LINEAR SYSTEMS AND CONTROL STRUCTURE SELECTION

BY

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To

My late father,

My mother,

My sister,

My uncle.

All my friends.

The memory of

Mr. Lackson K. Chishimba BEng, MSc, MPhil.

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DECLARATION

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ABSTRACT

This thesis is concerned with the development of concepts and results to facilitate study in two areas of control methodology. The two notions investigated are measures of controllability and observability and eigenstructure assignment. The link between these two areas is exposed, and it is demonstrated how the eigenvectors of a system play an important role in determining the degree of controllability and observability. The main concerns are issues dealing with the complexity of the instrumentation, and in particular the development of techniques that may assist in the development of methodology for sensor and actuator placement. The research involves the development of notions that help to structure a system on which control design is based. There are two areas of investigation. The first is the development of concepts and tools that aid in the selection and placement of sensors and actuators based on properties related to degrees of controllability and observability. The second is the investigation of the eigenstructure of a system and its properties, which enable the development of design procedures based on eigenstructure properties.

A study of existing measures of controllability and observability leads to new techniques which take into consideration the problem of coordinate transformations, which is often overlooked. It is shown that the degree of controllability is influenced by changes in the structure of the state feedback matrix, as well as how controllability properties can be determined from Plücker matrices of transfer function matrices. It is also shown that the energy required to move a system from one state to another is linked to the singular values of the output controllability grammian.

A review of the problem of eigenstructure assignment paves the way for the development of a new technique of assigning the closed loop eigenstructure. This is based on matrix fraction description algorithms, and stems from an algebraic description of the total system behaviour, leading to a systematic study of closed loop eigenvectors by using a parametric approach. A new algebraic characterisation of the family of closed loop eigenvectors and related input and output directions is shown. Closed loop system robustness to parameter variations is also considered, where it is shown that there is a link with the orthogonality of the matrix of eigenvectors. As a result, the notion of strong stability is introduced, where it is shown that the shape of the eigenframe plays a role in the system response by way of overshoots. The work develops concepts and results which are important steps in the development of an integrated methodology for input, output structure selection.

LIST OF SYMBOLS

$\varphi(\lambda)$	Characteristic polynomial
λ	Eigenvalue
I_n	Identity matrix
$A \in \mathbb{R}^{n\times n}$	Matrix of real elements with dimension $n\times n$
τ_i	Algebraic multiplicity of the λ_i eigenvalue
d_i	Geometric multiplicity of the λ_i eigenvalue
ρ	Rank of a matrix
$S(A,B,C,D)$	State space description
$\underline{x}(0)$	Initial condition
$e^{A(t-t_0)}$, $\Phi(t_f,t_0)$	State transition matrix
$G(s)=C(sI-A)^{-1}B+D$	Transfer function matrix
$\langle \bullet,\bullet \rangle$	Inner product
$*$	Convolution
$\mathbb{R}(A)$	Range space, column span of A
$\mathcal{N}_r(A)$	Right null space of A
$\mathcal{N}_l(A)$	Left null space of A
$\lVert\underline{x}\rVert_p$	p-norm
$\lVert\underline{x}\rVert_1$	1-norm
$\lVert\underline{x}\rVert_2$	2-norm
$\lVert\underline{x}\rVert_\infty$	Infinity norm
$\lVert A\rVert_F$	Frobenius norm
$\lVert A\rVert_p$	Matrix p-norm
$\lVert A\rVert_{\alpha,\beta}$	Subordinate norm
$\psi(A)$	Spectral radius
$\sigma_i(A)$	Singular values of A
$A_l^\perp$, $A_r^\perp$	Left, right annihilators of A
$G(s)\in\mathbb{R}^{m\times l}(s)$	Matrix of rational functions in s

$N(s) \in \mathbb{R}^{m\times l}[s]$	Matrix of polynomials in s
$\mathcal{U}$, $\mathcal{Y}$, $\mathcal{D}$	Input, output and disturbance spaces
$(\underline{x}_1, \underline{x}_2)$	Dot product of two vectors
$G \in \mathbb{C}^{k\times k}$	Matrix of complex numbers
$\operatorname{adj}(sI - A)$	Adjoint of $(sI - A)$
$\delta_M(H)$	McMillan degree of $H(s)$
$H(s) = H_0 + \hat{H}(s)$	Laurent series expression of $H(s)$
$H_0 = D,\ H_i = CA^{i-1}B,\ i = 1,\ 2,\ldots$	Markov parameters
$M_H(i, j)$	Hankel matrix
μ_H	Hankel condition number
$Q_{k,n}$	Set of lexicographically ordered strictly increasing sequences of k integers from 1, 2, …, n
$\underline{x}_{i_1} \wedge \ldots \wedge \underline{x}_{i_k} = \underline{x}_{\omega}{}^{\wedge}$	Exterior product
$\wedge^r\mathcal{V}$	r-th exterior power of $\mathcal{V}$
$C_r(H)$	r-th compound matrix of H
$\delta(t_f - \sigma)$	Dirac delta impulse function
$G_{OC}(t_0, t_f)$	Output controllability matrix
$\mathcal{B}$, $\mathcal{B}'$	Eigenbasis and dual eigenbasis
$\mathcal{T}(s_0)$	Transmission subspace of s_0
$\left\lvert "u*" u' \right\rvert$	Orthogonality test based on normalisation
$tr(A)$	Trace of $A = \sum_{i=1} A_{ii}$
$g_X(U, \underline{\lambda})$	Lagrangian operator
$\min\limits_{UU' = I_n} \lVert X - U \rVert_2$	Orthogonality test based on distance
$\underline{\xi}(s) = \left[\underline{x}(s)',\quad \underline{u}(s)',\quad \underline{y}(s)'\right]'$	Total behaviour vector
$Q_r(s)$	Behavioural representation
$T_r(s)$	Input – output representation

GLOSSARY

Almost zero	A local minimum of $\lVert \underline{p}(s) \rVert$, where $s = \sigma \pm j\omega$ $\underline{p}(s)$ is a set of polynomials [Kar., et al, 2].
Eigenframe	The matrix or set of eigenvectors.
Eigenstructure	Geometric properties of the eigenframe, e.g. skewness, orthogonality, etc.
Input decoupling zero	An uncontrollable mode [Ros., 1].
Measure of controllability	An algorithm that gauges the strength of presence of controllability. It is an attempt at measuring the distance of a given pair (A, B) from the family of uncontrollable pairs (A', B'). [Tar., 2]
Measure of observability	An algorithm that gauges the strength of presence of observability. It is an attempt at measuring the distance of a given pair (A, C) from the family of unobservable pairs (A', C'). [Tar., 2]
Output decoupling zero	An unobservable mode [Ros., 1].
Projective measures	Special measures of controllability and observability based on distance from singularity of the Plücker matrix of a given system [Kar., & Gia., 1].
Restricted input state pencil	$(sN - NA)$ Feedback free description of the controllability pencil [Kar., 3].
Restricted output state pencil	$(sM - AM)$ Output injection free description of the observability pencil [Kar., 3].
System mode	The triple of eigenvalue, right and left eigenvectors.
Weakly controllable	A system is referred to as weakly controllable if some appropriate distance function from the family of uncontrollable systems has a small value.
Weakly observable	A system is referred to as weakly observable if some appropriate distance function from the family of unobservable systems has a small value.

1 INTRODUCTION

Control systems engineers are concerned with understanding and controlling segments of the environment, called *systems*, to provide useful solutions to a variety of problems in engineering and society in general. A system can be defined as a collection of objects, or subsystems, which are related by interactions that produce various outputs in response to different inputs. Examples of such systems are chemical plants, aircraft, spacecraft, national economies, etc. The control problems associated with these systems may be the efficient production of a chemical product, automatic landing of an aircraft, a rendezvous with an artificial satellite in space, regulation of important economic variables, etc.

The ability to control a system is dependent on a valid mathematical model. However practical systems are inherently complicated and highly nonlinear. Therefore simplifications are made, such as the linearisation of the system. Error analysis can then be employed to provide information on how valid the linear mathematical model is as an approximation to the real system. Traditional control is based on the idea of a fixed model (if one is available) subject to certain interconnections of subsystems and selected input-output structures. However in many applications there is scope for using the selection of the input-output structure in terms of the components used for control purposes, namely sensors and actuators, to assign desirable properties, or to avoid the formulation of undesirable ones. This part of the design process is usually referred to as systems instrumentation and their placement. This also involves the study of the interaction between subprocesses and the overall system instrumentation.

The selection of locations of sensors and actuators on a system effectively shapes the system model by assigning the rules which couple the control variables (actuator variables) and measurement variables (command variables) to the internal variables of the system. Classical instrumentation deals with the development of techniques for acting upon certain variables and measuring, sensing, them. However it is not concerned with the crucial problem of their physical, geometric and spatial distribution. It is this distribution which shapes structurally the input and output coupling maps and in turn affects the resulting properties of the final model. Developing methodology for influencing the selection of the input and output maps is the subject of global or systems instrumentation [Kar., 1] and it is one of the main issues considered in this thesis. The distinct feature of the activities here is the study of the effects of input-output selection on the state space performance indicators which are linked to the degree of presence of certain system properties such as controllability, observability, input/output controllability, minimality, etc. This type of work is complementary to a framework that is based on assignment of desirable values of system invariants [Kar., & Gia., 1].

It is desirable that systems react in the way that they are designed to do, and that their behaviour can be in some way described by the properties exhibited by their mathematical models. To achieve this, information describing the system and the way it changes is needed. This is provided by a feedback control system, which calculates the difference between the measured variables and the desired output responses which in turn results in changes to the system to compensate for the subsequent error. Ideally, it is desirable to measure all of the variables, or states of a system to design a feedback scheme. If this is the case, then *state feedback* is being employed. Yet in practice, not all of the system states are available, and so the feedback has to be designed using the outputs of the system, called *output feedback*. The application of feedback affects the way the system behaves. It also affects the way in which a system can be controlled (*controllability*) and its states measured (*observability*). The mathematical model of the system also changes under feedback, and thus so do certain system behaviour indicators, such as eigenvalues and eigenvectors (collectively known as the *eigenstructure*).

State or output feedback affects the spectrum and the eigenframe of the closed loop state matrix. Techniques which have been primarily concerned with influencing the eigenframe of the resulting closed loop system are referred to as eigenstructure assignment methodologies. Although a lot of activity has taken place in this area most of it has relied on techniques aiming to achieve eigenvalue assignment and improve orthogonality of the set of eigenvectors. Orthogonality of the eigenframe is linked to robustness as far as parameter uncertainty. The role of eigenvectors in design is important for systems described by physical state variables where coordinate transformations are not allowed. This role is not well understood with the exception of the robustness concerns. Also the selection of the spectrum and the eigenframe are problems which are interlinked, but have not been treated as such. This thesis aims to contribute in developing the understanding of the eigenstructure by emphasising the role of skewed frames as causes for overshooting behaviour in overdamped systems, and by introducing new parameterisations which may provide additional tools for handling the simultaneous spectrum and eigenframe selection.

This thesis is primarily concerned with issues which relate in the development of the systems dimension of the instrumentation and in particular the elaboration of criteria which may assist in development of philosophy for sensor and actuator location. Thus the research carried out here deals with the development of methodology for structuring the system on which control design is eventually performed. The main areas under investigation are

1) Development of tools of methodology that will allow selection of locations of sensors and actuators based on properties such as degrees of controllability and observability.
2) Investigation of some new aspects in the area of the eigenstructure of a system and its properties

An important assumption underlying the first section of the work deals with systems which have physical variables. For such systems the study of problems of degrees of controllability and observability is justified. These problems do not make sense when dealing with general variables or when allowing arbitrary coordinate transformations. Although a number of tests for measuring relative degrees of controllability and observability have been already established, as such there is no unifying framework. The thesis contributes in the development of such a framework by

- Reviewing existing results and developing software
- Developing new criteria in areas such as
    a) Input-output controllability based on energy
    b) Degree of minimality based on properties of Hankel matrices
    c) Investigating the effect of feedback on the degrees of controllability and observability and introducing these notions in the open and closed loop sense as well as developing criteria for them.

Throughout this work emphasis is placed on the relationship between these properties and the state space parameters and this is important since through these problems the links between selection of sensors and actuators and their effect on system properties is established. These are prerequisites for the study of structure assignment problems, which however are not considered here.

The second cluster of problems deals with the significance of the eigenstructure of a system in control design, as well as the corresponding shaping of system properties. In fact the new view of the eigenstructure presented here establishes links with properties of degrees of controllability and observability considered before. Most of the studies carried out on the eigenstructure thus far have dealt with the development of algorithms for optimising the degree of orthogonality of the resulting frame. The approach taken here has been much more fundamental, and the following issues have been addressed:

- Development of two new forms of parameterisation of closed loop eigenvectors. The first is of an algebraic nature and is based on the properties of minimal bases of matrix pencils. The second is of parametric nature and provides a measure of pole mobility from the open loop to the closed loop.
- Study of the significance of degree of orthogonality for none overshooting free responses.
- Some interesting characteristic of the eigenstructure in relation to poles and zeros.

