ROBUST STABILISATION OF MULTIVARIABLE SYSTEMS: A SUPER-OPTIMISATION APPROACH

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Technical Comments

This text is typeset in $\LaTeX$ with a number of packages. For the style of the thesis, the usepackage *xthesis* was used and for the bibliography, the usepackage *amsalpha*. Further, any numerical analysis supported this work, was implemented in MATLAB 6.5 .

Abstract

The work aims to derive extended robust stability results for the case of unstructured uncertainty models of multivariable systems. More specifically, throughout the thesis, additive and coprime unstructured perturbation models are considered. Refined robust stabilisation problems of MIMO systems are defined and maximally robust controllers are synthesised in a state-space form. Unstructured perturbations which destabilise the feedback system for every optimal (maximally robust) controller are identified on the boundary of the optimal ball, i.e. the set of all admissible perturbations with norm equal to the maximum robust stability radius. Boundary perturbations are termed "uniformly destabilising" if they destabilise the closed-loop system for every optimal controller and it is shown that they all share a common characteristic, i.e. a projection of magnitude equal to the maximal robust stability radius, along a fixed direction defined by a pair of maximising vectors (scaled Schmidt pair) of a Hankel operator related to the problem. By imposing a directionality constraint it is shown that it is possible to increase the robust stability radius in every other direction by a subset of all optimal controllers.
In order to solve this problem, super-optimisation techniques are developed. Independently a natural extension of Hankel norm approximations, the so-called super-optimisation problem is posed and solved explicitly for the case of one-block problems in a state-space setting. It is thus shown that a subset of all maximally robust controllers, namely the class of super-optimal controllers, stabilises all perturbed plants within an extended stability radius $\mu^\star(\delta)$, subject to a directionality constraint.

In addition, the work is related to robust stabilisation subject to structured perturbations. The notions of structured robust stabilisation problem, and structured set approximation are defined in connection with the maximised set of permissible perturbations. It is further shown that $\mu^\star(\delta)$ can serve as an upper bound the structured robust stabilisation problem. The effect of $\mu^\star(\delta)$ as an upper bound depends on the compatibility between the two structures, the true structure and the artificial structure of the extended permissible set.

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Notation

$\mathcal{R}, \mathcal{C}, \mathcal{N}$	Sets of real, complex and natural numbers
$\mathcal{R}(s)$	field of rational functions in s with real coefficients
$\mathcal{R}[s]$	the set of real polynomials in the variable s
$\partial\mathcal{D}$	Boundary of set $\mathcal{D}$
sup, inf, max, min	supremum,infimum, maximum, minimum
$det(A)$, $trace(A)$,$rank(A)$	determinant, trace and rank of matrix A
$\rho(A)$	spectral radius of matrix A
$\lambda_i(A)$	i-th eigenvalue of A
$\overline{\lambda}(A)$ $(\overline{\sigma}(A))$	largest eigenvalue (singular value) of A
$\underline{\lambda}(A)$ $(\underline{\sigma}(A))$	smallest eigenvalue (singular value) of A
$In(A) = (\pi(A), \nu(A), \delta(A))$	Inertia of a complex matrix A is the ordered triple of the numbers of its positive, negative and zero eigenvalues.
A'	transpose of $A \in \mathcal{R}^{p \times m}$
A^*	complex conjugate transpose of $A \in \mathcal{C}^{p \times m}$ (or operator)
$G(s)^{\sim} = G(-\overline{s})^*$	the para-Hermitian conjugate of $G(s)$
$\mathcal{RL}_\infty$	Set of all proper real-rational matrix functions which are analytic on the imaginary axis
$\mathcal{RH}_\infty^+, \mathcal{RH}_\infty^-$	Sets of all proper real-rational matrix functions which are analytic in the closed RHP and closed LHP, repsectively
$\mathcal{RH}_\infty(k)$	Subset of $\mathcal{RL}_\infty$ consisting of all functions with no more than k poles in the RHP
$\mathcal{BH}_\infty$	$\mathcal{H}_\infty$-ball
Γ_G	Hankel operator associated with $G(s)$
$\sigma_i(\Gamma_G)$	i-th Hankel singular value of $G(s)$
$\mathcal{K}$	Family of all stabilising controllers
$\mathcal{S}$	Set of all stable closed-loop systems
$\mathcal{T}$	Set of all stable control sensitivity functions

Throughout this thesis matrix dynamical systems appear inside parenthesis so that they are distinguished from constant matrices which are denoted by square brackets.

Abbreviations

AAK	Adamyan, Arov and Krein
ARE	Algebraic Riccati Equation
CRSP	Constrained robust stabilisation problem
HNA	Hankel norm approximation
LFT	Linear fractional transformation
LMI	Linear matrix inequality
LTI	Linear time invariant
MIMO	Multiple input multiple output
MRSP	Maximal robust stabilisation problem
RCF (LCF)	Right (Left) Co-prime factorization
RHS (LHS)	Right (Left) hand side
SISO	Single input single output
SODP	Super-optimal distance problem
SSV	Structured singular value
SVD	Singular value decomposition

Introduction

Physical systems such as chemical processes, aerospace systems and power networks are observed to be in general non-linear, time-varying and highly complex. In the modelling process of large-scale multivariable systems, errors occur in the form of disturbances, inaccuracy of measurements, neglected or unmodelled dynamics, etc. In a simple design, the controller is synthesised so that it stabilises the mathematical model of the process, but obviously the model represents the real system only up to a certain degree of accuracy. Consequently, it is natural to assume that a simplified mathematical model which does not take into account all the above factors may be a poor indicator for controller design. Since a trade off arises between accuracy and simplicity of the model, the controller should work for the real system as well as for the model, i.e. it has to be robust against any errors introduced in the mathematical model. The objective of robust control is to take into account all modelling errors (inaccuracy of measurements, neglected or unmodelled dynamics, etc.) or disturbances, and design controllers which meet the required stability and performance criteria not only for one model but for a neighbourhood of models, inside which the real system is believed to lie.

In order to solve a robust control design problem various methods may be employed. Among them, the most mathematically sophisticated methods which search for optimal criteria are the $\mathcal{H}_\infty$, and Structured singular value (μ) design methods; the later being an extension of $\mathcal{H}_\infty$ optimal control. Further, other design methods are also used, depending on the application. The methods include Robust eigenvalue/eigenstructure assignment, Predictive control, Quantitative Feedback Theory (QFT), Fuzzy Logic, Multi-objective parameter tuning, etc. For a full discussion see [Gro97]. However, the most prominent and prolific method developed throughout the last decades has been proved to be $\mathcal{H}_\infty$-optimal control together with μ, which address systematically the effects of model uncertainty.

$\mathcal{H}_\infty$-optimal control is a frequency-domain optimisation method which was developed in response to the need for a synthesis procedure that explicitly addresses questions of modelling errors and unknown disturbances. $\mathcal{H}_\infty$ control is a natural extension to classical feedback theory for multivariable systems. The method's basic philosophy is to treat the worst case scenario, i.e. design a controller that stabilises the nominal plant for the worst-case perturbation that is likely to arise.

During the last four decades the $\mathcal{H}_\infty$ optimisation problem drew the attention initially of mathematicians and subsequently of control theorists, as it fitted well to the framework of engineering design. Various types of solution were developed arising from different fields of interest - some being mathematically elegant and others being more applicable computationally. Historically the problem was first solved using Nevanlinna-Pick algorithm an approach based on classical interpolation theory and complex analysis. In parallel the problem was formulated and solved in a more general setting, using the AAK theory [AAK71],[AAK78], which reduces the problem to a *general distance problem* [DC86],[Fra87],[GLD$^+$91]. The solution in the later formulation involves unitary dilations (see [Glo84] for the special case of one-block distance problem), which are tools adopted from operator theory and complex function theory. Other popular approaches are the J-spectral factorisation and the conjugate method of J-lossless factorisation [Kim97]; the later being related to interpolation theory. Furthermore, J-spectral factorisation was known to be related to LQ games and as consequence, $\mathcal{H}_\infty$ optimisation has been viewed and solved as a differential game [BB91] (a zero-sum game where the controller is treated as the minimising player and disturbance is the maximising player). Perhaps the most computationally tractable and theoretically fruitful method developed for solving $\mathcal{H}_\infty$ problems is the state-space approach by [DGKF89], which further led to the LMI formulation [DP00],[GNLC95] and its extensions to multi-objective optimisation, non-linear and time-varying control settings.

In connection to $\mathcal{H}_\infty$ optimisation problems, throughout this work, *super-optimisation* theory is considered. Essentially, this is an extension of $\mathcal{H}_\infty$ optimisation, as the objective of super-optimisation involves the minimisation of not only the largest singular value of the associated operator (which is the equivalent objective of $\mathcal{H}_\infty$ optimisation) but also of its subsequent singular values as well, in a hierarchical manner. The rationale behind this problem is to exploit all available degrees of freedom. Super-optimisation was first proposed and solved by Young [You86]. Although the initial motivation was esthetic rather practical, questions later arose on whether the methodology could also be applied to engineering problems. The problem was subsequently posed in a state-space framework by control theorists who developed algorithms for solving super-optimisation problems under various types of constraints [LHG87],[TGP88], [JL93]. The problem was also formulated and solved using polynomial methods, as reported in [KN89] and others. However, the most computationally powerful and elegant solution methodology is that developed in [JL93] were unitary dilations are considered. Although in this thesis the method is specialised to the one-block Nehari problem, it has been shown that super-optimal general distance problems can be addressed in this framework ([JL93]) together with the more general class of Hankel norm approximations ([HJ98]).

Throughout the thesis, a simple $\mathcal{H}_\infty$ problem is addressed, namely the robust stabilisation problem under the presence of various types of unstructured perturbation models [Glo86],[MG90]. Here we are interested to determine the largest possible region in uncertainty space guaranteed to be stabilised by a controller family, in terms of necessary and sufficient conditions. The first problem (MRSP) requires the solution of one-block Nehari approximation. This is subsequently extended to the multivariable case using a more refined direction-sensitive measure of robustness, using the theory of super-optimisation.

In the thesis we study the existing theory of robust stabilisation for LTI systems and develop a novel methodology which extend the known results using directionality information. The improved robust stability criteria derived here are based on the methods of [LCL$^+$84] and certain more recent generalisations. The work of [LCL$^+$84] goes back to 1984, i.e. to the early era of robust control. Throughout the eighties (and late seventies) the robust stabilisation problem was progressively linked to interpolation and approximation theories before taking its modern form. Due to the large impact of the approaches described in the seminal work of [Glo84] and [DC86] and other developments related to the structured singular value, little attention was given to the approach of [LCL$^+$84]. Essentially the main idea in [LCL$^+$84] which is followed in this work, is to improve the robustness tests by placing a weak restriction on the structure of the perturbation set. This restriction takes the form of a projection of the perturbation onto a subspace. In contrast to the methods of [LCL$^+$84], however, here the structural information, provided by the restriction, is used *a priori* (i.e. before a compensator is designed) and hence can be related directly to the directionality properties of robust stabilisation.

0.1 Summary of work

In this work an exposition of control theory related to $\mathcal{H}_\infty$- optimal design has been attempted. In particular, the problem of robust stabilisation under unstructured perturbation models is recast as one-block Nehari approximation problem, whose solution is studied in the first chapters of the thesis. The objectives of the present work are:

- To obtain necessary and sufficient conditions for robust stabilisation and characterise the set of all robustly stabilising controllers.

- To solve, explicitly, the maximally robust stabilisation problem under unstructured additive and coprime perturbations. As is shown in chapter 5, the solution of these problems, involves essentially a one-block Nehari optimal approximation, whose solution is described in chapter 4. The objective here is to use state-space analysis to reveal the underlying structure of the family of all optimally synthesised (maximally robust) controllers and the corresponding closed-loop systems.

- To derive necessary and sufficient conditions for extending the maximal robust stability radius under directionality information. Here a refined direction-sensitive measure of robust stabilisation appropriate for multivariable systems is introduced and optimised using super-optimisation theory.

- In order to derive stronger robust stability criteria as described in the previous objective, it is vital to solve the so-called super-optimal approximation problem. Here, a detailed and complete solution to the problem is developed using state-space techniques which removes all technical assumptions made in earlier approaches.

- To define proximity measures between different structured uncertainty sets in relation to the proposed methodology, and then extend the improved robust stabilisation results to the case of structured uncertainty models.

0.1.1 Contribution of thesis

In this section we summarise the main contribution of this work.

- The problem of super-optimisation has drawn the attention of control theorists and mathematicians for more than two decades. Here we present a computationally robust method for solving the one-block problem, using simple linear algebraic techniques. A state-space analysis is developed so that the structure of super-optimal decomposition becomes transparent and unnecessary ill-conditioning is avoided. The problem is solved under minimal possible assumptions.

- The results of this work show that a subclass of maximally robust controllers, namely the super-optimal controllers, guarantees robustness (in terms of stability) for a wider uncertainty set, i.e. they can stabilise additional perturbed plants compared to a general maximally robust controller, when the plant is subject to additive or coprime factor perturbations. The maximum permissible uncertainty set, characterised by a norm condition, consists of all perturbations lying inside the ball of maximal robust stability radius. It is shown that by imposing directionality constraints on the uncertainty set, super-optimal controllers guarantee the stabilisation of perturbations inside a set of a largest stability radius (in addition to perturbations guaranteed to be stabilised by optimal controllers). The extended robust stability radius is derived in closed-form as a function of a parameter δ which quantifies the directionality constraint.

- In many applications of robust control the choice of an appropriate model of uncertainty is an important issue. The formulation of coprime robust stabilisation problem removes some limitations of the additive and multiplicative perturbation models related to the number of RHP poles of the perturbed and nominal system. This motivates the generalisation of results (originally developed for the additive case) to this type of model. However, in contrast to the analysis for the additive case, our analysis of the coprime uncertainty model is carried out under the simplifying assumption that the largest Hankel singular value of the system constructed from the nominal coprime factors is simple. This is made for notational simplicity and may removed if required without serious technical difficulties.

- A practical problem faced by every designer in robust control is whether to model uncertainty in terms of its structure or as if it is not highly structured to avoid that by considering unstructured models. Hence, a trade-off appears between accuracy and conservatism which sometimes leads to either over-parameterised or moderate design. Here, we define an abstract approximation problem which aims to relate structured sets. Hence, it is shown that the methodology developed throughout this thesis can be used to approximate the robust stability radius of highly structured uncertainty sets by less structured sets.

0.2 Outline of thesis

In chapter 1 we outline main results of the mathematical framework that encompasses robust control theory. Aspects of functional analysis and operator theory are reviewed. The main theme of the chapter is the singular value decomposition of an operator on function spaces and its best approximation. Most of the material is covered in standard textbooks such as [Kre89], [Pow82],[Rud66],[Sut75],[You88] and [Pel03].

In a connection to the previous chapter, in chapter 2 we present the basic background theory of linear multivariable systems. The space of LTI systems is shown to be a normed vector space (in frequency and time domains) over which, under mild assumptions an algebra can be defined ($\mathcal{RH}_\infty$). Then the generalised regulator problem is addressed and important theory related to this work, is outlined. The main objective of this exposition is to define the framework of the stabilisation problem, studied in chapter 3.

The theory developed in chapter 3 involves stabilisation of LTI systems, i.e. necessary and sufficient conditions for the existence of stabilising controllers. The main result of this chapter is the "Youla" parametrisation of all stabilising controllers in terms of a free parameter in $\mathcal{H}_\infty$. The stabilisation problem is therefore recast as a convex optimisation problem via model-matching theory.

Hankel operators are defined within the rich mathematical theory reviewed in chapter 1. Over the three last decades Hankel operators have proved to be a major tool for robust control theory and $\mathcal{H}_\infty$ optimisation methods. In chapter 4, we study thoroughly the Hankel operator and its main properties (norm, singular values). Approximation-theory type distance problems which involve the computation of the Hankel norm and their relation to model reduction problems are also considered. In particular, a specific Nehari-type approximation problem as developed in [Glo84],[Glo89] is of special interest and hence an overview of these results is presented.

The main ideas of this work are developed in chapter 5. The theory behind the results of this chapter is mostly based in [GHJ00] and [Glo86]. An amalgamation of this theory with results included in previous chapters leads to the construction of a systematic procedure for solving the maximally robust stabilisation problem. A detailed and concrete state-space analysis illustrates the structure of the set of all optimal controllers.

Chapter 6 discusses the main points of the theory of super-optimisation, first introduced in [You86]. Super-optimisation is a natural extension of Hankel-norm approximations and hence this chapter is an extension of chapter 4. Here the one-block case is solved using the method developed for two and four block problems in [JL93] but with further state-space considerations. Utilising fully the structure of the one-block problem, the state-space solution to the problem is fully illuminated. All simplifying assumptions associated with the problem (e.g. multiplicity of the largest singular value, minimality, etc.) are removed. Finally the application of super-optimisation theory in the solution of the robust stabilisation problem in the matrix case is briefly discussed.

In Chapter 7 the extended stability criteria and radius for additive unstructured perturbations are derived in the framework of the solution of CRSP. It is shown that the set of (level-2) super-optimal controllers, a subclass of maximally robust controllers, offers improved robust stability properties. Indeed, the set of level-2 super-optimal controllers is precisely the class of controllers which guarantees robust stability inside the maximum extended uncertainty set. Further, the largest possible uncertainty set is explicitly characterised under directionality restrictions. Thereafter, its relation with other structures is briefly discussed.

The generalisation of results presented in chapter 7 results to the case of co-prime factor perturbations is the subject of chapter 8.

The main results and contribution of this work are summarised in the conclusions chapter. Further, novel research directions which aim to extend the existing theory are proposed.

Chapter 1

Mathematical Background of Robust Control

In this chapter the mathematical background is briefly introduced, along with the notation used in the thesis. Most of the material in this chapter is adapted from [Kre89], [Par04], [Pel03], [Rud66], [Sut75] and [You88].

1.1 Metric Spaces

The *Cartesian product* $X \times Y$ of two sets X and Y is defined as a set of all ordered pairs from X and Y, i.e. $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

Definition 1.1.1. *A metric space* (X, d) *is a set* X *together with a real valued function (metric)* $d : X \times X \to \mathcal{R}$ *satisfying*

(M1) $d(x, y) \geq 0$ for all $x, y \in X$ (non-negativity);

(M2) $d(x, y) = 0$ if and only if $x = y$ (non-degeneracy);

(M3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry);

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

Example 1.1.1. $X = \mathcal{R}$ equipped with $d(x, y) = |x - y|$ satisfies (M1)-(M4) (this is obvious from simple properties of the real line) and thus forms a metric space.

Example 1.1.2. $X = \mathcal{R}^n$ equipped with $d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$, for $1 \leq p < \infty$ form metric spaces. Here the triangle inequality (M4) is called Hölder's inequality.

Take for example the case $p = 1$, i.e. $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$. Then (M1)-(M3) are trivial and (M4) follows from the fact that,

$$\begin{aligned} |x_i - y_i| &\leq |x_i - z_i| + |z_i - y_i| \qquad \forall i = 1, 2, \ldots, n \\ \Rightarrow \sum_{i=1}^{n} |x_i - y_i| &\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| \end{aligned}$$

Further, by noticing that $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i| \leq \max_{1 \leq i \leq n} |x_i - z_i| + \max_{1 \leq i \leq n} |z_i - y_i|$ we conclude that $d_\infty(x,y) := \max_{1 \leq i \leq n} |x_i - y_i|$ defines another metric in $\mathcal{R}^n$.

Definition 1.1.2. (Induced metrics) If (X, d) is a metric space and $A \subset X$, then the restriction of d to $A \times A$ defines a metric on A. This restriction is defined by $d\mid_A (a,b) = d(a,b)$, $\forall a, b \in A$. So $d\mid_A : A \times A \to \mathcal{R}$ is a metric induced by d and $(A, d\mid_A)$ becomes a metric space.

Up to now all examples involved finite dimensional vector spaces (since x and y are finite dimensional vectors). Next, we define spaces of functions which are infinite dimensional.

Definition 1.1.3. (Spaces of bounded and continuous functions) Let $a < b$. Then we define

$$\mathbb{B}([a,b]) = \{f : [a,b] \to \mathcal{R} : f \text{ is bounded}\};$$

$$\mathbb{C}([a,b]) = \{f : [a,b] \to \mathcal{R} : f \text{ is continuous}\}.$$

Example 1.1.3. Take $f, g \in \mathbb{B}([a,b])$ and define the sup-metric, $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$ (i.e. the least upper bound). Then $(\mathbb{B}([a,b]), d)$, where d is the sup-norm, forms a metric space. Moreover, $\mathbb{C}([a,b]) \subset \mathbb{B}([a,b])$ and hence the sup-norm d induces a metric on $\mathbb{C}([a,b])$, which is also called the sup-metric but now $f, g \in \mathbb{C}([a,b])$. Note that on $\mathbb{C}([a,b])$ (but not on $\mathbb{B}([a,b])$) we may also define another metric by $d(f,g) = \int_a^b |f(x) - g(x)| dx$.

Remark 1.1.1. *If (X_1, ρ_1), (X_2, ρ_2) are metric spaces, there are several ways to define a metric on $X_1 \times X_2$, e.g. for $x = (x_1, x_2), y = (y_1, y_2)$, both in $X_1 \times X_2$,*

$$d_1(x,y) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2) \quad \text{or} \quad d_\infty(x,y) = \max\{\rho_1(x_1,y_1), \rho_2(x_2,y_2)\}$$

Definition 1.1.4. (Isometry) Let (X, d) and $(\widetilde{X}, \widetilde{d})$ be metric spaces. Then an isometry $f : X \to \widetilde{X}$ is a one-to-one correspondence such that

$$\widetilde{d}(f(x), f(y)) = d(x,y) \quad \forall\, x, y \in X$$

i.e. if the mapping f preserves distance.

Definition 1.1.5. *Suppose (X, d) is a metric space. Then a mapping $f : X \to X$ is called a contraction if there exists k, $0 \leq k \leq 1$, such that for every $x, y \in X$ we have $d(f(x), f(y)) \leq kd(x, y)$.*

1.2 Normed Spaces

Definition 1.2.1. *A normed vector space (sometimes called Pre-Banach) $(X, \|\cdot\|)$ is a vector space X equipped with a real valued function (norm) $\|\cdot\| : X \to \mathcal{R}$, satisfying*

(N1) $\|x\| \geq 0$ for all $x \in X$ (non-negativity);

(N2) $\|x\| = 0$ if and only if $x = 0$ (non-degeneracy);

(N3) $\|\alpha x\| = |\alpha|\|x\|$ (linearity);

(N4) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Here x and y are arbitrary vectors in X and α is an arbitrary scalar.

Remark 1.2.1. *It is easy to show that if $\|\cdot\|$ is a norm on a vector space then a function $d : X \times X \to \mathcal{R}$ defined by*

$$d(x, y) = \|x - y\| \geq 0$$

is a metric on X. It is obvious that conditions (M1) and (M2) hold for the above function, $d(x, y)$. Further,

$$d(y, x) = \|y - x\| = \|(-1)(x - y)\| = |-1|\|x - y\| = \|x - y\| = d(x, y)$$

and

$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

which prove that (M3) and (M4) are satisfied. In order to prove the latter inequality property (N4) was used.

Hence, all properties (N1)-(N4) are satisfied by the function d which is called the *canonical metric* induced by the given norm on the normed vector space X. Thus, all norms define metrics although not all metrics arise from norms, i.e. any normed vector space is a metric space.

1.2.1 Norms of finite-dimensional vectors and matrices

For vectors $x \in \mathcal{R}^n$ or $x \in \mathcal{C}^n$ the *Hölder* or p-norms are defined as follows

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, & 1 \leq p < \infty \\ \max_{1\leq i\leq n} |x_i|, & p = \infty \end{cases} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

An important property of the 2-norm is that it is invariant under unitary (orthogonal) transformations; e.g. if U is such that $UU^* = U^*U = I_n$ then it follows that $\|Ux\|_2^2 = x^*U^*Ux = x^*x = \|x\|_2^2$. Further, a useful relationship that holds for Hölder norms when $p = 1, 2, \infty$ is

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$$

Matrix norms are sometimes "induced" by vector norms. A matrix norm induced by the vector p-norms is defined for $A \in \mathcal{C}^{n\times m}$ as:

$$\|A\|_{p,q} := \sup_{x\neq 0} \frac{\|Ax\|_q}{\|x\|_p}$$

For the special cases where $p = q = 1$ or 2 or ∞ we have that

$$\|A\|_1 = \max_{1\leq i\leq n} \sum_{j=1}^m |A_{ij}|, \quad \|A\|_\infty = \max_{1\leq j\leq m} \sum_{i=1}^n |A_{ij}|, \quad \|A\|_2 = [\overline{\lambda}(AA^*)]^{1/2}$$

Besides induced matrix norms there exist other norms for matrices such as the *Schatten S_p-norms.* These non-induced norms are unitarily invariant. Let $\sigma_i(A)$, $1 \leq i \leq \min(m, n)$, be the singular values of A, i.e. the square roots of the eigenvalues of AA^*. Then

$$\|A\|_{S_p} := \left(\sum_{i=1}^m \sigma_i^p(A)\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

Considering the limit $p \to \infty$, we can also define

$$\|A\|_{S_\infty} := \sigma_{max}(A)$$

which is the same as the 2-induced norm of A. For $p = 1$ we obtain the *trace norm*

$$\|A\|_{S_1} = \sum_{i=1}^m \sigma_i(A)$$

and for $p = 2$ the resulting norm is also known as the *Frobenious norm* or the *Schatten 2-norm* or the *Hilbert-Schmidt norm* of A

$$\|A\|_F = \left(\sum_{i=1}^m \sigma_i^2(A)\right)^{\frac{1}{2}} = (Trace(AA^*))^{\frac{1}{2}} = (Trace(A^*A))^{\frac{1}{2}}$$

1.3 Inner product Spaces

The inner product is considered as a generalisation of the dot product $x \cdot y = |x||y| \cos\theta$, where x, y are real vectors (say e.g. in $\mathcal{R}^3$) and θ is the angle between them.

Definition 1.3.1. *An inner product space (or Pre-Hilbert space) $(X, \langle\cdot,\cdot\rangle)$ is a vector space X, over $\mathcal{R}$ or $\mathcal{C}$, together with a complex valued function (inner product) $\langle\cdot,\cdot\rangle : X \times X \to \mathcal{C}$ satisfying the following properties*

(I1) $\langle x, x\rangle > 0$ whenever $x \neq 0$;

(I2) $\langle x, y\rangle = \langle y, x\rangle^*$;

(I3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(I4) $\langle x + y, z\rangle \leq \langle x, z\rangle + \langle y, z\rangle$.

where α is a scalar and $x, y, z \in X$.

The inner product defined above induces a norm $\|x\| := \sqrt{\langle x, x\rangle}$ since all norm conditions (N1) to (N4) are satisfied. Further, it is a fact that in every vector space $(X, \langle\cdot,\cdot\rangle)$ the absolute value of the inner product of any two vectors $a, b \in X$ is less than or equal to the product of the norms of those two vectors, i.e.

Theorem 1.3.1 (Cauchy-Schwartz inequality). *Let $(X, \langle\cdot,\cdot\rangle)$ an inner product space. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\| \qquad \forall x, y \in X$$

where $\|x\| := \langle x, x\rangle^{1/2}$ and $\|y\| := \langle y, y\rangle^{1/2}$.

Proof. See [You88] or [Kre89]. □

Example 1.3.1. The space $\mathcal{R}^n$ with inner product defined by

$$\langle a, b\rangle := \sum_{i=1}^{n} a_i b_i \qquad \forall a, b \in \mathcal{R}^n$$

is an inner product space.

Example 1.3.2. $\mathcal{C}^{n\times m}$ with inner product defined by

$$\langle A, B\rangle := trace(A^* B) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^* b_{ij} \qquad \forall A, B \in \mathcal{C}^{n\times m}$$

is an inner product space.

1.4 Complete Spaces

Definition 1.4.1. (Cauchy sequences, completeness) A sequence $\{x_k\}$ in a normed space $(X, \|\cdot\|)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exist $N = N(\varepsilon) \geq 0$ such that

$$\|x_m - x_n\| \leq \varepsilon \qquad \forall\, m, n \geq N.$$

In other words, the above definition states that a sequence is *Cauchy* if it satisfies

$$\|x_m - x_n\| \overset{m,n\to\infty}{\longrightarrow} 0$$

A normed space is said to be complete if every Cauchy sequence in it converges; such spaces are called *Banach spaces.*

Definition 1.4.2. *A Banach space is a complete normed linear space (complete in the metric defined by the norm).*

Example 1.4.1. The simplest Banach spaces are the real line $(\mathcal{R})$ and the complex plane $(\mathcal{C})$ both equipped with the absolute value as a norm.

Definition 1.4.3. *A Hilbert space is a complete inner product space.*

Clearly Hilbert spaces are also Banach spaces; by definition, a Banach space where the norm can be derived from an inner product is a Hilbert space. Another fact that makes Hilbert spaces important is that the Euclidean space is a finite dimensional Hilbert space, which shows the geometric intuition offered by the Hilbert space. Thus, the notion of "orthogonality" carries over to Hilbert spaces:

Proposition 1.4.1. *Let X be a Hilbert space with a closed subspace $\mathcal{K}$. Then if $\mathcal{K}^\perp$ is the orthogonal complement of $\mathcal{K}$, i.e. $\mathcal{K}^\perp := \{x \in X : \langle x, k\rangle = 0, \ \forall\, k \in \mathcal{K}\}$, X has an orthogonal decomposition $X = \mathcal{K} \oplus \mathcal{K}^\perp$. Thus, any vector $x \in X$ decomposes uniquely as $x = k + k'$, for $k \in \mathcal{K}$ and $k' \in \mathcal{K}^\perp$. Further,*

$$\|x\|^2 = \|k\|^2 + \|k'\|^2$$

which extends Pythagoras theorem.

1.5 Isomorphism

Throughout the present work various spaces, depending on the assumptions made, are considered. However, what is common to all of them is that they consist of a set which is characterised by a *structure*. In the case of a metric space, the structure is clearly the metric, though when a vector space is considered, the structure of the space is described by the two algebraic operations, namely vector addition and scalar multiplication. Often the need for describing whether two spaces of the same kind are essentially identical or not, arises. The concept of isomorphism gives the answer when abstract spaces are involved. Roughly speaking, isomorphism is defined as a bijective mapping of a space X onto a space $\widetilde{X}$ (of the same kind) which preserves "structure".

Example 1.5.1. An isomorphism T of a vector space X onto a vector space $\widetilde{X}$ over the same field is a bijective mapping which preserves vector addition and scalar multiplication, i.e. for all $x, y \in X$ and scalar α

$$T(x+y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha T(x)$$

Remark 1.5.1. *Isomorphisms for normed spaces are vector space isomorphisms which also preserve norms.*

Definition 1.5.1. *If X is isomorphic with a subspace of a vector space Y then we say that X is embeddable in Y.*

1.6 Function Spaces

1.6.1 Lebesgue Integrable Spaces

For $1 \leq p < \infty$ we let $\mathcal{L}_p^{m\times n}(-\infty, \infty)$ denote the vector space of (Lebesgue) integrable matrix-valued functions mapping $\mathcal{R}$ to $\mathcal{C}^{m\times n}$, i.e.

$$\int_{-\infty}^{\infty} \|F(t)\|_{S_p}^p < \infty$$

The norm $\|\cdot\|_{S_p}$ used here is the usual Schatten p-norm.

When $p = 2$ we define the Hilbert space of matrix-valued, square (Lebesgue)-integrable functions on $\mathcal{R}$ with inner product

$$\langle F, G\rangle := \int_{-\infty}^{\infty} trace[F(t)^* G(t)]dt$$

The norm induced by this inner product is

$$\|F\|_{\mathcal{L}_2} = \left(\int_{-\infty}^{\infty} trace[F^*(t)F(t)]dt \right)^{\frac{1}{2}}$$

Similarly, we define by $\mathcal{L}_\infty^{m\times n}(-\infty,\infty)$ the set of all matrix valued functions $F:\mathcal{R}\rightarrow \mathcal{C}^{m\times n}$ such that $\|F(t)\|_{\mathcal{L}_\infty} < \infty$ where

$$\|F(t)\|_{\mathcal{L}_\infty} = \sup_{t\in\mathcal{R}} \|F(t)\|_{S_\infty} = \sup_{t\in\mathcal{R}} \sigma_{max}(F(t))$$

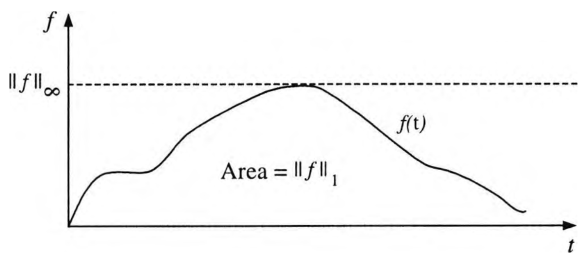

Figure 1.1: Geometric interpretation of $\mathcal{L}_p$ norms.

In general it is a fact that $\mathcal{L}_q \subseteq \mathcal{L}_p$, for $p > q$. The following simple example verifies this rule.

Example 1.6.1. *Consider the function* $f(t) = \frac{1}{\sqrt{1+t^2}}$ *for* $t \geq 0$ *and* $f(t) = 0$ *for* $t < 0$. *Then:*

$$\|f(t)\|_{\mathcal{L}_2} = \int_0^{\infty} f^2(t)dt = \int_0^{\infty} \frac{dt}{1+t^2} = [\tan(t)]_0^{\infty} = \frac{\pi}{2}$$

and

$$\|f(t)\|_\infty = \sup_{t\in\mathcal{R}} |f(t)| = 1$$

So, $f \in \mathcal{L}_2$ *and* $f \in \mathcal{L}_\infty$. *On the other hand consider the function* $g(t) = \frac{1}{\sqrt{1+|t|}}$. *Then,*

$$\|g(t)\|_{\mathcal{L}_2} = \int_{-\infty}^{\infty} g^2(t)dt = 2\int_0^{\infty} \frac{dt}{1+t} = 2[\ln(1+t)]_0^{\infty} = \infty$$

and

$$\|g(t)\|_\infty = \sup_{t\in\mathcal{R}} |g(t)| = 1$$

Hence, g *does not belong to* $\mathcal{L}_2$ *although* $g \in \mathcal{L}_\infty$.

Consider now, the *Fourier transform* of a function $f:\mathcal{R}\rightarrow\mathcal{C}^n$, $f\in\mathcal{L}_2(-\infty,\infty)$, defined as

$$\widehat{f}(j\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

and the *inverse fourier transform* of a function $\widehat{f} : j\mathcal{R} \to \mathcal{C}^n$ defined as

$$f(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(j\omega) e^{j\omega t} d\omega.$$

Then the following theorem links naturally the Lebesgue spaces, $\mathcal{L}_p(-\infty, \infty)$, with the restricted to the imaginary axis Lebesgue spaces, $\mathcal{L}_p(j\mathcal{R})$:

Theorem 1.6.1 (Plancherel). *Consider the functions as defined above, then*

1. *the map $\Phi : \mathcal{L}_2(-\infty, \infty) \to \mathcal{L}_2(j\mathcal{R})$, for any given $f, g \in \mathcal{L}_2(-\infty, \infty)$ defines an isometry, i.e. it preserves the inner product:*

$$\langle f, g \rangle_2 = \langle \Phi f, \Phi g \rangle_2$$

2. *the map $\Phi^{-1} : \mathcal{L}_2(j\mathcal{R}) \to \mathcal{L}_2(-\infty, \infty)$, for any given $\widehat{f}, \widehat{g} \in \mathcal{L}_2(j\mathcal{R})$ defines an isometry, i.e. it preserves the inner product:*

$$\langle \widehat{f}, \widehat{g} \rangle_2 = \langle \Phi^{-1} \widehat{f}, \Phi^{-1} \widehat{g} \rangle_2$$

Proof. see [DP00], [Rud66]. □

Further, Parseval's identity states that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(j\omega)\|_{S_2}^2 d\omega = \int_{-\infty}^{\infty} \|f(t)\|_{S_2}^2 dt$$

Hence, $\mathcal{L}_2(-\infty, \infty)$ and $\mathcal{L}_2(j\mathcal{R})$ are isometric spaces. Moreover, we say that $\mathcal{L}_\infty(-\infty, \infty)$ and $\mathcal{L}_\infty(j\mathcal{R})$ are isometric spaces since the norm of $\mathcal{L}_\infty(-\infty, \infty)$ is induced by norms from $\mathcal{L}_2(-\infty, \infty)$ to itself. This is a useful result which in the sequel will give us a stronger result, a way to connect Lebesgue spaces together with Hardy spaces.

1.6.2 Hardy Spaces

Suppose f is a complex function defined in $S \subset \mathcal{C}$. Then if for every $z_0 \in S$,

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (i.e. f is differentiable at z_0) we say that f is *analytic* (or *holomorphic*) in S. A matrix-valued function is analytic in S if every element of the matrix is analytic in S.

Definition 1.6.1. *The Hardy $\mathcal{H}_p^{m\times n} := \mathcal{H}_p^{m\times n}(\mathcal{C}_+)$ spaces are defined as the vector spaces of all $m \times n$ (matrix) complex-valued functions F which are analytic in the open right half complex plane ($\mathcal{C}_+$) equipped with norm*

$$\|F\|_{\mathcal{H}_p} := \left(\sup_{x<0} \int_{-\infty}^{\infty} \|F(s)\|_{S_p}^p \, dy\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty$$

$$\|F\|_{\mathcal{H}_\infty} := \sup_{s\in\mathcal{C}_+} \|F(s)\|_{S_p} < \infty , \qquad p = \infty$$

where $s = x + jy$, $x, y \in \mathcal{R}$ and $\|\cdot\|_{S_p}$ denotes the Schatten p-norm of F.

Of course, the above function F could be defined as a vector or even a scalar-valued function. If this is the case, then instead of using the *Schatten* class of norms inside the integrals we use *Hölder* norms. In the sequel, two important to our analysis $\mathcal{H}_p$ classes are discussed.

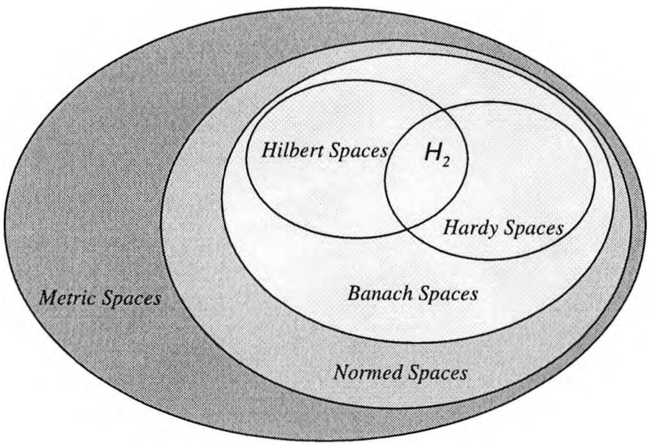

Figure 1.2: Spaces from a set theoretic point of view

Definition 1.6.2. *The Hardy $\mathcal{H}_2$ space is the space of all $m \times n$ complex-valued functions $F(s)$ defined as*

$$\mathcal{H}_2^{m\times n} := \{F(s) : \ F(s) \text{ is analytic in } \mathcal{C}_+, \ \|F\|_{\mathcal{H}_2} < \infty\}$$

where

$$\|F\|_{\mathcal{H}_2} = \left(\sup_{x>0} \int_{-\infty}^{\infty} \text{Trace}[F^*(s)F(s)]dy\right)^{\frac{1}{2}}$$

in which $s = x + jy$, $x, y \in \mathcal{R}$.

Definition 1.6.3. *The Hardy $\mathcal{H}_\infty$ is the space of all complex $m \times n$ functions $F(s)$*

$$\mathcal{H}_\infty^{m\times n} := \{F(s) : \ F(s) \text{ is analytic in } \mathcal{C}_+, \ \|F\|_{\mathcal{H}_\infty} < \infty\}$$

where

$$\|F\|_{\mathcal{H}_\infty} = \sup_{s\in\mathcal{C}_+} \sigma_{max}(F(s))$$

Theorem 1.6.2. (Maximum modulus) *A function f which is continuous inside a closed bounded set $D \subset \mathcal{C}$ as well as on its boundary ∂D and analytic inside D, attains its maximum on the boundary ∂D of D.*

Proof. see [Rud66]. □

According to the above theorem functions analytic in $\mathcal{C}_+$ and bounded over the $j\omega$-axis, attain their maximum on the $j\omega$-axis (i.e. its boundary), and hence

$$\|F\|_{\mathcal{H}_\infty} = \sup_{s\in\mathcal{C}_+} \sigma_{max}(F(s)) = \sup_{y\in\mathcal{R}} \sigma_{max}(F(jy)) = \|F\|_{\mathcal{L}_\infty}$$

and similarly,

$$\|F\|_{\mathcal{H}_2} = \|F\|_{\mathcal{L}_2}$$

Remark 1.6.1. *Heretofore, the capital letter denoting a Hardy or a Lebesgue norm, shall be omitted.*

1.7 Operator Theory

Suppose $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ and $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ are (real or complex) normed spaces. Then a linear bounded operator from a normed space $\mathcal{X}$ to a normed space $\mathcal{Y}$ is a linear mapping that satisfies

(Linearity) $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all $x_1, x_2 \in \mathcal{X}$ and $\alpha_1, \alpha_2 \in \mathbb{F}$;

(Boundedness) $\|Tx\|_\mathcal{Y} \leq k\|x\|_\mathcal{X}$ for any scalar $k > 0$ and for every $x \in \mathcal{X}$.

Here $\mathbb{F}$ denotes the field associated with vector space $\mathcal{X}$. If only the first condition holds then we call T a linear operator whereas if only the second condition is satisfied the operator is called bounded. However, throughout this work we consider the class of operators that satisfies both conditions and henceforth by operator we shall refer to a bounded linear mapping. Further, we define the *induced norm* of an operator T to be the least k such that the boundedness condition holds, that is

$$\|T\|_{\mathcal{X}\to\mathcal{Y}} := \sup_{x\in\mathcal{X}\,,\,x\neq 0} \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}$$

When the spaces $\mathcal{X}$ and $\mathcal{Y}$ are obvious from context, the induced norm of T will be simply denoted as $\|T\|$. Further, we shall denote the space of all bounded linear mappings from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and clearly $\mathcal{L}(\mathcal{X})$ will be the space of all bounded linear operators from $\mathcal{X}$ to itself. It can be verified that the induced norm defined above satisfies the properties of a norm (N1-N4).

Example 1.7.1. ***(Integration)*** *A linear operator T from $\mathbb{C}[a, b]$ into itself can be defined*

$$Tx(t) = \int_a^t x(\tau)d\tau \qquad t \in [a, b].$$

Example 1.7.2. ***(Multiplication)*** *Define another linear operator from $\mathbb{C}[a, b]$ into itself, by*

$$Tx(t) = tx(t).$$

Example 1.7.3. ***(Matrix)*** *A $m \times n$ matrix T defines an operator $T : \mathbb{F}^n \to \mathbb{F}^m$ by means of*

$$y = Tx, \qquad\qquad x \in \mathbb{F}^n,\ y \in \mathbb{F}^m$$

where $\mathbb{F}$ is a field.

Definition 1.7.1 ([Par04]). *The nonempty, compact subset of C, called the spectrum of an operator $T \in \mathcal{L}(\mathcal{X})$, where $\mathcal{X}$ is a Banach space, is defined as follows:*

$$spec(T) := \{\lambda \in C : T - \lambda I \text{ is not invertible}\}$$

Similarly, define the spectral radius of the operator T as

$$\rho(T) := \lim_{n\to\infty} \|T^n\|^{1/n} = \inf\{\|T^n\|^{1/n} : n \geq 1\}$$

In general, $\rho(T) \leq \|T\|$.

Definition 1.7.2 ([Par04]). *The adjoint of a bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$, between two Hilbert spaces $\mathcal{X}, \mathcal{Y}$ is defined by the equation:*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Clearly,

$$T^* : \mathcal{Y} \to \mathcal{X}$$

The following properties are well known. Consider operators T_1, T_2 and T_3 in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\alpha_1, \alpha_2 \in \mathcal{C}$. Then

1. $(\alpha_1 T_1 + \alpha_2 T_2)^* = \overline{\alpha}_1 T_1^* + \overline{\alpha}_2 T_2^*$

2. $(T^*)^* = T$

3. $\|T^*\| = \|T\|$

4. $(T_1 T_2)^* = T_1^* T_2^*$

Further, three important, in this content, classes of operators are:

1. the Hermitian or self adjoint operators T when $T = T^*$.

2. the Unitary operators T if $T^* = T^{-1}$, i.e. $TT^* = T^*T = I$

3. the Normal operators T if $T^*T = TT^*$.

Obviously, both Hermitian and unitary operators are normal.

Definition 1.7.3 (Maximising vectors [You86]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. A maximising vector for T is a non-zero vector $x \in \mathcal{X}$ such that*

$$\|Tx\| = \|T\|\|x\|$$

Thus a maximising vector for T is one at which T attains its norm.

1.7.1 Singular value decomposition of a matrix

Theorem 1.7.1. (SVD) *If $A \in \mathbb{F}^{m \times n}$, ($\mathbb{F} = \mathcal{R}$ or $\mathcal{C}$) then there exist unitary matrices*

$$U = \begin{bmatrix} u_1 & , & \dots & , & u_m \end{bmatrix} \in \mathbb{F}^{m \times m} \text{ and } V = \begin{bmatrix} v_1 & , & \dots & , & v_n \end{bmatrix} \in \mathbb{F}^{n \times n}$$

such that

$$A = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

where $r = rank(A)$. Further,

$$Im(A) = Im\left(\begin{bmatrix} u_1 & , & \dots & , & u_r \end{bmatrix}\right) \quad \text{and} \quad Ker(A) = Im\left(\begin{bmatrix} v_{r+1} & , & \dots & , & v_n \end{bmatrix}\right)$$

where $\Sigma_r = diag(\sigma_1, \ldots, \sigma_r)$, *with* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ *and* $r \leq min(m,n)$. *Here,* u_i *and* v_i *denote the i-th columns of matrices* U *and* V, *respectively.*

Proof. See [ZDG96]. □

Remark 1.7.1. From an operator theory point of view, the matrix A is considered as a linear map from the vector space $\mathbb{F}^n$ to the vector space $\mathbb{F}^m$. Keeping in mind the dyadic form of A and the fact that $v_i^* v_j = \delta_{ij}$ (since V is unitary) it follows that

$$Av_j = \left(\sum_{i=1}^{r} \sigma_i u_i v_i^* \right) v_j = \sigma_j u_j$$

So, v_j is mapped into $\sigma_j u_j$ by A. Moreover,

$$Av_j = \sigma_j u_j \Rightarrow A^* A v_j = \sigma_j^2 v_j \quad \text{and} \quad AA^* u_j = \sigma_j^2 u_j$$

which reveals that σ_j^2 is an eigenvalue of AA^* or A^*A, v_j is an eigenvector of A^*A and u_j is an eigenvector of AA^*.

Geometrically, the singular values of A are the principal lengths of a hyperellipsoid; in the case of two dimensions this is described in figure 1.3.

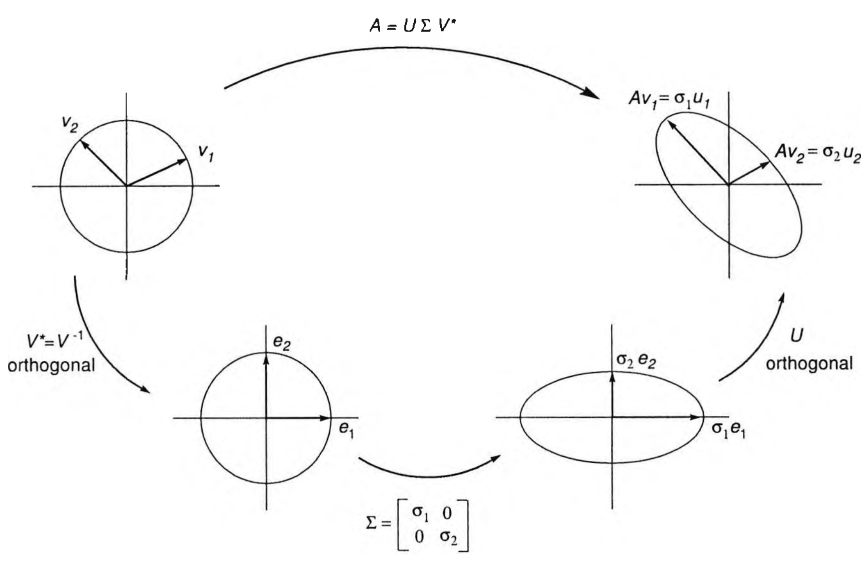

Figure 1.3: Singular values of A as a gain factor

1.7.2 The singular values of an operator

Definition 1.7.4. *([You88],[Pow82],[Pel03]) The singular values of an operator T between two Hilbert spaces, $\mathcal{X}$ and $\mathcal{Y}$, are defined as follows*

$$s_k(T) = \inf\{\|T - R\| : \ R \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \ \ rank(R) < k\} \qquad k \in \mathcal{N}/\{0\}$$

Clearly, $s_1(T) = \|T\|$. The numbers

$$s_1(T) \geq s_2(T) \geq \ldots \geq 0$$

are called the *s-numbers* or the singular values of T. Intuitively, $s_k(T)$ is the distance, with respect to the operator norm, of T from the set of operators of rank at most k in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$.

Remark 1.7.2. *In general, operators and in extension, their singular values are of infinite dimensions.*

The operator is said to be compact if and only if

$$\lim_{n \to \infty} s_n(T) = 0$$

If T is a compact operator from $\mathcal{X}$ to $\mathcal{Y}$, it admits a Schmidt expansion similar to that in Remark 1.5.1.

Definition 1.7.5 (Schmidt pair)). *([You88],[Pel03]) Let s be a singular value of an operator between two Hilbert spaces, i.e. $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then a Schmidt pair for T corresponding to s, is a pair (x, y) of non-zero vectors, with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, such that*

$$Tx = sy \quad and \quad T^*y = sx.$$

Obviously, a *singular vector* or *Schmidt vector* for T (where T is compact) corresponding to s is an eigenvector of T^*T corresponding to s^2. In particular, consider the subspaces

$$E_s^{(+)} = \{x \in \mathcal{X} : \ \ T^*Tx = s^2x\}, \ \ E_s^{(-)} = \{y \in \mathcal{Y} : \ \ TT^*y = s^2y\}$$

Vectors in $E_s^{(+)}$ are called *Schmidt vectors* of T and vectors in $E_s^{(-)}$ are called *Schmidt vectors* of T^*. Clearly, $x \in E_s^{(+)}$ if and only if $Tx \in E_s^{(-)}$, and we call the pair $\{x, y\}$ a Schmidt pair of T if it satisfies definition 1.7.5.

Corollary 1.7.1 (Schmidt Expansion). *([Pel03]) If T is a compact operator from a Hilbert space $\mathcal{X}$ to another Hilbert space $\mathcal{Y}$, it admits a Schmidt expansion*

$$Tx = \sum_{i \geq 0} s_i(T)\langle x, f_i\rangle g_i, \qquad x \in \mathcal{X}$$

where $\{f_i\}_{i\geq 0}$ is an orthonormal sequence in $\mathcal{X}$ and $\{g_i\}_{i\geq 0}$ is an orthonormal sequence in $\mathcal{Y}$.

Definition 1.7.6. *([You88])* **(Operator matrices)** *Let $T_{ij} \in \mathcal{L}(\mathcal{X}_j, \mathcal{Y}_i)$, $i, j = 1, 2$. The operator matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

is the operator from $\mathcal{X}_1 \oplus \mathcal{X}_2$ to $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} T_{11}x_1 + T_{12}x_2 \\ T_{21}x_1 + T_{22}x_2 \end{pmatrix}; \qquad x_i \in \mathcal{X}_i,\ i = 1, 2.$$

Operator matrices

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{Y}$$

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} : \mathcal{X} \to \mathcal{Y}_1 \oplus \mathcal{Y}_2$$

are defined analogously.

Remark 1.7.3. *In the case of an inner product space $\mathcal{V}$ if $\mathcal{X}$, $\mathcal{Y}$ are two subspaces of $\mathcal{V}$ such that for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have $\langle x, y\rangle = 0$, we say that $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal. Further, we say that $\mathcal{V}$ is the* orthogonal direct sum *of the two subspaces denoted as $\mathcal{X} \oplus \mathcal{Y}$, i.e. $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal and $\mathcal{X} + \mathcal{Y}$.*

Theorem 1.7.2 (Parrott's theorem). *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces with decompositions $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K}_1 \oplus \mathcal{K}_2$. Assume $T_{ij} : \mathcal{H}_j \to \mathcal{K}_i$, $(i, j = 1, 2)$, are bounded linear operators. Then, there exists an operator $\mathcal{Z} : \mathcal{K}_2 \to \mathcal{K}_2$ for which the operator*

$$Q_{\mathcal{Z}} := \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & \mathcal{Z} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2$$

is a contraction if and only if

$$\left\| \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} \right\| \leq 1 \quad \text{and} \quad \left\| \begin{pmatrix} T_{11} & T_{12} \end{pmatrix} \right\| \leq 1$$

Proof. See [Pel03]. □

1.8 Best approximation

The need to approximate a complicated function by a simpler function gives rise to approximation theory. Among the various methods, interpolation and best approximation are of the greatest interest as far as this work is concerned. The difference between best approximation and interpolation problems is that in best approximation there is no requirement for the approximation function to pass through the data values. Typically, best approximation problems with respect to norms which are not Hilbert space norms can hardly ever be solved explicitly.

Definition 1.8.1. (Best approximation) *Let $(\mathcal{X}, \|\cdot\|)$ be a normed space and suppose that any given $x \in \mathcal{X}$ is to be approximated by a $y \in \mathcal{Y}$, where $\mathcal{Y}$ is a fixed subspace of $\mathcal{X}$. Further, let δ denote the distance (the metric induced by the norm) from x to $\mathcal{Y}$. Then,*

$$\delta = \delta(x, \mathcal{Y}) = \inf_{y \in \mathcal{Y}} \|x - y\|$$

If there exists a $y_0 \in \mathcal{Y}$ such that

$$\|x - y_0\| = \delta$$

then y_0 is called a best approximation to x out of $\mathcal{Y}$.

Theorem 1.8.1 (Existence [Kre89].). *Let $(\mathcal{X}, \|\cdot\|)$ be a normed space with a finite dimensional subspace $\mathcal{Y}$. Then, for every $x \in \mathcal{X}$ there exists a best approximation to x out of $\mathcal{Y}$.*

Proposition 1.8.1 (Convexity [Kre89]). *In a normed space $(\mathcal{X}, \|\cdot\|)$ the set M of best approximations to a given point x from a subspace $\mathcal{Y}$ of $\mathcal{X}$ is convex.*

Definition 1.8.2 (Strict convexity). *A normed space with norm such that for all x, y of norm 1,*

$$\|x + y\| < 2$$

is called a strictly convex normed space.

Theorem 1.8.2 (Uniqueness [Kre89]). *In a strictly convex normed space $\mathcal{X}$ there is at most one best approximation to an $x \in \mathcal{X}$ out of a given subspace $\mathcal{Y}$.*

Proposition 1.8.2 ([Kre89]). *Hilbert space is strictly convex.*

Throughout this work we shall consider best approximation by analytic or meromorphic functions in the $\mathcal{L}_\infty$ norm. As discussed in the start of the paragraph, this is normally an untractable problem (since $\mathcal{L}_\infty$ is a Banach and not a Hilbert space), but under certain conditions it can be reduced to a mathematically tractable problem, which involves the rich theory of Hankel operators.

Chapter 2

The General $\mathcal{H}_\infty$ optimal control problem

In a SISO system the performance of the feedback loop depends on the variation of the loop gain over frequencies. However, extending this idea into the MIMO case is problematic since matrix systems do not have a unique gain; in fact, $\|G(s)u(s)\|$ depends on the direction of $u(s)$. Hence, a main difference between a scalar (SISO) system and a MIMO system is *directionality.* One possibility is to use eigenvalues to generalise the concept of gain; however these can only be computed for square systems and characterise system gain whenever the inputs and the outputs are in the same direction (eigenvector direction). Moreover, eigenvalues can be very sensitive to perturbations in the matrix elements. Hence, eigenvalues are a poor measure of gain. The singular value decomposition (SVD) provides a useful way of quantifying multivariable directionality, and we will see that most SISO results involving the absolute value (magnitude) may be generalised to the multivariable case by considering the maximum singular value of their transfer function evaluated over the imaginary axis (frequency response) [SP96],[Mac89],[FL88].

2.1 Signal and system spaces

In this context, a linear system G will be defined as a linear operator over a vector (linear) space; this is in turn defined over a field $\mathbb{F}$ (which for our purposes is either $\mathcal{R}$ or $\mathcal{C}$) and it is equipped with the usual operations of vector addition and scalar multiplication over that field. The linearity of G implies that:

$$\begin{cases} y_1 = G_1 u \\ y_2 = G_2 u \end{cases} \Rightarrow (G_1 + G_2)u = y_1 + y_2$$

$$y = Gu \Rightarrow (\lambda G)u = \lambda y$$

Definition 2.1.1 (Time-invariance). *[GL95] Let $y(t)$ be the response of a system G to input $u(t)$. If the response to the time-shifted input $u(t-T)$ is $y(t-T)$, the system is called time-invariant. Furthermore, if the system satisfies the linearity properties discussed above, then it is said to be linear time-invariant (LTI).*

By considering LTI systems we get concrete realisation of the input-output relation, i.e. if $g(t)$ is impulse response and $y(0) = 0$, then

$$y(t) = g(t) * u(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

where " $*$ " denotes the convolution operation.

Further, norms as measures, are vital for defining the notions of *stability* and *internal stability*. In that sense, we consider the class of all MIMO LTI systems (input-output mappings) $\mathcal{G}$ as a normed linear space which satisfies properties $(N1)$-$(N3)$ from chapter 1, definition 1.2.1. Since a linear system is a linear operator mapping elements from the input space to elements of the output space:

$$G : \mathcal{U} \to \mathcal{Y}$$

has induced norm given by

$$\|G\| = \sup_{\|u\|_{\mathcal{U}} \leq 1} \|Gu\|_{\mathcal{Y}}$$

which, in engineering terms, denotes the maximal possible gain whenever a nonzero input is applied. Thus, the system's norm is directly related with the type of input and output spaces considered as signal spaces. Throughout this work continuous-time systems are considered and hence signals are defined as functions in continuous time domain. Further, assuming that signals are square integrable with bounded energy, we define the input and output spaces to be *Lebesgue square integrable*, i.e. to belong to the class $\mathcal{L}_2(-\infty, \infty)$ (see chapter 1). Independently, if G is taken to be an operator in $\mathcal{L}_\infty$ then it is implied that it maps $\mathcal{L}_2(-\infty, \infty)$ input signals to $\mathcal{L}_2(-\infty, \infty)$ output signals (but not vice-versa). Further, recall from chapter 1, that the $\mathcal{L}_\infty$-norm can be written as $\mathcal{L}_2$-induced:

$$\|G\| = \|G(s)\|_{\mathcal{L}_\infty(j\mathcal{R})} = \sup_{\|u\|_{\mathcal{L}_2(-\infty,\infty)} \leq 1} \|Gu\|_{\mathcal{L}_2(-\infty,\infty)}$$

due to Parseval's theorem.

Remark 2.1.1. *The choice of these spaces is mainly due to the fact that they admit an "easy" way to calculate and optimise $\|G\|$ within a rich and well established mathematical framework which captures practical engineering issues and results to a reliable model of the physical (real) process.*

A pre requisite to any kind of feedback control design is stability. Hence, conditions of stability should be defined in this context. A system is called (input-output) *stable* if every bounded input signal produces a bounded output signal.

Definition 2.1.2 (External stability). *A system G is BIBO stable if*

$$\forall u \in \mathcal{L}_2[0,\infty) \Rightarrow y = Gu \in \mathcal{L}_2[0,\infty)$$

Hence, all LTI operators on $\mathcal{L}_2[0,\infty)$ are represented by functions in $\mathcal{H}_\infty$ [1].
Stability of LTI systems is described by analyticity of the transfer function in the right half plane of the complex domain. As this is defined in the frequency domain, we need to pose a time-domain analogue which preserves the norm (isometry). Thus, define the Hardy space $\mathcal{H}_\infty$ to be the space of all stable LTI systems, i.e. systems whose transfer function is analytic in the closed right-half plane, equipped with the following norm:

$$\|G\|_{\mathcal{H}_\infty} := \sup_{Re(s)>0} \overline{\sigma}[G(s)] = \sup_{\omega\in\mathcal{R}} \overline{\sigma}[G(j\omega)]$$

where the last equality is due to the maximum modulus theorem (see chapter 1). The subspace of all proper real-rational functions in $\mathcal{H}_\infty$ is denoted by $\mathcal{RH}_\infty$. Then if a system G is stable, $\|G\|_\infty$ is bounded.
In the same vein, signals in the Laplace domain belong to the space $\mathcal{H}_2$, if they are analytic in the open RHP, or $\mathcal{H}_2^\perp$ if they are analytic in the open LHP, respectively.
Frequently, a misunderstanding arises between the notions of stability and causality. In the following paragraph we distinguish clearly the two notions.

[1]Notice that this means that an LTI operator on $\mathcal{L}_2[0,\infty)$ is necessarily causal, a notion which we define later in the paragraph.

Definition 2.1.3 (Causality). *A system is called causal if the output up to time T depends only on the input up to time T, for every T.*

The Laplace transform formula together with its region of convergence (ROC) uniquely specify the time function and hence causality. The next example is constructed to support this argument.

Example 2.1.1. *Let $u(t)$ denote the one-sided step function $u(t) = 1$, for $t \geq 0$, and $u(t) = 0$ for $t < 0$. The SISO transfer function $\frac{1}{s+\alpha}$ is the Laplace transform of $e^{-\alpha t}u(t)$ and of $-e^{-\alpha t}u(-t)$. However, the regions of convergence for these time functions are different. When the causal exponential time function is considered, then the ROC of $g(s)$ is $Re(s) > -\alpha$. On the other hand, if we consider the anti-causal exponential time function, then the ROC of $g(s)$ is $Re(s) < -\alpha$.*

Remark 2.1.2. *For a causal LTI system, a necessary condition is that the region of convergence is to the right of the rightmost pole (in the s-domain) of the Laplace transform. Furthermore, a requirement for an LTI system to be causal and stable is that the region of convergence is to the right of the rightmost pole (in the s-domain) of the Laplace transform and all the poles are in the left-half plane.*

Remark 2.1.3. *The time domain analogue of $\mathcal{H}_\infty$ is $\mathcal{L}_\infty[0, \infty)$, which defines all causal systems. All anti-causal systems, i.e. systems whose impulse response lies in $\mathcal{L}_\infty(-\infty, 0]$ are isometric to $\mathcal{H}_\infty^-$, the set of all anti-stable LTI systems.*

Example 2.1.2 ([Kim97]). *Consider a transfer function*

$$G(s) = \frac{1}{s+\alpha}, \qquad \alpha > 0.$$

Then the infinity norm is calculated as:

$$\|G\|_\infty = \sup_\omega \frac{1}{\sqrt{\omega^2 + \alpha^2}} = \frac{1}{\alpha}$$

Example 2.1.3 ([Kim97]). *Consider a transfer function*

$$G(s) = \frac{s+\beta}{s+\alpha}, \qquad \alpha > 0.$$

Then,

$$\|G\|_\infty = \sup_\omega \sqrt{\frac{\omega^2 + \beta^2}{\omega^2 + \alpha^2}} = \left(\sup_\omega (1 - \frac{\alpha^2 - \beta^2}{\omega^2 + \alpha^2})\right)^{1/2} = \max\{1, \frac{|\beta|}{\alpha}\}$$

Along with the definition of a contractive operator given in chapter 1 (see def. 1.1.5 and section 1.7), contractive systems are defined:

Definition 2.1.4 (Contractive systems). *A system is called γ-contractive (or simply contractive if $\gamma = 1$) if its induced norm is less than or equal to γ, i.e. $||G|| \leq \gamma$. If $||G|| < 1$, the system is called strictly contractive.*

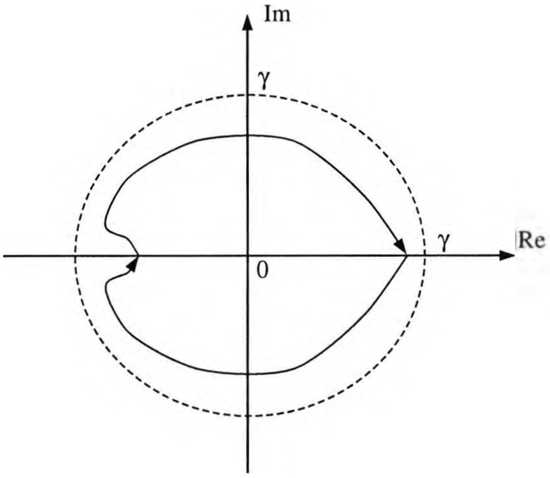

Figure 2.1: Nyquist diagram of a contractive system

Definition 2.1.5. *A system G is called γ-allpass if it satisfies*

$$GG^{\sim} = G^{\sim}G = \gamma^2 I$$

Hence, as a consequence, its induced norm $||G||$ is equal to γ.

From an engineering point of view, plotting the singular values of an all-pass system we see that they all have a constant value of 1, over all frequencies ($GG^{\sim} = I \Rightarrow \lambda_i(GG^{\sim}) = 1 \Rightarrow \sigma_i(G) = 1$).

2.2 State-space realisations of LTI systems

A linear transformation of a finite dimensional vector space into another finite dimensional vector space can be represented by means of different matrices, depending on the particular choice of bases in the vector spaces. Among all, there exist choices of bases which result in matrices of "standard" forms, called *canonical forms.* Such transformations preserve certain characteristics of the vector spaces (e.g. the rank of a matrix), called the invariants under the transformation. In control theory, however, motivation for such transformations usually arise from practical considerations. The study of an equivalence class, the so-called similarity transformations is of great interest in this particular work. In the sequel, standard results of such linear transformations are presented.

The realisation of a real-rational transfer-matrix $G(s) = C(sI - A)^{-1}B + D$ is written as

$$G \stackrel{s}{=} (A, B, C, D) \quad \text{or} \quad G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Proposition 2.2.1 (Equivalence). *Assume two LTI systems G_1 and G_2 have realisations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), respectively. Then G_1 and G_2 are said to be equivalent systems if and only if*

$$C_1 e^{A_1 t} B_1 = C_2 e^{A_2 t} B_2 \quad \text{and} \quad D_1 = D_2.$$

Proof. see [Ros70]. □

Definition 2.2.1. *A realisation (A, B, C, D) for a system G is called minimal if there does not exist a realisation for G with smaller state dimension. Then, the McMillan degree $deg(G)$ is equal to the dimension of the state vector in a minimal realisation of the system.*

Controllability and observability are two notions which play a very important role in the structural analysis of control systems. Controllability is concerned with the ability of "steering" the state $x(t)$ from an initial value $x(t_0)$ to the origin in finite time T, i.e. $x(T) = 0$, $T > t_0$ by means of an appropriate control $u(t)$, $t_0 \leq t \leq T$. Further, observability is concerned with the ability of determining in the "unforced" case $u(t) = 0$ (uniquely) the initial state $x(t_0)$ from knowledge of the system output $y(t)$, $t \in [t_0, T]$.

Proposition 2.2.2. *A system (A, B, C, D) with $deg(A) = n$, is controllable if and only if the controllability matrix*

$$\mathcal{C}_{AB} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has rank n. Further, the system is observable if and only if the observability matrix

$$\mathcal{O}_{CA} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank equal to n.

Proposition 2.2.3. *A system* G *admits a minimal realisation* (A, B, C, D) *if and only if* (A, C) *is observable and* (A, B) *is controllable.*

Proof. see [ZDG96]. ☐

Theorem 2.2.1 (Kalman canonical decomposition). *There always exists non-singular coordinate transformation* $\tilde{x} = Tx$ *so that every state-space representation* (A, B, C, D) *is equivalent to the following structure:*

$$\begin{bmatrix} \dot{\tilde{x}}_{co} \\ \dot{\tilde{x}}_{c\bar{o}} \\ \dot{\tilde{x}}_{\bar{c}o} \\ \dot{\tilde{x}}_{\overline{co}} \end{bmatrix} = \begin{bmatrix} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\bar{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\bar{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\overline{co}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{co} \\ \tilde{x}_{c\bar{o}} \\ \tilde{x}_{\bar{c}o} \\ \tilde{x}_{\overline{co}} \end{bmatrix} + \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \overline{C}_{co} & 0 & \overline{C}_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_{co} \\ \tilde{x}_{c\bar{o}} \\ \tilde{x}_{\bar{c}o} \\ \tilde{x}_{\overline{co}} \end{bmatrix} + Du$$

The state variables corresponding to the vector $\tilde{x}_{co}$ *are both controllable and observable,* $\tilde{x}_{c\bar{o}}$ *is controllable but unobservable,* $\tilde{x}_{\bar{c}o}$ *is observable but uncontrollable, and* $\tilde{x}_{\overline{co}}$ *is uncontrollable and unobservable.*

Proof. see [ZDG96]. ☐

Corollary 2.2.1. *Consider a realisation* (A, B, C, D) *corresponding to a system* G*. Then,*

1. *if the system is not controllable there exists an equivalent realisation*

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \qquad \tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \qquad (2.1)$$
$$\tilde{C} = CT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \qquad \tilde{D} = D$$

   in which $(\tilde{A}_{11}, \tilde{B}_1)$ *is controllable.*

2. *if the system is not observable there exists an equivalent realisation*

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \qquad \tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \qquad (2.2)$$
$$\tilde{C} = CT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \qquad \tilde{D} = D$$

in which $(\tilde{A}_{11}, \tilde{C}_1)$ *is observable.*

Definition 2.2.2. *The system* (A, B) *is denoted as stabilisable, if the matrix* $\tilde{A}_{22}$ *in the normal form (2.1) is stable.*

Definition 2.2.3. *The system* (A, C) *is denoted as detectable, if the matrix* $\tilde{A}_{22}$ *in the normal form (2.2) is stable.*

Proposition 2.2.4. *Controllability (stabilisability) and observability (detectability) remain invariant under similarity transformations.*

Proof. see [ZDG96]. □

Proposition 2.2.5. *The transfer function remains invariant under similarity transformations.*

Proof. see [ZDG96],[Ros70] □

Definition 2.2.4. *[MG90] Suppose* G *is a stable system with minimal state-space realisation* (A, B, C, D). *Then the associated controllability gramian,* W_c*, and observability gramian,* W_o*, are defined as:*

$$W_c := \int_0^\infty e^{At} BB' e^{A't} dt$$
$$W_o := \int_0^\infty e^{A't} C'C e^{At} dt$$

Further, these are the unique, positive definite solutions to the following Lyapunov equations:

$$AW_c + W_c A' + BB' = 0$$

$$A'W_o + W_o A + C'C = 0$$

respectively.

Proposition 2.2.6 (Balanced Realisation). *Suppose* W_c *and* W_o *satisfy the controllability and observability Lyapunov equations, respectively, of a realisation* (A, B, C, D) *of a stable system* G. *Then there exists an equivalent realisation* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *with gramians* $\tilde{W}_c$ *and* $\tilde{W}_o$*, under a similarity transformation* T *such that*

$$\tilde{W}_c := TW_cT' = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad \tilde{W}_o := (T^{-1})'W_oT^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ *are diagonal and positive definite.*

Further, if the original realisation is minimal then there exists a transformation such that $\tilde{W}_c = \tilde{W}_o = \Sigma$, *where* $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$. *Here, n is the McMillan degree of the system and* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ *are the Hankel singular values, defined by* $\sigma_i = \lambda_i^{\frac{1}{2}}(W_o W_c)$.

Proof. see [ZDG96]. □

Definition 2.2.5. *[ZDG96] Suppose* $G(s) \stackrel{s}{=} (A, B, C, D)$ *is minimal. Then the eigenvalues of A are called the poles of* $G(s)$.

Definition 2.2.6 (Multivariable zeros). *[BC85],[Kar01]*

1. *A complex number* $z_0 \in C$ *is called a system zero of the system realisation if the system matrix*

$$\begin{bmatrix} z_0 I - A & B \\ C & D \end{bmatrix} \tag{2.3}$$

is rank deficient. The system zeros are invariant under similarity transformations and constant linear feedback.

2. *Suppose that there exist complex numbers* z_{ci} *and* z_{oi} *which make the matrices*

$$\begin{bmatrix} z_{ci} I - A & -B \end{bmatrix}, \quad \begin{bmatrix} z_{oi} I - A \\ C \end{bmatrix}$$

rank deficient. Furthermore, suppose there exist numbers z_{coi} *for which both of these matrices are simultaneously rank deficient. All such numbers are called decoupling zeros and this set is subdivided into the sets of input-decoupling zeros* (z_{ci}), *output-decoupling zeros* (z_{co}) *and input-output decoupling zeros* (z_{coi}).

3. *The zeros of the transfer matrix* $G(s)$ *are called the transmission zeros and can be found from the Smith-McMillan form of* $G(s)$.

4. *The following identity holds:*

$$\{system\ zeros\} = \{transmission\ zeros\} \cup \{input\text{-}decoupling\ zeros\}$$
$$\cup \{output\text{-}decoupling\ zeros\} \text{ - } \{input\text{-}output\ decoupling\ zeros\}$$

Remark 2.2.1. *Systems with no decoupling zeros are said to be least order (minimal).*

Remark 2.2.2. *When the system matrix (2.3) is square and nonsingular, the zeros of the system are exactly the invariant zeros of the system.*

Remark 2.2.3. *Consider a system with realisation* (A, B, C, D)*. Then the input-decoupling zeros of* (A, B, C, D) *are uncontrollable eigenvalues of* A*, and the output-decoupling zeros are the unobservable eigenvalues of* A*. Note that in the case when* (A, B, C, D) *is controllable and observable, the zeros of the system, the invariant zeros, and the transmission zeros (zeros of the transfer matrix) all coincide.*

Heretofore, by *zeros* we shall referring to system zeros, unless stated otherwise.

2.3 Internal stability of feedback interconnections

Consider the feedback interconnection in figure 2.2 and define an extra set of three signals: $e_1 := v$, $e_2 := d + u$ and $e_3 := y_m := y + n$.

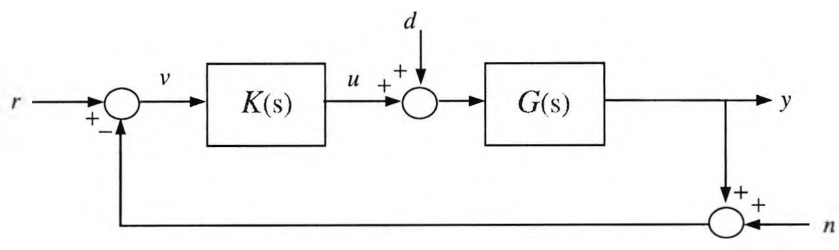

Figure 2.2: General feedback arrangement

Grouping the terms together:

$$\begin{pmatrix} I & 0 & I \\ -K & I & 0 \\ 0 & -G & I \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

Definition 2.3.1 (Well posedness). *The feedback system of figure 2.2 is well-posed if and only if the* 3×3 *matrix, above, is nonsingular for* $s = j\infty$*.*

The transfer matrix defining the input-output map is then given as:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} I & 0 & I \\ -K & I & 0 \\ 0 & -G & I \end{pmatrix}^{-1} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

Remark 2.3.1. *If the transfer functions of G and K are proper, well-posedness implies that the nine transfer functions from $(r, n, d) \to (e_1, e_2, e_3)$ exist and are proper.*

Definition 2.3.2. *The feedback system given in figure 2.2 is called internally stable if each of the nine transfer functions from $(r, n, d) \to (e_1, e_2, e_3)$ is stable.*

Remark 2.3.2 (External vs Internal stability). *It should be noted here that internal stability is a stronger requirement than (external) input-output stability as defined at the start of this chapter (definition 2.1.2), since it also takes into account the potential RHP pole-zero cancellations.*

The following example considers the SISO case of a feedback loop which is input-output stable but not internally stable.

Example 2.3.1. *[GL95] Consider the SISO plant and controller*

$$g(s) = \frac{-s}{s+1}, \quad k(s) = \frac{s+3}{s}$$

It can be observed that an unstable pole-zero cancellation in $Re(s) \geq 0$ occurs when the product $g(s)k(s)$ is formed. Further, the transfer function from r to e_1 is $(1 - g(s)k(s))^{-1} = \frac{s+1}{2(s+2)}$, which is stable. However, the closed-loop transfer function from r to e_2 is $k(s)(1 - g(s)k(s))^{-1} = \frac{(s+1)(s+3)}{2s(s+2)}$, which is unstable due to the closed-loop pole at the origin. Therefore, the feedback loop is not internally stable for this particular plant and controller.

