

Since $\mathrm{int}(\mathrm{cl}(\bigvee_{i\in F} f_i)) \leq \mathrm{cl}(\bigvee_{i\in F} f_i) = \bigvee_{i\in F} \mathrm{cl}(f_i)$, we have

$(\bigvee_{i\in F} \mathrm{cl}(f_i)) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Hence, g is almost compact.

(iii) Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a p-level regular cover of g . Then , for each $i\in J$, there exists a non-empty regular closed L-fuzzy set h_i such that $h_i \leq f_i$ and $(\bigvee_{i\in J} \mathrm{int}(h_i)) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. That is , $\{ \mathrm{int}(h_i) \}_{i\in J}$ is p-level open cover of g . Since g is almost compact, there is a finite subset F of J such that $(\bigvee_{i\in F} \mathrm{cl}(\mathrm{int}(h_i))) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Since each h_i is regular closed , we have

$(\bigvee_{i\in F} \mathrm{cl}(\mathrm{int}(h_i))) (x) = (\bigvee_{i\in F} h_i) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$. Thus, $(\bigvee_{i\in F} f_i) (x) \nleq p$ for all $x\in X$ with $g(x) \geq p'$ because $h_i \leq f_i$ for each $i\in J$. Hence, g is R-compact.

(iv) This follows immediately from the definitions.

Proposition 8.3.13 :

In extremally disconnected L-fuzzy topological spaces , the following are equivalent :

(i) near compactness **(ii)** θ-rigidity **(iii)** almost compactness

(iv) R-compactness **(v)** weak compactness

Proof : From Proposition 8.3.12 , it is sufficient to prove that if g is weakly compact in an extremally disconnected L-fts (X, τ) , then g is nearly compact.

Let $p \in pr(L)$ and $(f_i)_{i \in J}$ be a collection of regular open L-fuzzy sets with $(\vee_{i \in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Then , the family $\{ cl(f_i) \}_{i \in J}$ is a p-level regular cover of g. In fact , for each $i \in J$, $cl(f_i) \in \tau$ because (X, τ) is extremally disconnected. Since each f_i is open , by Remark 2.1.10 (i) , $cl(f_i)$ is regular closed . Moreover , we have $(\vee_{i \in J} int(cl(f_i)))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$ because $int(cl(f_i)) = f_i$.

So , by the weak compactness of g , there is a finite subset F of J such that $(\vee_{i \in F} cl(cl(f_i)))(x) = (\vee_{i \in F} cl(f_i))(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. On the other hand , we have $cl(f_i) = f_i$ for every $i \in J$ because $f_i = int(cl(f_i))$ and $cl(f_i) \in \tau$ for every $i \in J$. Therefore , $(\vee_{i \in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Hence , g is nearly compact .

Remark 8.3.14 :

We have been unable to prove whether R-compactness , weak compactness and θ-rigidity are good extensions of the corresponding notions in general topology.

CHAPTER IX

COMPACTNESS IN SMOOTH L-FUZZY TOPOLOGICAL SPACES

As we mentioned in the introduction as well as in Chapter II, various kinds of fuzzy topological spaces have appeared in the literature [e.g. 16, 37, 49, 79]. According to Šostak [84], in all these definitions, a fuzzy topology is a crisp subset of the fuzzy power set of a non-empty classical set satisfying the well known three axioms and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Thus, for every fuzzy subset of a fuzzy space it is precisely known, as in general topology, that whether it is open or not. Therefore, Šostak has introduced a new definition of fuzzy topology in 1985 [84], which we shall call here '**smooth fuzzy topology**'. The first aim of Šostak's approach is to consider a fuzzy topology to be a fuzzy subset on the fuzzy power set of an ordinary set. The second one is to allow fuzzy subsets to be open to some degree and this degree may range from 1 ('completely open fuzzy sets') to 0 ('completely non-open sets'). Later on he has developed the theory of smooth fuzzy topological spaces in [85, 86]. Meanwhile, several authors [17, 31,.77] have reintroduced the same definition and studied smooth fuzzy topological spaces independently.

In the present chapter, we shall study compactness, which we call " **smooth compactness** ", in smooth L-fuzzy topological spaces where L is a fuzzy lattice. " **α-Scott continuous functions** " from an ordinary topological space to a fuzzy lattice L with its Scott topology are introduced and studied. These functions turn out to be the natural tool to set up a 'goodness of extension' criterion for smooth L-fuzzy topological properties. Good definitions of smooth Hausdorffness, smooth compactness, ultra-smooth compactness, smooth countable compactness, smooth Lindelöfness and smooth local compactness in smooth L-fuzzy topological spaces are presented and some of their properties studied.

The structure of this chapter is as follows:

In the first section we present the basic definitions and results of smooth L-fuzzy topological spaces that here we shall deal with.

In Section 2, we first introduce a new class of functions from a topological space to a fuzzy lattice L with its Scott topology, called " α-Scott continuous functions " ($\alpha \in L$). Then, using this notion, we obtain a smooth L-fuzzy topology from a given ordinary topology and a functor between the category of ordinary topological spaces and the category of smooth L-fuzzy topological spaces and establish a " **goodness of extension** " criterion for smooth L-fuzzy topological properties. We also introduce smooth Hausdorffness and prove that it is a good extension of the Hausdorffness in general topology.

In the third section of this chapter, we focus on smooth compactness in smooth L-fuzzy topological spaces. We prove its goodness in the above sense and study some of its properties. Moreover, we define " **prime level spaces** " of a smooth L-fuzzy space and then characterize smooth compactness in terms of the ordinary compactness in such spaces. We also introduce ultra-smooth compactness and prove that it is a good extension.

The forth section is reserved for the notions of smooth countable compactness and smooth Lindelöfness in smooth L-fuzzy topological spaces. It is proved that they are good extensions of the corresponding concepts in general topology and some of their properties studied.

In the last section of this chapter, we introduce a good definition of smooth local compactness in smooth L-fuzzy topological spaces and study some of its properties.

9.1. Smooth L-fuzzy Topological Spaces

In the following, X will be a non-empty ordinary set and $L = L(\leq, \vee, \wedge, ')$ will denote a fuzzy lattice with a smallest element 0 and a largest element 1 ($0 \neq 1$) and with an order reversing involution $a \to a'$ ($a \in L$).

Definition 9.1.1 (Šostak [84]) :

A **smooth L-fuzzy topology** on X is a map $\Upsilon : L^X \to L$ satisfying the following three axioms :

(1) $\Upsilon(0) = \Upsilon(1) = 1$

(2) $\Upsilon(f \wedge g) \geq \Upsilon(f) \wedge \Upsilon(g)$ for every $f, g \in L^X$.

(3) $\Upsilon(\vee_{i \in J} f_i) \geq \wedge_{i \in J} \Upsilon(f_i)$ for every family $(f_i)_{i \in J}$ in L^X.

The pair (X, Υ) is called a **smooth L-fuzzy topological space** (for short, **smooth L-fts**). For every $f \in L^X$, $\Upsilon(f)$ is called **degree of openness** of the fuzzy subset f.

While an L-fuzzy topology on X (Definition 3.1.1) is an ordinary subset of L^X, a smooth L-fuzzy topology on X is a fuzzy subset of L^X.

Remark 9.1.2 (Šostak [84]) :

The intuitive motivation for this definition is as follows. Informally speaking, the first axiom states that the empty L-fuzzy set and the full L-fuzzy set are " absolutely open ". The axiom (2) states that the intersection of two

L-fuzzy sets is not " less open " than the minimum of " openness " of these L-fuzzy sets. Lastly, the axiom (3) requires that the degree of openness of the union of any family of L-fuzzy sets should be not less than the " smallest " degree of openness of these L-fuzzy sets.

Example 9.1.3 :

(i) Let (X,T) be an ordinary topological space. Then we can consider (X,T) as the smooth fuzzy topological space (X,Υ) where $\Upsilon = \chi_T : 2^X \to 2$, $(2 = \{0, 1\})$.

(ii) Let (X,τ) be an L-fuzzy topological space. Then we can consider (X,τ) to be a smooth fuzzy topological space as well, where $\Upsilon = \chi_\tau : L^X \to 2$.

Definition 9.1.4 (Šostak [85]) **:**

Let (X,Υ) be a smooth L-fts. The map $\Phi : L^X \to L$ defined by $\Phi(g) = \Upsilon(g')$ for every $f \in L^X$ is called the **degree of closedness** on X. $\Phi(g)$ is called the degree of closedness of the L-fuzzy set g.

From Definition 9.1.1 and Definition 9.1.4, it is easy to see that the mapping Φ has the following properties :

(1) $\Phi(0) = \Phi(1) = 1$

(2) $\Phi(f \vee g) \geq \Phi(f) \wedge \Phi(g)$ for every $f,g \in L^X$

(3) $\Phi(\wedge_{i\in J} f_i) \geq \wedge_{i\in J} \Phi(f_i)$ for every family $(f_i)_{i\in J}$ in L^X.

Definition 9.1.5 (Šostak [85]) :

Let (X,Υ) be a smooth L-fts and $Y \subset X$. The mapping $\Upsilon_Y : L^Y \to L$ defined by

$$\Upsilon_Y(g) = \vee\{\ \Upsilon(f) : f \in L^X \text{ and } f|_Y = g\ \}$$

is a smooth L-fuzzy topology on Y. The pair (Y,Υ_Y) is called a **smooth subspace** of (X,Υ).

Definition 9.1.6 (Šostak [84]) :

Let (X,Υ) and (Y,Υ^*) be smooth L-fuzzy topological spaces and $F : X \to Y$ be a function. The function F is called :

(a) smooth continuous if and only if for every $f \in L^Y$, $\Upsilon(F^{-1}(f)) \geq \Upsilon^*(f)$.

(b) smooth open if and only if for every $g \in L^X$, $\Upsilon^*(F(g)) \geq \Upsilon(g)$.

For more details on smooth L-fts's we refer to [84, 85, 86].

9.2. α-Scott Continuity and ' Goodness of Extension ' Criterion for Smooth Fuzzy Topological Properties

As we mentioned in Chapter II (Section 3), by using lower semi-continuous functions, Lowen [49, 51] has established a ' goodness of extension ' criterion for I-fuzzy topological properties. After that Warner [96] has extended this criterion to the L-fuzzy topological properties by using Scott continuous functions from a topological space to a fuzzy lattice with its Scott topology.