- Two variant eigenvector algorithms for eigenvector based assignment

The eigenstructure problem has been considered from a much more fundamental viewpoint and questions such as the effect of a selected spectrum on possible properties of resulting eigenframes have been considered but many important questions remain open.

Chapter 2 forms the mathematical foundations that are necessary for the coherent understanding of the problems and subproblems looked at in this thesis. The fundamentals of eigenvalue and eigenvector analysis are explained, as is the solution to time invariant autonomous and forced systems. Linear algebra is used extensively throughout the research, and a section detailing some of the essential notions connected to the results obtained in these studies is presented. Towards the end of this chapter, there is a section dealing with transfer functions, which is needed for the development of a new method of eigenstructure assignment presented at the end of the thesis.

Chapter 3 paves the way for the consideration of the first problem of the thesis and is primarily concerned with measurement and control problems for large scale systems, that starts off with a look at how control problems evolve from their conception through to the design stages. The definition of key terms such as controllability, observability, sensors, actuators and flexible structures lays down the basics that are to be developed in the following chapters. As an example of a large scale system, flexible structures are analysed where it is shown how state space models can be derived from a set of differential equations. State space models are highly relevant to the thesis, and are used for most of the new results obtained.

In Chapter 4, measures of controllability and observability are studied. This chapter examines the notions of measuring the degree of controllability and observability with a view to sensor and actuator placement. There is a section that comprises of a review of existing measures and a study of some new ones, which is proceeded by a comparison of some of the techniques in this field. A new measure based on Markov parameters is also considered, followed by a section dealing with open and closed loop degrees of controllability. A study of how exterior algebra and characteristics of Plücker matrices can be used to develop measures is also presented. This is conveniently followed by an account of a new method of input selection based on minimising the condition number of the controllability matrix, which concludes the chapter.

The link between the singular values of the output controllability matrix and the energy required to move a system from one state to another is the problem considered in Chapter 5. The output controllability grammian forms the crux of the investigations carried out here and is mathematically analysed. This is followed by a practical study concerning the importance of energy use in large scale systems, and ends with a few examples demonstrating the algorithms developed by using software written in MATLAB.

Chapters 3, 4 and 5 all deal with the first problem of measuring controllability and observability. The next three chapters all look at the second problem to be considered, eigenstructure assignment. Chapter 6 is an account of the basic concepts and background results in this widely covered area. The physical relevance of eigenvalues and eigenvectors are explained here in relation to rectilinear motions. The role of the eigenstructure on system performance is also covered here.

Sensitivity issues are dealt with in Chapter 7. A prime concern to control system designers is that of the closed-loop system robustness to parameter variations, external disturbances and system modelling errors. A major cause for concern is eigenvalue sensitivity to such perturbations, and it is part of the design process to minimise its detrimental effect. Therefore it is important that the eigenstructure to be assigned is formed with a view that the resulting system is as robust as possible. Thus the link between the frame of eigenvectors and system robustness is examined here. A short literature review is given that details some of the work done in measuring robustness. This is followed by a section examining the link between the orthogonality of the eigenframe and system robustness, with corresponding software demonstrating how the system response is affected by the nature of the eigenvector matrix. The notion of *strong stability* is also examined in relation to eigenstructure properties, and the chapter ends with a demonstration of some of the methods to measure the orthogonality of a matrix.

The final chapter of this thesis presents an algebraic description of the total system behaviour which in turn allows the study of closed loop eigenvectors in a systematic way by providing parameterisations. An algebraic characterisation of the total input, state and output behaviour in an implicit formulation is given based on properties of matrix fraction descriptions. The analysis provides a new algebraic characterisation of the family of closed loop eigenvectors and related input and output characteristics. This enables the derivation of a new method of eigenstructure assignment via state feedback, using minimal basis theory, and is demonstrated via an example. Also presented is a way to optimise the eigenframe, which contains the closed loop eigenvalues in order to guarantee maximum system robustness by making it as close to orthogonality as possible.

The thesis ends with conclusions and open issues encountered throughout the work.

The thesis makes novel contributions in the following areas:

- It provides a framework for the selection of sensors and actuators based on the effect of such selections on the degree of presence of a number of system properties in the resultant model.
- It develops energy criteria based on input, output controllability for evaluating the effect of input, output selection on such properties.
- It develops new tests for the selection of input and output structure based on tests characterising closeness to minimality and new algebraic characterisations of degrees of controllability and observability using the notions of almost zeros.
- It establishes new properties on the significance of the structure of eigenframes by examining their role in state overshoots and closed loop robustness.
- It provides two new parameterisations for closed loop eigenframes. One is based on the algebraic properties of minimal bases and the other on a

parameterisation characterising mobility from an open loop to a closed loop location.

The results enhance the range of techniques which may be used for the development of systematic procedures for selection of distribution of sensors and actuators, and for control design based on the eigenstructure assignment.

2 MATHEMATICAL ANALYSIS OF CONTROL SYSTEMS ENGINEERING

2.1 INTRODUCTION TO CONTROL SYSTEMS ENGINEERING

Engineering is concerned with understanding and controlling the materials and forces of nature in order to benefit mankind. Control systems engineers are concerned with understanding and controlling segments of the environment, called *systems*, to provide useful solutions that enhance the products of society. The effective control of systems requires that systems be understood and modelled. Furthermore control engineering must often consider the control of poorly understood systems such as chemical processes. Control engineering is based on the foundations of feedback theory and linear systems analysis, and is not limited to any engineering discipline. It is equally applicable to aeronautical, chemical, mechanical, environmental, civil and electrical engineering, or indeed a combination of these. This chapter will concentrate on the mathematical aspects of control systems engineering that are used throughout this thesis. The areas covered are eigenvalue-eigenvector analysis, the solution of linear time invariant systems, the geometrical and computational issues in the solution of linear systems and transfer function matrices.

2.2 EIGENVALUE-EIGENVECTOR ANALYSIS

2.2.1 CHARACTERISATION OF EIGENVALUES AND EIGENVECTORS

From a set of physical variables, differential equations can be developed and linearised which lead to the well known state space equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \quad \text{state equation} \\ y &= Cx + Du \quad \text{output equation} \end{aligned} \tag{2.1}$$

where A is the ($n \times n$) state matrix and describes the internal (homogenous) motion. B is the ($n \times m$) input matrix and describes how m inputs affect the n states. C, the ($p \times n$) output matrix, describes how n states contribute to p outputs. The ($p \times m$) D direct transmission matrix describes how m inputs are fed through to p outputs. These matrices possess certain characteristics that can be studied in order to determine how the system will behave. Such indicators of system behaviour are the eigenvalues and eigenvectors. Eigenvalues and eigenvectors play a prominent role in the way systems behave. They are important property indicators that can be derived from the systems matrices of equation (2.1). The significance of eigenvalues and eigenvectors in the way systems behave will be discussed later. This section will deal with the mathematical aspects. An eigenvalue is a root of the characteristic polynomial

$$\varphi(\lambda) = |\lambda I_n - A| = 0 \tag{2.2}$$

An eigenvector $\underline{u}_i$ that corresponds to an eigenvalue λ_i is a nontrivial solution of

$$[\lambda I_n - A]\underline{u} = 0 \tag{2.3}$$

where I_n is the identity matrix. The *algebraic multiplicity* of an eigenvalue is the multiplicity of the eigenvalue in $\varphi(\lambda)$. The columns of the eigenvector matrix U which consist of $\underline{u}_i$ have to be linearly independent. The inverse of U is V, and is the matrix of *left* or *dual* eigenvectors.

The *characteristic decomposition* is defined for the case of simple structure matrices as

$$A = U\Lambda V = \begin{bmatrix} \underline{u}_1 & \dots & \underline{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{v}_1' \\ \vdots \\ \underline{v}_n' \end{bmatrix} \tag{2.4}$$

where U is the eigenvector matrix, Λ is the spectral matrix which has as its diagonal the distinct eigenvalues, and V is the matrix of dual eigenvectors.

2.2.2 THE STRUCTURE OF EIGENVECTORS OF A MATRIX

Let the matrix $A \in \mathbb{R}^{n \times n}$, where the characteristic polynomial is defined as

$$\begin{aligned} \varphi(\lambda) = |\lambda I_n - A| &= \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0 \\ &= (\lambda - \lambda_1)^{\tau_1} \ldots (\lambda - \lambda_p)^{\tau_p} \end{aligned} \tag{2.5}$$

where τ_i is the algebraic multiplicity of the λ_i eigenvalue. The number of eigenvectors d_i for each λ_i is defined as the *geometric multiplicity* of the λ_i eigenvalue, and is computed using

$$d_i = n - \rho\{\lambda_i I_n - A\}, \ d_i \le \tau_i \tag{2.6}$$

ρ is the rank of a matrix. A matrix A where for every eigenvalue λ_i the algebraic and geometric multiplicities are equal is called *simple*. If for at least one eigenvalue λ_j $d_j < \tau_j$, it is called *nonsimple*. All simple $n \times n$ matrices have n eigenvectors, and all nonsimple square matrices have less than n eigenvectors.

Companion matrices, with the corresponding characteristic polynomial of (2.5), always have just one eigenvector for each eigenvalue, regardless of its algebraic multiplicity. The coefficients of the last row of the A matrix also correspond to the coefficients of the characteristic polynomial. Let $\lambda_1, \ \ldots, \ \lambda_k$ be the distinct eigenvalues of the companion matrix. Then the corresponding eigenvectors are linearly independent, i.e. U has full rank

$$U=\begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_k \\ \vdots & & \vdots \\ \lambda_1^{n-2} & \dots & \lambda_k^{n-2} \\ \lambda_1^{n-1} & \dots & \lambda_k^{n-1} \end{bmatrix} \text{ for } \lambda_i \neq \lambda_j \tag{2.7}$$

For A matrices with eigenvalues that are both real and complex, the characteristic decomposition takes on a form which consists of a complex spectral part and a real spectrum part.

2.2.3 GENERALISED EIGENVECTORS

[Ske., 1] Section 2.2.2 dealt with constructing eigenvector matrices for cases where the A matrix was simple and had distinct eigenvalues. But there are cases when A has a set of repeated eigenvalues. This is where the concept of *generalised eigenvectors* [Wil., 1] [Ske., 1] comes into effect. $\underline{u}_k$ is called the generalised eigenvector of rank k if

$$(A-\lambda I)^k \underline{u}_k = 0 \text{ and } (A-\lambda I)^{k-1} \underline{u}_k \neq 0 \tag{2.8}$$

From this a list of *generalised eigenvector chains* can be formed

$$\begin{aligned} A\underline{u}_k &= \lambda \underline{u}_k + \underline{u}_{k-1} \\ A\underline{u}_{k-1} &= \lambda \underline{u}_{k-1} + \underline{u}_{k-2} \\ &\vdots \\ A\underline{u}_3 &= \lambda \underline{u}_3 + \underline{u}_2 \\ A\underline{u}_2 &= \lambda \underline{u}_2 + \underline{u}_1 \\ A\underline{u}_1 &= \lambda \underline{u}_1 \end{aligned} \tag{2.9}$$

Due to the nature of the repeated eigenvalues, the spectral matrix is defined by the *Jordan canonical form*, *J*, where $J=U^{-1}AU$, and is a matrix of block diagonals. The upper block consists of the repeated eigenvalues, and its size is determined by the *index of annihilation*, *q*. If $A \in \mathbb{R}^{n \times n}$ is nonsimple then

$$\varphi(\lambda) = |\lambda I - A| = (\lambda - \lambda_1)^{\tau_1} \ldots (\lambda - \lambda_p)^{\tau_p}$$
$$\lambda_i \neq \lambda_j, \quad \tau_1 + \ldots + \tau_p = n \tag{2.10}$$

For the eigenvalue λ, define

$$(\lambda I - A)^i$$
$$\rho(\lambda I - A)^i = \rho_i, \quad i = 0, 1, \ldots, n \tag{2.11}$$

which has the following property

$$\rho_1 > \rho_2 > \ldots > \rho_{q-1} > \rho_q = \rho_{q+1} = \rho_{q+1} = \ldots \tag{2.12}$$

from which the *index of annihilation* of A at λ is

$$q \equiv \min j \text{ for which } \rho_j = \rho_{j+1} \tag{2.13}$$

2.3 Solution of Linear Time Invariant Systems

2.3.1 SOLUTION OF THE AUTONOMOUS SYSTEM

[Ant., & Mic., 1] Let an autonomous system be described by

$$S(A): \ \dot{\underline{x}} = A\underline{x}(t), \quad \underline{x}(0) = \underline{x}_0 \text{ : initial state} \tag{2.14}$$

The solution is given by

$$\underline{x}(t) = e^{At}\underline{x}_0 \tag{2.15}$$

Because the system is time invariant, the initial time is entirely arbitrary, and if $t_0 \neq 0$, then t can be replaced by t-t_0.

$$\underline{x}(t) = e^{A(t-t_0)}\underline{x}_0, \quad \underline{x}(t_0) = \underline{x}_0 \tag{2.16}$$

$\underline{x}(t)$ is defined as the *state trajectory* and $e^{A(t-t_0)}$ is called the *state transition matrix*. This may be split up into a function of eigenvalues and eigenvectors by implementing the *dyadic decomposition*, thus

$$\begin{aligned} & A = U\Lambda V \\ & \Rightarrow \underline{x}(t) = e^{At}\underline{x}_0 = Ue^{\Lambda t}V\underline{x}_0 \end{aligned} \tag{2.17}$$

As can be seen from (2.17), eigenvalues and eigenvectors can influence the way in which the state of a system responds.