Internal stability becomes a more complicated property when considering MIMO plants. For example, consider the transfer matrix, $H(s) = \begin{pmatrix} \frac{s+1}{s+2} & 0 \\ 0 & \frac{s+2}{s+1} \end{pmatrix}$ which has poles and zeros at the same location in $\mathcal{C}$, but not in the same input-output "channel". From a balanced realisation of $H(s)$, it may then shown that it has modes $\lambda = -1$ and $\lambda = -2$ which imply that the realisation is both controllable and observable and therefore minimal. Moreover, a loop having $H(s)$ as the closed-loop transfer function is internally stable.

Example 2.3.2. *Consider*

$$G(s) = \begin{pmatrix} \frac{1}{s} & \frac{1}{s} \\ \frac{2}{s+1} & \frac{1}{s} \end{pmatrix}, \quad K(s) = \begin{pmatrix} 1 & \frac{2s}{s-1} \\ 0 & -\frac{2s}{s-1} \end{pmatrix}$$

Then the unstable pole $s = 1$ of the controller does not appear in the product GK,

$$G(s)K(s) = \begin{pmatrix} \frac{1}{s} & 0 \\ \frac{2}{s+1} & \frac{2}{s+1} \end{pmatrix}$$

Further, one of the feedback-loop transfer matrices

$$K(I + GK)^{-1} = \begin{pmatrix} * & * \\ -\frac{4s^2}{(s-1)(s+1)(s+3)} & * \end{pmatrix}$$

has unstable poles and thus the feedback system is not internally stable.

Theorem 2.3.1 (Internal Stability). *[DC86] Consider a minimal realisation*

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Then there exists a proper real-rational transfer matrix K achieving internal stability of the feedback configuration of figure 2.2, if and only if the pair (A, B_2) is stabilisable and the pair (A, C_2) is detectable.

Proof. see [DC86]. □

In the sequel, a standard test for feedback loop stability, namely the Nyquist criterion for multivariable systems is discussed. Another important theorem which gives necessary and sufficient conditions for closed-loop stability is the small gain theorem which will be discussed in the following paragraph, where the standard $\mathcal{H}_\infty$ control problem is considered.

Theorem 2.3.2 (Nyquist stability). *[Ant01] Consider figure 2.2 where $d = n = 0$. Then given the square MIMO system $G(s)K(s)$, let Γ be the Nyquist plot of $det(I + G(s)K(s))$ [2]. Assuming that Γ does not pass through the origin, the number of unstable closed loop poles of the unity-feedback configuration in figure 2.2 is equal to the sum of*

- *the number of times Γ encircles the origin in a clockwise direction, plus*
- *the number of unstable open-loop poles of $G(s)K(s)$.*

[2] *In the case of positive feedback, i.e. in figure 2.3, we use the Nyquist plot of $det(I - G(s)K(s))$*

Obviously, the number of clockwise encirclements of the origin may be negative (due to unstable pole-zero cancellations).

As a result, the closed-loop system is stable if and only if, the number of unstable poles of $G(s)K(s)$ is equal to the number of anticlockwise encirclements of the origin by the Nyquist plot Γ.

Proof. For a complete treatise of MIMO Nyquist diagrams and homotopy arguments refer to [Vin01]. □

2.4 The standard $\mathcal{H}_\infty$ problem

Consider the following general feedback arrangement:

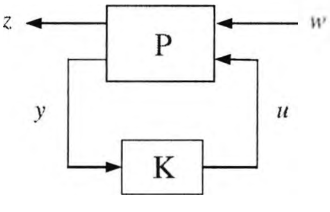

Figure 2.3: General plant

where w contains all exogenous inputs and model error outputs, the signal u is the controller output, the signal y is the controller input signal (measurements, references) and the signal z contains all the exogenous outputs. The overall control objective is to minimise the norm of the closed-loop transfer function between w an z by designing an appropriate controller K.

Let $P(s)$ be a partitioned system with a state-space realisation given by

$$P(s) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Then

$$P_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$$

is a state-space realisation of P_{ij}, for $i, j = 1, 2$. A linear fractional transformation of the partitioned system P and another system K, as appears above, is defined as

$$\mathcal{F}_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

which is the closed-loop transfer function, T_{wz}, from w to z. Next, we outline important results for the observable and controllable parts of a generalised plant and its relation with the system zeros. For further discussion we refer to [Kar01].

Proposition 2.4.1. *Consider the above generalised regulator problem. Then, if W_c and W_o are the controllability and observability gramians of P respectively,*

1. *the number of system zeros of $P_{12}(s)$ in C_+ = rank(W_c),*
2. *the number of system zeros of $P_{21}(s)$ in C_+ = rank(W_o),*
3. *every unobservable mode of the closed-loop system is a zero $P_{12}(s)$ and*
4. *every uncontrollable mode of the closed-loop system is a zero of $P_{21}(s)$.*

For (3) and (4), well posedness conditions must be satisfied.

Proof. see [LH87]. □

Example 2.4.1. *Formulation of the control diagram described by figure 2.2 in terms of a generalised plant. Here:*

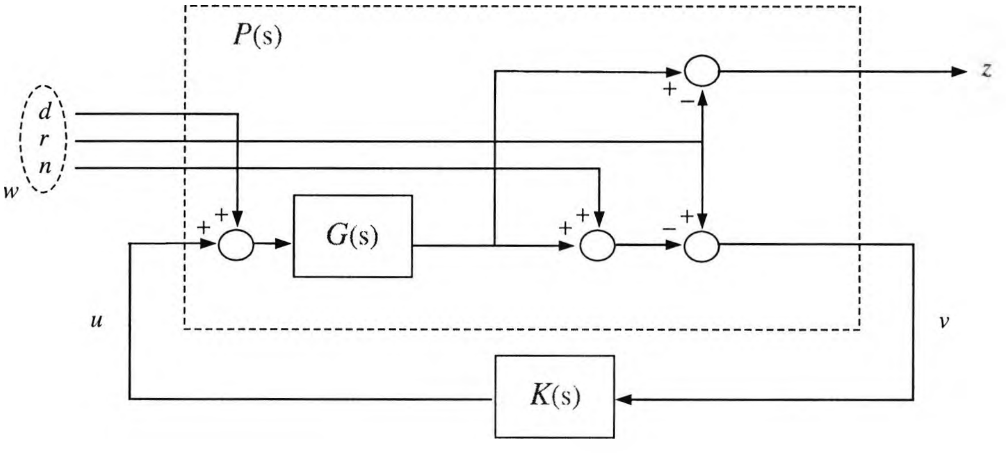

Figure 2.4: Equivalent representation of figure 2.2 where the error signal to be minimised is $z = y - r$ and the input to the controller is $v = r - y_m$ ([SP96]).

$$w := \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; \qquad z = e = y - r; \qquad v = r - y_m = r - y - n$$

Then,

$$z = y - r = Gu + Gd - r = Iw_1 - Iw_2 + 0w_3 + Gu$$

$$v = r - y_m = r - Gu - Gd - n = -Iw_1 + Iw_2 - Iw_3 - Gu$$

So, the transfer matrix from $\begin{pmatrix} w & u \end{pmatrix}'$ *to* $\begin{pmatrix} z & v \end{pmatrix}'$ *is equal to* $\mathcal{F}_l(P, K)$*, where*

$$P := \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} G & -I & 0 & G \\ -G & I & -I & -G \end{pmatrix}$$

Problem 2.4.1 (Standard $\mathcal{H}_\infty$ problem). *The $\mathcal{H}_\infty$-optimal regulation is the problem of determining a controller with transfer-matrix K that:*

1. *internally stabilises the closed loop system,*

2. *minimises the infinity norm $\|T\|_\infty$ of the closed-loop transfer matrix $T = \mathcal{F}_l(P, K)$ from the external input w to the control error z (see figure 2.3).*

Suppose that $P(s)$ have the realisation given in the start of the paragraph.

Assumption 2.4.1. *Suppose the following assumptions hold:*

(A1) *(A, B_2) is stabilisable and (A, C_2) is detectable.*

(A2) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ *has full column rank for all $\omega \in \mathcal{R}$.*

(A3) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ *has full row rank for all $\omega \in \mathcal{R}$.*

(A4) *D_{12} and D_{21} have full rank.*

The first assumption is made so that a stabilising controller exists. The other three assumptions ensure that P_{12} and P_{21} have full column and full row rank respectively, on the imaginary axis, including infinity.

If P_{12} and P_{21} are both square then the above problem is called of the *first kind.* However, if P_{12} or P_{21} is non-square then the problem is called of the *second kind* and further if the the case where both P_{12} and P_{21} are non-square occurs, then the problem is called of the *third kind.* The solution of the last two cases involves an iterative method to achieve optimality, the so-called γ iteration [DC86],[ZDG96].

(A5) Throughout this work we shall consider an extra assumption which relaxes the problem and avoids the γ-iteration. In particular, we assume that P_{12} and P_{21} are generically square.

Theorem 2.4.1 (Small Gain theorem). *Consider figure 2.3 and let $P \in \mathcal{RH}_\infty$ and $K \in \mathcal{RH}_\infty$. Then, the feedback loop is well-posed and internally stable for all K with:*

1. $\|K\|_\infty \leq \frac{1}{\gamma}$ *if and only if* $\|P\|_\infty < \gamma$

2. $\|K\|_\infty < \frac{1}{\gamma}$ *if and only if* $\|P\|_\infty \leq \gamma$

where $\gamma > 0$.

Proof. see [ZDG96],[GL95]. □

2.5 Summary

In this chapter we defined all important notions which are fundamental for the further development of our work in the consequent chapters. Initially, this chapter links the mathematical ideas presented in chapter 1 with the control framework which we follow in the sequel. In particular, the notions of signal and system spaces were related to function spaces and input-output stability was defined via the induced norm of the Hardy space $\mathcal{H}_\infty$. Further, it was aimed here to formulate the general $\mathcal{H}_\infty$ optimal control problem and give all necessary assumptions needed throughout this work. In the sequel, together with assumptions (2.4.1) we will always assume minimal realisations (with stabilisable and detectable parts) and internal stability to the feedback loops. Further, extensive use of similarity transformations will be made in order to derive, via Kalman decomposition, minimal realisations.

Chapter 3

Stabilising Controllers: Parametrisation

In this chapter stability conditions are described for feedback interconnections of LTI multivariable systems. A pre-requisite for stability of feedback systems is well posedness, a notion which was briefly discussed in the previous chapter. Assuming that this condition is satisfied, a controller that stabilises the closed loop system may then be designed such that the infinity norm of the closed loop remains bounded. Extending this idea and using the so-called "Youla parametrisation" [YJB76a],[YJB76b], the set of all stabilising controllers is characterised in a generalised regulator setting. The general feedback structure considered here is described in the figure below, or equivalently in figure 2.2 in chapter 2 (with $n = 0$ and positive feedback). Suppose that G (the plant) has p inputs and m outputs, then K (the controller) will have m inputs and p outputs.

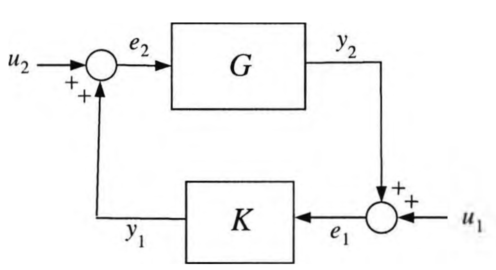

Figure 3.1: General feedback interconnection.

First, consider the equations corresponding to figure 3.1:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & G \\ K & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & I_p \\ I_m & 0 \end{pmatrix}}_{\triangleq F} \underbrace{\begin{pmatrix} K & 0 \\ 0 & G \end{pmatrix}}_{\triangleq T} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_p & -G \\ -K & I_m \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

The transfer function from the input signals to the output signals is described by

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} (I_m - KG)^{-1}K & (I_m - KG)^{-1}KG \\ (I_p - GK)^{-1}GK & (I_m - KG)^{-1}G \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} K(I_p - GK)^{-1} & K(I_p - GK)^{-1}G \\ (I_p - GK)^{-1}GK & (I_p - KG)^{-1}G \end{pmatrix}}_{\triangleq W(G,K)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

First note that $K(I_p - GK)^{-1} = (I_m - KG)^{-1}K$. To show this, note

$$\begin{aligned} & K(I_p - GK)^{-1} - (I_m - KG)^{-1}K = \\ & = (I_m - KG)^{-1}\{(I_m - KG)K - K(I_p - GK)\}(I_p - GK)^{-1} \\ & = (I_m - KG)^{-1}\{K - KGK - K + KGK\}(I_p - GK)^{-1} = 0 \end{aligned}$$

which proves the claim.

From the above analysis we see that the transfer function, mapping input signals to error signals, is equal to $(I_{p+m} - FT)^{-1}$. By assumption the matrix $(I_{p+m} - FT)$ is nonsingular and thus,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \underbrace{\begin{pmatrix} (I_p - GK)^{-1} & (I_p - GK)^{-1}G \\ K(I_p - GK)^{-1} & I_m - K(I_p - GK)^{-1}G \end{pmatrix}}_{\triangleq H(G,K)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Remark 3.0.1. *In order for $H(G,K)$ and $W(G,K)$ to be defined as proper transfer-functions we require $det(I - GK) = det(I - KG) \neq 0$, that is the feedback configuration at the above figures be well posed.*

In order to establish internal stability it is necessary and sufficient to prove that each of the four transfer function matrices of $H(G,K)$ are in $\mathcal{RH}_\infty$. An equivalent way to define internal stability is in terms of the transfer matrix from (u_1, u_2) to (y_1, y_2). To show the connection note that $T(I - FT)^{-1} = W \Rightarrow H = T^{-1}W$. Further,

$$\begin{aligned} H = (I - FT)^{-1} &\Rightarrow (I - FT)H = I \Rightarrow H - FTH = I \\ &\Rightarrow T = F^{-1}(H - I)H^{-1} \Rightarrow T^{-1} = H(H - I)^{-1}F \end{aligned}$$

Hence,

$$H = H(H-I)^{-1}FW \Rightarrow H - I = FW \Rightarrow H = I + FW$$
$$\Rightarrow W = F^{-1}(H-I) = F(H-I)$$

The only dynamical parts in the above relation are W and H, therefore

$$W \in \mathcal{RH}_\infty \Leftrightarrow H \in \mathcal{RH}_\infty$$

Remark 3.0.2. *Consider the generalised regulator feedback structure in figure 3.2. Consider also figure 3.1 where G has been substituted by G_{22} - the block (2,2) partition of G. Then, given a controller K the feedback system is internally stable if and only if the system in figure 3.2 is internally stable (see [DP00] lemma 5.4).*

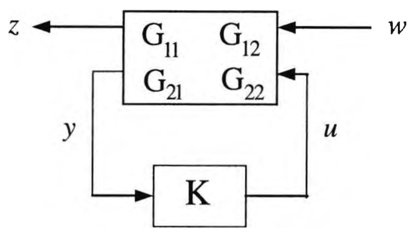

Figure 3.2: General feedback arrangement as lower LFT.

We say that G in figure 3.2 is stabilisable if there exists a (proper real-rational) K which stabilises it internally. Then K is said to be *admissible.* Note that an obvious non-stabilisable G (partitioned as G_{11} to G_{22}) is $G_{12} = G_{21} = G_{22} = 0$ and G_{11} unstable. If this is the case, then according to figure 3.2 the unstable part of the plant is not connected with u and y and so G_{11} is neither controllable from u nor observable from y. Hence, not every G is stabilisable.

Consider G partitioned as

$$G := \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Corollary 3.0.1 ([DP00]). *Suppose that (A, B_2, C_2), which corresponds to G_{22}, is stabilisable and detectable. Then the system in figure 3.1 is internally stable if and only if the transfer function from (u_1, u_2) to (e_1, e_2) is in $\mathcal{RH}_\infty$.*

Now consider the system equations corresponding to figure 3.2. Here the inputs of the plant are w and u and the outputs are z and y, respectively. So,

$$\dot{x} = Ax + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

and

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

On the other hand, the controller input and output signals are y and u, respectively. Thus,

$$\dot{x}_K = A_K x_K + B_K y$$
$$u = C_K x_K + D_K y$$

Combining the above equations we obtain the following state-space description of the closed loop system:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$\dot{x}_K = A_K x_K + B_K y$$

and

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w$$

Then, the A-matrix of the closed-loop realisation from w to z is

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

whenever the matrix $\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}$ is nonsingular (which is equivalent to well-posedness).

The next theorem establishes a connection between internal stability and the state-space model of the closed loop.

Theorem 3.0.1. *The system of figure 3.2 is internally stable if and only if $I - D_{22}D_K$ is invertible and A_{cl} is asymptotically stable, that is it has no eigenvalues in the closed right half plane.*

Proof. Straightforward from previous analysis. □

3.1 Coprime Factorisation over $\mathcal{RH}_\infty$

The set of rational functions $\mathcal{R}(s)$ has the algebraic structure of a field. This is not true for the set of stable rational functions because a stable rational function is not always stably invertible: the rational function $\frac{s-1}{s+1}$ is stable but has no inverse in the set of stable rational functions. The adequate structure for the description of the set of stable rational functions is that of a ring.

The main point of this section is that we can always write a proper real-rational transfer matrix as a ratio of two coprime stable proper real-rational transfer matrices. This powerful result is due to coprime factorisation theory which has been studied from system and operator theorists. In general, the subject has been studied over different rings (or rather algebras). However, throughout this work we shall consider the most prominent, to us, algebra which is $\mathcal{RH}_\infty$, the algebra of real-rational bounded analytic functions on the half plane.

Definition 3.1.1 ([MG90]). *Any square, invertible, transfer function matrix satisfying $U, U^{-1} \in \mathcal{RH}_\infty$ is called a unit in $\mathcal{RH}_\infty$.*

Definition 3.1.2 ([FO93]). *Let G be a proper (real) rational matrix-valued function. Then the factorisation*

1. *$G = NM^{-1}$ is called a right factorisation (RF) of G if N, M are stable proper (real) rational functions and M is invertible with proper inverse (M is square and $det(M) \neq 0$).*

   If N, M are right coprime, i.e. if there exists stable rational functions $\tilde{U}$,$\tilde{V}$ such that

   $$\tilde{V}M - \tilde{U}N = I,$$

   then the factorisation is called right coprime factorisation (RCF).

2. *$G = \tilde{M}^{-1}\tilde{N}$ is called a left factorisation (LF) of G if $\tilde{N}$, $\tilde{M}$ are stable proper (real) rational functions and $\tilde{M}$ is invertible with proper inverse ($\tilde{M}$ is square and $det(\tilde{M}) \neq 0$).*

   If $\tilde{N}$, $\tilde{M}$ are left coprime, i.e. if there exists stable rational functions U,V such that

   $$\tilde{M}V - \tilde{N}U = I,$$

   then the factorisation is called a left coprime factorisation (LCF).

Remark 3.1.1. *From the first Bezout identity we get that*

$$\tilde{V}M - \tilde{U}N = I \Rightarrow \begin{pmatrix} \tilde{V} & -\tilde{U} \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = I$$

In other words, N and M have the same number of columns and a statement equivalent to the above definition is that the matrix $\begin{pmatrix} M \\ N \end{pmatrix}$ *is left invertible in $\mathcal{RH}_\infty$. Similarly, the second Diophantine equation can be written as*

$$\tilde{M}V - \tilde{N}U = I \Rightarrow \begin{pmatrix} \tilde{M} & -\tilde{N} \end{pmatrix} \begin{pmatrix} V \\ U \end{pmatrix} = I$$

which reveals that $\tilde{M}$ and $\tilde{N}$ have the same number of rows and that equivalently to the LCF definition, the matrix $\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix}$ *is right invertible in $\mathcal{RH}_\infty$.*

Proposition 3.1.1. *Let $G = NM^{-1}$ be a, not necessarily coprime, right factorisation of G. If the McMillan degree of* $\begin{pmatrix} M \\ N \end{pmatrix}$ *equals the McMillan degree of G then N and M are right coprime.*
Dually, let $G = \tilde{M}^{-1}\tilde{N}$ be a, not necessarily coprime, left factorisation of G. If the McMillan degree of $\begin{pmatrix} -\tilde{N} & \tilde{M} \end{pmatrix}$ *equals the McMillan degree of G then $\tilde{N}$ and $\tilde{M}$ are left coprime.*

Proof. see [FO93]. □

It is possible to represent any proper real-rational transfer matrix function in terms of a pair of asymptotically stable, proper real-rational transfer matrices which are left, right or both left and right (doubly) coprime. The following result originally appeared in [NJB84] in a more general setting.

Proposition 3.1.2 ([NJB84],[DP00]). *Given a proper (real) rational matrix-valued function G, there exist both right and left coprime factorisations*

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

satisfying

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = I$$

for appropriate functions $U, V, \tilde{U}, \tilde{V}$ in $\mathcal{RH}_\infty$.

Here we reproduce an already known proof. This is done for continuation of arguments since the following construction is important for understanding the further development of the theory.

Proof. The proof is constructive. Assume

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with (A, B, C) stabilisable and detectable. The state space model of the realisation is

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where A, B, C, D are real matrices.

Now choose a real matrix F such that $A_F := A - BF$ is stable and define the vector $v := u + Fx$ and the matrix $C_F := C - DF$. Then the state space model can written as

$$\dot{x} = A_F x + Bv$$

$$u = (-F)x + v$$

$$y = C_F x + Dv$$

Now the transfer matrix from v to u is

$$M(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_F & B \\ \hline -F & I \end{array}\right] \Rightarrow M(s)^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline F & I \end{array}\right]$$

and that from v to y is

$$N(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_F & B \\ \hline C_F & D \end{array}\right]$$

Therefore,

$$u = Mv, \quad y = Nv$$

Then it is routine algebra to check that $y = NM^{-1}u$, i.e. $G = NM^{-1}$.

$$\begin{aligned} NM^{-1} &\stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline F & I \end{array}\right] \left[\begin{array}{c|c} A_F & B \\ \hline C_F & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_F & BF & B \\ 0 & A & B \\ \hline C_F & DF & D \end{array}\right] \stackrel{T}{=} \left[\begin{array}{cc|c} A_F & 0 & 0 \\ 0 & A & B \\ \hline C_F & C & D \end{array}\right] \\ &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] =: G \end{aligned}$$

where $T := \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$.

Similarly by choosing a matrix H so that $A_H := A - HC$ is stable and defining $B_H := B - HD$ we construct

$$\tilde{M}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_H & H \\ \hline -C & I \end{array}\right] \Rightarrow \tilde{M}(s)^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A & H \\ \hline C & I \end{array}\right]$$

$$\tilde{N}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_H & B_H \\ \hline C & D \end{array}\right]$$

Then it is easy to see that $G = \tilde{M}^{-1}\tilde{N}$, by using appropriate transformations:

$$\begin{aligned}
\tilde{M}(s)^{-1}\tilde{N}(s) &\stackrel{s}{=} \left[\begin{array}{c|c} A & H \\ \hline C & I \end{array}\right] \left[\begin{array}{c|c} A_H & B_H \\ \hline C & D \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & HC & HD \\ 0 & A-HC & B-HD \\ \hline C & C & D \end{array}\right] \stackrel{T_1}{=} \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A-HC & B-HD \\ \hline C & 0 & D \end{array}\right] \\
&\stackrel{T_2}{=} \left[\begin{array}{cc|c} A-HC & 0 & B-HD \\ 0 & A & B \\ \hline 0 & C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] =: G
\end{aligned}$$

where $T_1 := \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ and $T_2 := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Thus, we have obtained four matrices in $\mathcal{RH}_\infty$ satisfying the first condition $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$. Further, the second objective is met by defining the other four matrices as:

$$V(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_F & H \\ \hline C_F & I \end{array}\right], \quad U(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_F & H \\ \hline -F & 0 \end{array}\right]$$

$$\tilde{V}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_H & B_H \\ \hline F & I \end{array}\right], \quad \tilde{U}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_H & -H \\ \hline F & 0 \end{array}\right]$$

or, in a more compact form

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A-HC & B_H & H \\ \hline F & I & 0 \\ -C & -D & I \end{array}\right]$$

and

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A-BF & B & H \\ \hline -F & I & 0 \\ C_F & D & I \end{array}\right]$$

It can be easily shown that the product of the last two systems is equal to the unit matrix I, by removing all uncontrollable and unobservable modes. The computations are omitted. □

Corollary 3.1.1 (Existence [ZDG96]). *Let G be a proper real-rational matrix and $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be corresponding RCF and LCF over $\mathcal{RH}_\infty$. Then there exists a controller*

$$K_0 = U_0V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$$

with $U_0, V_0, \tilde{V}_0, \tilde{U}_0$ in $\mathcal{RH}_\infty$ such that

$$\begin{pmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U_0 \\ N & V_0 \end{pmatrix} = I$$

Furthermore, let F and H be such that $A - BF$ and $A - HC$ are stable. Then a particular set of state space realisations for these matrices can be given by

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A-BF & B & H \\ \hline -F & I & 0 \\ C-DF & D & I \end{array}\right]$$

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A-HC & B-HD & H \\ \hline F & I & 0 \\ -C & -D & I \end{array}\right]$$

Proof. From observer theory we apply state feedback and output injection to find a controller that achieves internal stability; for example

$$K_0 := \left[\begin{array}{c|c} A-BF-HC-HDF & H \\ \hline -F & 0 \end{array}\right]$$

Then, factorise K_0 as:

$$K_0 = U_0V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$$

i.e. in terms of doubly coprime factors. The result follows using proposition 3.1.2. □

In the sequel a complete characterisation of the set of all stabilising controllers K, with respect to a free parameter Q, is given. This will be the setting for formulating and solving optimisation problems in this work.

3.2 Parametrisation of all stabilising controllers

The main result of this section is summarised in the Theorem below:

Theorem 3.2.1 ([Fra87]). *The set of all (proper real-rational) controllers K stabilising G is parameterised by the formulae*

$$\begin{aligned} K &= (U + MQ)(V + NQ)^{-1} \\ &= (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) \end{aligned}$$

where $Q \in \mathcal{RH}_\infty$ and $det(I - DQ(\infty)) \neq 0$ $(D = G(\infty))$.

Proof. see [Fra87],[DP00]. □

Remark 3.2.1. *The proof of Theorem 3.2.1 is constructive and is based on figure 3.3. The dashed box describes the observer-based controller, which is connected with the free parameter Q. Note that the only restriction on $Q \in \mathcal{RH}_\infty$ is that the well-posedness condition $det(I - DQ(\infty)) \neq 0$ is satisfied. This condition is redundant if $G(s)$ is strictly proper.*

In the light of Theorem 3.2.1 the set of all stabilising controllers has the following (bilinear) form:

$$\begin{aligned} K &= (U + MQ)(V + NQ)^{-1} = (U + MQ)[V(I + V^{-1}NQ)]^{-1} \\ &= (U + MQ)(I + V^{-1}NQ)^{-1}V^{-1} = U(I + V^{-1}NQ)^{-1}V^{-1} + MQ(I + V^{-1}NQ)^{-1}V^{-1} \end{aligned}$$

Now rewrite $(I + V^{-1}NQ)^{-1}$ in the form $A + BQ(I + V^{-1}NQ)^{-1}$. Equating the two terms and post-multiplying by $I + V^{-1}NQ$, gives:

$$\begin{aligned} I &= A(I + V^{-1}NQ) + BQ \\ &= A + (B + AV^{-1}N)Q \end{aligned}$$

Hence,

$$A = I, \quad B = -AV^{-1}N = -V^{-1}N$$

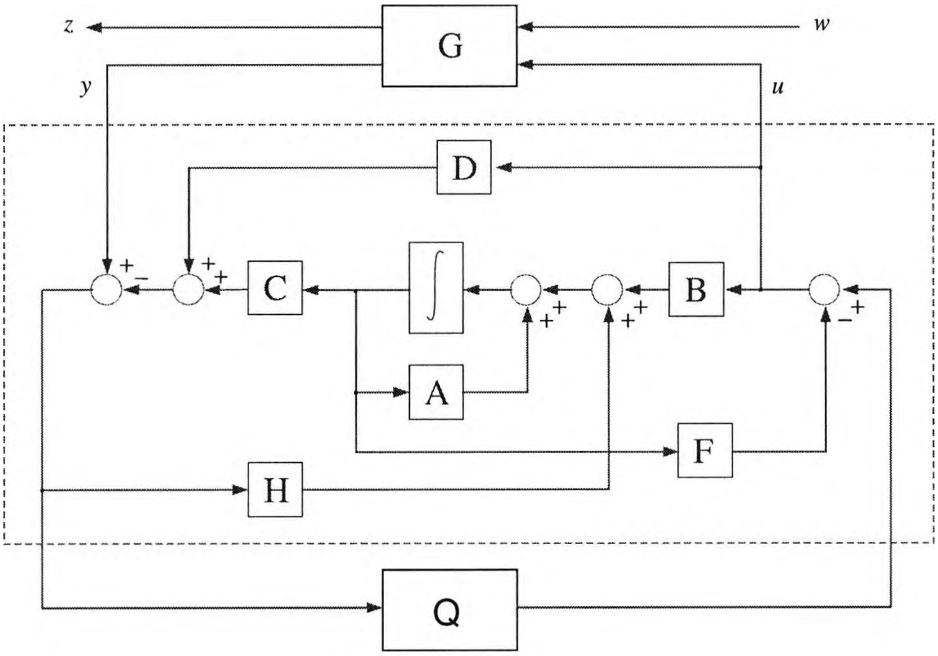

Figure 3.3: structure of (observer-based) stable controller.

By substitution:

$$
\begin{aligned}
K &= U\{I - V^{-1}NQ(I + V^{-1}NQ)^{-1}\}V^{-1} + MQ(I + V^{-1}NQ)^{-1}V^{-1} \\
&= UV^{-1} + (M - UV^{-1}N)Q(I + V^{-1}NQ)^{-1}V^{-1} \\
&= K_{11} + K_{12}Q(I - K_{22}Q)^{-1}K_{21}
\end{aligned}
$$

Consider now the transfer matrix K_o of the following form

$$
K_o = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BF - HC & H & B \\ \hline -F & 0 & I \\ -C & I & 0 \end{array}\right]
$$

Then, every stabilising controller K can be expressed as a lower linear fractional transformation of a transfer matrix K_o and a free parameter $Q \in \mathcal{RH}_\infty$ (assuming the well-posedness condition is satisfied - see remark 3.2.1).

Thus, the set of all real-rational stabilising controllers is

$$
\mathcal{K} = \{\mathcal{F}_l(K_o, Q) : Q \in \mathcal{RH}_\infty\}.
$$

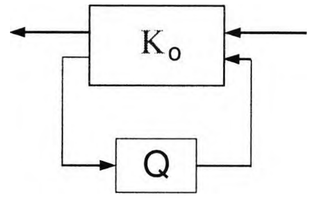

Figure 3.4: Controller K as a lower LFT interconnection.

3.3 Parametrisation of all stable Closed-loop transfer functions

The *model-matching* problem is formulated as shown in figure 3.5. Suppose $T_{11}(s)$, $T_{12}(s)$ and $T_{21}(s)$ are stable proper transfer functions. Then the model-matching problem is to find a stable transfer function $Q(s)$, such that it minimises the $\mathcal{H}_\infty$-norm of $T_{11} - T_{12}QT_{21}$. This is a hypothetical control problem in which T_{11} is interpreted as the model and $T_{12}QT_{21}$ describes a cascade connection of the plant and the controller. A distinction of the problem is made depending on the size of matrices T_{12} and T_{21}; either both T_{12} and T_{21} are square matrices, or one of these transfer matrices is non-square, or both of the transfer matrices are non-square. Then the problem is defined to be of the *first kind*, the *second kind* and the *third kind* respectively.

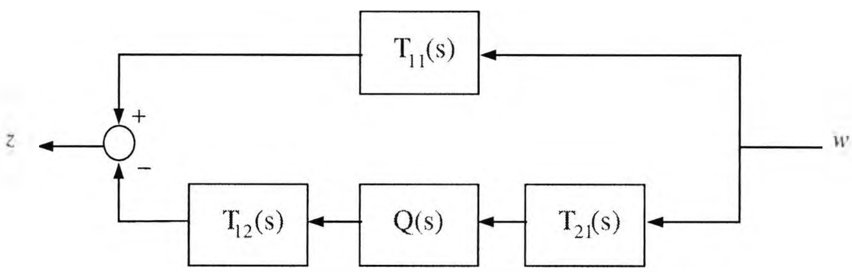

Figure 3.5: Model-matching problem.

Attempting to connect the theory of model-matching with the theory of previous paragraphs consider:

$$G := \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

in which by assumption (A, B_2, C_2) is stabilisable and detectable. Then there exist left and right coprime factorisations for the $(2,2)$ block,

$$G_{22} := N_2 M_2^{-1} = \tilde{M_2}^{-1} \tilde{N_2}$$

and appropriate $\mathcal{RH}_\infty$ transfer matrices which satisfy the doubly Bezout identity

$$\begin{pmatrix} \tilde{V_2} & -\tilde{U_2} \\ -\tilde{N_2} & \tilde{M_2} \end{pmatrix} \begin{pmatrix} M_2 & U_2 \\ N_2 & V_2 \end{pmatrix} = I \tag{3.1}$$

Every stabilising controller of G_{22} can be expressed as

$$\begin{aligned} K &= (U_2 + M_2 Q)(V_2 + N_2 Q)^{-1} \\ &= (\tilde{V_2} + Q\tilde{N_2})^{-1}(\tilde{U_2} + Q\tilde{M_2}) \end{aligned}$$

where $Q \in \mathcal{RH}_\infty$. Then, the principal aim of the generalised regulator problem (see chapter 2) is to minimise, in the infinity norm sense, the closed-loop transfer matrix $\mathcal{F}_l(G, K)$, i.e. $\min_K \|\mathcal{F}_l(G,K)\|_\infty$, provided the controller K stabilises the plant. Hence, the choice is to be made among the set of all stabilising controllers, which has already been characterised in terms of the parameter $Q \in \mathcal{H}_\infty$ above.

Further, it is possible to express the closed loop transfer function in terms of Q and $\mathcal{RH}_\infty$ matrices T_{11}, T_{12} and T_{21}, according to the following figure.

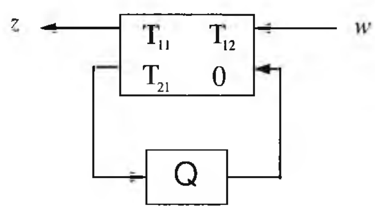

Figure 3.6: Closed loop as a lower LFT interconnection, in terms of parameter Q.

Hence, the problem can be recast in a model matching setting. The connection appears via the following theorem:

Theorem 3.3.1 (Model matching [Fra87]). *Consider the figures above. Then*

1. $T_{ij} \in \mathcal{RH}_\infty$ *for* $1 \leq i, j \leq 2$.

2. *Defining* K *as above (i.e. such as in Theorem 3.2.1), the transfer function from* w *to* z *is given by*

$$T = T_{11} - T_{12} Q T_{21}.$$

i.e. the closed-loop is affine in Q.

Proof. The proof again is constructive and can be found in various textbooks. However, the construction here is vital for constructing similar theoretical developments in the following chapters (chapter 5). Define,

$$T_{11} := G_{11} + G_{12}M_2\tilde{U}_2G_{21}$$
$$T_{12} := -G_{12}M_2$$
$$T_{21} := \tilde{M}_2G_{21}$$

Obviously, all matrices T_{ij}, for $1 \leq i,j \leq 2$ belong to $\mathcal{RH}_\infty$.

The closed loop transfer matrix is given by taking the following lower linear fractional transformation:

$$z = [G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}]w$$

Substitute $G_{22} = N_2M_2^{-1}$ and $K = (\tilde{V}_2 + Q\tilde{N}_2)^{-1}(\tilde{U}_2 + Q\tilde{M}_2)$ into $(I - KG_{22})^{-1}$:

$$\begin{aligned}(I - KG_{22})^{-1} &= [I - (\tilde{V}_2 + Q\tilde{N}_2)^{-1}(\tilde{U}_2 + Q\tilde{M}_2)N_2M_2^{-1}]^{-1} \\ &= [I - (\tilde{V}_2 + Q\tilde{N}_2)^{-1}(\tilde{U}_2N_2M_2^{-1} + Q\tilde{M}_2N_2M_2^{-1})]^{-1} \\ &= [I - (\tilde{V}_2 + Q\tilde{N}_2)^{-1}(\tilde{V}_2M_2M_2^{-1} - M_2^{-1} + Q\tilde{N}_2M_2M_2^{-1})]^{-1} \\ &= [I - (\tilde{V}_2 + Q\tilde{N}_2)^{-1}(\tilde{V}_2 + Q\tilde{N}_2 - M_2^{-1})]^{-1} \\ &= [I - I + (\tilde{V}_2 + Q\tilde{N}_2)^{-1}M_2^{-1})]^{-1} = M_2(\tilde{V}_2 + Q\tilde{N}_2)\end{aligned}$$

where in the third equality we used the blocks (1,1) and(2,1) of equation (3.1). Then,

$$(I - G_{22}K)^{-1}K = M_2(\tilde{U}_2 + Q\tilde{M}_2)$$

and the closed loop transfer function becomes

$$\begin{aligned}G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21} &= G_{11} + G_{12}M_2(\tilde{U}_2 + Q\tilde{M}_2)G_{21} \\ &= G_{11} + G_{12}M_2\tilde{U}_2G_{21} + (G_{12}M_2)Q(\tilde{M}_2G_{21}) \\ &=: T_{11} - T_{12}QT_{21}\end{aligned}$$

and the claim is proved. □

3.4 Summary

Throughout this chapter stability conditions for a wide class of general feedback interconnections were established. It was then shown how BIBO stability is linked to asymptotic stability of the closed-loop state-space model; assuming the well-posedness condition is satisfied, the condition of stabilisability and detectability of the state-space model is necessary and sufficient for internal stability. Next, using the theory of coprime factorisation over the ring of stable proper real rational matrices it was shown that $G \in \mathcal{RL}_\infty$ can always be factorised as $G = NM^{-1}$, where N, M are two stable proper real rational matrices (i.e. in $\mathcal{RH}_\infty$), independently of the inertia of its poles. Following this result, the existence of a controller in $\mathcal{RH}_\infty$ which stabilises the closed loop was established using observer-based methods with state feedback and output injection. Moreover, it was shown that the set of all stabilising controllers is parameterised in an LFT form ("Youla" or "Q" parametrisation), that is every stabilising controller of the feedback system of figure 3.2 can always be represented as an observer based controller connected with a free parameter $Q \in \mathcal{H}_\infty$. Concluding, via model matching, the set of all stable closed-loop transfer matrices is shown to admit a parametrisation in terms of Q. Then, in the light of the standard regulator problem (2.4.1), minimising $\|\mathcal{F}_l(P,K)\|_\infty$ over all K's which make the closed loop transfer function stable is equivalent to the minimisation of $\|T_{11} - T_{12}QT_{13}\|_\infty$, over all $Q \in \mathcal{H}_\infty$. This fact comes from the later paragraph of this chapter and the solution of the particular minimisation problem leads to the so-called Nehari approximation which is studied in the following chapter.

Chapter 4

Hankel operators in Robust Control

This chapter reviews the theory of Hankel operators, an important and wide class of operators, which was originally developed in the field of functional analysis and operator theory and which is also strongly related to modern control theory. Research on Hankel operators is still active in both areas, and thus the main objective of this chapter is to establish links between the two domains. In particular, it is shown that Hankel operators are successfully applied to model reduction theory as an approximation method, namely Hankel norm (best) approximation, and $\mathcal{H}_\infty$ control (Nehari problem). Throughout the chapter we shall consider a causal, bounded input-output operator G mapping $\mathcal{L}_2(-\infty,\infty)$ to $\mathcal{L}_2(-\infty,\infty)$ described by the state space convolution,

$$(Gu)(t) := \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau$$

where A is a Hurwitz matrix and (A,B,C) is a minimal realisation. Taking the Laplace transform of this equation implies that $G(s) = C(sI-A)^{-1}B$ is a strictly proper transfer function in $\mathcal{RH}_\infty$, i.e.

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \mathcal{RH}_\infty$$

Remark 4.0.1 (Causality and $\mathcal{H}_\infty$, [DP00]). *A system is called causal if the output up to time T depends only on the input up to time T, for every T.* [1] *Further, we say that a system G is stable if $y = Gu$ is in $\mathcal{L}_2[0,\infty)$ whenever $u \in \mathcal{L}_2[0,\infty)$. Hence, all LTI operators on $\mathcal{L}_2[0,\infty)$ are represented by functions in $\mathcal{H}_\infty$. Notice that this means that an LTI operator on $\mathcal{L}_2[0,\infty)$ is necessarily causal. On the contrary an LTI operator on $\mathcal{L}_2(-\infty,\infty)$ need not be causal; a time-invariant operator G mapping $\mathcal{L}_2(-\infty,\infty)$ to $\mathcal{L}_2(-\infty,\infty)$ is causal if and only if it maps $\mathcal{L}_2[0,\infty)$ to $\mathcal{L}_2[0,\infty)$, i.e.*

[1] *All real-time physical systems are causal because time moves forward. However, causality does not apply to systems processing recorded signals, e.g. taped sports game vs. live broadcast.*

it maps every function that is zero for negative time to a function which is also zero on the negative time axis.

Example 4.0.1. *The (bilateral) Laplace transform of both $e^{-\alpha t}u(t)$ (causal) and $-e^{-\alpha t}u(-t)$ (anti-causal) is $\frac{1}{s+\alpha}$ (where $u(t)$ denotes the unit step). However, the region of convergence (ROC) is different, implying that ROC must be known to uniquely determine the transfer function. For a causal system ROC is to the right of the rightmost pole (in the s-domain) of the Laplace transform; for a causal stable system ROC is to the right of the rightmost pole (in the s-domain) of the Laplace transform and all the poles are in the left-half plane.*

Decompose $\mathcal{L}_2(-\infty,\infty)$ as $\mathcal{L}_2(-\infty,0] \oplus \mathcal{L}_2[0,\infty)$. Then, a general LTI causal system G can be visualised by the following map:

$$\begin{pmatrix} y_- \\ y_+ \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} u_- \\ u_+ \end{pmatrix} = \begin{pmatrix} \mathcal{T}_G & 0 \\ \Gamma_G & \widetilde{\mathcal{T}}_G \end{pmatrix} \begin{pmatrix} u_- \\ u_+ \end{pmatrix}$$

where

$$\begin{cases} y_+ \in \mathcal{L}_2^p[0,\infty) \\ y_- \in \mathcal{L}_2^p(-\infty,0] \\ u_+ \in \mathcal{L}_2^m[0,\infty) \\ u_- \in \mathcal{L}_2^m(-\infty,0] \end{cases}$$

Note that causality implies $G_{12} = 0$. Further, $\mathcal{T}_G$ and $\widetilde{\mathcal{T}}_G$ are Toeplitz operators while Γ_G is a Hankel operator. Clearly, the Hankel operator maps "past" inputs (u_-) to "future" outputs (y_+) and so

$$\Gamma_G : \mathcal{L}_2^m(-\infty,0] \to \mathcal{L}_2^p[0,\infty)$$

or, equivalently

$$\begin{aligned} (\Gamma_G u)(t) := P_+ G u \mid_{\mathcal{L}_2(-\infty,0]} &= P_+ \int_{-\infty}^{0} C e^{A(t-\tau)} B u(\tau) d\tau \\ &= P_+ C e^{At} \int_{-\infty}^{0} e^{-A\tau} B u(\tau) d\tau \end{aligned}$$

for arbitrary inputs $u \in \mathcal{L}_2(-\infty,0]$. Here P_+ denotes the projection operator $\mathcal{L}_2(-\infty,\infty) \to \mathcal{L}_2[0,\infty)$. In order to establish a link between time and frequency domains we make the following observation.

Observation 4.0.1. *It is a fact that* $\mathcal{L}_\infty = \mathcal{H}_\infty^- \oplus \mathcal{H}_\infty^+$ *and thus a system* $G \in \mathcal{L}_\infty$ *can be decomposed into a strictly causal and an anti-causal part, i.e.*

$$G(s) = G_c(s) + G(\infty) + G_a(s).$$

Here $G_c \in \mathcal{H}_\infty$, $G_a \in \mathcal{H}_\infty^-$ *and* $G(\infty)$ *is the constant part of the system which in a state space realisation corresponds to its "D" matrix (direct feed-through matrix) and it can be absorbed in either* $G_c(s)$ *or* $G_a(s)$ *(here we absorb* D *into* G_a*). However for any* $u \in \mathcal{H}_2^\perp$

$$\Gamma_G u = P_+(Gu) = P_+(G_a u)$$

since the equivalent frequency-domain definition of the Hankel operator is

$$\Gamma_G : \mathcal{H}_2^\perp \to \mathcal{H}_2$$

Hence, the Hankel operator associated with a $G \in \mathcal{L}_\infty$ *depends only on the strictly causal part of* G*, that is if* $G \in \mathcal{H}_\infty^-$ *then* $\Gamma_G = 0$*. Therefore there is no loss of generality in taking* $G \in \mathcal{H}_\infty$ *as it was assumed at the beginning of this section.*

Remark 4.0.2. *An alternative definition of a Hankel operator in the time-domain, not adopted here but used by many researches, is to define it as a mapping*

$$\Gamma_G : \mathcal{L}_2^m[0,\infty) \to \mathcal{L}_2^p[0,\infty)$$

The equivalence of the two definitions follows from the following argument. Consider

$$y(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)d\tau$$

and set $v(\tau) = u(-\tau)$*. Substituting* $\xi := -\tau$,

$$y(t) = \int_{\infty}^{0} Ce^{A(t+\xi)}Bu(-\xi)(-d\xi) = \int_{0}^{\infty} Ce^{A(t+\xi)}Bv(\xi)d\xi$$

and so

$$\Gamma_G v : \quad \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$$

In this framework the Hankel operator is "induced" by the anti-causal part of the system. However, the original definition of the Hankel operator is more intuitively appealing from a signals/systems viewpoint and extends naturally to the theory presented in the next chapter.

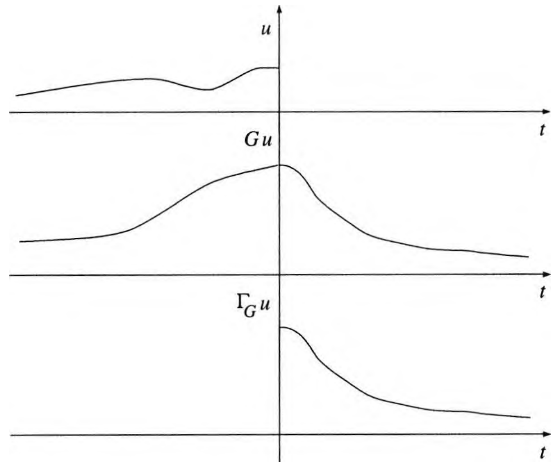

Figure 4.1: Hankel Operator of a system G in the time domain, given $u \in \mathcal{L}_2(-\infty, 0]$ (scalar case). Here $Gu \in \mathcal{L}_2(-\infty, \infty)$ and $\Gamma_G u$ is the projection of $Gu \in \mathcal{L}_2$ into $\mathcal{L}_2[0, \infty)$, by truncation.

In the sequel, it is shown how the notions of controllability and observability are connected with the above definition of the Hankel operator. The construction of two auxiliary operators, namely the controllability (Ψ_c) and observability (Ψ_o) operators, is largely motivated by the fact that the Hankel operator can be written as their composition.

4.1 Controllability and observability operators

Consider the autonomous LTI system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad x(0) = x_0 \in \mathcal{R}^n \\ y(t) &= Cx(t) \end{aligned}$$

Definition 4.1.1. *The observability operator is defined as follows*

$$\begin{aligned} &\Psi_o : \mathcal{R}^n \rightarrow \mathcal{L}_2[0, \infty) \\ &\Psi_o x_0 = Ce^{At} x_0 =: y(t) \qquad x_0 \in \mathcal{R}^n, t \geq 0. \end{aligned}$$

Clearly, the above definition shows that if $y(t)$ is known over an interval $[0, T]$ and the system is observable it is possible to determine the initial condition x_0 uniquely and hence every $x(t)$ for $t \in [0, T]$. Further, it can be shown that the unobservable space is the kernel of this operator, i.e. $\mathcal{N}_{CA} = ker(\Psi_o)$ (see [DP00]).

Remark 4.1.1. *The adjoint of the observability operator is given as:*

$$\Psi_o^* : \mathcal{L}_2[0,\infty) \to \mathcal{R}^n$$
$$\Psi_o^* f = \int_0^\infty e^{A^*\tau} C^* f(\tau) d\tau$$

where $f \in \mathcal{L}_2[0,\infty)$.

Recalling the observability gramian definition from chapter 2, it is easy to check that

$$W_o = \Psi_o^* \Psi_o$$

and so[2]

$$rank(W_o) = rank(\Psi_o^* \Psi_o) = rank(\Psi_o) = n - dim(ker(\Psi_o))$$

which is equal to the dimension of the observable subspace. We can also give the following geometric interpretation: The "observation energy" of the state x_0, that the output trajectory $y(t) = Ce^{At}x_0$ for $t \geq 0$ produces, is measured as:

$$\|y\|_2^2 := x(0)^* \left(\int_0^\infty e^{A^*t} C^* C e^{At} dt \right) x(0) = x_0^* W_o x_0$$

Thus the observability gramian reflects the effect of initial states on the "output energy" of the system when the input is zero. If W_o is nearly singular then there exist states which have low "observation energy" in the sense that $\|y\|_2$ is small [Wei02]. Define now:

$$\mathcal{E}_o := \left\{ W_o^{\frac{1}{2}} x_0 : x_0 \in \mathcal{R}^n \text{ and } \|x_0\| = 1 \right\}$$

Since W_o is positive semi-definite in general, and positive definite if and only if the system is observable, this set is an ellipsoid with the i-th eigenvector of $W_o^{\frac{1}{2}}$ giving the direction of the principal axis of the ellipsoid and the corresponding eigenvalue represents the length of each axis. The span of all eigenvectors corresponding to the zero eigenvalues of $W_o^{\frac{1}{2}}$ is precisely the unobservable space, $\mathcal{N}_{CA}$.

Dually, along similar lines the controllability characteristics of the system can now be defined:

Definition 4.1.2. *The controllability operator is defined as follows*

$$\Psi_c : \mathcal{L}_2(-\infty, 0] \to \mathcal{R}^n$$
$$(\Psi_c u)(t) = \int_{-\infty}^0 e^{-A\tau} Bu(\tau) d\tau = \int_0^\infty e^{A\tau} Bu(-\tau) d\tau =: x_0$$

[2]If T is a finite rank operator then $rank(T) = rank(T^*) = rank(T^*T)$. Further, if $T : \mathcal{R}^n \to \mathcal{U}$, where $\mathcal{U}$ is any Hilbert space, then $rank(A) = n - dim(ker(T))$.

Remark 4.1.2. *By definition Ψ_c is not defined on the full domain of G. However, it can be extended to the full space by defining $\mathcal{L}_2[0,\infty)$ to be in its null space.*

The above definition makes sense if it is considered as the response of a system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(-\infty) = 0$$

to an input function $u \in \mathcal{L}_2(-\infty, 0]$, where the output is the state trajectory. Then from the controllability operator definition it is easy to determine the input with minimum energy $\|u\|_2$ which drives the state to $x(0) = x_0$ at time zero. It also follows easily that the controllability gramian is given as the composition of controllability operator and its adjoint, i.e.

$$W_c = \Psi_c \Psi_c^*$$

Hence, note that[3]

$$rank(W_c) = rank(\Psi_c \Psi_c^*) = rank(\Psi_c) = dim(image(\Psi_c))$$

which is the dimension of the controllable subspace. Again, consider

$$\|u(t)\|_2^2 := x(0)^* \left(\int_0^\infty e^{A^*t} BB^* e^{At} dt \right)^{-1} x(0) = x(0)^* W_c^{-1} x(0)$$

and define the set

$$\mathcal{E}_c := \left\{ W_c^{-\frac{1}{2}} x : x \in \mathcal{R}^n, \|x\| = 1 \right\}$$

Geometrically this represents an ellipsoid, the so-called controllability ellipsoid, for which an analysis similar to the observability case applies (see [DP00]). Thus, the controllability gramian measures the "degree of controllability" of a given state.

The following equivalent definition of the Hankel operator follows from the above discussion.

Definition 4.1.3. (Hankel operator) *The Hankel operator of a system $G \in \mathcal{L}_\infty$, in terms of the observability and the controllability operators, is given as*

$$\Gamma_G = \Psi_o \Psi_c$$

At this point the concept of the balanced realisation of a system can be introduced. This is a minimal realisation in which both gramians are equal and take the form of a

[3]Take $T : \mathcal{X} \to \mathcal{Y}$, a mapping between two Hilbert spaces. Then the rank of T is defined by $rank(T) = dim(image(T))$.

diagonal matrix $\Sigma = diag(\sigma_1(\Gamma_G), \sigma_2(\Gamma_G), \ldots, \sigma_n(\Gamma_G))$. Here, $\sigma_1(\Gamma_G) \geq \sigma_2(\Gamma_G), \ldots \geq \sigma_n(\Gamma_G) > 0$ are called the Hankel singular values and as the name reveals they are the singular values of the Hankel operator Γ_G (see def. 1.7.4). Note that a balancing similarity transformation always exist for stable minimal systems. Further, note that the Hankel singular values are system invariants, that is whenever the basis of the state space is transformed (under similarity transformations) they remain unchanged.

Now suppose the system is balanced, so

$$W_c = W_o = \Sigma = diag(\sigma_1(\Gamma_G), \sigma_2(\Gamma_G), \ldots, \sigma_n(\Gamma_G))$$

as defined above. The minimal energy cost to reach the i-th state component

$$x_0 = e_i = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}'$$

is given by

$$e_i^* W_c^{-1} e_i = e_i^* \Sigma^{-1} e_i = \frac{1}{\sigma_i(\Gamma_G)}$$

whereas if the system is released from this state, the output energy will be

$$\|y\|_2^2 = e_i^* W_o e_i = e_i^* \Sigma e_i = \sigma_i(\Gamma_G)$$

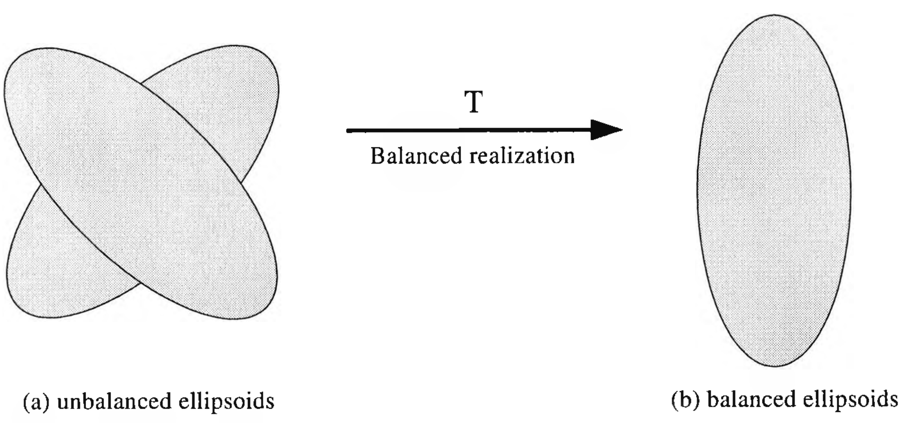

Figure 4.2: Observability and controllability ellipsoids for (a) unbalanced and (b) balanced system realisations.

Now, because of the non-increasing ordering of the Hankel singular values in a balanced realisation, the state components with low indices are "easy to observe" (output energy is large) and at the same time "easy to reach" (the minimal control energy needed to reach these states is small). The opposite conclusion applies to states with high indices.

4.2 Hankel norm approximation and Model reduction

In the case of linear time-invariant multivariable systems the dimension of the state space reflects the complexity of the system. So, the larger the dimension, the more difficult it is to design the system. In addition, modern control techniques such as $\mathcal{H}_\infty$ and LQG typically produce controllers of order at least equal to that of the plant (and usually higher because of the inclusion of weights). Hence, model reduction methods try to reduce the order of the model prior to controller design, or to reduce the controller at the final stage (or both). Reducing the model's order means that, if the dimension of the state space is too large, then it is desired to derive another system that has a realisation in a space of smaller dimension and whose input-output properties do not differ significantly from the properties of the initial system. If the system has a transfer function G which is a stable rational function then it admits a balanced realisation such that the dimension of the state space is equal to the McMillan degree of G. This is the most common case and hence the model reduction problem is normally posed in this framework.

Problem 4.2.1. *Given a high order linear time-invariant stable model G, find a low order stable approximation $\ddot{G}$, in the sense that*

$$\|G - \hat{G}\|_\infty$$

is minimum.

In general, this is considered to be an untractable problem, and hence it is recast as a Hankel norm approximation, which is another way of measuring the closeness of two transfer functions. The later method is physically well motivated and admits a satisfactory solution related to the original problem 4.2.1. In the sequel, characteristics, properties and norm bounds of the Hankel operator are discussed.

Theorem 4.2.1. (Kronecker) *Suppose G is a linear system with Hankel operator Γ_G, and suppose $rank(\Gamma_G)$ is finite. Then a minimal realisation of G has state-dimension equal to $rank(\Gamma_G)$. Equivalently, for $A \in \mathcal{R}^{n\times n}$,*

$$(A, B, C, D) \text{ is minimal } \Leftrightarrow rank(\Gamma_G) = n$$

Proof. First notice the fact that

$$rank(\Gamma_G) = rank(\Psi_o \Psi_c) = rank(\Psi_o^* \Psi_o \Psi_c \Psi_c^*) = rank(W_o W_c)$$

***only if* part ($\Leftarrow$):** Sylvester's inequality gives

$$rank(\Gamma_G) = rank(W_o W_c) \leq \min\{rank(W_o), rank(W_c)\}$$

Now, the rank of observability and controllability gramians are at most n (when rank$(W_o) = n$ then the system is observable and respectively controllable when rank$(W_c) = n$). So

$$n \leq \min\{rank(W_o), rank(W_c)\} \Rightarrow n = \min\{rank(W_o), rank(W_c)\}$$
$$\Rightarrow n = rank(W_o) = rank(W_c)$$

Hence, the system is controllable and observable.

***if* part ($\Rightarrow$):** The other Sylvester inequality gives

$$rank(\Gamma_G) = rank(W_o W_c) \geq rank(W_o) + rank(W_c) - n = n$$

by noticing that the system is observable and controllable, that is $rank(W_o) = rank(W_c) = n$.

□

Definition 4.2.1. *The induced norm by the Hankel operator (or simply Hankel norm) of a system $G \in \mathcal{RL}_\infty$ is defined as follows*

$$\|G\|_H = \|\Gamma_G\| = \max_{u \in \mathcal{L}_2(-\infty,\infty),\ u(t) \neq 0} \frac{\sqrt{\int_0^\infty \|y(\tau)\|_2^2 d\tau}}{\sqrt{\int_{-\infty}^0 \|u(\tau)\|_2^2 d\tau}}$$

A standard result from operator theory (see def. 1.7.4) is that $\|\Gamma_G\| = \sigma_1(\Gamma_G)$. By comparing the definitions of Hankel and $\mathcal{H}_\infty$ norms it follows immediately that the Hankel norm of a system is bounded above by its infinity norm. This is due to the fact that for arbitrary unit energy input in $\mathcal{L}_2(-\infty, 0]$, $\|\Gamma_G\|$ is the least upper bound on the energy of the future output and $\|G\|_\infty$ is the least upper bound on the energy of the total output. This is restated more formally in the following proposition:

Proposition 4.2.1. *The Hankel norm satisfies*

$$\|\Gamma_G\| \leq \|G\|$$

where $\|G\|$ is the induced norm from $\mathcal{L}_2(-\infty, \infty)$ to itself, i.e. $\|G\| = \|G\|_\infty$.

Proof. The projection P_+ has norm $\|P_+\|_\infty = 1$ ([Kre89], theorem 9.1-1). Hence,

$$\begin{aligned}\|\Gamma_G\| &= \|P_+ G\,|_{\mathcal{L}_2(-\infty,0]}\| \le \|P_+\|\|G\,|_{\mathcal{L}_2(-\infty,0]}\| \\ &= \|G\,|_{\mathcal{L}_2(-\infty,0]}\| \le \|G\|_\infty\end{aligned}$$

□

Observation 4.2.1. *Take F to be any anti-causal system; if $u \in \mathcal{L}_2(-\infty,0]$ then $(Fu)(t)$ is zero for $t > 0$. So, the future output remains unaffected by the addition of any anti-causal system and it is immediate from the above proposition that*

$$\|\Gamma_G\| \le \|G - F\|_\infty$$

However, as seen in the previous section the Hankel operator is intimately related to the observability and controllability gramians. Thus, its norm should be also somehow related to these two notions. The connection is made exact in the next proposition.

Proposition 4.2.2. *The Hankel norm of the system G is the induced-norm of its Hankel operator and satisfies*

$$\|\Gamma_G\| = (\lambda_{max}(W_o W_c))^{\frac{1}{2}}$$

In fact spec$(\Gamma_G^* \Gamma_G) =$ spec$(W_o W_c) \cup \{0\}$.

Proof. Γ_G is a finite rank operator (Kronecker's theorem) and it is bounded by its (induced) norm. Hence, it is compact ([You88]). Further, its adjoint and their product $\Gamma_G^* \Gamma_G$ are compact operators ([Kre89], theorem 8.2-5]), with the later being a self-adjoint and positive operator ($\langle \Gamma_G^* \Gamma_G x, x\rangle = \langle \Gamma_G x, \Gamma_G x\rangle = \|\Gamma_G x\|^2 \ge 0$).
It is a fact that $\|\Gamma_G\| = \|\Gamma_G^*\Gamma_G\|^{\frac{1}{2}} = (\rho(\Gamma_G^*\Gamma_G))^{\frac{1}{2}}$ ([DP00], prop.3.15-3.16 [4]). Also spec$(\Gamma_G^*\Gamma_G) =$ spec$(\Psi_c^*\Psi_o^*\Psi_o\Psi_c) =$ spec$(\Psi_o^*\Psi_o\Psi_c\Psi_c^*) \cup \{0\} =$ spec$(W_o W_c) \cup \{0\}$. The eigenvalues of $W_o W_c$ are real and positive, since spec$(W_o W_c) \cup \{0\} =$ spec$(W_c^{\frac{1}{2}} W_o W_c^{\frac{1}{2}})$ □

Example 4.2.1. *Suppose,*

$$G(s) = \frac{1-s}{2(s+1)} = \frac{1}{s+1} - \frac{1}{2} =: G_s(s) + G_a(s)$$

[4]In general, for any linear bounded operator T, $\|T\| \ge \rho(T)$. In the present case we have equality since the operator is self adjoint.

Clearly, from above, $\|G\|_\infty = \frac{1}{2}$ *and* $\|G_s\|_\infty = 1$*, which agrees with the triangle inequality* $\|G_s + G_a\|_\infty \leq \|G_s\|_\infty + \|G_a\|_\infty$*. Further, in order to compute the Hankel norm of* G *we isolate its strictly causal part which has a realisation:*

$$G_s(s) \stackrel{s}{=} \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right] =: \left[\begin{array}{c|c} a & b \\ \hline c & 0 \end{array}\right]$$

Computing the gramians from the Lyapunov controllability and observability equations:

$$2aW_c + b^2 \Rightarrow W_c = -\frac{b^2}{2a} = \frac{1}{2} \; (= W_o)$$

Then, $W_cW_o = \frac{1}{4}$ *and thus, according to proposition 4.2.2,*

$$\|G\|_H = \sqrt{\lambda_{max}(W_cW_o)} = \frac{1}{2}.$$

Hence, via this example, it is shown that the extreme case $\|\Gamma_G\| = \|G\|_\infty$ *can occur.*

Observation 4.2.2. *For any* $G, G_r \in \mathcal{L}_\infty$

$$\|G - G_r\|_\infty \geq \|\Gamma_{G-G_r}\| = \|\Gamma_G - \Gamma_{G_r}\|$$

The inequality is true due to the Proposition 4.2.1. In order to check the last equality, note that if

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & 0 \end{array}\right] \quad \text{and} \quad G_r \stackrel{s}{=} \left[\begin{array}{c|c} A_{G_r} & B_{G_r} \\ \hline C_{G_r} & 0 \end{array}\right]$$

then

$$G - G_r \stackrel{s}{=} \left[\begin{array}{cc|c} A_G & 0 & B_G \\ 0 & A_{G_r} & -B_{G_r} \\ \hline C_G & C_{G_r} & 0 \end{array}\right]$$

By definition, given an input $u(t)$*, the time domain Hankel operator of the last transfer matrix is*

$$\begin{aligned}(\Gamma_{G-G_r}u)(t) &:= \int_{-\infty}^{0} \begin{bmatrix} C_G & C_{G_r} \end{bmatrix} e^{\begin{bmatrix} A_G & 0 \\ 0 & A_{G_r} \end{bmatrix}(t-\tau)} \begin{bmatrix} B_G \\ -B_{G_r} \end{bmatrix} u(\tau)d\tau \\ &= \int_{-\infty}^{0} \begin{bmatrix} C_G & C_{G_r} \end{bmatrix} \begin{bmatrix} e^{A_G(t-\tau)} & 0 \\ 0 & e^{A_{G_r}(t-\tau)} \end{bmatrix} \begin{bmatrix} B_G \\ -B_{G_r} \end{bmatrix} u(\tau)d\tau \\ &= \int_{-\infty}^{0} C_G e^{A_G(t-\tau)} B_G u(\tau)d\tau - \int_{-\infty}^{0} C_{G_r} e^{A_{G_r}(t-\tau)} B_{G_r} u(\tau)d\tau =: (\Gamma_G u - \Gamma_{G_r} u)(t)\end{aligned}$$

Theorem 4.2.2. *Suppose G has a minimal realisation of order n. Then for any G_r of order $r < n$,*

$$\|G - G_r\| \geq \sigma_{r+1}(\Gamma_G)$$

where $\sigma_1(\Gamma_G) \geq \sigma_2(\Gamma_G) \geq \cdots \geq \sigma_n(\Gamma_G) > 0$ are the Hankel singular values of G.

Proof. Kronecker's theorem implies that $\text{rank}(\Gamma_G) = n$ and $\text{rank}(\Gamma_{G_r}) = r$. Recalling definition 1.7.4 (chapter 1), the singular values of a linear operator are defined as [5]

$$\sigma_r(G) = \inf\{\|G - X\| : rank(X) < r\} \qquad r \in \mathcal{N}/\{0\}$$

when G and X share the same input-output spaces and $\text{rank}(G) \geq r$. Take Γ_X such that $\text{rank}(\Gamma_X) = r$, then

$$\sigma_{r+1}(\Gamma_G) = \inf\{\|\Gamma_G - \Gamma_X\| : rank(\Gamma_X) = r < r+1\}$$

Hence, $\sigma_{r+1}(\Gamma_G) \leq \|\Gamma_G - \Gamma_X\|$ and so from observation 4.2.2

$$\|G - G_r\|_\infty \geq \|\Gamma_{G-G_r}\| = \|\Gamma_G - \Gamma_{G_r}\| \geq \sigma_{r+1}(\Gamma_G)$$

as required. □

4.3 SVD of a Hankel operator

According to the definition of the singular value decomposition (or Schmidt decomposition) of linear bounded operators (corollary 1.7.1), the SVD of Hankel operators is next defined.

Definition 4.3.1. *(Hankel SVD) The Schmidt decomposition of Γ_G is given by the following dyadic form*

$$\Gamma_G u = \sum_{i=1}^{n} \sigma_i \langle u, v_i \rangle w_i$$

in which $v_i \in \mathcal{L}_2(-\infty, 0]$ and $w_i \in \mathcal{L}_2[0, \infty)$ are sets of orthonormal functions. The pair (v_i, w_i) corresponding to the Hankel singular value $\sigma_i := \sigma_i(\Gamma_G)$ is called a Schmidt pair. From orthogonality it follows that

$$\Gamma_G v_i = \sigma_i w_i$$

$$\Gamma_G^* w_i = \sigma_i v_i$$

[5]Here σ_r denotes the r-th singular value of the linear operator and not the r-th Hankel singular value. However, for a completely stable system, the Hankel singular values coincide with that, induced by the operator, singular value.

Remark 4.3.1. *By taking the dyadic form of the above definition and setting* $u = v_j$, *then* $\Gamma_G v_j = \sum_{i=1}^{n} \sigma_i \langle v_j, v_i \rangle w_i$. *Orthonormality implies that* $\langle v_j, v_i \rangle = 1$ *whenever* $i = j$ *and zero otherwise. Hence,* $\Gamma_G v_i = \sigma_i w_i$. *By taking the adjoint of the dyadic form the second equation can be similarly proved.*

It is important from a control theoretic point of view to link the definition of Schmidt pairs with the notions of controllability and observability. This can be done as follows: Take $\Gamma_G v_i = \sigma_i w_i$ and pre-multiply by Γ_G^*. Substituting $\Gamma_G^* w_i = \sigma_i v_i$ it follows that $\Gamma_G^* \Gamma_G v_i = \sigma_i \Gamma_G^* w_i$, i.e. $\Gamma_G^* \Gamma_G v_i = (\sigma_i)^2 v_i$. Suppose now that $(\sigma_i)^2$ is a nonzero eigenvalue of $\Gamma_G^* \Gamma_G$. Further, consider the time-domain analog of $\Gamma_G^* \Gamma_G$, which equals $\Psi_c^* \Psi_o^* \Psi_o \Psi_c$. Then, there exists a nonzero $v_i \in \mathcal{L}_2[0, \infty)$ satisfying

$$\Psi_c^* \Psi_o^* \Psi_o \Psi_c v_i = (\sigma_i)^2 v_i$$

Pre-multiplying the above equation by Ψ_c and defining $x_i := \Psi_c v_i$,

$$W_c W_o x_i = (\sigma_i)^2 x_i$$

Hence, (using the spectrum argument in proposition 4.2.2) x_i is the eigenvector of $W_c W_o$ corresponding to the eigenvalue $(\sigma_i)^2$.

Remark 4.3.2. *Note that for a balanced realisation, i.e.* $W_c = W_o = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$, *then* $x_i = \sqrt{\sigma_i} e_i$, *where* e_i *is the i-th standard basis vector.*

Continuing the analysis in the time-domain, an implicit characterisation of time-domain Schmidt vectors can be obtained. We first write, $\Gamma_G^* = \Psi_c^* \Psi_o^*$, where

$$\begin{cases} \Psi_o : \mathcal{R}^n \to \mathcal{L}_2[0, \infty) \\ \Psi_o x_0 = Ce^{At} x_0 =: y(t), \quad (y \in \mathcal{L}_2[0, \infty)) \end{cases} \Rightarrow \begin{cases} \Psi_o^* : \mathcal{L}_2[0, \infty) \to \mathcal{R}^n \\ \Psi_o^* f = \int_0^\infty e^{A^* \tau} C^* f(\tau) d\tau \end{cases}$$

with $f \in \mathcal{L}_2[0, \infty)$, and

$$\begin{cases} \Psi_c : \mathcal{L}_2(-\infty, 0] \to \mathcal{R}^n \\ (\Psi_c u)(t) = \int_{-\infty}^0 e^{-A^* \tau} B u(\tau) d\tau = \int_0^\infty e^{A^* \tau} B u(-\tau) d\tau =: x_0 \end{cases} \Rightarrow \begin{cases} \Psi_c^* : \mathcal{R}^n \to \mathcal{L}_2(-\infty, 0] \\ \Psi_c^* x_0 = B^* e^{-A^* t} x_0 \end{cases}$$

Thus,

$$\begin{aligned} &\Gamma_G^* : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2(-\infty, 0] \\ &(\Gamma_G^* y)(t) = (\Psi_c^* \Psi_o^* y)(t) = B^* e^{-A^* t} \int_0^\infty e^{A^* \tau} C^* y(\tau) d\tau \\ &\qquad = B^* e^{-A^* t} \int_0^\infty e^{A^* \tau} C^* C e^{A \tau} x_0 d\tau = B^* e^{-A^* t} W_o x_0 \end{aligned}$$

In order to find the singular values of Γ_G, suppose that σ_i is a singular value with v the corresponding eigenvector of $\Gamma_G^*\Gamma_G$, i.e. $\Gamma_G^*\Gamma_G v = \sigma_i^2 v$. Let

$$y := \Gamma_G v = Ce^{At}x_0$$

where

$$x_0 = \int_{-\infty}^{0} e^{-A\tau}Bv(\tau)d\tau$$

Then,

$$\Gamma_G^*\Gamma_G v = \Gamma_G^* y = B^* e^{-A^*t}W_o x_0$$

which must be equal to $\sigma_i^2 v$. Hence,

$$B^*e^{-A^*t}W_o x_0 = \sigma_i^2 v \Rightarrow v(t) = B^* e^{-A^*t}W_o x_0 \sigma_i^{-2} \quad \in \mathcal{L}_2(-\infty, 0] \tag{4.1}$$

Take now w_i to be the i-th eigenvector of $\Gamma_G\Gamma_G^*$. Then , further simple computations show that

$$\begin{aligned}\Gamma_G\Gamma_G^* y = \Gamma_G B^* e^{-A^*t}W_o x_0 &= Ce^{At}\int_{-\infty}^{0} e^{-A\tau}BB^*e^{-A^*t}W_o x_0 d\tau \\ &= Ce^{At}W_c W_o x_0 = Ce^{At}\sigma_i^2 x_0\end{aligned}$$

which suggests that $y(t) := Ce^{At}x_0$ is the eigenvector of $\Gamma_G\Gamma_G^*$ corresponding to the i-th eigenvalue, σ_i^2. However, by assumption it is known that this eigenvector is w_i, so

$$w_i = Ce^{At}x_0. \tag{4.2}$$

Alternatively, the Schmidt vectors (in the Laplace domain) can be obtained via the following algorithm ([Fra87],[GL95],[ZDG96]):

Algorithm 4.3.1. (Schmidt pairs)

step 1. *Separate the system into causal and anti-causal parts*

$$G = G_c + G_a$$

step 2. *Find a minimal realisation of* G_c .

step 3. *Find the controllability and observability gramians by solving the following Lyapunov equations :*

$$AW_c + W_c A' + BB' = 0$$

$$W_o A + A'W_o + C'C = 0$$

step 4. *Find*

$$\gamma_0^2 = \lambda_{max}(W_c W_o)$$

step 5. *Find $x \in \mathcal{R}^n$: such that $W_c W_o x = \gamma_0^2 x$, $x \neq 0$, and define $\xi := \gamma_0^{-1} W_o x$. Then,*

$$W_c \xi = \gamma_0 x$$
$$W_o x = \gamma_0 \xi \ .$$

step 6. *Define*

$$v(s) := B'(sI + A')^{-1} \xi$$
$$w(s) := C(sI - A)^{-1} x \ .$$

Thus, the Laplace transform of a Schmidt pair can be written in terms of transfer matrix functions,

$$v(s) \stackrel{s}{=} \left[\begin{array}{c|c} -A' & \xi \\ \hline B' & 0 \end{array} \right] \in \mathcal{RH}_2^{\perp}, \quad w(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & x \\ \hline C & 0 \end{array} \right] \in \mathcal{RH}_2 \tag{4.3}$$

Remark 4.3.3. *If σ is a nonzero singular value of Γ_G of multiplicity one, then clearly the corresponding Schmidt pair is uniquely determined up to modulo scaling. However, there exist other maximising vectors (see definition 1.7.3), i.e. vectors for which Γ_G attains its norm. A particular construction which we shall consider later in chapter 6, is made in [JL93]. The latter and that in [LHG89], construct maximising vectors in $\mathcal{RH}_\infty$ and $\mathcal{RH}_\infty^-$ which form scaled version of Schmidt vectors in $\mathcal{RH}_2$ and $\mathcal{RH}_2^-$, respectively.*

Remark 4.3.4 (Multiplicity considerations). *Consider a stable system G with (A, B, C) a balanced realisation. Assume that the Hankel singular values are $\sigma_1 = \cdots = \sigma_r > \sigma_{r+1} \geq \cdots \geq \sigma_n > 0$. Further, define l to be the normal rank of the Laplace transform of the matrix formed by the r Schmidt vectors of Γ_G corresponding to σ_1. $G \stackrel{s}{=} (A, B, C)$ and hence $V(s)$ and $W(s)$ are given by*

$$V(s) = B'(sI + A')^{-1} \Xi \in \mathcal{RH}_2^{\perp, m \times r}, \quad \Xi = \sigma_1^{-1} P \left[\begin{array}{cccc} x_1 & x_2 & \ldots & x_r \end{array} \right]$$

and

$$W(s) = C(sI - A)^{-1} \Theta \in \mathcal{H}_2^{p \times r}, \quad \Theta = \left[\begin{array}{cccc} x_1 & x_2 & \ldots & x_r \end{array} \right]$$

where P and Q are the controllability and observability matrices of $G \stackrel{s}{=}(A, B, C)$ and the x_i's are r linearly independent eigenvectors of QP corresponding to the eigenvalue σ_1^2.