In this section, we introduce the concept of α-Scott continuity ($\alpha \in L$) in order to set up a ' goodness of extension ' criterion for smooth L-fuzzy topological properties.

Definition 9.2.1 (gradation of Scott continuity) :

Let (X,T) be an ordinary topological space and $\alpha \in L$. A function $f : (X,T) \to L$, where L has its Scott topology (Definition 1.1.20), is said to be **α-Scott continuous** if and only if for every $p \in pr(L)$ with $\alpha \nleq p$, $f^{-1}(\{t \in L : t \nleq p\}) \in T$.

In particular when $L = I$, then f is called **α-lower semi-continuous**. Thus, $f : (X,T) \to L$ is α-lower semi-continuous if and only if for every $p \in pr(I) = [0,1)$ with $\alpha > p$, $f^{-1}(\{t \in L : t \nleq p\}) = f^{-1}((p,1]) \in T$.

From the definition of Scott continuity and Definition 9.2.1, it is clear that if f is Scott continuous then f is α-Scott continuous for every $\alpha \in L$. Moreover, f is 1-Scott continuous iff f is Scott continuous. Naturally, every function from (X,T) to L is 0-Scott continuous.

α-Scott continuous functions will be used to obtain a smooth L-fuzzy topology from a given ordinary topology.

Lemma 9.2.2 :

Let $f, g : X \to L$ be two functions. For every $p \in pr(L)$, we have

$$(f \wedge g)^{-1}(\{t \in L : t \not\leq p\}) = f^{-1}(\{t \in L : t \not\leq p\}) \cap g^{-1}(\{t \in L : t \not\leq p\}).$$

Proof :

Let $x \in (f \wedge g)^{-1}(\{t \in L : t \not\leq p\})$. Then, $(f \wedge g)(x) \not\leq p \Rightarrow f(x) \wedge g(x) \not\leq p \Rightarrow f(x) \not\leq p$ and $g(x) \not\leq p$ $\Rightarrow x \in f^{-1}(\{t \in L : t \not\leq p\})$ and $x \in g^{-1}(\{t \in L : t \not\leq p\}) \Rightarrow x \in f^{-1}(\{t \in L : t \not\leq p\}) \cap g^{-1}(\{t \in L : t \not\leq p\})$. Hence, we have $(f \wedge g)^{-1}(\{t \in L : t \not\leq p\}) \subset f^{-1}(\{t \in L : t \not\leq p\}) \cap g^{-1}(\{t \in L : t \not\leq p\})$.

Now take $x \in f^{-1}(\{t \in L : t \not\leq p\}) \cap g^{-1}(\{t \in L : t \not\leq p\})$. Then, $f(x) \not\leq p$ and $g(x) \not\leq p$. Hence, $f(x) \wedge g(x) \not\leq p$ because p is prime. Thus, $(f \wedge g)(x) \not\leq p$ and so $x \in (f \wedge g)^{-1}(\{t \in L : t \not\leq p\})$. Hence, $f^{-1}(\{t \in L : t \not\leq p\}) \cap g^{-1}(\{t \in L : t \not\leq p\}) \subset (f \wedge g)^{-1}(\{t \in L : t \not\leq p\})$. Consequently, we have the desired equality.

Lemma 9.2.3 :

Let $(f_i)_{i \in J}$ be a family of functions from X to L. Then, for every $p \in L$

$$(\vee_{i \in J} f_i)^{-1}(\{t \in L : t \not\leq p\}) = \cup_{i \in J} f_i^{-1}(\{t \in L : t \not\leq p\}).$$

Proof :

$x\in(\bigvee_{i\in J}f_i)^{-1}(\{t\in L: t\nleq p\}) \Rightarrow (\bigvee_{i\in J}f_i)(x)\nleq p \Rightarrow \bigvee_{i\in J}f_i(x)\nleq p \Rightarrow \exists\, i\in J$ with $f_i(x)\nleq p \Rightarrow$ $\exists\, i\in J$ with $x\in f_i^{-1}(\{t\in L: t\nleq p\}) \Rightarrow x\in\bigcup_{i\in J}f_i^{-1}(\{t\in L: t\nleq p\})$.

Hence, we have $(\bigvee_{i\in J}f_i)^{-1}(\{t\in L: t\nleq p\})=\bigcup_{i\in J}f_i^{-1}(\{t\in L: t\nleq p\})$.

Lemma 9.2.4 :

Let (X,T) be an ordinary topological space.

(i) If $f, g : (X,T)\rightarrow L$ are α-Scott continuous and λ-Scott continuous respectively, then $f\wedge g : (X,T)\rightarrow L$ is $\alpha\wedge\lambda$-Scott continuous.

(ii) If $f_i : (X,T)\rightarrow L$ is α_i-Scott continuous ($i\in J$), then $\bigvee_{i\in J}f_i : (X,T)\rightarrow L$ is $\bigwedge_{i\in J}\alpha_i$-Scott continuous.

Proof :

(i) From the α-Scott continuity of f and the λ-Scott continuity of g, we have $f^{-1}(\{t\in L: t\nleq p\})\in T$ for all $p\in pr(L)$ with $\alpha\nleq p$ and $g^{-1}(\{t\in L: t\nleq p\})\in T$ for all $p\in pr(L)$ with $\lambda\nleq p$.

Hence, $f^{-1}(\{t\in L: t\nleq p\})\cap g^{-1}(\{t\in L: t\nleq p\})\in T$ for every $p\in pr(L)$ with $\alpha\wedge\lambda\nleq p$.

By Lemma 9.2.2, $(f\wedge g)^{-1}(\{t\in L: t\nleq p\})\in T$ for every $p\in pr(L)$ with $\alpha\wedge\lambda\nleq p$ and therefore $f\wedge g$ is $\alpha\wedge\lambda$-Scott continuous.

(ii) Let $p\in pr(L)$ with $\bigwedge_{i\in J}\alpha_i\nleq p$. Then, $\alpha_i\nleq p$ for every $i\in J$. Since $f_i : (X,T)\rightarrow L$ is α_i-Scott continuous for every $i\in J$, $f_i^{-1}(\{t\in L: t\nleq p\})\in T$ for every $i\in J$. Hence, $\bigcup_{i\in J}f_i^{-1}(\{t\in L: t\nleq p\})\in T$. By Lemma 9.2.3, $(\bigvee_{i\in J}f_i)^{-1}(\{t\in L: t\nleq p\})\in T$ for every $p\in pr(L)$ with $\bigwedge_{i\in J}\alpha_i\nleq p$ and therefore $\bigvee_{i\in J}f_i$ is $\bigwedge_{i\in J}\alpha_i$-Scott continuous.

Theorem 9.2.5 :

Let (X,T) be an ordinary topological space . The mapping $\omega_T : L^X \to L$ defined by

$$\omega_T(f) = \vee\{\alpha \in L : f \text{ is } \alpha\text{-Scott continuous}\} \quad \text{for every } f \in L^X ,$$

is a smooth L-fuzzy topology on X .

Proof :

(1) Since the constant functions **0** and **1** are Scott continuous , i.e. 1-Scott continuous , we have $\omega_T(\mathbf{0}) = \omega_T(\mathbf{1}) = 1$.

(2) Let $f, g \in L^X$. By the definition of ω_T ,

$\omega_T(f) = \vee\{ \alpha \in L : f$ is α-Scott continuous $\}$

$\omega_T(g) = \vee\{ \lambda \in L : g$ is λ-Scott continuous $\}$

$\omega_T(f \wedge g) = \vee\{ \gamma \in L : f \wedge g$ is γ-Scott continuous $\}$

Since L is completely distributive ,

$\omega_T(f) \wedge \omega_T(g) = \vee\{ \alpha \wedge \lambda \in L : f$ is α-Scott continuous , g is λ-Scott continuous $\}$.

From Lemma 9.2.4 (i) , we have

$\{ \alpha \wedge \lambda \in L : f$ is α-Scott cont. , g is λ-Scott cont .$\} \subseteq \{ \gamma \in L : f \wedge g$ is γ-Scott cont.$\}$.

Hence , we get $\omega_T(f \wedge g) \geq \omega_T(f) \wedge \omega_T(g)$.

(3) Let $\{ f_i : i \in J \} \subset L^X$. By the definition of ω_T ,

$\omega_T(f_i) = \vee\{ \alpha_i \in L : f_i$ is α_i-Scott continuous $\}$ for every $i \in J$.

$\omega_T(\vee_{i \in J} f_i) = \vee\{ \gamma \in L : \vee_{i \in J} f_i$ is γ-Scott continuous $\}$.

Since L is completely distributive , we have

$\wedge_{i\in J}\, \omega_T(f_i) = \wedge_{i\in J}(\vee\{ \alpha_i \in L : f_i \text{ is } \alpha_i\text{-Scott continuous}\})$

$= \vee\{\wedge_{i\in J}\alpha_i : f_i \text{ is } \alpha_i\text{-Scott continuous}\}$.

From Lemma 9.2.4 (ii), we have

$\{ \wedge_{i\in J}\alpha_i : f_i \text{ is } \alpha_i\text{-Scott continuous}\} \subseteq \{ \gamma\in L : \vee_{i\in J} f_i \text{ is } \gamma\text{-Scott continuous}\}$.

Hence, we get $\omega_T(\vee_{i\in J} f_i) \geq \wedge_{i\in J}\omega_T(f_i)$.

Consequently, the mapping ω_T is a smooth L-fuzzy topology on X.

In particular, when $L = I$ then the mapping $\omega_T : I^X \to I$ is defined by

$\omega_T(f) = \vee\{\alpha\in I : f \text{ is } \alpha\text{-lower semi-continuous}\}$ for every $f\in I^X$.

Definition 9.2.6 :

The smooth L-fuzzy topology ω_T obtained in Theorem 9.2.5 is called the **induced smooth L-fuzzy topology** and the smooth L-fts (X,ω_T) is called the **induced smooth L-fuzzy topological space**. Thus, a smooth L-fts (X,Υ) is an induced smooth L-fts if and only if there exists an ordinary topological space (X,T) such that $\omega_T = \Upsilon$.

Lemma 9.2.7 :

Let (X,T) be an ordinary topological space and $A\subset X$. Then, $A\in T$ if and only if $\omega_T(\chi_A) \neq 0$, where ω_T is the induced smooth L-fuzzy topology.

Proof :

Suppose that $A \in T$. Since the characteristic function of every open set is 1-Scott continuous, by the definition of ω_T, we have $\omega_T(\chi_A) = 1 \neq 0$.