2.3.2 SOLUTION TO FORCED SYSTEMS

[Ant., & Mic., 1] Forced systems are systems that are excited by an external force, in the form of an input or a disturbance. These are described by

$$S(A,B,C,D): \begin{cases} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{cases} \quad \underline{x}(0) = \underline{x}_0 \tag{2.18}$$

The corresponding solutions for the state and output trajectories are

$$\begin{aligned} & \underline{x}(t) = e^{At}\underline{x}_0 + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau \\ & \underline{y}(t) = Ce^{At}\underline{x}_0 + \int_0^t Ce^{A(t-\tau)}B\underline{u}(\tau)d\tau + D\underline{u}(t) \end{aligned} \tag{2.19}$$

The solutions defined above for the state and output trajectories each consist of two parts. The integral parts of (2.19) define the *forced response*, which are contributed from inputs and disturbances. The remaining part of the solutions are the *free response* contributions.

Taking Laplace transforms of the time domain solutions of (2.19), the following frequency domain representations are obtained

$$\begin{aligned} \underline{x}(s) &= (sI-A)^{-1}\underline{x}_0 + (sI-A)^{-1}B\underline{u}(s) \\ \underline{y}(s) &= C(sI-A)^{-1}\underline{x}_0 + \left\{C(sI-A)^{-1}B + D\right\}\underline{u}(s) \end{aligned} \tag{2.20}$$

From this the transfer function matrix is defined as

$$G(s) = C(sI-A)^{-1}B + D \tag{2.21}$$

2.3.3 MODAL FORM OF THE SOLUTION

Using the dyadic expansion, the spectral form of the state trajectory $\underline{x}(t)$ for a zero input response can be written as

$$\begin{aligned} &\underline{x}(t) = e^{At}\underline{x}(0) = e^{\lambda_1 t}\underline{u}_1\underline{v}_1'\underline{x}_0 + \ldots + e^{\lambda_n t}\underline{u}_n\underline{v}_n'\underline{x}_0 \\ &\Rightarrow \underline{x}_0(t) = \sum_{i=1}^{n} e^{\lambda_i t}\langle \underline{v}_i, \underline{x}_0\rangle \underline{u}_i \end{aligned} \tag{2.22}$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product. The spectral form of the state trajectory when $\underline{x}(0) = 0$ (initial condition) and for a nonzero input is

$$\begin{aligned} \underline{x}(t) &= \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau \\ &= \int_0^t \sum_{i=1}^{n} \underline{u}_i e^{\lambda_i(t-\tau)}\underline{v}_i' B\underline{u}(\tau)d\tau \end{aligned} \tag{2.23}$$

The total output response is

$$\underline{y}(t) = \sum_{i=1}^{n} \underline{\gamma}_i e^{\lambda_i t}\langle \underline{v}_i \cdot \underline{x}_0\rangle + \sum_{i=1}^{n} \underline{\gamma}_i e^{\lambda_i t} * \left\langle \underline{\beta}_i, \underline{u}(t)\right\rangle \tag{2.24}$$

where * denotes convolution and

$$CU = C\left[\underline{u}_1 \quad \cdots \quad \underline{u}_1\right] = \left[\underline{\gamma}_1 \quad \cdots \quad \underline{\gamma}_n\right]$$
$$VB = \begin{bmatrix} \underline{v}_1' \\ \vdots \\ \underline{v}_n' \end{bmatrix} B = \begin{bmatrix} \underline{\beta}_1' \\ \vdots \\ \underline{\beta}_n' \end{bmatrix} \tag{2.25}$$

The above equations once again show that the eigenvalues and eigenvectors (eigenstructure) are important factors that shape the way a system reacts when excited. One of the problems looked at later in the thesis deals with ways in which the eigenstructure of a system can be altered in order to guarantee a desired response and certain performance characteristics.

2.4 GEOMETRIC AND COMPUTATIONAL ISSUES IN THE SOLUTION OF LINEAR SYSTEMS

2.4.1 PROBLEM DEFINITION

The problem can be formulated as follows

Problem 2.1: Find $\underline{x} \in \mathbb{R}^n$ such that

$$A\underline{x} = \underline{b}, \quad A \in \mathbb{R}^{m \times n}, \quad \underline{b} \in \mathbb{R}^m$$

□

The problem is solvable if and only if

$$\underline{b} \in \mathcal{R}(A) = \operatorname{colsp}\{A\} \tag{2.26}$$

and the solution is uniquely defined if and only if

$$\mathcal{N}_r(A) = \{0\} \tag{2.27}$$

where $\mathcal{R}(A)$ is the *range space* or *column span* of A and $\mathcal{N}_r(A)$ is the *right null space* of A. However numerical issues do arise for this kind of problem. Finite precision arithmetic and inexact data give rise to problems such as [Horn, & Joh., 1]:

(i) If A and $\underline{b}$ are perturbed by a small amount, how is $\underline{x}$ affected?

(ii) How does *rank deficiency* of A affect the solution?

(iii)If $\underline{b} \notin \mathcal{R}(A)$, then how can $\underline{x}$ be determined so that $A\underline{x}$ is close to $\underline{b}$?

2.4.2 VECTOR NORMS

[Horn, & Joh., 1] A vector norm on $\mathbb{R}^n$ is a function $f:\mathbb{R}^n \rightarrow \mathbb{R}$ such that

(i) $f(x) \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n$ and $f(\underline{x}) = 0$ if and only if $\underline{x} = \underline{0}$

(ii) $f(\underline{x}+\underline{y}) \leq f(\underline{x}) + f(\underline{y}), \ \forall \ \underline{x}, \underline{y} \in \mathbb{R}^n$

(iii) $f(\alpha \underline{x}) = |\alpha| f(\underline{x}), \ \forall \ \alpha \in \mathbb{R}, \ \underline{x} \in \mathbb{R}^n$

Now if $f(\underline{x})$ is denoted by $\|\underline{x}\|$, different types of norms can be distinguished by using subscripts. The *Hölder*, or *p-norms* of $\underline{x} = [x_1, \quad x_2, \quad \ldots, \quad x_n]$ are

p-norm: $\|\underline{x}\|_p = \left(|x_1|^p + \ldots + |x_n|^p\right)^{1/p}, \ p \geq 1$

1-norm: $\|\underline{x}\|_1 = |x_1| + \ldots + |x_n|$

2-norm: $\|\underline{x}\|_2 = \left(|x_1|^2 + \ldots + |x_n|^2\right)^{1/2} = \left(\underline{x}'\underline{x}\right)^{1/2}$

∞-norm: $\|\underline{x}\|_\infty = \max_i |x_i|$

There are certain properties that are applicable to such norms. The *Cauchy-Schwartz* property is defined by

$$|\underline{x}'\underline{y}| \leq \|\underline{x}\|_2 \cdot \|\underline{y}\|_2 \tag{2.28}$$

The *invariance* property under orthogonal transformations is defined by

$$Q\text{: } Q'Q = I\text{: } \|Q\underline{x}\|_2^2 = \underline{x}'\underline{x} = \|\underline{x}\|_2^2 \tag{2.29}$$

The following norm relations are also applicable

$$\begin{aligned} \|\underline{x}\|_2 &\le \|\underline{x}\|_1 \le \sqrt{n}\|\underline{x}\|_2 \\ \|\underline{x}\|_\infty &\le \|\underline{x}\|_2 \le \sqrt{n}\|\underline{x}\|_\infty \\ \|\underline{x}\|_\infty &\le \|\underline{x}\|_1 \le n\|\underline{x}\|_\infty \end{aligned} \tag{2.30}$$

2.4.3 MATRIX NORMS

[Horn, & Joh., 1] A matrix norm assesses the size of matrices and it is a function defined on $\mathbb{R}^{m\times n}$ that satisfies the following

(i) $f(A)\ge 0\ \forall\ A\in\mathbb{R}^{m\times n}$ and $f(A)=0$, if and only if $A=0$

(ii) $f(A+B)\le f(A)+f(B),\ \forall\ A,B\in\mathbb{R}^{m\times n}$

(iii) $f(\alpha A)=|\alpha|\cdot f(A),\ \forall\ \alpha\in\mathbb{R},\ A\in\mathbb{R}^{m\times n}$

There are different types of matrix norms. The *Frobenius* norm is defined as

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|\right]^{1/2} \tag{2.31}$$

P-norms are defined as

$$\|A\|_P = \sup_{\underline{x}\ne 0}\frac{\|A\underline{x}\|_P}{\|\underline{x}\|_P} \tag{2.32}$$

Subordinate norms are characterised by

$$\|A\|_{\alpha,\beta} = \sup_{\underline{x}\neq 0} \frac{\|A\underline{x}\|_{\beta}}{\|\underline{x}\|_{\alpha}} = \sup_{\|\underline{x}\|_{\alpha}=1} \|A\underline{x}\|_{\beta} \tag{2.33}$$

There are several properties of matrix norms that should be considered. The first of these is concerned with *mutually consistent* norms. f_1, f_2 and f_3 norms defined on $\mathbb{R}^{m\times q}$, $\mathbb{R}^{m\times n}$, $\mathbb{R}^{n\times q}$ are mutually consistent if for all $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times q}$

$$f_1(AB) \leq f_2(A) f_3(B) \tag{2.34}$$

The following inequality properties also apply

$$\begin{aligned} \|A\underline{x}\|_p &\leq \|A\|_p \cdot \|\underline{x}\|_p \\ \|A\underline{x}\|_\beta &\leq \|A\|_{\alpha,\beta} \cdot \|\underline{x}\|_\alpha \end{aligned} \tag{2.35}$$

For orthogonal matrices Q and Z, the following *invariance* property applies

$$\begin{aligned} \|QAZ\|_F &= \|A\|_F \\ \|QAZ\|_2 &= \|A\|_2 \end{aligned} \tag{2.36}$$

For matrices of the type

$$\begin{aligned} A &= \left[a_{ij}\right] \in \mathbb{R}^{m\times n} \\ A &= \left[\underline{a}_1, \quad \ldots, \quad \underline{a}_n\right] = \begin{bmatrix} \underline{\alpha}_1' \\ \vdots \\ \underline{\alpha}_m' \end{bmatrix} \end{aligned} \tag{2.37}$$

the following computations can be applied

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}| = \max_j \left\{\|\underline{a}_1\|_1, \quad \ldots, \quad \|\underline{a}_n\|_1\right\} \tag{2.38}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = \max_i \left\{\|\underline{\alpha}_1\|_1, \quad \ldots, \quad \|\underline{\alpha}_m\|_1\right\} \tag{2.39}$$

There are also a number of relationships that can be considered. These are as follows

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2 \tag{2.40}$$

$$\max|a_{ij}| \leq \|A\|_2 \leq \sqrt{mn}\max|a_{ij}| \tag{2.41}$$

$$\|A\|_2 \leq \sqrt{\|A\|_1 \cdot \|A\|_\infty} \tag{2.42}$$

$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 < \sqrt{m}\|A\|_\infty \tag{2.43}$$

$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1 \tag{2.44}$$

For square matrices of the type $A \in \mathbb{R}^{n\times n}$ where $\lambda_i(A)$ represent the eigenvalues of A. the *spectral radius* is defined as

$$\psi(A) = \max_i |\lambda_i(A)| \tag{2.45}$$

$\psi(A)$ is not a matrix norm, but for any $\|\bullet\|$, $\psi(A) \leq \|A\|$.