In particular, if (A, B, C) is balanced (Remark 4.3.2), $P = Q = \mathrm{diag}(\sigma_1 I_r, \Sigma_2)$, and thus $\Xi = E_r$ and $\Theta = \sigma_1^2 E_r$ (where E_r denotes the first r-columns of the $n \times n$ unit matrix), so that

$$V(s) = \sigma_1^2 B'(sI + A')^{-1} E_r \in \mathcal{RH}_2^{\perp} \text{ and } W(s) = C(sI - A)^{-1} E_r \in \mathcal{RH}_2$$

Thus,

$$l := rank_{\mathcal{R}(s)} V^{\sim}(s) \geq \lim_{s \to \infty} [sV^{\sim}(s)] = rank\,(E_r' B) =: rank\,(B_1)$$

and

$$l := rank_{\mathcal{R}(s)} W(s) \geq \lim_{s \to \infty} [sW(s)] = rank\,(CE_r) =: rank\,(C_1)$$

where we partitioned $B' = \begin{bmatrix} B_1' & B_2' \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$. It is further shown in [Glo86] that these two inequalities are actually equalities. Thus $l \leq \min(p, m, r)$ and l can be easily determined from the balanced realisation of G.

The following example, constructed in MATLAB, illustrates the argument of Remark 4.3.4:

Example 4.3.1. *Consider a $G \in \mathcal{RH}_\infty$ with the following minimal balanced realisation:*

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] := \left[\begin{array}{ccc|cc} -1 & -1 & -\frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & \sqrt{10}/5 & (2\sqrt{10})/5 \\ -3 & -4 & -\frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & (2\sqrt{10})/5 & (4\sqrt{10})/5 \\ -\frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & -\frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & -\frac{1}{0.7} & 1 & 1 \\ \hline \sqrt{10}/5 & (2\sqrt{10})/5 & 1 & 0 & 0 \\ (2\sqrt{10})/5 & (4\sqrt{10})/5 & 1 & 0 & 0 \end{array}\right]$$

Then, the gramians are equal to

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

and we observe that the largest Hankel singular value of G ($\sigma_1 = \sigma_2 = 1$) has multiplicity 2. Further, we compute the Schmidt pair corresponding to the largest

Hankel singular value, according to algorithm 4.3.1 and remark 4.3.2.

$$V(s) \overset{s}{=} \left[\begin{array}{c|c} -A' & E_2 \\ \hline B' & 0 \end{array}\right] := \left[\begin{array}{ccc|cc} 1 & 3 & \frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & 1 & 0 \\ 1 & 4 & \frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & 0 & 1 \\ \frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & \frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & \frac{1}{0.7} & 0 & 0 \\ \hline \sqrt{10}/5 & (2\sqrt{10})/5 & 1 & 0 & 0 \\ (2\sqrt{10})/5 & (4\sqrt{10})/5 & 1 & 0 & 0 \end{array}\right]$$

and

$$W(s) \overset{s}{=} \left[\begin{array}{c|c} A & E_2 \\ \hline C & 0 \end{array}\right] := \left[\begin{array}{ccc|cc} -1 & -1 & -\frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & -1 & 0 \\ -3 & -4 & -\frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & -0 & 1 \\ -\frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & -\frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & -\frac{1}{0.7} & 0 & 0 \\ \hline \sqrt{10}/5 & (2\sqrt{10})/5 & 1 & 0 & 0 \\ (2\sqrt{10})/5 & (4\sqrt{10})/5 & 1 & 0 & 0 \end{array}\right]$$

Then, the normal rank of $V(s)$ and $W(s)$ is $l = rank(B_1) = rank(C_1) = 1 < r$. Further, taking the transfer functions of the Schmidt pair,

$$V(s) = \frac{1}{p_v(s)}\begin{pmatrix} s+0.3361 & 0 \\ 0 & s-0.5462 \end{pmatrix}\begin{pmatrix} 0.63246 & 1.2649 \\ 1.2649 & 2.5298 \end{pmatrix}\begin{pmatrix} s-2 & 0 \\ 0 & s+0.5 \end{pmatrix}$$

where $p_v(s) = (s-6.155)(s^2-0.2733s+0.2321)$, and

$$W(s) = \frac{1}{p_w(s)}\begin{pmatrix} s-0.3361 & 0 \\ 0 & s+0.5462 \end{pmatrix}\begin{pmatrix} 0.63246 & 1.2649 \\ 1.2649 & 2.5298 \end{pmatrix}\begin{pmatrix} s-2 & 0 \\ 0 & s+0.5 \end{pmatrix}$$

where $p_w(s) = (s+6.155)(s^2+0.2733s+0.2321)$. Then,

$$V^{\sim}(s)V(s) = \frac{1}{p(s)}\begin{pmatrix} s+2 & 0 \\ 0 & s-0.5 \end{pmatrix}\begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} s-2 & 0 \\ 0 & s+0.5 \end{pmatrix} = W^{\sim}(s)W(s)$$

where, $p(s) = p_v(s)p_w(s)$. The fact that $V^{\sim}(s)V(s) = W^{\sim}(s)W(s)$ is a characteristic property that all Schmidt pairs share ([Fra87]).

4.4 Nehari's Theorem

Throughout this section consider the problem of finding the distance (by means of the induced norm) from an $\mathcal{L}_\infty$ matrix function G to $\mathcal{H}_\infty^-$:

$$\text{dist}(G, \mathcal{H}_\infty^-) := \inf\{\|G-Q\|_\infty : Q \in \mathcal{H}_\infty^-\}.$$

i.e. we want to approximate, in the $\mathcal{L}_\infty-$norm sense, a given unstable (i.e. mixed pole-inertia) transfer matrix by an antistable one. A lower bound for the distance can be immediately obtained. Fix $Q \in \mathcal{H}_\infty^-$. Then

$$\|G - Q\|_\infty \geq \|P_+(G - Q)\mid_{\mathcal{L}_2(-\infty,0]}\| = \|\Gamma_G - \Gamma_Q\| = \|\Gamma_G\|$$

The last equality is due to the fact that $\Gamma_Q = 0$.

Surprisingly, it is shown that the infimum of the infinity norm attains the Hankel norm of G for a class of $Q \in \mathcal{H}_\infty^-$. The result was proved (in a more general context) during the seventies and it is due to Adamjan, Arov and Krein [AAK71],[AAK78]. It is known as the *AAK theorem* or equivalently in operator theory better known as *approximation by meromorphic (matrix-valued) functions* and it is considered to be the cornerstone of model reduction techniques involving Hankel norm approximation. The optimal solution to the general *AAK* problem is ([Glo89],[ZDG96], [GL95]) :

$$\inf_{\substack{Q\, \in \mathcal{H}_\infty^-(k) \subseteq \mathcal{L}_\infty \\ k\, <\, deg(G)}} \|G - Q\|_\infty = \inf_{\substack{Q_-\, \in \mathcal{H}_\infty^- \\ Q_+\, \in \mathcal{H}_\infty^+ \\ deg(Q_+)\, \leq\, k \\ k\, <\, deg(G)}} \|G - Q_- - Q_+\|_\infty = \inf_{\substack{Q_+\, \in \mathcal{H}_\infty^+ \\ deg(Q_+)\, \leq\, k}} \|G - Q_+\|_H = \sigma_{k+1}(\Gamma_G)$$

Note that $Q \in \mathcal{H}_\infty^-(k) \Rightarrow Q = Q_- + Q_+$, where $Q_- \in \mathcal{H}_\infty^-$, $Q_+ \in \mathcal{H}_\infty^+$ and $deg(Q_+) \leq k$. Obviously, if $deg(Q_+) > deg(G)$ (i.e. the Hankel operator Γ_{Q_+} can have higher rank than Γ_G) then the solution is trivial by selecting $Q_+ = G$.

Nehari (1957) first solved the special form of problem ($k = 0$) for the case of scalar discrete time systems. The solution is known as the *Nehari theorem* [Neh57] or simply, in functional analysis terms, *best approximation by analytic functions.* The Nehari theorem is restricted by the assumption that $G \in \mathcal{H}_\infty^+$, i.e. it involves the best approximation of a stable system G by an antistable system Q. Throughout the present work, the Nehari problem is considered for continuous time multivariable systems; thus the problem is posed as follows:

$$\inf_{Q \in \mathcal{H}_\infty^-} \|G - Q\|_\infty = \|\Gamma_G\| =: \sigma_1(\Gamma_G) \qquad \textbf{(Nehari's Theorem)}$$

The problem with inverse inertia operators involves approximating a $G \in \mathcal{H}_\infty^-$ by $Q \in \mathcal{H}_\infty^+$. This is a Nehari-type problem :

$$\inf_{Q \in \mathcal{H}_\infty^+} \|G - Q\|_\infty = \|\Gamma_{G(-s)}\|$$

Similarly for inverse (mixed) inertia operators, the AAK problem is formulated as :

$$\inf_{\substack{Q \in \mathcal{H}_\infty^+(k) \subseteq \mathcal{L}_\infty \\ k < deg(G(-s))}} \|G-Q\|_\infty = \inf_{\substack{Q_+ \in \mathcal{H}_\infty^+ \\ Q_- \in \mathcal{H}_\infty^- \\ deg(Q_-) \leq k \\ k < deg(G(-s))}} \|G-Q_- - Q_+\|_\infty = \inf_{\substack{Q_- \in \mathcal{H}_\infty^- \\ deg(Q_-) \leq k}} \|G(-s)-Q_-\|_H$$

Remark 4.4.1. *On certain occasions, in this work, the precise definition of the distance problem may vary depending on the nature of the application. In our framework the standard formulation of approximation problem will involve the minimisation of the infinity norm of $G+Q$ where $G \in \mathcal{RH}_\infty^-$ and $Q \in \mathcal{RH}_\infty^+$. The solution of this problem is exactly the same as the one already described (Nehari), with the only difference that Q is replaced by $-Q$ and thus, the state-space formulae are changed by means of changing the sign in the "B" and "D" part of Q. Note that the inertia properties of the transfer matrices remain the same. Further, we restrict ourselves to the case of real rational matrix functions $G(s)$ for which an explicit solution exists [Glo84],[Glo89].*

Theorem 4.4.1. (Suboptimal approximation) *Take $G \in \mathcal{RH}_\infty^+$. Then, there exists $J \in \mathcal{RH}_\infty^-$ such that*

(1) $G_a + J := \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} G+J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ *is γ-allpass, where* $\gamma > \sigma_1(\Gamma_G)$.

(2) $\|J_{22}\|_\infty < \gamma$.

Then, all suboptimal approximations $Q \in \mathcal{H}_\infty^-$ such that

$$\|G+Q\|_\infty \leq \gamma$$

are given by

$$Q = \mathcal{F}_l(J, \gamma^{-1}\mathcal{BH}_\infty^-)$$

The proof of this theorem constructs a square system J such that the "error" system $E := G_a + J$ is γ-allpass, i.e. $E^\sim(s)E(s) = \gamma^2 I$ $(> \sigma_1(\Gamma_G))$. From an engineering point of view this is the same as saying that the first (largest) singular value of E is constant at γ over all frequencies $\omega \in \mathcal{R} \cup \{\infty\}$. Plotting the singular values of E in a Bode diagram, it is clear that the largest singular value of E will be described dynamically by a flat line at a level 20 $\log_{10}(\gamma)$ db. Thus the first singular value, which is essentially the infinity norm, is minimised optimally to γ but sub-optimally in terms of $\sigma_1(\Gamma_G) < \gamma$.

The system J generates all $Q \in \mathcal{H}_\infty^-$ such that the error system $E := G + Q$ is γ-contractive. According to the proof all such Q can be written in terms of a lower linear fractional transformation of $J \in \mathcal{RH}_\infty^-$ and a Φ which is permitted to be anything γ^{-1}-contractive (i.e. in the $\gamma^{-1}\mathcal{H}_\infty$ ball). Noting that

$$G + Q = G + \mathcal{F}_l(J, \Phi) = G + J_{11} + J_{12}\Phi(I - J_{22}\Phi)^{-1}J_{21} = \mathcal{F}_l(G_a + J, \Phi)$$

it is clear that Q has the same input-output dimensions as G, on further noticing that J_{11} has the same input-output dimensions with G, J has the same input-output dimensions with G_a, an embedding of G. Consequently, if J is such that $G_a + J$ is γ-allpass then (due to theorem 4.3.3 [GL95]) the LFT of a γ-allpass system together with any γ^{-1}-contractive Φ will be a γ-contractive system, which in this case is the error system, E. In future, the system J will be referred as a *generator*, since it "generates" all suboptimal approximations Q.

Proof. Here it is not intended to get a full account of the numerous technicalities of the construction and hence the proof will only outline the main ideas presented in ([Glo89], Theorem 3.1) where the complete proof can be found. Although all γ-suboptimal Nehari extensions Q are characterised here, the fact that these generate the full solution set is not proved and the reader is referred to [Glo89].

Suppose, without loss of generality, that G has a balanced realisation (A, B, C) such that

$$A\Sigma + \Sigma A' + BB' = 0$$
$$A'\Sigma + \Sigma A + C'C = 0$$

Further, augment G such that

$$G_a := \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B & 0 \\ \hline C & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} A & B_a \\ \hline C_a & 0 \end{array}\right]$$

and define

$$J : \stackrel{s}{=} \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & D_e \end{array}\right]$$

where $\widehat{A}, \widehat{B}, \widehat{C}$ and D_e are yet to be defined. Then,

$$E := G_a + J \stackrel{s}{=} \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B_a \\ 0 & \widehat{A} & \widehat{B} \\ \hline C_a & \widehat{C} & D_e \end{array}\right]$$

Let

$$\Sigma_e := \begin{bmatrix} \Sigma & I \\ I & \Sigma\Gamma^{-1} \end{bmatrix}$$

to be the controllability grammian[6] of E, where $\Gamma := \Sigma^2 - \gamma^2 I$. Clearly, $\widehat{A}, \widehat{B}, \widehat{C}$ and D_e should be chosen such that Σ_e satisfies the Lyapunov controllability equation. However, this condition is guaranteed (implied) by the requirement that E is γ-allpass. The requirement is true if and only if the conditions below are met:

$$A_e\Sigma_e + \Sigma_e A_e' + B_e B_e' = 0$$

$$D_e D_e' = \gamma^2 I_{p+m}$$

$$D_e B_e' + C_e \Sigma_e = 0$$

Expanding the first and third equations gives:

$$\begin{bmatrix} A & 0 \\ 0 & \widehat{A} \end{bmatrix} \begin{bmatrix} \Sigma & I \\ I & \Sigma\Gamma^{-1} \end{bmatrix} + \begin{bmatrix} \Sigma & I \\ I & \Sigma\Gamma^{-1} \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & \widehat{A}' \end{bmatrix} + \begin{bmatrix} B_a \\ \widehat{B} \end{bmatrix} \begin{bmatrix} B_a' & \widehat{B}' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D_e \begin{bmatrix} B_a' & \widehat{B}' \end{bmatrix} + \begin{bmatrix} C_a & \widehat{C} \end{bmatrix} \begin{bmatrix} \Sigma & I \\ I & \Sigma\Gamma^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Then, it is immediate that

$$\widehat{C} := -C_a\Sigma - D_e B_a'$$

$$\widehat{B} := \Gamma^{-1}(\Sigma B_a + C_a' D_e)$$

$$\widehat{A} := -A' - \widehat{B} B_a'$$

Clearly, an appropriate D_e that satisfy the above would be:

$$D_e = \begin{bmatrix} 0 & \gamma I \\ \gamma I & 0 \end{bmatrix}$$

□

[6]Here, we construct "out of the air" a specific gramian. An approach with all possible Σ_e is considered in ([Glo84], lemma 8.2).

Properties of generator J

Observe that since $\|J_{22}\|_\infty < \gamma$ then $J_{12}(j\omega)$ is nonsingular (invertible) for every $\omega \in \mathcal{R} \cup \{\infty\}$ and in particular D_{12} is invertible. This is confirmed by the realisations above, since the system zeros of J_{12} are given as:

$$\lambda_i(\widehat{A} - \widehat{B}_2 D_{12}^{-1} \widehat{C}_1) = \lambda_i(-A'), \quad \forall i.$$

(see [Ros70]). Hence we deduce that J_{12} has no transmission zeros on the imaginary axis and hence $J_{12}(j\omega)$ has full rank over all frequencies $\omega \in \mathcal{R} \cup \{\infty\}$. Dually, the same argument applies to J_{21}.

It is a fact (see [GL95], lemma 4.1.2, p.137) that every unobservable mode of the natural realisation of $\mathcal{F}_l(E, \Phi)$ is a system zero of $E_{12} = J_{12}$ (i.e. the $(1,2)$-block of the first term that appears in the LFT), provided that Φ has a minimal realisation and that the closed-loop is well-posed. Similarly, every uncontrollable mode of the realisation of $\mathcal{F}_l(E, \Phi)$ is a system zero of $E_{21} = J_{21}$ (i.e. the $(2,1)$-block of the first term that appears in the LFT).

Now, by construction, $E = G_a + J$ satisfies $E^\sim E = EE^\sim = \gamma^2 I$. Expanding the later equation we get

$$\begin{pmatrix} (G + J_{11})^\sim & J_{21}^\sim \\ J_{12}^\sim & J_{22}^\sim \end{pmatrix} \begin{pmatrix} G + J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \gamma^2 I & 0 \\ 0 & \gamma^2 I \end{pmatrix} \tag{4.4}$$

Then from the (1,1) partition of equation (4.4), for each $\omega \in \mathcal{R} \cup \{\infty\}$,

$$\begin{aligned}
(G + J_{11})^\sim (G + J_{11})(j\omega) &= \gamma^2 I - (J_{21}^\sim J_{21})(j\omega) \\
\overline{\lambda}\{(G + J_{11})^\sim (G + J_{11})(j\omega)\} &= \overline{\lambda}\{\gamma^2 I - (J_{21}^\sim J_{21})(j\omega)\} \\
\overline{\lambda}\{(G + J_{11})^\sim (G + J_{11})(j\omega)\} &= \gamma^2 I - \underline{\lambda}\{(J_{21}^\sim J_{21})(j\omega)\} \\
\overline{\lambda}\{(G + J_{11})^\sim (G + J_{11})(j\omega)\} &\leq \gamma^2 I
\end{aligned}$$

But note that $J_{21}(j\omega)$ has full rank over all $\omega \in \mathcal{R} \cup \infty$, so $\underline{\lambda}(J_{21}^\sim J_{21})(j\omega) > 0$. Then,

$$\begin{aligned}
\overline{\lambda}(\{G + J_{11})^\sim (G + J_{11})(j\omega)\} &< \gamma^2 I \\
\|G + J_{11}\|_\infty &< \gamma
\end{aligned}$$

that is $G + J_{11}$ is γ-strictly contractive.

Next, consider the $(2,2)$ partition of the above matrix identity. This gives:

$$J_{22}^\sim J_{22} = \gamma^2 I - J_{12}^\sim J_{12}$$

Then a similar analysis as in partition $(1, 1)$ shows that

$$\|J_{22}\|_\infty < \gamma \Rightarrow \|E_{22}\|_\infty < \gamma$$

To conclude, E is γ-allpass and $E_{21}(j\omega)$ has full rank, and thus, via *(Theorem 4.3.3, [GL95])* it follows that $Q = \mathcal{F}_l(E, \Phi)$ is γ-strictly contractive, for any $\Phi \in \gamma^{-1}\mathcal{BH}_\infty^-$.

Now choose $\gamma = \sigma_1 := \sigma_1(\Gamma_G)$. According to Theorem 4.4.1, the suboptimal approximation gives $\|G + Q\|_\infty \leq \gamma$, which in this case makes the error system *contractive* (and not *strictly contractive*) in terms of σ_1:

$$\|G + Q\|_\infty \leq \sigma_1$$

Next consider the problem of characterising the set of all optimal approximations $Q \in \mathcal{H}_\infty^-$ such that the infimum :

$$\inf_{Q \in \mathcal{H}_\infty^-} \|G + Q\|_\infty = \sigma_1$$

is attained. The procedure is similar to the suboptimal approximation, but now a more refined treatment of the problem is needed since, for example, the multiplicity of the largest Hankel singular value becomes an issue here.

Theorem 4.4.2. (Optimal approximation) *Suppose $G \in \mathcal{RH}_\infty^+$. Then there exists an optimal approximation $Q \in \mathcal{H}_\infty^-$ such that*

$$\|G + Q\|_\infty = \|\Gamma_G\| =: \sigma_1$$

Further, an optimal Q may be chosen so that $Q \in \mathcal{RH}_\infty^-$.

Proof. The proof is constructive and it is presented via the following algorithm. □

Algorithm 4.4.1 ([Glo89]). *Without loss of generality we consider G to have a balanced realisation such that*

$$A\Sigma + \Sigma A' + BB' = 0$$

$$A'\Sigma + \Sigma A + C'C = 0$$

Here,

$$\Sigma := \begin{bmatrix} \sigma_1 I_r & 0 \\ 0 & \widehat{\Sigma} \end{bmatrix} > 0$$

where $\sigma_1 > \sigma_2 \geq \ldots \geq \sigma_{n-r} > 0$ are the Hankel singular values of G and r is the multiplicity of the largest Hankel singular value. Further, $\widehat{\Sigma} := diag(\sigma_2, \ldots, \sigma_{n-r})$.

1. *Partition the transfer matrix conformally, i.e.*

$$A = \begin{array}{c} \\ r \\ n-r \end{array}\begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array}, \quad B = \begin{array}{c} \\ r \\ n-r \end{array}\begin{array}{c} m \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{array}, \quad C = p \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} \end{array}$$

2. *The* $(1,1)$ *block from the Lyapunov equations of* G*, using the above conformal partitioning, give*

$$-A_{11} - A'_{11} = (\sigma_1)^{-2} B_1 B'_1 = (\sigma_1)^{-2} C'_1 C_1$$

Hence by [Glo89], lemma 2.2, there exists a $\sigma_1(\Gamma_G)$*-unitary matrix*

$$D_e := \begin{array}{c} \\ p \\ m-l \end{array}\begin{array}{c} \begin{array}{cc} m & p-l \end{array} \\ \begin{bmatrix} \sigma_1 D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \end{array}$$

where $l = rank(C_1) = rank(B_1)$ *(see remark 4.3.4), such that*

$$\begin{bmatrix} C'_1 & 0 \end{bmatrix} D_e + \begin{bmatrix} B_1 & 0 \end{bmatrix} = 0$$

3. *Using the same notation as in suboptimal approximation problem, define the augmented systems* G_a *and* J*. Then the corresponding error system is given by* $E := G_a + J$*, i.e.*

$$E \stackrel{s}{=} \begin{array}{c} \\ r \\ n\text{-}r \\ n\text{-}r \\ p \\ m\text{-}l \end{array}\begin{array}{c} \begin{array}{ccccc} r & n\text{-}r & n\text{-}r & m & p\text{-}l \end{array} \\ \left[\begin{array}{ccc|cc} A_{11} & A_{12} & 0 & B_1 & 0 \\ A_{21} & A_{22} & 0 & B_2 & 0 \\ 0 & 0 & \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline C_1 & C_2 & \widehat{C}_1 & \sigma_1 D_{11} & D_{12} \\ 0 & 0 & \widehat{C}_2 & D_{21} & 0 \end{array}\right] \end{array}$$

Further, define

$$\Sigma_e = \begin{array}{c} \\ r \\ n-r \\ n-r \end{array}\begin{array}{c} \begin{array}{ccc} r & n-r & n-r \end{array} \\ \begin{bmatrix} \sigma_1 I_r & 0 & 0 \\ 0 & \widehat{\Sigma} & I_{n-r} \\ 0 & I_{n-r} & \widehat{\Sigma}\Gamma^{-1} \end{bmatrix} \end{array}$$

to be the controllability gramian with $\Gamma := (\widehat{\Sigma})^2 - (\sigma_1)^2 I_{n-r}$.

4. Now, define

$$\begin{aligned}
\widehat{B}_1 &:= \Gamma^{-1}(\widehat{\Sigma}B_2 + \sigma_1 C_2' D_{11}) \\
\widehat{B}_2 &:= \Gamma^{-1}C_2' D_{12} \\
\widehat{A} &:= -A_{22}' - \widehat{B}_1 B_2' \\
\widehat{C}_1 &:= -C_2\widehat{\Sigma} - \sigma_1 D_{11} B_2' \\
\widehat{C}_2 &:= -D_{21}B_2' \\
\widehat{D} &:= \begin{bmatrix} \gamma D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix}
\end{aligned}$$

5. Then all Nehari extensions of G are given by $Q = \mathcal{F}_l(J, \Phi) = \mathcal{F}_l\left(\left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & \widehat{D} \end{array}\right], \Phi\right)$*, where* $\Phi \in (\sigma_1)^{-1}\mathcal{BH}_\infty^-$.

The above construction can be verified using the allpass equations of E. Since E is γ-allpass, $G+Q$ is σ_1–allpass, with $\|G+Q\|_\infty = \sigma_1$. Further, it is proved that all solutions are generated by the above LFT form, so now J acts as an optimal generator ([Glo89], theorem 4.1). The analysis is similar to that for sub-optimal approximations, but in this case E_{21} has full *row* rank whereas E_{12} has full *column* rank on the extended imaginary axis. The following proposition shows that the Schmidt vectors of the corresponding Hankel operator are intimately linked with optimal Nehari extensions and in the scalar case define the optimal Nehari extension uniquely.

Proposition 4.4.1 ([ZDG96]). *Suppose* $G \in \mathcal{RH}_\infty^+$ *such that*

$$\inf_{Q \in \mathcal{H}_\infty^+} \|G - Q\|_\infty = \sigma_1(\Gamma_G)$$

Then

$$(G(s) - Q(s))v(s) = (\sigma_1(\Gamma_G))w(s).$$

Further, in the case where $G(s)$ *is a scalar function (i.e. SISO system) then,*

$$Q(s) = G(s) - (\sigma_1(\Gamma_G))\frac{v(s)}{w(s)}$$

is the unique solution to the Nehari problem.

Proof. The proof is based on simple Hilbert space properties and it is given in [ZDG96]. Suppose $H(s) := (G(s) - Q(s))v(s)$; observe also that $\Gamma_G v \in \mathcal{RH}_2$ and

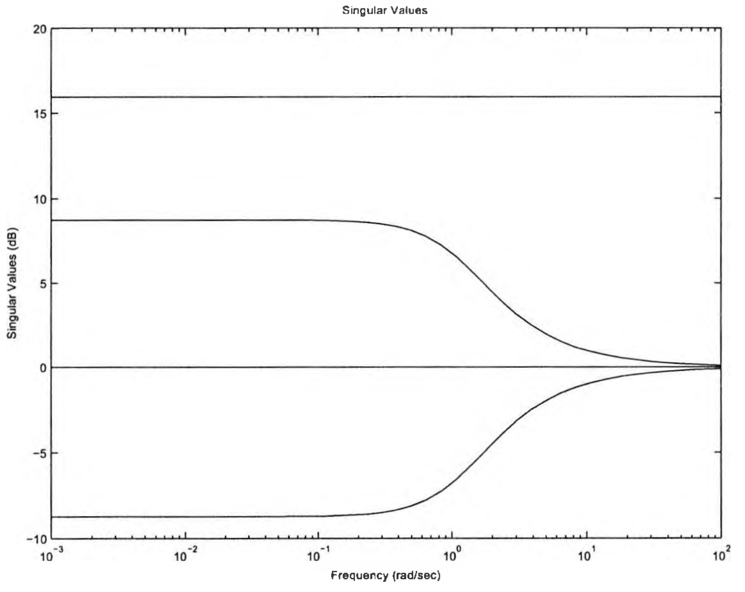

Figure 4.3: Typical singular values plot of an optimal error system $G+Q$ for a dynamic Φ; the largest singular value has a constant value, over all frequencies, which is the smallest it can be.

that $P_+H = P_+(Gv) = \Gamma_G v$. Then,

$$\begin{aligned}
0 &\leq \|H - \Gamma_G v\|_2^2 \\
&= \|H\|_2^2 + \|\Gamma_G v\|_2^2 - \langle H, \Gamma_G v\rangle - \langle \Gamma_G v, H\rangle \\
&\leq \|H\|_2^2 + \|\Gamma_G v\|_2^2 - \langle H, P_+\Gamma_G v\rangle - \langle P_+\Gamma_G v, H\rangle \\
&= \|H\|_2^2 + \|\Gamma_G v\|_2^2 - \langle \Gamma_G v, \Gamma_G v\rangle - \langle \Gamma_G v, \Gamma_G v\rangle \\
&= \|H\|_2^2 + \|\Gamma_G v\|_2^2 - \|\Gamma_G v\|_2^2 - \langle \Gamma_G v, \Gamma_G v\rangle \\
&= \|H\|_2^2 - \langle \Gamma_G v, \Gamma_G v\rangle = \|H\|_2^2 - \langle v, \Gamma_G^* \Gamma_G v\rangle \\
&= \|H\|_2^2 - (\sigma_1(\Gamma_G))^2 \langle v, v\rangle = \|H\|_2^2 - (\sigma_1(\Gamma_G))^2 \|v\|_2^2 \\
&= \|(G-Q)v\|_2^2 - (\sigma_1(\Gamma_G))^2\|v\|_2^2 \leq \|G-Q\|_\infty^2 \|v\|_2^2 - (\sigma_1(\Gamma_G))^2 \|v\|_2^2 \\
&= 0
\end{aligned}$$

Hence, $\|H - \Gamma_G v\|_2^2 = 0 \Rightarrow H = \Gamma_G v$, i.e. $(G(s) - Q(s))v(s) = \Gamma_G v(s) = (\sigma_1(\Gamma_G))w(s)$. □

In the case that G is scalar valued, Q is uniquely determined by $Q(s) = G(s) - \sigma_1(\Gamma_G)\frac{v(s)}{w(s)}$. Nevertheless, it should be noted that a complete characterisation of Q in terms of the Schmidt pair, in the multivariable case, is considered to be an open issue for further research.

4.5 Examples

The examples in this section summarise the main points of the previous paragraphs. In particular, the Nehari approximation is considered for both scalar and multivariable cases. At first, in the scalar case only the computation of Schmidt vectors is required in order to specify the (unique) Nehari extension, whereas in the multivariable case the algorithm 4.4.1 is followed without the need of computing a Schmidt pair. The first example makes use of proposition 4.4.1:

Example 4.5.1. *Find the Nehari extension, $Q \in \mathcal{RH}_\infty^-$ of the strictly proper $G \in \mathcal{RH}_\infty$ defined as follows*

$$G(s) := \frac{2\sqrt{2}s + 4}{s^2 + \sqrt{2}s + 1}$$

A corresponding state-space model is

$$A = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2\sqrt{2} & 4 \end{bmatrix}, \quad D = 0.$$

By computing its gramians:

$$W_c = \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}, \quad W_o = \begin{bmatrix} 6\sqrt{2} & 8 \\ 8 & 6\sqrt{2} \end{bmatrix}$$

it is easy to find the Hankel singular values,

$$(\sigma_1(\Gamma_G), \sigma_2(\Gamma_G)) = (\lambda_1(W_oW_c), \lambda_2(W_oW_c)) = (\sqrt{2}+1, \sqrt{2}-1).$$

Then, following algorithm (4.3.1),

$$x = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \xi = \begin{bmatrix} -2(\sqrt{2}-1) \\ 2(\sqrt{2}-1) \end{bmatrix}$$

and the Schmidt vectors corresponding to the largest Hankel singular value of G are

$$(v_1(s), w_1(s)) = \left(\frac{-2(\sqrt{2}-1)(s+1)}{s^2 - \sqrt{2}s + 1}, \frac{2(\sqrt{2}-1)(s-1)}{s^2 + \sqrt{2}s + 1} \right)$$

Following Proposition 4.4.1, the (unique) Nehari extension of the scalar valued system $G(s)$ is

$$\begin{aligned} Q(s) = G(s) - \sigma_1 \frac{v_1(s)}{w_1(s)} &= \frac{2\sqrt{2}s + 4}{s^2 + \sqrt{2}s + 1} - (\sqrt{2}+1) \\ &= \frac{(-\sqrt{2}-1)s^2 + (\sqrt{2}-2)s + (3-\sqrt{2})}{s^2 + \sqrt{2}s + 1} \end{aligned}$$

Example 4.5.2 ([Glo84]). *Find the Nehari extension of the following transfer function:*

$$G(s) := \frac{39s^2 + 105s + 250}{(s+2)(s+5)^2}$$

Obviously, $G \in \mathcal{RH}_\infty^+$ and hence, it is asked to find the distance from G to $\mathcal{RH}_\infty^-$. Firstly, take a balanced state-space realisation of $G(s)$,

$$A = \begin{bmatrix} -9 & 4 & -4 \\ 4 & -2 & 4 \\ 4 & -4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -6 & 2 & -1 \end{bmatrix}, \qquad D = 0$$

with gramians equal to

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Then, the Nehari extension of $G(s)$, $Q(s)$, is constructed using Algorithm 4.4.1 and has the following state-space realisation:

$$\widehat{A} = \frac{1}{3}\begin{bmatrix} 2 & 10 \\ -8 & 5 \end{bmatrix}, \quad \widehat{B} = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} 2 & 2.5 \end{bmatrix}, \qquad \widehat{D} = -2$$

It is observed that the Hankel singular values of Q are 1 and 0.5. Then,

$$Q(s) = \frac{-6s^2 + 13s - 90}{3s^2 - 7 + 30}$$

such that $G + Q$ is $\sigma_1(\Gamma_G)$-allpass, i.e.

$$\|G + Q\|_\infty = \|\Gamma_G\| = \sigma_1(\Gamma_G) = 2$$

Alternatively, a Q_{alt} such that $G - Q_{alt}$ is $\sigma_1(\Gamma_G)$-allpass can be constructed according to Remark 4.4.1 and is given by:

$$Q_{alt}(s) = \frac{6s^2 - 13s + 90}{3s^2 - 7 + 30}$$

4.6 Summary

In this chapter Hankel operators were defined and their role in control theory (e.g. observability and controllability operators) and model reduction was described. The need to approximate systems in the $\mathcal{H}_\infty$ norm brought Hankel norm approximation into control theorists attention, as an upper bound, since $\mathcal{H}_\infty$ norm approximation is a difficult problem not yet solved. A key point for understanding and solving the HNA is the Schmidt decomposition of Hankel operators which was extensively used by operator theorists. However, in the case of rational approximation an elegant solution, based only on state-space methods, was given by Glover [Glo84],[Glo89] which was studied towards the end of the chapter.

Nehari approximation is extensively used in the following chapter as an important tool to robustness synthesis, where the need arises to approximate anti-stable rational matrix functions (i.e. inverse inertia problems). As shown here, the same theory applies, but with opposite inertia considerations. It is then proved that the smallest Hankel singular value of the anti-stable part of the open loop system is a robustness measure. Further, in chapter 6 extensions of Nehari approximation of rational (anti-stable) matrix functions are developed for matrix-valued problems in the sense that all degrees of freedom are exploited to minimise additional objectives.

Chapter 5

Robust Stabilisation

In general, physical systems are typically highly complex, nonlinear and time-varying. However, it depends on the designer's judgement to some extent, whether to describe the real system by a complex model or make assumptions that relax the complexity of the system. From a platonic point of view a model can never represent exactly the true system but is only an approximation. Hence, uncertainty always arises in the modelling process with a trade off appearing between the degree of complexity and the degree of accuracy (in terms of approximation). Throughout this work, as discussed in previous chapters, only linear time-invariant systems are considered. This assumption is well suited to the present mathematical framework used for control design but, on the other hand, it should also somehow fit to a pragmatic description of the physical problem, so that the gap between model and reality is minimised.

Uncertainty in feedback systems appears mainly in the form of unmodelled or neglected dynamics, parameter variations or nonlinear effects. In order to restore accuracy (despite the assumptions of linearity and time-invariance), uncertainty can be treated as an LTI system for which the only *a priori* information is an upper bound of its "size". In this type of analysis, the "size" of the system is described by a metric induced by the operator (or in engineering terms the system's transfer function) and the degree of complexity is typically measured by the system's McMillan degree.

In order to introduce the main idea of this discussion, an example is presented next in which uncertainty arises in the form of parameter variations.

Example 5.0.1 (Parametric uncertainty [SP96]). *Consider the uncertain system*

$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3$$

At each frequency, parameter variation (inside the ranges specified above) defines *a region of complex numbers, $G_p(j\omega)$. In general, such uncertainty regions have complicated shapes with up to 2^n vertices for n uncertain parameters (see figure 5.1).*

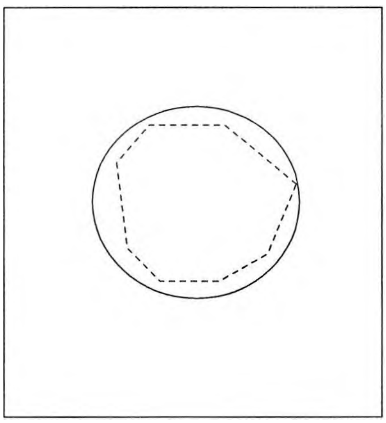

Figure 5.1: Disc approximation of original uncertainty region at a frequency ω_0.

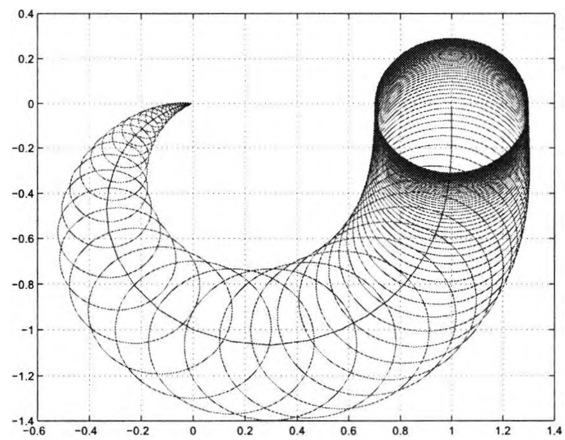

Figure 5.2: Typical Nyquist diagram of uncertain system approximated by discs.

Thus a good approximation to the original (complex) uncertainty region may be obtained by bounding it by a larger disc-shaped region using a frequency-by-frequency scheme. The disc is centred around the value of the chosen nominal plant (i.e. the plant for which uncertainty is equal to zero) at each frequency. Further, the uncertainty radius is defined by the vertex furthest from the centre. Generalising the idea in higher dimensions (i.e. MIMO systems), the uncertainty is modelled as a frequency dependent norm-bounded LTI system, $\Delta(j\omega)$. The norm constraint is described by $\overline{\sigma}(\Delta(j\omega)) < |W(j\omega)|$, for all ω, where $W(j\omega)$ characterises the assumed maximum model uncertainty at various frequencies[1]. However, the *filter* W is usually omitted since it is possible to normalise the uncertainty by defining a uniform bound of 1, at all frequencies, i.e. defining $\tilde{\Delta} := W^{-1}\Delta$ so that $\|\tilde{\Delta}(s)\|_\infty < 1$. The set (family) of all these perturbed systems can be visualised in figure 5.3; Each perturbation Δ, inside the uncertainty ball, results in a slightly different model. This ball is centred at the nominal plant and any perturbation lying inside the ball is a contractive system (operator). Summarising, the complete robust control model is the entire set description which captures the uncertain or unmodelled/neglected aspects of the

[1] $W(j\omega)$ is often estimated by experimental data and it cannot, by any means, be theoretically derived in a trivial sense.

assumed physical system. The size of the model set is constrained by setting a bound on its size, i.e. specifying the norm of Δ.

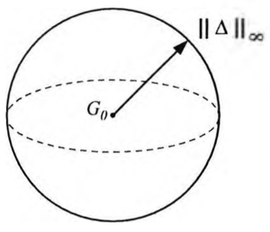

Figure 5.3: Set of uncertain systems

The difference between the model and the true system may be represented in several ways. For multivariable systems the three simplest and most commonly used perturbation models are described below:

1. $G_\Delta(s) = G_0(s) + \Delta_a(s)$

2. $G_\Delta(s) = (I + \Delta_m(s))G_0(s)$ or $G_\Delta(s) = G_0(s)(I + \Delta_m(s))$

3. $G_\Delta(s) = (\tilde{M}(s) + \Delta_M(s))^{-1}(\tilde{N}(s) + \Delta_N(s))$

where Δ_a represents an *additive* perturbation, Δ_m a *multiplicative* (or *proportional*) perturbation, defined at the system's outputs, and $[\Delta_M, \Delta_N]$ represent the factors of a *coprime* perturbation model. Here, G_0 is the nominal plant or the best estimate of the true plant. Further, some additional technical assumptions may need to be enforced in this description. These will be presented in detail later in the chapter.

Example 5.0.2 ([Mac89]). *Consider $\Delta_m = 0.1 = \Delta_a$. Then for the multiplicative uncertainty case :*

$$\|G_\Delta - G_0\|_\infty = \|\Delta_m G_0\|_\infty \leq \|\Delta_m\|_\infty \|G_0\|_\infty \leq 0.1\|G_0\|_\infty$$

i.e. the size of the perturbation is at most 10% of G_0. On the other hand taking $\Delta_a = 0.1$,

$$\|G_\Delta - G_0\|_\infty = \|\Delta_a\|_\infty \leq 0.1$$

i.e. the size of the perturbation in this case is less than the constant value of 0.1.

Remark 5.0.1. *Multiplicative uncertainty is simply a weighted form of additive uncertainty. In particular,*

$$(I + \Delta_m)G_0 = G_0 + \Delta_m G_0 =: G_0 + \Delta_a$$

by defining $\Delta_a := \Delta_m G_0$.

Consider now the following figures (next page) based on the stability analysis of chapter 3. Here it is shown that all three perturbation models defined above, can be written in the form of an upper LFT :

$$G_\Delta = \mathcal{F}_u(P, \Delta) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}$$

For example, it can be easily shown that the additive uncertainty model corresponds to:

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} := \begin{pmatrix} 0 & I \\ I & G_0(s) \end{pmatrix} \tag{5.1}$$

It follows in this case that:

$$\mathcal{F}_u(P, \Delta) := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} \equiv G_0 + \Delta$$

Further, the multiplicative uncertainty model is generated by

$$P(s) := \begin{pmatrix} 0 & G_0(s) \\ I & G_0(s) \end{pmatrix} \tag{5.2}$$

whereas the coprime uncertainty model can similarly obtained (see [MG90]). An interesting observation, worth noting, is that $P_{11}(s)$ (the $(1,1)$ block of the generalised plant) for the case of additive and multiplicative uncertainties is equal to 0, whereas on the other hand, in the case of coprime perturbation models $P_{11}(s)$ contains also a nonzero term.

In this general LFT framework we pose the following problem:

Problem 5.0.1 (Robust stabilisation). *Consider the general uncertain plant in figure 5.4(a). Then, find a controller such that the closed loop system is stable over a set of uncertainties*[2] Δ *which satisfy the norm bound* $\|\Delta\|_\infty < \epsilon$.

A necessary condition of robustness is internal stability. It is obvious that in order to stabilise the family of perturbed plants, in which the nominal plant is also included,

[2] *The problem above is defined for the general class of unstructured perturbations. Again, if we refer to the case of additive or multiplicative type of uncertainties, an extra constraint on the perturbation set should be considered. This "technical assumption" will be introduced later.*

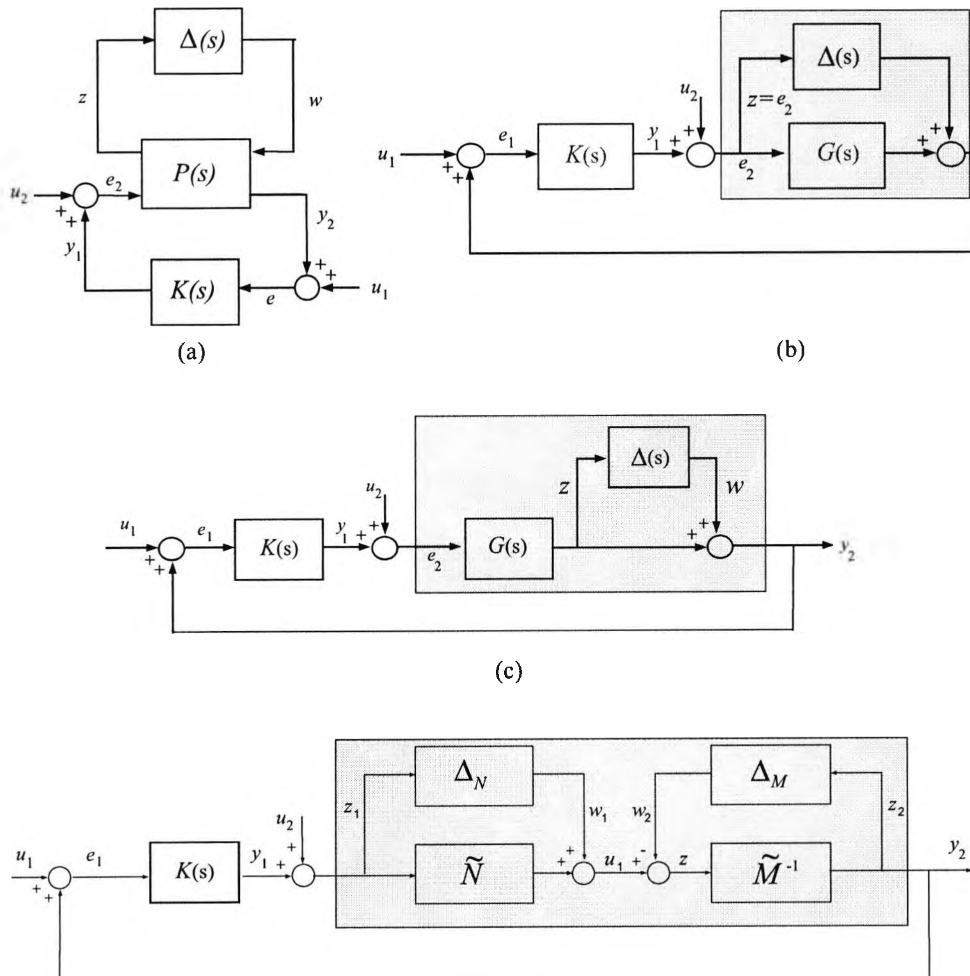

Figure 5.4: Feedback loop systems under unstructured uncertainties and the equivalent generalised plant.

internal stability of the nominal plant should be achieved, i.e. according to the notation of chapter 3, the feedback system $H(G, K)$ of figure 3.1 should be internally stable. Assuming that this condition is satisfied, one way of addressing robust stabilisation is by means of the small gain theorem (see chapter 2, Theorem 2.4.1) of the LFT interconnections of the perturbation and the closed-loop system. Since the perturbation is norm bounded, the nominal closed-loop systems should have norm less or equal than the inverse of this bound, so that the overall closed-loop of figure 5.4(a) is stable for every admissible perturbation.

For certain classes of unstructured uncertainty models the perturbed plant is constrained to have same number of RHP poles as the nominal plant. This condition applies to additive and multiplicative perturbation models and makes it necessary to prove robustness theorems via *homotopy* arguments (rather than via the small gain theorem).

5.1 Robust stability under additive perturbations

Assume $G \in \mathcal{RL}_\infty$. The closed loop system of figure 5.5 (with $\Delta = 0$) is internally stable if and only if it is well-posed, i.e. $\det(I - G(\infty)K(\infty)) \neq 0$ and the four transfer functions $(u_1, u_2) \rightarrow (e_1, e_2)$ given by

$$H(G,K) = \begin{bmatrix} (I-GK)^{-1} & (I-GK)^{-1}G \\ K(I-GK)^{-1} & I-K(I-GK)^{-1}G \end{bmatrix} \tag{5.3}$$

are all in $\mathcal{RH}_\infty$. Define $S \triangleq (I - GK)^{-1}$. Then we need $\det(I - GK)(\infty) \neq 0$ and $S, KS, SG, I - KSG \in \mathcal{RH}_\infty$, in order to ensure internal stability of the feedback system (see chapter 3, figure 3.1).

Now let $G_\Delta = G + \Delta$, as shown in figure 5.5, known as an additive uncertainty model. Here Δ is an LTI system and belongs to a class of perturbations for which there is no *a priori* information about its structure (i.e. on how the uncertainty is distributed among the elements Δ_{ij} of Δ), but only an upper bound of its "size" at each frequency (measured via the largest singular value).

Assuming that controller $K(s)$ stabilises internally $G(s)$, it is often required to determine whether for a specific $\epsilon > 0$ the closed-loop remains internally stable for

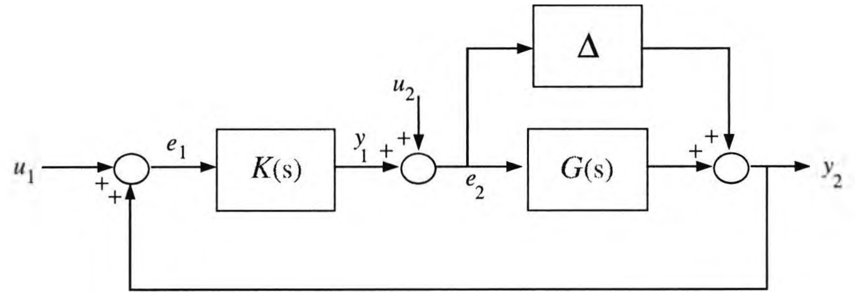

Figure 5.5: Closed-loop with additive uncertainty in the nominal plant

all Δ with norm $\|\Delta\| < \epsilon$. This is an *analysis* problem, in contrast to a *synthesis* problem which further aims to design a controller (or characterise the family of all controllers) with the corresponding robust-stability properties. In the sequel we define the robust stabilisation problem for the special case of additive perturbations and show how it can be recast as an $\mathcal{H}_\infty$ minimisation problem. The latter can be viewed as a suboptimal approximation problem in terms of the uncertainty ball (figure 5.3). The optimal solution to the problem is obtained when the stability radius is maximised (maximally robust stabilisation problem). Next we define the family of all permissible perturbations Δ and give certain technical conditions.

Definition 5.1.1 ([Glo86]). *A permissible perturbation, Δ, is one such that $\Delta \in \mathcal{D}_\epsilon$ where*

$$\mathcal{D}_\epsilon \triangleq \mathcal{D}_{\mathcal{S}_\epsilon} \cup \mathcal{D}_{\mathcal{U}_\epsilon}$$

and

$$\mathcal{D}_{\mathcal{S}_\epsilon} \triangleq \{\Delta \ : \Delta \in \mathcal{RH}_\infty; \quad \|\Delta\|_\infty < \epsilon\}$$
$$\mathcal{D}_{\mathcal{U}_\epsilon} \triangleq \{\Delta \ : \Delta \in \mathcal{RL}_\infty; \quad \eta(\mathcal{F}_u(P,0)) = \eta(\mathcal{F}_u(P,\Delta)) \quad \|\Delta\|_\infty < \epsilon\}$$

where P is the generalised (augmented) plant (5.1), and $\eta(.)$ denotes the number of closed RHP poles of a transfer function, counted in the McMillan degree sense.

Definition 5.1.2 ([Glo86]). *The feedback system in figure 5.5, denoted as $H(G,K)$, is ϵ-robustly stable if and only if $H(G+\Delta,K)$ is internally stable for all $\Delta \in \mathcal{D}_\epsilon$. Further, if there exists K such that $H(G,K)$ is ϵ-robustly stable then (G,ϵ) is said to be robustly stabilisable. Here, ϵ is referred to as the robust stability radius of the feedback system.*

Remark 5.1.1. *Note that G and $G + \Delta$ are required to have only the same number but not (necessarily) the same poles in the closed RHP.*

Remark 5.1.2 (Generalised Nyquist Criterion). *Assume a Δ permissible (see definition 5.1.1), such that $\Delta \in \mathcal{D}_\epsilon$ and further, let D_R be the Nyquist contour as in Theorem 2.3.2. Suppose now that $H(G,K)$ is ϵ-robustly stable. Then $H(G + \beta\Delta, K)$ is internally stable for every $\beta \in [0,1]$.*
Consider the contour

$$\Gamma_\beta = det\left[(I - (G + \beta\Delta)K)(s)\right], \quad s \in D_R, \beta \in [0,1]$$

As β varies between $\beta = 0$ and $\beta = 1$, Γ_β deforms continuously without crossing the origin, making $\eta(G) + \eta(K)$ anticlockwise encirclements around it (recall, straight from definition 5.1.1, $\eta(G) = \eta(G + \Delta)$), for each β. Thus,

$$det\left[(I - GK - \beta\Delta K)(s)\right] \neq 0, \quad s \in D_R, \beta \in [0,1]$$

Now, observe that

$$det(I - GK - \beta\Delta K) = det(I - GK) det(I - \beta\Delta K(I - GK)^{-1})$$

and $(I - GK)^{-1}$ is well-defined on D_R. Hence, we conclude that

$$det\left[(I - \beta\Delta T)(s)\right] \neq 0, \quad \forall s \in D_R, \beta \in [0,1]$$

where $T := K(I - GK)^{-1}$, and thus

$$det\left[(I - \Delta T)(s)\right] \neq 0, \quad \forall s \in D_R$$

To get the idea behind of the above definition note that we extend the notion of stability discussed before, in the sense that now our objective is to design a controller that not only stabilises the nominal plant but a whole family of plants. Of course it is assumed that the family contains the nominal system and hence by stabilising the whole family we also stabilise the nominal system. Consequently, we can say that robust stability is the ability of a closed-loop system to remain stable in the presence of modelling errors.

Theorem 5.1.1 ([MG90]). *Let $G \in \mathcal{RL}_\infty$, then $H(G,K)$ is ϵ-robustly stable if and only if $H(G,K)$ is internally stable and*

$$\|K(I - GK)^{-1}\|_\infty \leq \epsilon^{-1}$$

Proof. [MG90], Theorem 3.3. □

Theorem 5.1.1 implies that the robust stabilisation problem can be formulated as a (sub-optimal) $\mathcal{H}_\infty$ optimisation problem. However, by minimising the $\mathcal{H}_\infty$ norm of a transfer matrix we effectively minimise the maximum energy transfer between the energy of the input and output signals of the corresponding transfer function. In other words, if we choose a controller K that minimises $\|K(I-GK)^{-1}\|_\infty$, we then minimise the energy of the output signal e_2 (control effort) due to the external input signal u_1. The transfer function $K(I-GK)^{-1}$ is often referred as the "control sensitivity" of the feedback system.

Remark 5.1.3. *As initially discussed in the chapter, uncertainty size is a function depending on frequency. A uniform bound on uncertainty size can be accommodated via weighting functions which usually normalise the experimental estimates of the gain at several frequencies* ω. *In this framework, by introducing a weighting function* W *(which can be assumed to be in* $\mathcal{H}_\infty$*, without loss of generality), robust stability is imposed by requiring* $\|WK(I-GK)^{-1}\|_\infty \leq \epsilon^{-1}$. *The weight can be absorbed inside the generalised plant by re-defining:*

$$G' = \begin{pmatrix} WG_{11} & WG_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

the weights are appended to the original system. Hence, without loss of generality, using the sub-multiplicative property of the $\mathcal{H}_\infty$ *norm (Banach algebra property) it is always possible to restate the robust stabilisation problem in terms of weighting functions, using this technique.*

The following proposition shows that when solving the robust stabilisation problem we can assume without loss of generality that $G \in \mathcal{RH}_\infty^-$.

Proposition 5.1.1 ([Glo86]). *Assuming* $G \in \mathcal{RL}_\infty$ *with decomposition* $G = G_1 + G_2$ *such that* $G_1^\sim, G_2 \in \mathcal{RH}_\infty$, $G_1(\infty) = 0$, *then* (G, ϵ) *is robustly stabilisable if and only if*

$$\inf_{K_1} \|K_1(I - G_1K_1)^{-1}\|_\infty \leq \epsilon^{-1}$$

where the infimum is taken over all K_1 *that internally stabilise* $H(G_1, K_1)$ *and such that* $det(I - G_2K_1)(\infty) \neq 0$.

Proof. The proof is based on letting K_1 be such that $H(G_1, K_1)$ is robustly stable and defining $K := K_1(I + G_2K_1)^{-1}$. See [Glo86] for details. □

Remark 5.1.4. *Proposition 5.1.1 above implies that the solution to the robust stabilisation problem is effectively constrained only by the anti-stable part of the plant. For the special case that $G \in \mathcal{RH}_\infty \subset \mathcal{RL}_\infty$, the above proposition is simply proved by setting $K_1 = 0$. As the open-loop plant is already stable, obviously no controller is needed to stabilise it. Thus, to simplify the solution of the robust stabilisation problem we can assume without loss of generality $G \in \mathcal{RH}_\infty^-$.*

Theorem 5.1.1 gives necessary and sufficient conditions for robust stabilisability for the ϵ-ball of perturbations $\mathcal{D}_\epsilon$. The necessity part of the proof in [Vid85] proceeds via a homotopy argument based on the continuous deformation of the Nyquist plot of the nominal plant. To establish sufficiency, the existence of a boundary destabilising perturbations is proved ([Vid85]) by an explicit construction; in fact, it is shown that such destabilising perturbations (i.e. $\Delta \in \partial\mathcal{D}_\epsilon$, $\|\Delta\|_\infty = \epsilon$) can be assumed to be in $\partial\mathcal{D}_{S_\epsilon} \subseteq \partial\mathcal{D}_\epsilon$.

The problem of constructing boundary destabilising perturbations is formally posed below. Algorithm 5.1.1 which follows the problem is adapted from the proof in [Vid85].

Problem 5.1.1. *Suppose $T(s)$ is the transfer-function of the unperturbed (nominal) system in figure 5.5 corresponding to an ϵ-robust stabilising controller. Find $\Delta \in \mathcal{RH}_\infty$ such that*

$$det(I - \Delta(j\omega_0)T(j\omega_0)) = 0, \qquad \|\Delta\|_\infty = \frac{1}{\sigma_1(T(j\omega_0))}$$

for some frequency $\omega_0 \in [0, \infty)$.

The problem is solved by explicitly constructing such a destabilising perturbation. The method is presented via the next algorithm. For a more general setting of the problem see [Vid85], Theorem 4, pp. 273-279.

Algorithm 5.1.1 (Destabilising Perturbation). *Let $T(s) = K(I - GK)^{-1} \in \mathcal{RH}_\infty$ be the transfer function corresponding to a nominal system $G \in \mathcal{RH}_\infty^-$, with minimal realisation $T(s) = C_T(sI - A_T)^{-1}B_T + D_T$. Further, let $\overline{\sigma}(T(j\omega_0)) = \|T\|_\infty$. There are three possible cases: either $\omega_0 = 0$ or $\omega_0 = \infty$ or ω_0 is some finite frequency (which can be assumed positive). In the sequel, it is shown that the cases of $\omega_0 = 0$ and $\omega_0 = \infty$ can be grouped into one case:*

Case 1 : $\omega_0 = 0$ *or* $\omega_0 = \infty$. *If* $\omega_0 = 0$, *then as* A_T *is invertible (since* $T \in \mathcal{RH}_\infty$*):*

$$T(j\omega_0) = T(0) = -C_T A_T^{-1} B_T \in \mathcal{R}^{p\times m}.$$

If $\omega_0 = \infty$

$$T(j\omega_0) = T(\infty) = D_T \in \mathcal{R}^{p\times m}.$$

Then $T(j\omega_0)$ *has a SVD*

$$T(j\omega_0) = U\Sigma V'$$

with U,V *real orthogonal, i.e.* $U \in \mathcal{R}^{p\times p}$ *and* $V \in \mathcal{R}^{m\times m}$. *A destabilising additive perturbation in this case is constructed as follows:*

$$\Delta_{destab}(s) := -V \begin{bmatrix} \frac{1}{\sigma_1(T(j\omega_0))} & 0_{1\times(m-1)} \\ 0_{(p-1)\times 1} & 0_{(p-1)\times(m-1)} \end{bmatrix} U'$$

which is a real constant matrix.

Case 2 : $\omega_0 \in \mathcal{R}_+\backslash\{0\}$. *Let*

$$T(j\omega_0) = C_T(j\omega_0 I - A_T)^{-1}B_T$$

and consider its SVD :

$$T(j\omega_0) = U\Sigma V'$$

Define:

$$\Delta_{destab}(j\omega_0) := V \begin{bmatrix} \frac{1}{\sigma_1(T(j\omega_0))} & 0_{1\times(m-1)} \\ 0_{(p-1)\times 1} & 0_{(p-1)\times(m-1)} \end{bmatrix} U' = \frac{1}{\sigma_1} v_1 u_1'$$

So, at the critical frequency ω_0, $det(I - T(j\omega_0)\Delta_{destab}(j\omega_0)) = 0$. *However, in this case* U, V *may be complex (and thus also* Δ_{destab}, *as defined above).*

To construct a stable, real-rational destabilising perturbation we use an interpolation argument. Take the first column of V, v_1, *which is in general a complex-valued column vector. Write all non-real elements of* v_1 *in polar form:*

$$v_1 = \begin{bmatrix} \rho_{v_1} e^{j\theta_1} \\ \vdots \\ \rho_{v_n} e^{j\theta_n} \end{bmatrix}$$

and force $\theta_i \in (0, \pi)$, $\forall i \in \{1, 2, \ldots, n\}$, *by inverting, if necessary, the positive sign of* ρ_{v_i}, *i.e. set* $\overline{\rho}_{v_i} = -\rho_{v_i}$ *for all* i *such that* $\theta_i \notin (0, \pi)$ *(where* $\overline{\rho}_{v_i} = \{-1, 1\}$*). Otherwise* *set* $\overline{\rho}_{v_i} = \rho_{v_i}$. *Further, select appropriate* $\alpha_i > 0$ *such that* $\arg(j\omega - \alpha_i) - \arg(j\omega + \alpha_i) = \theta_i$. *Then, define*

$$v(s) := \begin{bmatrix} \overline{\rho}_{v_1} \cdot \frac{s-\alpha_1}{s+\alpha_1} \\ \vdots \\ \overline{\rho}_{v_n} \cdot \frac{s-\alpha_n}{s+\alpha_n} \end{bmatrix}$$

i.e. $v_i(s) = \overline{\rho}_{v_i} \frac{s-\alpha_i}{s+\alpha_i}$ *for all non-real entries of* v_1 *and* $v_i(s) = v_{1i}$ *for all* i *such that* v_{1i} *is real.*

Note that $v(s)$ *interpolates* v_1 *at* $s = j\omega_0$. *Geometrically this is described in figure 5.6. In order to select pole-zero pairs* $(-\alpha_i, \alpha_i)$ *with* $\alpha_i > 0$ *such that* $\arg(\frac{j\omega_0 - \alpha_i}{j\omega_0 + \alpha_i}) = \theta_i$ *note that as the location of* α_i *(and* $-\alpha_i$*) varies continuously over the positive (respectively negative) real axis,* θ_i *varies continuously in the interval* $(0, \pi)$. *Therefore, for any* $\theta_i \in (0, \pi)$ *there exists (exactly one) pole-zero pair* $(-\alpha_i, \alpha_i)$ *such that* $\arg(\frac{j\omega_0 - \alpha_i}{j\omega_0 + \alpha_i}) = \theta_i$. *Simple trigonometry gives*

$$\tan\left(\frac{\theta_i}{2}\right) = \frac{\alpha_i}{\omega_0} \Rightarrow \alpha_i = \omega_0 \tan\left(\frac{\theta_i}{2}\right)$$

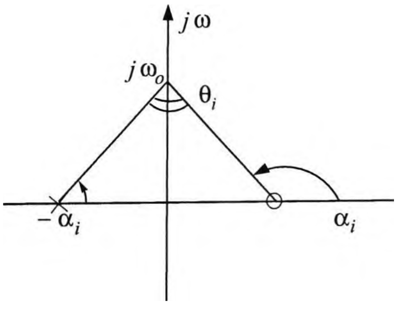

Figure 5.6: Construction of phase angles θ_i

Further, define β_i*'s and* ρ_{u_i}*'s in a similar manner so that*

$$u_1' = \begin{bmatrix} \rho_{u_1} e^{j\phi_1} \\ \vdots \\ \rho_{u_n} e^{j\phi_n} \end{bmatrix}$$

is interpolated by

$$u'(s) := \begin{bmatrix} \overline{\rho}_{u_1} \cdot \frac{s-\beta_1}{s+\beta_1} \\ \vdots \\ \overline{\rho}_{u_n} \cdot \frac{s-\beta_n}{s+\beta_n} \end{bmatrix}$$

at $s = j\omega_0$. Then, an additive destabilising perturbation in $\mathcal{RH}_\infty$, with $\|\Delta_{destab}\|_\infty = \frac{1}{\sigma_1}$ is defined as follows:

$$\Delta_{destab}(s) := -\frac{1}{\sigma_1(T(j\omega_0))} v(s)u'(s)$$

Remark 5.1.5. *By adding Δ_{destab} to the nominal plant, the number of unstable poles remain the same, in both cases. If the critical frequency is $\omega_0 = 0$ or $\omega_0 = \infty$, then Δ_{destab} is a constant real matrix and hence is irrelevant to the poles of $G + \Delta_{destab}$. On the other hand, if ω_0 is finite then Δ_{destab} is a stable dynamical system and thus, it does not affect the number of unstable poles of $G + \Delta_{destab}$ either.*

5.2 The maximally robust stabilisation problem

Up to this point, in terms of figure 5.3, necessary and sufficient robust stability conditions were given which characterise all perturbed plants inside the open ball of radius $\|T\|_\infty^{-1}$. Further, in the previous paragraph, an explicit construction of a destabilising perturbation (algorithm 5.1.1) proves the existence of such perturbations on the boundary of this ball. Next we consider the problem of maximising the robust-stability radius (in the case of additive perturbations) and of characterising the family of all controllers that guarantee this maximum robust stability margin. We first make the following definition.

Definition 5.2.1. *A controller K is maximally robust if $H(G, K)$ is ϵ-robustly stable for the maximum value of ϵ for which (G, ϵ) is robustly stabilisable.*

Following the above definition and proposition 5.1.1, the maximally robust stability problem for the case of additive unstructured perturbations can be stated as follows:

Problem 5.2.1 (MRSP). *Given any $G \in \mathcal{RH}_\infty^-$ such that $G(\infty) = 0$ find ϵ_o, the maximum value of ϵ, such that (G, K) is ϵ-robustly stable.*

A mathematical formulation to the above problem follows directly from Theorem 5.1.1:

$$\epsilon_o^{-1} = \gamma_o = \min_{K \in \mathcal{S}} \|K(I - GK)^{-1}\|_\infty \tag{5.4}$$

where $\mathcal{S}$ is the set of all stabilising controllers of G, i.e. all $K(s)$ for which $H(G, K) \in \mathcal{H}_\infty$. Note that $\epsilon_o = \frac{1}{\gamma_o}$ and it is defined as the *maximal robust stability* *radius.* The procedure we shall follow to solve (5.4) is described in four steps. At first we pose the MRSP in a generalised regulator framework (see chapter 2, problem 2.4.1), i.e. we embed the given nominal plant G_0 into an augmented system P, so that the control set-up of figure 5.5 is reformulated as a lower LFT of a generalised plant and the desired controller. Then we derive a coprime factorisation for G_0 via the solution of certain Diophantine equations, or equivalently via choosing appropriate state feedback and output injection matrices. Thereafter, by using the results presented in chapter 3, we parameterise the family of all stabilising controllers for MRSP and hence reformulate the original problem to a model-matching setting. The final part involves the reduction to a Nehari approximation problem, whose solution was outlined in chapter 4 (algorithm 4.4.1). As the procedure discussed here relies on a state-space analysis, for reasons of clarity we make the following assumption which does not involve any loss of generality.

Assumption 5.2.1. *Let G_1 be defined according to proposition 5.1.1 and assume without loss of generality that its realisation is balanced. For the sake of simplicity from now on we shall use the notation G and refer to the realisation of G_1 in proposition 5.1.1. Thus, take G to be an anti-stable system with minimal balanced realisation (A, B, C) which satisfies*

$$A'\Sigma + \Sigma A - C'C = 0$$

$$A\Sigma + \Sigma A' - BB' = 0$$

in which $\Sigma > 0$ is given by

$$\Sigma = \begin{bmatrix} \sigma_n & & & 0 \\ & \sigma_{n-1} & & \\ & & \ddots & \\ 0 & & & \sigma_1 \end{bmatrix}$$

and $\sigma_1 \geq \sigma_2 \geq \cdots > \sigma_{n-r} = \cdots = \sigma_n > 0$ are the Hankel singular values of G. Here, r denotes the multiplicity of the smallest Hankel singular value of G. In the sequel we use the fact that any realisation G can be factorised as the quotient of two stable (coprime) transfer matrices.

Generalised regulator framework. Now consider the generalised regulator problem which is described below.

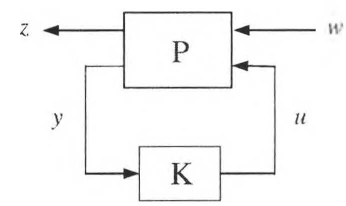

Figure 5.7: Generalised regulator problem

Here,

$$\begin{pmatrix} z(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \begin{pmatrix} w(s) \\ u(s) \end{pmatrix}$$

$$u(s) = K(s)y(s)$$

Using simple calculations it is verified that

$$z = \mathcal{F}_l(P, K)w$$

where

$$\mathcal{F}_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

is the *lower linear fractional transformation* of P and K. Recall from chapter 2, problem 2.4.1, that the generalised regulator problem is stated as follows:
Given P, find K (if it exists) such that:

1. The loop is internally stable

2. $\|\mathcal{F}_l(P, K)\|_\infty \leq \gamma$

for a chosen level $\gamma \in \mathcal{R}$. The first condition guarantees the stability of the nominal closed-loop system, a requirement which is fundamental for any feedback control system design and is related to the existence of the transfer functions appearing in (5.3), belonging to $\mathcal{RH}_\infty$. Now define

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \triangleq \begin{pmatrix} 0 & I \\ I & G(s) \end{pmatrix} \tag{5.5}$$

It is easy to check that the upper LFT interconnection of uncertainty Δ and the generalised plant $P(s)$ defined above is identical to the additive uncertainty model, i.e.

$$\mathcal{F}_u(P, \Delta) := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12} = G + \Delta$$

Further, considering the lower linear fractional transformation

$$\mathcal{F}_l(P,K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} = K(I - GK)^{-1}$$

where the RHS describes the closed-loop transfer function of the nominal plant G, whose infinity norm we want to minimise as in (5.4). The maximally robust stabilisation problem can be expressed in the general LFT framework, where the generalised plant has the form of (5.5). In particular, (5.4) is equivalent to

$$\min_{K \in \mathcal{S}} \|\mathcal{F}_l(P,K)\|_\infty = \gamma_o := \frac{1}{\epsilon_o} \tag{5.6}$$

The controller that minimises (5.6), which is not necessarily unique [3] must, in the first place, be a stabilising controller for the nominal plant G. Hence, the set of all maximally robust controllers forms in general a subset of all stabilising controllers.

Diophantine Equations and Coprime factorisation. Take $P_{22} := G \in \mathcal{RH}_\infty^-$ with left and right coprime factorisation

$$G = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$$

where $N, \widetilde{N}, M, \widetilde{M} \in \mathcal{RH}_\infty$. Then there always exist matrices $U, \widetilde{U}, V, \widetilde{V} \in \mathcal{RH}_\infty$ satisfying the following two Diophantine equations

$$\widetilde{V}M - \widetilde{U}N = I, \qquad \widetilde{M}V - \widetilde{N}U = I$$

or more generally,

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{5.7}$$

which will be referred to as the generalised Diophantine (Bezout) identities.

Proposition 5.2.1. *Let F,H be such that $\sigma(A - BF) \subseteq \mathcal{C}_-$ and $\sigma(A - HC) \subseteq \mathcal{C}_-$. Then,*

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BF & B & H \\ \hline -F & I & 0 \\ C & 0 & I \end{array}\right]$$

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - HC & B & H \\ \hline F & I & 0 \\ -C & 0 & I \end{array}\right]$$

[3] for MIMO plants generically it is not unique but a continuum infinite set of controllers

are stable state-space realisations of the coprime factors $M, N, \widetilde{M}, \widetilde{N}$ *and of* $U, V, \widetilde{U}, \widetilde{V}$ *satisfying the Diophantine equation. Now choose* $F = B'\Sigma^{-1}$ *and* $H = \Sigma^{-1}C'$*. Then* $A - BF$ *and* $A - HC$ *are asymptotically stable. Further, with this choice,* M *and* $\widetilde{M}$ *are inner.*

Proof. Assume without loss of generality that (A, B, C) is a minimal balanced realisation (see assumption 5.2.1). Then,

$$A - BF = A - BB'\Sigma^{-1} = (A\Sigma - BB')\Sigma^{-1} = -\Sigma A'\Sigma^{-1}$$

so that

$$\sigma(A - BF) \subseteq \mathcal{C}_-$$

Similarly,

$$A - HC = A - \Sigma^{-1}C'C = \Sigma^{-1}(\Sigma A - C'C) = -\Sigma^{-1}A'\Sigma$$

and so

$$\sigma(A - HC) \subseteq \mathcal{C}_-$$

Further,

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BB'\Sigma^{-1} & B & \Sigma^{-1}C' \\ \hline -B'\Sigma^{-1} & I & 0 \\ C & 0 & I \end{array}\right]$$

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - \Sigma^{-1}C'C & B & \Sigma^{-1}C' \\ \hline B'\Sigma^{-1} & I & 0 \\ -C & 0 & I \end{array}\right]$$

Hence,

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix}\begin{pmatrix} M & U \\ N & V \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - \Sigma^{-1}C'C & B & \Sigma^{-1}C' \\ \hline B'\Sigma^{-1} & I & 0 \\ -C & 0 & I \end{array}\right]\left[\begin{array}{c|cc} A - BB'\Sigma^{-1} & B & \Sigma^{-1}C' \\ \hline -B'\Sigma^{-1} & I & 0 \\ C & 0 & I \end{array}\right]$$

Then,

$$
\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix}
\begin{pmatrix} M & U \\ N & V \end{pmatrix}
=
\left[\begin{array}{cc|cc}
A-\Sigma^{-1}C'C & -BB'\Sigma^{-1}+\Sigma^{-1}C'C & B & \Sigma^{-1}C' \\
0 & A-BB'\Sigma^{-1} & B & \Sigma^{-1}C' \\
\hline
B'\Sigma^{-1} & -B'\Sigma^{-1} & I & 0 \\
-C & C & 0 & I
\end{array}\right]
$$

$$
\overset{T}{=}
\left[\begin{array}{cc|cc}
A-\Sigma^{-1}C'C & \begin{array}{c}(-BB'\Sigma^{-1}+\Sigma^{-1}C'C-A \\ +BB'\Sigma^{-1}+A-\Sigma^{-1}C'C)\end{array} & 0 & 0 \\
0 & A-BB'\Sigma^{-1} & B & \Sigma^{-1}C' \\
\hline
B'\Sigma^{-1} & 0 & I & 0 \\
-C & 0 & 0 & I
\end{array}\right]
$$

$$
=
\left[\begin{array}{cc|cc}
A-\Sigma^{-1}C'C & 0 & 0 & 0 \\
0 & A-BB'\Sigma^{-1} & B & \Sigma^{-1}C' \\
\hline
B'\Sigma^{-1} & 0 & I & 0 \\
-C & 0 & 0 & I
\end{array}\right]
$$

Removing the uncontrollable modes gives

$$
\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix}
\begin{pmatrix} M & U \\ N & V \end{pmatrix}
=
\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
$$

and thus verifies the generalised Bezout identities. Further,

$$
\begin{aligned}
NM^{-1} &= \left[\begin{array}{c|c} A-BB'\Sigma^{-1} & B \\ \hline C & 0 \end{array}\right]
\left[\begin{array}{c|c} A-BB'\Sigma^{-1} & B \\ \hline -B'\Sigma^{-1} & I \end{array}\right]^{-1} \\
&= \left[\begin{array}{c|c} A-BB'\Sigma^{-1} & B \\ \hline C & 0 \end{array}\right]
\left[\begin{array}{c|c} A-BB'\Sigma^{-1}+BB'\Sigma^{-1} & B \\ \hline B'\Sigma^{-1} & I \end{array}\right]^{-1} \\
&= \left[\begin{array}{cc|c} A-BB'\Sigma^{-1} & BB'\Sigma^{-1} & B \\ 0 & A & B \\ \hline C & 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A-BB'\Sigma^{-1} & BB'\Sigma^{-1}+A-BB'\Sigma^{-1}-A & 0 \\ 0 & A & B \\ \hline C & C & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = G
\end{aligned}
$$

□

Remark 5.2.1. *The particular selection of state feedback $F = B\Sigma^{-1}$ and output injection $H = \Sigma^{-1}C'$ in proposition 5.2.1 guarantees that $M, \widetilde{M}$ are inner ($MM^{\sim} = I$, $\widetilde{M}\widetilde{M}^{\sim} = I$ and $M, \widetilde{M} \in \mathcal{RH}_\infty$). This also follows via a routine state-space calculation which is omitted.*

Parametrisation of all Stabilising Controllers. It is well known (Youla parametrisation) that *every* stabilising controller can be written in the following bilinear form (see chapter 3, Theorem 3.2.1):

$$K = (U + MQ)(V + NQ)^{-1} = (U + MQ)[V(I + V^{-1}NQ)]^{-1}$$
$$= (U + MQ)(I + V^{-1}NQ)^{-1}V^{-1} = U(I + V^{-1}NQ)^{-1}V^{-1} + MQ(I + V^{-1}NQ)^{-1}V^{-1}$$

where $Q \in \mathcal{RH}_\infty$ and all other transfer matrices are as defined in chapter 3. Equivalently in an LFT form, the set of all stabilising controllers is given by $\mathcal{K} = \mathcal{F}_l(K_o, Q)$, where K_o is the *generator* of all stabilising controllers, given by

$$K_o := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BF - HC & H & B \\ \hline -F & 0 & I \\ -C & I & 0 \end{array}\right]$$

for appropriate matrices F and H (see chapter 3). Selecting F and H according to proposition 5.2.1, the generator of all real-rational stabilising controllers K takes the form

$$K_o = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BB'\Sigma^{-1} - \Sigma^{-1}C'C & \Sigma^{-1}C' & B \\ \hline -B'\Sigma^{-1} & 0 & I \\ -C & I & 0 \end{array}\right] \tag{5.8}$$

Hence, the set of all stabilising controllers is given by the set

$$\mathcal{K} := \{K : \ K = \mathcal{F}_l(K_o, Q) \ , \quad Q \in \mathcal{RH}_\infty\}$$

which depends on a parameter Q, varying freely in $\mathcal{H}_\infty$, and a fixed controller generator K_o as defined in (5.8). Hence, the above parametrisation transforms the original $\mathcal{H}_\infty$ optimisation problem over the class of stabilising controllers (problem 5.0.1 and Theorem 5.1.1) to an equivalent optimisation of an affine function of a parameter (Q) which varies freely over $\mathcal{H}_\infty$. Using the generalised regulator framework, already derived for the class of additive perturbations, we are now able to construct controllers K from the set $\mathcal{K}$ which robustly stabilise the plant G, in the sense that both criteria of (2.4.1) are satisfied for a subclass of $\mathcal{K}$. However, a complete characterisation of all maximally robust controllers, expressed directly in terms of the plant realisation, still requires some further investigation.