Now assume that $\omega_T(\chi_A) \neq 0$. Then,

$\omega_T(\chi_A) = \bigvee\{ \alpha \in L : \chi_A$ is α-Scott continuous $\} \neq 0$ implies that there exists an $\alpha \in L$ such that $\alpha \neq 0$ and χ_A is α-Scott continuous. Hence, for every $p \in pr(L)$ with $\alpha \nleq p$, $\chi_A^{-1}(\{t \in L : t \nleq p\}) = A \in T$.

Theorem 9.2.8 :

Let (X,T) and (Y,T^*) be ordinary topological spaces and $F : X \to Y$ be a function. If $F : (X,T) \to (Y,T^*)$ is continuous, then $F : (X,\omega_T) \to (Y,\omega_{T^*})$ is smooth continuous.

Proof :

Suppose that $F : (X,T) \to (Y,T^*)$ is continuous. We want to prove that $\omega_T(F^{-1}(f)) \geq \omega_{T^*}(f)$ for every $f \in L^Y$.

By the definition of ω_T and ω_{T^*}, we have

$\omega_T(F^{-1}(f)) = \bigvee\{\alpha \in L : F^{-1}(f)$ is α-Scott continuous from (X,T) to $L\}$

$\omega_{T^*}(f) = \bigvee\{ \alpha \in L : f$ is α-Scott continuous from (Y,T^*) to $L \}$.

To prove that $\omega_T(F^{-1}(f)) \geq \omega_{T^*}(f)$, it is sufficient to show that

$\{ \alpha \in L : f$ is α-Scott continuous from (Y,T^*) to $L\} \subseteq \{\alpha \in L : F^{-1}(f)$ is α-Scott continuous from (X,T) to $L \}$. Let $f : (Y,T^*) \to L$ be an α-Scott continuous function. Then, $f^{1}(\{t\in L: t\nleq p\})\in T^{*}$ for every $p\in pr(L)$ with $\alpha\nleq p$.

$(F^{-1}(f))^{-1}(\{t\in L: t\nleq p\}) = \{x\in X : F^{-1}(f)(x)\nleq p\} = \{x\in X : f(F(x))\nleq p\}$

$= \{x\in X: F(x)\in f^{1}(\{t\in L: t\nleq p\})\} = \{x\in X: x\in F^{-1}(f^{1}(\{t\in L: t\nleq p\}))\} = F^{-1}(f^{1}(\{t\in L: t\nleq p\}))$.

Since $F: (X,T)\to(Y,T^{*})$ is continuous and $f^{1}(\{t\in L: t\nleq p\})\in T^{*}$ for every $p\in pr(L)$ with $\alpha\nleq p$, we have $F^{-1}(f^{1}(\{t\in L: t\nleq p\}))\in T$ for every $p\in pr(L)$ with $\alpha\nleq p$. Hence, $(F^{-1}(f))^{-1}(\{t\in L: t\nleq p\})\in T$ for every $p\in pr(L)$ with $\alpha\nleq p$ and therefore $F^{-1}(f):(X,T)\to L$ is α-Scott continuous. Thus, we have

$\{\alpha\in L$: f is α-Scott continuous from (Y,T^{*}) to $L\} \subseteq \{\alpha\in L$: $F^{-1}(f)$ is α-Scott continuous from (X,T) to $L\}$. This completes the proof.

Remark 9.2.9 :

SFT will denote the category of smooth L-fuzzy topological spaces and smooth continuous functions between them, **TOP** will denote the category of ordinary topological spaces and continuous functions between them.

Define ω : TOP $\to$ SFT by $\omega(X,T) = (X,\omega_{T})$ for every ordinary topological space (X,T), where ω_{T} is the induced smooth L-fuzzy topology. Theorem 9.2.8 ensures that if $F:(X,T)\to(Y,T^{*})$ is a morphism in TOP then $\omega(F) = F : \omega(X,T) = (X,\omega_{T})\to(Y,\omega_{T^{*}}) = \omega(Y,T^{*})$ is a morphism in SFT. Thus, we get the functor, ω, from TOP into SFT.

Remark 9.2.10 :

By Theorem 9.2.5 and Theorem 9.2.8, we obtain a smooth L-fts from a given ordinary topological space and the functor ω from the category of ordinary topological spaces TOP into the category of smooth L-fuzzy topological spaces SFT. This provides a '**goodness of extension**' criterion for smooth L-fuzzy topological properties which will be used in the next section. A smooth L-fuzzy extension of a topological property is said to be **good** when it is possessed by (X,ω_T) if and only if the original property is possessed by (X,T).

Lemma 9.2.11 :

Let (X,Υ) be a smooth L-fts. Then, $\Psi(\Upsilon) = \{ A \subset X : \Upsilon(\chi_A) = 1 \}$ is an ordinary topology on X.

Proof : **(1)** Since $\Upsilon(0) = \Upsilon(1) = 1$, $X \in \Psi(\Upsilon)$ and $\emptyset \in \Psi(\Upsilon)$.

(2) Let $A, B \in \Psi(\Upsilon)$. Then, $\Upsilon(\chi_A) = 1$ and $\Upsilon(\chi_B) = 1$.

Since $\Upsilon(\chi_{A\cap B}) = \Upsilon(\chi_A \wedge \chi_B) \geq \Upsilon(\chi_A) \wedge \Upsilon(\chi_B) = 1$, $\Upsilon(\chi_{A\cap B}) = 1$ and hence $A\cap B \in \Upsilon$.

(3) Let $\{ A_i : i \in J \} \subset \Psi(\Upsilon)$. Then, $\Upsilon(\chi_{A_i}) = 1$ for every $i \in J$.

Since $\Upsilon(\chi_{\cup A_i}) = \Upsilon(\vee_{i\in J}\chi_{A_i}) \geq \wedge_{i\in J}\Upsilon(\chi_{A_i}) = 1$, $\Upsilon(\chi_{\cup A_i}) = 1$ and hence $\cup_{i\in J} A_i \in \Psi(\Upsilon)$.

Consequently, $\Psi(\Upsilon)$ is an ordinary topology on X.

Proposition 9.2.12 :

Let (X,Υ) and (Y,Υ^*) be smooth L-fuzzy topological spaces and $F : X \to Y$ be a function. If $F : (X,\Upsilon) \to (Y,\Upsilon^*)$ is smooth continuous, then $F : (X,\Psi(\Upsilon)) \to (Y,\Psi(\Upsilon^*))$ is continuous.

Proof :

Suppose that $F : (X,\Upsilon) \to (Y,\Upsilon^*)$ is smooth continuous and $B \in \Psi(\Upsilon^*)$. Then, $\Upsilon^*(\chi_B) = 1$. Since $F : (X,\Upsilon) \to (Y,\Upsilon^*)$ is smooth continuous, we have $\Upsilon(\chi_{F^{-1}(B)}) = \Upsilon(F^{-1}(\chi_B)) \geq \Upsilon^*(\chi_B) = 1$ and hence $\Upsilon(\chi_{F^{-1}(B)}) = 1$. By the definition of $\Psi(\Upsilon)$, $F^{-1}(B) \in \Psi(\Upsilon)$. Therefore, $F : (X,\Psi(\Upsilon)) \to (Y,\Psi(\Upsilon^*))$ is continuous.

Remark 9.2.13 :

Define $\Psi : SFT \to TOP$ by $\Psi(X,\Upsilon) = (X,\Psi(\Upsilon))$ for every smooth L-fts (X,τ), where $\Psi(\Upsilon)$ is the ordinary topology obtained in Lemma 9.2.11. Proposition 9.2.12 ensures that if $F : (X,\Upsilon) \to (Y,\Upsilon^*)$ is a morphism in SFT then $\Psi(F) = F : \Psi(X,\Upsilon) = (X,\Psi(\Upsilon)) \to (Y,\Psi(\Upsilon^*)) = \Psi(Y,\Upsilon^*)$ is a morphism in TOP. Thus, we get the functor, Ψ, from SFT into TOP.

Now we have two functors; $\omega : TOP \to SFT$ and $\Psi : SFT \to TOP$. Consider the composition functor $\Psi o \omega : TOP \to TOP$. By Lemma 9.2.7, we have $(\Psi o \omega)(T) = \Psi(\omega_T) = \{ A \subset X : \omega_T(\chi_A) = 1 \} = \{ A \subset X : A \in T \} = T$ for every $(X,T) \in TOP$. Thus, the composition functor $\Psi o \omega$ is the identity functor.

Now we introduce Hausdorffness property in smooth L-fuzzy topological spaces.

Definition 9.2.14 :

A smooth L-fuzzy topological space (X,Υ) is called **smooth Hausdorff** if and only if for every $p,q \in pr(L)$ and every pair x, y of distinct elements of X, there exist $f, g \in L^X$ with $\Upsilon(f) \nleq p$, $\Upsilon(g) \nleq q$ and $f(x) \nleq p$, $g(y) \nleq q$ and $(\forall z \in X)$ $f(z) = 0$ or $g(z) = 0$.

This may also be expressed as :

there exist $f, g \in L^X$ with $\Upsilon(f) \not\leq p$, $\Upsilon(g) \not\leq q$ and $x_p \in f$, $y_q \in g$ and $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$.

When Υ is crisp , i.e. $\Upsilon : L^X \to \{0,1\} \subset L$, then this definition becomes :

(X,τ) is (smooth) Hausdorff if and only if for every $p,q \in pr(L)$ and every pair x , y of distinct elements of X , there exist $f, g \in L^X$ with $\Upsilon(f) = 1$, $\Upsilon(g) = 1$ and $f(x) \not\leq p$, $g(y) \not\leq q$ and $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$.

Thus , in the crisp case of Υ , smooth Hausdorffness coincides with the Hausdorffness introduced by Warner and McLean (Definition 2.2.3) . Hence , smooth Hausdorffness is a generalisation of Definition 2.2.3 to smooth L-fts's .

The next theorem shows that smooth Hausdorffness is a good extension of the Hausdorffness property of general topological spaces .

Theorem 9.2.15 (The goodness of smooth Hausdorffness) :

Let (X,T) be an ordinary topological space . Then , (X,T) is Hausdorff if and only if the induced smooth L-fuzzy topological space (X, ω_T) is smooth Hausdorff.