2.4.4 SINGULAR VALUE DECOMPOSITION (SVD)

[Horn, & Joh., 1] For matrices of the type $A \in \mathbb{R}^{m\times n}$ there exist orthogonal matrices U and V ($U^tU = I_m$, $V^tV = I_m$), where

$$\begin{aligned} U &= [\underline{u}_1, \quad \ldots, \quad \underline{u}_m] \in \mathbb{R}^{m\times m} \\ V &= [\underline{v}_1, \quad \ldots, \quad \underline{v}_n] \in \mathbb{R}^{n\times n} \end{aligned} \tag{2.46}$$

such that

$$U'AV = \Sigma \Leftrightarrow A = U\Sigma V' \tag{2.47}$$

where

$$\Sigma = \begin{bmatrix} \Sigma^* \\ -- \\ 0 \end{bmatrix} \text{ if } m \geq n$$
$$\Sigma = \begin{bmatrix} \Sigma^* & \vdots & 0 \end{bmatrix} \text{ if } m \leq n \tag{2.48}$$

and

$$\Sigma^* = \operatorname{diag}\{\sigma_1, \quad \ldots, \quad \sigma_p\}$$
$$p = \min\{m, n\} \tag{2.49}$$
$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$$

$\sigma_i(A)$ is the i-th largest *singular value* of A. $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are the largest and smallest singular values of A respectively. The ratio of the largest singular value to the smallest is called the *condition number*. $\underline{u}_i$ and $\underline{v}_i$ represent the i-th left and right singular vectors of A respectively. The columns of U, $\underline{u}_i$, are the unit eigenvectors of AA^t. The columns of V, $\underline{v}_i$, are the unit eigenvectors of A^tA. The singular values can also be computed using eigenvalues using

$$\sigma_i(A) = \sqrt{\lambda_i(A'A)} = \sqrt{\lambda_i(AA')} \tag{2.50}$$

2.4.5 RANK, NULL SPACES AND SVD

[Horn, & Joh., 1] The rank of A, $\rho(A)$, is the number of nonzero singular values of $A \in \mathbb{R}^{m \times n}$, where

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\rho > 0 = \sigma_{\rho+1} = \ldots = \sigma_p \tag{2.51}$$

If

$$\begin{bmatrix} \underline{u}_1' \\ \vdots \\ \underline{u}_\rho' \\ \hdashline \underline{u}_{\rho+1}' \\ \vdots \\ \underline{u}_m' \end{bmatrix} A\left[\underline{v}_1 \quad \cdots \quad \underline{v}_\rho \;\vdots\; \underline{v}_{\rho+1} \quad \cdots \quad \underline{v}_n\right] = \begin{bmatrix} \sigma_1 & & & \vdots \\ & \ddots & & \vdots\; 0 \\ & & \sigma_r & \vdots \\ \hdashline & 0 & & \vdots\; 0 \end{bmatrix} \updownarrow (m-r)\times(n-r) \tag{2.52}$$

and by using the following definitions

$$\begin{aligned} \mathcal{N}_l(A) &\equiv \left\{\underline{y} \in \mathbb{R}^m : \; \underline{y}' A = \underline{0}\right\} \\ \mathcal{N}_r(A) &\equiv \left\{\underline{x} \in \mathbb{R}^n : \; A\underline{x} = \underline{0}\right\} \end{aligned} \tag{2.53}$$

where $\mathcal{N}_l(A)$ and $\mathcal{N}_r(A)$ are the left and right null spaces of A respectively, then

$$\begin{aligned} A_l^\perp &= \begin{bmatrix} \underline{u}_{\rho+1}' \\ \vdots \\ \underline{u}_m' \end{bmatrix} \in \mathbb{R}^{(m-r)\times m}, \quad A_l^\perp A = 0 \\ A_r^\perp &= \left[\underline{v}_{\rho+1}, \quad \ldots, \quad \underline{v}_n\right] \in \mathbb{R}^{n\times(n-r)}, \quad AA_r^\perp = 0 \end{aligned} \tag{2.54}$$

where $A_l^\perp$ and $A_r^\perp$ are the left and right *annihilators* of A respectively. The rows of $A_l^\perp$ define a basis matrix for $\mathcal{N}_l(A)$, $\dim\mathcal{N}_l(A) = m-r$. Likewise the columns of $A_r^\perp$ define a basis matrix for $\mathcal{N}_r(A)$, $\dim\mathcal{N}_r(A) = n-r$.

2.4.6 ALMOST RANK

In dealing with engineering system models with on the one hand the uncertainty about the true value of the parameters, and on the other hand round off computational errors, it may seem that trying to compute nongeneric values of invariants is an impossible task [Mit., et al, 1]. Given that any set of engineering data has a given numerical accuracy, it is clear that there is no point to try to compute with greater accuracy the indicators that help to determine the operation of the system than that of the original data. Thus an approximate solution has to be sought at some stage, before the procedure converges to some meaningless generic value.

Let $A = \{\underline{a}_1, \ \underline{a}_2, \ \ldots, \ \underline{a}_m\}$ be a set of m given vectors $\underline{a}_i \in \mathbb{R}^n$, i = 1, 2, ..., m. This set can be expressed in terms of a matrix $A = [\underline{a}_1, \ \underline{a}_2, \ \ldots, \ \underline{a}_m]^t \in \mathbb{R}^{m \times n}$. Dependence or independence of the set A is tested in terms of the rank of the associated matrix. If $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$, $r = \min(m, n)$ are the singular values of A, then for a given tolerance $\in$, the following may be defined [Mit., 1]:

Definition 2.1: The set A is $\in$-independent if $\sigma_i > \in$, i = 1, 2, ..., r, i.e. all the singular values are greater that $\in$.

□

Definition 2.2: The set is numerically $\in$-dependent if $\sigma_i > \in$, and $\sigma_j \leq \in$, for some i, j i.e. some singular values are greater than $\in$ and others are smaller than $\in$.

□

Definition 2.3: The set A is strongly $\in$-dependent if $\sigma_1 > \in$, $\sigma_i \leq \in$, i = 2, 3, ..., r i.e. the maximal singular value is greater than $\in$ and all the others are less than $\in$.

□

Definition 2.4: The set A is fuzzy $\in$-dependent if $\sigma_i \leq \in$, i = 1, 2, ..., r i.e. all the singular values are less than $\in$.

□

Since scaling affects the singular values of a matrix, the above definitions are more suitable when applied to a normalised set of vectors. When the vectors are normalised, strange situations of fuzzy $\in$-dependence may be avoided, which is mostly encountered when dealing with extremely low data values. If the normalisation process is numerically stable [Wil., 2], then the set A may always be assumed to be normalised. The definitions cited above are strongly linked to the notions of numerical $\in$-rank $(\rho_{\in}(A))$ of a matrix [Gol., & V Lo., 1].

Definition 2.5: The numerical $\in$-rank $(\rho_{\in}(A))$ of a matrix $A \in \mathbb{R}^{m \times n}$ is defined with respect to $\|\bullet\|_2$ by

$$\rho_{\in}(A) = \inf\{\rho(B) : \|A - B\|_2 \leq \in\} \tag{2.55}$$

□

A more simplified condition for the determination of the numerical $\in$-rank is given below.

Theorem 2.1: For a matrix $A \in \mathbb{R}^{m \times n}$ and a specified tolerance $\in$:

(i) $\rho_{\in}(A)$ = {number of singular values of A that are $> \in$}

(ii) $n_{\in}(A)$ = {number of singular values of A that are $\leq \in$}

(iii) $\rho_{\in}(A) = n - n_{\in}(A)$

□

The above theorem suggests one method for computing the numerical $\in$-rank via the singular value decomposition. This leads to the following remarks.

(i) The set A is $\in$-independent if and only if $\rho_{\in}(A) = r$

(ii) The set A is numerically $\in$-dependent if and only if $\rho_{\in}(A) < r$

(iii) The set A is strongly $\in$-dependent if and only if $\rho_{\in}(A) = 1$

The above definitions clearly provide a framework for defining the strength of presence of a system property in terms of the notion of the $\in$-numerical rank. The tests for evaluating the $\in$-rank may become substantial if the columns (rows) of the matrix under consideration are normalised [Mit., & Kar., 1].

2.5 Transfer Function Matrices

2.5.1 Transfer Function Matrices of Systems

The following equation is used to derive transfer functions from the state space matrices A, B, C and D.

$$G(s) = C(sI - A)^{-1} B + D \tag{2.56}$$

When the system is SISO (single input – single output), i.e. when the input matrix B has one column, and the output matrix C has one row, then $G(s)$ is simply a single transfer function in the form of a rational function which is a ratio of polynomials.

When the system is MIMO (multiple input – multiple output), i.e. when B and C have more than one column and row respectively, then $G(s)$ is a matrix where the elements are rational functions and are denoted by $G(s) \in \mathbb{R}^{m \times l}(s)$. Such matrices are often expressed in terms of polynomial matrices, i.e. $G(s) = N(s)D^{-1}(s)$, where $N(s) \in \mathbb{R}^{m \times l}[s]$ and $D(s) \in \mathbb{R}^{l \times l}[s]$ are numerator and denominator polynomial matrices respectively, and it is assumed that ($N(s)$, $D(s)$) are usually coprime [Kai., 1].

In this section, theory dealing with such matrices will be covered.

2.5.2 MATRIX PENCILS

For a system described by the state space model

$$S(A,B,C,D): \begin{cases} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{cases} \Leftrightarrow \begin{cases} (sI - A)\underline{x}(s) = B\underline{u}(s) + \underline{x}_0 \\ \underline{y}(s) = C\underline{x}(s) + D\underline{u}(s) \end{cases} \tag{2.57}$$

where $\underline{x}(0) = \underline{x}_0$, there are a family of matrices that can be obtained that describe the various coupling states of the system. Such matrices are called *pencils* [Ros., 1] [Kar., 3]. The *system matrix pencil* is derived using [Kar., & Lev., 1]

$$\begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \end{bmatrix} = \begin{bmatrix} \underline{x}_0 \\ -\underline{y}(s) \end{bmatrix} \tag{2.58}$$

where

$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \tag{2.59}$$

is the system matrix pencil. The *input-state pencil*, which describes the coupling between the input and the state, is denoted by $C(s)$ and is derived using

$$\begin{aligned} \begin{bmatrix} sI - A, & -B \end{bmatrix} \begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \end{bmatrix} &= \underline{x}_0 \\ C(s) &= \begin{bmatrix} sI - A, & -B \end{bmatrix} \end{aligned} \tag{2.60}$$

$R(s)$ is the *state-output pencil* for the case when $\underline{u}(t) = 0$. It describes the state-output coupling and is obtained by

$$\begin{aligned} \begin{bmatrix} sI - A \\ -C \end{bmatrix} \underline{x}(s) &= \begin{bmatrix} \underline{x}_0 \\ -\underline{y}(s) \end{bmatrix} \\ R(s) &= \begin{bmatrix} sI - A \\ -C \end{bmatrix} \end{aligned} \tag{2.61}$$

Finally the *state pencil*, $T(s)$, describes the internal mechanism of the system and is denoted by

$$\begin{aligned} (sI - A)\underline{x}(s) &= \underline{x}_0 \\ T(s) &= (sI - A) \end{aligned} \tag{2.62}$$

A matrix pencil is a special case of polynomial matrix where all the elements are polynomials of maximal degree 1. Generally if $F, G \in \mathbb{R}^{p \times r}$ then a matrix pencil is described by

$$sF - G \in \mathbb{R}^{p \times r}[s] \tag{2.63}$$

2.5.3 MATRIX FRACTION DESCRIPTIONS (MFDS)

[Kai., 1] Let $P(s) \in \mathbb{R}^{l \times m}[s]$ and its rank be denoted by $\rho(P(s)) = m$. A matrix $R(s) \in \mathbb{R}^{m \times m}[s]$ such that

$$P(s) = P'(s)R(s) \tag{2.64}$$

is called a *right matrix divisor* (RMD) [Kai., 1] of *P*(*s*). If $\overline{R}(s)$ is any other RMD and

$$R(s) = W(s)\overline{R}(s) \tag{2.65}$$

then *R*(*s*) is called a *right greatest matrix divisor* (RGMD) of *P*(*s*). If $\rho(P(s)) = l$, the notions of *left matrix divisors* (LMD) and *greatest left matrix divisors* (LGMD) are defined similarly [Kai., 1].

Let $P(s) = \left[\underline{p}_1(s), \quad \ldots, \quad \underline{p}_m(s)\right] \in \mathbb{R}^{l \times m}[s]$ and $\rho(P(s)) = m$. Then the set of degrees

$$I_P = \left\{\delta_i \colon\ \delta_i = \partial\left(\underline{p}_i(s)\right),\ i = \underline{m}\right\} \tag{2.66}$$

is defined as the *set of column degrees* and

$$c_P = \sum_{i=1}^{m} \delta_i \tag{2.67}$$

as the *column complexity* of *P*(*s*). *Row degrees* and *row complexity* are defined in a similar manner [Kai., 1]. If $\underline{p}_i(s) = \underline{p}_{i,h} s^{\delta_i} + \ldots + \underline{p}_{i,0}$, then

$$P(s) = \left[\underline{p}_{1,h}, \quad \ldots, \quad \underline{p}_{m,h}\right] diag\left\{s^{\delta_1}, \quad \ldots, \quad s^{\delta_m}\right\} + \hat{P}(s) \tag{2.68}$$

where the columns of $\hat{P}(s)$ have degrees less than δ_i. The matrix

$$P_h = \left[\underline{p}_{1,h}, \quad \ldots, \quad \underline{p}_{m,h}\right] := \left[P(s)\right]_h \in \mathbb{R}^{l\times m} \tag{2.69}$$

is referred to as the *high column coefficient* matrix of *P*(*s*). If $\rho(P_h) = m$, then *P*(*s*) is called *column reduced.* The *high row coefficient* matrix and *row reducedness* notions are defined similarly [Kai., 1].