Reduction to Nehari problem via Model-matching. A model matching problem is of the form: Find a matrix $Q \in \mathcal{RH}_\infty$ so that

$$Q \in \arg\min \|T_{11} - T_{12}QT_{21}\|_\infty$$

for $T_{ij} \in \mathcal{RH}_\infty$, $i, j = 1, 2$. The solution of such problems have already been studied in chapter 3. In connection with the previous paragraphs it can be shown that $\mathcal{F}_l(P, K) = \mathcal{F}_l(T, Q)$ for an appropriate selection of $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{pmatrix} \in \mathcal{RH}_\infty$, as shown in figure 5.8.

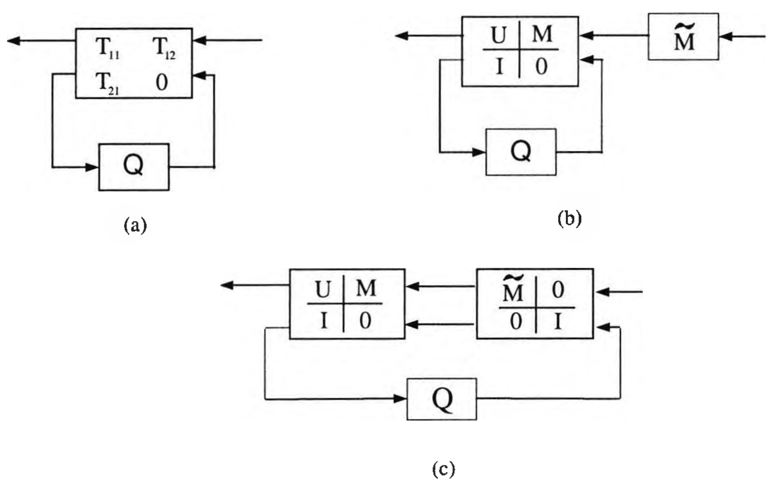

Figure 5.8: Equivalent block diagrams of model matching problem.

For the MRSP the three matrix functions T_{ij} are obtained (see [Fra87]) by:

$$T_{11} := P_{11} + P_{12}M\tilde{U}P_{21} = M\tilde{U}$$

$$T_{12} := P_{21}M = M$$

$$T_{21} := \widetilde{M}P_{21} = \widetilde{M}$$

Each T_{ij} belongs to $\mathcal{RH}_\infty$, a fact that follows immediately from the co-primeness properties of the factorisation. Then,

$$\begin{aligned} \mathcal{F}_l(P, K) &= \{T_{11} + T_{12}QT_{21} \quad : \ Q \in \mathcal{RH}_\infty\} \\ &= \{M\tilde{U} + MQ\widetilde{M} \quad : \ Q \in \mathcal{RH}_\infty\} \end{aligned} \tag{5.9}$$

Now, recall from equation (5.7) that

$$-M\tilde{U} + U\widetilde{M} = 0 \Rightarrow M\tilde{U} = U\widetilde{M}$$

Further, from remark 5.2.1 M and $\widetilde{M}$ are inner. Therefore,

$$\begin{aligned}\mathcal{F}_l(P,K) &= \{U\widetilde{M} + MQ\widetilde{M} \ : \ Q \in \mathcal{RH}_\infty\} \\ &= \{(U + MQ)\widetilde{M} \ : \ Q \in \mathcal{RH}_\infty\}\end{aligned}$$

Next we need to find the optimal, in the $\mathcal{H}_\infty$ sense, $Q \in \mathcal{RH}_\infty$ that solves the model-matching problem in this case. Consider the following $\mathcal{H}_\infty$ optimisation. Since M and $\widetilde{M}$ are inner:

$$\begin{aligned}\gamma_o &= \min_{Q\in\mathcal{RH}_\infty} \left\|(U + MQ)\widetilde{M}\right\|_\infty \\ &= \min_{Q\in\mathcal{RH}_\infty} \|U + MQ\|_\infty \\ &= \min_{Q\in\mathcal{RH}_\infty} \|M^{\sim}U + Q\|_\infty \\ &= \|M^{\sim}U\|_H\end{aligned} \tag{5.10}$$

and Q is a stable operator.

Remark 5.2.2 ([Fra87]). *If T_{12} and T_{21} are square but not inner, the transformation from a model-matching problem to a Nehari approximation problem can be achieved using two additional inner-outer factorisations [Pel03]: Bring in such factorisation $T_{12} = T_{12}^i T_{12}^o$ and $T_{21} = T_{21}^o T_{21}^i$, with T_{12}^i and T_{21}^i square inner and T_{12}^o, T_{21}^o square outer $\mathcal{H}_\infty$ functions (i.e. units). Then*

$$\begin{aligned}\|T_{11} - T_{12}QT_{21}\|_\infty &= \|T_{11} - T_{12}^i T_{12}^o Q T_{21}^o T_{21}^i\|_\infty \\ &= \|(T_{12}^i)^{\sim} T_{11} (T_{21}^i)^{\sim} - T_{12}^o Q T_{21}^o\|_\infty = \|(T_{12}^i)^{-1} T_{11} (T_{21}^i)^{-1} - T_{12}^o Q T_{21}^o\|_\infty\end{aligned}$$

Now the map $\mathcal{H}_\infty \to T_{12}^o \mathcal{H}_\infty T_{21}^o$, $Q \to \widehat{Q} := T_{12}^o Q T_{21}^o$ is a bijection, so

$$\min_{Q\in\mathcal{H}_\infty} \|T_{11} - T_{12}QT_{21}\|_\infty = \min_{\widehat{Q}\in\mathcal{H}_\infty} \|(T_{12}^i)^{\sim} T_{11} (T_{21}^i)^{\sim} - \widehat{Q}\|_\infty$$

which is a Nehari approximation problem. Clearly, the inner nature of M and $\widetilde{M}$ (guaranteed by the choice of F and H, remark 5.2.1) makes these two extra factorisations redundant.

Further, a simple state-space analysis shows that:

$$\begin{pmatrix} U & M \\ I & O \end{pmatrix} \begin{pmatrix} \widetilde{M} & 0 \\ 0 & I \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BF & H & B \\ \hline -F & 0 & I \\ 0 & I & 0 \end{array}\right] \left[\begin{array}{c|cc} A - HC & H & 0 \\ \hline -C & I & 0 \\ 0 & 0 & I \end{array}\right] = \left[\begin{array}{cc|cc} A - BF & -HC & H & B \\ 0 & A - HC & H & 0 \\ \hline -F & 0 & 0 & I \\ 0 & -C & I & 0 \end{array}\right] \tag{5.11}$$

and further,

$$\begin{pmatrix} U & M \\ I & O \end{pmatrix} \begin{pmatrix} \widetilde{M} & 0 \\ 0 & I \end{pmatrix} \equiv \begin{pmatrix} U\widetilde{M} & M \\ \widetilde{M} & 0 \end{pmatrix} \equiv \begin{pmatrix} M\widetilde{U} & M \\ \widetilde{M} & 0 \end{pmatrix} =: \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{pmatrix} \tag{5.12}$$

Summarising, we have shown that the maximally robust stability problem can be reformulated into a model-matching framework whose solution is equivalent to a Nehari approximation problem of the form

$$\min_{Q \in \mathcal{RH}_\infty} \|R + Q\|_\infty = \|R^\sim\|_H \;\; (= \|\Gamma_{R^\sim}\|)$$

The solution of Nehari problems has been previously studied. Here we want to approximate R ($:= M^\sim U$), an anti-stable system, by a stable system Q. As discussed earlier, there exists a generator J of all optimal solutions such that $\|R+Q\|_\infty = \sigma_1(\Gamma_R)$. Further, as already shown in chapter 4, all optimal approximations of R are given by

$$Q = \mathcal{F}_l(J, \gamma_o^{-1}\mathcal{BH}_\infty)$$

where $\gamma_o = \|R\|_H$. In the sequel, we derive the state-space model of R in terms of the realisation of G. Thereafter, we characterise all optimal Q's as an LFT interconnection and relate it to the maximally robust stabilisation problem.

Now,

$$R := M^\sim U \overset{s}{=} \left[\begin{array}{cc|c} -A' + \Sigma^{-1}BB' & -\Sigma^{-1}BB'\Sigma^{-1} & 0 \\ 0 & A - BB'\Sigma^{-1} & \Sigma^{-1}C' \\ \hline B' & -B'\Sigma^{-1} & 0 \end{array}\right]$$

Then by applying the transformation

$$T = \begin{bmatrix} I & -\Sigma^{-1} \\ 0 & I \end{bmatrix}$$

we have

$$\begin{aligned} R &\overset{T}{=} \left[\begin{array}{cc|c} -A' + \Sigma^{-1}BB' & 0 & -\Sigma^{-2}C' \\ 0 & A - BB'\Sigma^{-1} & \Sigma^{-1}C' \\ \hline B' & 0 & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} -A' + \Sigma^{-1}BB' & \Sigma^{-2}C' \\ \hline -B' & 0 \end{array}\right] \\ &\overset{T_1}{=} \left[\begin{array}{c|c} \Sigma(-A' + \Sigma^{-1}BB')\Sigma^{-1} & \Sigma^{-1}C' \\ \hline -B'\Sigma^{-1} & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} A & \Sigma^{-1}C' \\ \hline -B'\Sigma^{-1} & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_R & B_R \\ \hline C_R & 0 \end{array}\right] \end{aligned} \tag{5.13}$$

where $T_1 = \Sigma$. Clearly, the above realisation for R is minimal and balanced. Note that since G was assumed to be in $\mathcal{RH}_\infty^-$, so is R. By expanding the gramian (Lyapunov) equations, in assumption (5.2.1), it is straightforward to show that R has observability and controllability gramians equal to

$$\Sigma^{-1} = \begin{bmatrix} \sigma_n^{-1} I_r & 0 \\ 0 & \widehat{\Sigma}^{-1} \end{bmatrix}$$

where σ_n is the minimum Hankel singular value of $G(-s)$ and hence, $\sigma_n^{-1} = \gamma_o$ is the largest Hankel singular value of $R(-s)$, of (assumed) multiplicity r. Partition R conformally with Σ^{-1} as :

$$R \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11R} & A_{12R} & B_{1R} \\ A_{21R} & A_{22R} & B_{2R} \\ \hline C_{1R} & C_{2R} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & \sigma_n^{-1} C_1' \\ A_{21} & A_{22} & \widehat{\Sigma}^{-1} C_2' \\ \hline -\sigma_n^{-1} B_1' & -B_2' \widehat{\Sigma}^{-1} & 0 \end{array}\right] \tag{5.14}$$

Then,

$$Q = \mathcal{F}_l(J, \sigma_n \mathcal{BH}_\infty) \tag{5.15}$$

where the "Glover generator" of all optimal approximations is given by

$$J := \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C}_1 & \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{C}_2 & \widehat{D}_{21} & 0 \end{array}\right] \tag{5.16}$$

as in chapter 4, algorithm 4.4.1, whith

$$\begin{aligned} \widehat{A} &= -A_{22}' + \Gamma^{-1}(\widehat{\Sigma}^{-1} B_{2R} - C_{2R}' \widehat{D}_{11}) B_{2R}' \\ \widehat{B}_1 &= -\Gamma^{-1}(\widehat{\Sigma}^{-1} B_{2R} - C_{2R}' \widehat{D}_{11}) \\ \widehat{B}_2 &= \Gamma^{-1} C_{2R}' \widehat{D}_{12} \\ \widehat{C}_1 &= C_{2R} \widehat{\Sigma}^{-1} - \widehat{D}_{11} B_{2R}' \\ \widehat{C}_2 &= -\widehat{D}_{21} B_{2R}' \end{aligned} \tag{5.17}$$

Note that in (5.17) we substitute $-\widehat{\Sigma}^{-1}$ instead of $\widehat{\Sigma}^{-1}$, due to assumption 5.2.1; Recall in chapter 4, algorithm 4.4.1 the gramian Σ satisfies $A'\Sigma + \Sigma A + C'C = 0$ (i.e. different inertia). Further, straight substitution from the state-space description of R, gives

$$\begin{aligned}
\widehat{A} &= -A'_{22} + \Gamma^{-1}(\widehat{\Sigma}^{-2}C'_2 + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11})C_2\widehat{\Sigma}^{-1} \\
\widehat{B}_1 &= -\Gamma^{-1}(\widehat{\Sigma}^{-2}C'_2 + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}) \\
\widehat{B}_2 &= -\Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{12} \\
\widehat{C}_1 &= -B'_2\widehat{\Sigma}^{-2} - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} \\
\widehat{C}_2 &= -\widehat{D}_{21}C_2\widehat{\Sigma}^{-1}
\end{aligned} \tag{5.18}$$

and

$$\Gamma = (-\widehat{\Sigma}^{-1})^2 - (-\sigma_n^{-1})^2 I_r = \widehat{\Sigma}^{-2} - \sigma_n^{-2} I_r \tag{5.19}$$

Remark 5.2.3. *The matrix $\widehat{D}$ is chosen to be orthogonal with $\widehat{D}_{11} = \gamma CB^{\dagger}$ so that it satisfies the all-pass equations given in [Glo84]. In the framework of robust stabilisation we get:*

$$\begin{bmatrix} C'_{1R} & 0 \end{bmatrix} \widehat{D} + \begin{bmatrix} \sigma_n^{-1}B_{1R} & 0 \end{bmatrix} = 0$$

equivalently, in terms of the nominal plant:

$$\begin{bmatrix} -\sigma_n^{-1}B_1 & 0 \end{bmatrix} \begin{bmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & 0 \end{bmatrix} + \begin{bmatrix} \sigma_n^{-2}C'_1 & 0 \end{bmatrix} = 0$$

implying that $B_1\widehat{D}_{11} = \sigma_n^{-1}C'_1$ and that $B_1\widehat{D}_{12} = 0$. These results will be used extensively in the following state-space analysis.

It is now clear that if $R \in \mathcal{RH}_\infty^-$ then $J \in \mathcal{RH}_\infty$ and so $Q \in \mathcal{RH}_\infty$. Therefore by choosing among all $Q \in \mathcal{RH}_\infty$ those that are optimal, we can parameterise the desired set of all maximally robust controllers via (5.8).

5.2.1 Optimal closed-loop approximation

The MRSP in (5.4) involves the minimisation of the infinity norm of an appropriate closed-loop transfer-matrix, $T := K(I - GK)^{-1}$. Having already defined these optimal transfer-matrices rather implicitly, a more direct approach to the problem is attempted here. A cancellation analysis is carried out in order to understand completely, with the aid of loop transformations, the nature of the resulting optimal closed-loop systems. In particular, full characterisation of their structure and inertia properties are the main issues examined here.

Consider the model matching problem (5.9), described by figure 5.8, but further restrict

Q to have the form (5.15), i.e. to be an optimal approximation of the equivalent Nehari problem (5.10). Then, the optimal closed-loop transfer-matrix in terms of the optimal parameter Q_{opt} is :

$$\begin{aligned}
T_{\text{opt}} &= (U + MQ_{\text{opt}})\widetilde{M} \\
&= U\widetilde{M} + MQ_{\text{opt}}\widetilde{M} \\
&= U\widetilde{M} + M(J_{11} + J_{12}\Phi(I - J_{22}\Phi)^{-1}J_{21})\widetilde{M} \\
&= (U + MJ_{11} + MJ_{12}\Phi(I - J_{22}\Phi)^{-1}J_{21})\widetilde{M} \\
&= \mathcal{F}_l\left(\begin{pmatrix} U + MJ_{11} & MJ_{12} \\ J_{21} & J_{22} \end{pmatrix}, \Phi\right)\widetilde{M} = \mathcal{F}_l(T_{\text{gen}}, \Phi)\widetilde{M}
\end{aligned}$$

The term T_{gen} is easily decomposed as:

$$T_{\text{gen}} := \begin{pmatrix} U & M \\ I & 0 \end{pmatrix} \star \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} U + MJ_{11}(I - 0J_{11})^{-1} & MJ_{12} \\ J_{21} & \mathcal{F}_u(0, J) \end{pmatrix} = \begin{pmatrix} U + MJ_{11} & MJ_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

where $\star$ denotes the Redheffer product (see Appendix A). The decomposition is visualised in figure 5.9.

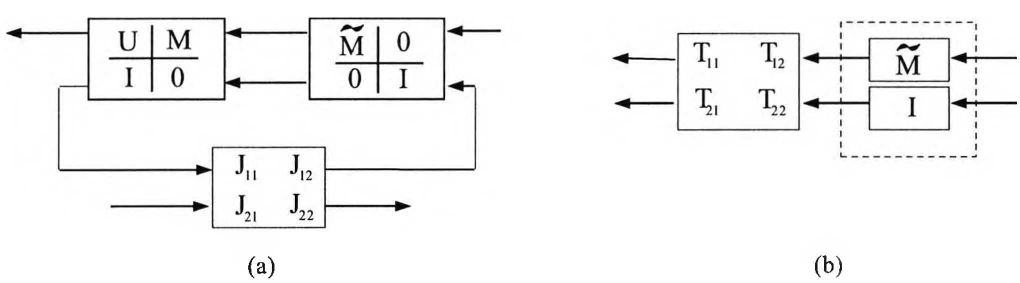

Figure 5.9: Optimal generator of closed-loops - equivalent interconnections.

Note that $\widetilde{M}$ is inner and Φ is anything contractive in $\sigma_n\mathcal{BH}_\infty$. Therefore, T_{gen} will generate all $\mathcal{H}_\infty$ optimal closed-loop systems, i.e. it is the generator of all $K(I-GK)^{-1}$ such that K internally stabilises the nominal plant and $\|K(I - GK)^{-1}\|_\infty = \gamma_o = \frac{1}{\sigma_n}$. Assume that the "Glover generator" has state-space realisation $J \stackrel{s}{=}(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$. Then the state equations of the star (Redheffer) interconnection are

$$\begin{cases} \dot{x} = (A - BB'\Sigma^{-1})x + \Sigma^{-1}C'u_1 + Bw \\ y_1 = -B'\Sigma^{-1}x + w \\ z = u_1 \end{cases} \qquad \begin{cases} \dot{\xi} = \widehat{A}\xi + \widehat{B}_1 z + \widehat{B}_2 u_2 \\ w = \widehat{C}_1\xi + \widehat{D}_{11}z + \widehat{D}_{12}u_2 \\ y_2 = \widehat{C}_2\xi + \widehat{D}_{21}z \end{cases}$$

using the realisation in (5.11) and hence, the state-space of $T_{\text{gen}}(s)$ is obtained as:

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A - BB'\Sigma^{-1} & B\widehat{C}_1 \\ 0 & \widehat{A} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} \Sigma^{-1}C' + B\widehat{D}_{11} & B\widehat{D}_{12} \\ \widehat{B}_1 & \widehat{B}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -B'\Sigma^{-1} & \widehat{C}_1 \\ 0 & \widehat{C}_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The generator of all optimal closed-loop transfer-matrices T_{gen} is:

$$\left[\begin{array}{ccc|cc} A_{11} - \sigma_n^{-1}B_1B_1' & A_{12} - B_1B_2'\widehat{\Sigma}^{-1} & -B_1B_2'\widehat{\Sigma}^{-2} - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} & \sigma_n^{-1}C_1' + B_1\widehat{D}_{11} & B_1\widehat{D}_{12} \\ A_{21} - \sigma_n^{-1}B_2B_1' & A_{22} - B_2B_2'\widehat{\Sigma}^{-1} & -B_2B_2'\widehat{\Sigma}^{-2} - B_2\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} & \widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11} & B_2\widehat{D}_{12} \\ 0 & 0 & -A_{22}' + \Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11})C_2\widehat{\Sigma}^{-1} & -\Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}) & -\Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{12} \\ \hline -\sigma_n^{-1}B_1' & -B_2'\widehat{\Sigma}^{-1} & -B_2'\widehat{\Sigma}^{-2} - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{11} & \widehat{D}_{12} \\ 0 & 0 & -\widehat{D}_{21}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{21} & 0 \end{array}\right]$$

Next, we eliminate all uncontrollable modes to reduce the state-dimension of T_{gen}. Consider the similarity transformation

$$T_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & \Gamma\widehat{\Sigma} \\ 0 & 0 & I \end{bmatrix}$$

Then,

$$\begin{aligned} B(2,1) &= \widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11} - \Gamma\widehat{\Sigma}\Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}) \\ &= \widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11} - \widehat{\Sigma}^{-1}C_2' - B_2\widehat{D}_{11} = 0 \end{aligned}$$

and

$$B(2,2) = B_2\widehat{D}_{12} - B_2\widehat{D}_{12} = 0$$

Moreover, straightforward computations give,

$$\begin{aligned} A(2,3) &= -\Gamma\widehat{\Sigma}A_{22}' + (\widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11})C_2\widehat{\Sigma}^{-1} - B_2B_2'\widehat{\Sigma}^{-2} - B_2\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - (A_{22} - B_2B_2'\widehat{\Sigma}^{-1})\widehat{\Sigma}\Gamma \\ &= -\Gamma\widehat{\Sigma}A_{22}' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} - B_2B_2'\widehat{\Sigma}^{-2} - A_{22}\widehat{\Sigma}\Gamma + B_2B_2'\Gamma \\ &= -\Gamma\widehat{\Sigma}A_{22}' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} - A_{22}\widehat{\Sigma}\Gamma + B_2B_2'(\Gamma - \widehat{\Sigma}^{-2}) \\ &= (\sigma_n^{-2}I - \widehat{\Sigma}^{-2})\widehat{\Sigma}A_{22}' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} - A_{22}\widehat{\Sigma}(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) - \sigma_n^{-2}B_2B_2' \\ &= \sigma_n^{-2}\widehat{\Sigma}A_{22}' - \widehat{\Sigma}^{-1}A_{22}' + \widehat{\Sigma}^{-1}(\widehat{\Sigma}A_{22} + A_{22}'\widehat{\Sigma})\widehat{\Sigma}^{-1} - A_{22}\widehat{\Sigma}^{-1} - \sigma_n^{-2}B_2B_2' + \sigma_n^{-2}A_{22}\widehat{\Sigma} \\ &= \sigma_n^{-2}\{\widehat{\Sigma}A_{22}' + A_{22}\widehat{\Sigma} - B_2B_2'\} = 0 \end{aligned}$$

Further,

$$
\begin{aligned}
A(1,3) &= -B_1B_2'\widehat{\Sigma}^{-2} - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - (A_{12} - B_1B_2'\widehat{\Sigma}^{-1})\widehat{\Sigma}\Gamma = B_1B_2'(\Gamma - \widehat{\Sigma}^{-2}) - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - A_{12}\widehat{\Sigma}\Gamma \\
&= \sigma_n^{-2}B_1B_2' - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - A_{12}\widehat{\Sigma}\Gamma = \sigma_n^{-2}B_1B_2' - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - A_{12}\widehat{\Sigma}(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) \\
&= \sigma_n^{-2}B_1B_2' - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - A_{12}\widehat{\Sigma}^{-1} + \sigma_n^{-2}A_{12}\widehat{\Sigma} = \sigma_n^{-2}(-\sigma_nA_{21}') - A_{12}\widehat{\Sigma}^{-1} - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} \\
&= -\sigma_n^{-1}A_{21}' - A_{12}\widehat{\Sigma}^{-1} - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} = -\sigma_n^{-1}A_{21}' - A_{12}\widehat{\Sigma}^{-1} + \sigma_n^{-1}C_1'C_2\widehat{\Sigma}^{-1} \\
&= \sigma_n^{-1}\{-A_{21}'\widehat{\Sigma} - \sigma_nA_{12} + C_1'C_2\}\widehat{\Sigma}^{-1} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
C(1,3) &= -B_2'\widehat{\Sigma}^{-2} - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} + B_2'\widehat{\Sigma}^{-1}\widehat{\Sigma}\Gamma = -B_2'\widehat{\Sigma}^{-2} + B_2'(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} \\
&= -\sigma_n^{-2}B_2' - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1}
\end{aligned}
$$

Note that (last line in partition $A(1,3)$) we used the fact that $B_1\widehat{D}_{11} = -\sigma_n^{-1}C_1'$, due to remark 5.2.3. Then T_{gen} is equal to

$$
\left[\begin{array}{ccc|cc}
A_{11} - \sigma_n^{-1}B_1B_1' & A_{12} - B_1B_2'\widehat{\Sigma}^{-1} & 0 & 0 & 0 \\
A_{21} - \sigma_n^{-1}B_2B_1' & A_{22} - B_2B_2'\widehat{\Sigma}^{-1} & 0 & 0 & 0 \\
0 & 0 & -A_{22}' + \Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11})C_2\widehat{\Sigma}^{-1} & -\Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}) & -\Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{12} \\
\hline
-\sigma_n^{-1}B_1' & -B_2'\widehat{\Sigma}^{-1} & -\sigma_n^{-2}B_2' - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{11} & \widehat{D}_{12} \\
0 & 0 & -\widehat{D}_{21}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{21} & 0
\end{array}\right]
$$

Removing all uncontrollable modes, the optimal closed-loop transfer-function, T_{gen}, is given by

$$
\left[\begin{array}{c|cc}
-A_{22}' + \Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11})C_2\widehat{\Sigma}^{-1} & -\Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}) & -\Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{12} \\
\hline
-\sigma_n^{-2}B_2' - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{11} & \widehat{D}_{12} \\
-\widehat{D}_{21}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{21} & 0
\end{array}\right]
$$

or equivalently, the generator of all optimal closed-loop approximations is

$$
T_{\text{gen}} \stackrel{s}{=} \left[\begin{array}{c|cc}
\widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\
\hline
-\sigma_n^{-2}B_2' - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} & \widehat{D}_{11} & \widehat{D}_{12} \\
\widehat{C}_2 & \widehat{D}_{21} & 0
\end{array}\right] \tag{5.20}
$$

in the sense that all optimal closed-loops have the form $T_{\text{opt}} = \mathcal{F}_l(T_{\text{gen}}, \Phi)\widetilde{M}$, where $\Phi \in \sigma_n\mathcal{BH}_\infty$ and $\widetilde{M}$ is a known all-pass matrix function. Note that $deg(T_{\text{gen}}) \leq n - r$ where n is the McMillan degree of G and r is the multiplicity of the largest singular value of Γ_G.

5.2.2 Maximally robust controllers

Having obtained a state-space formulation for the generator of all optimal closed-loop transfer-matrices, we proceed to characterise the family of all maximally robust controllers, using a state-space approach, i.e. to derive a closed-form state-space model of all maximally robust controllers, described by the figure below. Using the *LFT* interconnection derived for the family of all stabilising controllers, the set of all maximally robust controllers is obtained by setting Q equal to Q_{opt}, the solution set of the equivalent Nehari approximation problem.

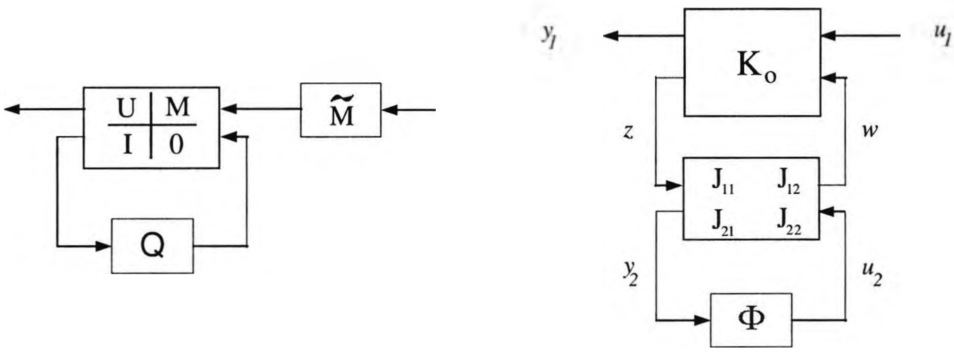

Figure 5.10: Equivalent block diagram representation of all maximally robust controllers

The above figure describes the set of all maximally robust controllers,

$$\mathcal{K}_{\text{opt}} := \mathcal{F}_l(K_o, \mathcal{F}_l(J, \sigma_n \mathcal{BH}_\infty))$$

where $\Phi \in \sigma_n \mathcal{BH}_\infty$, i.e. anything contractive inside the σ_n-ball. The generator of all $K \in \mathcal{K}_{\text{opt}}$, is defined by

$$K_{\text{gen}} := K_o \star J$$

and thus, according to the RHS interconnection of figure 5.10, satisfies the following set of equations:

$$\begin{cases} \textit{(i)} : \dot{x} = (A - BB'\Sigma^{-1} - \widehat{\Sigma}^{-1}C'C)x + \Sigma^{-1}C'u_1 + Bw \\ \textit{(ii)} : y_1 = -B'\Sigma^{-1}x + w \\ \textit{(iii)} : z = -Cx + u_1 \end{cases} \quad \begin{cases} \textit{(iv)} : \dot{\xi} = \widehat{A}\xi + \widehat{B}_1 z + \widehat{B}_2 u_2 \\ \textit{(v)} : w = \widehat{C}_1\xi + \widehat{D}_{11}z + \widehat{D}_{12}u_2 \\ \textit{(vi)} : y_2 = \widehat{C}_2\xi + \widehat{D}_{21}z \end{cases} \tag{5.21}$$

Now substitute 5.21(iii) into 5.21(iv):

$$w = \widehat{C}_1\xi - \widehat{D}_{11}Cx + \widehat{D}_{11}u_1 + \widehat{D}_{12}u_2 \tag{5.22}$$

Substituting (5.22) into 5.21(i) we get

$$\dot{x} = (A - BB'\Sigma^{-1} - \Sigma^{-1}C'C)x + (\Sigma^{-1}C')u_1 + B\widehat{C}_1\xi - B\widehat{D}_{11}Cx + B\widehat{D}_{11}u_1 + B\widehat{D}_{12}u_2$$

$$\Rightarrow \dot{x} = (A - BB'\Sigma^{-1} - \Sigma^{-1}C'C - B\widehat{D}_{11}C)x + B\widehat{C}_1\xi + (\Sigma^{-1}C' + B\widehat{D}_{11})u_1 + (B\widehat{D}_{12})u_2 \tag{5.23}$$

Also substituting 5.21(iii) into 5.21(iv):

$$\dot{\xi} = \widehat{A}\xi - \widehat{B}_1Cx + \widehat{B}_1u_1 + \widehat{B}_2u_2$$

or equivalently

$$\dot{\xi} = -\widehat{B}_1Cx + \widehat{A}\xi + \widehat{B}_1u_1 + \widehat{B}_2u_2$$

Rewrite 5.21(ii) as

$$y_1 = -B'\Sigma^{-1}x + \widehat{C}_1\xi - \widehat{D}_{11}Cx + \widehat{D}_{11}u_1 + \widehat{D}_{12}u_2$$

$$\Rightarrow y_1 = (-\widehat{D}_{11}C - B'\Sigma^{-1})x + \widehat{C}_1\xi + \widehat{D}_{11}u_1 + \widehat{D}_{12}u_2$$

So 5.21(vi) becomes

$$y_2 = (-\widehat{D}_{21}C)x + \widehat{C}_2\xi + \widehat{D}_{21}u_1$$

Hence, the equivalent state-space description, of K_{gen} is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A - BB'\Sigma^{-1} - \widehat{\Sigma}^{-1}C'C - B\widehat{D}_{11}C & B\widehat{C}_1 \\ -\widehat{B}_1C & \widehat{A} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} \Sigma^{-1}C' + B\widehat{D}_{11} & B\widehat{D}_{12} \\ \widehat{B}_1 & \widehat{B}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\widehat{D}_{11}C - B'\Sigma^{-1} & \widehat{C}_1 \\ -\widehat{D}_{21}C & \widehat{C}_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

i.e,

$$K_{\text{gen}} \overset{s}{=} \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

where,

$$A_K = \begin{bmatrix} A_{11} - \sigma_n^{-1}B_1B_1' - \sigma_n^{-1}C_1'C_1 - B_1\widehat{D}_{11}C_1 & A_{12} - B_1B_2'\widehat{\Sigma}^{-1} - \sigma_n^{-1}C_1'C_2 - B_1\widehat{D}_{11}C_2 & B_1\widehat{C}_1 \\ A_{21} - \sigma_n^{-1}B_2B_1' - \widehat{\Sigma}^{-1}C_2'C_1 - B_2\widehat{D}_{11}C_1 & A_{22} - B_2B_2'\widehat{\Sigma}^{-1} - \widehat{\Sigma}^{-1}C_2'C_2 - B_2\widehat{D}_{11}C_2 & B_2\widehat{C}_1 \\ -\widehat{B}_1C_1 & -\widehat{B}_1C_2 & \widehat{A} \end{bmatrix}$$

$$B_K = \begin{bmatrix} \sigma_n^{-1}C_1' + B_1\widehat{D}_{11} & B_1\widehat{D}_{12} \\ \widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11} & B_2\widehat{D}_{12} \\ \widehat{B}_1 & \widehat{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11} & B_2\widehat{D}_{12} \\ \widehat{B}_1 & \widehat{B}_2 \end{bmatrix}$$

$$C_K = \begin{bmatrix} -\widehat{D}_{11}C_1 - \sigma_n^{-1}B_1' & -\widehat{D}_{11}C_2 - B_2'\widehat{\Sigma}^{-1} & \widehat{C}_1 \\ -\widehat{D}_{21}C_1 & -\widehat{D}_{21}C_2 & \widehat{C}_2 \end{bmatrix}$$

$$D_K = \begin{bmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & 0 \end{bmatrix}$$

Next, apply the similarity transformation

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & \Gamma\widehat{\Sigma} \\ 0 & 0 & I \end{bmatrix}$$

Then, a cancellation analysis shows that the generator of all maximally robust controllers has McMillan degree of at most $n - r$. This agrees with a standard result of $\mathcal{H}_\infty$ theory [HLG93]. Straightforward algebra, shows that

$$B_K(2,1) = \widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11} - (\widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11}) = 0$$
$$B_K(2,2) = B_2\widehat{D}_{12} - B_2\widehat{D}_{12} = 0$$

and

$$\begin{aligned}
A_K(2,3) =& -\Gamma\widehat{\Sigma}A_{22}' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} + B_2\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - B_2B_2'\widehat{\Sigma}^{-2} - B_2\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - A_{22}\Gamma\widehat{\Sigma} \\
&+ B_2B_2'\Gamma + \widehat{\Sigma}^{-1}C_2'C_2\Gamma\widehat{\Sigma} + B_2\widehat{D}_{11}C_2\Gamma\widehat{\Sigma} + (\widehat{\Sigma}^{-1}C_2'C_2 + B_2\widehat{D}_{11}C_2)(-\Gamma\widehat{\Sigma}) \\
=& -(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I)\widehat{\Sigma}A_{22}' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} - B_2B_2'\widehat{\Sigma}^{-2} - A_{22}(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I)\widehat{\Sigma} + B_2B_2'(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) \\
&+ \widehat{\Sigma}^{-1}C_2'C_2(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I)\widehat{\Sigma} + B_2\widehat{D}_{11}C_2\Gamma\widehat{\Sigma} + (\widehat{\Sigma}^{-1}C_2'C_2 + B_2\widehat{D}_{11}C_2)(-\Gamma\widehat{\Sigma}) \\
=& -\widehat{\Sigma}^{-1}A_{22}' + \sigma_n^{-2}\widehat{\Sigma}A_{22}' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} - B_2B_2'\widehat{\Sigma}^{-2} - A_{22}\widehat{\Sigma}^{-1} + \sigma_n^{-2}A_{22}\widehat{\Sigma} + B_2B_2'\widehat{\Sigma}^{-2} \\
&- \sigma_n^{-2}B_2B_2' + \widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma}^{-1} - \sigma_n^{-2}\widehat{\Sigma}^{-1}C_2'C_2\widehat{\Sigma} + B_2\widehat{D}_{11}C_2\Gamma\widehat{\Sigma} + (\widehat{\Sigma}^{-1}C_2'C_2 + B_2\widehat{D}_{11}C_2)(-\Gamma\widehat{\Sigma}) \\
=& \widehat{\Sigma}^{-1}C_2'C_2[\widehat{\Sigma}^{-2} - \sigma_n^{-2}I]\widehat{\Sigma} + B_2\widehat{D}_{11}C_2\Gamma\widehat{\Sigma} + (\widehat{\Sigma}^{-1}C_2'C_2 + B_2\widehat{D}_{11}C_2)(-\Gamma\widehat{\Sigma}) \\
=& (\widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11})C_2\Gamma\widehat{\Sigma} - (\widehat{\Sigma}^{-1}C_2'C_2 + B_2\widehat{D}_{11}C_2)\Gamma\widehat{\Sigma} = 0
\end{aligned}$$

Further,

$$A_K(2,2) = A_{22} - B_2B_2'\widehat{\Sigma}^{-1} - \widehat{\Sigma}^{-1}C_2'C_2 - B_2\widehat{D}_{11}C_2 + \widehat{\Sigma}(\widehat{\Sigma}^{-2}C_2'C_2 + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}C_2) = A_{22} - B_2B_2'\widehat{\Sigma}^{-1}$$
$$A_K(2,1) = A_{21} - \sigma_n^{-1}B_2B_1' - \widehat{\Sigma}^{-1}C_2'C_1 - B_2\widehat{D}_{11}C_1 + (\widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11})C_1 = A_{21} - \sigma_n^{-1}B_2B_1'$$

and

$$\begin{aligned}
A_K(1,3) =& -B_1B_2'\widehat{\Sigma}^{-2} - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - (A_{12} - B_1B_2'\widehat{\Sigma}^{-1} - \sigma_n^{-1}C_1'C_2 - B_1\widehat{D}_{11}C_2)\Gamma\widehat{\Sigma} \\
=& -B_1B_2'\widehat{\Sigma}^{-2} - B_1\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - A_{12}\Gamma\widehat{\Sigma} + B_1B_2'\Gamma + \sigma_n^{-1}C_1'C_2\Gamma\widehat{\Sigma} + B_1\widehat{D}_{11}C_2\Gamma\widehat{\Sigma} \\
=& -B_1B_2'\widehat{\Sigma}^{-2} - A_{12}(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I)\widehat{\Sigma} + B_1B_2'(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) + \sigma_n^{-1}C_1'C_2(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I)\widehat{\Sigma} \\
&+ B_1\widehat{D}_{11}C_2(-\widehat{\Sigma}^{-2} + \Gamma)\widehat{\Sigma} \\
=& -A_{12}\widehat{\Sigma}^{-1} + \sigma_n^{-2}A_{12}\widehat{\Sigma} - \sigma_n^{-2}B_1B_2' + \sigma_n^{-1}C_1'C_2\widehat{\Sigma}^{-1} - \sigma_n^{-3}C_1'C_2\widehat{\Sigma} - \sigma_n^{-2}B_1\widehat{D}_{11}C_2\widehat{\Sigma} \\
=& -A_{12}\widehat{\Sigma}^{-1} + \sigma_n^{-2}A_{12}\widehat{\Sigma} - \sigma_n^{-2}(A_{12}\widehat{\Sigma} + \sigma_nA_{21}') + \sigma_n^{-1}(A_{21}'\widehat{\Sigma} + \sigma_nA_{12})\widehat{\Sigma}^{-1} \\
=& -A_{21}\widehat{\Sigma}^{-1} + \sigma_n^{-2}A_{12}\widehat{\Sigma} - \sigma_n^{-2}A_{12}\widehat{\Sigma} - \sigma_n^{-1}A_{21}' + \sigma_n^{-1}A_{21}' + A_{12}\widehat{\Sigma}^{-1} = 0
\end{aligned}$$

Also,

$$\begin{aligned}
A_K(3,3) =& -A_{22}' + \Gamma^{-1}\widehat{\Sigma}^{-2}C_2'C_2\widehat{\Sigma}^{-1} + \Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - \Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11})C_2\Gamma\widehat{\Sigma} \\
=& -A_{22}' + \Gamma^{-1}\widehat{\Sigma}^{-2}C_2'C_2\widehat{\Sigma}^{-1} + \Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{11}C_2\widehat{\Sigma}^{-1} - \Gamma^{-1}\widehat{\Sigma}^{-2}C_2'C_2\Gamma\widehat{\Sigma} - \Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{11}C_2\Gamma\widehat{\Sigma} \\
=& -A_{22}' + \Gamma^{-1}\widehat{\Sigma}^{-2}C_2'C_2(\widehat{\Sigma}^{-2} - \Gamma)\widehat{\Sigma} + \Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{11}C_2(\widehat{\Sigma}^{-2} - \Gamma)\widehat{\Sigma} \\
=& -A_{22}' + \sigma_n^{-2}\Gamma^{-1}\widehat{\Sigma}^{-2}C_2'C_2\widehat{\Sigma} + \sigma_n^{-2}\Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{11}C_2\widehat{\Sigma} \\
=& -A_{22}' + \sigma_n^{-2}\Gamma^{-1}\widehat{\Sigma}^{-1}[\widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11}]C_2\widehat{\Sigma}
\end{aligned}$$

Finally,

$$\begin{aligned}
C_K(1,3) &= -B_2'\widehat{\Sigma}^{-2} - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} + (\widehat{D}_{11}C_2 + B_2'\widehat{\Sigma}^{-1})\widehat{\Sigma}\Gamma \\
&= -B_2'\widehat{\Sigma}^{-2} - \widehat{D}_{11}C_2\widehat{\Sigma}^{-1} + \widehat{D}_{11}C_2\widehat{\Sigma}(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) + B_2'(\widehat{\Sigma}^{-2} - \sigma_n^{-2}I) \\
&= -\sigma_n^{-2}\widehat{D}_{11}C_2\widehat{\Sigma} - \sigma_n^{-2}B_2' = -\sigma_n^{-2}(B_2' + \widehat{D}_{11}C_2\widehat{\Sigma}) \\
C_K(2,3) &= -\widehat{D}_{21}C_2\widehat{\Sigma}^{-1} + \widehat{D}_{21}C_2\Gamma\widehat{\Sigma} = \widehat{D}_{21}C_2[\Gamma - \widehat{\Sigma}^{-2}]\widehat{\Sigma} = -\sigma_n^{-2}\widehat{D}_{21}C_2\widehat{\Sigma}
\end{aligned}$$

In conclusion, all maximally robust controllers are generated by the following system:

$$K_{\text{gen}} \stackrel{s}{=} \left[\begin{array}{c|cc} -A_{22}' + \sigma_n^{-2}\Gamma^{-1}\widehat{\Sigma}^{-1}(\widehat{\Sigma}^{-1}C_2' + B_2\widehat{D}_{11})C_2\widehat{\Sigma} & -\Gamma^{-1}(\widehat{\Sigma}^{-2}C_2' + \widehat{\Sigma}^{-1}B_2\widehat{D}_{11}) & -\Gamma^{-1}\widehat{\Sigma}^{-1}B_2\widehat{D}_{12} \\ \hline -\sigma_n^{-2}(B_2' + \widehat{D}_{11}C_2\widehat{\Sigma}) & \widehat{D}_{11} & \widehat{D}_{12} \\ -\sigma_n^{-2}\widehat{D}_{21}C_2\widehat{\Sigma} & \widehat{D}_{21} & 0 \end{array}\right]$$

or in terms of the "Glover generator" (5.16):

$$K_{\text{gen}} \stackrel{s}{=} \left[\begin{array}{c|cc} -A_{22}' - \sigma_n^{-2}\hat{B}_1\hat{B}_2'\widehat{\Sigma}^2 & \hat{B}_1 & \hat{B}_2 \\ \hline \sigma_n^{-2}\hat{C}_2\widehat{\Sigma}^2 & \widehat{D}_{11} & \widehat{D}_{12} \\ -\sigma_n^{-2}\widehat{D}_{21}C_2\widehat{\Sigma} & \widehat{D}_{21} & 0 \end{array}\right] \tag{5.24}$$

It is now clear that the McMillan degree of K_{gen} is at most $n - r$ from which the existence of optimal controller with this degree bound follows (set the contraction Φ equal to a constant matrix). The main results are now summarised in the following corollary:

Corollary 5.2.1. *Problem 5.4 or equivalently 5.6 has a continuum of solutions given by the set of all maximally robust controllers,*

$$K_{opt} = \mathcal{F}_l(K_{\text{gen}}, \Phi)$$

where K_{gen} is the generator of all maximally robust controllers with state-space description given by (5.24) and Φ is any σ_n-contraction, i.e. $\Phi \in \epsilon_o\mathcal{BH}_\infty$. In other words, connecting such a controller in the feedback loop 5.5 effects to the minimisation of the norm of the control sensitivity function ($u_1 \to e_2$) and all optimal functions are parameterised as

$$T_{\text{opt}} = \mathcal{F}_l(T_{\text{gen}}, \Phi)\widetilde{M}$$

where T_{gen} is the generator of all optimal control sensitivity functions and has state-space description as given in (5.20). Here, $\widetilde{M}$ is a known allpass function.

Further, we demonstrate all major results of this chapter in the following example:

Example 5.2.1. *Consider the following* 3×3 *anti-stable system:*

$$G \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] := \left[\begin{array}{ccc|ccc} \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The realisation is balanced with gramian

$$\Sigma = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.75 \end{bmatrix}$$

and Hankel singular values $\{\sigma_3, \sigma_2, \sigma_1\} = \{0.25, 0.25, 0.75\}$ *(according to the notation of assumption 5.2.1). Note that the multiplicity of* 0.25 *is two* ($\sigma_3 = \sigma_2$)*. The generator of all maximally robust controllers as found in (5.24), is*

$$K_{\text{gen}} \overset{s}{=} \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] := \left[\begin{array}{c|cccc} 1.1667 & -\frac{1}{8} & -\frac{1}{8} & 0 & 0 \\ \hline 16 & -4 & 0 & 0 & 0 \\ 16 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

Take now a random stable 1×1 *system* Φ,

$$\Phi \overset{s}{=} \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right] := \left[\begin{array}{ccc|c} -0.6048 & 0.1720 & 0.3032 & -0.5883 \\ 0.0168 & -0.0352 & -0.0044 & 2.1832 \\ -0.1567 & 0.0466 & 0.0478 & -0.1364 \\ \hline 0.0003 & 0.0030 & 0.0002 & 0 \end{array}\right]$$

such that $\|\Phi\|_\infty = 0.22727 < 0.25$*. Then, we can construct one maximally robust controller,* $K_{opt} = \mathcal{F}_l(K_{\text{gen}}, \Phi)$*, for which the corresponding optimal closed loop* $T_{opt} = K_{opt}(I - GK_{opt})^{-1}$ *has Hankel singular values as plotted in the figure below.*
From the plot, it is clear that the norm of T_{opt} *is equal to 12db or,*

$$\|T_{opt}\|_\infty = 10^{\frac{12}{20}} \simeq 4 = \frac{1}{\sigma_n}$$

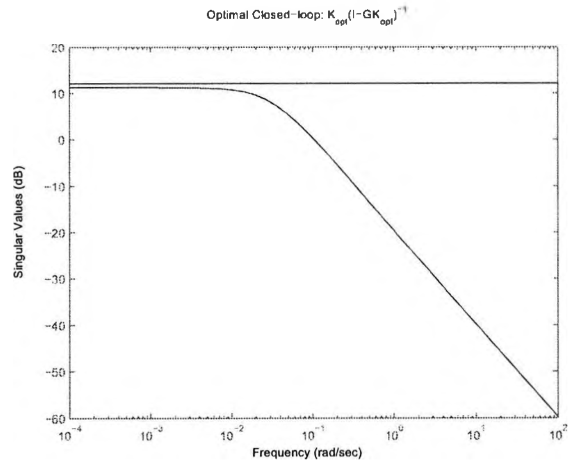

Figure 5.11: Singular values of Optimal Closed-loop.

Further, we construct the following perturbation according to algorithm 5.1.1:

$$\Delta \stackrel{s}{=} \left[\begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array}\right] := \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 0.1347 & 0.2106 & 0 \\ 0 & -17.6753 & 0 & -4.7629 & 0 & 0 \\ 0 & 0 & -0.7242 & 0 & -0.3050 & 0 \\ \hline 0 & -0.8619 & -0.8619 & -0.1161 & -0.1815 & 0 \\ 1.0143 & -0.5071 & -0.5071 & -0.0683 & -0.1068 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The perturbation has norm $\epsilon^\star = 0.25$ *and as shown by the generalised Nyquist plot*

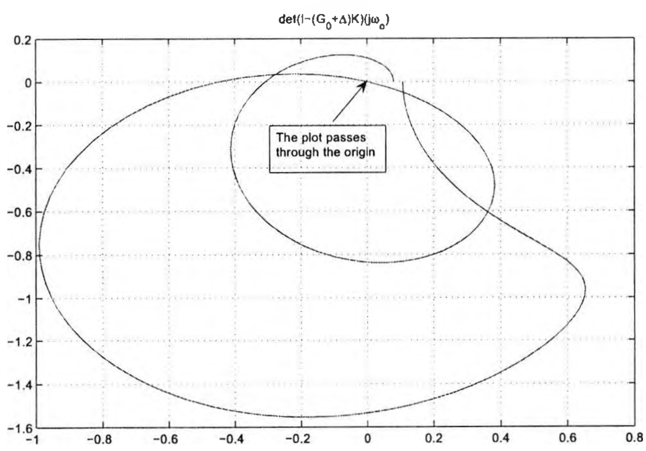

Figure 5.12: Plot of $det(I - (G + \Delta)K)$, *over all frequencies* $\omega \in \mathcal{R}$.

displayed in the following figure it is a destabilising perturbation when the above optimal *controller is used. It is further observed that all four transfer functions of (5.3) have two marginally unstable poles, (i.e. placed on the $j\omega$ axis). Note that this agrees with Nyquist plot of $det[I - (G_0 + \Delta)K]$ (figure 5.12) which can be seen to pass through the origin.*

5.3 Summary

In this chapter, a discussion of model uncertainty was the launching point to formulate the robust stabilisation problem. The main aim of this chapter was to derive solution criteria for the maximally robust stabilisation problem for additive perturbations using a state-space approach.

In particular, it was shown that under assumption 5.2.1 the maximally robust stabilisation problem can be reformulated into an $\mathcal{H}_\infty$ synthesis problem, i.e. one involving the design of a stabilising controller K that minimises the infinity norm of the closed-loop system $T_{wz} = \mathcal{F}_l(P, K)$. Here, P is the generalised plant defined directly from the nominal plant G, assumed anti-stable with no loss of generality. Solving this minimisation, defines the maximum norm $\|\Delta\|_\infty$ of the perturbations that can be stabilised. This is a well known problem, whose solution was derived in detail. The solution follows from the fact that the set of all stabilising controllers can be parameterised in bilinear form (or equivalently via a lower LFT) in terms of a free parameter $Q \in \mathcal{H}_\infty$, i.e.

$$K = \mathcal{F}_l(K_0, Q)$$

Here, K_0 was obtained from the right and left coprime factorisations of G satisfying two Diophantine equations. The advantage of this parametrisation is that the initial, hard-to-solve, $\mathcal{H}_\infty$ optimisation problem can be reduced to a convex optimisation problem, by applying model-matching theory. Until this point, the theory was in the standard framework and had been outlined in previous chapters. Next the set of all stable closed-loop functions was considered. It was shown that this can be expressed in affine form, i.e.

$$T_{wz} := \mathcal{F}_l(P, K) = \mathcal{F}_l\left(P, \mathcal{F}_l(K_0, Q)\right) = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21}$$

with $Q \in \mathcal{H}_\infty$ a free parameter.

The optimisation hence involves a model matching problem and for this application (maximally robust stabilisation) the matrices T_{12} and T_{21} are square and have full rank. In such cases, the problem is said to be of the *first kind* [LH87]. The optimisation can now be reformulated as a Nehari-type approximation problem:

$$\min_K \|T_{zw}\|_\infty = \min_{Q\in\mathcal{H}_\infty} \|R+Q\|_\infty = \|\Gamma_R\| =: \sigma_1(\Gamma_R) =: \frac{1}{\sigma_n(\Gamma_G)}$$

where $R \in \mathcal{RH}_\infty^-$. From theory outlined in chapter 4, all optimal extensions Q are given in terms of the following LFT:

$$Q = \mathcal{F}_l(J, \Phi)$$

where $J \in \mathcal{RH}_\infty$ "generates" all optimal approximations whenever connected with any $(\sigma_1(\Gamma_R))^{-1}$ -contractive (or equivalently $\sigma_n(\Gamma_G)$-contractive) system Φ. The generator of all optimal closed-loop systems was subsequently expressed as an affine map of $\Phi \in \sigma_n(\Gamma_G)\mathcal{BH}_\infty$. Finally, the maximum robust stability radius was obtained in terms of the smallest Hankel singular value of the plant, i.e.

$$\epsilon_o = \sigma_n(\Gamma_G)$$

The contribution made throughout this chapter is a detailed state-space analysis for both the optimal closed-loop transfer-matrices and all maximally robust controllers, showing in particular through various cancellations the existence of optimal controllers of state dimension not exceeding $n-r$, where n is the McMillan degree of the nominal plant [HLG93]. Further, the overall study resulted in a thorough and concrete analysis, which is directly implementable, giving rise to elegant formulae and a clear overview of the maximally robust stabilisation problem.

Concluding, the problem of uniqueness should somehow be restored. How can we choose a controller with some "additional" robust characteristics within this optimal set? Is this controller unique? By setting $\Phi = 0$ it is very well known that the controller satisfies some extra performance characteristics, i.e. guarantees the minimisation of an entropy integral (e.g. see [LH87]). However, the issue examined in this thesis is that of extending the robust stability and hence a complete answer to the above questions is given in the next two chapters where *super-optimisation* is introduced and linked to the maximally robust stabilisation problem.

Chapter 6

Superoptimisation

"...On the assumption that God is an engineer as well as geometer, I am inclined to expect that the stronger minimisation condition, seeming so mathematically "right", will have physical significance ..."

N.J. Young, 1986.

In Nehari approximation problems we seek to minimise

$$\inf_{Q \in \mathcal{H}_\infty^{+,\,p\times m}} \|R+Q\|_\infty \tag{6.1}$$

where $R \in \mathcal{RL}_\infty^{p\times m}$ (or $R \in \mathcal{RH}_\infty^{-,\,p\times m}$ without loss of generality). Throughout this chapter we study the matrix case $\min(p,m) > 1$. Further, depending on the kind of application Q, may be further constrained to have a zero block row and/or column. Then the problem is said to be a *two-block* or a *four-block* distance problem. In this thesis we consider only *one-block* problems. The motivation initially arose from the fact that MRSP is a one-block problem as well, but also later, it is shown that the structure of one-block problems permits a deeper and thorough state-space analysis of this independent problem (super-optimisation), which is one of the novelties of the particular chapter.

By introducing the new notation $s_1^\infty(R) = \|R\|_\infty$ the approximation problem posed in (6.1) above can be rewritten as:

$$s_1(R) := \inf_{Q \in \mathcal{H}_\infty^{+,\,p\times m}} s_1^\infty(R+Q) \tag{6.2}$$

where $s_1(R)$ will be referred to as the optimal level of R. The set of all optimal approximations of R is defined by

$$\mathcal{S}_1(R) := \{Q \in \mathcal{H}_\infty^{+,\,p\times m} : s_1^\infty(R+Q) = s_1\} \tag{6.3}$$

Note that $s_1(R) := \sigma_1(R^\sim)$ is the Hankel norm of $R^\sim$, $R \in \mathcal{RH}_\infty^{-,p\times m}$. Since, in general, the solution of this problem is not unique, we can define a stronger version of optimality, by requiring that the sequence of the suprema (taken over $\omega \in \mathcal{R} \cup \{\infty\}$) of all singular values of the "error" system $(R+Q)(j\omega)$ is minimised lexicographically. This stronger version of the problem was first proposed by Young and was defined as *super-optimisation.* The main motivation, arising from esthetic considerations, was to restore uniqueness to the solution of the matrix Nehari problem, by showing in [You86] the existence of a unique super-optimal approximation Q_{sup}. Nevertheless, in the present work and also others (e.g. [PF85]) it is argued that super-optimisation fits naturally within the modern robust control-theoretic framework, and can be used to define hierarchical optimisation problems in which additional performance and stability objectives can be addressed [PF85], [GHJ00].

Problem definition. A formal definition of the problem follows. Firstly, define

$$s_i^\infty(R) := \sup_{\omega\in\mathcal{R}} \sigma_i[R(j\omega)], \qquad i = 1, 2, \ldots, \min(p, m).$$

If p and m are both greater than 1, then we define recursively the first and subsequent super-optimal levels of R as

$$s_i(R) := \inf_{Q\in\mathcal{S}_{i-1}(R)} s_i^\infty(R+Q) \quad i = 1, 2, \ldots, \min(p, m) \tag{6.4}$$

and the set of all i-th level super-optimal approximations of R as

$$\mathcal{S}_i(R) := \{Q \in \mathcal{S}_{i-1}(R) : s_i^\infty(R+Q) = s_i(R)\} \quad i = 1, 2, \ldots, \min(p, m).$$

In other words, we seek among all super-optimal approximations at the $(i-1)$-th level $\mathcal{S}_{i-1}(R)$ a set for which $s_i(R)$ is minimised (it turns out that the infimum in (6.4) is always attained). This set is not a singleton in general (apart from the case of $i = \min(p, m)$), but forms a subset of all $(i-1)$-th level super-optimal approximations of R, $\mathcal{S}_{i-1}(R)$. Due to the lexicographic nature of the problem, it is clear that every element of $\mathcal{S}_i(R)$ is also an element of $\mathcal{S}_{i-1}(R)$, i.e. that the super-optimal approximation sets nest as:

$$\mathcal{S}_0(R) \supseteq \mathcal{S}_1(R) \supseteq \ldots \supseteq \mathcal{S}_i(R) \supseteq \ldots \supseteq \mathcal{S}_{\min(p,m)}(R)$$

Note that for $i = 1$, (6.4) is taken to be a Nehari extension problem and hence we define $\mathcal{S}_0(R) := \mathcal{H}_\infty^{+,\,p\times m}$. The super-optimal approximation problem ([SODP]) considered in this thesis can be formally defined as follows:

Problem 6.0.1. [SODP]. *Given a* $G \in \mathcal{RH}_{\infty}^{-,\, p\times m}$, *find the (unique) matrix-function* $Q_{sup} \in \mathcal{H}_{\infty}^{+,\, p\times m}$ *which minimises the sequence*

$$s^{\infty}(G+Q) = (s_1^{\infty}(G+Q), s_2^{\infty}(G+Q), \ldots, s_k^{\infty}(G+Q))$$

with respect to the lexicographic ordering, where $k = \min(p, m)$.

The approach followed here involves the reduction of the lexicographic minimisation into a hierarchy of ordinary $\mathcal{H}_{\infty}$-optimisation (Nehari-extension) problems of progressively reduced input-output dimensions, whose solution is well known in the literature [Glo84], [Glo89], [ZDG96], [GL95] and has been presented in chapter 4 of the thesis.

6.1 The 1-block Super-Optimal Distance Problem

At this point the work is organised in two parts. In the first part transfer function approach is followed. The solution of optimal and suboptimal Nehari approximations are restated in a more abstract setting than in chapter 4; subsequently a new *block-diagonal* generator of all optimal Nehari extensions is presented. A crucial difference here is that the generator is constructed with the aid of rational all-pass matrix functions and is reminiscent of the partial singular value decomposition of constant matrices. This analysis is carried out in the first part of the chapter. The later part involves a concrete state-space analysis which reveals the structure of the diagonal form of the generator and solves the super-optimal optimisation problem in a hierarchical setting. The chapter concludes with the presentation of numerous examples which support the derived results.

In contrast to other parallel solutions of the problem reported in the literature the main contribution of the present work is as follows:

(i) The solution is derived in a concrete state-space setting with minimal assumptions (no minimality or balanced form of the realisation of the system which is approximated is assumed and the largest Hankel singular value of the associated Hankel operator is permitted to have arbitrary multiplicity). The analysis allows for the derivation of generically minimal realisations of the super-optimal approximation which establishes tight McMillan degree bounds of the solutions and removes potential ill-conditioned numerical procedures at the intermediate steps of the algorithm. In particular, all assumptions involved in previous results, e.g. McMillan degree bounds, interlacing inequalities between Hankel singular values and super-optimal levels, existence of solutions of certain ARE's, etc. are removed and these results (with suitable modifications) are shown to carry over to the solution of the general problem.

(ii) In contrast to existing techniques ([LHG89],[TGP88]), the method does not depend explicitly on the diagonalising properties of Schmidt pairs of a sequence of Hankel operators generated during the construction process. Thus, several unnecessary preliminary scaling steps are eliminated, together with certain conceptual difficulties related to the multiplicity of Hankel singular values. The present approach ([JL93]) depends on a conceptually simple matrix dilation technique and the interplay between the optimal and suboptimal Nehari generators.

6.1.1 The two-level super-optimal approximation problem and its solution

The approach for solving the SODP adopted in this work is based on all-pass dilation techniques. First the system to be approximated, $R(s)$, is embedded in an all-pass system $H(s)$ of higher dimensions (note that $R(s)$ is taken to lie in $\mathcal{H}_\infty^-$ for compatibility with existing $\mathcal{H}_\infty$ optimal-control literature). This acts as a "generator" of the optimal solution set of the Nehari extension problem, as all solutions can be obtained via a LFT of $H(s)$ with the ball of $\mathcal{H}_\infty$ of radius s_1^{-1} (i.e. the set of all stable s_1^{-1}-contractions) [Glo89]. Next, a sub-block of the optimal generator $H(s)$ is dilated to define a new square all-pass system $\overline{H}(s)$, of lower dimensions compared to those of $H(s)$. Exploiting the all-pass nature of $H(s)$ and $\overline{H}(s)$ and the fact that they share a common block, two diagonalising transformations of $H(s)$ can be defined from certain sub-blocks of $H(s)$ and $\overline{H}(s)$. The diagonalisation is analogous to the partial singular-value decomposition of constant matrices and makes the minimisation of the second super-optimal level transparent.

First, the general solution of the optimal Nehari-extension problem is given under minimal assumptions:

Theorem 6.1.1 (Optimal Nehari approximation). *Consider* $R \in \mathcal{RH}_\infty^{-,\, p\times m}$ *with realisation* $R \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ *where* $\lambda(A) \subseteq C_+$. *Then there exists* $Q_a \in \mathcal{RH}_\infty^{(p+m-l)\times(p+m-l)}$ *such that all* $Q \in \mathcal{H}_\infty^{+,\, p\times m}$ *such that* $\|R+Q\|_\infty = \|R^\sim\|_H = s_1$ *(Nehari optimal approximations of R) are given by*

$$Q = \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_\infty^{(p-l)\times(m-l)})$$

Here $l \le min(p, m, r)$ *is defined in remark 6.1.2, where* r *denotes the multiplicity of the largest singular value of* Γ_R. *Further,*

$$Q_a := \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_q & B_{q1} & B_{q2} \\ \hline C_{q1} & D_{11} & D_{12} \\ C_{q2} & D_{21} & 0 \end{array}\right] \tag{6.5}$$

The corresponding "error" system is given by

$$H := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \left(\begin{array}{c|c} R+Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right) \stackrel{s}{=} \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \tag{6.6}$$

where $\|H_{22}\|_\infty < s_1$ *and* $Q_{ij} \in \mathcal{H}_\infty$, *for* $i, j \in \{1, 2\}$. *Further,* $HH^\sim = H^\sim H = s_1^2 I$ *and the following set of equations is satisfied ("all-pass" equations):*

$$\begin{aligned} P_H Q_H &= Q_H P_H = s_1^2 I \\ D_H D_H' &= D_H' D_H = s_1^2 I \\ A_H' Q_H + Q_H A_H + C_H' C_H &= 0 \\ A_H P_H + P_H A_H' + B_H B_H' &= 0 \\ D_H' C_H + B_H' Q_H &= 0 \\ D_H B_H' + C_H P_H &= 0 \end{aligned} \tag{6.7}$$

Here P_H *and* Q_H *are the gramians of the realisation of H given in (6.6).*

Proof. See [Glo84] where detailed formulae are included; see also [JL93] and [GLD+91]. □

Remark 6.1.1. *The realisation of R need not be assumed minimal. However, we require that* $\lambda(A) \subseteq C_+$. *If R has McMillan degree n, it can be shown [Glo86] that* Q_a *given in (6.5) has degree* $n - r$; *in addition,* $\sigma_i(Q_a) = \sigma_{i+r}(R^\sim)$, $i = 1, 2, \ldots, n-r$ *[Glo86], [GL95].*

Remark 6.1.2. *Integer parameter l which is used to define the input and output dimension of Q_{22} is the normal rank of the Laplace transform of the matrix formed by the r Schmidt vectors of $\Gamma_{R^\sim}$ corresponding to σ_1. In the notation of Theorem 6.1.1 $R^\sim = (-A', C', -B')$ and hence $U(s)$ and $V(s)$ are given as*

$$U(s) = -C(sI - A)^{-1}\Xi \in \mathcal{RH}_2^{\perp, m\times r}, \;\; \Xi = \sigma_1^{-1}P\begin{bmatrix} x_1 & x_2 & \dots & x_r \end{bmatrix}$$

and

$$V(s) = -B'(sI + A')^{-1}\Theta \in \mathcal{H}_2^{p\times r}, \;\; \Theta = \begin{bmatrix} x_1 & x_2 & \dots & x_r \end{bmatrix}$$

where P and Q are the controllability and observability matrices of $R = (A, B, C)$ and the x_i's are r linearly independent eigenvectors of QP corresponding to the eigenvalue σ_1^2. In particular, if (A, B, C) is balanced, $P = Q = -\operatorname{diag}(\sigma_1 I_r, \Sigma_2)$, and thus $\Xi = -E_r$ and $\Theta = \sigma_1^{-2}E_r$ (where E_r denotes the first r-columns of the $n \times n$ unit matrix), so that $U(s) = C(sI - A)^{-1}E_r \in \mathcal{H}_2^\perp$ and $V(s) = -s_1^2B'(sI + A')^{-1}E_r \in \mathcal{H}_2$. Thus,

$$rank_{\mathcal{R}(s)}U^\sim(s) \geq \lim_{s\to\infty}[sU^\sim(s)] = rank\,(CE_r)$$

and

$$rank_{\mathcal{R}(s)}V(s) \geq \lim_{s\to\infty}[sV(s)] = rank\,(E_r'B)$$

It is shown in [Glo86] that these two inequalities are actually equalities; further, the normal rank of $U(s)$ and $V(s)$ is equal, since Rank (CE_r) = Rank $(E_r'B)$, as can be verified by the equality $E_r'C'CE_r = E_r'BB'E_r$, which follows easily from the all-pass equations (6.7). Thus $l \leq \min(p, m, r)$ and l can be easily determined from the balanced realisation of R.

Remark 6.1.3. *In the present work, the gramians of H are not considered to be balanced. The above set of equations is known as the set of "all-pass" equations. Partitioning conformally with (6.5), these can be written in full (for easy future* *reference) as:*

$$
\begin{array}{ll}
\textbf{\textit{(i)}} & \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} = \begin{bmatrix} s_1^2 I & 0 \\ 0 & s_1^2 I \end{bmatrix} \\
\textbf{\textit{(ii)}} & \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \begin{bmatrix} D_{11}' & D_{21}' \\ D_{12}' & 0 \end{bmatrix} = \begin{bmatrix} s_1^2 I & 0 \\ 0 & s_1^2 I \end{bmatrix} = \begin{bmatrix} D_{11}' & D_{21}' \\ D_{12}' & 0 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \\
\textbf{\textit{(iii)}} & \begin{bmatrix} A' & 0 \\ 0 & A_q' \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} + \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} C' & 0 \\ C_{q1}' & C_{q2}' \end{bmatrix} \begin{bmatrix} C & C_{q1} \\ 0 & C_{q2} \end{bmatrix} = 0 \\
\textbf{\textit{(iv)}} & \begin{bmatrix} A & 0 \\ 0 & A_q \end{bmatrix} \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} + \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} \begin{bmatrix} A' & 0 \\ 0 & A_q' \end{bmatrix} + \begin{bmatrix} B & 0 \\ B_{q1} & B_{q2} \end{bmatrix} \begin{bmatrix} B' & B_{q1}' \\ 0 & B_{q2}' \end{bmatrix} = 0 \\
\textbf{\textit{(v)}} & \begin{bmatrix} D_{11}' & D_{21}' \\ D_{12}' & 0 \end{bmatrix} \begin{bmatrix} C & C_{q1} \\ 0 & C_{q2} \end{bmatrix} + \begin{bmatrix} B' & B_{q1}' \\ 0 & B_{q2}' \end{bmatrix} \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} = 0 \\
\textbf{\textit{(vi)}} & \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \begin{bmatrix} B' & B_{q1}' \\ 0 & B_{q2}' \end{bmatrix} + \begin{bmatrix} C & C_{q1} \\ 0 & C_{q2} \end{bmatrix} \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} = 0
\end{array}
\tag{6.8}
$$

Next, using $H_{22} = Q_{22} \in \mathcal{H}_\infty^{+,(m-l)\times(p-l)}$ (with $\|Q_{22}\| < s_1$ from Theorem 6.1.1), we construct an s_1-allpass matrix function $\overline{H}$, corresponding to a new system $\widehat{R} \in \mathcal{H}_\infty^{-,(p-l)\times(m-l)}$ defined from its $(1,1)$ block. It is shown that $\overline{H}$ acts as a s_1-suboptimal Nehari generator of $\widehat{R}$, i.e. that the LFT of $\overline{H}$ with the s_1^{-1}-ball of $\mathcal{H}_\infty$ generates the set

$$\{\Psi \in \mathcal{H}_\infty^{(p-l)\times(m-l)} : \|\hat{R} + \Psi\| \leq s_1\}$$

Using this structure, it is possible to construct all level-two super-optimal approximations of R, which lie inside the set of all optimal approximations, Q, of R. By choosing all Q inside the subset, the corresponding "error" systems $R + Q$ will now minimise the first as well as the second super-optimal levels of R, i.e. this subset defines the super-optimal approximations of R with respect to the first two levels. The method can be repeated using a recursive procedure until all degrees of freedom have been exhausted.

The construction of $\overline{H}$ is based on the following proposition, first stated at a transfer function level. A state-space construction of $\overline{H}$ follows, proving that it acts as an s_1-suboptimal Nehari generator of the anti-stable projection of its $(1,1)$ block.

Proposition 6.1.1. *Let H_{22} be defined in theorem 6.1.1 with $\|H_{22}\|_\infty < s_1$. Then,*

1. *There exists a square transfer matrix $\overline{H}_{21} \in \mathcal{RH}_\infty^{(m-l)\times(m-l)}$ such that $\overline{H}_{21}\overline{H}_{21}^\sim = s_1^2 I - H_{22}H_{22}^\sim$ and $\overline{H}_{21}^{-1} \in \mathcal{RH}_\infty^{(m-l)\times(m-l)}$.*

2. *There exists a square transfer matrix $\overline{H}_{12} \in \mathcal{RH}_\infty^{(p-l)\times(p-l)}$ such that $\overline{H}_{12}^\sim\overline{H}_{12} = s_1^2 I - H_{22}^\sim H_{22}$ and $\overline{H}_{12}^{-1} \in \mathcal{RH}_\infty^{(p-l)\times(p-l)}$.*

3. *The system*

$$\overline{H} = \begin{pmatrix} \overline{H}_{11} & \overline{H}_{12} \\ \overline{H}_{21} & \overline{H}_{22} \end{pmatrix} := \begin{pmatrix} -\overline{H}_{12}H_{22}^\sim\overline{H}_{21}^\sim & \overline{H}_{12} \\ \overline{H}_{21} & H_{22} \end{pmatrix}$$

is in $\mathcal{RL}_\infty^{(p+m-2l)\times(m+p-2l)}$ and is s_1-allpass. Further, let $-\overline{H}_{12}H_{22}^\sim\overline{H}_{21}^\sim = \hat{R} + \overline{Q}_{11}$ where $\hat{R} \in \mathcal{RH}_\infty^-$ and $\overline{Q}_{11} \in \mathcal{RH}_\infty^+$. Then $\|\hat{R}^\sim\|_H < s_1$.

Proof. For parts (1) and (2) see [ZDG96], Corollary 13.22. The proof follows from a detailed construction involving elements from the theory of algebraic Riccati equations and spectral factorisation, which is briefly discussed in appendix B. The proof that $\overline{H}$ is in $\mathcal{L}_\infty$ and is s_1-allpass follows from [Glo86] and can be verified directly by showing that $\overline{H}\overline{H}^\sim = s_1^2 I$. Finally, to show that $\|\hat{R}^\sim\|_H < s_1$, note that since $\overline{H}_{12}$ (or $\overline{H}_{21}$) is a unit of $\mathcal{H}_\infty$ and $\overline{H}$ is s_1-allpass, then $\|\overline{H}_{11}\|_\infty < s_1$. Write $\overline{H}_{11} = \hat{R} + \overline{Q}_{11}$ where $\hat{R} \in \mathcal{H}_\infty^-$ and $\overline{Q}_{11} \in \mathcal{H}_\infty^+$. Then, using Nehari's theorem

$$\|\hat{R}^\sim\|_H = \inf_{X\in\mathcal{H}_\infty^-} \|\hat{R}^\sim + X\|_\infty \le \|\hat{R}^\sim + \overline{Q}_{11}^\sim\|_\infty = \|\overline{H}_{11}^\sim\|_\infty < s_1$$

which completes the proof. □

Remark 6.1.4. *Since $s_1 = \sigma_1(R^\sim)$ the inequality of part (3) says that $\sigma_1(\hat{R}) < \sigma_1(R^\sim)$. As shown later in this section this can be strengthened to $\sigma_1(\hat{R}) < \sigma_{r+1}(R^\sim)$, where r is the multiplicity of the largest Hankel singular value of $R^\sim$.*

A detailed state-space construction of $\overline{H}$ and its properties are given in Theorem 6.1.2 below.