Proof :

Suppose that (X,T) is Hausdorff and let $x,y \in X$ with $x \neq y$. Then there exist $G,H \in T$ with $x \in G$, $y \in H$ and $G \cap H = \varnothing$. Let $p , q \in pr(L)$. Then $p \neq 1$ and $q \neq 1$. Since the characteristic function of every open set is 1-Scott continuous , we have $\omega_T(\chi_G) = 1 \nleq p$ and $\omega_T(\chi_H) = 1 \nleq q$. We also have $\chi_G(x) = 1 \nleq p$, $\chi_H(y) = 1 \nleq q$ and $(\forall z \in X)$ $\chi_G(z) = 0$ or $\chi_H(z) = 0$ because $G \cap H = \varnothing$. Thus, (X, ω_T) is smooth Hausdorff.

Now suppose that (X, ω_T) is smooth Hausdorff. Let $x,y \in X$ with $x \neq y$ and let $p,q \in pr(L)$. Then, there exist $f,g \in L^X$ with $\omega_T(f) \nleq p$, $\omega_T(g) \nleq q$ and $f(x) \nleq p$, $g(y) \nleq q$ and $(\forall z \in X)$ $f(z) = 0$ or $g(z) = 0$. Since $\omega_T(f) \nleq p$ and $\omega_T(g) \nleq q$, we have $f^{-1}(\{t \in L : t \nleq p\}) \in T$ and $g^{-1}(\{t \in L : t \nleq q\}) \in T$. We also have $x \in f^{-1}(\{t \in L : t \nleq p\})$, $y \in g^{-1}(\{t \in L : t \nleq q\})$ and $f^{-1}(\{t \in L : t \nleq p\}) \cap g^{-1}(\{t \in L : t \nleq q\}) = \varnothing$. Hence, (X,T) is Hausdorff.

Proposition 9.2.16 :

Let (X, Υ) be a smooth L-fts and $Y \subset X$. If (X, Υ) is smooth Hausdorff then the smooth subspace (Y, Υ_Y) is smooth Hausdorff as well.

Proof :

Suppose that (X, Υ) is smooth Hausdorff and let $x,y \in Y$ with $x \neq y$ and $p,q \in pr(L)$. From the smooth Hausdorffness of (X, Υ),

there exist $f,g \in L^X$ with $\Upsilon(f) \nleq p$, $\Upsilon(g) \nleq q$ and $f(x) \nleq p$, $g(y) \nleq q$ and $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$. We know that $\Upsilon_Y : L^Y \to L$ is defined by

$\Upsilon_Y(h) = \bigvee\{ \Upsilon(f) : f \in L^X \text{ and } f|_Y = h \}$ (Definition 9.1.5).

Now let $f^* = f|_Y$ and $g^* = g|_Y$. Then $f^*, g^* \in L^Y$ and by the definition of Υ_Y, $\Upsilon_Y(f^*) \nleq p$ and $\Upsilon_Y(g^*) \nleq q$. We also have $f^*(x) \nleq p$, $g^*(y) \nleq q$ and $(\forall z \in Y)$ $f^*(z)=0$ or $g^*(z) = 0$. Hence, (Y, Υ_Y) is smooth Hausdorff.

9.3. Smooth Compactness

In this section, we introduce the notion of compactness in smooth L-fuzzy topological spaces. We study some of its properties and prove that according to the goodness of extension criterion established in the previous section this concept is a good extension of the ordinary compactness in general topology.

Definition 9.3.1 :

Let (X,Υ) be a smooth L-fts and let $g \in L^X$. The L-fuzzy subset g is said to be **smooth compact** if and only if for every prime element p of L and every collection $(f_i)_{i\in J}$ of L-fuzzy sets with $\Upsilon(f_i) \nleq p$ for every $i \in J$ and $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there is a finite subset F of J such that $(\vee_{i\in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Particularly, when the whole space $X(=1)$ is smooth compact then (X,Υ) is called smooth compact smooth L-fuzzy topological space.

When Υ is crisp, i.e. $\Upsilon : L^X \to \{0,1\}$, then this definition becomes:

g is (smooth) compact if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i\in J}$ of L-fuzzy sets with $\Upsilon(f_i) = 1$ for every $i \in J$ and $(\vee_{i\in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there is a finite subset F of J such that $(\vee_{i\in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Thus, in the crisp case of Υ, smooth compactness coincides with the compactness introduced by Warner and McLean (Definition 3.1.2) and extended for arbitrary L-fuzzy sets by Kudri (Definition 3.1.3). That is, smooth compactness is a generalisation of Definition 3.1.2 and Definition 3.1.3 to smooth L-fuzzy topological spaces.

When $L = I$ the definition for whole space becomes:

A smooth I-fts (X,Υ) is smooth compact if and only if for every $\alpha\in[0,1)$ and every collection $(f_i)_{i\in J}$ of I-fuzzy sets with $\Upsilon(f_i)>\alpha$ for every $i\in J$ and $(\bigvee_{i\in J}f_i)(x)>\alpha$ for all $x\in X$, there is a finite subset F of J such that $(\bigvee_{i\in F}f_i)(x)>\alpha$ for all $x\in X$. In this case, if Υ is crisp, i.e. $\Upsilon : I^X \to \{0,1\}\subset I$, then the smooth compactness for smooth I-fts's coincides with Lowen's definition of strong compactness [51].

Theorem 9.3.2 :

Let (X,Υ) be a smooth L-fts and $g\in L^X$. The L-fuzzy set g is smooth compact if and only if for every $\alpha\in M(L)$ and every collection $(h_i)_{i\in J}$ of L-fuzzy sets with $\Phi(h_i)\not\leq\alpha'$ for every $i\in J$ and $(\bigwedge_{i\in J}h_i)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$, there is a finite subset F of J such that $(\bigwedge_{i\in F}h_i)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$, where Φ is the degree of closedness on X (Definition 9.1.4).

Proof : This follows immediately from the definition.

Lemma 9.3.3 :

Let (X,Υ) be a smooth L-fts and let $p\in pr(L)$. Then, the family

$$\{ f^{-1}(\{t\in L : t\not\leq p\} : f\in L^X \text{ with } \Upsilon(f)\not\leq p \}$$

forms a base for some ordinary topology on X .

Proof : This is trivial and therefore omitted .

Notation :

Let (X,Υ) be a smooth L-fts . For a prime element p of L , the ordinary topology obtained in the previous lemma will be denoted by Υ_p .

Definition 9.3.4 :

Let (X,Υ) be a smooth L-fts . For a prime element p of L , the ordinary topological space (X,Υ_p) will be called a **prime level space** of the smooth L-fuzzy topological space (X,Υ) .

The next theorem shows that smooth compactness is characterized in terms of ordinary compactness in prime level spaces .

Theorem 9.3.5 :

Let (X,Υ) be a smooth L-fts and $g\in L^X$. The L-fuzzy set g is smooth compact if and only if for every $p\in pr(L)$, $G_p = \{ x\in X : g(x)\geq p' \}$ is compact in the prime level space (X,Υ_p) .

Proof :

__Necessity__ : Let $p\in pr(L)$ and $(O_i)_{i\in J}$ be a basic open covering of G_p in (X,Υ_p). Then , $G_p \subset \bigcup_{i\in J} O_i$ and for each $i\in J$, there exists $f_i\in L^X$ with $\Upsilon(f_i)\nleq p$ such that $O_i = f_i^{-1}(\{t\in L: t\nleq p\}$. Hence , $(\bigvee_{i\in J} f_i)(x)\nleq p$ for all $x\in G_p$, i.e. for all $x\in X$ with

$g(x)\geq p'$. By the smooth compactness of g, there is a finite subset F of J such that $(\vee_{i\in F}f_i)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. Thus, we have $G_p \subset \cup_{i\in J} O_i$ and therefore G_p is compact in (X,Υ_p).

<u>**Sufficiency**</u>: Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a collection of L-fuzzy sets with $\Upsilon(f_i)\nleq p$ for every $i\in J$ and $(\vee_{i\in J}f_i)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. Then, $G_p\subset\cup_{i\in J}f_i^{-1}(\{t\in L:t\nleq p\}$ and so the family $\{ f_i^{-1}(\{t\in L:t\nleq p\} : i\in J$ and $f_i\in L^X$ with $\Upsilon(f_i)\nleq p \}$ is an open covering of G_p in (X,Υ_p). From the compactness of G_p, there is a finite subset F of J such that $G_p \subset \cup_{i\in F} f_i^{-1}(\{t\in L:t\nleq p\}$. Hence, $(\vee_{i\in F}f_i)(x)\nleq p$ for all $x\in G_p$, i.e. for all $x\in X$ with $g(x)\geq p'$. Consequently, g is smooth compact.

Corollary 9.3.6 :

A smooth L-fts (X,Υ) is smooth compact if and only if every prime level space of (X,Υ) is compact, i.e. for every $p\in pr(L)$, the ordinary topological space (X,Υ_p) is compact.

Proof : This follows from the previous theorem considering the whole space $X(=1)$ instead of g.

Lemma 9.3.7 :

Let (X,T) be an ordinary topological space. For every $p\in pr(L)$, $(\omega_T)_p = T$, where ω_T is the induced smooth L-fuzzy topology on X.

Proof :

Let $p \in pr(L)$. Take $A \in (\omega_T)_p$. By the definition of $(\omega_T)_p$, there is a family $\{ f_i^{-1}(\{t \in L : t \nleq p\} : i \in J \text{ and } f_i \in L^X \text{ with } \omega_T(f_i) \nleq p \}$ such that $A = \bigcup_{i \in J} f_i^{-1}(\{t \in L : t \nleq p\})$. Since $\omega_T(f_i) \nleq p$ for every $i \in J$, by the definition of ω_T, we have $f_i^{-1}(\{t \in L : t \nleq p\}) \in T$ for every $i \in J$. Hence, $A \in T$ and so $(\omega_T)_p \subset T$. Now take $A \in T$. Since the characteristic function of every open set is 1-Scott continuous, we have $\omega_T(\chi_A) = 1 \nleq p$. Hence, $A = \chi_A^{-1}(\{t \in L : t \nleq p\}) \in (\omega_T)_p$. Thus, we have $T \subset (\omega_T)_p$. Consequently, we have the desired equality.

The next theorem shows that smooth compactness is a good extension of the compactness in general topology.

Theorem 9.3.8 (The goodness of smooth compactness) :

Let (X,T) be an ordinary topological space. Then (X,T) is compact if and only if the induced smooth L-fts (X, ω_T) is smooth compact.