A matrix $P(s) \in \mathbb{R}^{l\times m}[s]$ with $\rho(P(s)) = m$ is called *right irreducible*, or *least degree*, if all RMDs are $\mathbb{R}[s]$-unimodular, i.e. $|P(s)| = c \in \mathbb{R}$, $c \neq 0$. A *left irreducible* matrix is defined in a similar manner. $P(s) \in \mathbb{R}^{l\times m}[s]$ with $\rho(P(s)) = m$ (or *l*) is called a *minimal basis* [For., 1] if it is

1. Right (left) irreducible
2. Column reduced

If $P_r := \left\{P_i(s) \in \mathbb{R}^{l_i\times m}[s],\ i \in v\right\}$ is a set of matrices then the matrix

$$T_P^r(s) := \begin{bmatrix} P_1(s) \\ \vdots \\ P_v(s) \end{bmatrix} \in \mathbb{R}^{l\times m}[s] \tag{2.70}$$

is called a *matrix representative* of P_r, where $l = \sum_{i=1}^{v} l_i$. If $\rho(T_P^r(s)) = m$, then P_r is *right regular*. If P_r is right regular and $T_P^r(s)$ is right irreducible, then the set P_r is called *right coprime*. *Left coprimeness* is defined in a similar way [Kai., 1].

Suppose that $G(s) \in \mathbb{R}^{l\times m}(s)$ and $\rho(G(s)) = \min\{l, m\}$. It is well established that *G*(*s*) can always be factorised in a nonunique way as

$$G(s) = \widetilde{D}^{-1}(s)\widetilde{N}(s) = N(s)D^{-1}(s) \tag{2.71}$$

$\widetilde{D}^{-1}(s)\widetilde{N}(s)$ and $N(s)D^{-1}(s)$ are *left* and *right matrix fraction descriptions* respectively. If $\left[\widetilde{D}^{-1}(s),\widetilde{N}(s)\right]$ and $\left[N(s),D^{-1}(s)\right]$ are left and right coprime respectively, the corresponding MFDs are referred to as *coprime* [Kai., 1].

2.5.4 THE PLÜCKER MATRIX

Let a polynomial matrix be described by $A(s)=\left[\underline{a}_1(s),\quad \underline{a}_2(s)\right]\in \mathbb{R}[s]$, where

$$A(s)=\left[\underline{a}_1(s),\quad \underline{a}_2(s)\right]=\begin{array}{c}\begin{bmatrix} 1 & s \\ s+1 & -1 \\ 0 & s+2 \\ s & 1 \end{bmatrix}\end{array}\left.\begin{array}{c}1\\2\\3\\4\end{array}\right\}\text{row numbers}$$

The exterior product is defined by $\underline{a}_1(s)\wedge\underline{a}_2(s)$, which are the minors of $A(s)$ in lexicographical order, i.e.

$$\underline{a}_1(s)\wedge\underline{a}_2(s)=\begin{bmatrix} -s^2-s-1 \\ s+2 \\ -s^2+1 \\ s^2+3s+2 \\ 2s+1 \\ -s^2-2s \end{bmatrix}\begin{array}{c}1,2\\1,3\\1,4\\2,3\\2,4\\3,4\end{array}$$

This can then be split into a coefficient matrix and a parameter matrix, as follows

$$P=\begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}$$

where P is called the Plücker matrix [Kar., & Gia., 2].

2.6 SUMMARY

The mathematical contents of this chapter are necessary in order to provide cohesion to the concepts investigated throughout this thesis. The eigenvalue-eigenvector analysis of Section 2.2 lays the foundation for the eigenstructure assignment to be covered in Chapters 6, 7 and 8. The solution of linear time invariant systems study is particularly relevant to the output controllability of Chapter 5. The study of geometric and computational issues in the solution of linear systems is especially essential for the measures of controllability and observability studied in Chapters 3 and 4. The study of matrix fraction descriptions covered in Section 2.5 provides the necessary background for a new method of eigenstructure assignment developed in Chapter 8.

3 MEASUREMENT AND CONTROL PROBLEMS FOR LARGE SCALE SYSTEMS

3.1 INTRODUCTION

Large scale systems always pose stern challenges throughout the design process. For such systems the selection of suitable sets of inputs and outputs is a problem that is not automatically resolved by the specifications and nature of the problem, but imposes new design issues. The control and measurement of the numerous variables on such systems are the principal concerns of control engineers, and a formulation of such problems has to be addressed. The role of the control engineer is essential throughout the evolution of the design of the system. Control aspects can be facilitated by using the notion of controllability, and similarly measurement issues can be aided by observability. Flexible structures are a type of system that is contained within the family of large scale systems, and as a paradigm will be used because for such systems the issues (such as controllability and observability) covered in this thesis are highly applicable to them. This chapter will first take a look at the formulation of control problems and the design process. Following pertinent definitions, a mathematical analysis of flexible structures will be carried out, and will end with a look at related controllability and observability issues.

3.2 DEVELOPMENT OF CONTROL PROBLEMS

3.2.1 FORMULATION OF CONTROL PROBLEMS

There are essentially two bodies of knowledge which can be attributed to the way engineering systems operate and react. *Systems analysis* provides a reasoning to the behavioural characteristics of system responses. *Control theory* is aimed at providing mechanisms as to how a system can be physically altered in order to change the response in a desired manner [Ske., 1]. These two tasks are no doubt related, especially where modification to the dynamics of a system is concerned. This can be done in two ways, either to modify the system parameters, or to change the forcing functions (more commonly termed as inputs) using differential equations.

A typical control problem involves the computation of an adequate forcing function so that the responses behave in an acceptable way [Ske., 1]. If this forcing function is specified as a function of time, then it is adequate enough for an open loop control law to be employed. Yet if it is a function of the system responses, then a closed loop policy is used. A *controller* is the device used to calculate the desired forcing function with respect to the system responses. As a precursor to solving control problems, it is necessary to study physical sciences (i.e. electricity, mechanics, aerodynamics, chemistry, etc.). This paves the way for control engineers to use the application of known physical laws in order to develop mathematical models of engineering systems. This in turn raises two questions. The first is how to develop the mathematical model of a physical system. The second is what to do with the model once it has been derived. This thesis concentrates on the latter, specifically with state space models, yet it is evident that these two questions are not unrelated. This is because it is impossible to know what level of detail is needed in the model prior to knowing the accuracy that needs to be achieved of the controlled performance, and the nature of the control inputs required to attain this performance. Thus the control problem can be restated, where an appropriate forcing function must be found using a given or developed model. Consequences may arise from the type of control policy chosen, since in feedback control, regulating the forcing function is dependent on the type of model chosen to describe the physical system [Ske., 1].

The efficiency of the controller designed to implement a particular control law is highly dependent on the model of the system developed. Thus the accuracy of the model characterises the ability of the developed controller to achieve the control objectives. Inaccurate system models will consequently hamper the attempts of the control engineer to produce an efficient controller.

For the purposes of the research to be carried out here, traditional control techniques are not the main area of interest. Instead focus will be placed on problems which affect the final model used for control, from which the control potential can be determined. Large scale systems are an area where such problems are realistic.

3.2.2 THE SYSTEM AND ITS EVOLUTION IN DESIGN

Controller design is not the only aspect dependent on a mathematical system model. The design process does indeed begin with a model, but from there decisions are taken that will contribute to the gradual shaping of the final structural characteristics. Yet structural properties and thus performance, operability, etc. characteristics evolve in a complex manner, as operational targets may change throughout the duration of the project [Kar., 2]. The crux of the design procedure is thus to evolve the model along paths that avoid undesirable structural characteristics, and to specify where it is possible to assign desirable ones. The principal structure assigning activity areas are those of *Process Synthesis* and *Global Process Instrumentation.* The first of these activities deals with the structuring of the "internal mechanism" of the system and the second with the building of bridges between the internal mechanism and the "system environment." An overall system, such as a flexible structure, may be represented in simple terms by the diagram in Figure 3.1 [Kar., 2]. The internal mechanism of the process is the set of all the independent internal variables, signals and attributes referred to as the **internal space,** $\mathcal{X}$, irrespective of whether they can be measured (or acted upon) and the relationships between them. These relationships express the physical laws describing the phenomena associated with the interconnection of subsystems and manifest themselves in the **internal map** f of the system. The process system environment refers to the three set spaces of the signals $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{D}$ which are referred to as the **input space**, **output space** and **disturbance space** respectively.

$\mathcal{U}$ is the space of all the external, arbitrarily assignable signals which may be applied to the system. $\mathcal{Y}$ is the space of all possible signals that may be measured,

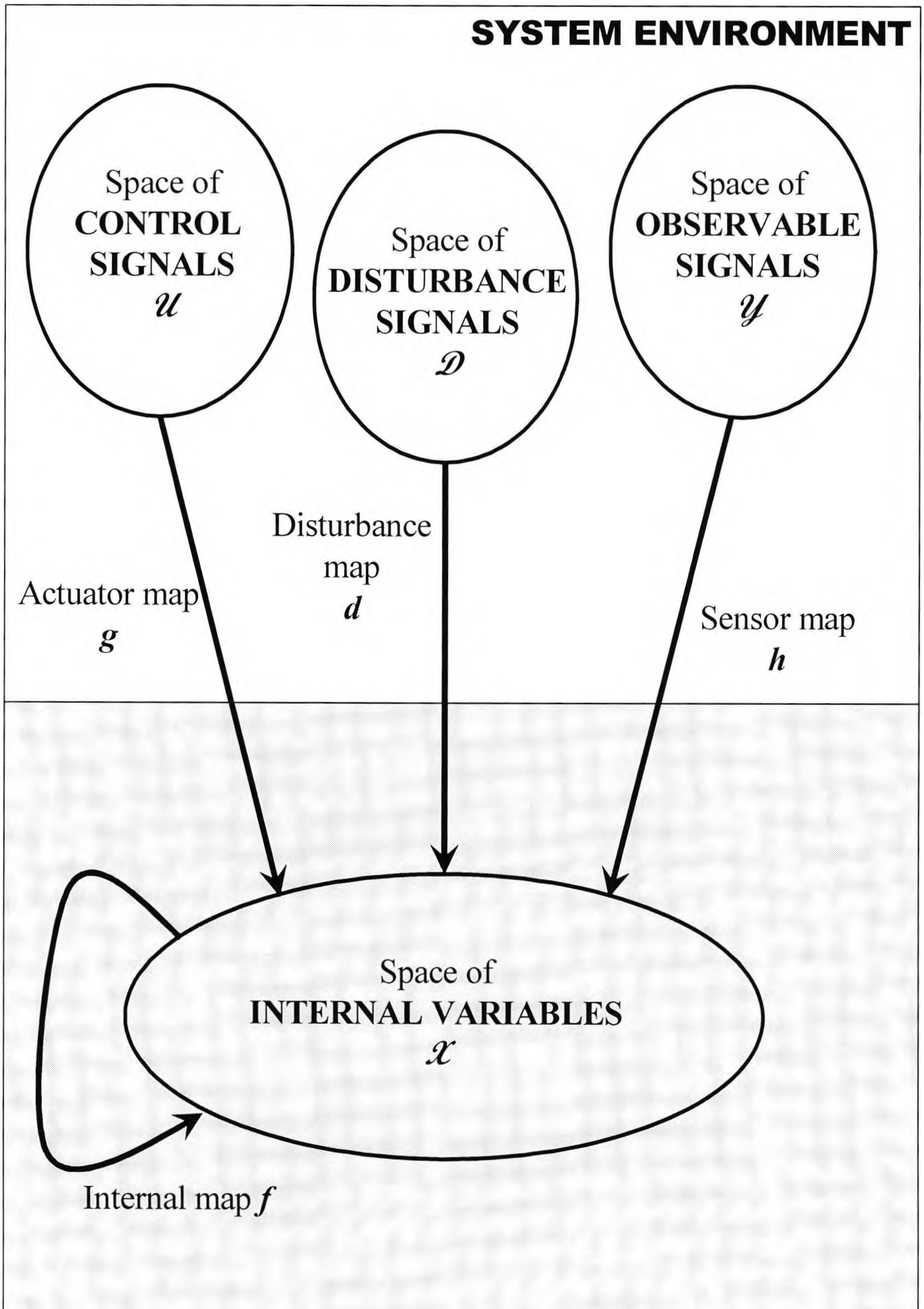


***Figure 3.1** Representation of overall system*

and $\mathscr{D}$ is the space of all disturbances that may affect the system. As a result of instrumentation, two maps (or functions) can be constructed [Kar., 2]. The first map is denoted by $\boldsymbol{g}$ and expresses the coupling of the input space to the internal variable spaces and it is called the **input** or **actuator map**, since it is the result of selecting *actuators*. The second map, $\boldsymbol{h}$, expresses the coupling of the internal variables to the environment and is called the **output** or **sensor map** since it is the result of selecting *sensors*. The coupling of disturbances to the internal mechanism is expressed by the map $\boldsymbol{d}$. These disturbances may be measurable or unmeasurable and are referred to as the **disturbance map** [Kar., 2].