Theorem 6.1.2. *Consider*

$$H_{22} = Q_{22} \stackrel{s}{=} \left[\begin{array}{c|c} A_q & B_{q2} \\ \hline C_{q2} & 0 \end{array}\right] \in \mathcal{H}_\infty^{+,(m-l)\times(p-l)}, \quad \|Q_{22}\|_\infty < s_1$$

defined in Theorem 6.1.1. Then there exist unique stabilising solutions $\overline{P}_2$ and $\overline{Q}_2$ to the following algebraic Riccati equations:

$$\begin{aligned} A_q\overline{P}_2 + \overline{P}_2A_q' + B_{q2}B_{q2}' + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 = 0 \\ A_q'\overline{Q}_2 + \overline{Q}_2A_q + C_{q2}'C_{q2} + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 = 0 \end{aligned} \tag{6.9}$$

respectively. Define:

$$\overline{R} := \overline{Q}_2\overline{P}_2 - s_1^2 I \tag{6.10}$$

Then $\overline{R}$ is non-singular. Further, there exists a $\overline{Q}_a \in \mathcal{H}_\infty^{+,(p+m-2l)\times(p+m-2l)}$ with realisation

$$\overline{Q}_a := \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{21} & Q_{22} \end{pmatrix} \overset{s}{=} \left[\begin{array}{c|cc} A_q & \overline{B}_{q1} & B_{q2} \\ \hline \overline{C}_{q1} & 0 & s_1 I \\ C_{q2} & s_1 I & 0 \end{array}\right] \tag{6.11}$$

where

$$\begin{aligned} \overline{C}_{q1} &= -s_1^{-1}B_{q2}'\overline{Q}_2 \\ \overline{B}_{q1} &= -s_1^{-1}\overline{P}_2C_{q2}' \end{aligned} \tag{6.12}$$

so that $\overline{Q} = \mathcal{F}_l(\overline{Q}_a, s_1^{-1}\,\mathcal{B}\mathcal{H}_\infty^{(p-l)\times(m-l)})$ is the set of all s_1−suboptimal Nehari extensions of a system $\widehat{R} \in \mathcal{H}_\infty^{-,(p-l)\times(m-l)}$ defined as:

$$\widehat{R} \overset{s}{=} \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & 0 \end{array}\right] \tag{6.13}$$

in which

$$\begin{aligned} \widehat{A} &= -(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2})' = -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 \\ \widehat{B} &= -s_1^{-1}C_{q2}' \\ \widehat{C} &= s_1^{-1}B_{q2}'\overline{R} \end{aligned} \tag{6.14}$$

The corresponding "error system"

$$\overline{H} = \widehat{R}_a + \overline{Q}_a = \begin{pmatrix} \widehat{R} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{21} & Q_{22} \end{pmatrix} \tag{6.15}$$

is s_1-allpass and has a realisation

$$\overline{H} := \begin{pmatrix} \overline{H}_{11} & \overline{H}_{12} \\ \overline{H}_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} \widehat{R} + \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{21} & Q_{22} \end{pmatrix} \overset{s}{=} \left[\begin{array}{cc|cc} \widehat{A} & 0 & \widehat{B} & 0 \\ 0 & A_q & \overline{B}_{q1} & B_{q2} \\ \hline \widehat{C} & \overline{C}_{q1} & 0 & s_1 I \\ 0 & C_{q2} & s_1 I & 0 \end{array}\right] \tag{6.16}$$

which satisfies the following set of all-pass equations:

$$
\begin{aligned}
A'_{\overline{H}}Q_{\overline{H}} + Q_{\overline{H}}A_{\overline{H}} + C'_{\overline{H}}C_{\overline{H}} &= 0 \\
A_{\overline{H}}P_{\overline{H}} + P_{\overline{H}}A'_{\overline{H}} + B_{\overline{H}}B'_{\overline{H}} &= 0 \\
D'_{\overline{H}}C_{\overline{H}} + B'_{\overline{H}}Q_{\overline{H}} &= 0 \\
D_{\overline{H}}B'_{\overline{H}} + C_{\overline{H}}P'_{\overline{H}} &= 0 \\
D_{\overline{H}}D'_{\overline{H}} = D'_{\overline{H}}D_{\overline{H}} &= s_1^2 I \\
P_{\overline{H}}Q_{\overline{H}} = Q_{\overline{H}}P_{\overline{H}} &= s_1^2 I
\end{aligned}
\tag{6.17}
$$

in which $Q_{\overline{H}}$ and $P_{\overline{H}}$ are the gramians of the realisation of $\overline{H}$ given in (6.16).

Proof. The proof is based on [Glo84]; see also [JL93] and [GLD+91] for a more general setting. Here we outline the sequence of logical arguments. The existence of solutions of the two Riccati equations (6.9) follows from standard theory of spectral factorisation and the bounded real-lemma (see Lemma 6.1.1 in the next section) and relies on the fact that $\|Q_{22}\|_\infty < s_1$. Details and additional properties of the two solutions are included in Appendix B. Since the two stabilising solutions are chosen, $\hat{A}$ defined in equation (6.14) is anti-stable and thus $\hat{R} \in \mathcal{H}_\infty^-$. Systems $\overline{Q}_a$ and $\hat{R}$ correspond to the stable and anti-stable projections of $\overline{H}$ given in Proposition 6.1.1 which also shows that $\overline{H}$ is s_1-all pass. For a state-space based proof one needs to verify the all-pass equations given in (6.17) and expanded in (6.18) below; this is straightforward using the realisations given in Theorem 6.1.1 and the two Riccati equations (6.9). To show that $\overline{R}$ is non-singular, first note that $\overline{P}_2$ and $\overline{Q}_2$ are the controllability and observability gramians, respectively, of the realisation of $\overline{Q}_a$ given in equation (6.11), so that $\sigma_1^2(\overline{Q}_a) = \lambda_{\max}(\overline{P}_2\overline{Q}_2)$. A standard argument (e.g. see the early part of the proof of Theorem 6.1.4 which does not rely on any state-space arguments) shows that $\sigma_1(\overline{Q}_a) \leq \sigma_{r+1}(\hat{R}) < \sigma_1(R^\sim) = s_1$. Thus $\rho(\overline{P}_2\overline{Q}_2) < s_1^2$ and thus $\overline{R}$ is nonsingular. Finally, the fact that $\overline{Q}_a$ generates all s_1-suboptimal Nehari extensions of $\hat{R}$ follows from the inertia properties of A and $\hat{A}$ and the all pass-nature of $\overline{H}$ [Glo86]; the proof reduces to showing that the invariant zeros of the realisations of $\overline{Q}_{12}$ (or $\overline{Q}_{21}$) given in (6.16) lie in the open right-half plane, which follows readily by a simple calculation using the fact that $\lambda(\hat{A}) \subseteq \mathcal{C}_+$. □

Remark 6.1.5. *Expanding the compact form of the all-pass equations given in Theorem*

6.1.2 we get

$$\begin{aligned}
&(i) && \begin{bmatrix} \widehat{A}' & 0 \\ 0 & A_q' \end{bmatrix} \begin{bmatrix} \overline{Q}_1 & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix} + \begin{bmatrix} \overline{Q}_1 & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix} \begin{bmatrix} \widehat{A} & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} \widehat{C}' & 0 \\ \overline{C}_{q1}' & C_{q2}' \end{bmatrix} \begin{bmatrix} \widehat{C} & \overline{C}_{q1} \\ 0 & C_{q2} \end{bmatrix} = 0 \\
&(ii) && \begin{bmatrix} \widehat{A} & 0 \\ 0 & A_q \end{bmatrix} \begin{bmatrix} \widehat{P}_1 & I \\ I & \overline{P}_2 \end{bmatrix} + \begin{bmatrix} \widehat{P}_1 & I \\ I & \overline{P}_2 \end{bmatrix} \begin{bmatrix} \widehat{A}' & 0 \\ 0 & A_q' \end{bmatrix} + \begin{bmatrix} \widehat{B} & 0 \\ \overline{B}_{q1} & B_{q2} \end{bmatrix} \begin{bmatrix} \widehat{B}' & \overline{B}_{q1}' \\ 0 & B_{q2}' \end{bmatrix} = 0 \\
&(iii) && \begin{bmatrix} 0 & s_1 I \\ s_1 I & 0 \end{bmatrix} \begin{bmatrix} \widehat{C} & \overline{C}_{q1} \\ 0 & C_{q2} \end{bmatrix} + \begin{bmatrix} \widehat{B}' & \overline{B}_{q1}' \\ 0 & B_{q2}' \end{bmatrix} \begin{bmatrix} \overline{Q}_1 & -\overline{R}' \\ -\overline{R} & \overline{Q}_2 \end{bmatrix} = 0 \\
&(iv) && \begin{bmatrix} 0 & s_1 I \\ s_1 I & 0 \end{bmatrix} \begin{bmatrix} \widehat{B}' & \overline{B}_{q1}' \\ 0 & B_{q2}' \end{bmatrix} + \begin{bmatrix} \widehat{C} & \overline{C}_{q1} \\ 0 & C_{q2} \end{bmatrix} \begin{bmatrix} \widehat{P}_1 & I \\ I & \overline{P}_2 \end{bmatrix} = 0 \\
&(v) && \begin{bmatrix} \overline{Q}_2 \overline{R}^{-\prime} & I \\ I & \overline{P}_2 \end{bmatrix} \begin{bmatrix} \overline{P}_2 \overline{R} & -\overline{R}' \\ I & \overline{Q}_2 \end{bmatrix} = \begin{bmatrix} s_1^2 I & 0 \\ 0 & s_1^2 I \end{bmatrix}
\end{aligned} \tag{6.18}$$

where $\widehat{P}_1 = \overline{Q}_2 \overline{R}^{-\prime}$ *and* $\overline{Q}_1 = \overline{P}_2 \overline{R}$.

In the sequel, a significant result involving the diagonal form of super-optimal approximation is derived. The proofs of the following theorems combine all results derived in this chapter up to this point.

Diagonalisation with multiplicity considerations. The following theorem constructs a diagonalising transformation of $\overline{H}$ and solves the level-two SODP.

Theorem 6.1.3. *Let* H *and* $\overline{H}$ *be as defined in Theorems 6.1.1 and 6.1.2, respectively. Then*

$$\|R^{\sim}\|_H = s_1(R) = s_2(R) = \ldots = s_l(R) > s_{l+1}(R) = \|\hat{R}^{\sim}\|_H$$

Further,

$$\mathcal{S}_1(R) = \mathcal{S}_2(R) = \ldots = \mathcal{S}_l(R) = \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_\infty^{(p-l)\times(m-l)})$$

and

$$\mathcal{S}_{l+1}(R) = \mathcal{F}_l[Q_a, \mathcal{F}_u(\overline{Q}_a^{-1}, \mathcal{S}_1(\widehat{R}))] \subseteq \mathcal{S}_1(R)$$

where Q_a *and* $\overline{Q}_a$ *are defined in Theorems 6.1.1 and 6.1.2.*

Proof. We adapt the proof of [JL93] Theorem 3 to our setting. First note that since $HH^{\sim} = H^{\sim}H = s_1^2 I$ and $\overline{H}\overline{H}^{\sim} = \overline{H}^{\sim}\overline{H} = s_1^2 I$, it follows that

$$H_{11}H_{21}^{\sim} = -H_{21}H_{22}^{\sim}, \qquad \overline{H}_{11} = -\overline{H}_{12}\overline{H}_{22}^{\sim}\overline{H}_{21}^{-\sim}, \tag{6.19}$$

$$\overline{H}_{21}\overline{H}_{21}^{\sim} = s_1^2 I - H_{22}H_{22}^{\sim} = H_{21}H_{21}^{\sim} \tag{6.20}$$

and

$$\overline{H}_{12}^{\sim}\overline{H}_{12} = s_1^2 I - H_{22}^{\sim}H_{22} = H_{12}^{\sim}H_{12} \tag{6.21}$$

Define

$$V_{\perp} := H_{12}\overline{H}_{12}^{-1} \qquad \text{and} \qquad W_{\perp} := H_{21}^{\sim}\overline{H}_{21}^{-\sim} \tag{6.22}$$

Then (6.20) implies that

$$V_{\perp}^{\sim}V_{\perp} = I_{p-l} \qquad \text{and} \qquad W_{\perp}^{\sim}W_{\perp} = I_{m-l} \tag{6.23}$$

It can be readily verified from a state-space calculation (see next section) that $V_{\perp} \in \mathcal{H}_{\infty}^{+,(p-l)\times p}$ and $W_{\perp} \in \mathcal{H}_{\infty}^{-,(m-l)\times m}$. Thus there exist complementary inner and co-inner factors, respectively, such that

$$V := \begin{pmatrix} v & V_{\perp} \end{pmatrix} \in \mathcal{H}_{\infty}^{+,\, p\times p} \qquad \text{and} \qquad W := \begin{pmatrix} w & W_{\perp} \end{pmatrix} \in \mathcal{H}_{\infty}^{-,\, m\times m}$$

are square-inner and square anti-inner, respectively [ZDG96], [GL95]. Thus, using (6.19) and the definitions (6.22), we obtain

$$\begin{aligned} V_{\perp}^{\sim}H_{12} &= \overline{H}_{12}^{-\sim}H_{12}^{\sim}H_{12} = \overline{H}_{12}^{-\sim}\overline{H}_{12}^{\sim}\overline{H}_{12} = \overline{H}_{12} \\ H_{21}W_{\perp} &= H_{21}H_{21}^{\sim}\overline{H}_{21}^{-\sim} = \overline{H}_{21}\overline{H}_{21}^{\sim}\overline{H}_{21}^{-\sim} = \overline{H}_{21} \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} V_{\perp}^{\sim}H_{11}W_{\perp} &= V_{\perp}^{\sim}H_{11}H_{21}^{\sim}\overline{H}_{21}^{-\sim} = -V_{\perp}^{\sim}H_{12}H_{22}^{\sim}\overline{H}_{21}^{-\sim} \\ &= -\overline{H}_{12}H_{22}^{\sim}\overline{H}_{21}^{-\sim} = \overline{H}_{11} \end{aligned} \tag{6.25}$$

It follows that

$$\begin{pmatrix} V^{\sim} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} W & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} v^{\sim}H_{11}w & v^{\sim}H_{11}W_{\perp} & v^{\sim}H_{12} \\ V_{\perp}^{\sim}H_{11}w & \overline{H}_{11} & \overline{H}_{12} \\ H_{21}w & \overline{H}_{21} & \overline{H}_{22} \end{pmatrix} \tag{6.26}$$

Now, since V and W are all-pass and H is s_1-allpass, the system on the RHS of equation (6.26) is s_1-allpass. But since $\overline{H}$ is also s_1-allpass (Theorem 6.1.2), we have that $v^{\sim}H_{11}W_{\perp} = 0$, $v^{\sim}H_{12} = 0$, $V_{\perp}^{\sim}H_{11}w = 0$, $H_{21}w = 0$, and $v^{\sim}H_{11}w$ is s_1-allpass and can be written as $v^{\sim}H_{11}w = s_1\alpha(s)$, for some $l \times l$ all-pass matrix-function $\alpha(s)$ (generically $l = 1$ and hence $\alpha(s)$ is scalar). Taking linear fractional transformations with the set $s_1^{-1}\ \mathcal{BH}_\infty^{(p-l)\times(m-l)}$ and using the results of Theorem 6.1.2 and Theorem 6.1.1 shows that:

$$V^{\sim}[\mathcal{F}_l(H, s_1^{-1}\ \mathcal{BH}_\infty^{(p-l)\times(m-l)})]W = \begin{pmatrix} s_1\alpha & 0 \\ 0 & \mathcal{F}_l(\overline{H}, s_1^{-1}\ \mathcal{BH}_\infty^{(p-l)\times(m-l)}) \end{pmatrix} \tag{6.27}$$

or equivalently,

$$V^{\sim}[R + \mathcal{S}_1(R)]W = \begin{pmatrix} s_1\alpha & 0 \\ 0 & \hat{R} + \mathcal{S}(\hat{R}, s_1) \end{pmatrix} \tag{6.28}$$

Since $\alpha(s) \in \mathcal{R}^{l\times l}(s)$ and is all-pass (in fact anti-inner as shown in the next section), it follows that:

$$\|R^{\sim}\|_H = s_1(R) = s_2(R) = \ldots = s_l(R) > s_{l+1}(R) = \|\hat{R}^{\sim}\|_H$$

and

$$\mathcal{S}_1(R) = \mathcal{S}_2(R) = \ldots = \mathcal{S}_l(R) = \mathcal{F}_l(Q_a, s_1^{-1}\ \mathcal{BH}_\infty^{(p-l)\times(m-l)})$$

which is the set of all optimal Nehari extensions of R. Further, since all optimal Nehari extensions of $\hat{R}$ are also s_1-suboptimal extensions of $\hat{R}$, i.e. $\mathcal{S}_1(\hat{R}) \subseteq \mathcal{S}(\hat{R}, s_1)$, it follows that

$$s_{l+1}(R) = s_1(\widehat{R}) = \|R^{\sim}\|_H$$

and

$$\begin{aligned} R + \mathcal{S}_2(R) &= \begin{pmatrix} v & V_\perp \end{pmatrix} \begin{pmatrix} s_1\alpha & 0 \\ 0 & \widehat{R} + \mathcal{S}_1(\widehat{R}) \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix} \\ &= \begin{pmatrix} v & V_\perp \end{pmatrix} \begin{pmatrix} s_1\alpha & 0 \\ 0 & \widehat{R} + \overline{Q} \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix} + \begin{pmatrix} v & V_\perp \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{S}_1(\widehat{R}) - \overline{Q} \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix} \\ &= R + Q_{11} + V_\perp(\mathcal{S}_1(\widehat{R}) - \overline{Q})W_\perp^{\sim} \end{aligned} \tag{6.29}$$

by observing that

$$V^{\sim}H_{11}W = \begin{pmatrix} s_1\alpha & 0 \\ 0 & \overline{H}_{11} \end{pmatrix} \Rightarrow R + Q_{11} = V \begin{pmatrix} s_1\alpha & 0 \\ 0 & \widehat{R} + \overline{Q}_{11} \end{pmatrix} W^{\sim}$$

Using the definitions of of $V_\perp$ and $W_\perp^{\sim}$ in (6.22) and cancelling R from both sides of equation (6.29), we can write:

$$\begin{aligned}
\mathcal{S}_2(R) &= Q_{11} + Q_{12}\overline{Q}_{12}^{-1}(\mathcal{S}_1(\widehat{R}) - \overline{Q})\overline{Q}_{21}^{-1}Q_{21} \\
&= \mathcal{F}_l\left(\begin{pmatrix} Q_{11} - Q_{12}\overline{Q}_{12}^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{21} & Q_{12}\overline{Q}_{12}^{-1} \\ \overline{Q}_{21}^{-1}Q_{21} & 0 \end{pmatrix}, \mathcal{S}_1(\widehat{R})\right) \\
&=: \mathcal{F}_l(K, \mathcal{S}_1(\widehat{R}))
\end{aligned}$$

where

$$K := \begin{pmatrix} Q_{11} - Q_{12}\overline{Q}_{12}^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{21} & Q_{12}\overline{Q}_{12}^{-1} \\ \overline{Q}_{21}^{-1}Q_{21} & 0 \end{pmatrix} = \mathcal{F}_l(Q_a, \overline{Q}_a^{-1})$$

using a series of calculations (see appendix C). This completes the proof. □

"Interlacing" inequalities of super-optimal levels. The following Theorem establishes bounds on the super-optimal levels. The proof is similar to a parallel result in [LHG89], but the assumption involving the multiplicity of the largest Hankel singular value of $R^\sim$ is removed.

Theorem 6.1.4 (Super-optimal level bounds). *The $(l+1)$-th super-optimal level is bounded above by the $(r+1)$-th Hankel singular value of $R^\sim$, i.e.*

$$\sigma_1(\hat{R}^\sim) = s_{l+1}(R) \leq \sigma_{r+1}(R^\sim) < s_1(R) = s_2(R) = \ldots = s_l(R) = \sigma_1(R^\sim)$$

Proof. The proof follows from the following sequence of inequalities:

$$\begin{aligned}
\sigma_{i+r}(R^\sim) &= \sigma_i(Q_a) \qquad\qquad i = 1, 2, \ldots, n-r \\
&= \inf_{\Psi \in \mathcal{H}_\infty^-(i-1)} \|Q_a + \Psi\|_\infty \\
&= \inf_{\Psi \in \mathcal{H}_\infty^-(i-1)} \|R + Q_a + \Psi\|_\infty \\
&\geq \inf_{\Psi \in \mathcal{H}_\infty^-(i-1)} \left\| \begin{pmatrix} V_\perp^\sim & 0 \\ 0 & I \end{pmatrix} (R + Q_a + \Psi) \begin{pmatrix} W_\perp & 0 \\ 0 & I \end{pmatrix} \right\|_\infty \\
&\geq \inf_{\widehat{\Psi} \in \mathcal{H}_\infty^-(i-1)} \|\widehat{R}_a + \overline{Q}_a + \widehat{\Psi}\|_\infty \\
&\geq \inf_{\widehat{\Psi} \in \mathcal{H}_\infty^-(i-1)} \|\overline{Q}_a + \widehat{\Psi}\|_\infty \\
&= \sigma_i(\overline{Q}_a)
\end{aligned}$$

where the set $\mathcal{H}_\infty^-(i-1)$ is already defined in chapter 4. The first equality follows from Theorem 6.1.1. The second equality is a statement of the AAK Theorem [Glo86], while the third equality holds since $R \in \mathcal{H}_\infty^-$ and can be absorbed in Ψ. The first inequality follows from the fact that $V_\perp$ and $W_\perp$ are contractive, while the second inequality follows from Theorem 6.1.3 and the fact that $V_\perp^\sim$ and $W_\perp$ are both in $\mathcal{RH}_\infty^-$. Finally, the third inequality follows from the fact that $\widehat{R} \in \mathcal{RH}_\infty^-$, while the last equality is a restatement of the AAK Theorem.

Setting $i = 1$ in the above inequality shows that $\sigma_{r+1}(R^\sim) \geq \sigma_1(\overline{Q}_a)$. Now, using (6.18), it follows that

$$\sigma_i^2(\widehat{R}^\sim) = \lambda_i(\widehat{P}_1\overline{Q}_1) = \lambda_i(\overline{Q}_2\overline{R}^{-'}\overline{P}_2\overline{R}) = \lambda_i(\overline{Q}_2\overline{P}_2) = \sigma_i^2(\overline{Q}_a)$$

and so $\widehat{R}^\sim$ and $\overline{Q}_a$ have identical Hankel singular values. In particular, $s_{l+1}(R) = \sigma_1(\widehat{R}^\sim) \leq \sigma_{r+1}(R^\sim)$ using the result of Theorem 6.1.3. □

Remark 6.1.6. *The result of Theorem 6.1.4 may be propagated to establish upper bounds for the subsequent super-optimal levels $s_i(R)$, $i > l+1$.*

Remark 6.1.7. *The early part of the proof (which does not rely on any state-space based arguments) may be used to show that $\sigma_1(\overline{Q}_a) \leq \sigma_{r+1}(R^\sim) < \sigma_1(R^\sim) = s_1$, from which it follows immediately that $\overline{R}$ defined in Theorem 6.1.2 is non-singular.*

6.1.2 State-space analysis

In this section we develop a state-space analysis of the solution to the super-optimal distance problem. At this point we shall use some background material which is presented in appendix B and it is related to algebraic Riccati equations and spectral factorisations. The main results are derived from the "Bounded-real lemma" presented in appendix B.

Lemma 6.1.1. *Let $G \in \mathcal{RH}_\infty$ with $G(s) = C(sI - A)^{-1}B$ and assume that (A, B) and (C, A) are stabilisable and detectable, respectively. Then, the following conditions are equivalent:*

1. $\|G\|_\infty < \gamma$

2. *The Hamiltonian* $H = \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & A' \end{bmatrix}$ *has no pure imaginary eigenvalues*

3. $H \in dom(Ric)$

Proof. 1 $\Leftrightarrow$ 2. See [ZDG96], lemma 4.7.

2 $\Leftrightarrow$ 3. See [ZDG96], Theorem 13.6. □

As an immediate consequence of the above Lemma we get the following result:

Proposition 6.1.2. *The algebraic Riccati equations (6.9) (Theorem 6.1.2) have (unique) positive-semidefinite stabilising solutions $\overline{P}_2$ and $\overline{Q}_2$ respectively.*

Proof. Since A_q is asymptotically stable, the conditions of stabilisability and detectability of Lemma 6.1.1 are trivially satisfied. Further, the fact that $\|Q_{22}\|_\infty < s_1$ (see Theorem 6.1.1) shows that the two Hamiltonian associated with equations (6.9) are free of imaginary axis eigenvalues and that (unique) stabilising solutions $\overline{P}_2$ and $\overline{Q}_2$ to these two equations exist. The fact that $\overline{P}_2 \geq 0$ and $\overline{Q}_2 \geq 0$ follows from [ZDG96]. □

Our next result shows that the two Riccati equations (6.9) are intimately related.

Proposition 6.1.3. *Let $\overline{P}_2$ be the stabilising solution of **Ric1**,*

$$A_q\overline{P}_2 + \overline{P}_2A_q' + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 + B_{q2}B_{q2}' = 0$$

so that $\lambda(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}) \subseteq C_-$ and its associated Hamiltonian

$$H_1 = \begin{bmatrix} A_q' & s_1^{-2}C_{q2}'C_{q2} \\ -B_{q2}B_{q2}' & -A_q \end{bmatrix} \tag{6.30}$$

*Let also $\overline{Q}_2$ be the stabilising solution of **Ric2**:*

$$A_q'\overline{Q}_2 + \overline{Q}_2A_q + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 + C_{q2}'C_{q2} = 0$$

so that $\lambda(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2) \subseteq C_-$ and its associated Hamiltonian

$$H_2 = \begin{bmatrix} A_q & s_1^{-2}B_{q2}B_{q2}' \\ -C_{q2}'C_{q2} & -A_q' \end{bmatrix} \tag{6.31}$$

Then H_1 and H_2 have identical spectra. In particular there exists a similarity transformation $\overline{R}'$ so that

$$(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}) = \overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)(\overline{R}')^{-1} \tag{6.32}$$

where $\overline{R}$ is defined (6.10).

Proof. Take

$$T = \begin{bmatrix} 0 & s_1^{-1}I \\ s_1 I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} s_1 I & 0 \\ 0 & s_1^{-1}I \end{bmatrix}$$

Note that $T = T^{-1}$. Then by inspection, $-TH_2T^{-1} = H_1$ and hence the first claim is true (since we know from Appendix B that the spectrum of a Hamiltonian is symmetrical with respect to $j\omega$-axis). Define

$$T_P := \begin{bmatrix} I & 0 \\ -\overline{P}_2 & I \end{bmatrix} \quad \Rightarrow \quad T_P^{-1} = \begin{bmatrix} I & 0 \\ \overline{P}_2 & I \end{bmatrix}$$

and observe that

$$\begin{bmatrix} I & 0 \\ -\overline{P}_2 & I \end{bmatrix} \begin{bmatrix} A'_q & s_1^{-2}C'_{q2}C_{q2} \\ -B_{q2}B'_{q2} & -A_q \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{P}_2 & I \end{bmatrix}$$
$$= \begin{bmatrix} A'_q + s_1^{-2}C'_{q2}C_{q2}\overline{P}_2 & s_1^{-2}C'_{q2}C_{q2} \\ 0 & -(A_q + s_1^{-2}\overline{P}_2C'_{q2}C_{q2}) \end{bmatrix} =: \widehat{H}_1$$

Similarly, define

$$T_Q := \begin{bmatrix} I & 0 \\ -\overline{Q}_2 & I \end{bmatrix} \quad \Rightarrow \quad T_Q^{-1} = \begin{bmatrix} I & 0 \\ \overline{Q}_2 & I \end{bmatrix}$$

so that

$$\begin{bmatrix} I & 0 \\ -\overline{Q}_2 & I \end{bmatrix} \begin{bmatrix} A_q & s_1^{-2}B_{q2}B'_{q2} \\ -C'_{q2}C_{q2} & -A'_q \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{Q}_2 & I \end{bmatrix}$$
$$= \begin{bmatrix} A_q + s_1^{-2}B_{q2}B'_{q2}\overline{Q}_2 & s_1^{-2}B_{q2}B'_{q2} \\ 0 & -(A'_q + s_1^{-2}\overline{Q}_2B_{q2}B'_{q2}) \end{bmatrix} =: \widehat{H}_2$$

Thus,

$$\begin{cases} H_1 = -TH_2T^{-1} \\ \widehat{H}_1 = T_PH_1T_P^{-1} \\ \widehat{H}_2 = T_QH_2T_Q^{-1} \Rightarrow H_2 = T_Q^{-1}\widehat{H}_2T_Q \end{cases}$$

Using the three above equations

$$H_1 = -TT_Q^{-1}\widehat{H}_2T_QT^{-1} \Rightarrow \widehat{H}_1 = -T_PTT_Q^{-1}\widehat{H}_2T_QT^{-1}T_P^{-1}$$

Further,

$$\widehat{H}_1(T_QT^{-1}T_P^{-1})^{-1} = -T_PTT_Q^{-1}\widehat{H}_2 \Rightarrow \widehat{H}_1T_PTT_Q^{-1} = -T_PTT_Q^{-1}\widehat{H}_2$$

and

$$(T_PTT_Q^{-1})^{-1}\widehat{H}_1 = -\widehat{H}_2T_QT^{-1}T_P^{-1} \Rightarrow \widehat{H}_1(T_PTT_Q^{-1}) = -(T_PTT_Q^{-1})\widehat{H}_2 \tag{6.33}$$

with

$$T_PTT_Q^{-1} = \begin{bmatrix} I & 0 \\ -\overline{P}_2 & I \end{bmatrix} \begin{bmatrix} 0 & s_1^{-1}I \\ s_1I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{Q}_2 & I \end{bmatrix} = s_1^{-1} \begin{bmatrix} \overline{Q}_2 & I \\ -\overline{R}' & -\overline{P}_2 \end{bmatrix}$$

and

$$T_QT^{-1}T_P^{-1} = \begin{bmatrix} I & 0 \\ -\overline{Q}_2 & I \end{bmatrix} \begin{bmatrix} 0 & s_1^{-1}I \\ s_1I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{P}_2 & I \end{bmatrix} = s_1^{-1} \begin{bmatrix} \overline{P}_2 & I \\ -\overline{R} & -\overline{Q}_2 \end{bmatrix}$$

Writing equation (6.33) in full:

$$\begin{bmatrix} A_q' + s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & s_1^{-2}C_{q2}'C_{q2} \\ 0 & -(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}) \end{bmatrix} \begin{bmatrix} \overline{Q}_2 & I \\ -\overline{R}' & -\overline{P}_2 \end{bmatrix}$$
$$= \begin{bmatrix} -\overline{Q}_2 & -I \\ \overline{R}' & \overline{P}_2 \end{bmatrix} \begin{bmatrix} A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2 & s_1^{-2}B_{q2}B_{q2}' \\ 0 & -(A_q + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}') \end{bmatrix}$$

From the $(2,1)$ partition of the above equation, we have $(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2})\overline{R}' = \overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)$. So,

$$(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}) = \overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)(\overline{R}')^{-1}$$

which proves the second claim. □

Remark 6.1.8. *Note that this proposition implies that the "A" matrices of the state space realisations of $V_\perp$ and $W_\perp$, defined in (6.22), have the same spectrum.*

Proposition 6.1.4. *Define*

$$V_\perp := H_{12}\overline{H}_{12}^{-1} \quad \text{and} \quad W_\perp := H_{21}^{\sim}\overline{H}_{21}^{-\sim}$$

Then, $V_\perp$ and $W_\perp^{\sim}$ have, the following realisations:

$$V_\perp \overset{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline C_{q1} - s_1^{-1}D_{12}\overline{C}_{q1} & s_1^{-1}D_{12} \end{array}\right]$$

and

$$W_\perp^{\sim} \overset{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}\overline{B}_{q1}C_{q2} & B_{q1} - s_1^{-1}\overline{B}_{q1}D_{21} \\ \hline s_1^{-1}C_{q2} & s_1^{-1}D_{21} \end{array}\right]$$

with corresponding controllability and observability gramians:

$$Y_v = -(\overline{R}')^{-1}\overline{P}_2, \qquad X_v = Q_2 - \overline{Q}_2$$
$$Y_w = P_2 - \overline{P}_2, \qquad X_w = -\widehat{P}_1.$$

In particular, the following matrix inequalities hold: $P_2 \geq \overline{P}_2 \geq 0$ *and* $Q_2 \geq \overline{Q}_2 \geq 0$.

Proof. See appendix C. □

Complementary inner factorisation. The matrix functions $V_\perp$ and $W_\perp^\sim$ which were constructed in proposition 6.1.4 are parts of inner matrix functions. Theorem 6.1.3 relies on the construction of two inner complements v and $w^\sim$ so that $\begin{pmatrix} v & V_\perp \end{pmatrix}$ and $\begin{pmatrix} w^\sim \\ W_\perp^\sim \end{pmatrix}$ are square inner. To find realisations for v and w, we can apply Lemma 13.31 from [ZDG96] which uses the gramians of the realisations of $V_\perp$ and $W_\perp^\sim$. This is outlined next, together with concrete realisations of v and $w^\sim$.

Corollary 6.1.1. *Let $V_\perp, W_\perp^\sim$ be as defined in proposition 6.1.4. Then there exists a complementary inner factor of v and a complementary co-inner factor of w, respectively, such that*

$$V(s) := \begin{pmatrix} v & V_\perp \end{pmatrix}(s), \qquad W(s) := \begin{pmatrix} w^\sim \\ W_\perp^\sim \end{pmatrix}(s)$$

are square inner. Further, $V \in \mathcal{RH}_\infty^{-,\, p\times p}$ and $W \in \mathcal{RH}_\infty^{+,\, m\times m}$. Concrete realisations of $v^\sim$ and w are given as:

$$v^\sim \overset{s}{=} \left[\begin{array}{c|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & C_{q1}' + s_1^{-2}\overline{Q}_2 B_{q2} D_{12}' \\ \hline (D_{12}^\perp)' C_{q1}(Q_2 - \overline{Q}_2)^\dagger & (D_{12}^\perp)' \end{array}\right]$$

and

$$w \overset{s}{=} \left[\begin{array}{c|c} -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & (\overline{P}_2 - P_2)^\dagger B_{q1} D_{21}^\perp \\ \hline -B_{q1}' - s_1^{-2}D_{21}'C_{q2}\overline{P}_2 & D_{21}^\perp \end{array}\right]$$

respectively.

Proof. This follows immediately from Lemma 13.31 in [ZDG96]. □

Observation 6.1.1. *Along with Remark (4.3.3), the pair (v, w) as constructed in corollary 6.1.1 forms a scaled Schmidt pair corresponding to the largest Hankel singular value of $R^\sim$. Observe that $v^\sim, w \in \mathcal{RH}_\infty$ (i.e. they have a "D" matrix), where Schmidt vectors by definition belong to $\mathcal{RH}_2$ and $\mathcal{RH}_2^-$, respectively (i.e. strictly proper).*

Inertia properties of all-pass function $\alpha(s)$. In the final part of this section we develop a state space realisation of the allpass system $\alpha(s)$ defined in the proof of Theorem 6.1.3 and show that it is anti-inner. The proof is based on a lengthy state space calculation which reveals numerous pole-zero cancellations. We first need the following two results.

Proposition 6.1.5. *Let Q, P be the observability and the controllability gramians, respectively, of a system having state space realisation $G \stackrel{s}{=} (A, B, C)$. Then, (i) $\mathcal{N}(Q) \subseteq \mathcal{N}(C)$ and (ii) $\mathcal{N}(P) \subseteq \mathcal{N}(B')$.*

Proof. (i) Let $\underline{\xi}_o \in Ker(Q), \underline{\xi}_o \neq 0$. Then, $Q\xi_o = 0$. Consider the Lyapunov equation:

$$A'Q + QA + C'C = 0 \Rightarrow \xi_o'(A'Q + QA + C'C)\xi_o = 0 \Rightarrow C\xi_o = 0$$

and hence $\mathcal{N}(Q) \subseteq \mathcal{N}(C)$. A similar argument proves part (ii). □

Proposition 6.1.6. *In previously defined notation:*

(i) $\left[I - (Q_2 - \overline{Q}_2)^\dagger (Q_2 - \overline{Q}_2)\right] C_{q1}' D_{12}^\perp = 0$, *and*

(ii) $\left[I - (\overline{P}_2 - P_2)^\dagger (\overline{P}_2 - P_2)\right] B_{q1} D_{21}^\perp = 0$.

Proof. (i) First note that from Proposition 6.1.4 $(Q_2 - \overline{Q}_2)$ is the observability gramian of $(A_q + s_1^{-2} B_{q2} B_{q2} \overline{Q}_2, C_{q1} + s_1^{-2} D_{12} B_{q2}' \overline{Q}_2)$. It follows, using Proposition 6.1.5 that $\mathcal{N}[Q_2 - \overline{Q}_2] \subseteq \mathcal{N}[C_{q1} + s_1^{-2} D_{12} B_{q2}' \overline{Q}_2]$, or equivalently, $\mathcal{R}[C_{q1}' + s_1^{-2} \overline{Q}_2 B_{q2} D_{12}'] \subseteq \mathcal{R}[Q_2 - \overline{Q}_2]$. Thus,

$$\mathcal{R}[(C_{q1}' + s_1^{-2} \overline{Q}_2 B_{q2} D_{12}') D_{12}^\perp] = \mathcal{R}[C_{q1}' D_{12}^\perp] \subseteq \mathcal{R}[C_{q1}' + s_1^{-2} \overline{Q}_2 B_{q2} D_{12}']$$

and hence $\mathcal{R}[C_{q1}' D_{12}^\perp] \subseteq \mathcal{R}[Q_2 - \overline{Q}_2]$. The result now follows on noting that $\left[I - (Q_2 - \overline{Q}_2)^\dagger (Q_2 - \overline{Q}_2)\right]$ projects orthogonally onto $\mathcal{N}[Q_2 - \overline{Q}_2]$. Part (ii) follows dually on noting that $P_2 - \overline{P}_2$ is the controllability gramian of the realisation of $W_\perp^\sim$ given in Proposition 6.1.4. □

Proposition 6.1.7. *The s_1-allpass system $s_1\alpha(s) \in \mathcal{RL}_\infty^{l \times l}$ defined in the proof of Theorem 6.1.3 can be written as a parallel system interconnection $s_1\alpha(s) = \alpha_1(s) + \alpha_2(s)$,*

$$s_1\alpha(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A & 0 & B_{\alpha_1} \\ 0 & -A_q' - s_1^{-2} C_{q2}' C_{q2} \overline{P}_2 & B_{\alpha_2} \\ \hline C_{\alpha_1} & C_{\alpha_2} & (D_{12}^\perp)' D_{11} D_{21}^\perp \end{array}\right]$$

in which

$$B_{\alpha_1} := BD_{21}^{\perp} + P_3(\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp}$$
$$B_{\alpha_2} := (\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp}$$
$$C_{\alpha_1} := -(D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}Q_3' + (D_{12}^{\perp})'C$$
$$C_{\alpha_2} := -(D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}\overline{R}$$

In particular, $\alpha \in \mathcal{RH}_{\infty}^{-,\,l\times l}$ *and* $deg(\alpha) \leq 2n - r$.

Proof. see Appendix C. □

6.2 Examples

Throughout this section some examples of level-two super-optimal approximations are considered. Most of the examples considered here are pathological cases with interesting properties. The first example is carefully constructed to illustrate the non-generic case of remark 6.1.2 and example 4.3.1.

Example 6.2.1. *Consider the following anti-stable system*[1] *with realisation:*

$$R(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right] := \left[\begin{array}{ccc|cc} 1 & 1 & \frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & \sqrt{10}/5 & (2\sqrt{10})/5 \\ 3 & 4 & \frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & (2\sqrt{10})/5 & (4\sqrt{10})/5 \\ \frac{1}{1+0.7}\frac{3\sqrt{10}}{5} & \frac{1}{1+0.7}\frac{6\sqrt{10}}{5} & \frac{1}{0.7} & 1 & 1 \\ \hline \sqrt{10}/5 & (2\sqrt{10})/5 & 1 & 0 & 0 \\ (2\sqrt{10})/5 & (4\sqrt{10})/5 & 1 & 0 & 0 \end{array}\right]$$

where $B = C'$. *Then, the realisation is balanced with gramians equal to*

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

and so the multiplicity on the largest Hankel singular value is equal to $r = 2$. *On the other hand,* $l = rank(B_1) = rank(C_1) = 1 < r$. *In addition, we construct the generator of all* s_1*-suboptimal Nehari extensions,*

$$\bar{Q}_a(s) = \frac{1}{s + 0.6443}\begin{pmatrix} 0.045176 & s + 0.5112 \\ s + 0.5112 & 0.39216 \end{pmatrix}$$

[1] *Note that* $R(s)$ *is essentially* $G(-s)$ *of example 4.3.1, in chapter 4 (i.e. inverse inertia problem). Hence, the Schmidt vectors found in example 4.3.1 correspond here to the largest singular value of* $\Gamma_{R(-s)}$.

and find the appropriate functions

$$\hat{R}(s) = \frac{0.34698}{s - 0.5112}, \quad \hat{Q} = 0.5888$$

such that $\|\hat{R}_{aug}(s) + \bar{Q}_a(s)\|_\infty < s_1$ *and* $\|\hat{R}(s) + \hat{Q}\|_\infty = s_2$. *The super-optimal approximation is given by*

$$Q_{sopt} = \frac{1}{s + 0.5112}\begin{pmatrix} 0.47153(s + 0.5165) & 0.26424(s + 0.6258) \\ 0.26424(s + 0.6258) & 0.86788(s + 0.933) \end{pmatrix}$$

so that

$$V^{\sim}E_{sopt}W = V^{\sim}(R + Q_{sopt})W = \begin{pmatrix} \frac{(s+0.5112)(s+6.155)}{(s-0.5112)(s-6.155)} & 0 \\ 0 & \frac{0.33941(s+0.5112)}{s-0.5112} \end{pmatrix}$$

The resulting super-optimal singular values are plotted in figure 6.1.

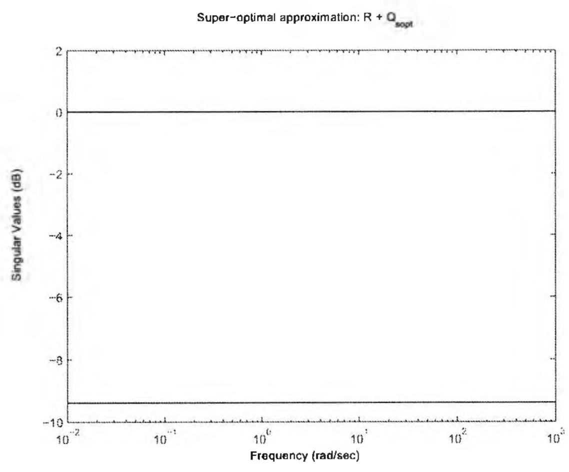

Figure 6.1: Super-optimisation in terms of the first two distinct super-optimal levels - Example 6.2.1.

Here,

$$\{s_1(R + Q_{sopt}), s_2(R + Q_{sopt})\} = \{1, 0.3394\}$$

The above example shows that the proposed method works in the pathological case of $r > l$. A misleading argument based on a fallacy would be that since $r = 2$, we minimise the first three $(r + 1)$ super-optimal levels. However, in this example we look at the non-generic case where $l(= 1) < r$, which implies that the $l + 1$ $(= 2)$ first super-optimal levels are effectively minimised. Next, we consider another example, this time with a simple largest Hankel singular value of $R(s)$.

Example 6.2.2 (Pseudo-Diagonal). *Suppose, we have the following anti-stable system in the pseudo-diagonal form:*

$$R(s) := \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} \frac{4}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} \frac{1/\sqrt{2}}{s-2} & \frac{4\sqrt{2}}{s-1} \\ -\frac{1/\sqrt{2}}{s-2} & \frac{4\sqrt{2}}{s-1} \end{pmatrix}$$

with realisation

$$R(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ \hline \sqrt{2} & 1/(\sqrt{2}) & 0 & 0 \\ \sqrt{2} & -1/(\sqrt{2}) & 0 & 0 \end{array}\right]$$

This realisation is balanced with gramians:

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}$$

Clearly, $s_1 = 2$ *and* $s_2 = 0.25(= 1/4)$. *Further,*

$$\bar{Q}_a(s) = \frac{1}{s + 2.063} \begin{pmatrix} 0.015873 & 2(s+2) \\ 2(s+2) & 1.0159 \end{pmatrix}$$

with $Q_{22}(s) = \frac{1.0159}{(s+2.063)}$. *Following the main steps of the procedure presented in this chapter, we obtain* $\hat{R}(s) = \frac{1}{s-2}$ *and* $\hat{Q} = 0.25$. *Then, in this case,*

$$Q_{sopt} = \begin{pmatrix} \frac{\sqrt{2}}{8} & \sqrt{2} \\ -\frac{\sqrt{2}}{8} & \sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} \frac{1}{4} & 2 \\ -\frac{1}{4} & 2 \end{pmatrix}$$

so that

$$E_{sopt} = R + Q_{sopt} = \begin{pmatrix} \frac{\sqrt{2}(s+2)}{8(s-2)} & \frac{\sqrt{2}(s+1)}{(s-1)} \\ -\frac{\sqrt{2}(s+2)}{8(s-2)} & \frac{\sqrt{2}(s+1)}{(s-1)} \end{pmatrix}$$

and

$$V^{\sim} E_{sopt} W = V^{\sim}(R + Q_{sopt})W = \begin{pmatrix} -2\frac{(s+1)}{(s-1)} & 0 \\ 0 & \frac{1}{4}\frac{(s+2)}{(s-2)} \end{pmatrix}$$

The super-optimal singular values are plotted in figure 6.2.

In this case,

$$\{s_1(R + Q_{sopt}), s_2(R + Q_{sopt})\} = \{2, 0.25\}$$

as expected.

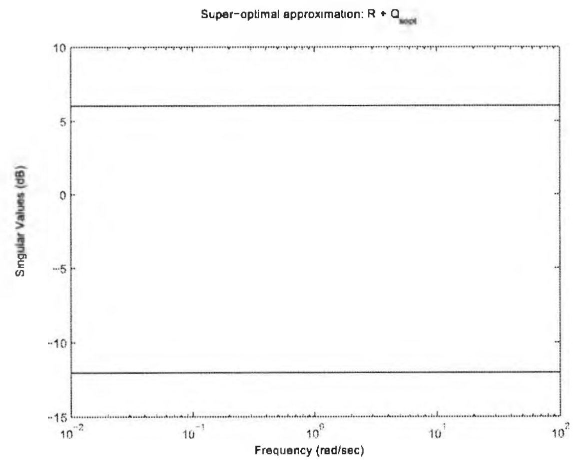

Figure 6.2: Super-optimisation in terms of the first two distinct super-optimal levels - Example 6.2.2.

Remark 6.2.1. *Since the left and right matrices multiplying* $\begin{pmatrix} \frac{4}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix}$ *are orthogonal, the super-optimal approximation can be obtained directly by solving two independent scalar Nehari extension problems, i.e.*

$$\min_{q_1 \in \mathcal{H}_\infty} \|\frac{4}{s-1} + q_1\|_\infty = 2$$

with $q_1^{opt} = 2$ *so that* $e_1 = \frac{4}{s-1} + 2 = 2\frac{s+1}{s-1}$, *and*

$$\min_{q_2 \in \mathcal{H}_\infty} \|\frac{1}{s-2} + q_2\|_\infty = \frac{1}{4}$$

with $q_2^{opt} = \frac{1}{4}$ *so that* $e_2 = \frac{1}{s-2} + \frac{1}{4} = \frac{1}{4}\frac{s+2}{s-2}$. *The super-optimal solution must then have the form*

$$Q_{sopt} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{8} & \sqrt{2} \\ -\frac{\sqrt{2}}{8} & \sqrt{2} \end{bmatrix}$$

which agrees with the solution obtained by the general algorithm presented above.

The following example is a diagonal system having same super-optimal levels with the previous, pseudo-diagonal system. It is interesting though to compare the optimal and super-optimal Nehari extensions.

Example 6.2.3 (Diagonal). *Here consider the diagonal system*

$$R(s) := \begin{pmatrix} \frac{4}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ \hline 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The realisation again, as in the above example, is balanced with corresponding gramians:

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0.25 \end{bmatrix}$$

Clearly, $s_1 = 2$ *and* $s_2 = 0.25 (= 1/4)$*. In this case,*

$$\bar{Q}_a = \frac{1}{s + 2.063} \begin{pmatrix} 0.015873 & 2(s+2) \\ 2(s+2) & 1.0159 \end{pmatrix}$$

and again $Q_{22}(s) = \frac{1.0159}{s+2.063}$, $\hat{R}(s) = \frac{1}{s-2}$ *and* $\hat{Q} = 0.25$*. However,*

$$Q_{sopt} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

so that

$$E_{sopt} = R + Q_{sopt} = \begin{pmatrix} \frac{2(s+1)}{(s-1)} & 0 \\ 0 & \frac{(s+2)}{4(s-2)} \end{pmatrix}$$

and

$$\{s_1(R + Q_{sopt}), s_2(R + Q_{sopt})\} = \{2, 0.25\}$$

which are equal to Hankel norms of the diagonal elements and also coincide with the super-optimal singular values of the pseudo-diagonal example.

Next, an anti-stable (strictly proper) system, randomly generated in MATLAB, is considered:

Example 6.2.4. *Suppose,*

$$R(s) \stackrel{s}{=} \left[\begin{array}{cc|cc} 9.2328 & 3.3876 & 0.1764 & -1.8379 \\ 7.4865 & 4.5641 & 0.0894 & -0.9135 \\ \hline 0.0102 & 0.5162 & 0 & 0 \\ 1.8339 & 0.7364 & 0 & 0 \\ 0.2137 & -0.1840 & 0 & 0 \end{array}\right]$$

with

$$\Sigma = \begin{bmatrix} 0.1846 & 0 \\ 0 & 0.0923 \end{bmatrix}$$

Here $s_1 = 0.1846$ *and* $s_2 = 2.1887 \times 10^{-4}$. *In addition,*

$$Q_{22}(s) = \begin{pmatrix} \frac{0.0011036}{s+2.88} & \frac{-0.0006093}{s+2.88} \end{pmatrix}$$

and

$$\hat{R}(s) = \begin{pmatrix} \frac{0.0011036}{s-2.88} \\ \frac{-0.0006093}{s-2.88} \end{pmatrix}$$

so that

$$\hat{R}_{aug} + \bar{Q}_a = \begin{pmatrix} \frac{0.0011036}{s-2.88} & 0 & 0 \\ \frac{-0.0006093}{s-2.88} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{0.18461}{s+2.88} & 0 \\ 0 & 0 & 0.18461 \\ 0.18461 & \frac{0.0011036}{s+2.88} & \frac{-0.0006093}{s+2.88} \end{pmatrix}$$

Then, $\hat{Q} = 0.1032$. *Further,*

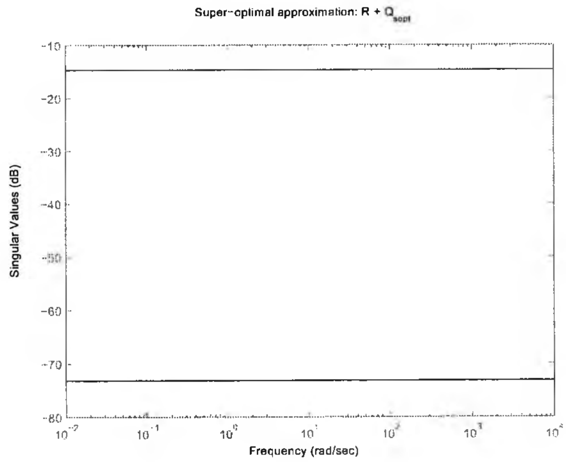

Figure 6.3: Super-optimisation in terms of the first two distinct super-optimal levels - Example 6.2.4.

$$Q_{sopt} = \frac{1}{s+2.88} \begin{pmatrix} 0.00029007(s-25.2) & -0.00099203(s-83.12) \\ 0.017534(s+3.878) & -0.18252(s+3.884) \\ 0.0019413(s+6.167) & -0.021276(s+6.021) \end{pmatrix}$$

So,

$$E_{sopt} = \begin{pmatrix} \frac{0.00029007(s+111)(s+13.85)(s+1.345)}{(s-12.45)(s+2.88)(s-1.348)} & \frac{-0.00099203(s+383.4)(s+12.44)(s+1.348)}{(s-12.45)(s+2.88)(s-1.348)} \\ \frac{0.017534(s+12.48)(s+1.36)(s-1.552)}{(s-12.45)(s+2.88)(s-1.348)} & \frac{-0.18252(s+12.45)(s+1.348)(s-1.558)}{(s-12.45)(s+2.88)(s-1.348)} \\ \frac{0.0019413(s-10.96)(s+12.93)(s+1.349)}{(s-12.45)(s+2.88)(s-1.348)} & \frac{-0.021276(s-11.01)(s+12.44)(s+1.348)}{(s-12.45)(s+2.88)(s-1.348)} \end{pmatrix}$$

and

$$V^{\sim} E_{sopt} W = \begin{pmatrix} \frac{0.00019161(s+2.891)(s+2.868)}{(s-2.88)(s+2.88)} \\ \frac{-0.00010579(s+2.895)(s+2.864)}{(s-2.88)(s+2.88)} \end{pmatrix}$$

6.3 Summary - Connection with robust control

Super-optimisation is essentially a problem of hierarchical optimisation, which involves a nested optimisation problem of the same form, but progressively reduced input-output (and state) dimension. Here we consider approximations of proper (real) rational matrix functions and thus the generic number of nested minimisation problems is equal to $\min(p, m)$, after which the available degrees of freedom are exhausted. Throughout the chapter we considered approximations of the first two distinct super-optimal levels which is a generic step for solving the full super-optimisation problem via a recursive procedure ([JL93]). Note that similar to Nehari approximations in chapter 4, minimisation of each super-optimal level requires the solution of one optimal and one suboptimal Nehari approximation problem, progressively of reduced dimensions. The method presented here does not make any use of Schmidt vectors and is totally based on state-space methods, involving the solution of Riccati inequalities, which make the proposed algorithm computationally robust compared to other existing methods. Of course, since closed-form state-space realisations have been derived, it is preferable to dispense with the intermediate steps completely and assemble directly the updated super-optimal solution at each step of the algorithm.

Super-optimal controllers. Super-optimisation has many potential applications in robust control, however, in this thesis we focus on robust stabilisation. This problem has already been discussed in chapter 5, and its optimal solution was derived in terms of the smallest Hankel singular value of the anti-stable part of the nominal plant. In particular, in the optimal version of the problem, we seek to minimise the infinity norm of an appropriate closed-loop transfer-matrix or equivalently to maximise the size of the uncertainty set, i.e. the radius of an open $\mathcal{H}_\infty$-ball in which all permissible perturbations lie. However, in the matrix case, the associated Nehari problem can have many optimal solutions and, in general, the optimal robust stability radius is tight only for the "weakest perturbation direction", i.e. the direction that stability is most easily lost, when an optimal (maximally robust) controller is chosen. As shown in the following chapter, in the case of additive perturbations this direction is described by the Schmidt vectors of a Hankel operator associated with the problem, corresponding to the smallest Hankel singular value of the nominal plant (see chapter 5).

In a connection to super-optimisation, by minimising the two largest closed-loop singular values in a lexicographic fashion, it is possible to extend the robust stability further in certain directions, and thus guarantee the stabilisation of a wider uncertainty set. This topic will be analysed in full in the next chapter.

Chapter 7

Robust Stabilisation Under Additive Perturbations

The ideas developed throughout this chapter are based on those in [GHJ00] but here, they are extended in various directions. The problem of extending the maximal robust stability is reduced to a frequency-by-frequency type of argument and thus, it is shown that the notions of rank-reducing and destabilising perturbations for constant matrices and dynamical systems, respectively, are intimately related. A key result for establishing the connection between the two notions is the distance to singularity of a complex matrix subject to structured constraints, which is derived in terms of the first two (distinct) singular values of the associated matrix. Note that the first two (distinct) singular values of a level-two super-optimal matrix function, at any given frequency $\omega_o \in \mathcal{R}$ are the first two distinct super-optimal levels, (s_1, s_{l+1}). Further, stability of the feedback system is lost, if at a given frequency the Nyquist criterion is violated. In the matrix case, by imposing directionality constraints along the direction defined by the largest singular value, a finer measure of "distance to singularity" can be defined which subsequently can be used in the dynamical system case to extend the robust stability radius for additive perturbations (using the available degrees of freedom which are present in the multivariable case). The optimum solution is shown to be associated with the solution of a super-optimal Nehari extension problem; thus, the results of the last two chapters form the basis for the further developments reported here.

7.1 Introduction

At this point, for reasons of clarity, we outline the formal notation used; this is based on results of the previous two chapters.

Assumption 7.1.1. Consider a nominal plant $G \in \mathcal{RH}_{\infty}^{-,p\times m}$, with balanced realisation $G \stackrel{s}{=} (A, B, C)$ of McMillan degree n and with a *smallest* (nonzero) Hankel singular value of multiplicity r. Further, $\Delta \in \mathcal{RL}_{\infty}$ denote the uncertainty around the nominal plant, which is assumed to be unstructured; i.e. there is no a priori information for Δ, except for a frequency bound on its norm, which we denote[1] as $\|\Delta\|_{\infty} < \epsilon$. Then, suppose that the uncertainty enters into the model of G additively and that all permissible perturbations satisfy the technical assumption made in chapter 5:

$$\eta(G) = \eta(G + \Delta) \tag{7.1}$$

where $\eta(\cdot)$ denotes the number of poles in the right half of the complex plane, counted in a McMillan degree sense. Note that in the next chapter, where we consider other types of perturbation, assumption 7.1 is removed. □

Under these assumptions recall that the transfer functions of all stable control-sensitivity functions described by figure 5.5 (problem 2.4.1, proposition 5.1.1) belong to the set

$$\mathcal{T} = \{K(I - GK)^{-1} : K \in \mathcal{K}\}$$

where $\mathcal{K}$ denotes the set of all internally stabilising controllers of G. Further, as shown earlier, there exists a subset of controllers, $\mathcal{K}_1(G)$, which minimises the $\mathcal{H}_{\infty}$-norm of $T \in \mathcal{T}$, i.e. for any $K \in \mathcal{K}_1$,

$$\mathcal{T}_1 = \{T \in \mathcal{T} : \|T\|_{\infty} = r_1^{-1}\} \subseteq \mathcal{T} \tag{7.2}$$

where r_1 is the maximum robust stability radius. Equation (7.2) describes the solution set of the maximally robust stabilisation problem (MRSP), under additive perturbations. Note that explicit state-space formulae were derived in chapter 5 for all optimal closed-loop approximations ($\mathcal{T}_1$) and the family of all maximally robust controllers ($\mathcal{K}_1$). The problem of characterising the set $\mathcal{T}_1$ in (7.2) involves the following

[1] it is important to notice that the size (i.e. norm) of Δ is bounded by a *strict* inequality.

Nehari approximation:

$$r_1^{-1} := \inf_{T \in \mathcal{T}} \|T\|_\infty = \inf_{Q \in \mathcal{H}_\infty} \|R + Q\|_\infty$$

where $R \in \mathcal{RH}_\infty^-$ is defined in (5.10) and (5.13). A key property of R is its balanced realisation, which can be obtained directly from a balanced realisation of G. In chapter 5 it was proved that each Hankel singular value of R is equal to the inverse of the corresponding Hankel singular value of plant G (multiplicities included). Hence,

$$r_1^{-1} = \inf_{Q \in \mathcal{H}_\infty} \|R + Q\|_\infty = \sigma_1(\Gamma_R) = (\sigma_n(\Gamma_G))^{-1}$$

Essentially, this is a level-1 super-optimisation problem (in terms of the definition of chapter 6) and hence the set of all optimal control-sensitivity functions admits the following diagonal decomposition:

$$\mathcal{T}_1 = Y(s) \begin{pmatrix} r_1^{-1} a(s) & 0 \\ 0 & (\hat{R} + \mathcal{Q})(s) \end{pmatrix} X(s) \tag{7.3}$$

where $\hat{R} \in \mathcal{RH}_\infty^{-,(p-l)\times(m-l)}$, X and Y are appropriate square inner matrices to be defined in the sequel, $a(s) \in \mathcal{H}_\infty^{-,l\times l}$ (where $l \leq r$ is defined in chapter 6) is anti-inner and

$$\mathcal{Q} = \left\{ Q \in \mathcal{H}_\infty : \|\hat{R} + Q\|_\infty \leq r_1^{-1} \right\}.$$

where $\mathcal{Q}$ is the set of all r_1^{-1}-suboptimal approximations of $\hat{R}$. Note that in terms of the notation in chapter 6, $s_1(R) := r_1^{-1}$ and $\mathcal{S}_1 := \mathcal{Q}$.

Along similar lines with the definitions of Chapter 6, we define the set of i-th super-optimal control-sensitivity transfer functions by $\mathcal{T}_i$ if any $T \in \mathcal{T}_i$ minimises lexicographically the sequence $\{s_1(T), s_2(T), \ldots, s_i(T)\}$. Clearly, $\mathcal{T}_i \subseteq \mathcal{T}_{i-1} \subseteq \cdots \subseteq \mathcal{T}_1 \subseteq \mathcal{T}$. In the same way, the corresponding to $\mathcal{T}_i$ set of controllers is denoted by $\mathcal{K}_i$. However, if multiplicity $l > 1$ occurs in, say, the first super-optimal level, then we say that $\mathcal{K}_2$ is the corresponding set of controllers corresponding to $\mathcal{T}_{l+1}$ (i.e. we do not count multiplicities on the indexing of controller sets).

Lemma 7.1.1 (Level-2 Superoptimisation). *The set $\mathcal{T}_{l+1}$ can be parameterised as*

$$\mathcal{T}_{l+1} = Y_1(s) \begin{pmatrix} s_1 a(s) & 0 & 0 \\ 0 & s_{l+1} b(s) & 0 \\ 0 & 0 & \check{R}(s) + \mathcal{S}_{l+1} \end{pmatrix} X_1(s)$$

where X_1, Y_1 are square inner, $a(s), b(s)$ are anti-inner functions and $\mathcal{S}_{l+1} = \left\{ S \in \mathcal{H}_\infty : \|\check{R}(s) + S\|_\infty \leq s_{l+1} \right\}$, for some $\check{R} \in \mathcal{H}_\infty^-$.

Proof. In the light of (5.9) and (5.10), the set of all optimal control sensitivity matrix functions $\mathcal{T}_1$ takes the following equivalent forms:

$$\begin{aligned}
\mathcal{T}_1 &= \left\{ \left[U + M\mathcal{F}_l(Q_a, s_1^{-1}\mathcal{BH}_\infty) \right] \widetilde{M} \right\} \quad \left(= \mathcal{F}_l(T_{gen}, s_1^{-1}\mathcal{BH}_\infty)\widetilde{M} \right) \\
&= \left\{ U\widetilde{M} + M\mathcal{F}_l(Q_a, s_1^{-1}\mathcal{BH}_\infty)\widetilde{M} \right\} \\
&= M \left\{ M^{\sim}U + \mathcal{F}_l(Q_a, s_1^{-1}\mathcal{BH}_\infty) \right\} \widetilde{M} \\
&= M \left\{ R(s) + \mathcal{Q}_{opt}(s) \right\} \widetilde{M}
\end{aligned}$$

and hence $\mathcal{Q}_{opt}$ is chosen to be the set of all optimal Nehari extensions, in the sense that for any $Q_{opt} \in \mathcal{Q}_{opt}$ we have $\|R(s) + Q_{opt}(s)\|_\infty = s_1$. Now, using the notation of chapter 6 (see Theorem 6.1.1) $\mathcal{T}_1$ can be written as

$$\begin{aligned}
\mathcal{T}_1 &= M\left\{ R + \mathcal{Q}_{opt} \right\} \widetilde{M} \\
&= MV \begin{pmatrix} s_1\alpha(s) & 0 \\ 0 & \hat{R}(s) + \mathcal{F}_l(\bar{Q}_a, s_1^{-1}\mathcal{BH}_\infty) \end{pmatrix} W^{\sim}\widetilde{M} \\
&= MV \begin{pmatrix} s_1\alpha(s) & 0 \\ 0 & \hat{R}(s) + \mathcal{S}_1 \end{pmatrix} W^{\sim}\widetilde{M} =: Y(s) \begin{pmatrix} s_1\alpha(s) & 0 \\ 0 & \hat{R}(s) + \mathcal{S}_1 \end{pmatrix} X(s)
\end{aligned}$$

for some $\hat{R}(s)$ such that $\|\hat{R}(s) + \mathcal{S}_1\|_\infty \leq s_1$. Now, $\mathcal{T}_{l+1} \subseteq \mathcal{T}_1$, and it is formed whenever we restrict ourselves to a subset of $\mathcal{S}_1$, i.e. whenever the set of all s_{l+1}-optimal Nehari extensions $\{\hat{Q} : \|\hat{R}(s) + \hat{Q}\|_\infty = s_{l+1}\}$ is considered in the place of $\mathcal{S}_1$ (recall $s_{l+1} < s_1$). Then for all super-optimal extensions $\mathcal{Q}_{sopt}$ (with respect to the first two levels),

$$\begin{aligned}
\mathcal{T}_{l+1} &= M\left\{ R + \mathcal{Q}_{sopt} \right\} \widetilde{M} \\
&= MV \begin{pmatrix} s_1 a(s) & \mathbf{0} \\ \mathbf{0} & V_1 \begin{pmatrix} s_{l+1}b(s) & 0 \\ 0 & \check{R}(s) + \mathcal{S}_{l+1} \end{pmatrix} W_1^{\sim} \end{pmatrix} W^{\sim}\widetilde{M} \\
&= MV \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & V_1 \end{array} \right) \left(\begin{array}{c|cc} s_1 a(s) & 0 & 0 \\ \hline 0 & s_{l+1}b(s) & 0 \\ 0 & 0 & \check{R}(s) + \mathcal{S}_{l+1} \end{array} \right) \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & W_1^{\sim} \end{array} \right) W^{\sim}\widetilde{M} \\
&=: Y_1(s) \begin{pmatrix} s_1 a(s) & 0 & 0 \\ 0 & s_{l+1}b(s) & 0 \\ 0 & 0 & \check{R}(s) + \mathcal{S}_{l+1} \end{pmatrix} X_1(s)
\end{aligned}$$

for some $\check{R}(s)$ such that $\|\check{R}(s) + \mathcal{S}_{l+1}\|_\infty \leq s_{l+1}$. This result is based on a recursive application of the methodology described in chapter 6 for the first two (distinct) super-optimal levels. □

Remark 7.1.1. *An important observation that can be made between the form (7.3) and that of lemma 7.1.1 is that the transfer matrices $X(s)$ and $X_1(s)$ (and respectively $Y(s)$ and $Y_1(s)$) share the same first l columns (respectively, rows). This follows immediately from the proof of Lemma 7.1.1 above ($\mathcal{T}_{l+1} \subseteq \mathcal{T}_1$). In addition these columns are essentially the maximising vectors of Γ_R (see corollary 6.1.1), scaled by the inner matrix functions $\widetilde{M}$ and M, respectively.*

Problem Definition. The set of all permissible perturbations corresponding to optimal controllers is given by

$$\mathcal{D}_{r_1}(G) = \{\Delta \in \mathcal{RL}_\infty : \|\Delta\|_\infty < r_1, \quad \eta(G) = \eta(G + \Delta)\}. \tag{7.4}$$

Further, we define the boundary of this set,

$$\partial\mathcal{D}_{r_1}(G) = \{\Delta \in \mathcal{RL}_\infty : \|\Delta\|_\infty = r_1, \quad \eta(G) = \eta(G + \Delta)\}. \tag{7.5}$$

Here r_1 is the robust stability radius, i.e. $r_1 := \bar{\sigma}(\Gamma_{M^\sim U}) = \underline{\sigma}(\Gamma_G)$ (recall $R(s) := M^\sim U(s)$ - see chapter 5). This corresponds to the maximisation of the size of the non-destabilising uncertainty set (measured as a norm), i.e. the radius of an open $\mathcal{H}_\infty$-ball in which all permissible non-destabilising perturbations lie. However, in the multivariable case, this maximisation of robustness is tight only for the "weakest perturbation direction", i.e. the direction that stability is most easily lost.

In a connection to super-optimisation, it is possible to impose a tighter optimisation criterion with the objective of minimising the singular values of the control-sensitivity function in a lexicographic fashion. Hence, we pose the following problem:

Problem 7.1.1. *Given a nominal plant and an uncertainty set as defined in assumption 7.1.1, how much (if possible) can the maximal robust stability be extended along different directions by making a selection among the continuum of all optimal (maximally robust) controllers? In what sense, if any, can the super-optimal controller considered to be the "best"? What is the description of the extended uncertainty set, guaranteed to be stabilised by this controller?* □

A motivating example showing that there exist permissible non-destabilising perturbations, lying outside the r_1-ball, for certain choices of optimal controllers follows. The example is special a case of that in [Nym95], for the case where uncertainty enters to the model in terms of an additive perturbation.

Example 7.1.1 (Extended set of permissible perturbations). *Assume* $G \in \mathcal{RH}_\infty^{-,p\times m}$ *with super-optimal decomposition of the corresponding closed-loop transfer function,* $T \in \mathcal{T}_k$*:*

$$UTV = \begin{pmatrix} s_1 a_1(s) & 0 & 0 & 0 \\ 0 & s_{l+1} a_2(s) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & s_k a_k(s) \end{pmatrix}$$

where $k = \min(p, m, n)$ *and* $a_i(s)$ *are all-pass functions. Set* $\epsilon = s_1^{-1}$ *such that the plant is* ϵ*-robustly stabilisable by* $K \in \mathcal{K}_k$*, due to Theorem 5.1.1. Then let*

$$\Lambda = \begin{bmatrix} I_l & 0 & 0 & 0 \\ 0 & \frac{s_1}{s_{l+1}} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{s_1}{s_k} I_{n-(k-1)} \end{bmatrix}$$

Further, define the following class of perturbations:

$$\mathcal{D}_\epsilon^e \triangleq \mathcal{D}_{\mathcal{S}_\epsilon} \cup \mathcal{D}_{V\mathcal{S}_\epsilon} \cup \mathcal{D}_{V\mathcal{U}_\epsilon}$$

with

$$\begin{aligned} \mathcal{D}_{\mathcal{S}_\epsilon} &\triangleq \{\Delta \ : \Delta \in \mathcal{RH}_\infty; \quad \|\Delta\|_\infty < \epsilon\} \\ \mathcal{D}_{V\mathcal{S}_\epsilon} &\triangleq \{\Delta \ : \Delta \in V\mathcal{RH}_\infty; \quad \|\Lambda^{-1}V^{-1}\Delta\|_\infty < \epsilon\} \\ \mathcal{D}_{V\mathcal{U}_\epsilon} &\triangleq \{\Delta \ : \Delta \in \mathcal{RL}_\infty; \quad \eta(G) = \eta(G+\Delta), \quad \|\Lambda^{-1}V^{-1}\Delta\|_\infty < \epsilon\} \end{aligned}$$

where $\eta(.)$ *denotes the number of closed RHP poles of a transfer function, counted in the McMillan degree sense. Note that the set* $\mathcal{D}_{\mathcal{S}_\epsilon}$ *remains as in definition 5.1.1, chapter 5. On the other hand, observe that* Λ^{-1} *is contractive so that*

$$\|\Lambda^{-1}V^{-1}\Delta\|_\infty \leq \|\Delta\|_\infty < \epsilon$$

Then, we say that the set $\mathcal{D}_{V\mathcal{U}_\epsilon}$ *is weaker than* $\mathcal{D}_{\mathcal{U}_\epsilon}$ *(see definition 5.1.1), which implies that*

$$\mathcal{D}_\epsilon \subseteq \mathcal{D}_\epsilon^e$$

and the super-optimal controller guarantees stabilisation of all $\Delta \in \mathcal{D}_\epsilon^e$*.* □

In general, Nyman gives a rather implicit description of the extended set of permissible perturbations. In contrast, this work aims to characterise the maximum possible extended set, in terms of directionality properties and define the maximum robust stability radius along different directions in a form which is more useful for controller design. At first, in the following paragraph, it is shown that there exist perturbations which are destabilising for *every* optimal closed-loop transfer function, $\mathcal{T}$, and have norm equal to $\epsilon := (s_1(\mathcal{T}))^{-1}$, i.e. destabilising perturbations which lie on the boundary of the optimal (maximum-radius) open ball of perturbations which are guaranteed to be stabilised by every optimal-controller. Such (uniformly) *destabilising* perturbations can be chosen to be real-rational. Thereafter, we prove that perturbations of this type share a common characteristic - they all have the same worst case "projection" in a specific direction, specified by the *maximising vectors* (scaled Schmidt pair) of the Hankel operator related to the problem.

7.2 Uniformly Destabilising Perturbations

In the introductory part of this chapter the existence of permissible perturbations outside ball of radius r_1 was established (example 7.1.1). On the other hand, we have not yet identified minimum norm perturbations which destabilise an optimal closed-loop transfer function. Given *any specific* optimal controller $K \in \mathcal{K}_1$, it is clear from the optimal solution of the MRSP that at least *one* $\Delta_1 \in \partial\mathcal{D}_{r_1}(G)$ can be chosen such that $(G + \Delta_1, K_1) \notin \mathcal{S}$ (see algorithm 5.1.1). The next lemma establishes a stronger result, i.e. the existence of $\Delta_1 \in \partial\mathcal{D}_{r_1}(G)$ for which $(G+\Delta_1, K_1) \notin \mathcal{S}$ for *every* $K \in \mathcal{K}_1$. This generalises a corresponding lemma of [GHJ00]:

Lemma 7.2.1 (Existence). *[GHJ00] There exist $\Delta \in \partial\mathcal{D}_{r_1}(G)$ such that $(G+\Delta, K) \notin \mathcal{S}$ for every $K \in \mathcal{K}_1$. Furthermore, Δ can be chosen to be stable real-rational matrix functions.*

Proof. The proof is adapted from [GHJ00]. It is constructive and it is based on the fact that every optimal closed-loop system (optimal control sensitivity transfer function) can be written in the form (7.3). Hence define, at any frequency $\omega_o \in \mathcal{R}$, the complex matrix:

$$\Delta_o = X^{\sim}(j\omega_o)\begin{bmatrix} r_1 a(j\omega_o)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^{\sim}(j\omega_o) \in \mathcal{C}^{m\times p}$$

Now, following the technique of Algorithm 5.1.1 we can use an "interpolation" argument to construct $\Delta(s) \in \mathcal{RH}_\infty$ of norm $\|\Delta(s)\| = r_1^{-1}$ such that $\Delta(j\omega_o) = \Delta_o$ (details are omitted). Further, for <u>every</u> $K \in \mathcal{K}_1 (\Leftrightarrow T \in \mathcal{T}_1)$ we can write

$$det(I - \Delta(j\omega_o)T(j\omega_o)) = det(I - \Delta_o T(j\omega_o)) = 0$$

and hence

$$det(I - (G + \Delta_o)K)(j\omega_o) = 0 \Rightarrow H(G + \Delta_o, K) \notin \mathcal{S}$$

□

All perturbations (not necessarily real-rational) as constructed in lemma 7.2.1 will be called *uniformly destabilising*. Formally, we give the following definition:

Definition 7.2.1 (Uniformly destabilizing perturbations). *Any* $\Delta \in \partial\mathcal{D}_{r_1}(G)$ *which destabilizes* (G, K) *for every* $K \in \mathcal{K}_1$ *is called a uniformly destabilizing perturbation.* □

Remark 7.2.1. *It is crucial to note that any perturbation within this class is destabilising for every optimal control-sensitivity transfer function* $T \in \mathcal{T}_1$. *Moreover, all frequencies are "equally critical", in the sense that the generalised Nyquist theorem can be made to fail at an arbitrary frequency* ω_o, *i.e.* $det(I - \Delta T(j\omega_o)) = 0$ *for <u>every</u>* $T \in \mathcal{T}_1$. □

Now define $x^T(s)$ and $y(s)$, to be the first l-rows and l-columns of $X(s)$ and $Y(s)$, respectively, as defined in equation (7.3). Then, in the following lemma we prove that every (boundary) uniformly destabilising perturbation must have a projection of magnitude r_1 along a certain "worst" direction, defined by $y(j\omega)$ and $x^T(j\omega)$, $\omega \in \mathcal{R}$.

Lemma 7.2.2 (Directionality). *[GHJ00] Let* $\Delta \in \partial\mathcal{D}_{r_1}(G)$ *be a (uniformly) destabilising perturbation of* G *for every* $K \in \mathcal{K}_1$. *Then, there exists an* $\omega_o \in \mathcal{R}$, *such that*

$$\|x^T(j\omega_o)\Delta(j\omega_o)y(j\omega_o)\| = r_1 \tag{7.6}$$

Proof. Suppose $\Delta \in \partial\mathcal{D}_{r_1}(G)$ is a uniformly destabilising perturbation, i.e. that $(G_0 + \Delta, K) \notin \mathcal{S}$ for every $K \in \mathcal{K}_1$. Let $\beta \in [0, 1]$ and consider the family of Nyquist plots Γ_β obtained by mapping the standard contour D_R via $det[I - (G + \beta\Delta)K]$ for a fixed $K \in \mathcal{K}_1$. Since $(G_0 + \beta\Delta, K) \in \mathcal{S}$ for every $\beta \in [0, 1)$ the contours Γ_β do not cross the origin and encircle it $\eta(G) + \eta(K)$ times in the counter-clockwise direction. Since $(G_0 + \Delta, K) \notin \mathcal{S}$, by continuity of deformation Γ_β we must have that $0 \in \Gamma_1$, i.e. that there exists $\omega_o \in \mathcal{R}$ such that

$$det[I - (G + \Delta)K](j\omega_o) = 0$$

or equivalently,

$$det[I - GK - \Delta K](j\omega_o) = 0 \Leftrightarrow det[I - \Delta K(I - GK)^{-1}](j\omega_o) = 0$$

since $det[I - GK](j\omega_o) \neq 0$. Thus,

$$det[I - \Delta T](j\omega_o) = 0$$

where $T = K(I - GK)^{-1}$.