Proof :

By Corollary 9.3.6 and Lemma 9.3.7, we have that (X, ω_T) is smooth compact iff for every $p \in pr(L)$, the prime level space $(X, (\omega_T)_p) = (X,T)$ is compact.

Lemma 9.3.9 :

Let (X, Υ) be a smooth L-fts. Then, the family

$$\{ f^{-1}(\{t \in L : t \nleq p\} : p \in pr(L) \text{ and } f \in L^X \text{ with } \Upsilon(f) \nleq p \} \cup \{X\}$$

is a subbase for some ordinary topology, $\varepsilon(\Upsilon)$, on X.

Proof: This is straightforward.

Definition 9.3.10:

A smooth L-fts (X,Υ) is said to be **ultra-smooth compact** if and only if the ordinary topological space $(X,\varepsilon(\Upsilon))$ is compact.

It is clear that every ultra-smooth compact smooth L-fts is smooth compact.

Theorem 9.3.11 (The goodness of ultra-smooth compactness):

Let (X,T) be an ordinary topological space. Then (X,T) is compact if and only if the induced smooth L-fts (X, ω_T) is ultra-smooth compact.

Proof:

Suppose that (X,T) is compact. We need to prove that the ordinary topological space $(X,\varepsilon(\omega_T))$ is compact. Let $(A_i)_{i\in J}$ be a subbasic open cover of $(X,\varepsilon(\omega_T))$. Then each A_i is of the form $f_i^{-1}(\{t\in L:t\nleq p_i\}$ for some $p_i\in pr(L)$ and $f_i\in L^X$ with $\omega_T(f)\nleq p_i$. Hence, by the definition of ω_T, $f_i^{-1}(\{t\in L:t\nleq p_i\}\in T$ and therefore $(A_i)_{i\in J}$ is an open cover of (X,T). Since (X,T) is compact, $(A_i)_{i\in J}$ has a finite subcover. Hence, $(X,\varepsilon(\omega_T))$ is compact.

Now suppose that (X,ω_T) is ultra-smooth compact. Then (X,ω_T) is smooth compact. From the goodness of smooth compactness (Theorem 9.3.8), (X,T) is compact.

Proposition 9.3.12 :

Any smooth L-fuzzy topology on a finite set is smooth compact.

Proof : This follows immediately from Corollary 9.3.6 and the fact that every ordinary topology on a finite set is compact.

Proposition 9.3.13 :

In a smooth L-fts, every L-fuzzy set with finite support is smooth compact.

Proof :

Let (X,Υ) be a smooth L-fts and let g be an L-fuzzy set with finite support and $p \in pr(L)$. By Theorem 9.3.5, it is sufficient to prove that $G_p = \{x \in X: g(x) \geq p'\}$ is compact in the prime level space (X,Υ_p). Since the support of g is finite and $p' \neq 0$, the subset G_p of X is finite and therefore it is compact in (X,Υ_p).

Proposition 9.3.14 :

Let (X,Υ) be a smooth L-fts and let $g,h \in L^X$. If g and h are smooth compact, then $g \vee h$ is smooth compact as well.

Proof :

Let $p \in pr(L)$, $G_p = \{ x \in X : g(x) \geq p' \}$, $H_p = \{ x \in X : h(x) \geq p' \}$ and $K_p = \{x \in X : (g \vee h)(x) \geq p' \}$. Then, we have $K_p = G_p \cup H_p$ because p is prime. Since g and h are smooth compact, by Theorem 9.3.5, G_p and H_p are compact in the prime level space (X,Υ_p). Hence, $K_p = G_p \cup H_p$ is compact in (X,Υ_p). Thus, by Theorem 9.3.5, $g \vee h$ is smooth compact.

Proposition 9.3.15 :

Let (X,Υ) be a smooth L-fts and let $g,h \in L^X$. If g is smooth compact and $\Phi(h) = 1$, then $g \wedge h$ is smooth compact, where Φ is the degree of closedness on X (Definition 9.1.4) .

Proof :

Let $p \in pr(L)$, $G_p = \{x \in X: g(x) \geq p'\}$, $H_p = \{x \in X: h(x) \geq p'\}$ and $N_p = \{x \in X: (g \wedge h)(x) \geq p'\}$. From the smooth compactness of g, by Theorem 9.3.5, G_p is compact in (X,Υ_p). Now we are going to show that H_p is closed in (X,Υ_p). We have

$X \setminus H_p = \{ x \in X : h(x) \not\geq p' \} = \{ x \in X : h'(x) \not\leq p \} = (h')^{-1}(\{t \in L: t \not\leq p\})$.

Since $\Phi(h) = \Upsilon(h') = 1$, by the definition of Υ_p, $X \setminus H_p \in \Upsilon_p$ and hence H is closed in (X,Υ_p). Since G_p is compact and H_p is closed in the ordinary topological space (X,Υ_p), $N_p = G_p \cap H_p$ is compact in (X,Υ_p) and therefore $g \wedge h$ is smooth compact .

Corollary 9.3.16 :

Let (X,Υ) be a smooth L-fts . If $g \in L^X$ is smooth compact, then every L-fuzzy set h with $\Phi(h) = 1$ and $h \leq g$ is smooth compact .

Proof : This is an immediate consequence of the previous proposition .

Proposition 9.3.17 :

Let (X,Υ) and (Y,Υ^*) be smooth L-fuzzy topological spaces and $F : X \rightarrow Y$ be a function . If $F : (X,\Upsilon) \rightarrow (Y,\Upsilon^*)$ is smooth continuous, then for every $p \in pr(L)$, $F : (X,\tau_p) \rightarrow (Y,\tau_p^*)$ is continuous .

Proof :

Let $p \in pr(L)$ and let B be a basic open set in (Y,Υ_p^*). Then, there is an $f \in L^Y$ with $\Upsilon^*(f) \nleq p$ such that $B = f^{-1}(\{t \in L: t \nleq p\})$. Since F is smooth continuous, we have $\Upsilon(F^{-1}(f)) \geq \Upsilon^*(f)$ and hence $\Upsilon(F^{-1}(f)) \nleq p$ because $\Upsilon^*(f) \nleq p$. Moreover, we have the following

$F^{-1}(B) = \{ x \in X : f(F(x)) \nleq p \} = \{ x \in X : F^{-1}(f)(x)) \nleq p \} = (F^{-1}(f))^{-1}(\{t \in L: t \nleq p\})$.

So, $F^{-1}(B)$ is open in the prime level space (X,Υ_p). This means that $F:(X,\Upsilon_p) \rightarrow (Y,\Upsilon_p^*)$ is continuous.

Proposition 9.3.18 :

Let (X,Υ) and (Y,Υ^*) be smooth L-fuzzy topological spaces and $F:(X,\Upsilon) \rightarrow (Y,\Upsilon^*)$ be a smooth continuous surjection. If (X,Υ) is smooth compact, then (Y,Υ^*) is smooth compact as well.

Proof :

Let (X,Υ) be smooth compact and $p \in pr(L)$. Then, by Corollary 9.3.6, the prime level space (X,Υ_p) of (X,Υ) is compact. Since $F:(X,\Upsilon) \rightarrow (Y,\Upsilon^*)$ is smooth continuous surjection, by the previous proposition, $F:(X,\Upsilon_p) \rightarrow (Y,\Upsilon_p^*)$ is continuous surjection and so the prime level space (Y,Υ_p^*) of (Y,Υ^*) is compact and therefore, by Corollary 9.3.6, (Y,Υ^*) is smooth compact.

Definition 9.3.19 :

An element p of a complete lattice L is called **completely prime** if and only if for every family $(a_i)_{i\in J}$ in L with $\wedge_{i\in J} a_i \leq p$, there is some $i\in J$ such that $a_i \leq p$. It is evident that every completely prime element is prime.

Example 9.3.20 :

Let X be a set and let L be the power set of X. Then, L is a complete lattice and for an $x\in X$, $X\setminus\{x\}$ is a completely prime element in L.

Lemma 9.3.21 :

Let L be a fuzzy lattice in which 0 is completely prime and let (X,Υ) be a smooth L-fts. For an L-fuzzy set f, $\Upsilon(f) \neq 0$ if and only if for every $x_p \in pr(L^X)$ with $x_p \in f$, there exists $g\in L^X$ such that $\Upsilon(g) \neq 0$ and $x_p \in g \leq f$.

Proof : <u>Necessity</u> : This is obvious.

<u>Sufficiency</u>: By the hypothesis, we have $f = \vee\{ g\in L^X : \Upsilon(g) \neq 0 \text{ and } x_p \in g \leq f\}$. Hence, $\Upsilon(f) \geq \wedge\{ \Upsilon(g) : \Upsilon(g) \neq 0 \text{ and } x_p \in g \leq f \}$. Since 0 is completely prime, $\wedge\{\Upsilon(g) : \Upsilon(g) \neq 0 \text{ and } x_p \in g \leq f \} \neq 0$ and therefore $\Upsilon(f) \neq 0$.

Theorem 9.3.22 :

Let L be a fuzzy lattice in which 0 is completely prime, let (X,Υ) be a smooth L-fts and $F\subset X$. If (X,Υ) is smooth Hausdorff and χ_F is smooth compact, then the degree of closedness of χ_F is greater than 0, i.e. $\Phi(\chi_F) \neq 0$.

Proof :

To prove that $\Phi(\chi_F) = \Upsilon(\chi_F) \neq 0$, by the previous lemma, it is sufficient to show that for every $x_p \in pr(L^X)$ with $x_p \in \chi_F$, there exists $g \in L^X$ such that $\Upsilon(g) \neq 0$ and $x_p \in g \leq \chi_F$. Let $x_p \in pr(L^X)$ with $x_p \in \chi_F$. Then, $\chi_F(x) \nleq p$ and hence $x \in F'$. For all $y \in F$, by the smooth Hausdorffness of (X, Υ), there are $f_y, g_y \in L^X$ with $\Upsilon(f_y) \nleq p$ and $\Upsilon(g_y) \nleq p$ such that $x_p \in g_y$, $y_p \in f_y$ and $(\forall z \in X)$ $f_y(z) = 0$ or $g_y(z) = 0$. Thus, $(\bigvee_{y \in F} g_y)(z) \nleq p$ for every $z \in F$ and $\Upsilon(g_y) \nleq p$ for every $y \in F$. From the smooth compactness of χ_F, there are $g_{y_1}, g_y, \ldots, g_{y_n}$ such that $(\bigvee^n_{i=1} g_y)(z) \nleq p$ for every $z \in F$. Let $g = \bigwedge^n_{i=1} g_{y_i}$. Then, $\Upsilon(g) \geq \bigwedge^n_{i=1} \Upsilon(g_{y_i})$ and hence $\Upsilon(g) \nleq p$ because $\Upsilon(g_{y_i}) \nleq p$ for every $i \in \{1,2,\ldots,n\}$. We also have $x_p \in g$ and $g \leq \chi_F$. This completes the proof.