The instrumentation of a system basically involves the selection of measurement variables (outputs) and actuation variables (inputs). It consists of two aspects [Kar., 1], the first of which deals with the problem of measurement, or implementation of an action upon given variables. The second aspect stems from designing an instrumentation scheme for a given process (classification and selection of input and output variables), and expresses the attempt of the "observer" (designer) to build bridges with the internal mechanism of the process in order to observe it and/or act upon it. What is considered as the final system, on which control systems design is to be performed, is the product of the interaction of the internal mechanism with the specification of the overall instrumentation scheme. It is vital that the designer deals with the issues arising in the selection of actuation systems and measurement variables which should aid in the fabrication of a system and control laws to create a framework for global instrumentation. This is ultimately linked to the problem of the selection of input and output schemes [Kar., 1].

In order to satisfy a set of control laws, a control engineer must decide whether a suitable controller can be implemented, or whether some of the system parameters have to be altered. Parameter changes could handicap other areas of the physical system, such as weight gain or loss of speed. Therefore a balance is needed between modifying the structural design of a system and increasing the sophistication of the controller. As control requirements become more demanding in modern applications it may be necessary to add more control variables since the aspects that need to be controlled may be uncontrollable from a single device. This in turn consolidates the need for instruments to measure the action and responses on certain parts of the physical system. These additional components add weight and increase the cost of a project, so ways have to be found to optimise the number of devices to meet the control criteria, which has to be achieved using a mathematical model.

One of the first questions that a designer of a control scheme for a large scale system, such as flexible space structures, faces is how many components - *actuators* and *sensors* – are needed for the mission objectives to be met, and where to place them on the structure (spatial distribution), as well as analysing their dynamical behaviour. Such issues affect the controllability and observability properties. In order to assist the designer in making such choices, the proceeding chapters will discuss and present certain methodologies. Existing and new measures of controllability, which are defined as quantitative indications of how well the system can be controlled with a given set of actuators, will be presented. Similarly, measures of observability will also be looked at, and are defined as quantitative indications of how well a system can be observed given a set of sensors. This work is stimulated by the fact that in the thoroughness of control theory there is little provision for a quantitative measure of controllability or observability. Controllability (observability) is merely a binary concept, either a system is controllable (observable), or it is not. What is required, and what will be investigated later in the thesis, is a quantitative indication of how well a system can be controlled by a given set of actuators, i.e. a fundamental measure which does not depend on the design of the system. Likewise, a quantitative measure of how well a system can be observed by a given set of sensors will also examined. The spatial distribution of sensors and actuators affects the input, output model properties and thus the potential for control design and it is a fundamental issue in integrated instrumentation and control.

As a background for the following chapters, certain control and measurement problems, specifically *controllability* and *observability* will be addressed here. Chapter 4 will examine existing methods of measuring a degree of controllability (observability) and present new methodologies. As a paradigm, a physical system known as a *flexible structure*, will be used because for such systems the control structure selection problem is fundamental to them due to the high number of states involved. This family includes large space structures, rotating machinery, vibrational systems, wind turbines, dynamics of aircraft wings, dynamics of bridges and buildings, power systems etc.

3.3 DEFINITIONS

3.3.1 CONTROLLABILITY

If a particular part of a system needs to be controlled, it must be determined whether a desired objective can be achieved by manipulating the chosen control variables. The general property of being able to transfer a system from any given state to another via a suitable choice of control functions can be defined as controllability. Given a linear, system described by

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, the following definitions of controllability can be made.

Definition 3.1: A linear system is state controllable at t_0 if it is possible to find an input function $u(t)$, defined over the time of interest, that will transfer the initial state $x(t_0)$ to the origin in finite time.

□

Definition 3.2: A system is said to be *completely controllable* if, for any initial time t_0, any initial state $x(t_0) = x_0$ and any given final state x_f, there exists a finite time $t_f > t_0$ and a control $u(t)$, $t_0 \leq t \leq t_f$, such that $x(t_f) = x_f$.

□

Equivalent mathematical conditions for a system to be completely controllable are given in the following theorem.

Theorem 3.1: [Bar. & Cam., 1] [Kai., 1] [Ros., 1] A system is said to be completely controllable if and only if one of the following equivalent conditions holds.

(i) $rank\left[B, \quad AB, \quad \ldots, \quad A^{n-1}B\right] = n$

(ii) $rank\left[\lambda I_n - A, \quad -B\right] = n, \quad \forall \lambda \in \mathbb{C}$

(iii) $\underline{v}_i'(\lambda_i I_n - A) = 0$ and $\underline{v}_i' B \neq 0 \ \forall \ \lambda_i \in \mathbb{C}$

where $\underline{v}_i'$ is the left eigenvector corresponding to the mode λ_i.

□

The controllability property plays an important role in many control problems, such as stabilisation of an unstable system by feedback, or optimal control. Although controllability is a binary property, the degree of controllability is a measure worth investigating since it affects the solution/behaviour of a variety of problems/systems. However the degree of controllability is a problem, where system dynamics could hamper the control effort needed.

3.3.2 OBSERVABILITY

Closely related to the concept of controllability is that of observability. This is defined as the possibility of determining the state of a system by measuring only the outputs. For the system described by the differential equations given in (3.1), the following definition of observability can be made.

Definition 3.3: [Bar. & Cam., 1] [Kai., 1] A system is said to be *completely observable* if, for any initial time t_0, and any initial state $x(t_0) = x_0$, there exists a finite time $t_f > t_0$ such that knowledge of $u(t)$ and $y(t)$ for $t_0 \leq t \leq t_f$ is sufficient to determine x_0 uniquely. There is no loss of generality in assuming $u(t) \equiv 0$ throughout the interval.

□

A characterisation of the observability property is given in the following theorem.

Theorem 3.2: [Ros., 1] A system is said to be *completely observable* if and only if one of the following equivalent conditions holds.

(i) $$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

(ii) $$rank\begin{bmatrix} \lambda I_n - A \\ -C \end{bmatrix} = n, \ \forall \lambda \in \mathbb{C}$$

(iii) $$(\lambda_i I_n - A)\underline{u}_i = 0 \text{ and } C\underline{u}_i \neq 0, \ \forall\ \lambda_i \in \mathbb{C}$$

where $\underline{u}_i$ is the right eigenvector corresponding to the mode λ_i.

□

As with controllability, observability is also a property of a binary nature and the system dynamics plays a part in determining the degree to which the outputs can be measured, and this needs further investigation.

3.3.3 CONTROLLABILTY AND OBSERVABILITY ISSUES

While analysis of system behaviour in the frequency domain may convey many insights into the way processes react in a simple graphical manner, there are a variety of characteristics which cannot be determined in using such approaches. Controllability and observability fall into this category. The concepts of controllability and observability deal with the ability of a control system to measure and control the states of a given system.

Frequency domain analysis techniques involve the assumption that the response of the system can be completely determined by its transfer function for zero initial conditions, or in other words, that the system is both controllable and observable. However it has been shown [Kal., 1] from early development of state space analysis that this assumption cannot be strongly validated. In fact, generally, a system can be viewed as comprising of four sub-systems: one which is both controllable and observable, one which is controllable but not observable, one which is observable but not controllable, and one which is neither controllable nor observable. It is possible that the transfer function of a single input, single output system is of a lower degree than the corresponding state space dimension which indicates that the system contains uncontrollable or unobservable states. This leads to the conclusion that the transfer function of a system does not enable the state response to be determined as a whole.

The simple fact that a system has uncontrollable and/or unobservable states does not necessarily present a problem. If all the eigenvalues of A are in the left half plane, then any initial conditions in the uncontrollable and unobservable states will decay to zero over time and the system will be stable. The poles of the uncontrollable and/or unobservable states can however be in the right half plane, thus making these states unstable. A system whose uncontrollable states with are stable is referred to as *stabilisable* [Kal., 1]. A system which has stable, unobservable states is called *detectable*. Uncontrollable and/or unobservable states can arise from a variety of situations. The most common problems arise due to poor actuator and sensor placement.

3.3.4 FLEXIBLE STRUCTURES

A significant paradigm, where the problem of selecting inputs and outputs becomes highly essential is that of flexible structures. Large *flexible space structures* are characterised by very light damping, a very low frequency range and a large number of elastic modes, which is a consequence of their large size and weight [Jos., 1]. Gawronski [Gaw., 1], describes a flexible structure as a linear system with oscillatory properties that is characterised by a strong amplification of a harmonic signal for certain frequencies (resonant frequencies), and whose transfer function poles are complex conjugate, typically with small real parts. An aircraft wing is a classical example of a flexible structure. The wings of an aircraft tend to bend upwards, centred at the edge where it joins onto the fuselage, whilst in flight. Whilst in the air, the wing also flaps about slightly. These oscillations are determined by the nature of the complex poles attributed to the structure. The small real parts define the envelope of the oscillations. Although this description of a flexible structure is widely accepted, there seems to be some dispute about what properties are possessed by such a structure, whether it is linear, or whether the poles have to have small real parts (i.e. a small damping property), or whether all its poles must be complex conjugate.

Control problems regarding flexible space structures stem from disturbances (from on board operations, positioning manoeuvres, mico-meteorite impacts, etc.) [Ton., & Mel., 1], which excite the flexible modes of these structures. Therefore there is need to develop a control scheme to enhance the damping of the structure and to provide active vibration suppression of these unwanted motions. For such systems the location of sensors and actuators is crucial for the development of control schemes. The study of the dynamics and control of flexible structures has been an area of interest for many years. The majority of studies conducted on flexible space structures address the problem of precision of attitude control and the introduction of active damping elements to control the vibrations and shape distortions which result from the inherent flexibility of such a structure. An area that requires further research is the problem of sensor and actuator placement on a large flexible space structure. Because of weight, cost and control performance considerations, there is the need to determine an optimal number of sensors and actuators and where to strategically place them. The problem of choosing actuator (sensor) locations for the control of large flexible space structures is an important area of research [Kim & Jun., 1].

In the study undertaken here, a flexible structure provides the motivation for some of the issues that are addressed and will be defined as a finite-dimensional, controllable, and observable linear system with complex poles and with small damping characteristics. A linear system that fits this definition may be represented by the second order matrix differential equation

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= Bu \\ y &= C_q q + C_v \dot{q} \end{aligned} \tag{3.2}$$

where q is the displacement vector, u the input vector, y the output vector, M the mass matrix, D the damping matrix, K the stiffness matrix, B the input matrix, C_q the output displacement matrix and C_v is the output velocity matrix. Typically, the mass matrix is positive definite, which means that all its eigenvalues are positive, and that all its principal minors have positive determinants [Ske., 1]. The stiffness and damping matrices are typically positive semidefinite, as all their eigenvalues are either positive or zero and all the principal minors of such matrices have either positive or zero determinants [Ske., 1]. Such a description can always be transformed to a state space one by an appropriate realisation, from which controllability and observability properties can be established.

3.3.5 ACTUATORS AND SENSORS

The physical elements used for observability and controllability are *sensors* and *actuators* respectively. A sensor is placed at a particular point of a system in order for a measurement to be taken that signals whether it is operating within acceptable parameters. An actuator implements the control action that is required to bring the system to a certain desired state. The optimal positioning of actuators and sensors is an important aspect in the active control of certain engineering systems like satellites and other aerospace applications. It is therefore vital to consider the positioning of sensors and actuators in order to maximise their effectiveness. Proper positioning of the sensors will improve the ability of the observer to observe the states of the system. Carefully situated actuators will increase their control effect on the response modes of the structure. Work could be carried out in order to examine the effect that the placement of sensors and actuators has on the dynamics of a system, and their effect on the system controllability and observability. There is also a danger in overcrowding a system with sensors and actuators, as this would add to a more complicated system, both mathematically and financially. With the application of flexible structures, overburdening a spacecraft for example with too many components, may have a detrimental effect on its weight distribution. It is therefore vital to minimise the number of sensors and actuators used, and to investigate a way in which this can be achieved. This type of investigation can be hindered by the fact that the freedom of choice of locations may be limited, making it essential for the accurate and minimal placement of sensors and actuators.

3.4 Representation and properties of flexible structures

3.4.1 INTRODUCTION

So far a number of definitions have been given which pave the way for a deeper analysis to be conducted into the controllability and observability properties of flexible structures. It is clear that the degree to which a system can be controlled is vital in the early design stages, and thus a measurement index is needed in order to limit the number of actuators needed to perform a specific control function. Likewise a measurement of observability is needed to optimise the number of sensors required. The remainder of this chapter will delve deeper into the mathematical analysis of flexible structures, controllability and observability.