Furthermore, all $T \in \mathcal{T}_1$ admit parametrisation as in equation (7.3) where $\|\hat{R}+Q\|_\infty \leq r_1^{-1}$, and $Q \in \mathcal{S}_1$. Define $\Phi(s) := (\hat{R} + Q)(s)$. Then it is always possible to choose a $T \in \mathcal{T}_1$ which admits the parametrisation of equation (7.3) and satisfies[2] $\|\Phi\|_\infty < r_1^{-1}$. This is fixed by the choice of $Q \in \mathcal{S}_1$. Recall now (see chapter 5) that this Q parameterise the controller (Youla parametrisation)

$$K = (U + MQ)(V + NQ)^{-1}$$

For the choice of $Q \in \mathcal{S}_1$ we made above, it is clear that $K \in \mathcal{K}_1$. So if Δ destabilises G for every $K \in \mathcal{K}_1$, it is also destabilising for the controller above which depends on the value of Q, determined by the approximation $\|\Phi\|_\infty < r_1^{-1}$. Since the Nyquist deformation argument presented earlier applies to every $K \in \mathcal{K}_1$, it also applies to the specific controller in $\mathcal{K}_1$ chosen above, i.e. the one corresponding to $\|\Phi\| < s_1^{-1}$. Thus, there exists a frequency $\omega_o \in \mathcal{R}$ such that

$$\det\left\{ I_m - X(j\omega_o)\Delta(j\omega_o)Y(j\omega_o)\begin{bmatrix} r_1^{-1}a(j\omega_o) & 0 \\ 0 & \Phi(j\omega_o)\end{bmatrix}\right\} = 0 \qquad (7.7)$$

due to equation (7.3). Next, define

$$\tilde{\Delta} = \begin{bmatrix} \tilde{\Delta}_{11} & \tilde{\Delta}_{12} \\ \tilde{\Delta}_{21} & \tilde{\Delta}_{22}\end{bmatrix} := \begin{bmatrix} x_1^T(j\omega_o) \\ X_\perp^T(j\omega_o)\end{bmatrix} \Delta(j\omega_o)\begin{bmatrix} y_1(j\omega_o)a(j\omega_o) & Y_\perp(j\omega_o)\end{bmatrix}$$

[2] In general for such Φ, $\|\Phi\|_\infty \leq r_1^{-1}$.

Note that $\begin{bmatrix} x_1^T(j\omega_o) \\ X_\perp^T(j\omega_o) \end{bmatrix}$ and $\begin{bmatrix} y_1(j\omega_o)a(j\omega_o) & Y_\perp(j\omega_o) \end{bmatrix}$ are partitions of the evaluation of $X(s)$ and $Y(s)$ (the later involves an extra term $a(j\omega_o)$) at a given frequency ω_o, i.e. they are complex matrices. Then, (7.7) is written

$$\det\left\{\begin{bmatrix} I_l - r_1^{-1}\tilde{\Delta}_{11} & -\tilde{\Delta}_{12}\Phi(j\omega_o) \\ -r_1^{-1}\tilde{\Delta}_{21} & I_{m-l} - \tilde{\Delta}_{22}\Phi(j\omega_o) \end{bmatrix}\right\} = 0$$

Assume now for contradiction that $I_l - r_1^{-1}\tilde{\Delta}_{11}$ is non-singular. Then, a Schur-type argument gives,

$$\det\left\{I_l - r_1^{-1}\tilde{\Delta}_{11}\right\}\det\left\{\left(I_{m-l} - \tilde{\Delta}_{22}\Phi(j\omega_o)\right) - r_1^{-1}\tilde{\Delta}_{21}\left(I_l - r_1^{-1}\tilde{\Delta}_{11}\right)^{-1}\tilde{\Delta}_{12}\Phi(j\omega_o)\right\} = 0$$

Further,

$$\det\left\{I_{m-l} - \left(\tilde{\Delta}_{22} + r_1^{-1}\tilde{\Delta}_{21}\left(I_l - r_1^{-1}\tilde{\Delta}_{11}\right)^{-1}\tilde{\Delta}_{12}\right)\Phi(j\omega_o)\right\} = 0$$
$$\Rightarrow \det\left\{I_{m-l} - \mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\Phi(j\omega_o)\right\} = 0$$

However, $\|\Phi\|_\infty < r_1^{-1}$, $\|x^T\|_\infty = \|y\|_\infty = \|a\|_\infty = 1$ and $\|\Delta\|_\infty = r_1$, by initial assumption. Then, $\|\tilde{\Delta}\| \leq r_1$ and so from basic LFT properties (see [GL95], Theorem 4.3.1) we conclude that $\|\mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\| \leq r_1$. Hence,

$$\left\|\mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\Phi(j\omega_o)\right\| \leq \left\|\mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\right\|\|\Phi(j\omega_o)\| < 1$$

i.e.

$$1 - \bar{\sigma}\left[\mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\Phi(j\omega_o)\right] > 0 \Rightarrow \underline{\sigma}\left[I_{m-l} - \mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\Phi(j\omega_o)\right] > 0$$
$$\Rightarrow det\left\{I_{m-l} - \mathcal{F}_u\left(\tilde{\Delta}, r_1^{-1}I_l\right)\Phi(j\omega_o)\right\} \neq 0$$

which contradicts our earlier assumption. Thus,

$$\det\left\{I_l - r_1^{-1}\tilde{\Delta}_{11}\right\} = 0 \Rightarrow \exists\lambda_i(I_l - r_1^{-1}\tilde{\Delta}_{11}) = 0$$
$$\Rightarrow \exists\lambda_i(\tilde{\Delta}_{11}) = r_1 \Rightarrow \bar{\lambda}(\tilde{\Delta}_{11}) \geq r_1 \Rightarrow \rho(\tilde{\Delta}_{11}) \geq r_1$$

but recall that $\|x^T(j\omega_o)\| = \|y(j\omega_o)\| = \|a(j\omega_o)\| = 1$ and $\|\Delta(j\omega_o)\| = r_1$. Then from above arguments

$$r_1 = \|x_1^T(j\omega_o)\|\|\Delta(j\omega_o)\|\|y_1(j\omega_o)\|\|a(j\omega_o)\| \geq \|(x_1^T\Delta y_1 a)(j\omega_o)\|$$
$$\geq \rho\left(x_1^T(j\omega_o)\Delta(j\omega_o)y_1(j\omega_o)a(j\omega_o)\right) \geq r_1$$

So,

$$\rho(x_1^T(j\omega_o)\Delta(j\omega_o)y_1(j\omega_o)a(j\omega_o)) = r_1 \tag{7.8}$$

Further,

$$\|x_1^T(j\omega_o)\Delta(j\omega_o)y_1(j\omega_o)a(j\omega_o)\| = r_1$$

or, equivalently

$$\|x^T(j\omega_o)\Delta(j\omega_o)y(j\omega_o)\| = r_1 \tag{7.9}$$

since $a(j\omega_o)$ is unitary. □

Remark 7.2.2. *In [GHJ00] condition (7.6) is interpreted as a "directionality" property. Consider the vector space of complex valued matrices $A \in \mathcal{C}^{m\times n}$ over $\mathcal{C}$. It is easy to show that in this space we can define an inner product (see chapter 1):*

$$\langle A, B\rangle = trace(B^*A)$$

which induces the Frobenious norm:

$$\langle A, A\rangle = \|A\|_F^2 = trace(A^*A) = \sum_{j=1}^{n}\sum_{i=1}^{m}|a_{ij}|^2$$

The Cauchy-Schwartz inequality then gives:

$$|\langle A, B\rangle| \leq \|A\|_F\|B\|_F$$

and we can think of $|\langle A, B\rangle|$ as the magnitude of the projection of A along the "direction" B. In particular, when B has rank one, $B = uv^*$ for two vectors u and v, we can write $|\langle A, uv^*\rangle| = |trace(vu^*A)| = |u^*Av|$. Thus, imposing the constraint $|u^*Av| \leq \phi$ can be interpreted as limiting the magnitude of the projection of A along the direction defined by $B = uv^*$. □

In Lemma 7.2.2 it is shown that the condition

$$\|x^T(j\omega)\Delta(j\omega)y(j\omega)\| = r_1$$

for some $\omega \in \mathcal{R}$, is necessary for a $\Delta \in \partial\mathcal{D}_{r_1}(G)$ to be destabilising for every $K \in \mathcal{K}_1$. As a consequence of this result, a sufficient condition that a boundary permissible perturbation is not uniformly destabilising is that

$$\|x^T(j\omega)\Delta(j\omega)y(j\omega)\| < r_1 \tag{7.10}$$

for some $\omega \in \mathcal{R}$, or equivalently

$$\|x^T \Delta y\|_\infty < r_1$$

Moreover a tighter condition that a boundary perturbation should satisfy in order not to be uniformly destabilising, is together with the restriction (7.10) to impose a further restriction on the spectral radius of $x^T \Delta y$, i.e.

$$\rho(x^T \Delta y a) < r_1 \tag{7.11}$$

Note that here the all-pass matrix function $a(s)$ is included. Constraints (7.10) and (7.11)) effectively impose structure on the set of additive perturbations and may be used to investigate the possible increase of the robust stability radius along different directions. Clearly, note that if we restrict $\rho(x^T \Delta y a)(= \|x^T \Delta y a\|_\infty) = \|x^T \Delta y\|_\infty \leq \phi$, for any $\phi < r_1$, then the second condition (7.11) becomes redundant, in the sense that $\|A\| = \phi \Rightarrow \rho(A) \leq \phi$. The choice of constraint gives rise to two different problems with the later (i.e. considering both restrictions (7.10) and (7.11) simultaneously) having a more technically complicated solution. Initially, we ignore condition (7.11) and develop tractable stability conditions for the first case, i.e. characterise the largest possible permissible uncertainty set that satisfies (7.10) and the corresponding robust stability radius.

Example 7.2.1. *Take a perturbation $\Delta := X(s)^\sim \Delta_1(s) Y(s)^\sim$, where*

$$\Delta_1(s) := \begin{pmatrix} \frac{1}{s_1} a(s)^\sim & 0 \\ 0 & 0 \end{pmatrix}$$

and with $X(s), Y(s)$ and $a(s)$ as defined in equation (7.3) and assume that $\eta(G+\Delta) = \eta(G)$. We observe that $\|\Delta_1\|_\infty = \frac{1}{s_1} =: r_1$ and hence, Δ lies on the boundary of the r_1-ball. Further, Δ is destabilising ($\det(I - \Delta T) = 0$). The "directionality" property of this perturbation is obtained as:

$$\begin{aligned} \|x^T \Delta y\|_\infty &= \|x^T X^\sim \Delta_1 Y^\sim y\|_\infty \\ &= \left\| \begin{bmatrix} I_l & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{s_1} a(s)^\sim & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} I_l \\ 0 \end{bmatrix} \right\|_\infty = \frac{1}{s_1} = r_1 \end{aligned}$$

Now define the perturbation $\hat{\Delta} := X^\sim \Delta_2 Y^\sim$ where

$$\Delta_2 := \begin{pmatrix} \phi\, a(s)^\sim I_l & 0 \\ 0 & r_1 I_{n-l} \end{pmatrix}$$

Take $\phi < r_1$. Again, we note that $\|\hat{\Delta}\|_\infty = r_1$ and so $\hat{\Delta}$ is on the boundary of r_1-ball. However, it can be shown that $\hat{\Delta}$ now is not a destabilising perturbation. Check that

$$\begin{aligned} \|x^T \Delta y\|_\infty &= \|x^T X \Delta_2 Y^\sim y\|_\infty \\ &= \left\| \begin{bmatrix} I_l & 0 \end{bmatrix} \begin{pmatrix} \phi\, a(s)^\sim I_l & 0 \\ 0 & r_1\, I_{n-l} \end{pmatrix} \begin{bmatrix} I_l \\ 0 \end{bmatrix} \right\|_\infty = \phi < r_1. \end{aligned}$$

which shows that the necessary (directionality) condition fails. □

Up to this point, it has been shown that there exist both permissible perturbations outside the r_1-ball and uniformly destabilising perturbations on the boundary of the ball, the later possessing an identical projection along a particular worst direction, defined at an arbitrary frequency. Hence, it may be deduced that r_1-ball is not necessarily the largest possible set of permissible perturbations, if the degrees of freedom of the optimal controller set are taken into account. In order to characterise the "extended" set, it is natural to impose a restriction to its structure by considering perturbations which have a projection of magnitude less than r_1 along the worst direction, defined by $\{x^T, y\}$ uniformly in frequencies $\omega \in \mathcal{R}$. A natural way to describe this condition is in terms of an arbitrary but fixed parameter $\delta \in (0, 1]$, such that

$$\|x^T \Delta y\|_\infty \leq r_1(1 - \delta) \tag{7.12}$$

is always satisfied for the new perturbation set. Thus, the extended set of permissible perturbations is defined as:

$$\mathcal{E}(\delta, \mu) = \left\{ \Delta \in \mathcal{D}_\mu(G) \ : \ \|x^T \Delta y\|_\infty \leq r_1(1 - \delta) \right\} \tag{7.13}$$

where

$$\mathcal{D}_\mu(G) = \{\Delta \in \mathcal{RL}_\infty : \|\Delta\|_\infty < \mu, \quad \eta(G) = \eta(G + \Delta)\}$$

according to equation (7.4). Note that the "direction" constraint is essentially a form of structure that the extended set is equipped with. Then the following stronger version of the MRSP is posed:

Problem 7.2.1 (structured robust stability radius). *Given $\delta \in [0, 1]$, find $\mu^\star(\delta) = \sup \mu$ so that every*

$$\Delta \in \mathcal{E}(\delta, \mu) \cup \mathcal{D}_{r_1}(G)$$

is guaranteed to be stabilised for some $K \in \mathcal{K}_1$ and characterise the set of optimal controllers K.

Intuitively, we expect $\mu^\star(\delta) \geq r_1$, for every $\delta \in (0, 1]$ and further that $\mu^\star(\delta_1) \geq \mu^\star(\delta_2)$ for $\delta_1 \geq \delta_2$, i.e. if we impose a tighter (structured) constraint $\|x^T \Delta y\| \leq r_1(1-\delta)$ the constrained robust stability radius $\mu^\star(\delta)$ should increase. In the sequel, we consider two problems which arise from the above definitions: The first is intended to determine and characterise $\mu^\star(\delta)$. This problem involves matrix distance to singularity, a notion which will be discussed in the following section. Secondly, it will be shown that $\mu^\star(\delta)$ is an increasing function of the gap between the two largest singular values of an associated Hankel operator. Since the first super-optimal level is fixed for all $K \in \mathcal{K}_1$, the problem reduces to the minimisation of the second largest singular value (uniformly in frequency), i.e. super-optimisation. For both problems a closed-form solution will be provided, as shown in the following sections.

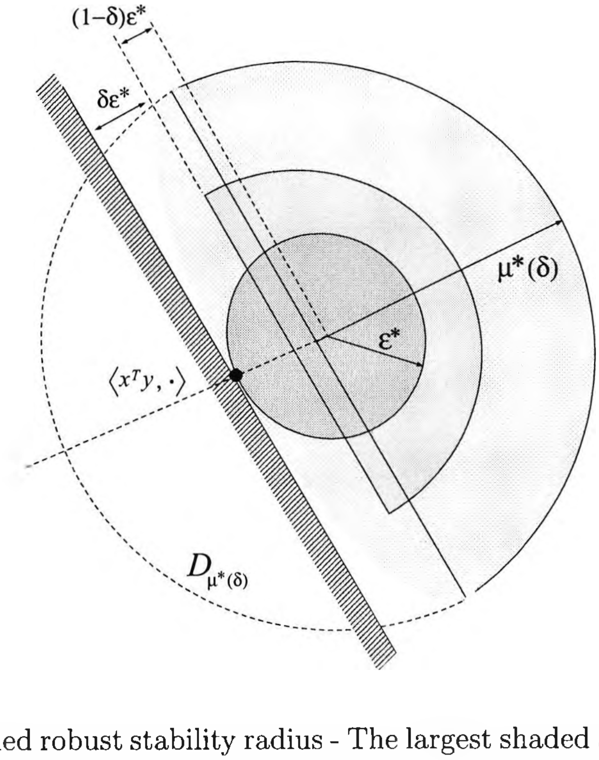

Figure 7.1: Extended robust stability radius - The largest shaded area is the set $\mathcal{E}(\delta, \mu^\star)$.

Parametric constraint. It can be argued that depending on the value of δ, the strength of the imposed structural constraint varies. Of course we have defined implicitly that $\mu^\star \geq r_1$, since all $\Delta \in \mathcal{D}_{r_1}$ need to be stabilised and hence for any $\delta \in [0, 1]$, the set of controllers that maximise $\mu^\star(\delta)$ is a subset of the set of all optimal controllers, $\mathcal{K}_1$. Formally, we shall denote this subset by $\mathcal{K}_1^\delta$, and so

$$\mathcal{K}_1^\delta \subseteq \mathcal{K}_1$$

Note that if $\delta = 0$ the constraint $\|x^T \Delta y\| \leq r_1$ defining the set $\mathcal{E}(\delta, \mu)$ is redundant since $\|\Delta\|_\infty \leq r_1$ and $\|x\|_\infty = \|y\|_\infty = 1$ and therefore $\mathcal{E}(0, \mu) = \mathcal{D}_\mu$. Thus, in this case, the maximisation problem takes the form: $\sup \mu$, such that all $\Delta \in \mathcal{D}_{r_1}(G)$ are stabilised by some $K \in \mathcal{K}_1$, so that the optimal solution is $\mu^\star(0) = r_1$ and $\mathcal{K}_1^0 = \mathcal{K}_1$. Further, since

$$\{\Delta : \|x^T \Delta y\|_\infty \leq r_1(1-\delta_1)\} \supseteq \{\Delta : \|x^T \Delta y\|_\infty \leq r_1(1-\delta_2)\}$$

whenever $0 \leq \delta_1 \leq \delta_2 \leq 1$, we have that $\mathcal{K}_1^{\delta_1} \subseteq \mathcal{K}_1^{\delta_2}$. In the sequel it is shown that the sets $\mathcal{K}_1^\delta$, $\delta \in (0, 1]$ are identical and coincide with the set $\mathcal{K}_2$, the set of all super-optimal controllers with respect to the first two levels.

As the constraint in (7.10) suggests, the robust stability of perturbed plants inside the extended permissible uncertainty set ("structured robust stability radius" ball of problem 7.2.1) can be examined on a frequency by frequency basis to ensure that the generalised Nyquist criterion (Theorem 2.3.2) is not violated. When looking at the closed-loop transfer function at each frequency independently, the robust stabilisation problem becomes a constant distance to singularity problem. In the sequel we pose problem 7.2.1 in frequency-by-frequency framework.

Problem 7.2.2 (Constrained Maximum Robust Stability). *Find the set of controllers which supremise $\mu(\delta)$ under the constraint that $G+\Delta$ is robustly stabilisable for all $\Delta \in \mathcal{D}_{r_1}(G) \cup \mathcal{E}(\delta, \mu)$. Equivalently, determine*

$$\begin{aligned} d(\phi) = \sup \{ d : det(I_m - \Delta(j\omega) T(j\omega)) \neq 0, \\ \text{for all } \Delta \in \{\|\Delta(j\omega)\| < d\} \cap \{\|x^T(j\omega)\Delta(j\omega) y(j\omega)\| \leq \phi\}\} \end{aligned}$$

over all frequencies $\omega \in \mathcal{R}$. Here we take $\phi := r_1(1-\delta)$, $\delta \in (0, 1]$. □

Remark 7.2.3. *The second part of the problem is essential to the solution of the first part. In particular, we search for the maximum norm that permissible perturbations are allowed to have, such that the generalised Nyquist criterion is not violated. In addition, all such perturbations should satisfy the constraint (7.10), for all frequencies $\omega \in \mathcal{R}$.*□

The approach we follow to solve problem 7.2.2 is to consider first an equivalent problem at a given frequency $\omega \in \mathcal{R}$, i.e. a simplified problem involving distance to singularity of complex matrices. In particular, the simplified problem is now in "relative" form. The problem can be easily recast into an "absolute" form whose theory is studied intensively in the next section.

Lemma 7.2.3 (Relative to absolute distance to singularity). *Suppose that $T \in \mathcal{C}^{p\times m}$ has singular value decomposition,*

$$T = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V' = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}'$$

with $\Sigma = diag(\sigma_1, \ldots, \sigma_l, \sigma_{l+1}, \ldots, \sigma_t)$, $\sigma_1 = \ldots = \sigma_l > \sigma_{l+1} \geq \ldots \geq \sigma_t > 0$. Let V_l and U_l be the first l-columns of V and U, respectively, and let $\phi < \sigma_1^{-1}$ be given. Define

$$\begin{cases} \mathcal{B}_d^{m\times p} = \{E \in \mathcal{C}^{m\times p} : \|E\| < d\}, \\ \mathcal{P}(\phi) = \{E \in \mathcal{C}^{m\times p} : \|V_l' E U_l\| \leq \phi\}, \end{cases} \tag{7.14}$$

and

$$d(\phi) = \sup \{d : \det(I_m - ET) \neq 0 \text{ for all } E \in \mathcal{B}_d^{m\times p} \cap \mathcal{P}(\phi)\} \tag{7.15}$$

Then the later maximisation problem can be recast as

$$d(\phi) = \sup \left\{d : \det(\Sigma^{-1} - \tilde{E}) \neq 0 \text{ for all } \tilde{E} \in \{\tilde{E} \in \mathcal{B}^{t\times t}_{\sigma_1^{-1}} : \|\tilde{E}_{11}\| < \phi\}\right\} \tag{7.16}$$

where $\tilde{E}_{11}$ denotes the $(1,1)$ leading $l \times l$ sub-block of $\tilde{E} := V_1' E U_1$. Equivalently,

$$d(\phi) = \sup \left\{d : \det(\Sigma^{-1} - \tilde{E}) = 0 \text{ for all } \tilde{E} \in \{\tilde{E} \in \mathcal{B}^{t\times t}_{\sigma_1^{-1}} : \|\tilde{E}_{11}\|_\infty < \phi\}\right\} \tag{7.17}$$

and the minimum is attained.

Proof. This is a straightforward generalisation of a parallel result in [GHJ00]. □

Remark 7.2.4. *Condition $\|x^T \Delta y\| \leq \phi$, essentially enforces a structure on the perturbation set. The objective is to maximise the size of the set $\mathcal{E}(\delta, \mu)$, and hence the magnitude of the non-destabilising permissible perturbation set, on the basis of this structure. Later we shall see that only a partial characterisation of this structure is needed.* □

At this point, it is interesting to visualise the effect of this restriction to the perturbation set. We consider the following example:

Example 7.2.2. *Consider the unstable block-diagonal system:*

$$G = \begin{pmatrix} \frac{0.1}{s-1} & 0 & 0 \\ 0 & \frac{-1/\sqrt{2}}{s-2} & \frac{2\sqrt{2}}{s-1} \\ 0 & \frac{1/\sqrt{2}}{s-2} & \frac{2/\sqrt{2}}{s-1} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \sqrt{0.1} & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ \hline \sqrt{0.1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \end{array}\right]$$

The realisation of G is balanced since the corresponding gramians are equal to:

$$\Sigma = \begin{bmatrix} \frac{1}{20} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So, the smallest Hankel singular value of G is equal to $\frac{1}{20}$. Further, the scaled Schmidt pair of the corresponding system R (see definition) is

$$\{v, w^{\sim}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right\}$$

and hence, using appropriate parts of the doubly coprime factorisation of G we deduce that

$$\{y, x^T\} := \{Mv, w^{\sim}\widetilde{M}\} = \left\{ \begin{pmatrix} \frac{s-1}{s+1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{s-3}{s+3} & 0 & 0 \end{pmatrix} \right\}$$

which are all-pass functions. Then, at an arbitrary frequency

$$\|x^T \Delta y\| < \phi \Rightarrow \left\| \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| \leq \phi \Rightarrow \|\Delta_{11}\| \leq \phi$$

where Δ is any permissible perturbation. So, in this case the restriction involves directly the norm of the $(1,1)$ element of Δ. □

Remark 7.2.5. *If in the above example we assume that $\|\Delta\| = r_1$ and that $\phi := (1-\delta)r_1$, $\delta \in [0,1)$, then $(1-\delta)$ is essentially the percentage reduction of $\|\Delta\|$ relative to the maximal unstructured robust stability radius r_1 .* □

The analysis in the example above assumes constant matrices, i.e. perturbations that are evaluated at each frequency $\omega_o \in \mathcal{R}$. This is the framework followed in the next paragraph as well, and it is based purely on linear algebraic arguments. This is motivated by the fact that all frequencies are considered "equally critical", whenever optimal controllers are employed. In order to solve problem 7.2.2, it is crucial to understand first the simplified version of the problem (involving complex matrices) and hence the next paragraph outlines some important results for "structured distance to singularity" problems.

7.3 Constrained Distance to Singularity

Constrained distance to singularity will provide us the optimal structure that a perturbation must have (optimal in the sense of the parametric restriction) so that it has the minimum possible norm and in the same time it is rank-reducing. First, we outline some standard results in matrix theory and distance to singularity problems.

Problem 7.3.1. *Consider the square matrix*

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_n \end{bmatrix}$$

such that $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n$. *Then find for* $\Delta \in \mathcal{C}^{n \times n}$

$$\gamma = \min_{\Delta} \|\Delta\| \qquad (= \min \bar{\sigma}(\Delta))$$
$$\text{s.t. } det(\Sigma - \Delta) = 0$$

□

Remark 7.3.1. *Any* Δ *of the form*

$$\Delta = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \Delta_{22} \end{bmatrix}$$

makes

$$\Sigma - \Delta = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Sigma} - \Delta_{22} \end{bmatrix}$$

which is clearly rank deficient. Here $\hat{\Sigma}$ is a conformal partition of Σ. The norm of Δ is equal to σ_1 for all Δ such that $\|\Delta_{22}\| \leq \sigma_1$. □

This is a standard result of (absolute) distance to singularity, known as the Eckard-Young theorem. A problem related to that outlined in the previous section is to consider the case when we restrict the largest singular value of Δ to be less or equal than ϕ which is *strictly* less than the smallest singular value of Σ, σ_1. The solution to the problem is well-known [LCL+84] and is summarised next:

Lemma 7.3.1 (constrained distance to singularity). *Let A be a square non-singular complex matrix which has a singular value decomposition $A = U\Sigma V'$, where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $0 < \sigma_1 < \sigma_2 \leq \cdots \leq \sigma_n$ and denote by u_1 and v_1 the columns of U and V, respectively, corresponding to the smallest singular value, σ_1. Then all E which minimise*

$$\gamma = \min\{\|E\| : det(A - E) = 0, \quad |\langle u_1 v_1, E\rangle| \leq \phi\}$$

are given by

$$E = U \begin{bmatrix} \phi & \nu & 0 \\ \nu' & -\phi & 0 \\ 0 & 0 & P_s \end{bmatrix} V'$$

where P_s is arbitrary except for the constraint

$$\|P_s\| \leq \sqrt{\sigma_1\sigma_2 + \phi(\sigma_1 - \sigma_2)} \tag{7.18}$$

and ν is given by

$$\nu = \sqrt{(\phi + \sigma_2)(\sigma_1 - \phi)}e^{j\theta}, \quad \theta \in [0, 2\pi).$$

The minimum value of $\gamma(\phi)$ is given by the righthand side of (7.18).

Proof. see [LCL+84], [JH+06], [GHJ00]. □

Example 7.3.1 ([LCL+84]). *Let A be given by*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

then find the minimum norm matrix E so that $det(A - E) = 0$.

If E is unconstrained,

$$E = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & & E_s \end{array}\right]$$

where $\|E_s\| \leq 1$ but otherwise E_s is arbitrary.

Consider now the case where we impose a constraint on e_{11}:

$$|e_{11}| \leq \phi := \frac{1}{2} \qquad\qquad (< \sigma_1(A) = 1)$$

Then, all optimal rank-reducing matrices E have the form

$$E = \left[\begin{array}{cc|c} \frac{1}{2} & \frac{3}{2}e^{j\theta} & 0 \\ \frac{3}{2}e^{-j\theta} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & e_{33} \end{array}\right]$$

where $|e_{33}| < \|E\| = \frac{\sqrt{10}}{2} \simeq 1.58$ and θ is arbitrary.

A full treatment of structured distance to singularity problems under various constraints which generalise the above results are found in [JH+06]. In particular, we are interested for the case where the largest singular value has multiplicity greater than one. The proof of the following theorem is quite technical and involves a sequence of lemmas which can be found together with their proofs in the appendix D.

Assumption 7.3.1. *Suppose that $T \in \mathcal{C}^{n\times n}$ has a singular value decomposition*

$$T = U\Sigma V' = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 I_l & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix}, \qquad \Sigma_2 := diag(\sigma_{l+1}, \ldots, \sigma_n)$$

with $U_1, V_1 \in \mathcal{C}^{n\times l}$. Assume that $\sigma_1 = \cdots = \sigma_l > \sigma_{l+1} \geq \cdots \geq \sigma_n > 0$, and define

$$A := \Sigma^{-1} = diag(A_1, A_2)$$

where $A_1 = a_1 I_l := \sigma_1^{-1} I_l$ and $A_2 := diag(a_{l+1}, \ldots, a_n)$ with $0 < a_1 = \cdots = a_l < a_{l+1} \leq \cdots \leq a_n$. Further, define $E_1 = \begin{bmatrix} I_l & 0_{l\times(n-l)} \end{bmatrix}$.

Then the next theorem holds:

Theorem 7.3.1 (Distance to singularity). *Let everything be defined as in assumption 7.3.1. Then, the minimum distance to singularity*

$$\gamma = \min\left\{\|\Delta\| : det(A - \Delta) = 0,\ \ \|E_1' \Delta E_1\| \leq \phi < \underline{\sigma}(A)\right\}$$

is given by

$$\gamma = \sqrt{a_{l+1}a_1 - (a_{l+1} - a_1)\phi}$$

Further, all perturbations in the set

$$\mathcal{D}_\phi := \left\{\Delta \in \mathcal{C}^{n\times n} : \|\Delta\| = \gamma, \;\; det(A - \Delta) = 0, \;\; \|E_1'\Delta E_1\| \leq \phi < a_1\right\}$$

have the following structure:

$$\Delta = \begin{bmatrix} \Delta_{11} & \gamma\Delta_{31}^{\gamma}\hat{\Delta}_2 \\ \gamma\hat{\Delta}_3\Delta_{13}^{\gamma} & \gamma^2\mathcal{F}_l\left\{\hat{\Delta}, \Delta_{33}^{\gamma}\right\} \end{bmatrix}$$

where $\Delta_{11} \in \mathcal{C}^{l\times l}$, $\|\Delta_{11}\| \leq \phi$ *but is otherwise arbitrary, and*

$$\Delta_1^{\gamma} := \begin{bmatrix} \Delta_{11} & \Delta_{13}^{\gamma} \\ \Delta_{31}^{\gamma} & \Delta_{33}^{\gamma} \end{bmatrix}$$

is any $n \times n$ *optimal* γ*-completion of* Δ_{11}. *Further,*

$$\hat{\Delta} = \begin{bmatrix} 0_l & \hat{\Delta}_2 \\ \hat{\Delta}_3 & \hat{\Delta}_4 \end{bmatrix}$$

is an optimally structured perturbation for the distance to singularity problem where $\hat{\Delta}$ *is constrained to have its first* $l \times l$ *block equal to zero, as defined in Lemma D.0.1 in appendix.*

Proof. See appendix D. The proof involves a series of lemmas. □

Example 7.3.2. *Take* $\Delta_{11} = \phi I$, $\phi < \gamma$. *Then choosing* $\Delta_{13}^{\gamma}, \Delta_{31}^{\gamma}, \Delta_{33}^{\gamma}$ *to be real symmetric, one possible* γ*-completion of* Δ_{11} *is the following:*

$$\Delta_1^{\gamma} = \begin{bmatrix} \phi I & \left(\sqrt{\gamma^2 - \phi^2}\right) I \\ -\left(\sqrt{\gamma^2 - \phi^2}\right) I & \phi I \end{bmatrix} =: \begin{bmatrix} \Delta_{11} & \Delta_{13}^{\gamma} \\ \Delta_{31}^{\gamma} & \Delta_{33}^{\gamma} \end{bmatrix}$$

Then it can be verified that $\|\Delta_1^{\gamma}\| = \gamma$.

Similarly, we could choose

$$\Delta_1^{\gamma} = \begin{bmatrix} \phi I & \left(\sqrt{\gamma^2 - \phi^2}\right) I \\ \left(\sqrt{\gamma^2 - \phi^2}\right) I & -\phi I \end{bmatrix} =: \begin{bmatrix} \Delta_{11} & \Delta_{13}^{\gamma} \\ \Delta_{31}^{\gamma} & \Delta_{33}^{\gamma} \end{bmatrix}$$

Keeping the first choice from above, we now take

$$\hat{\Delta} = \begin{bmatrix} 0_l & \hat{\Delta}_2 \\ \hat{\Delta}_3 & \hat{\Delta}_4 \end{bmatrix} := \left[\begin{array}{cc|cc} 0 & 0 & \sqrt{a_3} & 0 \\ 0 & 0 & 0 & 0 \\ \hline \sqrt{a_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that

$$E = \begin{bmatrix} \Delta_{11} & \gamma\Delta_{31}^{\gamma}\hat{\Delta}_2 \\ \gamma\hat{\Delta}_3\Delta_{13}^{\gamma} & \gamma^2\mathcal{F}_l\left\{\hat{\Delta}, \Delta_{33}^{\gamma}\right\} \end{bmatrix} = \left[\begin{array}{cc|cc} \phi & 0 & -\sqrt{\gamma^2-\phi^2} & 0 \\ 0 & \phi I_{l-1} & 0 & 0 \\ \hline \sqrt{\gamma^2-\phi^2} & 0 & \phi & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, $\|E\| = \gamma$ *and* $det(A - E) = 0$.

The second choice in the above example will be used extensively in the next section. In the sequel, we state one of the main results of this chapter:

Corollary 7.3.1 (Relative to absolute distance to singularity). *Assume everything as defined in Lemma 7.2.3 and Theorem 7.3.1. Then,*

$$d(\phi) = \sqrt{\frac{1}{\sigma_1\sigma_{l+1}} - \phi\left(\frac{1}{\sigma_{l+1}} - \frac{1}{\sigma_1}\right)}$$

all $E \in \mathcal{P}(\phi)$ *such that* $det(I_m - ET) = 0$ *and* $\|E\| = d(\phi)$ *are of the form*

$$E = V \left[\begin{array}{cc|c} \Delta_{11} & \gamma\Delta_{31}^{\gamma}\hat{\Delta}_2 & 0 \\ \gamma\hat{\Delta}_3\Delta_{13}^{\gamma} & \gamma^2\mathcal{F}_l\left\{\hat{\Delta}, \Delta_{33}^{\gamma}\right\} & 0 \\ \hline 0 & 0 & E_{44} \end{array}\right] U'$$

where $\|E_{44}\| \leq d(\phi)$ *and*

$$\left\| \begin{bmatrix} \Delta_{11} & \gamma\Delta_{31}^{\gamma}\hat{\Delta}_2 \\ \gamma\hat{\Delta}_3\Delta_{13}^{\gamma} & \gamma^2\mathcal{F}_l\left\{\hat{\Delta}, \Delta_{33}^{\gamma}\right\} \end{bmatrix} \right\| = d(\phi)$$

Hence if the two (distinct) largest singular values of a complex matrix T are known then the minimum norm of the set of all rank-reducing perturbations (i.e. distance to singularity) and the optimal structure of this set, are *a priori* known. The nature of this structure depends on the multiplicity of the largest singular value of T and other constraints that are imposed, e.g. on the spectral radius.

7.4 Extended robust stability radius

In the previous paragraph an "optimal" structure to the set of all permissible rank-reducing perturbations was determined such that it minimises the distance to singularity of complex matrices which have multiplicity greater than 1 on their largest singular value. Further, Problem 7.2.2, Lemma 7.2.3 and Corollary 7.3.1 linked the problem of robust stability under the constraint uncertainty set (7.13) with the distance to singularity problem, in a frequency-by-frequency Nyquist-type of argument. In a connection with those results, in this paragraph we show that the extended robust stability radius is a function of the two (distinct) largest super-optimal singular values of the closed-loop transfer function and so it is maximised by choosing a super-optimal controller $K \in \mathcal{K}_2$. This arises naturally from the fact that the two largest (distinct) singular values of any closed-loop $T \in \mathcal{T}_{l+1}$ evaluated at any frequency $\omega_0 \in \mathcal{R}$ (i.e. the complex matrix $T(j\omega_0)$) are equal to the first two super-optimal levels, s_1 and s_{l+1}.

Proposition 7.4.1. *Given a nominal plant G as in 7.1.1 and any $\Delta \in \mathcal{D}_{r_1} \cup \mathcal{E}(\delta, \mu(\delta))$ where*

$$\mu(\delta) := \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)}, \quad \delta \in [0, 1]$$

and

$$\mathcal{E}(\delta, \mu(\delta)) = \left\{\Delta \in \mathcal{D}_{\mu(\delta)}(G) \ : \ \|x^T \Delta y\|_\infty \leq r_1(1-\delta)\right\}$$

in which

$$\mathcal{D}_{\mu(\delta)}(G) = \{\Delta \in \mathcal{L}_\infty : \|\Delta\|_\infty < \mu(\delta), \quad \eta(G) = \eta(G + \Delta)\}$$

then $(G + \Delta, K) \in \mathcal{S}$, for each $K \in \mathcal{K}_2$.

Essentially the proposition states that any perturbed plant inside the $\mu(\delta)$-ball (for any value of $\delta \in [0, 1]$) is stabilisable by every super-optimal controller $K \in \mathcal{K}_2$.

Proof. Recall that $T = K(I - GK)^{-1}$ and choose a $K \in \mathcal{K}_2$ so that $T \in \mathcal{T}_{l+1}$. It is well known (Lemma 7.1.1) that any $T \in \mathcal{T}_{l+1}$ admits the following decomposition

$$T = Y_1(s) \begin{pmatrix} s_1 a(s) & 0 & 0 \\ 0 & s_{l+1} b(s) & 0 \\ 0 & 0 & \check{R}(s) + \mathcal{S}_{l+1} \end{pmatrix} X_1(s)$$

where X_1, Y_1 are square inner, $a(s), b(s)$ are all-pass functions and $\|\check{R}(s) + \mathcal{S}_{l+1}\|_\infty \leq s_{l+1}$. Obviously, $\mathcal{K}_2 \subseteq \mathcal{K}_1$ and so $\mathcal{T}_{l+1} \subseteq \mathcal{T}_1$ which implies that $(G + \Delta, K) \in \mathcal{S}$ for any $\Delta \in \mathcal{D}_{r_1}$. Hence, we only examine the case where $\Delta \in \mathcal{E}(\delta, \mu(\delta))$.
For a fixed value of $\delta \in (0, 1]$, let

$$\frac{1}{s_1} \leq \|\Delta\|_\infty < \mu(\delta), \quad \|x^T \Delta y\|_\infty \leq \phi < \frac{1}{s_1} \tag{7.19}$$

where $\phi := r_1(1-\delta)$. Assume now for contradiction that such Δ is destabilising. Then there exists a frequency $\omega_0 \in \mathcal{R}$ so that

$$det(I - \xi\Delta(j\omega_0)T(j\omega_0)) = 0, \quad 0 < \xi \leq 1 \tag{7.20}$$

Of course, $T(j\omega_0)$ is a complex matrix which, in the light of the above decomposition, has its two largest (distinct) singular values equal to s_1 and s_{l+1}, respectively. Further, considering the constraint (7.19) and equation (7.20), the problem can be viewed as in Lemma 7.2.3. Then, corollary 7.3.1 states that the minimum norm of a destabilising perturbation is equal to $\mu(\delta)$ which implies that (7.20) is true only for $\|\xi\Delta(j\omega)\| \geq \mu(\delta)$. Therefore, equation (7.20) gives a contradiction and any perturbation $\Delta \in \mathcal{D}_{r_1} \cup \mathcal{E}(\delta, \mu(\delta))$ is stabilisable by each $K \in \mathcal{K}_2$. □

Later on, in Theorem 7.4.1, it will be shown that $\mu(\delta)$ given in proposition 7.4.1 is the maximum extended robust stability radius $\mu^\star(\delta)$, and that $\mathcal{K}_2 \equiv \mathcal{K}_1^\delta$, for each $\delta \in (0,1]$. Now, consider the following problem:

Problem 7.4.1. *Given that the original plant has multiplicity r (greater than one) on its smallest Hankel singular value, construct a $\Delta \in \mathcal{RH}_\infty$ ($\eta(G) = \eta(G+\Delta)$), s.t.*

(i) $\|\Delta\|_\infty = \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)}$,

(ii) $\|x^T\Delta y\|_\infty \leq r_1(1-\delta)$,

(iii) $(G+\Delta, K)$ *is unstable* $\forall\ K \in \mathcal{K}_2$.

where l is the multiplicity of s_1 (see chapter 6). □

The solution of the above problem is constructive and it is outlined by the following algorithm (whose proof follows in Proposition 7.4.2). First, recall Lemma 7.1.1 in which $T \in \mathcal{T}_{l+1}\ \forall\ K \in \mathcal{K}_2$ is written in the form

$$T = Y_1(s)\begin{pmatrix} s_1 a(s) & 0 & 0 \\ 0 & s_{l+1}b(s) & 0 \\ 0 & 0 & (\check{R}+\check{Q})(s) \end{pmatrix} X_1(s)$$

in which $Y_1(s)$ and $X_1(s)$ are square inner matrices, $a(s), b(s)$ are all-pass functions and $\|\check{R}+\check{Q}\|_\infty \leq s_{l+1}$. In addition, $Y_1(s)$ and $Y(s)$ ($X(s)$ and $X_1(s)$, respectively) share the same first l-columns (rows), which are denoted by $y(s)(x^T(s))$, where $X(s)$

and Y are defined in equation (7.3). To fix notation, let $T \in \mathcal{T}_{l+1} \subseteq \mathcal{RH}_\infty^{p\times m}$, $Y_1 \in \mathcal{RH}_\infty^{p\times p}$, $X_1 \in \mathcal{RH}_\infty^{m\times m}$ with $Y_1 Y_1^\sim = I_p$ and $X_1 X_1^\sim = I_m$. Further, $a \in \mathcal{RH}_\infty^{-,l\times l}$ and $b \in \mathcal{RH}_\infty^{-,1\times 1}$ with $aa^\sim = I_l$ and $bb^\sim = 1$ so that $R \in \mathcal{RH}_\infty^{-,(p-l-1)\times(p-l-1)}$ with $\min\{p-l-1, m-l-1\} > 0$.

Algorithm 7.4.1. *The algorithm consists of five steps:*

step 1 *Define the all-pass function*

$$Y_2 := Y_1 \begin{pmatrix} a(s) & 0 & 0 \\ 0 & b(s) & 0 \\ 0 & 0 & I_{p-l-1} \end{pmatrix}$$

step 2 *Compute left and right co-prime factorisations of the columns of $X_1^\sim$ and the rows of $Y_2^\sim$, respectively; both with inner denominators.*

$$X_1^\sim = \begin{pmatrix} n_1 & n_2 & \cdots & n_m \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_m \end{pmatrix} =: N_1 D_1$$

and

$$Y_2^\sim = \begin{pmatrix} \tilde{d}_1 & 0 & 0 & 0 \\ 0 & \tilde{d}_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tilde{d}_p \end{pmatrix} \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \vdots \\ \tilde{n}_p \end{pmatrix} =: D_2 N_2$$

where $N_1, N_2, d_i^{-1}, \tilde{d}_i^{-1} \in \mathcal{RH}_\infty$, N_1, N_2 are square inner and $d_i, \tilde{d}_i$ are scalar all-pass functions.

step 3 *Pick any $\omega_o \in \mathcal{R}$. Then write for each $i = \{1, 2, \ldots, l, l+1\}$:*

$$d_i(j\omega_o) = e^{j\phi_i} \quad \text{and} \quad \tilde{d}_i(j\omega_o) = e^{j\tilde{\phi}_i}$$

where $-\pi \le \phi_i, \tilde{\phi}_i < \pi$.
(recall that $|d_i(j\omega_o)| = 1$, since the $d_i(s)$ are chosen to be scalar all-pass)

step 4 *Define two diagonal inner matrices*

$$A_1 = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_{l+1} \end{bmatrix}, \; A_2 = \begin{bmatrix} \tilde{\alpha}_1 & & \\ & \ddots & \\ & & \tilde{\alpha}_{l+1} \end{bmatrix}$$

such that $\alpha_i(\tilde{\alpha}_i)$ are stable, all-pass functions that interpolate $d_i^{-1}(\tilde{d}_i^{-1})$. Thus,

case 1: *If* $0 < \phi_i < \pi$ $(0 < \tilde{\phi}_i < \pi)$ *then set*

$$\alpha_i = \frac{s-\beta_i}{s+\beta_i}, \qquad \left(\tilde{\alpha}_i = \frac{s-\tilde{\beta}_i}{s+\tilde{\beta}_i}\right)$$

where $\arg\left(\frac{j\omega_o-\beta_i}{j\omega_o+\beta_i}\right) = \phi_i > 0$, $\left(\arg\left(\frac{j\omega_o-\tilde{\beta}_i}{j\omega_o+\tilde{\beta}_i}\right) = \tilde{\phi}_i > 0\right)$

case 2: *If* $-\pi < \phi_i < 0$ $(-\pi < \tilde{\phi}_i < 0)$ *then set*

$$\alpha_i = -\frac{s-\beta_i}{s+\beta_i}, \qquad \left(\tilde{\alpha}_i = \frac{-(s-\tilde{\beta}_i)}{s+\tilde{\beta}_i}\right)$$

where $\arg\left(\frac{j\omega_o-\beta_i}{j\omega_o+\beta_i}\right) = \pi + \phi_i > 0$, $\left(\arg\left(\frac{j\omega_o-\tilde{\beta}_i}{j\omega_o+\tilde{\beta}_i}\right) = \pi + \tilde{\phi}_i > 0\right)$

case 3: *If* $\phi_i = 0$ $(\tilde{\phi}_i = 0)$ *or* $\phi_i = -\pi$ $(\tilde{\phi}_i = -\pi)$ *then set* $\alpha_i = 1$ *or* $\alpha_i = -1$, *respectively* $(\tilde{\alpha}_i = 1$ *or* $\tilde{\alpha}_i = -1)$.

step 5 *Denote the matrix consisting of the first* $l+1$ *columns (rows) of* $N_1(N_2)$ *by* $N_{11}(N_{21})$. *Then define the destabilising (to all optimal closed-loop transfer functions) perturbation:*

$$\Delta = N_{11}A_1 \left[\begin{array}{cc|c} \phi I_l & & \nu_o \\ & & 0 \\ \hline \nu_o & 0 & -\phi \end{array}\right] A_2 N_{21}$$

where $\phi = \frac{1-\delta}{s_1} = r_1(1-\delta)$; $\nu_o = \sqrt{\gamma^2 - \phi^2}$ *and*

$$\gamma(\phi) = \sqrt{\frac{1}{s_1 s_{l+1}} - \left(\frac{1}{s_{l+1}} - \frac{1}{s_1}\right)\phi}$$

Then,

$$\nu_o = \sqrt{\left(\frac{1}{s_1} - \phi\right)\left(\frac{1}{s_{l+1}} + \phi\right)}$$

□

Remark 7.4.1 (on step 2). *The (scaled) Schmidt vectors corresponding to the first super-optimal level are matrix functions (because in general their multiplicity* $l > 1$*). However, the columns are linearly independent to each other. Hence, the vector nature of each co-prime factorisation, follows.* □

Remark 7.4.2 (on step 4). *The interpolating functions, as constructed in step 4, are needed for the proof of problem 7.4.1, part (iii). These are constructed to be stable so that* $\Delta \in \mathcal{RH}_\infty$*; so* $\beta_i > 0, \tilde{\beta}_i > 0$ *for all i.* □

The rationale behind this algorithm is to construct a dynamic perturbation which interpolates all dynamic parts of every optimal control sensitivity matrix function and hence effectively reduces the problem to a constant distance to singularity problem. In previous analysis it was shown that all optimal control sensitivity transfer matrices have the same l-largest singular values, which remain constant over all frequencies and are equal to the first super-optimal level, $s_1(G)$. Similarly, all $(l+1)$-th level super-optimal approximations also have their $(l+1)$ singular values constant over all frequencies. Knowing the first two distinct singular values of a constant matrix the problem becomes equivalent to constructing a minimum norm rank-reducing perturbation.

Proposition 7.4.2. *Let $\Delta(s)$ be constructed according to Algorithm 7.4.1. Then $\Delta(s)$ has the following properties:*

(i) $\Delta(s) \in \mathcal{RH}_\infty$

(ii) $\|\Delta\|_\infty = \gamma(\phi)$

(iii) $\|x^T \Delta y\|_\infty = \phi$

(iv) $det[I_m - \Delta(j\omega_o)T(j\omega_o)] = 0 \; \forall \; T \in \mathcal{T}$.

Proof. See Appendix E. □

Consequently, the algorithm constructs a destabilising perturbation for *all* $(l+1)$-super-optimal control sensitivity functions (see Appendix E). Further, this particular perturbation lies on the boundary of $\gamma(\phi)$-ball which reveals the fundamental similarity between this construction and the construction of a destabilising perturbation to all optimal control sensitivity functions (which lie on the boundary of r_1-ball).

Theorem 7.4.1. (CMRS) [GHJ00] *Let $\mathcal{T}_1 \subseteq \mathcal{H}_\infty^{(p)\times m}$ be as defined in equation (7.3). Let x^T and y be the first row and column of X and Y, respectively, and define $\mathcal{D}_{r_1}$ and $\mathcal{E}(\delta,\mu)$ as in equations (7.4) and (7.13), respectively, for some fixed $\delta \in [0,1]$. Let $\mu^\star(\delta)$ be the supremum of μ such that there exists a K for which $(G+\Delta, K) \in \mathcal{S}$ for every $\Delta \in \mathcal{D}_{r_1} \cup \mathcal{E}(\delta,\mu)$. Then, for each δ, the supremum of extended robust stability is given by*

$$\mu^\star(\delta) = \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)} \geq r_1$$

where s_1 and s_{l+1} are the first two distinct super-optimal levels of $\mathcal{T}$ with $s_1 = r_1^{-1}$.

Proof. The construction of destabilising perturbation in algorithm 7.4.1 together with Proposition 7.4.1 show that the constrained maximum robust stability radius for any $K \in \mathcal{K}_2$ is given by $\mu^\star(\delta)$. Next, consider $K \in \mathcal{K}_1 \backslash \mathcal{K}_2$. In this case, there exists a frequency ω_0 so that $T = K(I - GK)^{-1}$ has its first two (largest) distinct singular values equal to s_1 (of multiplicity l) and $\overline{\sigma}(\hat{R}(j\omega_0) + \overline{Q}(j\omega_0))$, with $s_1 > \overline{\sigma}(\hat{R}(j\omega_0) + \overline{Q}(j\omega_0)) > s_{l+1}$ (since $T \notin \mathcal{T}_{l+1}$). Then, using a procedure similar to Proposition 7.4.2, it is always possible to construct a perturbation $\Delta \in \mathcal{RH}_\infty^{m\times p}$ in the interior of $\mathcal{E}(\delta, \mu^\star(\delta))$ such that

$$\|\Delta\|_\infty = \sqrt{\phi^2 + \nu_1^2} < \mu^\star(\delta), \quad \|x^T \Delta y\|_\infty = \phi$$

where

$$\nu_1 := \sqrt{\left(\frac{1}{\overline{\sigma}(\hat{R}(j\omega_0) + \overline{Q}(j\omega_0))} + \phi\right)\left(\frac{1}{s_1} - \phi\right)}$$

for which $(G + \Delta, K) \notin \mathcal{S}$. Thus, $\mu^\star(\delta)$ is the supremum of the constrained robust stability radius among all $K \in \mathcal{K}_1$ (and also among all $K \in \mathcal{K}$, since every perturbation inside $\mathcal{D}_{r_1}$ is required to be stabilised). □

The next results are immediate from the theorem:

Proposition 7.4.3. *The following three statements hold:*

1. *For each $0 < \delta \leq 1$,*

$$(G + \Delta, K) \in \mathcal{S} \text{ for every } \Delta \in \mathcal{D}_{r_1} \cup \mathcal{E}(\delta, \mu^\star(\delta)), \text{ if and only if, } K \in \mathcal{K}_2.$$

2. (a) *For the (extreme) case where $\delta = 0$,*

$$\mathcal{E}(0, \mu^\star(0)) = \mathcal{D}_{r_1}$$

   (b) *For each $K \in \mathcal{K}_2$,*

$$(G + \Delta, K) \in \mathcal{S} \text{ for every } \Delta \in \bigcup_{\delta\in[0,1]} \mathcal{E}(\delta, \mu^\star(\delta)).$$

3. *Let σ_n and σ_{n-r-1} denote the two smallest (distinct) Hankel singular values of $G(-s)$ with $\sigma_{n-r-1} > \sigma_r$. Then, an immediate lower bound is given by*

$$\mu^\star(\delta) \geq \sqrt{\delta\sigma_n\sigma_{n-r-1} + (1-\delta)\sigma_n^2}.$$

Proof. Part (1.) follows immediately from Proposition 7.4.1 and Theorem 7.4.1. Part (2a.) is immediate consequence of the definitions. Further, part (2b.) is proved by using part (2a.) and part (1.). Part (3.) follows from the fact that $s_1 = \sigma_n^{-1}$ and $s_{l+1} \leq \sigma_{n+r}^{-1}$ (see chapter 6, Theorem 6.1.4). □

To summarise, the main result is that for any parameter $\delta \in (0, 1]$, the maximum constrained robust stability radius is $\mu^\star(\delta)$ and is achieved by choosing any $K \in \mathcal{K}_2$. In fact, all perturbed systems inside this ball are stabilisable by any controller K *if and only if* $K \in \mathcal{K}_2$.

Further, we may think of δ as the amount of "structure" imposed to the problem. Clearly, the "more" structure we impose, the largest is the structured stability radius (note that as expected $\mu^\star(\delta)$ is an increasing function of δ). However, no matter what δ is (in the interval $0 < \delta \leq 1$), $\mu^\star(\delta)$ is achieved by the set of super-optimal controllers $\mathcal{K}_2$, and hence each super-optimal controller $K \in \mathcal{K}_2$ guarantees to stabilise all $\Delta \in \bigcup_{\delta\in[0,1]} \mathcal{E}(\delta, \mu^\star(\delta))$ (note that $D_{r_1} = \mathcal{E}(0, \mu^\star(0))$).

Computation of δ. In simulation experiments the need of specifying a $\delta \in [0, 1]$ such that a Δ is constructed with $\|\Delta\|_\infty = \mu^\star(\delta)$ and $\|x^T \Delta y\|_\infty = \frac{1-\delta}{s_1}$ appears to be not trivial. Therefore, in the sequel a method for deriving an appropriate pair $\{\delta, \Delta\}$ is presented:

Consider first a perturbation Δ and assume that for an unspecified δ, it satisfies $\|\Delta\|_\infty = \mu^\star(\delta)$ and $\|x^T \Delta y\|_\infty = \frac{1-\delta}{s_1}$. Then:

$$\begin{aligned}\|\Delta\|_\infty &= \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)} = \sqrt{\frac{1}{s_1^2}\left(\frac{\delta s_1}{s_{l+1}} + 1 - \delta\right)} \\ &= \frac{1}{s_1}\sqrt{1 + \delta\left(\frac{s_1}{s_{l+1}} - 1\right)} = \frac{1}{s_1}\sqrt{1 + \delta\left(\frac{s_1 - s_{l+1}}{s_{l+1}}\right)} = \frac{1}{s_1}\sqrt{1 + \delta\psi}\end{aligned}$$

where $\psi := \frac{s_1 - s_{l+1}}{s_{l+1}}$. Thus,

$$\frac{\|x^T \Delta y\|_\infty}{\|\Delta\|_\infty} = \frac{1-\delta}{\sqrt{1+\delta\psi}} =: \gamma$$

Solving in terms of δ,

$$\frac{(1-\delta)^2}{1+\delta\psi} = \gamma^2 \Leftrightarrow \delta^2 - 2\delta + 1 = \gamma^2 + \gamma^2\delta\psi$$

i.e.

$$\delta^2 - (2 + \gamma^2\psi)\delta + 1 - \gamma^2 = 0$$

with solutions

$$\delta_{1,2} = \frac{(2+\gamma^2\psi) \pm \sqrt{(2+\gamma^2\psi)^2 - 4(1-\gamma^2)}}{2}$$
$$= 1 + \frac{\gamma^2\psi}{2} \pm \frac{1}{2}\sqrt{4\gamma^2\psi + \gamma^4\psi^2 + 4\gamma^2}$$

By selecting the solution

$$\delta^\star = 1 + \frac{\gamma^2\psi}{2} - \frac{1}{2}\sqrt{4\gamma^2\psi + \gamma^4\psi^2 + 4\gamma^2}$$

it is guaranteed that $\delta^\star \in [0, 1]$. Further, by defining

$$\Delta_{new} = \frac{\Delta}{\|\Delta\|}\frac{\sqrt{1+\delta^\star\psi}}{s_1}$$

it is guaranteed, by construction, that $\|\Delta_{new}\|_\infty = \mu^\star(\delta^\star)$, $\|x^T\Delta_{new}y\|_\infty = \frac{1-\delta^\star}{s_1}$.

7.5 Examples

In this section we present examples which verify certain aspects of the derived theoretical results. As it was shown before (Lemma 7.2.1 and Lemma 7.2.2) there exist boundary perturbations Δ which are destabilising for every optimal controller. The examples given below verify some of the derived results.

Example 7.5.1. *Consider a nominal plant with the following balanced realisation:*

$$G_0 \stackrel{s}{=} \left[\begin{array}{cc|cc} 2.9688 & -6.3584 & 1.5299 & 0.9490 \\ 4.6752 & 2.9189 & -1.7270 & -1.8414 \\ \hline -1.7570 & -0.3720 & 0 & 0 \\ 0.3923 & 2.4970 & 0 & 0 \end{array}\right]$$

and gramians equal to

$$\Sigma = \begin{bmatrix} 0.5459 & 0 \\ 0 & 1.0917 \end{bmatrix}$$

so that, $\sigma_n = 0.5459 =: s_1^{-1}$.

One maximally robust controller is computed as:

$$K_{opt} \stackrel{s}{=} \left[\begin{array}{c|cc} -16.1058 & -1.7104 & 1.1866 \\ \hline 10.9709 & 1.5194 & -0.3393 \\ 9.3899 & 0.9425 & -0.2105 \end{array}\right]$$

The observer-based generator of all stabilising controllers is:

$$K_o \stackrel{s}{=} \left[\begin{array}{cc|cccc} -8.906 & -5.33 & -3.219 & 0.7188 & 1.53 & 0.949 \\ 11.22 & -8.757 & -0.3407 & 2.287 & -1.727 & -1.841 \\ \hline -2.803 & 1.582 & 0 & 0 & 1 & 0 \\ -1.739 & 1.687 & 0 & 0 & 0 & 1 \\ 1.757 & 0.372 & 1 & 0 & 0 & 0 \\ -0.3923 & -2.497 & 0 & 1 & 0 & 0 \end{array}\right]$$

and further

$$M \stackrel{s}{=} \left[\begin{array}{cc|cc} -2.969 & -2.338 & 1.53 & 0.949 \\ 12.72 & -2.919 & -1.727 & -1.841 \\ \hline -2.803 & 1.582 & 1 & 0 \\ -1.739 & 1.687 & 0 & 1 \end{array}\right], \quad \widetilde{M} \stackrel{s}{=} \left[\begin{array}{cc|cc} -2.969 & -9.35 & -3.219 & 0.7188 \\ 3.179 & -2.919 & -0.3407 & 2.287 \\ \hline -1.757 & -0.372 & 1 & 0 \\ 0.3923 & 2.497 & 0 & 1 \end{array}\right]$$

are inner factors of the coprime factorisation of G_0, which are needed in later analysis. The second super-optimal level is $s_2 = 0.13955$. Further the unique level-2 super-optimal controller is:

$$K_{sopt} \stackrel{s}{=} \left[\begin{array}{cccc|cc} -2.9688 & -2.3376 & 6.4237 & 0 & 0 & 0 \\ 3.1792 & -12.5940 & -8.8623 & 0 & -4.7201 & 3.1716 \\ -3.4707 & -3.5208 & -6.1439 & 0 & -1.7176 & 1.1541 \\ 0 & 0 & 0 & -6.1439 & 0 & 0 \\ \hline 0 & 3.1676 & 2.9015 & 0 & 1.5033 & -0.4111 \\ 0 & 2.2834 & 2.0916 & 0 & 0.9683 & -0.0947 \end{array}\right]$$

The maximising vectors Mv and $w^{\sim}\widetilde{M}$ are found to be:

$$x \stackrel{s}{=} \left[\begin{array}{ccc|cc} -6.1439 & 3.4707 & 3.5208 & -1.7176 & 1.1541 \\ 0 & -2.9688 & -9.3504 & -3.2188 & 0.7188 \\ 0 & 3.1792 & -2.9189 & -0.3407 & 2.2872 \\ \hline -5.5319 & 1.800 & 0.9072 & -0.9760 & 0.2179 \end{array}\right]$$

and

$$y \stackrel{s}{=} \left[\begin{array}{ccc|c} -2.9688 & -2.3376 & 6.4237 & -1.8003 \\ 12.7168 & -2.9189 & -8.8623 & 2.4382 \\ 0 & 0 & -6.1439 & 3.4272 \\ \hline -2.8027 & 1.5819 & 2.9015 & -0.8498 \\ -1.7385 & 1.6867 & 2.0916 & -0.5271 \end{array}\right]$$

Then the following perturbation can be constructed which destabilises the feedback loop above when the optimal controller K_{opt} is employed:

$$\Delta \stackrel{s}{=} \left[\begin{array}{cc|cc} -1.1907 & 1.2646 & 0.1982 & 0.0943 \\ -0.0437 & -0.2103 & -1.3281 & 0.4195 \\ \hline -0.1045 & -0.0807 & 0 & 0 \\ -0.1284 & -0.0959 & 0 & 0 \end{array}\right]$$

Further, the perturbation was constructed such that $\|x^T \Delta y\|_\infty = \frac{1-\delta}{s_1} = 0.19531 < 0.54585 =: \frac{1}{s_1}$ *and* $\|\Delta\|_\infty = \mu(\delta) = 1.6182$, *where* $\delta = 0.6422$. *The first table shows the poles of the four closed-loop transfer functions (see equation (5.3)), when the maximally robust controller K_{opt} is connected. A necessary and sufficient condition of internal stability is that these four transfer-functions have stable poles. However, note that there exists one unstable pole corresponding to each transfer function and hence the feedback loop is unstable.*

$p_i(H_{11})$	$p_i(H_{12})$	$p_i(H_{21})$	$p_i(H_{22})$
-2.858+5.4717i	-2.858+5.4717i	-2.858+5.4717i	-2.858+5.4717i
-2.858-5.4717i	-2.858-5.4717i	-2.858-5.4717i	-2.858-5.4717i
-6.2408+0i	-6.2408+0i	-6.2408+0i	-6.2408+0i
-1.4833+0i	-1.4833+0i	-1.4833+0i	-1.4833+0i
0.0023569+0i	**0.0023569+0i**	**0.0023569+0i**	**0.0023569+0i**

On the other hand the same perturbation is stabilised by the super-optimal controller K_{sopt}. The second table shows the poles of the four closed-loop transfers when the super-optimal controller is connected to the loop; in this case all poles are stable.

$p_i(H_{11})$	$p_i(H_{12})$	$p_i(H_{21})$	$p_i(H_{22})$
-2.858+5.4719i	-2.858+5.4719i	-2.9439+5.4522i	-2.9439+5.4522i
-2.858-5.4719i	-2.858-5.4719i	-2.9439-5.4522i	-2.9439-5.4522i
-0.013915+0i	-0.013915+0i	-2.858+5.4719i	-2.858+5.4719i
-1.4785+0i	-1.4785+0i	-2.858-5.4719i	-2.858-5.4719i
-6.1735+0i	-6.1735+0i	-0.013915+0i	-0.013915+0i
-2.9439+5.4522i	-2.9439+5.4522i	-1.4785+0i	-1.4785+0i
-2.9439-5.4522i	-2.9439-5.4522i	-6.1735+0i	-6.1735+0i
-6.1439+0i	-6.1439+0i	-6.1439+0i	-6.1439+0i

Hence, the particular perturbation Δ constructed above is destabilising to the optimal closed-loop, i.e. $(G_0 + \Delta, K_{opt}) \notin \mathcal{S}$, *where on the other hand* $(G_0 + \Delta, K_{sopt}) \in \mathcal{S}$.

In the previous example we considered the simple case where the smallest Hankel singular value of the nominal plant has multiplicity one. In the following example we study the case where the smallest Hankel singular value of the nominal plant has multiplicity larger than one (two) and the constraint posed on the perturbation is specified in terms of its norm (directionality). Again we follow a state-space analysis.

Example 7.5.2. *Consider a nominal plant with the following balanced realisation:*

$$G_0 \stackrel{s}{=} \left[\begin{array}{ccc|ccc} 1.1737 & -1.3187 & 1.7698 & 0.5584 & -0.4715 & 1.3261 \\ 2.0211 & 0.5016 & 0.6891 & 0.9819 & -0.1061 & 0.0662 \\ -1.1173 & -1.2526 & 1.1179 & -0.1396 & -1.6694 & 1.2492 \\ \hline 0.7688 & 0.6729 & -1.7194 & 0 & 0 & 0 \\ -1.2924 & -0.0312 & -0.7927 & 0 & 0 & 0 \\ 0.1770 & 0.7253 & -0.8845 & 0 & 0 & 0 \end{array}\right]$$

and gramians equal to

$$\Sigma = \begin{bmatrix} 0.9767 & 0 & 0 \\ 0 & 0.9767 & 0 \\ 0 & 0 & 1.9533 \end{bmatrix}$$

and thus, $\sigma_n = 0.9767 =: s_1^{-1}$.

One maximally robust controller is computed as:

$$K_{opt} \stackrel{s}{=} \left[\begin{array}{c|ccc} -3.0487 & -0.8789 & 0.8039 & -0.0779 \\ \hline -3.6984 & -0.6687 & -0.0607 & -0.7684 \\ 1.7906 & 0.1414 & -0.2886 & 0.0061 \\ -0.3942 & -0.2618 & 0.9208 & 0.1890 \end{array}\right]$$

Further, we compute

$$M \stackrel{s}{=} \left[\begin{array}{ccc|ccc} -1.1737 & -2.0211 & 0.5587 & 0.5584 & -0.4715 & 1.3261 \\ 1.3187 & -0.5016 & 0.6263 & 0.9819 & -0.1061 & 0.0662 \\ -3.5396 & -1.3781 & -1.1179 & -0.1396 & -1.6694 & 1.2492 \\ \hline -0.5717 & -1.0054 & 0.0715 & 1 & 0 & 0 \\ 0.4827 & 0.1086 & 0.8547 & 0 & 1 & 0 \\ -1.3578 & -0.0678 & -0.6395 & 0 & 0 & 1 \end{array}\right]$$

and

$$\widetilde{M} \stackrel{s}{=} \left[\begin{array}{ccc|ccc} -1.1737 & -2.0211 & 2.2347 & 0.7872 & -1.3233 & 0.1813 \\ 1.3187 & -0.5016 & 2.5052 & 0.6889 & -0.0319 & 0.7426 \\ -0.8849 & -0.3445 & -1.1179 & -0.8803 & -0.4058 & -0.4528 \\ \hline 0.7688 & 0.6729 & -1.7194 & 1 & 0 & 0 \\ -1.2924 & -0.0312 & -0.7927 & 0 & 1 & 0 \\ 0.1770 & 0.7253 & -0.8845 & 0 & 0 & 1 \end{array}\right]$$

which are inner factors of the doubly coprime factorisation of the nominal plant G_0. *Moreover, the second super-optimal level is* $s_2 = 0.1216$ *and the unique super-optimal controller is:*

$$K_{sopt} \stackrel{s}{=} \left[\begin{array}{ccccc|ccc} -1.1737 & -2.0211 & 0.5587 & -0.8580 & 0 & 0 & 0 & 0 \\ 1.3187 & -0.5016 & 0.6263 & -0.9619 & 0 & 0 & 0 & 0 \\ -0.8849 & -0.3445 & -2.4324 & -0.6730 & 0 & -1.2463 & 1.2835 & -0.2137 \\ 1.7285 & 0.6730 & -0.8559 & -1.5560 & 0 & -0.8115 & 0.8357 & -0.1391 \\ 0 & 0 & 0 & 0 & -1.5560 & 0 & 0 & 0 \\ \hline 0 & 0 & -1.8151 & -0.9292 & 0 & -0.6757 & -0.0640 & -0.7620 \\ 0 & 0 & 0.7703 & 0.3943 & 0 & 0.0606 & -0.3266 & 0.0794 \\ 0 & 0 & -0.2258 & -0.1156 & 0 & -0.2875 & 0.9087 & 0.2124 \end{array}\right]$$

The corresponding maximising vectors (scaled Schmidt pair) are:

$$x \stackrel{s}{=} \left[\begin{array}{cccc|ccc} -1.5560 & -1.7285 & -0.6730 & 0.8559 & -0.8115 & 0.8357 & -0.1391 \\ 0 & -1.1737 & -2.0211 & 2.2347 & 0.7872 & -1.3233 & 0.1813 \\ 0 & 1.3187 & -0.5016 & 2.5052 & 0.6889 & -0.0319 & 0.7426 \\ 0 & -0.8849 & -0.3445 & -1.1179 & -0.8803 & -0.4058 & -0.4528 \\ \hline 2.1154 & 1.0755 & 0.9413 & -1.5230 & 0.7148 & -0.3219 & 0.6208 \\ -1.5328 & -1.0658 & 0.3062 & -1.1109 & -0.0000 & 0.8877 & 0.4603 \end{array}\right]$$

and

$$y \stackrel{s}{=} \left[\begin{array}{cccc|cc} -1.1737 & -2.0211 & 0.5587 & -0.8580 & -0.5603 & 1.4067 \\ 1.3187 & -0.5016 & 0.6263 & -0.9619 & -0.9852 & 0.0953 \\ -3.5396 & -1.3781 & -1.1179 & -0.6730 & 0.0397 & 1.6976 \\ 0 & 0 & 0 & -1.5560 & -2.8761 & 0.6933 \\ -0.5717 & -1.0054 & 0.0715 & -0.9292 & -0.9966 & 0 \\ \hline 0.4827 & 0.1086 & 0.8547 & 0.3943 & 0.0782 & -0.3040 \\ -1.3578 & -0.0678 & -0.6395 & -0.1156 & 0.0250 & 0.9527 \end{array}\right]$$

Pick $\delta = 0.3609$. *The following perturbation is constructed such that* $\|x^T \Delta y\|_\infty = \frac{1-\delta}{s_1} = 0.6242 < 0.9767 =: \frac{1}{s_1}$ *and* $\|\Delta\|_\infty = \mu(\delta) = 1.8729$.

$$\Delta \stackrel{s}{=} \left[\begin{array}{ccc|ccc} -1.4651 & 3.6437 & 0.1326 & -0.5491 & 1.1746 & -1.9955 \\ 0.6349 & -3.1794 & -0.5857 & 0.0431 & -0.8775 & -0.3038 \\ 0.5241 & -4.3411 & -1.3983 & -0.4441 & 0.6327 & 1.6165 \\ \hline -0.0383 & -0.1137 & -0.0359 & 0 & 0 & 0 \\ -0.0171 & 0.0589 & 0.1457 & 0 & 0 & 0 \\ -0.0329 & -0.0127 & -0.1445 & 0 & 0 & 0 \end{array}\right]$$

The table below shows that this perturbation is destabilising for the optimal controller K_{opt}. The table summarises the poles of the four closed-loop transfer functions. Among them there is one unstable pole for all transfer functions and hence the closed-loop is unstable.

$p_i(H_{11})$	$p_i(H_{12})$	$p_i(H_{21})$	$p_i(H_{22})$
-3.9522+0i	-3.9522+0i	-1.1718+2.3745i	-1.1718+2.3745i
-3.0238+0i	-3.0238+0i	-1.1718-2.3745i	-1.1718-2.3745i
-1.1718+2.3745i	-1.1718+2.3745i	-3.0238+0i	-3.0238+0i
-1.1718-2.3745i	-1.1718-2.3745i	-3.9522+0i	-3.9522+0i
0.00045493+0i	**0.00045493+0i**	-0.50675+0.20291i	-0.50675+0.20291i
-0.50675+0.20291i	-0.50675+0.20291i	-0.50675-0.20291i	-0.50675-0.20291i
-0.50675-0.20291i	-0.50675-0.20291i	**0.00045493+0i**	**0.00045493+0i**

However, when connecting the super-optimal controller K_{sopt}, as theory suggests, the perturbed closed-loop is stabilised because the perturbation lies inside the ball of the extended permissible perturbation set. In this case, as shown in the table below, the four closed-loop transfer functions have stable poles.

$p_i(H_{11})$	$p_i(H_{12})$	$p_i(H_{21})$	$p_i(H_{22})$
-3.9498+0i	-3.9498+0i	-1.2457+2.3798i	-1.2457+2.3798i
-2.9929+0i	-2.9929+0i	-1.2457-2.3798i	-1.2457-2.3798i
-1.1718+2.3741i	-1.1718+2.3741i	-0.30174+0i	-0.30174+0i
-1.1718-2.3741i	-1.1718-2.3741i	-1.1718+2.3741i	-1.1718+2.3741i
-0.017093+0i	-0.017093+0i	-1.1718-2.3741i	-1.1718-2.3741i
-0.50228+0.19514i	-0.50228+0.19514i	-3.9498+0i	-3.9498+0i
-0.50228-0.19514i	-0.50228-0.19514i	-2.9929+0i	-2.9929+0i
-1.2457+2.3798i	-1.2457+2.3798i	-0.017093+0i	-0.017093+0i
-1.2457-2.3798i	-1.2457-2.3798i	-0.50228+0.19514i	-0.50228+0.19514i
-0.30174+0i	-0.30174+0i	-0.50228-0.19514i	-0.50228-0.19514i
-1.556+0i	-1.556+0i	-1.556+0i	-1.556+0i

7.6 Spectral Radius Constraint

Throughout this chapter, mainly for simplicity reasons, we selected to constrain the norm of the perturbation set along a specified worst direction (equation 7.10). This is essentially a structural constraint, which was proved to be efficient for the purposes of extending the robust stability radius in other directions. However, Lemma (7.2.2) suggests that there are more degrees of freedom that can be exploited, by means of imposing alternative-type constraints. One possibility is to formulate the directionality constraint simultaneously in terms of a norm-constraint and a spectral radius constraint (7.11). The analysis of this problem is similar to the previous case of CRSP except for additional technical complications in the solution of the corresponding distance to singularity argument.

In addition, formulating appropriate bounds in terms of the spectral radius constraint appears to be difficult problem in comparison to the norm constraint (which in the case of block-diagonal uncertainty models is equivalent to the solution of an LMI [JH+06]). For this reason we will not pursue this problem further on the thesis and the efficient solution to the problem in terms of simultaneous norm and spectral radius constraints will be a topic of future research.

7.7 Structures and the extended robust stability

One of the major results of this chapter (Theorem and Proposition 7.4.1) is the fact that any super-optimal controller guarantees stability not only to every perturbed plant inside the $\epsilon^\star$-ball, but further to all (additively) perturbed plants which lie outside the $\epsilon^\star$-ball and inside the $\mu^\star(\delta)$-ball provided the corresponding perturbations possess a given "structure" specified by parameter δ. In other words, the class of the later perturbations has norm greater or equal to $\epsilon^\star$ and the perturbations are "structured" because they are constrained along a certain direction (see (7.13)). We should note here that $\epsilon^\star$-ball is the optimal ball and hence the uncertainty set cannot be extended norm-wise (as $\epsilon^\star$ would not be optimal if this was the case). However, the only way to include perturbations of norm greater than $\epsilon^\star$ in the non-destabilising permissible uncertainty set (and hence extend the uncertainty set) is by imposing constraints on the structure of perturbations (7.12). Hence, δ quantifies the imposed structure in relative terms, i.e. as a percentage of $\epsilon^\star$. Clearly a super-optimal controller is at least as good as any other optimal controller for all uncertainties inside the $\epsilon^\star$ ball. If, however, the uncertainty set is poorly modelled and the true plant happens to lie outside the $\epsilon^\star$ optimal ball, then there is a possibility that the super-optimal controller will stabilise the true plant. However, this is an "unstructured uncertainty" point of view. In the following paragraph we discuss the same problem from the viewpoint of how close this set can approximate sets of structured perturbations.