9.4. Smooth Countable Compactness and Smooth Lindelöfness

In this section we introduce and study countable compactness and Lindelöf property in smooth L-fuzzy topological spaces. We also prove that these notions are good extensions of the corresponding properties in general topology.

Definition 9.4.1 :

Let (X,Υ) be a smooth L-fts and let $g \in L^X$. The L-fuzzy subset g is said to be **smooth countably compact** if and only if for every prime element p of L and every countable collection $(f_i)_{i \in J}$ of L-fuzzy sets with $\Upsilon(f_i) \not\leq p$ for every $i \in J$ and $(\bigvee_{i \in J} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$, there is a finite subset F of J such that $(\bigvee_{i \in F} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

If g is the whole space, then we say that the smooth L-fts (X,Υ) is smooth countably compact.

When Υ is crisp, this definition reduces to the countable compactness in L-fuzzy topological spaces (Definition 3.2.2 (i)).

Definition 9.4.2 :

Let (X,Υ) be a smooth L-fts and let $g \in L^X$. The L-fuzzy subset g is said to be **smooth Lindelöf** if and only if for every prime element p of L and every collection $(f_i)_{i \in J}$ of L-fuzzy sets with $\Upsilon(f_i) \not\leq p$ for every $i \in J$ and $(\bigvee_{i \in J} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$, there is a countable subset N of J such that

$(\bigvee_{i\in N} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

If g is the whole space, then we say that the smooth L-fts (X,Υ) is smooth Lindelöf.

When Υ is crisp, this definition reduces to the Lindelöfness in L-fuzzy topological spaces (Definition 3.2.2 (ii)).

Theorem 9.4.3 (The goodness of smooth countable compactness):

Let (X,T) be an ordinary topological space. Then (X,T) is countably compact if and only if the induced smooth L-fts (X, ω_T) is smooth countably compact.

Proof :

Suppose that (X,T) is countably compact. Let $p \in pr(L)$ and $(f_i)_{i\in J}$ be a countable collection of L-fuzzy sets with $\omega_T(f_i) \not\leq p$ for every $i \in J$ and $(\bigvee_{i\in J} f_i)(x) \not\leq p$ for all $x \in X$. Then, $f_i^{-1}(\{t \in L : t \not\leq p_i\} \in T$ for every $i \in J$ and $X = \bigcup_{i\in J} f_i^{-1}(\{t \in L : t \not\leq p_i\}$. Hence, $(f_i^{-1}(\{t \in L : t \not\leq p_i\})_{i\in J}$ is a countable open cover of (X,T). From the countable compactness of (X,T), there is a finite subset F of J such that $X = \bigcup_{i\in F} f_i^{-1}(\{t \in L : t \not\leq p_i\}$. So, we have that $(\bigvee_{i\in F} f_i)(x) \not\leq p$ for all $x \in X$. Hence, (X, ω_T) is smooth countably compact.

Now suppose that (X, ω_T) is smooth countably compact. Let $p \in pr(L)$ and $(A_i)_{i\in J}$ be a countable open cover of (X,T). Then, $(\bigvee_{i\in J} \chi_{A_i})(x) = 1 \not\leq p$ for all $x \in X$. In addition, since each χ_A is 1-Scott continuous, $\omega_T(\chi_{A_i}) = 1 \not\leq p$ for every $i \in J$. From the smooth countable compactness of (X, ω_T), there is a finite subset F of J such that $(\bigvee_{i\in F} \chi_{A_i})(x) \not\leq p$ for all $x \in X$. Thus, $X = \bigcup_{i\in F} A_i$ and therefore (X,T) is countably compact.

Theorem 9.4.4 (The goodness of smooth Lindelöfness) :

Let (X,T) be an ordinary topological space. Then (X,T) is Lindelöf if and only if the induced smooth L-fts (X, ω_T) is smooth Lindelöf.

Proof : This is similar to the proof of Theorem 9.4.3 .

Theorem 9.4.5 :

Let (X,Υ) be a smooth L-fts and $g \in L^X$.

(i) The L-fuzzy set g is smooth countably compact if and only if for every $\alpha \in M(L)$ and every countable collection $(h_i)_{i\in J}$ of L-fuzzy sets with $\Phi(h_i) \not\leq \alpha'$ for every $i \in J$ and $(\bigwedge_{i\in J} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there is a finite subset F of J such that $(\bigwedge_{i\in F} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

(ii) The L-fuzzy set g is smooth Lindelöf if and only if for every $\alpha \in M(L)$ and every collection $(h_i)_{i\in J}$ of L-fuzzy sets with $\Phi(h_i) \not\leq \alpha'$ for every $i \in J$ and $(\bigwedge_{i\in J} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there is a countable subset N of J such that $(\bigwedge_{i\in N} h_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof : This follows easily from the definitions.

Theorem 9.4.6 :

Let (X,Υ) be a smooth L-fts and $g \in L^X$.

(i) The L-fuzzy set g is smooth countably compact if and only if for every $p \in pr(L)$, $G_p = \{ x \in X : g(x) \geq p' \}$ is countably compact in the prime level space (X,Υ_p).

(ii) The L-fuzzy set g is smooth Lindelöf if and only if for every $p \in pr(L)$, $G_p = \{ x \in X : g(x) \geq p' \}$ is Lindelöf in the prime level space (X, Υ_p).

Proof : These are very similar to the proof of Theorem 9.3.5 .

Corollary 9.4.7 :

(i) A smooth L-fts (X, Υ) is smooth countably compact if and only if every prime level space of (X, Υ) is countably compact .

(ii) A smooth L-fts (X, Υ) is smooth Lindelöf if and only if every prime level space of (X, Υ) is Lindelöf .

Proof : These follow from the previous theorem .

Proposition 9.4.8 :

Every smooth compact L-fuzzy set is both smooth countably compact and Lindelöf .

Proof : This follows directly from the definitions .

Proposition 9.4.9 :

Let (X, Υ) be a smooth L-fts and let $g \in L^X$ be smooth Lindelöf . Then , g is smooth countably compact if and only if g is smooth compact .

Proof : This follows easily from the definitions .

Proposition 9.4.10 :

Let (X, Υ) be a smooth L-fts and let $g, h \in L^X$. If g and h are smooth countably compact (smooth Lindelöf) , then $g \vee h$ is smooth countably compact (smooth Lindelöf) as well .

Proof : This is similar to the proof of Proposition 9.3.14 .

Proposition 9.4.11 :

Let (X,Υ) be a smooth L-fts and let $g,h \in L^X$. If g is smooth countably compact (smooth Lindelöf) and $\Phi(h) = 1$, then $g \wedge h$ is smooth countably compact (smooth Lindelöf) .

Proof : This is similar to the proof of Proposition 9.3.15 .

Proposition 9.4.12 :

Let (X,Υ) and (Y,Υ^*) be smooth L-fuzzy topological spaces and let $F:(X,\Upsilon) \to (Y,\Upsilon^*)$ be a smooth continuous surjection . If (X,Υ) is smooth countably compact (smooth Lindelöf), then (Y,Υ^*) is smooth countably compact (smooth Lindelöf) as well .

Proof : This is similar to the proof of Proposition 9.3.18 .

9.5. Smooth Local Compactness

In this section, we introduce and study smooth local compactness in smooth L-fuzzy topological spaces. We also prove that it is a good extension of the local compactness in general topology.

Definition 9.5.1 :

A smooth L-fts (X,Υ) is said to be **smooth locally compact** if and only if for every $x_p \in pr(L^X)$, there exist an $f \in L^X$ with $\Upsilon(f) \nleq p$ and an L-fuzzy set k of the form

$$k(z)=\begin{cases} e & \text{if } z \in K \subseteq X \\ 0 & \text{otherwise} \end{cases}$$

for some $e \in L$, such that χ_K is smooth compact and $x_p \in f \leq k$.

In particular, when Υ is crisp, i.e. $\Upsilon : L^X \to \{0,1\} \subset L$ then smooth local compactness reduces to the local compactness in L-fts's (see Definition 3.2.4). Clearly, every smooth compact smooth L-fts is smooth locally compact.

The next theorem shows that smooth local compactness is a good extension of the local compactness in general topology.

Theorem 9.5.2 (The goodness of smooth local compactness) :

Let (X,T) be a topological space. Then, (X,T) is locally compact if and only if the induced smooth L-fts (X,ω_T) is smooth locally compact.

Proof :

Necessity : Let $x \in X$ and $p \in pr(L)$. By the local compactness of (X,T), there are an $A \in T$ and a compact subset U in (X,T) such that $x \in A \subseteq U$. Then, $\omega_T(\chi_A) = 1 \nleq p$ and $\chi_A \leq \chi_U$. Let $k = \chi_U$ and $f = \chi_A$. Since U is compact in (X,T), by the goodness of smooth compactness, χ_U is smooth compact in (X, ω_T). We also have $x_p \in f \leq k$ and hence (X, ω_T) is smooth locally compact.

Sufficiency: Let $x \in X$ and $p \in pr(L)$. By the smooth local compactness of (X, ω_T), there are an $f \in L^X$ with $\omega_T(f) \nleq p$ and an L-fuzzy set k of the form

$$k(z) = \begin{cases} e \in L & \text{if } z \in K \subset X \\ 0 & \text{otherwise} \end{cases} \qquad \text{such that } \chi_K \text{ is smooth compact and } x_p \in f \leq k.$$

Since $\omega_T(f) \nleq p$, we have $A = f^{-1}(\{t \in L : t \nleq p\}) \in T$. Let $U = k^{-1}(\{t \in L : t \nleq p\}) = \{z \in X : k(z) \nleq p\}$. Then, $U = K$ if $e \nleq p$ and $U = \emptyset$ if $e \leq p$. Since χ_K is smooth compact in (X, ω_T), by the goodness of smooth compactness, K is compact in (X,T). Thus, in both case U is compact in (X,T). We also have $x \in A \subset U$ and hence (X,T) is locally compact.