3.4.2 STATE SPACE REPRESENTATIONS

Following on from the definition of a flexible structure [Gaw., 1] from Section 3.2.3, in order to proceed with the analysis, equation (3.2) has to be rewritten in the state space form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.3}$$

The matrices A, B and C form the state-space representation of the system, and x is the state vector. Both the state representation and state vector are not unique, which means that the same input-output relationship can be obtained for different states. In order to obtain a state space representation of equation (3.2), it is rewritten into the form below, assuming that M is non-singular, and thus invertible

$$\begin{aligned} &\ddot{q} + M^{-1}D\dot{q} + M^{-1}Kq = M^{-1}B_o u \\ &y = C_q q + C_v \dot{q} \end{aligned} \tag{3.4}$$

The state vector is defined as $x^T = \left[q^T \; \dot{q}^T\right]$, where the first component is the system displacement, and the second is the system velocity. By manipulating equation (3.4), the following minimal state space representation is obtained

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ M^{-1}B_o \end{bmatrix}, \quad C = \begin{bmatrix} C_q & C_v \end{bmatrix} \tag{3.5}$$

where the dimension of the state model n is twice the number of its degree of freedom, n_2, i.e., $n = 2n_2$ [Gaw., 1].

Due to the unacceptably high order, n, of such engineering systems that could be described by equation (3.5), this type of representation is hardly used. Alternatively, equation (3.2) is represented in a modal form, by using a modal matrix Φ (n_2 x p), $p \leq n_2$, which consists of p eigenvectors (mode shapes) of a structure ϕ_i, $i = 1, \ldots, p$

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \end{bmatrix}, \; p \leq n_2 \tag{3.6}$$

The modes of equation (3.6) diagonalise the mass and stiffness matrices M and K to give

$$M_m = \Phi^T M \Phi$$
$$K_m = \Phi^T K \Phi \tag{3.7a}$$

where M_m and K_m are the corresponding diagonalised matrices, assuming they are diagonalisable, and are of dimension $p \times p$. The number of modes, p, is usually much smaller than the number of degrees of freedom, n_2, so a substantial reduction in matrix dimension is achieved. Similarly, if the damping matrix is diagonalisable, a proportional damping matrix is obtained

$$D_m = \Phi^T D\Phi \tag{3.7b}$$

Continuing with deriving the modal form for equation (3.2), a new variable q_m ($p\times 1$) is introduced

$$q = \Phi q_m \tag{3.8}$$

Then, if (3.2) is left-multiplied by Φ^T, equation (3.4) can be rewritten in the form

$$\begin{aligned} &\Phi^T M\Phi\ddot{q}_m + \Phi^T D\Phi\dot{q}_m + \Phi^T K\Phi q_m = \Phi^T B_o u \\ &y = C_q\Phi q_m + C_v\Phi\dot{q}_m \end{aligned} \tag{3.9a}$$

or equivalently by substituting the equations of (3.7)

$$\begin{aligned} &M_m\ddot{q}_m + D_m\dot{q}_m + K_m q_m = \Phi^T B_o u \\ &y = C_q\Phi q_m + C\Phi\dot{q}_m \end{aligned} \tag{3.9b}$$

and then by multiplying by M_m^{-1}

$$\begin{aligned} &\ddot{q}_m + M_m^{-1}D_m\dot{q}_m + M_m^{-1}K_m q_m = M_m^{-1}\Phi^T B_o u \\ &y = C_q\Phi q_m + C_v\Phi\dot{q}_m \end{aligned} \tag{3.9c}$$

To obtain the final form of the modal model, it is necessary to make an analogy with the standard second order transfer function of the form $\dfrac{K_{ss}\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$. This is achieved by denoting $M_m^{-1}K_m = \Omega_m^2$, where Ω_m = $diag(\omega_i)$, which is a square diagonal matrix of dimension p that consists of natural frequencies ω_i (rad/sec). Also, by setting $M_m^{-1}D_m = 2\mathrm{Z}\Omega$, where Z = $diag(\xi_i)$ is the modal damping coefficient matrix which consists of ξ_i (damping coefficient of the ith mode), the final version of the modal model is derived from (3.9c)

$$\begin{aligned} &\ddot{q}_m + 2Z\Omega_m \dot{q}_m + \Omega_m^2 q_m = M_m^{-1}\Phi^T B_o u \\ &y = C_q \Phi q_m + C_v \Phi \dot{q}_m \end{aligned} \tag{3.10}$$

Before (3.10) can be written as a set of first order equations, the state variable $x^T = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} = \begin{bmatrix} q_m^T & \dot{q}_m^T \end{bmatrix}$ is defined, giving [Gaw., 1]

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = -\Omega_m^2 x_1 - 2Z\Omega_m x_2 + M_m^{-1}\Phi B_o u \\ &y = C_q \Phi x_1 + C_v \Phi x_2 \end{aligned} \tag{3.11}$$

These equations can also be represented in the matrix form of (3.3), where [Gaw., 1]

$$\begin{aligned} &A = \begin{bmatrix} 0 & I \\ -\Omega_m^2 & -2Z\Omega_m \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M_m^{-1}\Phi^T B_o \end{bmatrix} \\ &C = \begin{bmatrix} C_q \Phi & C_v \Phi \end{bmatrix} \end{aligned} \tag{3.12}$$

The matrices of equation (3.12) form the state space representation in modal co-ordinates of the initial set of differential equations, and is a realisation stemming from the original differential model of (3.4). Here, x_1 is the vector of modal displacements, and x_2 is the vector of modal velocities. Its dimension is $2p$ (where p is the number of the structure's eigenvectors), compared to the state space representation of equation (3.5), whose dimension is $2n_2$, $2p << 2n_2$. Controllability and observability properties can be established from the state space matrices represented by (3.12). Although the state space matrices of (3.12) represent a possible realisation of a flexible structure, controllability and observability properties are equally applicable to any state space system.

3.4.3 STATE SPACE MODAL REPRESENTATIONS

The state space representation derived in Section 3.3.1 was obtained in the modal coordinate form, q_m. However, they are not *modal state* equations. Such a representation contains the three matrices A, B, and C in a special, unique form, characterised by the block diagonal matrix A

$$A = diag(A_i), \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n2} \end{bmatrix} \tag{3.13}$$
$$C = \begin{bmatrix} C_1 & C_2 & \cdots & C_{n2} \end{bmatrix}$$

where $i = 1, 2, \ldots, n_2$. The diagonal A_i blocks are in four different forms [Gaw., 1], assuming distinct, complex eigenvalues:

Modal Form 1

$$A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\zeta_i\omega_i \end{bmatrix} \tag{3.14a}$$

Modal Form 2

$$A_i = \begin{bmatrix} -\zeta_i\omega_i & \omega_i \\ -\omega_i & -\zeta_i\omega_i \end{bmatrix} \tag{3.14b}$$

Modal Form 3

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix} \tag{3.14c}$$

Modal Form 4

$$A_i = \begin{bmatrix} -\zeta_i\omega_i + j\omega_i\delta_i & 0 \\ 0 & -\zeta_i\omega_i - j\omega_i\delta_i \end{bmatrix} \tag{3.14d}$$

where $j = \sqrt{-1}$ and $\delta_i = \sqrt{1-\zeta_i^2}$. The ith state component, x_{ik}, corresponding to the ith block of the kth modal form is as follows [Gaw., 1]

$$x_{i1} = \begin{bmatrix} q_{mi} \\ \dot{q}_{mi}/\omega_i \end{bmatrix}, \quad x_{i2} = \begin{bmatrix} q_{mi} \\ q_{moi} \end{bmatrix}$$
$$x_{i3} = \begin{bmatrix} q_{mi} \\ \dot{q}_{mi} \end{bmatrix}, \quad x_{i4} = \begin{bmatrix} q_{mi} - jq_{moi} \\ q_{mi} + jq_{moi} \end{bmatrix} \tag{3.15}$$

where q_{mi} and $\dot{q}_{mi}$ are the ith modal displacement and velocity respectively, with $q_{moi} = \zeta_i q_{mi} + \dot{q}_{mi} / \omega_i$.

Modal forms 1, 2 and 3 all consist of real numbers, whereas modal form 4 is a complex representation and creates unnecessary numerical difficulties. Modal form 3 is very similar to the modal coordinate A matrix in equation (3.12). In fact it is obtained by simply rearranging the columns of A and C and the rows of A and B. As a result of this rearrangement, the state vector $x^T = \begin{bmatrix} q_m^T & \dot{q}_m^T \end{bmatrix}$, which contains the modal displacements and the modal velocities, is transformed to the new state $x_n^T = \begin{bmatrix} q_{m1} & \dot{q}_{m1} & q_{m2} & \dot{q}_{m2} & \cdots & q_{mn} & \dot{q}_{mn} \end{bmatrix}$, where the modal displacement for each component stays next to its velocity. The transformation is carried out using the matrix R [Gaw., 1]

$$R = \begin{bmatrix} 0 & e_1 \\ e_1 & 0 \\ 0 & e_2 \\ e_2 & 0 \\ \vdots & \vdots \\ 0 & e_{n2} \\ e_{n2} & 0 \end{bmatrix} \tag{3.16}$$

where e_i is an n_2 row vector, where all its elements are zero, except the ith which is equal to one.

If A_k denotes the state matrix A in the modal form k, where $k = 1, 2, 3$, or 4, then the transformation matrix, R_{kl}, that transforms the state variable x_k into x_l, $x_l = R_{kl}x_k$, $k, l = 1, 2, 3$, or 4 is

$$R_{kl} = \operatorname{diag}(R_{kli}) \tag{3.17}$$

If small damping is assumed, i.e. $\zeta_i << 1,\ i = 1,\cdots,n_2$, the following state transformations are obtained

$$\begin{aligned} R_{12i} &= \begin{bmatrix} 1 & 0 \\ \zeta_i & 1 \end{bmatrix},\ R_{13i} = \begin{bmatrix} 1 & 0 \\ 0 & \omega_i \end{bmatrix},\ R_{14i} = \begin{bmatrix} 1 - j\zeta_i & -j \\ 1 + j\zeta_i & j \end{bmatrix} \\ R_{23i} &= \begin{bmatrix} 1 & 0 \\ -\xi_i\omega_i & \omega_i \end{bmatrix},\ R_{24i} = \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix},\ \mathrm{R}_{34i} = \begin{bmatrix} 1 - j\zeta_i & -j/\omega_i \\ 1 + j\zeta_i & j/\omega_i \end{bmatrix} \end{aligned} \tag{3.18}$$

The remaining transformations can be derived from those above by simply using

$$R_{kpi} = R_{pki}^{-1} \tag{3.19}$$

or by noting that

$$R_{kpi} = R_{lpi}R_{kli} \tag{3.20}$$

where $l, k, p = 1, 2, 3$ or 4.

3.4.4 EXAMPLE OF A FLEXIBLE STRUCTURE

An example of a simple flexible structure is a three-mass system [Gaw., 1], as shown below in Figure 3.2. Its simplicity enables easy analysis and straightforward interpretation. The system has three masses m_1, m_2 and m_3, stiffnesses k_1, k_2, k_3 and k_4. It also has a damping matrix which is a linear combination of the stiffness and mass matrices, $D = \alpha K + \beta M$, where $\alpha > 0$, $\beta > 0$ are constants

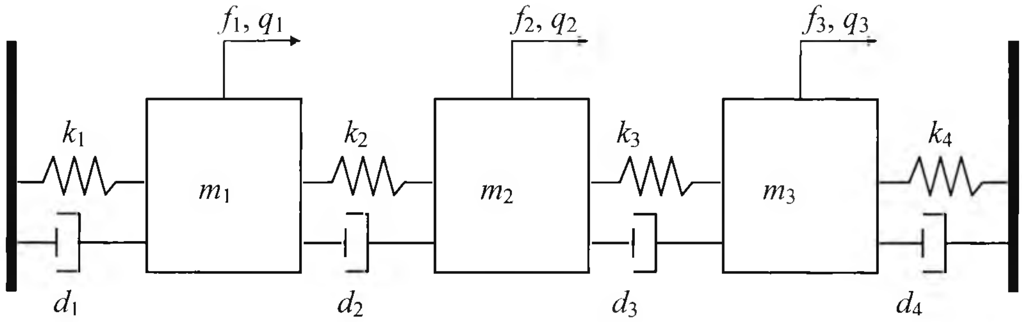


Figure 3.2 *Simple flexible structure*

3.5 CONTROLLABILITY AND OBSERVABILITY GRAMMIANS

3.5.1 INTRODUCTION

Controller design is heavily dependent on the controllability and observability properties of a plant. Controllability is a property of the plant's input that excites the total system dynamics. Basically, it is the ability of the input actuators to excite all the states of a system. Observability is a property of the plant's output, and it is the ability of the output to sense all the states. However, controllability and observability cannot provide sufficient information for controller design by themselves. For example, parts of a system may be weakly controllable (weakly influenced by the plant input), making it impossible to decide which controller to use, yet these same states could be strongly observable at the output. Similarly, weakly observable states could be strongly controllable by the input actuators. However, if the system states are both weakly controllable and weakly observable, they can be ignored in the controller design. The joint controllability and observability properties of a system can be characterised by the Hankel singular values, and has been developed by Moore [Moore, 1].