7.7.1 Structured set approximation

The structured maximally robust stabilisation problem is posed as follows:

Problem 7.7.1. *Maximise the robust stability radius of G_0, over the family of all stabilising controllers, corresponding to the permissible uncertainty set of all additive perturbations possessing a structure. Equivalently, we write the problem as:*

$$\max_K \{r \ : (G_0 + \Delta, K) \in \mathcal{S}, \ \ \forall \Delta \in \mathcal{D}_r^{structure}\}$$

where

$$\mathcal{D}_r^{structure} = \{\Delta : \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty < r, \ \eta(G_0) = \eta(G_0 + \Delta), \text{ and } \Delta \text{ preserves the defined structure on its elements}\}$$

or in the light of definition 5.1.1,

$$\mathcal{D}_r^{structure} = \{\Delta : \Delta \in \mathcal{D}_r \cap \mathcal{D}^{structure}\}$$

where $\mathcal{D}^{structure}$ denotes the set of all Δ with a given structure. □

Examples of $\mathcal{D}_r^{structure}$ are normalised perturbations of diagonal form (structured singular value μ, [PD93]), perturbations with norm bounds on blocks of their elements (the generalised μ, see [CFN96a], [CFN96b]), or any other set with a spatial structure. Let r_0 be a maximiser of problem 7.7.1. Then obviously,

$$r_0 \geq \epsilon^\star := \sigma_n(\Gamma_{G_0})$$

Theorem 7.7.1. *Consider a (normalised) structured uncertainty set*

$$\mathrm{B}\mathcal{D}^{structure} = \{\Delta \in \mathcal{D}^{structure} : \|\Delta\|_\infty \leq 1\} \subseteq \mathcal{H}_\infty$$

where $\mathcal{D}^{structure}$ denotes an arbitrary structure (e.g. block diagonal,etc.).

Suppose that

$$\phi_o = \max_{\Delta \in \frac{1}{s_1}\mathbf{B}\mathcal{D}^{structure}} \|x^T \Delta y\|_\infty$$

and set $\delta_o := 1 - \phi_o$. Then $\mu^\star(\delta_o)$ is a **lower bound** *of the maximum structured robust stability radius (relative to the given structure), i.e.*

$$\mu^\star(\delta_o) \leq \sup\{\mu : \exists K s.t. (G_0 + \Delta, K) \in \mathcal{S},\ \forall \Delta \in \mathcal{D}^{structure}, \|\Delta\| < \mu\}$$

Further, any $K \in \mathcal{K}_2$ guarantees a structured robust stability radius of at least $\mu^\star(\delta_o)$, and $\mu^\star(\delta_o) > \epsilon^\star$ (the maximal unstructured robust stability radius) provided that $\delta_o \neq 0$.

Proof. Follows immediate from theorem 7.4.1 and proposition 7.4.3 since:

$$\{\Delta \in \mathcal{D}^{structure} : \|\Delta\| < \mu^\star(\delta_o)\} \subseteq \mathcal{E}(\delta_o, \mu^\star(\delta_o))$$

and all $\Delta \in \mathcal{E}(\delta_o, \mu^\star(\delta_o))$ are stabilised by every $K \in \mathcal{K}_2$. □

Note that for comparison (compatibility) reasons, in the above theorem we scale by $\frac{1}{s_1}$ the normalised set. In that line, the theorem essentially states that the extended robust stability, $\mu^\star(\delta_o)$, serves as a better upper bound the structured robust stabilisation problem, than the maximal robust stability $\epsilon^\star$. In the case where a designer ignores the structure of the uncertainty (because the problem is unsolvable, difficult or for any other reason) the extended robust stability radius is then closer to the structured robust stability radius. In addition, if the structure is somehow "compatible" with the artificial structure imposed in problem 7.2.2 then this upper bound can be tight. Now, some examples supporting the ideas discussed above are presented.

Compatibility issues

Throughout this chapter, we considered the following three sets:

$$\mathcal{D}_r^0 = \left\{\Delta := \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} : \|\Delta\|_\infty \leq \frac{1}{s_1}\right\}$$

$$\mathcal{D}_r^\delta = \left\{\Delta := \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} : \|\Delta_{11}\|_\infty \leq \frac{1-\delta}{s_1} \text{ and } \|\Delta\|_\infty \leq \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)}\right\}$$

$$\mathcal{D}_r^1 = \left\{\Delta := \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} : \|\Delta\|_\infty \leq \frac{1}{s_1 s_{l+1}}\right\}$$

As it was shown before, $\mathcal{D}_r^0 \subseteq \mathcal{D}_r^\delta$ and $\mathcal{D}_r^1 \subseteq \mathcal{D}_r^\delta$. The question examined in the following examples, is how compatible are the above sets (whose stability radius is well known) with a structured set.

Example 7.7.1. *Assume that $G(s)$ is a diagonal system with multiplicity 2 on its smallest Hankel singular value and that the first two (distinct) super-optimal levels are $s_1 = 0.5$, $s_{l+1} = 0.2$. Because of the diagonal form of the system we make the extra assumption that the Schmidt pair corresponding to the first super-optimal level (s_1) are equal to $x^T = y' = \begin{bmatrix} I_2 & 0 \end{bmatrix}$, times an all-pass function. Further, suppose the uncertainty set is described by additive perturbations Δ with norm bound $\|\Delta\|_\infty < \frac{1}{s_1}$. Then the maximal robust stability radius is given by $\epsilon^\star = \frac{1}{s_1}$. However, suppose the uncertainty is actually of the form*

$$G_{true}(s) = G(s) + \left(\begin{array}{cc|c} \sin\theta & \cos\theta & * \\ -\cos\theta & \sin\theta & * \\ \hline * & * & * \end{array}\right)$$

where "$*$" denotes terms which remain unstructured. Then

$$\|x^T \Delta y\|_\infty = \left\| \begin{bmatrix} I_2 & 0 \end{bmatrix} \left(\begin{array}{cc|c} \sin\theta & \cos\theta & * \\ -\cos\theta & \sin\theta & * \\ \hline * & * & * \end{array}\right) \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \right\|_\infty = \left\| \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \right\|_\infty = 1$$

Solving in terms of δ the equation:

$$\|x^T \Delta y\|_\infty = \frac{1-\delta}{s_1}$$

we define $\delta := 1 - s_1$. Then, the extended robust stability radius is

$$\mu^\star(\delta) = \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)} = \sqrt{\frac{1 + s_{l+1} - s_1}{s_1 s_{l+1}}} = 3.5 \geq \epsilon^\star$$

Remark 7.7.1. *From theory, the Schmidt pair is always point-wise orthogonal which supports the choice in the above example. Assuming that $x^T = y' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$ we would get $\delta = 0.1910$ and hence $\mu^\star(\delta) = 2.2685$, still larger than $\epsilon^\star$.* □

Example 7.7.2 (Complex diagonal structure $\mathcal{DH}_\infty^+$). *Consider the unit ball of $\mathcal{DH}_\infty^+$,*

$$\mathrm{B}\mathcal{DH}_\infty^+ = \left\{ \Delta = \begin{pmatrix} \delta_1(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_n(s) \end{pmatrix} : \ \delta_i \in \mathcal{H}_\infty^{+,1\times 1},\ \|\delta_i\|_\infty \leq 1 \right\}$$

Assume $s_1(R) > s_2(R) > 0$, $l = 1$ and let $x_i^T(s)$ and $y_i(s)$ denote the i-th element of x^T and y, respectively. Let also $z_i(s) = x_i^T(s)y_i(s)$. Then

$$\max_{\Delta \in \frac{1}{s_1}\mathbf{B}\mathcal{DH}_\infty^+} \|x^T \Delta y\|_\infty = \max_{\Delta \in \frac{1}{s_1}\mathbf{B}\mathcal{DH}_\infty^+} \|\sum_{i=1}^n \delta_i x_i^T y_i\|_\infty$$
$$= \max_{|\delta_i| \le \frac{1}{s_1}} \|\sum_{i=1}^n \delta_i z_i\|_\infty$$

The maximum is attained on the boundary. Hence, write now each δ_i in a polar form $\delta_i = \frac{1}{s_1}e^{j\phi_i}$.

$$\max_{|\delta_i| \le \frac{1}{s_1}} \|\sum_{i=1}^n \delta_i z_i\|_\infty = \max_{\phi_i \in [0,2\pi)} \max_{\omega \in \mathcal{R}} |\sum_{i=1}^n \frac{1}{s_1} e^{j\phi_i} z_i(j\omega)|$$
$$= \frac{1}{s_1} \max_{\phi_i \in [0,2\pi)} \max_{\omega \in \mathcal{R}} |\sum_{i=1}^n e^{j\phi_i} z_i(j\omega)|$$
$$= \frac{1}{s_1} \max_{\omega \in \mathcal{R}} \max_{\phi_i, \in [0,2\pi)} |\sum_{i=1}^n e^{j\phi_i} z_i(j\omega)|$$
$$= \frac{1}{s_1} \max_{\omega \in \mathcal{R}} \left\{ |\sum_{i=1}^n z_i(j\omega)| \right\} =: \frac{\phi_o}{s_1}$$

Note that the Cauchy-Schwartz inequality implies that:

$$|\sum_{i=1}^n z_i(j\omega)|^2 = |\sum_{i=1}^n x_i^T(j\omega)y_i(j\omega)|^2 \le \left(\sum_{i=1}^n |x_i^T(j\omega)|^2\right)\left(\sum_{i=1}^n |y_i(j\omega)|^2\right) = 1$$

and hence $\phi_o \le 1$. Setting $\delta_o = 1 - \phi_o$ shows that

$$\mu^\star(\delta_o) = \sqrt{\frac{1}{s_1}\left(\frac{\delta_o}{s_2} + \frac{1-\delta_o}{s_1}\right)}$$

is a lower bound of the maximal robust stability radius relative to $\mathcal{DH}_\infty^+$. Note that $\mu^\star(\delta_o) > \epsilon^\star = s_1^{-1}$ for every $\delta_o \neq 0$ and that $\gamma_{max} := \max_{\omega \in \mathcal{R}} \{|\sum_{i=1}^n z_i(j\omega)|\}$ can be easily obtained (e.g. graphically). □

The examples above show that *a priori* information about the structure of the perturbations set implies better robustness. Then, of course, it is only arguable of how "robust" results can someone achieve using the artificial structure in (7.12) (i.e. how tight as an upper bound), compared to the robust stabilisation problem were the true structure is fully exploited. In this line, it should be noted that links between the complex structured singular value - a highly structured set - and the discussed structure were reported in [GHJ00] and [JH+06], for the constant matrix case.

7.8 Summary

In this chapter we considered additive perturbation models where the smallest Hankel singular value of the nominal plant had multiplicity $l > 1$. It is known from previous chapters that the maximal robust stability is achieved by designing controllers which minimise the infinity norm of the control sensitivity transfer function; these functions are described by the term "optimal" and the problem of maximising the robust stability radius is known as the maximally robust stabilisation problem. The solution to the later problem involves a Nehari approximation whose solutions are obtained via an all-pass dilation technique.

In this chapter it was proved that on the boundary of the ball there exist perturbations which are destabilising for all optimal feedback systems and therefore are called *uniformly destabilising*. Further, *all* such perturbations have a projection of magnitude equal to the maximal robust stability radius (r_1) along a worst direction determined by the maximal Schmidt vectors of the associated Hankel operator. It is proved that all frequencies are equally critical, in the sense that the Nyquist criterion can be violated at an arbitrary frequency $\omega \in \mathcal{R}$ by constructing appropriate boundary perturbations. Hence, the only way to extend the uncertainty set is on the basis of its structure. It is shown that by imposing a parametric constraint on this projection, $\|x^T \Delta y\|_\infty \leq r_1(1-\delta)$, $\delta \in [0,1)$, i.e. partially characterising an "optimal" structure to define permissible perturbations, the uncertainty set can be extended to a ball with radius up to

$$\mu^\star(\delta) = \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_{l+1}} + \frac{1-\delta}{s_1}\right)} \geq r_1$$

This is the maximum possible extended robust stability region of perturbed plants. It was shown that every super-optimal controller $K \in \mathcal{K}_2$ guarantees stability for any perturbation lying inside the $\mu^\star(\delta)$-ball. Finally, the chapter has presented an interpretation of this result in the case of perturbations with arbitrary structure and a method which guarantees robust stabilisation in this case using super-optimal controllers.

Chapter 8

Robust Stabilisation Under Coprime Perturbations

Until this point we have considered uncertainty on the nominal plant to arise as an additive perturbation. Of course, modelling uncertainty is not a trivial task and thus there are several types of unstructured uncertainty used in the modelling process, depending on the application. A popular type for modelling uncertainty is to consider all admissible perturbed plants, around the nominal plant, expressed in terms of stable co-prime factors. Although this may seem to be an "artificial" way for modelling uncertainty, it possesses certain advantages over other perturbation types. Its main advantage over additive and multiplicative types of uncertainty is the complete removal of technical assumption (7.1) which means that the nominal and perturbed plants are allowed to have different number of poles in the right half plane. Co-prime factors uncertainty in the framework of robust stabilisation problem was first studied in [Vid85] and thereafter the theory was successfully applied to various problems by [MG90]. The theoretical results developed throughout this chapter aim to derive stronger solutions to the maximally robust stabilisation problem in the multivariable case.

8.1 Introduction

The control setup we are interested in is shown in figure 8.1. Consider the following generalised plant:

$$P = \left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right) := \left(\begin{array}{c|c} 0 & I \\ \widetilde{M}^{-1} & G \\ \hline \widetilde{M}^{-1} & G \end{array}\right) \tag{8.1}$$

where $P_{22} := G$ denotes the plant and admits normalised lcf and rcf:

$$G = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$$

respectively. Recall that the coprime factors satisfy the following Diophantine equations:

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{8.2}$$

where $N, \widetilde{N}, M, \widetilde{M}, U, \widetilde{U}, V, \widetilde{V} \in \mathcal{RH}_\infty$. Further, it is assumed here without loss of generality that the coprime factors are normalised and hence,

$$\widetilde{M}\widetilde{M}^{\sim} + \widetilde{N}\widetilde{N}^{\sim} = I, \quad M^{\sim}M + N^{\sim}N = I \tag{8.3}$$

are satisfied as well.

Definition 8.1.1. *A perturbation on the nominal plant G is said to be permissible for the control setup in figure 8.1, if it can be written as $\Delta := \begin{pmatrix} \Delta_N & \Delta_M \end{pmatrix} \in \mathcal{D}_{\mathcal{S}_\epsilon}$ where*

$$\mathcal{D}_{\mathcal{S}_\epsilon} := \{\Delta : \Delta \in \mathcal{RH}_\infty, \quad \|\Delta\|_\infty < \epsilon\}.$$

□

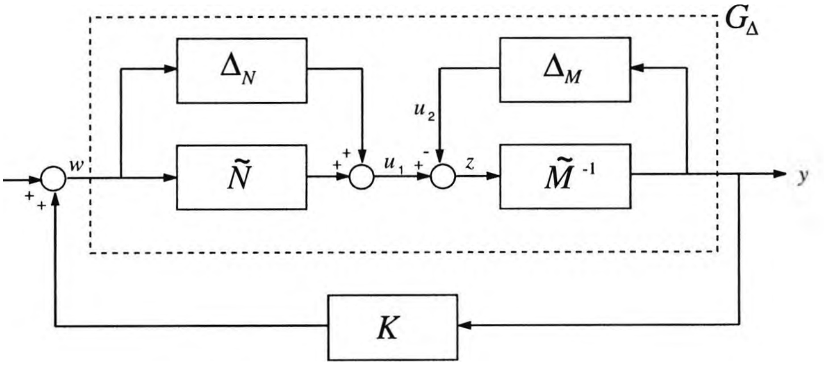

Figure 8.1: Closed-loop system under stable perturbations

It can be verified that the perturbed plant G_Δ of figure 8.1 is the same with the upper LFT of the generalised plant ([MG90], [ZDG96], [GL95]):

$$\mathcal{F}_u(P, \Delta) = (\widetilde{M} + \Delta_M)^{-1}(\widetilde{N} + \Delta_N)$$

as shown in figure 5.4(a), of chapter 5. Further, the lower LFT is

$$\mathcal{F}_l(P,K) = \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1}\widetilde{M}^{-1}$$

Definition 8.1.2. *Consider the (nominal) feedback system (G,K) in Figure 8.1 with $\Delta = 0$. If (G,K) is internally stable we say that K stabilises G and write $K \in \mathcal{K}$, or equivalently $(G,K) \in \mathcal{S}$. Further, (G,K) is said to be ϵ-robustly stable if and only if $(G_\Delta, K) \in \mathcal{S}$ for every $\Delta \in \mathcal{D}_{S_\epsilon}$.* □

In the framework of generalised regulator problem 2.4.1, the next theorem gives necessary and sufficient conditions for the coprime perturbations robust stabilisation problem:

Theorem 8.1.1 (Robust Stabilisation). *[Vid85], [MG90] Let $G \in \mathcal{RL}_\infty$ admit left and right coprime factorisations $G = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$, respectively. Then (G,K) is ϵ-robustly stable if and only if $(G,K) \in \mathcal{S}$ and $\|T\|_\infty \le \epsilon^{-1}$ where*

$$T = \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1}\widetilde{M}^{-1}$$

is the corresponding closed-loop system.

Proof. See [MG90], Theorem 3.3. □

It follows immediately that the robust stability radius ϵ is maximised by solving:

$$\epsilon^\star = \left(\inf_{K \in \mathcal{K}} \|T\|_\infty \right)^{-1}$$

If a controller K stabilises G then it is well-known that it can be written in the bilinear form $K = (U + MQ)(V + NQ)^{-1}$, where $Q \in \mathcal{H}_\infty$ and all other terms are defined as in equation (8.2). Then, the closed-loop transfer function is equal to

$$\begin{aligned} T &= \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1}\widetilde{M}^{-1} \\ &= \begin{pmatrix} K \\ I \end{pmatrix} (V + NQ) = \begin{pmatrix} U + MQ \\ V + NQ \end{pmatrix} \\ &= \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q \end{aligned}$$

Following the standard procedure developed in chapter 5 ([Fra87], [Glo86]) the maximum robust stabilisation problem can be reduced to a Nehari approximation, by considering, the following sequence of norm preserving transformations:

$$
\begin{aligned}
\|T\|_\infty &= \left\| \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q \right\|_\infty =: \|T_1 + T_2 Q\|_\infty \\
&= \left\| T_1 + \begin{pmatrix} T_2 & T_\perp \end{pmatrix} \begin{pmatrix} Q \\ 0 \end{pmatrix} \right\|_\infty \\
&= \left\| \begin{pmatrix} T_2^\sim \\ T_\perp^\sim \end{pmatrix} T_1 + \begin{pmatrix} Q \\ 0 \end{pmatrix} \right\|_\infty
\end{aligned}
$$

where we defined $T_2 := \begin{pmatrix} M \\ N \end{pmatrix}$, $T_\perp := \begin{pmatrix} -\widetilde{N}^\sim \\ \widetilde{M}^\sim \end{pmatrix}$ and have used equation (8.2). Thus:

$$
\begin{aligned}
\|T\|_\infty &= \left\| \begin{pmatrix} M^\sim & N^\sim \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} Q \\ 0 \end{pmatrix} \right\|_\infty \\
&= \left\| \begin{pmatrix} M^\sim U + N^\sim V \\ I \end{pmatrix} + \begin{pmatrix} Q \\ 0 \end{pmatrix} \right\|_\infty =: \gamma
\end{aligned} \tag{8.4}
$$

so that

$$
\|M^\sim U + N^\sim V + Q\|_\infty = \sqrt{\gamma^2 - 1}
$$

and

$$
\gamma_{opt} := \inf_{Q \in \mathcal{H}_\infty} \|T\|_\infty = \sqrt{(\|M^\sim U + N^\sim V\|_H^2 + 1)} = (\epsilon^\star)^{-1} \tag{8.5}
$$

using the Nehari theorem.

Hence, the computation of the maximal stability radius involved in the co-prime factor robust stabilisation problem reduces to a Nehari approximation problem. A state-space parametrisation of all optimal solutions to this problem is well known (see [Glo84]). This solution proceeds via the derivation of a state-space realisation of $R := M^\sim U + N^\sim V$.

Proposition 8.1.1. *[MG90] Assume $G \in \mathcal{RL}_\infty$ with a (minimal) state-space realisation:*

$$
G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right]
$$

Then G has an essentially unique normalised right and left co-prime factor represen-

tation $G = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$ *(i.e. unique up to multiplication by unitary matrices from the right and left, respectively). Further,*

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - BB'X & B & ZC' \\ \hline -B'X & I & 0 \\ C & 0 & I \end{array}\right]$$

and

$$\begin{pmatrix} \widetilde{V} & -\widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - ZC'C & B & ZC' \\ \hline B'X & I & 0 \\ -C & 0 & I \end{array}\right]$$

where X *and* Z *are the unique stabilising solutions of the algebraic Riccati equations:*

$$A'X + XA - XBB'X + C'C = 0$$

and

$$AZ + ZA' - ZC'CZ + BB' = 0$$

respectively.

Proof. See chapter 3, Proposition 3.1.2. Also, see [MG90], Propositions 2.21 and 2.22. □

Straight substitution from the Lemma above shows that:

$$R \stackrel{s}{=} \left[\begin{array}{c|c} -A' + XBB' & -(I + XZ)C' \\ \hline B' & 0 \end{array}\right] \tag{8.6}$$

after using an appropriate similarity transformation to remove the unobservable part. Thus $R \in \mathcal{RH}_\infty^-$. The following theorem gives equivalent conditions for robust stabilisability.

Theorem 8.1.2. *A controller* K *stabilises* $G = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$ *and satisfies*

$$\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1}\widetilde{M}^{-1} \right\|_\infty \leq \gamma$$

if and only if either condition 1 *or* 2 *below holds:*

1. $\|R\|_H \leq \sqrt{\gamma^2 - 1}$
2. $\left\| \begin{pmatrix} -\widetilde{N}^\sim \\ \widetilde{M}^\sim \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} \right\|_\infty \leq \sqrt{1 - \gamma^{-2}}$

Proof. The equivalence with condition 2 is proved in [MG90]. Then using norm preserving transformations the first claim can also be proved. □

8.2 Optimal and Super-optimal approximations

As shown in the previous section the set of all internally stabilising controllers $\mathcal{K}$ and the corresponding set of all stable closed-loop systems $\mathcal{T}$ are parameterised as:

$$\mathcal{K} = \{(U + MQ)(V + NQ)^{-1} : Q \in \mathcal{H}_\infty\}$$

and

$$\mathcal{T} = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q : Q \in \mathcal{H}_\infty \right\}$$

respectively. In order to parameterise the set of all optimal (maximally robust) controllers $\mathcal{K}_1 \subseteq \mathcal{K}$ and the corresponding set of optimal closed-loop systems $\mathcal{T}_1 \subseteq \mathcal{T}$, we need to solve a Nehari approximation problem, defined in equation (8.4). The set of all optimal solutions is parameterised in the following theorem.

Assumption 8.2.1. *To simplify notation it is assumed throughout this chapter that the largest singular value of Γ_R is simple (non-repeated).*

Theorem 8.2.1 (Optimal Nehari approximation). *Consider $R \in \mathcal{RH}_\infty^{-;\, p\times m}$ with realisation $(A_R, B_R, C_R, 0)$ defined in equation (8.6) with $\lambda(A_R) \subseteq \mathcal{C}_+$. Then there exists $Q_a \in \mathcal{RH}_\infty^{(p+m-1)\times(p+m-1)}$ such that all $Q \in \mathcal{H}_\infty^{p\times m}$, for which $\|R + Q\|_\infty = \|\Gamma_R\| = s_1$ (Nehari optimal approximations of R), are given by*

$$Q = \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_\infty^{(p-1)\times(m-1)})$$

The corresponding "error" system is given by

$$H := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \left(\begin{array}{c|c} R + Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right) \tag{8.7}$$

where $\|H_{22}\|_\infty < s_1$. Further, $HH^\sim = H^\sim H = s_1^2 I_{p+m-1}$.

Proof. See (as in chapter 6) [Glo84]; see also [JL93] for a more general setting. □

It then follows from Theorem 8.2.1 that

$$\mathcal{K}_1 = \{(U + MQ)(V + NQ)^{-1} : Q \in \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_\infty)\}$$

and

$$\mathcal{T}_1 = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q : Q \in \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_\infty) \right\} \tag{8.8}$$

A more revealing parametrisation of $\mathcal{T}_1$ for our purposes can be obtained via the method used to construct super-optimal approximations (see chapter 6 and 7). Before stating this parametrisation we need the following result.

Theorem 8.2.2. *Consider everything as defined in Theorem 8.2.1. Then*

(i) There exist square inner matrix functions:

$$V = \begin{pmatrix} v & V_\perp \end{pmatrix} \quad \text{and} \quad W^\sim = \begin{pmatrix} w^\sim \\ W_\perp^\sim \end{pmatrix}$$

such that

$$\begin{pmatrix} V^\sim & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} s_1\alpha & 0 \\ 0 & \bar{H} \end{pmatrix}$$

where $\alpha(s)$ is scalar anti-inner.

(ii) $\bar{H}$ can be decomposed as

$$\bar{H} = \hat{R} + \bar{Q}_a := \begin{pmatrix} \hat{R} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{pmatrix}$$

where $\hat{R} \in \mathcal{RH}_\infty^{-,(p-1)\times(m-1)}$ and $\bar{Q}_{ij} \in \mathcal{RH}_\infty$.

(iii) $s_2(R) = \|\Gamma_{\hat{R}}\|$.

(iv) All s_1-suboptimal approximations of $\hat{R}$ are generated as $\bar{Q} = \mathcal{F}_l(\bar{Q}_a, s_1^{-1}\, \mathcal{BH}_\infty^{(p-1)\times(m-1)})$.

Proof. See chapter 6, Theorem 6.1.3. □

It follows from the definition of R and equation (8.4) that generating the optimal approximations of T implicitly requires the parametrisation of all optimal approximation of R. The following theorem exploits Theorems 8.2.1 and 8.2.2 and gives a pseudo-diagonal decomposition of the set $\mathcal{T}_1$:

Theorem 8.2.3 (Optimal and Super-Optimal Decompositions). *Consider everything as defined in Theorems 8.2.1 and 8.2.2. Then*

(i) The set of all optimal closed-loop transfer functions, $\mathcal{T}_1$ can be parameterised as:

$$\mathcal{T}_1 = Y \begin{pmatrix} s_1 a(s) & 0 \\ 0 & \hat{R} + \mathcal{F}_l(\bar{Q}_a, s_1^{-1}\, \mathcal{BH}_\infty^{(p-1)\times(m-1)}) \\ 1 & 0 \\ 0 & I \end{pmatrix} X$$

where

$$Y := \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \quad \text{and } X = W^{\sim}$$

are square all-pass.

(ii) *The set of all level-2 super-optimal closed-loop transfer functions, $\mathcal{T}_2$ can be parameterised as:*

$$\mathcal{T}_2 = Y_1 \begin{pmatrix} s_1 a(s) & 0 & 0 \\ 0 & s_2 b(s) & 0 \\ 0 & 0 & \check{R} + \mathcal{S}_2(R) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_m \end{pmatrix} X_1$$

where Y_1 (X_1) has common first column (row) with Y (X). Further, both Y_1 and X_1 are square all-pass and $\mathcal{S}_2(R)$ and $\check{R}$ are as defined in Lemma 7.1.1.

(iii) *The first two super-optimal levels of $\mathcal{T}$ are $(\sqrt{s_1^2+1}, \sqrt{s_2^2+1})$ where (s_1, s_2) are the first two super-optimal levels of R.*

Proof. See appendix F. □

Corollary 8.2.1. *The maximum robust stability radius is given by*

$$\epsilon^\star = \frac{1}{\sqrt{s_1^2+1}}$$

Proof. Immediate from Theorems 8.1.1 and 8.2.3. □

8.3 Uniformly destabilising perturbations

It is well known that co-primeness (as a property) of a perturbed plant is eventually lost outside the r_1-ball, where r_1 is the maximum robust stability radius ([Vid85],[MG90]); i.e. the minimal distance $\delta = \|\Delta_N \;\; \Delta_M\|$ for which $(N + \Delta_N, M + \Delta_M)$ fails to be a co-prime pair cannot be smaller than r_1. This is natural as such perturbed systems cannot be stabilised by any compensator, as they have a pole/zero cancellation in the RHP. Moreover, it is well known that in this case $\delta > r_1$ is general. Of course, under such perturbations the plant is the *nearest unstabilisable* plant and in [MG90], p.59, a simple example verifies this brief discussion. In particular, Vidyasagar constructs such perturbations in [Vid85], pp. 280-281. At the present work we do not intend to discuss any further this phenomenon. Nevertheless, we are interested in characterising all *uniformly destabilising* perturbations which, as it will be shown later, exist on the boundary of r_1-ball and which cannot be stabilised by any optimal controller. Formally we define,

Definition 8.3.1. *A $\Delta \in \partial\mathcal{D}_{S_{\epsilon^*}}$ is called uniformly destabilising if $(G+\Delta, K) \notin \mathcal{S}$ for every $K \in \mathcal{K}_1$.* □

Example 8.3.1 (Destabilising perturbation). *Suppose $p = m = n = 2$, and assume a $T \in \mathcal{T}_1$. Recall from Lemma 8.2.3 that such T admits factorisation $T = Y\hat{T}X$, s.t. $XX^{\sim} = Y^{\sim}Y = I$. Take*

$$\hat{\Delta} = \left(\begin{array}{cc|cc} a(s)^{\sim}\frac{s_1}{s_1^2+1} & 0 & \frac{1}{s_1^2+1} & 0 \\ \frac{1}{s_1^2+1} & 0 & -a(s)\frac{s_1}{s_1^2+1} & 0 \end{array}\right)$$

Clearly, $\|\hat{\Delta}\|_\infty = \frac{1}{\sqrt{s_1^2+1}}$, i.e. it lies on the boundary of r_1-ball. Define $\Delta := X^{\sim}\hat{\Delta}Y^{\sim}$ and pick any frequency $\omega_o \in \mathcal{R}$ so that,

$$det(I - \Delta T)(j\omega_o) = det(I - X^{\sim}\hat{\Delta}Y^{\sim}Y\hat{T}X)(j\omega_o)$$
$$= det(I - XX^{\sim}\hat{\Delta}Y^{\sim}Y\hat{T})(j\omega_o) = det(I - \hat{\Delta}\hat{T})(j\omega_o)$$

but

$$(\hat{\Delta}\hat{T})(j\omega_o) =$$
$$\frac{1}{s_1^2+1}\begin{pmatrix} s_1 a(j\omega_o)^{\sim} & 0 & 1 & 0 \\ 1 & 0 & -s_1 a(j\omega_o) & 0 \end{pmatrix}\begin{pmatrix} s_1 a(j\omega_o) & 0 \\ 0 & \left(\hat{R} + \mathcal{S}_1(R)\right)(j\omega_o) \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So,

$$det(I - \Delta T)(j\omega_o) = det\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Although this Δ may not be permissible (as $\Delta \notin \mathcal{H}_\infty$ in general) it is possible to "interpolate" Δ at ω_o, i.e. define $\tilde{\Delta}(s) \in \mathcal{RH}_\infty$, $\tilde{\Delta}(j\omega_o) = \Delta(j\omega_o)$ and repeat the above argument to show that generalised Nyquist criterion is violated and hence Δ is a destabilising perturbation for T. □

The point of example 8.3.1 suggests that there exist destabilising perturbations to every optimal closed-loop transfer matrix on the boundary of the r_1-ball. The next Lemma establishes this formally and shows that such perturbations can be chosen to be real-rational. The proof of the Lemma (which is omitted) relies on a direct construction of such perturbations using the techniques of [Vid85] (chapter 7). The construction reveals that all frequencies are "equally critical", in the sense that such perturbations can be constructed so that the generalised Nyquist stability criterion of the open-loop perturbed system can be violated at an arbitrary frequency (including zero and infinity).

Lemma 8.3.1 (Existence). *There exists* $\Delta = \begin{pmatrix} \Delta_N & \Delta_M \end{pmatrix} \in \partial\mathcal{D}_{S_{\epsilon^*}}$ *such that* $\{(\widetilde{M}+\Delta_M)^{-1}(\widetilde{N}+\Delta_N), K\} \notin \mathcal{S}$ *for every* $K \in \mathcal{K}_1$. *Furthermore,* Δ *can be chosen to be a stable real-rational matrix function.*

Proof. See Lemma 7.2.1, chapter 7. □

Lemma 8.3.2. *Consider the two vectors:*

$$\xi(s) = \begin{pmatrix} M(s) \\ N(s) \end{pmatrix} v(s) \text{ and } \psi(s) = \begin{pmatrix} -\widetilde{N}^{\sim}(s) \\ \widetilde{M}^{\sim}(s) \end{pmatrix} w(s)$$

where $v(s)$ *and* $w(s)$ *are the first columns of* $V(s)$ *and* $W(s)$*, respectively, defined in Theorem 8.2.2. Then*

- *(i)* $\xi(s), \psi(s) \in \mathcal{RL}_\infty$, $\xi^{\sim}(s)\xi(s) = \psi^{\sim}(s)\psi(s) = 1$ *and* $(\xi(s), \psi(s))$ *are point-wise orthogonal, i.e.* $\psi^{\sim}(s)\xi(s) = 0$.
- *(ii) Let* $y_{sc}(s) := \frac{1}{\sqrt{s_1^2+1}}(s_1 a(s)\xi(s) + \psi(s))$. *Then* $y_{sc}(s) \in \mathcal{RL}_\infty$ *and* $y_{sc}^{\sim}(s)y_{sc}(s) = 1$.

Proof. *(i)* The result follows immediately from the fact that

$$\begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} M^{\sim} & N^{\sim} \\ -\widetilde{N} & \widetilde{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

(see equations (8.2) and (8.3)).

(ii) The result follows since

$$\begin{aligned} y_{sc}^{\sim} y_{sc} &= \frac{1}{s_1^2+1}(s_1 a^{\sim}\xi^{\sim} + \psi^{\sim})(s_1 a\xi + \psi) \\ &= \frac{1}{s_1^2+1}(s_1^2\xi^{\sim}\xi + \psi^{\sim}\psi) = 1 \end{aligned}$$

using the fact that $a^{\sim}(s)a(s) = a(-s)a(s) = 1$ since $a(s)$ is scalar anti-inner and using the point-wise orthogonality of $(\xi(s), \psi(s))$ established in part (i). $\square$

The next Lemma shows that a necessary condition for a $\Delta \in \partial\mathcal{D}_{\mathcal{S}_{\epsilon^\star}}$ to be uniformly destabilising is that it is aligned with a particular direction at an arbitrary frequency. For simplicity we continue to assume that $s_1(R) > s_2(R) > 0$ (according to assumption 8.2.1).

Lemma 8.3.3. *Let $\Delta \in \partial\mathcal{D}_{\mathcal{S}_{\epsilon^\star}}$ be a uniformly destabilising perturbation of G. Then, there exists an $\omega_o \in \mathcal{R} \cup \{\infty\}$ such that*

$$\|x^T \Delta y_{sc}\|_\infty = x^T(j\omega_o)\Delta(j\omega_o)y_{sc}(j\omega_o) = \epsilon^\star$$

Proof. Take any $\Phi \in \mathcal{H}_\infty$ with $\|\Phi\| < s_1$ so that $\Phi \in s_1^{-1}\mathcal{BH}_\infty^{(p-1)\times(m-1)}$ and define the controller $K = (U + MQ)(V + NQ)^{-1}$ where $Q = \mathcal{F}_l(Q_a, \Phi)$. Further consider the corresponding closed-loop system

$$T = Y \begin{pmatrix} s_1 a(s) & 0 \\ 0 & \Psi \\ 1 & 0 \\ 0 & I \end{pmatrix} X$$

as defined in Theorem 8.2.3. Here we define $\Psi = \hat{R} + \mathcal{F}_l(\bar{Q}_a, \Phi)$. Clearly $K \in \mathcal{K}_1$ and $T \in \mathcal{T}_1$. Since $\beta\Delta$ is an admissible stabilising perturbation for every $\beta \in [0, 1)$ and $(G_\Delta, K) \notin \mathcal{S}$, there exists $\omega_o \in \mathcal{R} \cup \{\infty\}$ such that $\det(I - \Delta(j\omega_o)T(j\omega_o)) = 0$. Hence, the matrix

$$I - \Delta(j\omega_o)Y(j\omega_o) \begin{bmatrix} s_1 a(j\omega_o) & 0 \\ 0 & \Psi(j\omega_o) \\ 1 & 0 \\ 0 & I \end{bmatrix} X(j\omega_o)$$

is singular. Partition $X(j\omega_o)$ and $Y(j\omega_o)$ as follows:

$$\begin{bmatrix} x_1^T \\ X_\perp \end{bmatrix} =: X(j\omega_o)$$

and

$$
\begin{aligned}
Y(j\omega_o) &= \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} v & V_\perp & 0 & 0 \\ 0 & 0 & w & W_\perp \end{pmatrix} (j\omega_o) \\
&= \left(\begin{pmatrix} M \\ N \end{pmatrix} \begin{pmatrix} v & V_\perp \end{pmatrix} \middle| \begin{pmatrix} -\widetilde{N}^{\sim} \\ \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} w & W_\perp \end{pmatrix} \right) (j\omega_o) \\
&=: \left[\begin{array}{cc|cc} y_1 & Y_{1\perp} & y_2 & Y_{2\perp} \end{array} \right]
\end{aligned}
$$

Note that $y_1 = \xi(j\omega_o)$ and $y_2 = \psi(j\omega_o)$ where $\xi(s)$ and $\psi(s)$ are defined in Lemma 8.3.2. Further, define:

$$
\begin{aligned}
\tilde{\Delta}(j\omega_o) &= \begin{pmatrix} \tilde{\delta}_{11} & \tilde{\delta}_{12} & \tilde{\delta}_{13} & \tilde{\delta}_{14} \\ \tilde{\delta}_{21} & \tilde{\delta}_{22} & \tilde{\delta}_{23} & \tilde{\delta}_{24} \end{pmatrix} \\
&=: \begin{pmatrix} x_1^T \\ X_\perp \end{pmatrix} \Delta(j\omega_o) \left(\begin{array}{cc|cc} y_1 a(j\omega_o) & Y_{1\perp} & y_2 & Y_{2\perp} \end{array} \right)
\end{aligned}
$$

Next, introduce a suitable permutation matrix P which interchanges the second and third block columns of $\tilde{\Delta}(j\omega_o)$, i.e. define

$$
\hat{\Delta} = \tilde{\Delta}(j\omega_o)P = \begin{pmatrix} \tilde{\delta}_{11} & \tilde{\delta}_{13} & \tilde{\delta}_{12} & \tilde{\delta}_{14} \\ \tilde{\delta}_{21} & \tilde{\delta}_{23} & \tilde{\delta}_{21} & \tilde{\delta}_{24} \end{pmatrix}
$$

Clearly $\|\tilde{\Delta}(j\omega_o)\| = \|\hat{\Delta}\| = \epsilon^\star$. Then, singularity of $I - \Delta(j\omega_o)T(j\omega_o)$ is now equivalent to:

$$
\det \left\{ I_m - \begin{pmatrix} \tilde{\delta}_{11} & \tilde{\delta}_{12} & \tilde{\delta}_{13} & \tilde{\delta}_{14} \\ \tilde{\delta}_{21} & \tilde{\delta}_{22} & \tilde{\delta}_{23} & \tilde{\delta}_{24} \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & \Psi(j\omega_o) \\ 1 & 0 \\ 0 & I \end{pmatrix} \right\} = 0
$$

or

$$
\det \left\{ \begin{pmatrix} 1 - \tilde{\delta}_{11}s_1 - \tilde{\delta}_{13} & -(\tilde{\delta}_{12}\Phi(j\omega_o) + \tilde{\delta}_{14}I) \\ -(\tilde{\delta}_{21}s_1 + \tilde{\delta}_{23}) & I_{m-1} - \tilde{\delta}_{22}\Phi(j\omega_o) - \tilde{\delta}_{24}I \end{pmatrix} \right\} = 0
$$

Assume now for contradiction that

$$
1 - \tilde{\delta}_{11}s_1 - \tilde{\delta}_{13} \neq 0 \tag{8.9}
$$

Then, expanding the determinant,

$$
\begin{aligned}
&(1 - \tilde{\delta}_{11}s_1 - \tilde{\delta}_{13})\det\{I_{m-1} - \tilde{\delta}_{22}\Psi(j\omega_o) - \tilde{\delta}_{24}I - \\
&(\tilde{\delta}_{21}s_1 + \tilde{\delta}_{23})(1 - \tilde{\delta}_{11}s_1 - \tilde{\delta}_{13})^{-1}(\tilde{\delta}_{12}\Psi(j\omega_o) + \tilde{\delta}_{14}I)\} = 0
\end{aligned}
$$

or by using the permutation introduced above, we can write in LFT terms:

$$\det \left\{ I_{m-1} - \mathcal{F}_u \left(\hat{\Delta}, \begin{pmatrix} s_1 \\ 1 \end{pmatrix} \right) \begin{pmatrix} \Psi(j\omega_o) \\ I \end{pmatrix} \right\} = 0 \tag{8.10}$$

Now, from standard LFT contractive properties:

$$\left\| \mathcal{F}_u \left(\hat{\Delta}, \begin{pmatrix} s_1 \\ 1 \end{pmatrix} \right) \right\| \leq \epsilon^\star = \frac{1}{\sqrt{s_1^2 + 1}}$$

Also, $\|\Psi(j\omega_o)\| < s_1$ by assumption, which implies that

$$\left\| \begin{pmatrix} \Psi(j\omega_o) \\ I \end{pmatrix} \right\| = \sqrt{\|\Psi(j\omega_o)\|^2 + 1} < \sqrt{s_1^2 + 1} = (\epsilon^\star)^{-1}$$

and so,

$$\left\| \mathcal{F}_u \left(\hat{\Delta}, \begin{pmatrix} s_1 \\ 1 \end{pmatrix} \right) \begin{pmatrix} \Psi(j\omega_o) \\ I \end{pmatrix} \right\| < 1$$

which contradicts the singularity of the matrix in (8.10). Thus contrary to the initial assumption,

$$1 - \tilde{\delta}_{11} s_1 - \tilde{\delta}_{13} = 0 \Rightarrow \tilde{\delta}_{11} s_1 + \tilde{\delta}_{13} = 1$$

or by direct substitution,

$$\begin{aligned} x_1^T \Delta(j\omega_o) \left(y_1 a(j\omega_o) s_1 + y_2 \right) &= 1 \\ \Rightarrow x^T(j\omega_o) \Delta(j\omega_o) y_{sc}(j\omega_o) &= \frac{1}{\sqrt{s_1^2 + 1}} = \epsilon^\star \end{aligned}$$

where $x^T(s)$ denotes the first row of $X(s)$ and $y_{sc}(s)$ is defined in Lemma 8.3.2. Thus $\|x^T(s)\Delta(s)y_{sc}(s)\|_\infty \geq \epsilon^\star$. However, since $x^\sim x = y_{sc}^\sim y_{sc} = 1$ and $\|\Delta\|_\infty = \epsilon^\star$ we conclude that $\|x^T \Delta y_{sc}\|_\infty = \epsilon^\star$. □

Remark 8.3.1. *We can interpret the condition $x^T(j\omega_o)\Delta(j\omega_o)y_{sc}(j\omega_o) = \epsilon^\star$ as follows: Define an inner product over $\mathcal{C}^{p\times m}$ (the space of $p \times m$ complex matrices) as:*

$$\langle A, B \rangle = trace(B^\star A)$$

whenever $A, B \in \mathcal{C}^{p\times m}$. Then we can write:

$$\begin{aligned} x^T(j\omega_o)\Delta(j\omega_o)y_{sc}(j\omega_o) &= trace\left(y_{sc}(j\omega_o) x^T(j\omega_o) \Delta(j\omega_o) \right) \\ &= \langle \Delta(j\omega_o), E_o \rangle = \epsilon^\star \end{aligned}$$

where $E_o := x(-j\omega_o)y'_{sc}(-j\omega_o)$, which means that Δ has a projection of $\epsilon^\star$ in the direction defined by E_o. □

8.4 Extended robust stability radius

Lemma 8.3.3 shows that all uniformly destabilising perturbations Δ are constrained to have a projection equal to $\epsilon^\star$ along the *fixed* direction $x(-j\omega_o)y'_{sc}(-j\omega_o)$ at some frequency ω_o. This means that it is impossible to extend the robust stability radius along this direction, using a subset of all maximally robust controller $\mathcal{K}_1$ (assume that we still want to stabilise all $\Delta \in \mathcal{D}_{\mathcal{S}_{\epsilon^\star}}$). Moreover, all frequencies are equally critical, in the sense that we can construct uniformly destabilising perturbations such that the generalised Nyquist criterion is violated at an arbitrary frequency. Thus, we can only hope to extend the robust stability radius (beyond $\epsilon^\star$) at directions other than $\langle \cdot, x(-j\omega)y'_{sc}(-j\omega)\rangle$, $\omega \in \mathcal{R} \cup \{\infty\}$.

To motivate the formulation of an optimisation problem which allows us to extend the robust stability radius in all directions (other than the "most critical" direction), consider the following "distance to singularity" problem:

Let A be a $n \times n$ complex non-singular matrix with singular value decomposition $A = U\Sigma V^* = \sum_{i=1}^{n} \sigma_i u_i v_i^*$ with $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{n-2} \geq \sigma_{n-1} > \sigma_n > 0$. What is the minimum norm perturbation $\|E\|$ such that $A - E$ is singular? It is well known that the unique solution is given by the rank 1 matrix $E_o = \sigma_n u_n v_n^*$ so that $\|E_o\| = \sigma_n$. Thus in this case $u_n^* E_o v_n = \sigma_n$ or $\langle u_n v_n^*, E_o\rangle = \sigma_n$. Thus E_o has a projection σ_n in the most critical direction $\langle u_m v_n^*, \cdot\rangle$. Suppose now that we constrain the magnitude of the projection of allowable perturbations in this direction, i.e. impose the restriction that

$$|\langle u_n v_n^*, E_o\rangle| \leq \phi$$

for some non-negative constant $\phi \leq \sigma_n$. Since now the new minimum-norm singularising perturbation cannot have a projection of magnitude σ_n in the most-critical direction, we expect the constrained optimal distance to singularity $\gamma(\phi)$ to be larger than σ_n; further, the tighter the constraint (ϕ decreases), the more $\gamma(\phi)$ should deviate from σ_n. The full solution to the problem is provided by Lemma 7.3.1, in chapter 7.

Remark 8.4.1. *In Lemma 7.3.1, σ_n and σ_{n-1} are fixed and so the constrained distance to singularity $\gamma(\phi)$ is a function only of ϕ. Suppose that somehow we could influence the level of σ_{n-1}, assuming that σ_n and ϕ are fixed. Then, in order to maximise $\gamma(\phi)$, we would have to maximise σ_{n-1}, i.e. make the gap $\sigma_{n-1} - \sigma_n$ as large as possible, an observation which motivates super-optimisation used later in the section.* □

Motivated by the above result we proceed as follows: Suppose we impose a structure on the permissible uncertainty set, by defining the set:

$$\mathcal{E}(\delta, \mu) = \left\{\Delta \in \mathcal{D}_{S_\mu} \ : \ \|x^T \Delta y_{sc}\|_\infty \leq (1-\delta)\epsilon^\star\right\}$$

where

$$\mathcal{D}_{S_\mu} = \{\Delta \in \mathcal{H}_\infty : \|\Delta\|_\infty < \mu\}$$

Then we formulate the following optimisation problem:

Problem 8.4.1 (Constrained maximum robust stabilisation). *For a fixed δ, $0 \leq \delta \leq 1$, find all K that solve:*

$$\max\{\mu : (G_\Delta, K) \in \mathcal{S} \textit{ for all } \Delta \in \mathcal{E}(\delta, \mu) \cup \mathcal{D}_{S_{\epsilon^\star}}\}$$

and the corresponding maximum value $\mu = \mu^\star(\delta)$. □

Remark 8.4.2. *(i) Note that since we still require that all $\Delta \in \mathcal{D}_{S_{\epsilon^\star}}$ are stabilised, the set of optimal controllers which solve CMRS must be a subset of $\mathcal{K}_1$. (ii) When $\delta = 0$ the constraint $\|x^T \Delta y_{sc}\|_\infty \leq (1-\delta)\epsilon^\star$ is redundant (i.e. no structure is imposed) and thus $\mathcal{E}(0, \mu) = \mathcal{D}_{S_\mu}$; hence in this case the solution to the CMRS problem is trivial and is given by $\mathcal{K}_{opt} = \mathcal{K}_1$ and $\mu^\star(0) = \epsilon^\star$.* □

The solution of the CMRS problem is summarised in the last theorem of this chapter. Note that (s_1, s_2) denote the first two super-optimal levels of R and we assume that $s_1 > s_2$. Further, $\mathcal{K}_1$ denotes the set of all optimal (maximally robust) controllers and $\mathcal{K}_2$ the set of all super-optimal controllers with respect to the first two levels, so that $\mathcal{K}_2 \subseteq \mathcal{K}_1$. First consider a parametrisation of the families of optimal and super-optimal (level-2) feedback loops, in terms of the first super-optimal singular values.

Theorem 8.4.1 (Closed Loop Decompositions - Alternative Form). *Consider the decompositions in Theorem 8.2.3. Then,*

(i) The set of all optimal closed-loop systems may be parameterised as:

$$\mathcal{T}_1 = \Theta(\hat{Q}; s) \begin{pmatrix} \sqrt{s_1^2 + 1} & 0 \\ 0 & M(\hat{Q}; s) \end{pmatrix} X(s)$$

i.e. in terms of its first super-optimal singular value. Here $\Theta(\hat{Q}; s) \in \mathcal{L}_\infty^{(p+m)\times(m)}$ satisfies $\Theta^\sim\Theta = I_m$ and $X(s)$ is square allpass, as previously defined in Theorem 8.2.3.

Further, as $\hat{Q}$ varies in the set

$$\mathcal{S}_1 = \{\hat{Q} \in \mathcal{H}_\infty : \|\hat{R} + \hat{Q}\|_\infty \leq s_1\}$$

$M(\hat{Q}; s)$ is well-defined and satisfies $1 \leq \underline{\sigma}(M(\hat{Q}; j\omega)) \leq \|M(\hat{Q}; s)\|_\infty \leq \sqrt{s_1^2 + 1}$ for all $\omega \in \mathcal{R}$.

(ii) The set of all level-2 super-optimal closed-loop systems may be parameterised as:

$$\mathcal{T}_2 = \Theta_1(\check{Q}; s) \begin{pmatrix} \sqrt{s_1^2+1} & 0 & 0 \\ 0 & \sqrt{s_2^2+1} & 0 \\ 0 & 0 & M(\check{Q}; s) \end{pmatrix} X_1(s)$$

i.e. in terms of its first two distinct super-optimal singular values. Here $\Theta(\check{Q}; s) \in \mathcal{L}_\infty^{(p+m)\times(m)}$ satisfies $\Theta^\sim\Theta = I_m$ and $X_1(s)$ is square allpass, as previously defined in Theorem 8.2.3. Further, as $\check{Q}$ varies in the set

$$\mathcal{S}_2 = \{\check{Q} \in \mathcal{H}_\infty : \|\check{R} + \check{Q}\|_\infty \leq s_2\}$$

$M(\check{Q}; s)$ is well-defined and satisfies $1 \leq \underline{\sigma}(M(\check{Q}; j\omega)) \leq \|M(\check{Q}; s)\|_\infty \leq \sqrt{s_2^2 + 1}$ for all $\omega \in \mathcal{R}$.

Remark 8.4.3. *The above decompositions are reminiscent of partial singular value decompositions for constant matrices. The term $M(\hat{Q}; s)$ (and similarly $M(\check{Q}; s)$) appearing in the diagonal in the above form is essentially a spectral factor of $I + (\hat{R} + \hat{Q})^\sim(\hat{R} + \hat{Q})$, following the notation of Theorem 8.2.3. Hence it can always be assumed a minimum-phase stable system (i.e. $M(\hat{Q}; s), M(\hat{Q}; s)^{-1} \in \mathcal{RH}_\infty$) without loss of generality.*

Proof. (i) For any $\hat{Q} \in \mathcal{S}_1$ perform the spectral factorisation:

$$\Phi := I + (\hat{R} + \hat{Q})^\sim(\hat{R} + \hat{Q}) = M^\sim(\hat{Q}; s)M(\hat{Q}; s) \tag{8.11}$$

which is well-defined since $\Phi(j\omega) > 0$, for every $\omega \in \mathcal{R}$ (e.g. see [ZDG96], Corollary

13.20). Routine algebra verifies that

$$\mathcal{T}_1 = Y(s)\begin{pmatrix} \frac{s_1 a(s)}{\sqrt{s_1^2+1}} & 0 \\ 0 & (\hat{R}+\hat{Q})M^{-1}(\hat{Q};s) \\ \frac{1}{\sqrt{s_1^2+1}} & 0 \\ 0 & M^{-1}(\hat{Q};s) \end{pmatrix}\begin{pmatrix} \sqrt{s_1^2+1} & 0 \\ 0 & M(\hat{Q};s) \end{pmatrix} X(s)$$

$$= Y(s)\begin{pmatrix} s_1 a(s) & 0 \\ 0 & \hat{R}+\hat{Q} \\ 1 & 0 \\ 0 & I \end{pmatrix} X(s)$$

where $\hat{Q} \in \mathcal{S}_1$. Further, in terms of the above form writing $\Theta(\hat{Q}, s) = Y(s)P(\hat{Q}, s)$ we get that $P^{\sim}(\hat{Q}, s)P(\hat{Q}, s) = I_m$ and hence $\Theta^{\sim}(\hat{Q}, s)\Theta(\hat{Q}, s) = I_m$. Setting $s = j\omega$ in (8.11) gives:

$$\Phi(j\omega) := I + (\hat{R}+\hat{Q})^*(\hat{R}+\hat{Q})(j\omega) = M^*(\hat{Q}; j\omega)M(\hat{Q}; j\omega)$$

and so

$$\sigma_i^2\left[M(\hat{Q}; j\omega)\right] = \lambda_i\left[M^*(\hat{Q}; j\omega)M(\hat{Q}; j\omega)\right] = 1 + \sigma_i^2\left[(\hat{R}+\hat{Q})(j\omega)\right] \geq 1$$

and the claim is proved.

(ii) This is essentially identical to the proof of part *(i)* but using the form of $\mathcal{T}_2$ given in Theorem 8.2.3. Here

$$\Theta := \begin{pmatrix} \vartheta_1(s) & \vartheta_2(s) & \Theta_\perp(\check{Q}, s) \end{pmatrix}$$

$$= Y_1(s)\begin{pmatrix} \frac{s_1 a(s)}{\sqrt{s_1^2+1}} & 0 & 0 \\ 0 & \frac{s_1 b(s)}{\sqrt{s_2^2+1}} & 0 \\ 0 & 0 & (\check{R}+\check{Q})M^{-1}(\check{Q};s) \\ \frac{1}{\sqrt{s_1^2+1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{s_2^2+1}} & 0 \\ 0 & 0 & M^{-1}(\hat{Q};s) \end{pmatrix}$$

□

As shown in the proof of Theorem 8.4.1, both $\Theta(\hat{Q}; s)$ and $\Theta_1(\hat{Q}; s)$ have their first column common:

$$\vartheta_1 = \frac{1}{\sqrt{s_1^2+1}}\left(y_1 s_1 a(s) + y_2\right)$$

This vector coincides with y_{sc} (as defined in Lemma 8.3.2) which essentially shows how this construction is connected with the imposed directionality constraint. Moreover, ϑ_2 is formed in a similar line and hence, the first two columns of $\Theta_1(\hat{Q}; s)$ are clearly independent of the choice of $\check{Q} \in \mathcal{S}_2$. Exploiting this and the fact that the spectral factor (in the super-optimal decomposition) has norm less than s_2, it is possible to apply the already known theory of chapter 7, based on distance to singularity arguments (recall Theorem 7.4.1 and set l=1) and thus derive refined robust stability properties.

Theorem 8.4.2. *In previously defined notation the following statements hold:*

1. *For each* $\delta \in [0, 1]$,

$$\mu^\star(\delta) = \sqrt{\frac{1}{\sqrt{s_1^2+1}}\left(\frac{\delta}{\sqrt{s_2^2+1}} + \frac{1-\delta}{\sqrt{s_1^2+1}}\right)} \geq \epsilon^\star$$

with equality only in the case $\delta = 0$. *Here* $\sqrt{s_1^2+1}$ *and* $\sqrt{s_2^2+1}$ *are the first two (distinct) super-optimal levels of* $\mathcal{T}$ *with* $\sqrt{s_1^2+1} = (\epsilon^\star)^{-1}$.

2. *For each* $0 < \delta \leq 1$ *the following two statements are equivalent:*

   (a) $\left((\widetilde{M} + \Delta_M)^{-1}(\widetilde{N} + \Delta_N), K\right) \in \mathcal{S}$ *for every* $\Delta \in \mathcal{D}_{\mathcal{S}_{\epsilon^\star}} \cup \mathcal{E}(\delta, \mu^\star(\delta))$,

   (b) $K \in \mathcal{K}_2$.

3. (a) $\mathcal{E}(0, \mu^\star(0)) = \mathcal{D}_{\mathcal{S}_{\epsilon^\star}}$,

   (b) *for each* $K \in \mathcal{K}_2$,

$$\left((\widetilde{M} + \Delta_M)^{-1}(\widetilde{N} + \Delta_N), K\right) \in \mathcal{S}$$

   for every $\Delta \in \bigcup_{\delta \in [0,1]} \mathcal{E}(\delta, \mu^\star(\delta))$.

Proof. By exploiting the structure of the set of super-optimal closed-loop approximations in Theorem 8.4.1, the proof becomes identical to those of Theorem 7.4.1 and Proposition 7.4.3. □

As expected the constrained robust stability radius $\mu^\star(\delta)$ is a strictly increasing function of δ with $\mu^\star(0) = \epsilon^\star$. Moreover, for a fixed $\delta \neq 0$ and s_1, $\mu^\star(\delta)$ increases as s_2 is reduced. Thus, we may expect a significant increase in $\mu^\star(\delta)$ when the gap between the largest two singular values is significantly large. Further, for each $\delta \neq 0$ the set of optimal controllers is the same, namely $\mathcal{K}_2$. Thus each super-optimal controller guarantees the stability of all perturbations in the union of the sets $\bigcup_{\delta\in[0,1]} \mathcal{E}(\delta, \mu^\star(\delta))$ which contains the the ball of radius $\epsilon^\star$ as a subset.

8.5 Summary

Throughout the chapter the robust stabilisation problem was posed in terms of perturbed plants which admit coprime factorisation. This is called the coprime robust stabilisation problem and its solution involves a series of similar arguments to those presented in chapter 7, i.e. robust stabilisation under additive perturbations. The main difficulty here was to define an appropriate scaling of the maximal Schmidt pair of a sequence of Hankel operators so that the set of all optimal (and super-optimal) closed loop systems admit a "pseudo-diagonal" decomposition where in the diagonal entries the super-optimal singular values of T appear. For this an extra step involving spectral factorisations (depending on the set of all suboptimal approximations) required. In particular, all super-optimal singular values of T are expressed in terms of the super-optimal singular values of the associated system R. Hence, it is shown that the maximal robust stability radius is

$$\epsilon^\star = \frac{1}{\sqrt{s_1^2(R)+1}} = \frac{1}{s_1(T)}$$

Further, using the appropriately scaled Schmidt vectors, it is shown that all destabilising perturbations of norm $\epsilon^\star$ have a projection of the same magnitude along a particular worst-direction. Hence it is possible to extend the uncertainty set which is non-destabilising for a subset of optimal controllers, in the same way as in chapter 7, i.e. by imposing a parametric constraint on the allowable projection of perturbations along this direction. The problem is then reduced to a constrained maximal robust stabilisation problem whose solution has been derived in chapter 7, and hence the proof is based on the same arguments. The extended robust stability radius is given in terms of the first two closed-loop super-optimal singular values

$$\mu^\star(\delta) = \sqrt{\frac{1}{\sqrt{s_1^2+1}}\left(\frac{\delta}{\sqrt{s_2^2+1}} + \frac{1-\delta}{\sqrt{s_1^2+1}}\right)} \geq \epsilon^\star$$

Moreover, as in chapter 7, it is proved that any perturbed plant lying inside the $\mu^\star(\delta)$-ball is stabilisable by each level-2 super-optimal controller $\mathcal{K}_2$. Note that in contrast to chapter 7, throughout this chapter the assumption of non-repeated singular values was made mainly to avoid messy indexing. In view of the theory developed in chapter 7, this assumption can be removed without any serious technical difficulties.

As a final comment, the similarity of results in chapters 7 and 8 motivates future research on a unified approach to the robust stabilisation problem under various unstructured perturbation models (including weighting functions), which can be described by a general LFT framework.

Conclusion

In control theory, many design problems may be recast as $\mathcal{H}_\infty$ problems, one of which is central to the present work, namely the maximally robust stabilisation problem for classes of unstructured perturbations. Initially, when no uncertainty enters into the model, it is desired to design a controller that stabilises the nominal plant. In $\mathcal{H}_\infty$ terms, we seek to find a stabilising controller that minimises the $\mathcal{H}_\infty$-norm of the closed-loop transfer-function. This can be achieved using the model-matching theory and Youla parametrisation, a methodology which characterises the family of all stabilising controllers and it is studied in chapter 3. Nevertheless, uncertainty always appears in real processes. By choosing the additive uncertainty scheme, in chapter 5 the robust stabilisation problem is posed. Using elements from approximation theory (Nehari approximation) which are developed in chapter 4, an explicit solution to the problem is given, with emphasis to the extreme case of destabilising perturbations lying on the boundary of the uncertainty ball (figure 5.3). It is further proved in chapters 7 and 8 that all such perturbations have a worst "projection" along the same direction. In chapter 5 an explicit expression for the maximum robust stability radius is derived by solving the maximally robust stabilisation problem and characterising the optimal solution set by means of a linear fractional map. All optimal closed-loops and maximally robust controllers are further obtained in a closed (state-space) form.

Independently, the problem of super-optimisation was considered, motivated by the fact that it is a form of hierarchical optimisation which restores uniqueness of solutions of the Nehari approximation problem in the matrix case (chapter 6). Exploiting the first two (distinct) super-optimal levels it is proved in chapter 7 that the maximally robust stabilisation problem admits a larger class of permissible perturbations, when a certain structure is imposed (7.12), (7.13). As a result this "superset" of non-destabilising perturbations is characterised explicitly and conditions for robust stability are derived (theorem 7.4.1). Then, the class of super-optimal controllers is identified as the subset of all maximally robust controllers which guarantees robust stability for this extended set of non-destabilising perturbations. All results of chapter 7 involve the additive unstructured perturbation model (assumption 7.1.1) whereas equivalent results in chapter 8 are derived in the set up of the coprime unstructured perturbations model (assumption 8.2.1).

Another point of view of the extended robust stability was presented in chapter 7. An abstract notion of structured set approximation was defined, in terms of the artificially structured set in (7.13) and supported via numerical examples. The idea is to extend the results to robust stabilisation problems involving arbitrary uncertainty structures and obtain tight bounds of the structured robust stabilisation using the developed technique.

Future Directions

- It is intended to generalise the work of the last two chapters in a general LFT framework which considers all kinds of unstructured perturbations. Weighting (performance) matrices will be also considered, which may lead to formulation of two-block or four-block problems. Then a direct comparison of the different unstructured uncertainty models in a (real) control design will show the quantitative difference of the robust stability radius which may occur between the models.

- In chapter 7 we defined an "artificial" structure to the extended permissible uncertainty set (equation (7.13)) and showed by examples that if other structures are compatible with this set, then the extended robust stability radius (theorem 7.4.1) can be a tight upper bound to the structured robust stabilisation problem. More effort is needed in this direction, by extending the idea of structured set approximation (problem 7.7.1) for other structures and quantifying the gap between these sets using this upper bound.

- As a special case of the above point, in [GHJ00] and [JH$^+$06] it was argued that this method can be used to compute a tractable upper bound of the structured singular value, by avoiding the D-iteration. However, this idea has only been explored for constant matrices and we intend to extend it to the dynamic case.

- The extended robust stability properties offered by super-optimal controllers can be used for closed-loop system identification. This involves the redefinition of the nominal plant such that it is in the "centre" of the extended set with radius $\mu^\star(\delta)$. Connections between closed-loop identification and $\mathcal{H}_\infty$-control are already established (see [GBC+99],[Hja05],[DdH05]) which motivates the present objective.

- Links with $\mathcal{H}_\infty$-based loop-shaping methods ([MG90]) related to directionality can be explored. Note that this method relies on the re-definition of the open-loop system using weighting functions and defines the controller via the solution of a maximally robust coprime-factor $\mathcal{H}_\infty$ controller (for the weighted system). The objective used here may be applied to define a more refined measure of robust stability on the case of structured uncertainty.

- The problem of simultaneous $\mathcal{H}_\infty$ stabilisation has been treated by many authors, among them those in [SGKP02], where the problem is tackled by defining an appropriate nominal plant via the solution of a two-block Nehari extension problem, for which a maximally robust controller is subsequently design in the hope that all plants which need to be stabilised are enclosed inside the guaranteed robust stability region. Again the extended robust stability radius provided by the method proposed in this work is potentially useful for deriving stronger simultaneous stabilisation conditions.

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Appendix A

Linear Fractional Transformations

Linear fractional transformation are also known as *Möbius transformations*. Let $P \in \mathcal{R}^{(m_1+m_2)\times(p_1+p_2)}$ be partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Then we define two classes of *linear fractional transformations* (LFTs) as the maps

$$\mathcal{F}_l(P, \cdot) : \mathcal{R}^{p_2 \times m_2} \longrightarrow \mathcal{R}^{m_1 \times p_1}, \quad \mathcal{F}_u(P, \cdot) : \mathcal{R}^{p_1 \times m_1} \longrightarrow \mathcal{R}^{m_2 \times p_2}$$

defined by

$$\mathcal{F}_l(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

$$\mathcal{F}_l(P, K) := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}$$

for some matrices K,Δ of appropriate dimensions. Here, the existence of the inverses is assumed to be well-defined.

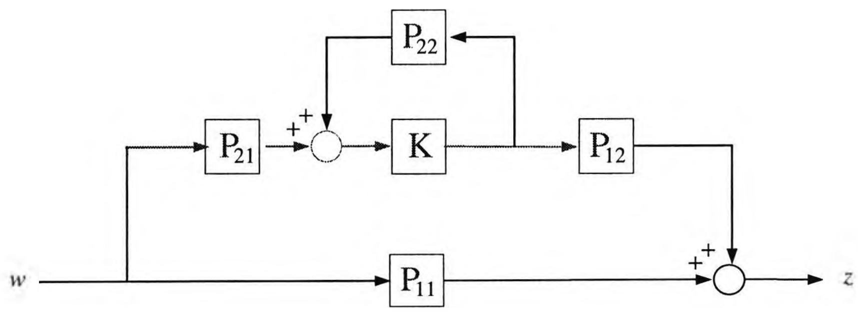

Figure A.1: A lower LFT interconnection representing a transfer function from w to z.

Example A.0.1. *Consider the transfer matrix*

$$G(s) = D + C(sI - A)^{-1}B = D + C\frac{1}{s}(I - A\frac{1}{s}I)^{-1}B = \mathcal{F}_u(P, \frac{1}{s}I)$$

where $P_{11} := A$*,* $P_{12} := B$*,* $P_{21} := C$ *and* $P_{22} := D$*.*

Another important class of LFTs is the *Redheffer* or *star product.* For appropriately partitioned matrices P, K such that $(I - P_{22}K_{11})$ is invertible, we define

$$P \star K := \begin{bmatrix} \mathcal{F}_l(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{12}(I - P_{22}K_{11})^{-1}P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}$$

Appendix B

Algebraic Riccati Equations

Let A, Q and R be real n-by-n matrices with Q and R symmetric. The *Algebraic Riccati equation* (ARE) is then defined by the matrix equation:

$$A'X + XA + XRX + Q = 0$$

Associated with the above equation, the *Hamiltonian matrix* is defined by

$$H := \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix} \in \mathcal{R}^{2n \times 2n}$$

Further, define

$$J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

where $J' = J^{-1}$ and so $J^2 = -I$. It follows easily that

$$J^{-1}HJ = -JHJ = -H'$$

and hence H is similar to $-H'$. Therefore, the spectrum of H is symmetric with respect to the imaginary axis. Now, observe that ARE is a quadratic equation and thus it may have many solutions in $\mathcal{C}^{n \times n}$. One of the solutions is when X is real and $A + RX$ is stable (stabilising solution) - if such a solution exists then it is unique. To see this assume for contradiction that there exist X_1 and X_2 such that $A + RX_1$ and $A + RX_2$ are both stable. Then

$$A'X_1 + X_1A + X_1RX_1 + Q = 0$$

and

$$A'X_2 + X_2A + X_2RX_2 + Q = 0$$

Subtracting the two equations gives:

$$A'X_1 + X_1A + X_1RX_1 - A'X_2 - X_2A - X_2RX_2 = 0$$

or

$$(X_1 - X_2)(A + RX_1) + (A + RX_2)'(X_1 - X_2) = 0$$

since R, X_i are symmetric (with $i = 1, 2$). Now both $A + RX_1$ and $A + RX_2$ are stable by assumption, so the only solution of the above equation is when $X_1 - X_2 = 0$. Hence $X_1 = X_2$ and the claim of uniqueness is proved.

Summarising, the spectrum of a Hamiltonian matrix, H, is symmetric with respect to the imaginary axis. Further, under mild assumptions, among all solutions of the ARE there exists a unique solution such that $A + RX$ is stable, which from now on will be denoted by $X = Ric(H)$. It is crucial to realise that if X is the stabilising solution then H has no eigenvalues on the imaginary axis. This can be shown by observing that under a similarity transformation

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} A + RX & R \\ 0 & -(A + RX)' \end{bmatrix}$$

Then, it is clear that

$$\lambda(H) = \lambda(A + RX) \cup \lambda(-(A + RX))$$

For necessary and sufficient conditions for the existence of a stabilising solution and its proof refer to [ZDG96], [Kim97] and [Fra87].

Lemma B.0.1 (Bounded Real Lemma). *Let $G \in \mathcal{RH}_\infty$ with $G(s) = C(sI - A)^{-1}B$ and assume (A,B), (C,A) are stabilisable and detectable, respectively. Then, the following conditions are equivalent*

1. $\|G\|_\infty < \gamma$
2. *The Hamiltonian* $H = \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & A' \end{bmatrix}$ *has no pure imaginary eigenvalues*
3. $H \in dom(Ric)$

Proof. $1 \Leftrightarrow 2$. See [ZDG96], lemma 4.7.

$2 \Leftrightarrow 3$. See [ZDG96], Theorem 13.6. □

Remark B.0.1. *Condition 3 suggests that if (A, B) is stabilisable and (C, A) is detectable then the unique stabilising solution of* ARE

$$A'X + XA + XBB'X + C'C = 0$$

(i.e. there exists X which stabilises $\widehat{A} := A + \gamma^{-2}BB'X$) is positive semi-definite [Kim97], corollary 3.11.

Appendix C

Super-optimisation

C.1 Proof of proposition 6.1.4

Proposition. Define

$$V_\perp := H_{12}\overline{H}_{12}^{-1} \quad \text{and} \quad W_\perp := H_{21}^{\sim}\overline{H}_{21}^{-\sim}$$

Then, $V_\perp$ and $W_\perp^{\sim}$ have, the following realisations:

$$V_\perp \stackrel{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline C_{q1} - s_1^{-1}D_{12}\overline{C}_{q1} & s_1^{-1}D_{12} \end{array}\right]$$

and

$$W_\perp^{\sim} \stackrel{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}\overline{B}_{q1}C_{q2} & B_{q1} - s_1^{-1}\overline{B}_{q1}D_{21} \\ \hline s_1^{-1}C_{q2} & s_1^{-1}D_{21} \end{array}\right]$$

with corresponding controllability and observability gramians:

$$Y_v = -(\overline{R}')^{-1}\overline{P}_2, \qquad X_v = Q_2 - \overline{Q}_2$$

$$Y_w = P_2 - \overline{P}_2, \qquad X_w = -\widehat{P}_1.$$

In particular, the following matrix inequalities hold: $P_2 \geq \overline{P}_2$ and $Q_2 \geq \overline{Q}_2$.

Proof. The proof is broken into separate sections. The first part is verified by simple state-space calculations. This is followed by deriving the gramians for the derived realisations of $V_\perp$ and $W_\perp^{\sim}$.

1.State-space realisations:

$$V_\perp = Q_{12}\overline{Q}_{12}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A_q & B_{q2} \\ \hline C_{q1} & D_{12} \end{array}\right] \left[\begin{array}{c|c} A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline -s_1^{-1}\overline{C}_{q1} & s_1^{-1}I \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A_q & -s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ 0 & A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline C_{q1} & -s_1^{-1}D_{12}\overline{C}_{q1} & s_1^{-1}D_{12} \end{array}\right]$$

$$\stackrel{T}{=} \left[\begin{array}{cc|c} A_q & 0 & 0 \\ 0 & A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline C_{q1} & C_{q1} - s_1^{-1}D_{12}\overline{C}_{q1} & s_1^{-1}D_{12} \end{array}\right]$$

where $T := \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$. By removing the uncontrollable terms,

$$V_\perp \stackrel{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}B_{q2}\overline{C}_{q1} & s_1^{-1}B_{q2} \\ \hline C_{q1} - s_1^{-1}D_{12}\overline{C}_{q1} & s_1^{-1}D_{12} \end{array}\right]$$

Similarly,

$$W_\perp = Q'_{21}\overline{Q}_{21}^{-\sim} \stackrel{s}{=} \left[\begin{array}{c|c} -A'_q & C'_{q2} \\ \hline -B'_{q1} & D'_{21} \end{array}\right] \left[\begin{array}{c|c} -A'_q + s_1^{-1}C'_{q2}\overline{B}'_{q1} & s_1^{-1}C'_{q2} \\ \hline s_1^{-1}\overline{B}'_{q1} & s_1^{-1}I \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -A'_q & s_1^{-1}C'_{q2}\overline{B}'_{q1} & s_1^{-1}C'_{q2} \\ 0 & -A'_q + s_1^{-1}C'_{q2}\overline{B}'_{q1} & s_1^{-1}C'_{q2} \\ \hline -B'_{q1} & s_1^{-1}D'_{21}\overline{B}'_{q1} & s_1^{-1}D'_{21} \end{array}\right]$$

$$\stackrel{T}{=} \left[\begin{array}{cc|c} -A'_q & 0 & 0 \\ 0 & -A'_q + s_1^{-1}C'_{q2}\overline{B}'_{q1} & s_1^{-1}C'_{q2} \\ \hline -B'_{q1} & s_1^{-1}D'_{21}\overline{B}'_{q1} - B'_{q1} & s_1^{-1}D'_{21} \end{array}\right]$$

$$= \left[\begin{array}{c|c} -A'_q + s_1^{-1}C'_{q2}\overline{B}'_{q1} & s_1^{-1}C'_{q2} \\ \hline s_1^{-1}D'_{21}\overline{B}'_{q1} - B'_{q1} & s_1^{-1}D'_{21} \end{array}\right]$$

2a. Controllability and Observability gramian of $V_\perp$: Take the controllability gramian of $V_\perp$. That is

$$(A_q - s_1^{-1}B_{q2}\overline{C}_{q1})Y_v + Y_v(A'_q - s_1^{-1}\overline{C}'_{q1}B'_{q2}) + s_1^{-2}B_{q2}B'_{q2} = 0$$

or

$$(A_q + s_1^{-2}B_{q2}B'_{q2}\overline{Q}_2)Y_v + Y_v(A'_q + s_1^{-2}\overline{Q}B_{q2}B'_{q2}) + s_1^{-2}B_{q2}B'_{q2} = 0$$

Multiply from the left by $\overline{R}'$ and multiply from the right by $\overline{R}$.

$$\overline{R}'(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2)Y_v\overline{R} + \overline{R}'Y_v(A_q' + s_1^{-2}\overline{Q}B_{q2}B_{q2}')\overline{R} + s_1^{-2}\overline{R}'B_{q2}B_{q2}'\overline{R} = 0$$

From proposition 6.1.3, we have

$$(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2) = (\overline{R}')^{-1}(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2})\overline{R}'$$

Thus,

$$(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2})\overline{R}'Y_v\overline{R} + \overline{R}'Y_v\overline{R}(A_q' + s_1^{-2}C_{q2}'C_{q2}\overline{P}_2) + s_1^{-2}\overline{R}'B_{q2}B_{q2}'\overline{R} = 0 \quad \text{(C.1)}$$

Compare this with the all-pass equation:

$$\widehat{A}_q'\overline{Q}_1 + \overline{Q}_1\widehat{A} + \widehat{C}'\widehat{C} = 0$$

written out in full as:

$$(-A_q - s_1^{-2}\overline{P}_2C_{q2}'C_{q2})\overline{Q}_1 + \overline{Q}_1(-A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2) + s_1^{-2}\overline{R}'B_{q2}B_{q2}'\overline{R} = 0 \quad \text{(C.2)}$$

Subtracting (C.1) from (C.2) shows that

$$Y_v = -(\overline{R}')^{-1}\overline{P}_2$$

since the matrix $A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}$ is asymptotically stable (i.e. its spectrum lies in the open left half plane).