Proposition 9.5.3 :

If a smooth L-fts (X,Υ) is smooth locally compact, then each prime level space of (X,Υ) is locally compact.

Proof :

Let (X,Υ) be smooth locally compact and let $p \in pr(L)$. We are going to show that the ordinary topological space (X,Υ_p) is locally compact. Take $x \in X$. From the smooth local compactness of (X,Υ), there are an $f\in L^X$ with $\Upsilon(f)\nleq p$ and an L-fuzzy set k of the form

$$k(z) = \begin{cases} e\in L & \text{if } z\in K\subseteq X \\ 0 & \text{otherwise} \end{cases} \qquad \text{such that } \chi_K \text{ is smooth compact and } x_p\in f\leq k.$$

Since $\Upsilon(f)\nleq p$, by the definition of (X,Υ_p), $A=f^{1}(\{t\in L:t\nleq p\})\in\Upsilon_p$. Since χ_K is smooth compact in (X,Υ), by Theorem 9.3.5, $\{ z\in X : \chi_K(z)\geq p' \} = K$ is compact in the prime level space (X,Υ_p). We also have $x\in A$ and $A\subset K$. Hence, (X,Υ_p) is locally compact.

Theorem 9.5.4 :

Let (X,Υ) and (Y,Υ^*) be smooth L-fuzzy topological spaces and $F:(X,\tau)\rightarrow(Y,\tau^*)$ be a both smooth open and smooth continuous surjection. If (X,Υ) is smooth locally compact, then (Y,Υ^*) is also smooth locally compact.

Proof :

Let $p\in pr(L)$ and $y\in Y$ with $F(x)=y$. By the smooth local compactness of (X,Υ), there exist an $f\in L^X$ with $\Upsilon(f)\nleq p$ and an L-fuzzy set k of the form

$$k(z) = \begin{cases} e\in L & \text{if } z\in K\subseteq X \\ 0 & \text{otherwise} \end{cases} \qquad \text{such that } \chi_K \text{ is smooth compact in } (X,\tau) \text{ and } x_p\in f\leq k.$$

Since F is smooth open and $\Upsilon(f)\nleq p$, we have $\Upsilon^*(F(f))\nleq p$. We also have

$$F(k)(u) = \begin{cases} e & \text{if } u \in F(K) \subseteq Y \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi_{F(K)} = F(\chi_K).$$

Since χ_K is smooth compact in (X,Υ) and F is smooth continuous, $\chi_{F(K)}$ is smooth compact in (Y,Υ^*). In addition, we have $y_p \in F(f) \leq F(k)$. Consequently, (Y,Υ^*) is smooth locally compact.

Theorem 9.5.5 :

Let (X,Υ) be a smooth L-fts and $Y \subset X$ with $\Phi(\chi_Y) = 1$. If (X,Υ) is smooth locally compact, then the smooth subspace (Y,Υ_Y) is smooth locally compact.

Proof :

Let $p \in pr(L)$ and $y \in Y$. From the smooth local compactness of (X,Υ), there exist an $f \in L^X$ with $\Upsilon(f) \not\leq p$ and an L-fuzzy set k of the form

$$k(z) = \begin{cases} e \in L & \text{if } z \in K \subseteq X \\ 0 & \text{otherwise} \end{cases} \quad \text{such that } \chi_K \text{ is smooth compact in } (X,\tau) \text{ and } y_p \in f \leq k.$$

Let $f_Y = f|_Y$ and $k_Y(z) = \begin{cases} e & \text{if } z \in K \cap Y \\ 0 & \text{if } z \in Y \setminus K \end{cases}$. Then, $y_p \in f_Y \leq k_Y$ and $\Upsilon_Y(f_Y) \not\leq p$.

In fact, since $\Upsilon(f) \not\leq p$, we have $\Upsilon_Y(f_Y) = \vee\{\Upsilon(g) : g \in L^X \text{ with } g|_Y = f_Y\} \not\leq p$. Moreover, since χ_K is smooth compact in (X,Υ) and $\Phi(\chi_Y) = 1$, by Proposition9.3.15, $\chi_{K\cap Y} = \chi_K \wedge \chi_Y$ is smooth compact in (X,Υ). Then, $\chi_{K\cap Y}$ is smooth compact in (Y,Υ_Y). Hence, (Y,Υ_Y) is smooth locally compact. ■

REFERENCES

[1] A.A. Allam and A.A. Zahran , On fuzzy δ-continuity and α-near compactness in fuzzy topological spaces , Fuzzy Sets and Systems 50(1992) 103-112.

[2] S.P. Arya and N. Singal , Some forms of continuity , Journal of Fuzzy Mathematics 2(3) (1994) 723-731 .

[3] K.K. Azad , On fuzzy semicontinuity , fuzzy almost continuity and fuzzy weakly continuity , J. Math. Anal. and Appl. 82 (1981) 14-32 .

[4] R.N. Bhaumik and A. Mukherjee , Completely lower semi continuous functions , Mathematics Education 26(2) (1992) 66-69 .

[5] R.N. Bhaumik and A. Mukherjee , Completely induced fuzzy topological spaces , Fuzzy Sets and Systems 47(1992) 387-390 .

[6] R.N. Bhaumik and A. Mukherjee, Some more results on completely induced fuzzy topological spaces , Fuzzy Sets and Systems 50(1992) 113-117 .

[7] R.N. Bhaumik and A. Mukherjee, On connections between properties of a topological space and its completely induced fuzzy space , Journal of Fuzzy Mathematics 1(3) (1993) 593-600 .

[8] A.S. Bin Shahna , On fuzzy strong semicontinuity and fuzzy precontiuity , Fuzzy Sets and Systems 44 (1991) 303-308 .

[9] G. Birkoff , Lattice Theory , Amer. Math. Soc. Colloq. Pub. , vol. 25, Providence , R.I. , 1984 .

[10] T.S. Blyth and M.F. Janowitz , Residuation Theory , Pergamon Press , Oxford , 1972 .

[11] N. Bourbaki , Elements of Mathematics , General Topology I , Addison-Wesley , Reading , MA , 1966 .

[12] F. Cammaroto and T. Noiri , A note on weakly-compact spaces , Indian J. Pure Appl. Math. 16(12) (1985) 1472-1477 .

[13] F. Cammaroto and T. Noiri , On R-compact Spaces , Matematik Vesnik 41 (1989) 141-147 .

[14] J.J. Chadwick , A generalised form of compactness in fuzzy topological spaces , J. Math. Anal. and Appl. 162 (1991) 92-110 .

[15] J.J. Chadwick , Relative compactness and compactness of general subsets in an I-topological spaces , Questiones Mathematicae 14 (1991) 491-507 .

[16] C.L. Chang , Fuzzy topological spaces , J. Math. Anal. and Appl. 24(1968) 182-190 .

[17] K.C. Chattopadhyay , R.N.Hazra and S.K. Samanta , Gradation of openness: fuzzy topology , Fuzzy Sets and Systems 49(1992) 237-242 .

[18] K.C. Chattopadhyay , S.K. Samanta , Fuzzy topology : Fuzzy closure operator , fuzzy compactness and fuzzy connectedness , Fuzzy Sets and Systems 54(1993) 207-212 .

[19] D. Çoker and A.H. Eş , On fuzzy S-closed spaces , Doğa Mat. 11(1987) 145-152 .

[20] D. Çoker and H. Eş , Weak compactness in fuzzy topological spaces, Hacettepe Bulletin of Natural Sciences and Engineering 16 (1987) 37-45.

[21] S. Dang and A. Behera, Fuzzy almost α-continuous functions, Journal of Fuzzy Mathematics 3(2) (1995) 291-300 .

[22] P. Das , Fuzzy semi-closure , Journal of Fuzzy Mathematics 2(3) (1994) 713-722 .

[23] P. Das and P. Das , Bifuzzy topology redefined , Journal of Fuzzy Mathematics 3(2) (1995) 331-336 .

[24] A. Di Concilio and G. Gerla , Almost compactness in fuzzy topological spaces , Fuzzy Sets and Systems 13(1984) 187-192 .

[25] R.F. Dickman , Jr. and J.R. Porter , θ-perfect and θ-absolutely closed functions , Illinois J. Math 21 (1977) 42-60 .

[26] M.K. El Gayyar , E.E. Kerre and A.A. Ramadan , Almost compactness and near compactness in smooth topological spaces , Fuzzy Sets and Systems 62(1992) 193-202 .

[27] A.H. Eş , Almost compactness and near compactness in fuzzy topological spaces , Fuzzy Sets and Systems 22(1987) 289-295.

[28] A.H. Eş and D. Çoker , On several types of degrees of fuzzy compactness in fuzzy topological spaces in Šostak's sense , Journal Fuzzy Mathematics 3(3) (1995) 481-491 .

[29] J. Fang , On completely induced fuzzy topological spaces , Journal of Fuzzy Mathematics 3(4) (1995) 851-861 .

[30] T.E. Gantner , R.C. Steinlage and R.H. Warren , Compactness in fuzzy topological spaces , J. Math. Anal. and Appl. 62(1978) 547-562 .

[31] G. Gerla , On fuzzy convergence , Information Sciences 39(1986) 269-284.

[32] G. Gierz et al. , A Compendium of Continuous Lattices , Springer-Verlag, Berlin Heidelberg New York , 1980 .

[33] J.A. Goguen , L-fuzzy sets , J. Math. Anal. and Appl. 18(1967) 145-174.

[34] J.A. Goguen , The fuzzy Tychonoff theorem , J. Math. Anal. and Appl. 18(1967) 145-174 .

[35] S. Gottwald , Fuzzy points and local properties of fuzzy topological spaces, Fuzzy Sets and Systems 5 (1981) 199-201 .

[36] R.N. Hazra , S.K. Samanta and K.C. Chattopadhyay , Fuzzy topology redefined , Fuzzy Sets and Systems 45(1992) 79-82 .

[37] B. Hutton , Products of fuzzy topological spaces , Topology and Its Applications 11(1980) 59-67 .

[38] P.T. Johnstone , Stone Spaces , Cambridge University Press , 1982 .

[39] S.R.T. Kudri , Compactness in L-fuzzy topological spaces , Fuzzy Sets and Systems 67(1994) 329-336 .

[40] S.R.T. Kudri and M.W. Warner , L-fuzzy local compactness , Fuzzy Sets and Systems 67(1994) 337-345 .

[41] S.R.T. Kudri , Paracompactness in L-fuzzy topological spaces , Fuzzy Sets and Systems 70(1995) 119-123 .

[42] S.R.T. Kudri , Countability in L-fuzzy topology , Fuzzy Sets and Systems 71(1995) 241-249 .

[43] S.R.T. Kudri and M.W. Warner , Some good L-fuzzy compactness-related concepts and their properties I , Fuzzy Sets and Systems 76(1995) 141-155.

[44] S.R.T. Kudri and M.W. Warner, Some good L-fuzzy compactness-related concepts and their properties II , Fuzzy Sets and Systems 76(1995) 168-177.

[45] S.R.T. Kudri , L-fuzzy compactness and related concepts . PhD Thesis: City University , Department of Mathematics , 1995 .

[46] S.R.T. Kudri and M.W. Warner , L-fuzzy weak local compactness , Fuzzy Sets and Systems 81(1996) 275-279 .

[47] S.R.T. Kudri , Semicompactness and S^*-closedness in L-fuzzy topological spaces (submitted to Fuzzy Sets and Systems) .

[48] Y. Liu and M. Luo, Induced spaces and fuzzy Stone-Čech compactifications, Scientia Sinica (Series A) 30(10) (1987) 1034-1044 .

[49] R. Lowen , Fuzzy topological spaces and fuzzy compactness , J. Math. Anal. and Appl. 56(1976) 621-633 .

[50] R. Lowen , Compact Hausdorff fuzzy topological spaces are topological , Topology and Its Applications 12(1981) 65-74 .

[51] R. Lowen , A comparison of different compactness notions in fuzzy topological spaces , J. Math. Anal. and Appl. 64(1978) 446-454 .

[52] R. Lowen and P. Wuyts , Concerning the constants in fuzzy topology , J. Math. Anal. and Appl. 129(1988) 256-268 .

[53] Luo Maokang , Lattice-valued semicontinuous mappings and induced topologies , Acta Mathematica Sinica , New Series 6(3) (1990) 193-205 .

[54] S.N. Maheshwari and S.S. Thakur , On α-compact spaces , Bulletin of The Institute of Mathematics Academia Sinica 13(4) (1985) .

[55] S.R. Malghan and S.S. Benchalli , Open maps , closed maps and local compactness in fuzzy topological spaces , J. Math. Anal. and Appl. 99(1984) 338-349 .

[56] H.W. Martin , Weakly induced fuzzy topological spaces , J. Math. Anal. and Appl. 78(1980) 634-639 .

[57] A.S. Mashhour , G.H. Ghanim and M.A. Fath Alla , α-separation axioms and α-compactness in fuzzy topological spaces , Rocky Mountain Journal Mathematics 16(2) (1986) 591-600 .

[58] R.G. McLean and M.W. Warner , Local theory and fuzzy topology , Fuzzy Sets and Systems 54(1993) 91-97 .

[59] G. Meng , Lowen's compactness in L-fuzzy topological spaces , Fuzzy Sets and Systems 53(1993) 329-335 .

[60] G. Meng , On the sum of L-fuzzy topological spaces , Fuzzy Sets and Systems 59(1993) 65-67.

[61] G. Meng and S. Chen , New characterizations of three kinds of compact sets in L-fuzzy topological spaces , Journal of Fuzzy Mathematics 3(3) (1995) 691-697 .

[62] Y. Mingsheng , A new approach for fuzzy topology (I) , Fuzzy Sets and Systems 39 (1991) 303-321 .

[63] M. Mrsevic , I.L. Reilly and M.K. Vamanamurthy , On semi-regularization topologies , J. Austral. Math. Soc. (Series A) 38(1985) 40-54 .

[64] M.N. Mukherjee and S.P. Sinha , Almost compact fuzzy topological spaces, Mat. Vesnik 41(1989) 89-97.

[65] M.N. Mukherjee and S.P. Sinha , On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces , Fuzzy Sets and Systems 32(1989) 103-114 .

[66] M.N. Mukherjee and S.P. Sinha , On some near-fuzzy continuous functions between fuzzy topological spaces , Fuzzy Sets and Systems 34 (1990) 245-254 .

[67] M.N. Mukherjee and B. Ghosh , On nearly compact and θ-rigid fuzzy sets in fuzzy topological spaces , Fuzzy Sets and Systems 43 (1991) 57-68 .

[68] M.N. Mukherjee and S. Malakar , On unification of some fuzzy covering axioms , Bulletin of The Institute of Mathematics , Academia Sinica 22(1) (1994) 75-81 .

[69] S. Nanda , On fuzzy topological spaces , Fuzzy Sets and Systems 19(1986) 193-197 .

[70] S. Nanda , Strongly compact fuzzy topological spaces , Fuzzy Sets and systems 42(1991) 259-262 .

[71] O. Njastad , On some classes of nearly open sets , Pacific Journal of Mathematics 15(3) (1965) 961-970 .

[72] J.H. Park and H.Y. Ha , Fuzzy weakly pre-irresolute and strongly pre-irresolute mapping , Journal of Fuzzy Mathematics 4(1) (1996) 131-140.

[73] Y. Peng , Some characterization theorems on lattice-valued induced spaces, Fuzzy Sets and Systems 55(1993) 209-217 .

[74] Z. Petricevic , Weak forms of continuity in fuzzy topology , Matematik Vesnik 42(1990) 35-44 .

[75] P.M. Pu and Y.M. Liu , Fuzzy Topology I ; Neighborhood structure of a fuzzy point and Moore-Smith convergence , J. Math. Anal. and Appl. 76(1980) 571-599 .

[76] P.M. Pu and Y.M. Liu , Fuzzy Topology II ; Product and quotent spaces, J. Math. Anal. and Appl. 77(1980) 20-37 .

[77] A.A. Ramadan , Smooth topological spaces , Fuzzy Sets and Systems 48(1992) 371-375 .

[78] S.E. Rodabaugh , Point-set lattice-theoretic topology , Fuzzy Sets and Systems 40(1991) 297-345 .

[79] R.D. Sarma and N. Ajmal , A category of fuzzy topological spaces , Journal of Fuzzy Mathematics 4(1) (1993) 849-864 .

[80] R.D. Sarma and N. Ajmal , Category N and functions with closed fuzzy graph , Fuzzy Sets and Systems 63(1994) 219-226 .

[81] R.D. Sarma and N. Ajmal , A new approach to fuzzy topological spaces and various fuzzy topological notions defined by nets , Fuzzy Sets and Systems 73(1995) 389-394 .

[82] M.K. Singal and N. Prakash , Fuzzy preopen sets and fuzzy preseparation axioms , Fuzzy Sets and Systems 44(1991) 273-281 .

[83] M.K. Singal and N. Rajvanshi , Regularly open sets in fuzzy topological spaces , Fuzzy Sets and Systems 50(1992) 343-353 .

[84] A.P. Šostak , On a fuzzy topological structure , Suppl. Rend. Circ. Mat. Palermo Ser. II 11 (1985) 89-103 .

[85] A.P. Šostak, Two decades of fuzzy topology : basic ideas, notions, and results, Russian Math. Surveys 44 : 6 (1989) 125-186 .

[86] A.P. Šostak, On some modifications of fuzzy topology, Mat. Vesnik 41(1989) 51-64 .

[87] R. Srivastava, S.N. Lal and A.K. Srivastava, Fuzzy T_1-topological spaces, J. Math. Anal. and Appl. 102(1984) 442-448 .

[88] R. Srivastava, On separation axioms in a newly defined fuzzy topology Fuzzy Sets and Systems 62(1994) 341-346 .

[89] L.A. Steen and J.A. Seebach, Jr., Counterexamples in Topology, Holt, Rinehart and Winston, Inc., 1970.

[90] R.C. Steinlage, Wallman compactification of L-fuzzy topological spaces, Proceedins of Sixth IFSA World Congress (1995) 459-462 .

[91] S.S. Thakur and R.K. Saraf, α-compact fuzzy topological spaces, Mathematica Bohemica 120(3) (1995) .

[92] G.J. Wang, A new fuzzy compactness defined by fuzzy nets, J. Math. Anal. and Appl. 94(1983) 1-23 .

[93] G.J. Wang, Generalised topological molecular lattices, Sci. Sinica 8(1984) 785-798 .

[94] G.J. Wang, Theory of topological molecular lattices, Fuzzy Sets and Systems 47(1992) 351-376 .

[95] Wang Geping and Hu Lanfang, On induced fuzzy topological spaces, J. Math. Anal. and Appl. 108(1985) 495-506 .

[96] M.W. Warner, Fuzzy topology with respect to continuous lattices, Fuzzy Sets and Systems 35(1990) 85-91 .

[97] M.W. Warner , Frame-fuzzy points and membership , Fuzzy Sets and Systems 42(1991) 335-344 .

[98] M.W. Warner and R.G. McLean , On compact Hausdorff L-fuzzy spaces, Fuzzy Sets and Systems 56(1993) 103-110 .

[99] M.W. Warner , On fuzzy compactness , Proceedings of Fifth IFSA World Congress (1993) 336-340 .

[100] M.W. Warner , On fuzzy compactness , Fuzzy Sets and Systems 80(1996) 15-22 .

[101] M.D. Weiss , Fixed points , separation , and induced topologies for fuzzy sets , J. Math. Anal. and Appl. 50(1975) 142-150 .

[102] C.K. Wong , Covering properties of fuzzy topological spaces , J. Math. Anal. and Appl. 43(1973) 697-704 .

[103] C.K. Wong , Fuzzy points and local properties of fuzzy topology , J. Math. Anal. and Appl. 46(1974) 316-328 .

[104] T.H. Yalvaç , Fuzzy sets and functions on fuzzy spaces , J. Math. Anal. and Appl. 126(1987) 409-423 .

[105] L.A. Zadeh , Fuzzy Sets , Information Control 8(1965) 338-353 .

[106] A.M. Zahran , Strongly compact and P-closed fuzzy topological spaces , Journal of Fuzzy Mathematics 3(1) (1995) 97-102 .

[107] A.M. Zahran , On some near fuzzy continuous functions on fuzzy topological spaces , Journal of Fuzzy Mathematics 3(1) (1995) 209-219 .

[108] D. Zhao , The N-compactness in L-fuzzy topological spaces , J. Math. Anal. and Appl. 128(1987) 64-79 .