Recall first the following Lyapunov equation, derived from all-pass equations $(6.8(iii))_{2,2}$,

$$A_q'Q_2 + Q_2A_q + C_{q1}'C_{q1} + C_{q2}'C_{q2} = 0$$

Further, from all-pass equation $(6.18(i))_{2,2}$ we get the Riccati equation

$$A_q'\overline{Q}_2 + \overline{Q}_2A_q + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 + C_{q2}'C_{q2} = 0$$

Now, let the observability gramian of the derived realisation of $V_\perp$ be X_v. Then:

$$X_v(A_q - s_1^{-1}B_{q2}\overline{C}_{q1}) + (A_q' - s_1^{-1}\overline{C}_{q1}'B_{q2}')X_v + (C_{q1}' - s_1^{-1}\overline{C}_{q1}'D_{12}')(C_{q1} - s_1^{-1}D_{12}\overline{C}_{q1}) = 0$$

Substituting

$$\overline{C}_{q1} = -s_1^{-1}B_{q2}'\overline{Q}_2$$

gives:

$$X_v(A_q + s_1^{-2}B_{q2}B_{q2}'\overline{Q}_2) + (A_q' + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}')X_v$$
$$+ (C_{q1}' + s_1^{-2}\overline{Q}_2B_{q2}D_{12}')(C_{q1} + s_1^{-2}D_{12}B_{q2}'\overline{Q}_2) = 0$$

which can be expanded as:

$$X_vA_q + s_1^{-2}X_vB_{q2}B_{q2}'\overline{Q}_2 + A_q'X_v + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'X_v + C_{q1}'C_{q1}$$
$$+ s_1^{-2}\overline{Q}_2B_{q2}D_{12}'C_{q1} + s_1^{-2}C_{q1}'D_{12}B_{q2}'\overline{Q}_2 + s_1^{-4}\overline{Q}_2B_{q2}D_{12}'D_{12}B_{q2}'\overline{Q}_2 = 0$$

But from $(6.8(ii))_{22}$ we get that $D_{12}'D_{12} = s_1^2I$. Hence,

$$X_vA_q + s_1^{-2}X_vB_{q2}B_{q2}'\overline{Q}_2 + A_q'X_v + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'X_v + C_{q1}'C_{q1}$$
$$+ s_1^{-2}\overline{Q}_2B_{q2}D_{12}'C_{q1} + s_1^{-2}C_{q1}'D_{12}B_{q2}'\overline{Q}_2 + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 = 0$$

In the sequel we show that $X_v = Q_2 - \overline{Q}_2$ is the unique solution of the above equation. The term on the left hand side of the equation can be written as:

$$(Q_2 - \overline{Q}_2)A_q + s_1^{-2}(Q_2 - \overline{Q}_2)B_{q2}B_{q2}'\overline{Q}_2 + A_q'(Q_2 - \overline{Q}_2)$$
$$+ s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'(Q_2 - \overline{Q}_2) + C_{q1}'C_{q1} + s_1^{-2}\overline{Q}_2B_{q2}D_{12}'C_{q1}$$
$$+ s_1^{-2}C_{q1}'D_{12}B_{q2}'\overline{Q}_2 + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2$$

or, equivalently as:

$$Q_2A_q - \overline{Q}_2A_q + s_1^{-2}Q_2B_{q2}B_{q2}'\overline{Q}_2 - s_1^{-1}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 + A_q'Q_2 - A_q'\overline{Q}_2$$
$$+ s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'Q_2 - s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2 + C_{q1}'C_{q1} + s_1^{-2}\overline{Q}_2B_{q2}D_{12}'C_{q1}$$
$$+ s_1^{-2}C_{q1}'D_{12}B_{q2}'\overline{Q}_2 + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'\overline{Q}_2$$

By subtracting the Riccati from the Lyapunov equation we get

$$s_1^{-2}Q_2B_{q2}B_{q2}'\overline{Q}_2 + s_1^{-2}\overline{Q}_2B_{q2}B_{q2}'Q_2 + s_1^{-2}\overline{Q}_2B_{q2}D_{12}'C_{q1} + s_1^{-2}C_{q1}'D_{12}B_{q2}'\overline{Q}_2$$
$$= s_1^{-2}\overline{Q}_2B_{q2}(B_{q2}'Q_2 + D_{12}'C_{q1}) + s_1^{-2}(Q_2B_{q2} + C_{q1}'D_{12})B_{q2}'\overline{Q}_2$$
$$= 0 =: \text{RHS}$$

using all-pass equation $(6.8(v))_{22}$.

2b. Controllability and Observability gramians of $W_\perp^{\sim}$: First note that

$$W_\perp \stackrel{s}{=} \left[\begin{array}{c|c} -A_q' + s_1^{-1}C_{q2}'\overline{B}_{q1}' & s_1^{-1}C_{q2}' \\ \hline s_1^{-1}D_{21}'\overline{B}_{q1}' - B_{q1}' & s_1^{-1}D_{21}' \end{array}\right]$$

implies that

$$W_{\perp}^{\sim} \stackrel{s}{=} \left[\begin{array}{c|c} A_q - s_1^{-1}\overline{B}_{q1}C_{q2} & B_{q1} - s_1^{-1}\overline{B}_{q1}D_{21} \\ \hline s_1^{-1}C_{q2} & s_1^{-1}D_{21} \end{array}\right]$$

Hence, the controllability gramian Y_w of this realisation satisfies:

$$(A_q - s_1^{-1}\overline{B}_{q1}C_{q2})Y_w + Y_w(A_q' - s_1^{-1}C_{q2}'\overline{B}_{q2}') + (B_{q1} - s_1^{-1}\overline{B}_{q1}D_{21})(B_{q1}' - s_1^{-1}D_{21}'\overline{B}_{q1}') = 0$$

Substituting

$$\overline{B}_{q1} := -s_1^{-1}\overline{P}_2C_{q2}'$$

this can be written as:

$$\begin{aligned} &(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2})Y_w + Y_w(A_q' + s_1^{-2}C_{q2}'C_{q2}\overline{P}_2) + B_{q1}B_{q1}' \\ &\quad + s_1^{-2}B_{q1}D_{21}'C_{q2}\overline{P}_2 + s_1^{-2}\overline{P}_2C_{q2}'D_{21}B_{q1}' + s_1^{-4}\overline{P}_2C_{q2}'D_{21}D_{21}'C_{q2}\overline{P}_2 = 0 \end{aligned}$$

From all-pass equation $(6.8(ii))_{2,2}$ we get that $D_{21}'D_{21} = D_{21}D_{21}' = s_1^2I$. When this is substituted in the above equation we get:

$$\begin{aligned} &A_qY_w + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}Y_w + Y_wA_q' + s_1^{-2}Y_wC_{q2}'C_{q2}\overline{P}_2 + B_{q1}B_{q1}' \\ &\quad + s_1^{-2}B_{q1}D_{21}'C_{q2}\overline{P}_2 + s_1^{-2}\overline{P}_2C_{q2}'D_{21}B_{q1}' + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 = 0 \end{aligned}$$

Assume $Y_w = P_2 - \overline{P}_2$. Next we show that with this assumption Y_w satisfies the above equation.

$$\begin{aligned} \text{LHS} :=\ & A_qP_2 - A_q\overline{P}_2 + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}P_2 - s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 + P_2A_q' - \overline{P}_2A_q' \\ & + s_1^{-2}P_2C_{q2}'C_{q2}\overline{P}_2 - s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 + B_{q1}B_{q1}' + s_1^{-2}B_{q1}D_{21}'C_{q2}\overline{P}_2 \\ & + s_1^{-2}\overline{P}_2C_{q2}'D_{21}B_{q1}' + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 \\ =\ & A_qP_2 + P_2A_q' - A_q\overline{P}_2 - \overline{P}_2A_q' - s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 + B_{q1}B_{q1}' \\ & + s_1^{-2}\overline{P}_2C_{q2}'(C_{q2}P_2 + D_{21}B_{q1}') + s_1^{-2}(P_2C_{q2}' + B_{q1}D_{21}')C_{q2}\overline{P}_2 \end{aligned}$$

Now, $(6.8(vi))_{2,2}$ shows that $B_{q1}D_{21}' + P_2C_{q2}' = 0$. Thus,

$$A_qP_2 + P_2A_q' - A_q\overline{P}_2 - \overline{P}_2A_q' - s_1^{-2}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 + B_{q1}B_{q1}' = 0 =: \text{RHS}$$

The last equation is derived by subtracting the all-pass equation $(6.18(ii))_{2,2}$ from $(6.8(iv))_{2,2}$. This gives:

$$A_qP_2 + P_2A_q' + B_{q1}B_{q1}' - A_q\overline{P}_2 - \overline{P}_2A_q' - \overline{B}_{q1}\overline{B}_{q1}' = 0$$

Using $\overline{B}_{q1} := -s_1^{-1}\overline{P}_2C_{q2}'$ this can be written as:

$$A_qP_2 + P_2A_q' - A_q\overline{P}_2 - \overline{P}_2A_q' - s_1^{-1}\overline{P}_2C_{q2}'C_{q2}\overline{P}_2 + B_{q1}B_{q1}' = 0$$

Next we find the observability gramian of the realisation of $W_{\perp}^{\sim}$. This is the unique solution of the Lyapunov equation:

$$(A_q' - s_1^{-1}C_{q2}'\overline{B}_{q1}')X_w + X_w(A_q - s_1^{-1}\overline{B}_{q1}C_{q2}) + s_1^{-1}C_{q2}'C_{q2} = 0$$

or, equivalently

$$(A_q' + s_1^{-2}C_{q2}'C_{q2}\overline{P}_2)X_w + X_w(A_q + s_1^{-2}\overline{P}_2C_{q2}'C_{q2}) + s_1^{-1}C_{q2}'C_{q2} = 0$$

Now by definition,

$$\overline{Q}_1 := \overline{P}_2\overline{R} \quad \text{and} \quad \widehat{P}_1 := \overline{Q}_2(\overline{R}')^{-1}$$

Further, the all-pass equation $(6.18(ii))_{11}$,

$$\widehat{A}\widehat{P}_1 + \widehat{P}_1\widehat{A}' + \widehat{B}\widehat{B}' = 0$$

implies that

$$(-A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2)\widehat{P}_1 + \widehat{P}_1(-A_q - s_1^{-2}\overline{P}_2C_{q2}'C_{q2}) + s_1^{-2}C_{q2}'C_{q2} = 0$$

Therefore, $X_w = -\widehat{P}_1$. □

C.2 Proof of proposition 6.1.7

Proposition. The s_1-allpass system $s_1\alpha(s) \in \mathcal{RL}_\infty^{l\times l}$ defined in the proof of Theorem 6.1.3 can be written as a parallel system interconnection $s_1\alpha(s) = \alpha_1(s) + \alpha_2(s)$,

$$s_1\alpha(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A & 0 & B_{\alpha_1} \\ 0 & -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & B_{\alpha_2} \\ \hline C_{\alpha_1} & C_{\alpha_2} & (D_{12}^{\perp})'D_{11}D_{21}^{\perp} \end{array}\right]$$

in which

$$B_{\alpha_1} := BD_{21}^{\perp} + P_3(\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp}$$

$$B_{\alpha_2} := (\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp}$$

$$C_{\alpha_1} := -(D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}Q_3' + (D_{12}^{\perp})'C$$

$$C_{\alpha_2} := -(D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}\overline{R}$$

In particular, $\alpha \in \mathcal{H}_\infty^{-,l\times l}$ and $\deg(\alpha) \leq 2n - r$.

Proof. The proof follows a sequence of detailed state-space calculations. First,

$$v^{\sim}H_{11} \overset{s}{=} \left[\begin{array}{c|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & C_{q1}' + s_1^{-2}\overline{Q}_2 B_{q2}D_{12}' \\ \hline (D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger} & (D_{12}^{\perp})' \end{array}\right] \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_q & B_{q1} \\ \hline C & C_{q1} & D_{11} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & C_{q1}'C + Z_1 C & C_{q1}'C_{q1} + Z_1 C_{q1} & C_{q1}'D_{11} \\ 0 & A & 0 & B \\ 0 & 0 & A_q & B_{q1} \\ \hline (D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger} & (D_{12}^{\perp})'C & (D_{12}^{\perp})'C_{q1} & (D_{12}^{\perp})'D_{11} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & C_{q1}'C + Z_1 C & C_{q1}'C_{q1} + Z_1 C_{q1} & C_{q1}'D_{11} \\ 0 & A & 0 & B \\ 0 & 0 & A_q & B_{q1} \\ \hline (D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger} & (D_{12}^{\perp})'C & (D_{12}^{\perp})'C_{q1} & (D_{12}^{\perp})'D_{11} \end{array}\right]$$

in which $Z_1 = s_1^{-2}\overline{Q}_2 B_{q2}D_{12}'$, using $D_{12}'D_{11} = 0$. Further, using a similarity transformation T:

$$T := \begin{bmatrix} I & 0 & Q_2 - \overline{Q}_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we can write:

$$v^{\sim}H_{11} \overset{T}{=} \left[\begin{array}{cc|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & C_{q1}'C + Z_1 C & (Q_2 - \overline{Q}_2)B_{q1} + C_{q1}'D_{11} \\ 0 & A & B \\ \hline (D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger} & (D_{12}^{\perp})'C & (D_{12}^{\perp})'D_{11} \end{array}\right]$$

$$=: \left[\begin{array}{cc|c} \Phi_1 & \Phi_2 & \Phi_5 \\ 0 & A & B \\ \hline \Phi_3 & \Phi_4 & \Phi_6 \end{array}\right]$$

We next form:

$$v^{\sim}H_{11}w = \left[\begin{array}{cc|c} \Phi_1 & \Phi_2 & \Phi_5 \\ 0 & A & B \\ \hline \Phi_3 & \Phi_4 & \Phi_6 \end{array}\right] \left[\begin{array}{c|c} -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & (\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp} \\ \hline -B_{q1}' - s_1^{-2}D_{21}'C_{q2}\overline{P}_2 & D_{21}^{\perp} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} \Phi_1 & \Phi_2 & \Phi_5(-B_{q1}' - s_1^{-2}D_{21}'C_{q2}\overline{P}_2) & \Phi_5 D_{21}^{\perp} \\ 0 & A & B(-B_{q1}' - s_1^{-2}D_{21}'C_{q2}\overline{P}_2) & BD_{21}^{\perp} \\ 0 & 0 & -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2 & (\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp} \\ \hline \Phi_3 & \Phi_4 & \Phi_6(-B_{q1}' - s_1^{-2}D_{21}'C_{q2}\overline{P}_2) & \Phi_6 D_{21}^{\perp} \end{array}\right]$$

Now,

$$A(1,1) = -A_q' - s_1^{-2}\overline{Q}_2 B_{q2} B_{q2}'$$
$$A(1,2) = C_{q1}'C + s_1^{-2}\overline{Q}_2 B_{q2} D_{12}' C$$
$$= -A_q' Q_3' - Q_3' A - s_1^{-2}\overline{Q}_2 B_{q2} B_{q2}' Q_3$$

by using the all-pass equations $(6.8(v))_{2,1}$ and $(6.8(iii))_{2,1}$. In addition,

$$A(1,3) = -(Q_2 - \overline{Q}_2)B_{q1}B_{q1}' - s_1^{-2}(Q_2 - \overline{Q}_2)B_{q1}D_{21}'C_{q2}\overline{P}_2 - C_{q1}'D_{11}B_{q1}'$$
$$- s_1^{-2}C_{q1}'D_{11}D_{21}'C_{q2}\overline{P}_2$$
$$= -(Q_2 - \overline{Q}_2)B_{q1}B_{q1}' - s_1^{-2}(Q_2 - \overline{Q}_2)B_{q1}D_{21}'C_{q2}\overline{P}_2 - C_{q1}'D_{11}B_{q1}'$$

on noticing that $D_{11}D_{21}' = 0$ (from all-pass $(6.8(ii))_{(1,2)}$). Moreover,

$$A(2,3) = -BB_{q1}' - s_1^{-2}BD_{21}'C_{q2}\overline{P}_2$$
$$A(3,3) = -A_q' - s_1^{-2}C_{q2}'C_{q2}\overline{P}_2$$
$$B(1) = (Q_2 - \overline{Q}_2)B_{q1}D_{21}^{\perp} + C_{q1}'D_{11}D_{21}^{\perp}$$
$$B(2) = BD_{21}^{\perp}$$
$$B(3) = (\overline{P}_2 - P_2)^{\dagger}B_{q1}D_{21}^{\perp}$$
$$C(1) = (D_{12}^{\perp})'C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}$$
$$C(2) = (D_{12}^{\perp})'C$$
$$C(3) = -(D_{12}^{\perp})'D_{11}B_{q1}' - s_1^{-2}(D_{12}^{\perp})'D_{11}D_{21}'C_{q2}\overline{P}_2$$
$$= -(D_{12}^{\perp})'D_{11}B_{q1}'$$

The expression for $C(3)$ is due to the fact that $D_{11}D_{21}' = 0$ (from all-pass $(6.8(ii))_{(1,2)}$). Apply now, the similarity transformation

$$T = \begin{bmatrix} I & Q_3' & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} I & -Q_3' & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Then, we have that $A(1,2) = -A(1,1)Q_3' + A(1,2) + Q_3'A(2,2) = 0$. Further,

$$A(1,3) = A(1,3) + Q_3'A(2,3)$$
$$= -(Q_2 - \overline{Q}_2)B_{q1}B_{q1}' - s_1^{-2}(Q_2 - \overline{Q}_2)B_{q1}D_{21}'C_{q2}\overline{P}_2$$
$$- C_{q1}'D_{11}B_{q1}' + Q_3'(-BB_{q1}' - s_1^{-2}BD_{21}'C_{q2}\overline{P}_2)$$
$$= -(Q_2 - \overline{Q}_2)B_{q1}B_{q1}' - s_1^{-2}(Q_2 - \overline{Q}_2)B_{q1}D_{21}'C_{q2}\overline{P}_2$$
$$- C_{q1}'D_{11}B_{q1}' - Q_3'BB_{q1}' - s_1^{-2}Q_3'BD_{21}'C_{q2}\overline{P}_2)$$

Next note that the all-pass equation $(D'_H C_H + B'_H Q_H = 0)_{(1,2)}$ implies that:

$$B'Q_3 + B_{q1}Q_2 = -D'_{11}C_{q1} - D'_{21}C_{q2}$$
$$\Rightarrow Q'_3 B = -Q'_2 B_{q1} - C'_{q1}D_{11} - C'_{q2}D_{21}$$
$$\Rightarrow Q'_3 B B'_{q1} = -Q'_2 B_{q1} B'_{q1} - C'_{q1}D_{11}B'_{q1} - C'_{q2}D_{21}B'_{q1}$$
$$\Rightarrow Q'_3 B D'_{21} = -Q'_2 B_{q1} D'_{21} - C'_{q1}D_{11}D'_{21} - C'_{q2}D_{21}D'_{21} = Q'_2 P'_2 C'_{q2} - s_1^2 C_{q2}$$

Note from all-pass (6.8(ii)) that $D_{11}D'_{21} = 0$ and that $D_{21}D'_{21} = s_1^2 I$. Further, taking $(D_H B'_H + C_H P_H = 0)'_{(2,2)}$ we deduce that $BD'_{21} = P'_2 C'_{q2}$. Thus,

$$\begin{aligned}
A(1,3) &= -(Q_2 - \overline{Q}_2)B_{q1}B'_{q1} - s_1^{-2}(Q_2 - \overline{Q}_2)B_{q1}D'_{21}C_{q2}\overline{P}_2 - C'_{q1}D_{11}B'_{q1} \\
&\quad + Q'_2 B_{q1}B'_{q1} + C'_{q1}D_1 B'_{q1} + C'_{q2}D_{21}B'_{q1} - s_1^{-2}Q'_2 P'_2 C'_{q2}C_{q2}\overline{P}_2 \\
&\quad + C'_{q2}C_{q2}\overline{P}_2 \\
&= \overline{Q}_2 B_{q1}B'_{q1} - s_1^{-2}(Q_2 - \overline{Q}_2)B_{q1}D'_{21}C_{q2}\overline{P}_2 + C'_{q2}D_{21}B'_{q1} \\
&\quad - s_1^{-2}Q'_2 P'_2 C'_{q2}C_{q2}\overline{P}_2 + C'_{q2}C_{q2}\overline{P}_2 \\
&= \overline{Q}_2 B_{q1}B'_{q1} - s_1^{-2}Q_2 B_{q1}D'_{21}C_{q2}\overline{P}_2 + s_1^{-2}\overline{Q}_2 B_{q1}D'_{21}C_{q2}\overline{P}_2 \\
&\quad + C'_{q2}D_{21}B'_{q1} - s_1^{-2}Q'_2 P'_2 C'_{q2}C_{q2}\overline{P}_2 + C'_{q2}C_{q2}\overline{P}_2 \\
&= \overline{Q}_2 B_{q1}B'_{q1} - s_1^{-2}Q_2 B_{q1}D'_{21}C_{q2}\overline{P}_2 + s_1^{-2}\overline{Q}_2 B_{q1}D'_{21}C_{q2}\overline{P}_2 \\
&\quad - C'_{q2}C_{q2}P_2 - s_1^{-2}Q'_2 P'_2 C'_{q2}C_{q2}\overline{P}_2 + C'_{q2}C_{q2}\overline{P}_2
\end{aligned}$$

by observing that $D_{21}B'_{q1} = -C_{q2}P_2$ (from all-pass $(6.8(vi))_{(22)}$). Now, take the last two terms of the above equation:

$$\begin{aligned}
-s_1^{-2}Q'_2 P'_2 C'_{q2}C_{q2}\overline{P}_2 + C'_{q2}C_{q2}\overline{P}_2 &= s_1^{-2}(s_1^2 I - Q'_2 P'_2)C'_{q2}C_{q2}\overline{P}_2 \\
&= s_1^{-2}(Q'_3 P_3)C'_{q2}C_{q2}\overline{P}_2 \\
&= -s_1^{-2}Q'_3 B D'_{21}C_{q2}\overline{P}_2
\end{aligned}$$

where all-pass $(6.8(vi))_{21}$ gives $D_{21}B' = -C_{q2}P'_3 \Rightarrow P_3 C'_{q2} = -BD'_{21}$. Thus, we have

$$\begin{aligned}
A(1,3) &= \overline{Q}_2 B_{q1}B'_{q1} - s_1^{-2}Q_2 B_{q1}D'_{21}C_{q2}\overline{P}_2 + s_1^{-2}\overline{Q}_2 B_{q1}D'_{21}C_{q2}\overline{P}_2 \\
&\quad - C'_{q2}C_{q2}P_2 - s_1^{-2}Q'_3 B D'_{21}C_{q2}\overline{P}_2 \\
&= \overline{Q}_2 B_{q1}B'_{q1} + (-s_1^{-2}Q_2 B_{q1} + s_1^{-2}\overline{Q}_2 B_{q1} - s_1^{-2}Q'_3 B)D'_{21}C_{q2}\overline{P}_2 \\
&\quad - C'_{q2}C_{q2}P_2
\end{aligned}$$

by taking the term $D'_{21}C_{q2}\overline{P}_2$ as common factor. Now, recall from all-pass $(6.8(v))_{(12)}$ $B'Q_3 + B'_{q1}Q_2 + D'_{11}C_{q1} + C'_{q2}D_{21} = 0$ or equivalently $Q'_3 B + Q_2 B_{q1} = -C'_{q1}D_{11} - C'_{q2}D_{21}$.

In addition, from all-pass equations we get that $D_{11}D_{21}' = 0$ and that $D_{21}D_{21}' = s_1^2 I$. Hence, we have

$$\begin{aligned} A(1,3) =& \overline{Q}_2 B_{q1} B_{q1}' - C_{q2}' C_{q2} P_2 + (-Q_2 B_{q1} + \overline{Q}_2 B_{q1} - Q_3' B) s_1^{-2} D_{21}' C_{q2} \overline{P}_2 \\ =& \overline{Q}_2 B_{q1} B_{q1}' - C_{q2}' C_{q2} P_2 + s_1^{-2} \overline{Q}_2 B_{q1} D_{21}' C_{q2} \overline{P}_2 + C_{q2}' C_{q2} \overline{P}_2 \end{aligned}$$

but $B_{q1}D_{21}' = -P_2 C_{q2}'$ and $B_{q1}' B_{q1} = -A_q P_2 - P_2 A_q' - B_{q2}' B_{q2}$. Substituting,

$$\begin{aligned} A(1,3) &= \overline{Q}_2 B_{q1} B_{q1}' - C_{q2}' C_{q2} P_2 + s_1^{-2} \overline{Q}_2 B_{q1} D_{21}' C_{q2} \overline{P}_2 + C_{q2}' C_{q2} \overline{P}_2 \\ &= \overline{Q}_2(-A_q P_2 - P_2 A_q' - B_{q2}' B_{q2}) - C_{q2}' C_{q2} P_2 - s_1^{-2} \overline{Q}_2 P_2 C_{q2}' C_{q2} \overline{P}_2 \\ &\quad + C_{q2}' C_{q2} \overline{P}_2 \\ &= \overline{Q}_2(-A_q P_2 - P_2 A_q' - B_{q2}' B_{q2} - s_1^{-2} P_2 C_{q2}' C_{q2} \overline{P}_2) - C_{q2}' C_{q2} P_2 \\ &\quad + C_{q2}' C_{q2} \overline{P}_2 \\ &= \overline{Q}_2((\overline{P}_2 - P_2) A_q' + A_q(\overline{P}_2 - P_2) + s_1^{-2}(\overline{P}_2 - P_2) C_{q2}' C_{q2} \overline{P}_2) \\ &\quad - C_{q2}' C_{q2} P_2 + C_{q2}' C_{q2} \overline{P}_2 =: \Phi \end{aligned}$$

since $-B_{q2} B_{q2}' = A_q \overline{P}_2 + \overline{P}_2 A_q' + s_1^{-2} \overline{P}_2 C_{q2}' C_{q2} \overline{P}_2$. Further,

$$\begin{aligned} CT^{-1} &= \begin{bmatrix} Z_2 & (D_{12}^{\perp})' C & -(D_{12}^{\perp})' D_{11} B_{q1}' \end{bmatrix} \begin{bmatrix} I & -Q_3' & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} Z_2 & -Z_2 Q_3' + (D_{12}^{\perp})' C & -(D_{12}^{\perp})' D_{11} B_{q1}' \end{bmatrix} \end{aligned}$$

where $Z_2 := (D_{12}^{\perp})' C_{q1} (Q_2 - \overline{Q}_2)^{\dagger}$ and

$$\begin{aligned} TB &= \begin{bmatrix} I & Q_3' & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} (Q_2 - \overline{Q}_2) B_{q1} D_{21}^{\perp} + C_{q1}' D_{11} D_{21}^{\perp} \\ B D_{21}^{\perp} \\ (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \end{bmatrix} \\ &= \begin{bmatrix} (Q_2 - \overline{Q}_2) B_{q1} D_{21}^{\perp} + C_{q1}' D_{11} D_{21}^{\perp} + Q_3' B D_{21}^{\perp} \\ B D_{21}^{\perp} \\ (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \end{bmatrix} \\ &= \begin{bmatrix} (Q_2 B_{q1} - \overline{Q}_2 B_{q1} + C_{q1}' D_{11} + Q_3' B) D_{21}^{\perp} \\ B D_{21}^{\perp} \\ (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \end{bmatrix} \\ &= \begin{bmatrix} (-C_{q2}' D_{21} - \overline{Q}_2 B_{q1}) D_{21}^{\perp} \\ B D_{21}^{\perp} \\ (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \end{bmatrix} = \begin{bmatrix} -\overline{Q}_2 B_{q1} D_{21}^{\perp} \\ B D_{21}^{\perp} \\ (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \end{bmatrix} \end{aligned}$$

So, $v^{\sim}H_{11}w$ has a realisation:

$$\left[\begin{array}{ccc|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & 0 & \Phi & -\overline{Q}_2 B_{q1} D_{21}^{\perp} \\ 0 & A & -BB_{q1}' - s_1^{-2} B D_{21}' C_{q2}\overline{P}_2 & B D_{21}^{\perp} \\ 0 & 0 & -A_q' - s_1^{-2} C_{q2}' C_{q2}\overline{P}_2 & (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \\ \hline Z_2 & -Z_2 Q_3' + (D_{12}^{\perp})' C & -(D_{12}^{\perp})' D_{11} B_{q1}' & (D_{12}^{\perp})' D_{11} D_{21}^{\perp} \end{array}\right] \stackrel{T}{=}$$

$$\left[\begin{array}{ccc|c} -A_q' - s_1^{-2}\overline{Q}_2 B_{q2}B_{q2}' & 0 & 0 & 0 \\ 0 & A & -BB_{q1}' - s_1^{-2} B D_{21}' C_{q2}\overline{P}_2 & B D_{21}^{\perp} \\ 0 & 0 & -A_q' - s_1^{-2} C_{q2}' C_{q2}\overline{P}_2 & (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \\ \hline Z_2\overline{Q}_2 & -Z_2 Q_3' + (D_{12}^{\perp})' C & C(3) & (D_{12}^{\perp})' D_{11} D_{21}^{\perp} \end{array}\right]$$

where we have used the transformation

$$T = \begin{bmatrix} I & 0 & \overline{Q}_2(\overline{P}_2 - P_2) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} I & 0 & -\overline{Q}_2(\overline{P}_2 - P_2) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Also,

$$\begin{aligned} C(3) &= -(D_{12}^{\perp})' C_{q1}(Q_2 - \overline{Q}_2)^{\dagger}\overline{Q}_2(\overline{P}_2 - P_2) - (D_{12}^{\perp})' D_{11} B_{q1}' \\ B(1) &= -\overline{Q}_2 B_{q1} D_{21}^{\perp} + \overline{Q}_2(\overline{P}_2 - P_2)^{\dagger}(\overline{P}_2 - P_2) B_{q1} D_{21}^{\perp} \\ &= -\overline{Q}_2 \left\{ I - (\overline{P}_2 - P_2)^{\dagger}(\overline{P}_2 - P_2) \right\} B_{q1} D_{21}^{\perp} = 0 \end{aligned}$$

due to corollary 6.1.6. Further,

$$\begin{aligned} A(1,3) &= \overline{Q}_2(\overline{P}_2 - P_2)\left[-A_q' - s_1^{-2} C_{q2}' C_{q2}\overline{P}_2\right] - \left[-A_q' - s_1^{-2}\overline{Q}_2 B_{q2} B_{q2}'\right]\overline{Q}_2(\overline{P}_2 - P_2) \\ &\quad + \overline{Q}_2(\overline{P}_2 - P_2) A_q' + \overline{Q}_2 A_q(\overline{P}_2 - P_2) + s_1^{-2}\overline{Q}_2(\overline{P}_2 - P_2) C_{q2}' C_{q2}\overline{P}_2 \\ &\quad + C_{q2}' C_{q2}(\overline{P}_2 - P_2) \\ &= 0 \end{aligned}$$

Hence, $v^{\sim}H_{11}w$ has a realisation:

$$\left[\begin{array}{cc|c} A & -BB_{q1}' - s_1^{-2} B D_{21}' C_{q2}\overline{P}_2 & B D_{21}^{\perp} \\ 0 & -A_q' - s_1^{-2} C_{q2}' C_{q2}\overline{P}_2 & (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \\ \hline -Z_2 Q_3' + (D_{12}^{\perp})' C & -Z_2\overline{Q}_2(\overline{P}_2 - P_2) - (D_{12}^{\perp})' D_{11} B_{q1}' & (D_{12}^{\perp})' D_{11} D_{21}^{\perp} \end{array}\right]$$

Finally, applying the similarity transformation $T := \begin{bmatrix} I & P_3 \\ 0 & I \end{bmatrix}$ shows that:

$$s_1\alpha(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A & 0 & B_{\alpha_1} \\ 0 & -A_q' - s_1^{-2} C_{q2}' C_{q2}\overline{P}_2 & B_{\alpha_2} \\ \hline C_{\alpha_1} & C_{\alpha_2} & (D_{12}^{\perp})' D_{11} D_{21}^{\perp} \end{array}\right]$$

where

$$
\begin{aligned}
B_{\alpha_1} &:= BD_{21}^{\perp} + P_3(\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \\
B_{\alpha_2} &:= (\overline{P}_2 - P_2)^{\dagger} B_{q1} D_{21}^{\perp} \\
C_{\alpha_1} &:= -(D_{12}^{\perp})' C_{q1}(Q_2 - \overline{Q}_2)^{\dagger} Q_3' + (D_{12}^{\perp})' C \\
C_{\alpha_2} &:= -(D_{12}^{\perp})' C_{q1}(Q_2 - \overline{Q}_2)^{\dagger} \overline{R}
\end{aligned}
$$

This completes the proof. □

C.3 Proof of proposition C.3.1

Proposition C.3.1. $K = \mathcal{F}_l(Q_a, \overline{Q}_a^{-1})$, *where*

$$
K := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & 0 \end{pmatrix} = \begin{pmatrix} Q_{11} - Q_{12}\overline{Q}_{12}^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{21} & Q_{12}\overline{Q}_{12}^{-1} \\ \overline{Q}_{21}^{-1}Q_{21} & 0 \end{pmatrix}
$$

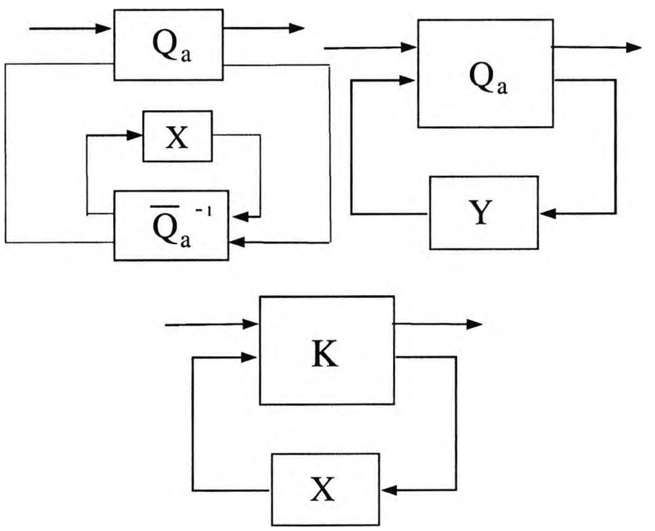

Figure C.1: Sketch of proof of proposition 6.1.5.

Proof. The argument is summarised in figure 6.1. The proof is carried out at a transfer function level. Take

$$
K := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & 0 \end{pmatrix} = \begin{pmatrix} Q_{11} - Q_{12}\overline{Q}_{12}^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{21} & Q_{12}\overline{Q}_{12}^{-1} \\ \overline{Q}_{21}^{-1}Q_{21} & 0 \end{pmatrix}
$$

Then,

$$\begin{aligned}\mathcal{F}_l(K, \mathrm{X}) &= K_{11} + K_{12}\mathrm{X}K_{21} \\ &= Q_{11} - Q_{12}\overline{Q}_{12}^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{21} + Q_{12}\overline{Q}_{12}^{-1}\mathrm{X}\overline{Q}_{21}^{-1}Q_{21} \\ &= Q_{11} + Q_{12}\overline{Q}_{12}^{-1}(\mathrm{X} - \overline{Q}_{11})\overline{Q}_{21}^{-1}Q_{21}\end{aligned}$$

Let $\mathrm{Y} := \mathcal{F}_u(\overline{Q}_a^{-1}, \mathrm{X})$. Then

$$\mathcal{F}_l(Q_a, \mathcal{F}_u(\overline{Q}_a^{-1}, \mathrm{X})) = \mathcal{F}_l(Q_a, \mathrm{Y}) = Q_{11} + Q_{12}\mathrm{Y}(I - Q_{22}\mathrm{Y})^{-1}Q_{21}$$

Hence, we want to prove that

$$\overline{Q}_{12}^{-1}(\mathrm{X} - \overline{Q}_{11})\overline{Q}_{21}^{-1} = \mathrm{Y}(I - Q_{22}\mathrm{Y})^{-1}$$

First we obtain an expression for $\overline{Q}_a^{-1}$. Partitioning $\overline{Q}_a^{-1}$ conformally $\overline{Q}_a$:

$$\overline{Q}_a^{-1}\overline{Q}_a = \begin{pmatrix} \Upsilon & \Phi \\ \Psi & \Omega \end{pmatrix} \begin{pmatrix} \overline{Q}_{11} & \overline{Q}_{12} \\ \overline{Q}_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Expanding the above equation gives:

$$\begin{aligned} &\textbf{(S5):} && \Upsilon\overline{Q}_{11} + \Phi\overline{Q}_{21} = I \\ &\textbf{(S6):} && \Upsilon\overline{Q}_{12} + \Phi Q_{22} = 0 \Rightarrow \Upsilon = -\Phi Q_{22}\overline{Q}_{12}^{-1} \\ &\textbf{(S7):} && \Psi\overline{Q}_{11} + \Omega\overline{Q}_{21} = 0 \Rightarrow \Omega = -\Psi\overline{Q}_{11}\overline{Q}_{21}^{-1} \\ &\textbf{(S8):} && \Psi\overline{Q}_{12} + \Omega Q_{22} = I \end{aligned}$$

Substituting (S6) into (S5)

$$\begin{aligned} -\Phi Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11} + \Phi\overline{Q}_{21} = I \Rightarrow \Phi(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11}) = I \\ \Rightarrow \Phi = (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1} \\ \Rightarrow \Upsilon = -(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1} \end{aligned}$$

Further, substituting (S7) into (S8)

$$\begin{aligned} \Psi\overline{Q}_{12} = -\Psi\overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22} = I \Rightarrow \Psi = (\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})^{-1} \\ \Rightarrow \Omega = -(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1} \end{aligned}$$

So, the inverse of $\overline{Q}_a$ is

$$\overline{Q}_a^{-1} = \left(\begin{array}{c|c} -(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1} & (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1} \\ \hline (\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})^{-1} & -(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1} \end{array}\right)$$

Now let

$$\mathrm{Y} := \mathcal{F}_u(\overline{Q}_a^{-1}, \mathrm{X}) = -(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})^{-1}\overline{Q}_{11}\overline{Q}_{21}^{-1} +$$
$$+ (\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})^{-1}\mathrm{X}[I + (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}]^{-1}(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}$$

which is equivalent to

$$(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})\mathrm{Y}(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11}) =$$
$$= - \overline{Q}_{11}\overline{Q}_{21}^{-1}(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11}) + \mathrm{X}[I + (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}]^{-1}$$

and thus

$$(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})\mathrm{Y}(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})[I + (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}] =$$
$$= - \overline{Q}_{11}\overline{Q}_{21}^{-1}(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})[I + (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}] + \mathrm{X}$$
$$= - \overline{Q}_{11}\overline{Q}_{21}^{-1}[\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11} + Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}] + \mathrm{X}$$
$$= - \overline{Q}_{11}\overline{Q}_{21}^{-1}[\overline{Q}_{21} + Q_{22}\overline{Q}_{12}^{-1}(\mathrm{X} - \overline{Q}_{11})] + \mathrm{X}$$
$$= \mathrm{X} - \overline{Q}_{11} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22}\overline{Q}_{12}^{-1}(\mathrm{X} - \overline{Q}_{11})$$

Summarising,

$$(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})\mathrm{Y}(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})[I + (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}]$$
$$= [I - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22}\overline{Q}_{12}^{-1}](\mathrm{X} - \overline{Q}_{11})$$

Noticing that

$$(\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})[I + (\overline{Q}_{21} - Q_{22}\overline{Q}_{12}^{-1}\overline{Q}_{11})^{-1}Q_{22}\overline{Q}_{12}^{-1}\mathrm{X}] = \overline{Q}_{21} + Q_{22}\overline{Q}_{12}^{-1}(\mathrm{X} - \overline{Q}_{11})$$

(shown above) we get,

$$(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})\mathrm{Y}[\overline{Q}_{21} + Q_{22}\overline{Q}_{12}^{-1}(\mathrm{X} - \overline{Q}_{11})] = [I - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22}\overline{Q}_{12}^{-1}](\mathrm{X} - \overline{Q}_{11})$$

multiplying form the right both sides by $(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12}$ gives

$$(\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})\mathrm{Y}[\overline{Q}_{21}(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12} + Q_{22}] = (\overline{Q}_{12} - \overline{Q}_{11}\overline{Q}_{21}^{-1}Q_{22})$$
$$\Rightarrow \mathrm{Y}[\overline{Q}_{21}(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12} + Q_{22}] = I$$
$$\Rightarrow [\overline{Q}_{21}(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12} + Q_{22}] = \mathrm{Y}^{-1}$$
$$\Rightarrow [\overline{Q}_{21}(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12} + Q_{22}]\mathrm{Y} = I$$
$$\Rightarrow [\overline{Q}_{21}(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12}]\mathrm{Y} = I - Q_{22}\mathrm{Y}$$
$$\Rightarrow \mathrm{Y}(I - Q_{22}\mathrm{Y})^{-1} = (\overline{Q}_{21}(\mathrm{X} - \overline{Q}_{11})^{-1}\overline{Q}_{12})^{-1}$$

which proves the initial claim. □

Appendix D

Distance to singularity

Throughout this section we summarise important results in distance to singularity matrix problems (for a complete treatment see [JH+06]). A distance to singularity type of problem involves in finding matrix Δ such that

$$det(A - \Delta) = 0$$

under certain constraints that Δ may have. Here we are interested in deriving the minimum distance to singularity which is usually referred to as the minimum (induced) norm of a matrix Δ that is also rank reducing to A. This is an "absolute" distance to singularity problem and without loss of generality, A may be considered diagonal:

$$\begin{aligned} det(A - \Delta) = 0 \Leftrightarrow det(U\Sigma V - \Delta) = 0 \Leftrightarrow det\left(U(\Sigma - U'\Delta V')V\right) = 0 \\ \Leftrightarrow det(U)det(\Sigma - U'\Delta V')det(V) = 0 \Leftrightarrow det(\Sigma - U'\Delta V') = 0 \end{aligned}$$

by considering the singular value decomposition $A = U\Sigma V$. Observe that $\|U'\Delta V'\| = \|\Delta\|$. Throughout the chapter we examine distance to singularity problems where the rank-reducing perturbation has structural constraints in terms of its norm and spectral radius. Further, we assume multiplicity on the smallest singular value of A, greater than one, motivated by the theory developed in chapter 7.

The first result outlined here concerns the case when the rank reducing perturbation is constrained to have a zero first block.

Problem D.0.1. *Let*

$$A = \begin{array}{c c} & \begin{array}{cc} m & n\text{-}m \end{array} \\ \begin{array}{c} m \\ n\text{-}m \end{array} & \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] \end{array} = \left[\begin{array}{cc|cc} a_1 I_{m_1} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ \hline 0 & 0 & a_3 I_{m_3} & 0 \\ 0 & 0 & 0 & A_{44} \end{array}\right] \in \mathcal{R}^{n\times n}$$

and assume that $a_1 < \underline{\sigma}(A_{22})$, $0 < a_3 < \underline{\sigma}(A_{44})$ *and* $a_1 < a_3$. *Further, define* $E_1 = \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix}$. *Find the minimum norm of all rank-reducing perturbations* Δ, *which are constrained to have their first* $m \times m$ *block equal to zero. Equivalently, solve the following problem:*

$$\gamma_{0_{m\times m}} := \min\{\|\Delta\| : det(A - \Delta), \quad E_1'\Delta E_1 = 0_{m\times m}\}$$

Then determine the optimal structure of such perturbations. □

The solution of the problem is given via the following lemma.

Lemma D.0.1. *Consider everything as defined in problem D.0.1. Then the structured distance to singularity is*

$$\gamma_{0_{m\times m}} = \sqrt{a_1 a_3} \quad \left(=: \sqrt{\underline{\sigma}(A_1)\underline{\sigma}(A_2)}\right)$$

All optimal rank-reducing to matrix A perturbations have the following form:

$$\Delta = W \left[\begin{array}{c|c} 0_m & \hat{\Delta}_2 \\ \hline \hat{\Delta}_3 & \hat{\Delta}_4 \end{array}\right] W' = W \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & \sqrt{a_1 a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_{13} & \Delta_{14} \\ 0 & 0 & 0 & 0 & \Delta_{23} & \Delta_{24} \\ \hline \sqrt{a_1 a_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_{31} & \Delta_{32} & 0 & \Delta_{33} & \Delta_{34} \\ 0 & \Delta_{41} & \Delta_{42} & 0 & \Delta_{43} & \Delta_{44} \end{array}\right] W'$$

where $W = diag(W_1, I_{m_2}, W_3, I_{m_4}) \in \mathcal{C}^{n\times n}$ *is unitary and*

$$\left\| \begin{bmatrix} 0 & 0 & \Delta_{13} & \Delta_{14} \\ 0 & 0 & \Delta_{23} & \Delta_{24} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} \end{bmatrix} \right\| \leq \sqrt{a_1 a_3}$$

Proof. The proof is identical with that in [JH+06], where a scaled version of the problem is considered. □

The following lemma is an auxiliary result needed in later analysis.

Lemma D.0.2. *Let* $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ *and* U *be complex matrices and assume that for any* $a_1 > 0$, $(a_1 I - H_{11}U)^{-1}$, H_{12}^{-1} *and* H_{21}^{-1} *exist. If* H *is* γ*-unitary for some* $\gamma > 0$, H_{11} *is square,* $a_1 I - H_{11}$ *is nonsingular and* $\|H_{11}\| \leq a_1$ *then*

$$\|\mathcal{F}_u(H, a_1 I)\| = \|(a_1^{-1}I - H_{11})^{-1}(\gamma^2 I - a_1^{-1}H_{11})\|$$

Proof. The proof is identical with that in [JH+06], where a scaled version of the problem is considered. □

Lemma D.0.3. *Let*

$$A = \left[\begin{array}{c|c} a_1 I_m & 0 \\ \hline 0 & A_2 \end{array}\right] = \left[\begin{array}{c|ccc} a_1 I_m & 0 & 0 & 0 \\ \hline 0 & a_{m+1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{array}\right]$$

$0 < a_1 < a_{m+1} \leq \cdots \leq a_n$. *Further define* E_1 *as define in problem D.0.1 and let* $\Delta_{11} \in \mathcal{C}^{m\times m}$ *be given. Assuming that* $\|\Delta_{11}\| \leq \phi < a_1$ *and that* $(a_1 I - \Delta_{11})$ *is nonsingular, then*

$$\min_{\substack{det(A-\Delta)=0 \\ E_1'\Delta E_1 = \Delta_{11}}} \|\Delta_{11}\| = \min_{\substack{\|(\gamma^2 I - a_1\Delta_{11})(a_1 I - \Delta_{11})^{-1}\| = a_{m+1} \\ \gamma > a_1}} \gamma$$

Sketch of proof. Here we adopt standard all-pass dilation theory in order to construct γ-unitary completions for the given block Δ_{11}. Note the importance of conditions $\|\Delta_{11}\| < a_1$ and $det(a_1 I - \Delta_{11}) \neq 0$, so that the (off diagonal term of the) dilated matrices have full rank. Hence, we construct matrices Δ_1^γ, Δ_0^γ of dimensions $2m \times 2m$ and $2n \times 2n$, respectively, such that $\|\Delta_1^\gamma\| = \|\Delta_0^\gamma\| = \gamma$ (and which also share the same first block, Δ_{11}).

Further, we construct the upper LFT's of the inverse of dilation Δ_0^γ and A, Δ, respectively:

$$X_A^\gamma := \mathcal{F}_u\left[(\Delta_0^\gamma)^{-1}, A\right]$$

and

$$\Phi_\Delta^\gamma := \mathcal{F}_u\left[(\Delta_0^\gamma)^{-1}, \Delta\right]$$

and observe that

$$\|\Phi_\Delta^\gamma\| = \gamma^{-1}$$

from Lemma D.0.2. Further, after some algebra, it can be shown that the first $m \times m$ block of Φ_Δ^γ is equal to zero.

Then the idea is to show that the original problem

$$\min_{\substack{det(A-\Delta)=0 \\ \gamma = \|\Delta_{11}\| \\ E_1'\Delta E_1 = \Delta_{11}}} \gamma$$

is equivalent to solving a distance of singularity of the form given in problem D.0.1, i.e.

$$\min_{\substack{det(X_A^\gamma - \Phi_\Delta^\gamma) = 0 \\ \|\Phi_\Delta^\gamma\| = \gamma^{-1} \\ E_1' \Phi_\Delta^\gamma E_1 = 0_{m\times m}}} \gamma$$

where the optimal rank-reducing perturbation is constrained to have zeros in its first $m \times m$ block. The solution of this problem is already known and hence using Lemma D.0.2 it is shown that the problem is equivalent to

$$\min_{\substack{\|(\gamma^2 I - a_1\Delta_{11})(a_1 I - \Delta_{11})^{-1}\| = a_{m+1} \\ \gamma > a_1}} \gamma$$

and thus the claim is proved. For a complete proof see [JH+06], Lemma 3.7. □

Remark D.0.1. *Lemma D.0.3 gives the minimum norm of a rank-reducing perturbation to a diagonal matrix A which has multiplicity greater than one, on its smallest singular value. Here, as before, the perturbation is constrained to have norm largest than the smallest singular value of A. Further, from Lemma D.0.3 it is possible to obtain implicitly the structure of the optimally rank-reducing perturbation. Note that if* γ_o *denotes the optimal distance to singularity, then*

$$\Phi_\Delta^{\gamma_o} = \mathcal{F}_u\left[(\Delta_0^{\gamma_o})^{-1}, \Delta\right] \Leftrightarrow \Delta = \mathcal{F}_u\left[\Delta_0^{\gamma_o}, \Phi_\Delta^{\gamma_o}\right] \tag{D.1}$$

directly from Lemma D.0.2.

Remark D.0.2. *Lemma D.0.3 requires (assumes) the a priori knowledge of a* Δ_{11}. *In [JH+06], Lemma 3.11 generalises the result for* <u>*any*</u> $\Delta_{11} \in \mathcal{C}^{m\times m} : \|\Delta_{11}\| \leq a_1$ *and such that* $(a_1 I - \Delta_{11})^{-1}$ *exists and is bounded. However, here we shall not need this generalisation.*

Lemma D.0.4. *Assume everything defined as in problem D.0.1 and consider all perturbations* $\Delta \in \mathcal{C}^{n\times n}$, *such that* $E_1'\Delta E_1 \in \mathbf{\Delta}^m_{\phi_1,\phi_2}$:

$$\mathbf{\Delta}^m_{\phi_1,\phi_2} = \{E_1'\Delta E_1 \in \mathcal{C}^{m\times m}, \rho(E_1'\Delta E_1) \leq \phi_1, \|E_1'\Delta E_1\| \leq \phi_2\}$$

where $0 \leq \phi_1 \leq \phi_2 \leq a_1$. *Then every optimally rank-reducing perturbation* Δ *which is constrained such that* $E_1'\Delta E_1 \in \mathbf{\Delta}^m_{\phi_1,\phi_2}$ *is similar to the following structure:*

$$E_1'\Delta E_1 := \Delta^m_{\phi_1,\phi_2} = \begin{cases} 0, & j < i; \\ \phi_1, & j = i; \\ (-\frac{\phi_1}{\phi_2})^{j-i-1}\frac{\phi_2^2-\phi_1^2}{\phi_2}, & j > i. \end{cases}$$

i.e., for every optimal Δ, its $m \times m$ block $E_1'\Delta E_1$ is always similar to the particular upper triangular Toeplitz matrix. Further the structured distance to singularity

$$\gamma_{\Delta^m_{\phi_1,\phi_2}} = \min\{\|\Delta\| : det(A - \Delta) = 0, E_1'\Delta E_1 \in \mathbf{\Delta}^m_{\phi_1,\phi_2}\}$$

remain invariant under the transformation and it is given by

$$\gamma_{\phi_1,\phi_2} = \sqrt{a_1^2 + \zeta_{\Delta^m_{\phi_1,\phi_2}}}$$

where

$$\zeta^{-1}_{\Delta^m_{\phi_1,\phi_2}} = \bar{\lambda}\left(\begin{bmatrix} \frac{a_1\left\{(a_1 I - \Delta^m_{\phi_1,\phi_2})^{-1} + (a_1 I - \Delta^{m}_{\phi 1,\phi 2}{}')^{-1}\right\}}{a_{m+1}^2 - a_1^2} & -\frac{(a_1 I - \Delta^m_{\phi_1,\phi_2})^{-1}}{\sqrt{a_{m+1}^2 - a_1^2}} \\ -\frac{(a_1 I - \Delta^{m}_{\phi 1,\phi 2}{}')^{-1}}{\sqrt{a_{m+1}^2 - a_1^2}} & 0 \end{bmatrix}\right)$$

Proof. Too technical and not of present interest, therefore omitted. For complete proof see [JH⁺06]. □

Appendix E

Robust stabilisation under additive perturbations

E.1 Proof of Proposition 7.4.2

Proposition. Let $\Delta(s)$ be constructed according to Algorithm 7.4.1. Then $\Delta(s)$ has the following properties:

(i) $\Delta(s) \in \mathcal{RH}_\infty$

(ii) $\|\Delta\|_\infty = \gamma(\phi)$

(iii) $\|x^T \Delta y\|_\infty = \phi$

(iv) $det[I_m - \Delta(j\omega_o)T(j\omega_o)] = 0 \ \forall \ T \in \mathcal{T}$.

Proof. *(i)* Follows immediately since N_{11}, N_{21}, A_1 and A_2 are all $\mathcal{RH}_\infty$ functions.
(ii) Write

$$\hat{\Delta} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\phi I_l} & \begin{matrix}\nu_o \\ 0\end{matrix} \\ \hline \nu_o & 0 & -\phi \end{array}\right]$$

where $\nu_o = \sqrt{\gamma^2 - \phi^2}$. Then,

$$\begin{aligned} \hat{\Delta}^2 &= diag\{\phi^2 + \nu_o^2, \phi^2 + \nu_o^2, \ldots, \phi^2 + \nu_o^2, \phi^2, \phi^2 + \nu_o^2\} \\ &= diag\{\gamma^2, \gamma^2, \ldots, \gamma^2, \phi^2, \gamma^2\} \end{aligned}$$

Also noting that $N_{11}^\sim N_{11} = I$; $N_{21} N_{21}^\sim = I$, we have

$$\begin{aligned} \Delta\Delta^\sim &= N_{11} A_1 \hat{\Delta} A_2 N_{21} N_{21}^\sim A_2^\sim \hat{\Delta} A_1^\sim N_{11}^\sim \\ &= N_{11} A_1 \hat{\Delta}^2 A_1^\sim N_{11}^\sim \end{aligned}$$

Thus,

$$
\begin{aligned}
\|\Delta\|_\infty^2 &= \max_{\omega\in\mathcal{R}} \lambda_{max}\left[N_{11}(j\omega)A_1(j\omega)\hat{\Delta}^2A_1^*(j\omega)N_{11}^*(j\omega)\right] \\
&= \max_{\omega\in\mathcal{R}} \lambda_{max}\left[\hat{\Delta}^2A_1^*(j\omega)N_{11}^*(j\omega)N_{11}(j\omega)A_1(j\omega)\right] \\
&= \max_{\omega\in\mathcal{R}} \lambda_{max}\left[\hat{\Delta}^2\right] = \lambda_{max}\left[\hat{\Delta}^2\right] = \gamma^2
\end{aligned}
$$

and hence $\|\Delta\|_\infty = \gamma$.

(iii) Note that $X_1X_1^\sim = I_m$. Hence, writing $X_1^\sim = N_1D_1$, implies that:

$$
X_1\begin{pmatrix} n_1 & n_2 & \cdots & n_m \end{pmatrix}\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_m \end{pmatrix} = I_m
$$

or, equivalently,

$$
\left(\begin{array}{c} x^T \\ \hline X^\perp \end{array}\right)\left(\begin{array}{ccc|ccc} n_1 & \cdots & n_l & n_{l+1} & \cdots & n_m \end{array}\right)\left(\begin{array}{ccc|ccc} d_1 & & & & & \\ & \ddots & & & & \\ & & d_l & & & \\ \hline & & & d_{l+1} & & \\ & & & & \ddots & \\ & & & & & d_m \end{array}\right) = \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & I_{m-l} \end{array}\right)
$$

Then considering the upper blocks of the above equation:

$$
x^T\left(\begin{array}{ccc|ccc} n_1 & \cdots & n_l & n_{l+1} & \cdots & n_m \end{array}\right)\left(\begin{array}{ccc|ccc} d_1 & & & & & \\ & \ddots & & & & \\ & & d_l & & & \\ \hline & & & d_{l+1} & & \\ & & & & \ddots & \\ & & & & & d_m \end{array}\right) = \left(\begin{array}{c|c} I_l & 0_{m-l} \end{array}\right)
$$

Take N_{11} to be the first $l+1$-columns of N_1. Then,

$$
x^T\left(\begin{array}{ccc|c} n_1 & \cdots & n_l & n_{l+1} \end{array}\right)\left(\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & \\ & & d_l & \\ \hline & & & d_{l+1} \end{array}\right) = \left(\begin{array}{c|c} I_l & 0 \end{array}\right)
$$

i.e.,

$$x^T N_{11}(s) = \left(\begin{array}{ccc|c} d_1^{-1}(s) & & & \\ & \ddots & & 0 \\ & & d_l^{-1}(s) & \end{array}\right)$$

or

$$x^T N_{11} A_1(s) = \left(\begin{array}{ccc|c} d_1^{-1}(s) & & & \\ & \ddots & & 0 \\ & & d_l^{-1}(s) & \end{array}\right) \left(\begin{array}{ccc|c} \alpha_1(s) & & & \\ & \ddots & & \\ & & \alpha_l(s) & \\ \hline & & & \alpha_{l+1}(s) \end{array}\right)$$

by picking appropriate interpolating $A_1(s)$ as defined in algorithm 7.4.1. Then, note that

$$x^T N_{11} A_1(j\omega_o) = \begin{matrix} l & 1 \\ \end{matrix} \atop \left[\begin{array}{cc} I_l & 0 \end{array}\right] \tag{E.1}$$

Dually,

$$A_2 N_{21} y(s) = \left(\begin{array}{ccc|c} \tilde{\alpha}_1(s) & & & \\ & \ddots & & \\ & & \tilde{\alpha}_l(s) & \\ \hline & & & \tilde{\alpha}_{l+1}(s) \end{array}\right) \left(\begin{array}{c} \tilde{n}_1(s) \\ \vdots \\ \tilde{n}_l(s) \\ \hline \tilde{n}_{l+1}(s) \end{array}\right) y(s)$$

and

$$Y_2 = \left(\begin{array}{c|c} y(s)a(s) & * \end{array}\right) = \left(\begin{array}{ccc} \tilde{n}_1^{\sim}(s) & \cdots & \tilde{n}_p^{\sim}(s) \end{array}\right) \left(\begin{array}{ccc} \tilde{d}_1 & & \\ & \ddots & \\ & & \tilde{d}_p \end{array}\right)$$

where "*" denotes irrelevant terms to the present analysis. Then

$$y(s)a(s) = \left(\begin{array}{ccc} \tilde{n}_1^{\sim}(s) & \cdots & \tilde{n}_l^{\sim}(s) \end{array}\right) \left(\begin{array}{ccc} \tilde{d}_1^{\sim} & & \\ & \ddots & \\ & & \tilde{d}_l^{\sim} \end{array}\right)$$

and so,

$$A_2 N_{21} y a(s) = \left(\begin{array}{ccc|c} \tilde{\alpha}_1(s) & & & \\ & \ddots & & \\ & & \tilde{\alpha}_l(s) & \\ \hline & & & \tilde{\alpha}_{l+1}(s) \end{array}\right) \left(\begin{array}{c} I_l \\ \hline 0_{1,\,l} \end{array}\right) \left(\begin{array}{ccc} \tilde{d}_1^{\sim} & & \\ & \ddots & \\ & & \tilde{d}_l^{\sim} \end{array}\right)$$

or

$$A_2 N_{21} y a(s) = \left(\begin{array}{cccc} \tilde{\alpha}_1 \tilde{d}_1^{-1} & & & \\ & \tilde{\alpha}_2 \tilde{d}_2^{-1} & & \\ & & \ddots & \\ & & & \tilde{\alpha}_l \tilde{d}_l^{-1} \\ \hline & & 0_{1,l} & \end{array} \right)$$

and hence

$$A_2(j\omega_o) N_{21}(j\omega_o) y(j\omega_o) a(j\omega_o) := \begin{bmatrix} I_l \\ 0 \end{bmatrix} \begin{array}{c} l \\ 1 \end{array} \tag{E.2}$$

so that after some computations

$$x^T \Delta y a(s) = x^T N_{11} A_1 \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\phi I_l} & \begin{array}{c} \nu_o \\ 0 \end{array} \\ \hline \nu_o & 0 & -\phi \end{array} \right] A_2 N_{21} y a(s)$$

$$= \phi \begin{pmatrix} \alpha_1 d_1^{-1} \tilde{\alpha}_1 \tilde{d}_1^{-1} & & & \\ & \alpha_2 d_2^{-1} \tilde{\alpha}_2 \tilde{d}_2^{-1} & & \\ & & \ddots & \\ & & & \alpha_l d_l^{-1} \tilde{\alpha}_l \tilde{d}_l^{-1} \end{pmatrix}$$

and hence

$$\|x^T \Delta y\|_\infty = \|x^T \Delta y a(s)\|_\infty = \phi$$

(iv)

$$I - \Delta T = I - N_{11} A_1 \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\phi I_l} & \begin{array}{c} \nu_o \\ 0 \end{array} \\ \hline \nu_o & 0 & -\phi \end{array} \right] A_2 N_{21} Y_2 \begin{pmatrix} s_1 I_l & & \\ & s_2 & \\ & & \check{R} + \check{Q} \end{pmatrix} X_1$$

So,

$$\begin{aligned} det[I_m - \Delta(j\omega_o) T(j\omega_o)] &= det[I_m - N_{11} A_1 \hat{\Delta} A_2 N_{21} Y_2 \Pi X_1] \\ &= det[I_m - X_1 N_{11} A_1 \hat{\Delta} A_2 N_{21} Y_2 \Pi] \end{aligned}$$

Now by construction,

$$X_1 N_{11} A_1(j\omega_o) = \begin{array}{c} \\ l{+}1 \\ m{-}l{-}1 \end{array} \begin{array}{c} l{+}1 \\ \begin{bmatrix} I_{l+1} \\ 0 \end{bmatrix} \end{array}$$

and

$$A_2N_{21}Y_2(j\omega_o) = \begin{array}{c} \\ l+1 \end{array}\begin{array}{c} \begin{array}{cc} l+1 & p\text{-}l\text{-}1 \end{array} \\ \begin{bmatrix} I_{l+1} & 0 \end{bmatrix} \end{array}$$

so that,

$$X_1N_{11}A_1\hat{\Delta}A_2N_{21}Y_2\Pi = \begin{bmatrix} I_{l+1} \\ 0 \end{bmatrix} \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\phi I_l} & \begin{matrix} \nu_o \\ 0 \end{matrix} \\ \hline \nu_o & 0 & -\phi \end{array}\right] \begin{bmatrix} I_{l+1} & 0 \end{bmatrix} \left[\begin{array}{cc|c} s_1 I_l & 0 & 0 \\ 0 & s_{l+1} & 0 \\ \hline 0 & 0 & * \end{array}\right]$$

$$= \left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\phi s_1 I_l} & \begin{matrix} \nu_o s_{l+1} \\ 0 \end{matrix} & 0 \\ \cline{1-3} \nu_o s_1 & 0 & -\phi s_{l+1} & \\ \hline \multicolumn{3}{c|}{0} & 0 \end{array}\right] \tag{E.3}$$

The instability of the closed-loop under such perturbation is verified by (E.3) and the following argument which involves determinants:

$$det(I_m - \Delta T) = det\left\{\left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{I_l - s_1\phi I_l} & \begin{matrix} -s_{l+1}\nu_o \\ 0 \end{matrix} & 0 \\ \cline{1-3} -s_1\nu_o & 0 & 1 + s_{l+1}\phi & \\ \hline \multicolumn{3}{c|}{0} & I_{m-l-1} \end{array}\right]\right\}$$

and this is equal to zero if and only if

$$det\left\{\left[\begin{array}{cc|c} \multicolumn{2}{c|}{I_l - s_1\phi I_l} & \begin{matrix} -s_{l+1}\nu_o \\ 0 \end{matrix} \\ \hline -s_1\nu_o & 0 & 1 + s_{l+1}\phi \end{array}\right]\right\} = 0 \tag{E.4}$$

In order to prove (E.4) we use the following Schur argument, which is true for matrices A, B, C, D (see [Ant01]):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}$$

Hence, in order to show that the determinant of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is equal to zero, it suffices to show that $A - BD^{-1}C$ looses rank, or in this problem framework,

$$
\begin{aligned}
&(1-s_1\phi)I_l - \begin{bmatrix} -s_{l+1}\nu_o \\ 0 \end{bmatrix} (1+s_{l+1}\phi)^{-1} \begin{bmatrix} -s_1\nu_o & 0 \end{bmatrix} \\
=&(1-s_1\phi)I_l - \begin{bmatrix} s_1 s_{l+1}\nu_o^2(1+s_{l+1}\phi)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
=&\begin{bmatrix} (1-s_1\phi) - s_1 s_{l+1}\nu_o^2(1+s_{l+1}\phi)^{-1} & 0 \\ 0 & (1-s_1\phi)I_{l-1} \end{bmatrix}
\end{aligned}
$$

looses rank. Indeed, substituting $\nu_o^2 = \gamma^2 - \phi^2$:

$$
\begin{aligned}
&1 - s_1\phi - s_1 s_{l+1}\nu_o^2(1+s_{l+1}\phi)^{-1} \\
=&(1+s_{l+1}\phi)^{-1}\left\{1 + s_{l+1}\phi - s_1\phi - s_1 s_{l+1}\phi^2 - s_1 s_{l+1}\gamma^2 + s_1 s_{l+1}\phi^2\right\} \\
=&(1+s_{l+1}\phi)^{-1}\left\{1 + s_{l+1}\phi - s_1\phi - s_1 s_{l+1}\left(\frac{1}{s_1 s_{l+1}} - \frac{1}{s_{l+1}}\phi + \frac{1}{s_1}\phi\right)\right\} = 0
\end{aligned}
$$

using the fact that $\gamma = \sqrt{\frac{1}{s_1 s_{l+1}} - \frac{1}{s_{l+1}}\phi + \frac{1}{s_1}\phi}$. Therefore,

$$
det(I_m - \Delta(j\omega_0)T(j\omega_0)) = 0 \Rightarrow det(I_m - ((G+\Delta)(j\omega_0)K(j\omega_0))) = 0
$$

□

Appendix F

Robust stabilisation under coprime perturbations

F.1 Proof of Theorem 8.2.3

Theorem. Consider everything as defined in Theorems 8.2.1 and 8.2.2. Then

(i) The set of all optimal closed-loop transfer functions, $\mathcal{T}_1$ can be parameterised as:

$$\mathcal{T}_1 = Y \begin{pmatrix} s_1\alpha(s) & 0 \\ 0 & \hat{R} + \mathcal{F}_l(\bar{Q}_a, s_1^{-1}\,\mathcal{BH}_\infty^{(p-1)\times(m-1)}) \\ 1 & 0 \\ 0 & I \end{pmatrix} X$$

where

$$Y := \begin{pmatrix} M & -\widetilde{N}^\sim \\ N & \widetilde{M}^\sim \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \quad \text{and } X = W^\sim$$

are square all-pass.

(iii) The first two super-optimal levels of $\mathcal{T}$ are $(\sqrt{s_1^2+1}, \sqrt{s_2^2+1})$ where (s_1, s_2) are the first two super-optimal levels of R. Hence $\epsilon^* = \frac{1}{\sqrt{s_1^2+1}}$.

Proof of (i). Consider everything as defined in Theorems 8.2.1 and 8.2.2. Then write,

$$\mathcal{T}_1 = \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} R + \mathcal{F}_l(Q_a, s_1^{-1}\mathcal{BH}_{\infty}^{-}) \\ I \end{pmatrix}$$

$$= \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} \mathcal{F}_l(H, s_1^{-1}\mathcal{BH}_{\infty}^{-}) \\ I \end{pmatrix}$$

$$= \begin{pmatrix} M \\ N \end{pmatrix} \mathcal{F}_l(H, s_1^{-1}\mathcal{BH}_{\infty}^{-}) + \begin{pmatrix} -\widetilde{N}^{\sim} \\ \widetilde{M}^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} M \\ N \end{pmatrix} (H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21}) + \begin{pmatrix} -\widetilde{N}^{\sim} \\ \widetilde{M}^{\sim} \end{pmatrix}$$

where $U \in s_1^{-1}\mathcal{BH}_{\infty}$. Further, we isolate the all-pass part so that

$$\mathcal{T}_1 = \left(\begin{pmatrix} M \\ N \end{pmatrix} H_{11} + \begin{pmatrix} -\widetilde{N}^{\sim} \\ \widetilde{M}^{\sim} \end{pmatrix} \right) + \begin{pmatrix} M \\ N \end{pmatrix} H_{12}U(I - H_{22}U)^{-1}H_{21}$$

$$= \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} H_{11} \\ I \end{pmatrix} + \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} H_{12} \\ 0 \end{pmatrix} U(I - H_{22}U)^{-1}H_{21}$$

$$= \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \mathcal{F}_l \left\{ \left(\begin{array}{c|c} H_{11} & H_{12} \\ I & 0 \\ \hline H_{21} & H_{22} \end{array} \right), s_1^{-1}\mathcal{BH}_{\infty} \right\}$$

We know that:

$$\begin{pmatrix} V^{\sim} & 0 \\ 0 & I \end{pmatrix} H \begin{pmatrix} W & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} s_1^2 a(s) & 0 \\ 0 & \overline{H} \end{pmatrix}$$

$$\Rightarrow H = \begin{pmatrix} v & V_{\perp} & 0 \\ 0 & 0 & I \end{pmatrix} \left(\begin{array}{c|cc} s_1^2 a(s) & 0 & 0 \\ \hline 0 & \overline{H}_{11} & \overline{H}_{12} \\ 0 & \overline{H}_{21} & H_{22} \end{array} \right) \begin{pmatrix} w^{\sim} & 0 \\ W_{\perp}^{\sim} & 0 \\ 0 & I \end{pmatrix}$$

Thus,

$$H_{11} = \begin{pmatrix} v & V_{\perp} \end{pmatrix} \begin{pmatrix} (s_1)^2 a(s) & 0 \\ 0 & \overline{H}_{11} \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_{\perp}^{\sim} \end{pmatrix}$$

and

$$H_{12} = V_{\perp}\overline{H}_{12}, \quad H_{21} = \overline{H}_{21}W_{\perp}^{\sim}, \quad H_{22} = H_{22}$$

Substituting,

$$\mathcal{T}_1 = \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \left\{ \begin{pmatrix} H_{11} \\ I \end{pmatrix} + \begin{pmatrix} H_{12} \\ 0 \end{pmatrix} U(I - H_{22}U)^{-1}H_{21} \right\}$$

$$\mathcal{T}_1 = \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \Xi$$

where

$$\Xi := \left\{ \left(\frac{\begin{pmatrix} v & V_\perp \end{pmatrix} \begin{pmatrix} s_1^2 a(s) & 0 \\ 0 & \overline{H}_{11} \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix}}{\begin{pmatrix} w & W_\perp \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix}} \right) + \begin{pmatrix} V_\perp \overline{H}_{12} \\ \hline 0 \end{pmatrix} U(I - H_{22}U)^{-1} \overline{H}_{21} W_\perp^{\sim} \right\}$$

Taking common factors $\begin{pmatrix} v & V_\perp & 0 & 0 \\ 0 & 0 & w & W_\perp \end{pmatrix}$ from left and $\begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix}$ from right, we get

$$\mathcal{T}_1 = \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \left\{ \begin{pmatrix} v & V_\perp & 0 & 0 \\ 0 & 0 & w & W_\perp \end{pmatrix} \begin{pmatrix} s_1^2 a(s) & 0 \\ 0 & \overline{H}_{11} \\ \hline I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} v & V_\perp & 0 & 0 \\ 0 & 0 & w & W_\perp \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \overline{H}_{12} U(I - H_{22}U)^{-1} \overline{H}_{21} \\ \hline 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix} \right\}$$

or

$$\mathcal{T}_1 = \begin{pmatrix} M & -\widetilde{N}^{\sim} \\ N & \widetilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} v & V_\perp & 0 & 0 \\ 0 & 0 & w & W_\perp \end{pmatrix} \begin{pmatrix} s_1 a(s) & 0 \\ 0 & \mathcal{F}_l\left(\overline{H}, s_1^{-1} \mathcal{B}\mathcal{H}_\infty\right) \\ \hline I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix}$$

Proof of (ii). Immediate from a recursive argument of part *(i)* proof.

Proof of (iii). Recall that

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}(s) = \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} a(s)s_1 & 0 \\ 0 & \overline{H} \end{pmatrix} \begin{pmatrix} W^{\sim} & 0 \\ 0 & I \end{pmatrix}(s)$$

where we partition

$$V = \begin{pmatrix} v & V_\perp \end{pmatrix}, \quad W^{\sim} = \begin{pmatrix} w^{\sim} \\ W_\perp^{\sim} \end{pmatrix}$$

Then,

$$\begin{cases} H_{11}^{\sim}v(s) = s_1 a(s)^{\sim}w(s) \\ H_{11}w(s) = s_1 a(s)v(s) \end{cases} \Rightarrow \begin{cases} \|H_{11}^{\sim}v(s)\|_2 = s_1\|w(s)\|_2 \\ \|H_{11}w(s)\|_2 = s_1\|v(s)\|_2 \end{cases}$$

and so $\|v\|_2 = \|w\|_2$ (recall $\|a(s)\| = 1$). Further,

$$H_{11}^{\sim}H_{11}w(s) = s_1 a(s) H_{11}^{\sim}v(s) = s_1^2 a(s)a^{\sim}(s)w(s) = s_1^2 w(s) \tag{F.1}$$

and hence w is a singular vector of H_{11}. Respectively,

$$H_{11}H_{11}^{\sim}v(s) = s_1^2 v(s) \tag{F.2}$$

Straight from definition, take a $T_{opt} \in \mathcal{T}_1$. Then,

$$\begin{aligned} T_{opt}^{\sim}T_{opt} &= \begin{pmatrix} (R+Q_{opt})^{\sim} & I \end{pmatrix} \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix}^{\sim} \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} R+Q_{opt} \\ I \end{pmatrix} \\ &= \begin{pmatrix} (R+Q_{opt})^{\sim} & I \end{pmatrix} \begin{pmatrix} R+Q_{opt} \\ I \end{pmatrix} = (R+Q_{opt})^{\sim}(R+Q_{opt}) + I \end{aligned}$$

and so

$$\begin{aligned} & T_{opt}^{\sim}T_{opt}w(s) = (R+Q_{opt})^{\sim}(R+Q_{opt})w(s) + w(s) \\ \Rightarrow\ & T_{opt}^{\sim}T_{opt}w(s) = (s_1^2+1)w(s) \end{aligned}$$

Hence

$$\|T_{opt}\|_H = \sqrt{s_1^2+1}$$

On the other hand,

$$\begin{aligned} & T_{opt}^{\sim}T_{opt}w(s) = (s_1^2+1)w(s) \Rightarrow T_{opt}T_{opt}^{\sim}T_{opt}w(s) = (s_1^2+1)T_{opt}w(s) \\ \Rightarrow & (T_{opt}T_{opt}^{\sim}) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} R+Q_{opt} \\ I \end{pmatrix} w(s) = (s_1^2+1) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} R+Q_{opt} \\ I \end{pmatrix} w(s) \\ \Rightarrow & (T_{opt}T_{opt}^{\sim}) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} (R+Q_{opt})w(s) \\ w(s) \end{pmatrix} = (s_1^2+1) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} (R+Q_{opt})w(s) \\ w(s) \end{pmatrix} \\ \Rightarrow & (T_{opt}T_{opt}^{\sim}) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} s_1 a(s)v(s) \\ w(s) \end{pmatrix} = (s_1^2+1) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} s_1 a(s)v(s) \\ w(s) \end{pmatrix} \\ \Rightarrow & (T_{opt}T_{opt}^{\sim}) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} = (s_1^2+1) \begin{pmatrix} M & -\tilde{N}^{\sim} \\ N & \tilde{M}^{\sim} \end{pmatrix} \begin{pmatrix} v(s) \\ w(s) \end{pmatrix} \end{aligned}$$

□