One way of determining the level of participation of a state variable to a system has been developed by Skelton [Ske., 1]. Instead of using controllability and observability properties, Skelton associated each state variable's participation with a cost. If the participation is strong, the cost is high. If it is weak, then the cost is low. There are disadvantages to this method. Firstly, the cost evaluation of a particular mode is made at the output of the system, whilst the input is not taken into consideration. Also, if two states are strongly coupled, their individual costs are not reflected in their participation to the output of the system. A more convenient way of analysing state variable participation to a system is by using *observability* and *controllability grammians*.

Before analysing the controllability and observability grammians, there follows a short mathematical examination of one of the most useful tools that has widespread applications in several areas of matrix theory, i.e. *grammians*.

Definition 3.3: [Gant., 1] Let $(\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n) \in \mathbb{R}^n$ be a set of vectors. The matrix

$$G = \begin{bmatrix} (\underline{x}_1 \cdot \underline{x}_1) & (\underline{x}_1 \cdot \underline{x}_2) & \cdots & (\underline{x}_1 \cdot \underline{x}_m) \\ (\underline{x}_2 \cdot \underline{x}_1) & (\underline{x}_2 \cdot \underline{x}_2) & \cdots & (\underline{x}_2 \cdot \underline{x}_m) \\ \vdots & \vdots & \ddots & \vdots \\ (\underline{x}_m \cdot \underline{x}_1) & (\underline{x}_m \cdot \underline{x}_2) & \cdots & (\underline{x}_m \cdot \underline{x}_m) \end{bmatrix} \in \mathbb{R}^{m \times m} \tag{3.21}$$

is called the *gram matrix* of the vectors $(\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n)$ and the determinant $|G|$ is called the *grammian* of the vectors $(\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n)$. The grammian of linearly independent vectors is always positive, and that of linearly dependent vectors is zero. Negative grammians do not exist.

□

The grammian has an important ability in the sense that it can provide an indication about the degree of linear dependence of vectors.

Theorem 3.3: [Gant., 1] The vectors $(\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n)$ are linear independent if and only if their grammian is not equal to zero. This is known as **gram's criterion**. If any principal minor of the grammian is zero, then the grammian is zero.

□

Gram matrices are related with positive definite matrices.

Theorem 3.4: [Horn & Jon., 1] Let $G \in \mathbb{C}^{k \times k}$ be a given matrix of the vectors $\{\underline{w}_1, \ \underline{w}_2, \ \ldots, \ \underline{w}_n\} \subset \mathbb{C}^n$ with respect to a given inner product (.,.), and let $W = [\underline{w}_1, \ \underline{w}_2, \ \ldots, \ \underline{w}_k] \in \mathbb{C}^{n \times k}$. Then

(a) G is positive definite

(b) G is non-singular if and only if the vectors $\underline{w}_1, \ \underline{w}_2, \ \ldots, \ \underline{w}_k$ are independent

(c) There exists a positive definite matrix $A \in \mathbb{C}^{n \times n}$ such that $G = W^* A W$

(d) $\rho(G) = \rho(W)$ is the maximum number of independent vectors in the set $\{\underline{w}_1, \ \underline{w}_2, \ \ldots, \ \underline{w}_n\}$

□

The remainder of this section will concentrate on controllability and observability grammians and their different types. Each of the considered grammians take into account the theorems and definitions that have been discussed above.

3.5.2 PRINCIPAL GRAMMIANS

Controllability and observability grammians are a convenient form of characterising the properties of a system's observability and controllability. They can be defined by taking into consideration a stable linear, observable and controllable system with state space matrices A, B and C. λ_i is the ith eigenvalue of A, and the condition $\lambda_i + \lambda_k \neq 0$ is applied for every i, k = 1,..., n. Therefore the controllability and observability grammians are defined respectively [Kai., 1] as follows

$$
\begin{aligned}
W_c(t) &= \int_0^t \exp(A\tau) BB^T \exp(A^T \tau) d\tau \\
W_o(t) &= \int_0^t \exp(A^T \tau) C^T C \exp(A\tau) d\tau
\end{aligned}
\tag{3.22}
$$

They can also be determined by using the following differential equations

$$\begin{aligned}\dot{W}_c &= AW_c + W_cA^T + BB^T \\ \dot{W}_o &= A^TW_o + W_oA + C^TC\end{aligned} \tag{3.23}$$

The stationary solutions of the above equations are derived in the limiting case for $t \to \infty$. Thus, for a stable system, one obtains $\dot{W}_c = \dot{W}_o = 0$, and the grammians are determined from the following Lyapunov equations

$$\begin{aligned}AW_c + W_cA^T + BB^T = 0 \\ A^TW_o + W_oA + C^TC = 0\end{aligned} \tag{3.24}$$

If A is stable, then the solutions of equation (3.24) are *positive definite*. The system defined by (3.1) is completely controllable and completely observable if and only if the respective grammians of (3.22) are positive definite from time $t = 0$ up to the final time t.

Grammians are dependent on the system coordinates and when a state is linearly transformed, $\bar{x} = Rx$, the resulting grammian transformation is

$$\begin{aligned}\overline{W}_c &= R^{-1}W_cR^{-T} \\ \overline{W}_o &= R^TW_oR\end{aligned} \tag{3.25}$$

The eigenvalues of the grammian product do not change when linear transformation takes place, as

$$\lambda_i\left(\overline{W}_c\overline{W}_o\right) = \lambda_i\left(R^{-1}W_cR^{-T}R^TW_oR\right) = \lambda_i\left(R^{-1}W_cW_oR\right) = \lambda_i\left(W_cW_o\right) \tag{3.26}$$

Although the rank of W_c and W_o is not affected by the transformation matrix R, their respective norms are.

3.5.3 TIME LIMITED GRAMMIANS

The Lyapunov equations of (3.24) define the stationary grammians, but those defined over the finite time interval $[0, t]$ can also be determined. Assuming that the response to an excited system is measured within the time interval $T = [t_1, t_2]$, $\infty > t_2 > t_1 \geq 0$, then on rearranging the equations of (3.22), the grammians over the time interval T are defined as follows

$$\begin{aligned} W_c(t) &= \int_{t_1}^{t_2} \exp(A\tau)BB^T \exp(A^T\tau)d\tau \\ W_o(t) &= \int_{t_1}^{t_2} \exp(A^T\tau)C^T C \exp(A\tau)d\tau \end{aligned} \tag{3.27}$$

Again, for a stable A matrix, these grammians are positive definite, which means that $W_c(T) > 0$, $W_o(T) > 0$ if $t_2 > t_1$. They can be derived from the stationary grammians W_c and W_o in the following way [Gaw. & Jua., 1], [Gaw., 1]

$$\begin{aligned} W_c(T) &= W_c(t_1) - W_c(t_2) \\ W_o(T) &= W_o(t_1) - W_o(t_2) \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} W_c(t) &= S(t)W_c S^T(t) \\ W_o(t) &= S^T(t)W_o S(t) \end{aligned} \tag{3.29}$$

where

$$S(t) = -\exp(At) \tag{3.30}$$

where W_c and W_o are the unlimited time grammians, and are the solutions of equation (3.24).

3.5.4 USE OF GRAMMIANS

The control of large scale systems, such as flexible space structures, is often studied in a centralised framework [Will., & Xu, 1], where the outputs of all sensors on the structure are fed back to all of the actuators. Yet for practical implementations, a decentralised arrangement is much more feasible. Such systems have only a specified number of sensors that are connected to each particular actuator, which result in a set of independent local controllers. But this in turn creates a drawback when it comes to the ability to shift closed loop poles (Chapters 6, 7 and 8), as this may be more feasible via a centralised feedback scheme [Will., & Xu, 1]. The ability to shift closed loop poles using a decentralised control configuration for a given control effort has been studied [Leven., et al, 1], [Will., & Xu, 1]. From this research stems the ability to determine the *degree of controllability* of each mode of a system, and is essentially a centralised control problem. Results have been obtained in this area of research with flexible structure applications [Greg., 1], [Skel., et al, 2] where the natural frequencies are widely spaced. For flexible structures where the natural frequencies are very close to each other, the problem of determining the degree of controllability is more complicated [Josh., 1]. For such cases the degrees of controllability for each mode are obtained by analysing the singular values of the controllability grammian. Yet for flexible space structures, the nature of their decentralised control schemes makes use of the controllability grammian redundant as it is purely an open loop property. In order to allow the analysis to proceed, a closed loop controllability grammian has been studied [Will., & Xu, 1], defined for a flexible space structure with a constant output feedback configuration, for changes in the gain matrix and expressions for the degree of controllability are established. This study motivates further investigations into the area of degrees of controllability (Chapter 4), singular values of the output controllability grammian (Chapter 5), and pole placement/eigenvalues assignment (Chapter 6).

Other types of grammians also exist. These are *band limited* grammians [Gaw., 1] and are applicable to systems that operate in the frequency domain. These grammians will not be considered here, as this thesis is concerned with systems of a linear, time invariant nature.

3.5.5 GRAMMIAN ASSIGNMENT

It has already been established that for a stable system to be controllable or observable, then its respective controllability or observability grammians have to be positive definite. The grammians of (3.22) and (3.23) contain the system matrices A, B (for controllability) and C (for observability). This leads to the issue of appropriately selecting the system matrices so as to ensure that the grammians are positive definite.

For control system purposes it is useful to have a tool for modifying or shaping system controllability and observability properties. This can be done in two ways. The first, and most common, is by introducing a feedback loop [Wic. & Dec., 1] in order to modify the system properties. The second method is to determine sensor and actuator locations which contribute to the best observability/controllability properties. The latter method is addressed as the *grammian assignment problem* [Gaw., 1].

The problem is stated as follows

Problem 3.1: Let a system be described by a stable state matrix A, but with unknown actuator/sensor locations (i.e. B and C are unknown). For a positive definite matrix W, find a state space representation of a system such that its grammians are equal to W.

□

A possible solution to this problem [Gaw., 1] is to find matrices B_1 and C_1 and a non-singular transformation R, such that the grammians of (A_1, B_1, C_1) are equal to W, and $A_1 = R^{-1}AR$. Depending on what is to be determined (sensors, actuators, or both), Problem 3.1 can be divided into three separate problems [Gaw., 1].

Problem 3.2: Given A, B, find C_1 and the transformation R such that $W_{c1} = W_{o1} = W$ for the representation (A_1, B_1, C_1), where $A_1 = R^{-1}AR$ and $B_1 = R^{-1}B$.

□

Problem 3.3: Given A, C, find B_1 and the transformation R such that $W_{c1} = W_{o1} = W$ for the representation (A_1, B_1, C_1), where $A_1 = R^{-1}AR$ and $C_1 = CR$.

□

Problem 3.4: Given A, find B_1 and C_1 and the transformation R such that $W_{c1} = W_{o1} = W$ for the representation (A_1, B_1, C_1), where $A_1 = R^{-1}AR$.

□

Note that the matrices B_1 and C_1 include not only the actuator and sensor locations, but also the gain at each location. For the location only problem, the entries of B_1 and C_1 would be either 1 or 0.

The solution to the problems posed may or may not exist, since not every positive definite grammian can be obtained through the sensor or actuator placement. Yet they are worth further investigation, and there has been scant consideration of such problems in the literature.

3.6 SUMMARY

The concepts of controllability and observability were reviewed in this chapter, where definitions of these quantitative system properties were given. Systems to which they can be applied to are of a large scale nature, i.e. flexible structures, which have also been defined, along with the system components that are linked to the act of carrying out a control action and measuring the result of a control action (actuators and sensors). A mathematical study of flexible structures was presented in Section 3.3, where it was shown how a mathematical model in both the state space and modal coordinates can be derived from the differential equations of (3.4). The mathematical means of examining controllability and observability through grammians in the time domain was examined in Section (3.4). The chapter concluded with a mathematical analysis of how grammians and system models can be used to link controllability and observability properties to flexible structures.

Systems are said to be balanced when their controllability and observability grammians are equal [Moore., 1]. It is vital that for the purpose of flexible structure testing and control that investigations into possible sensor and actuator locations are carried out. The locations of these components have an impact on the dynamic behaviour and closed loop performance, and this has to be evaluated. Yet in practical situations there is a lack of the freedom of choice for the locations which creates a design problem. Such a problem is the determination of the locations of the sensors/actuators of an open loop system in order to meet the specified controllability and observability requirements. This problem has been addressed as the *grammian assignment* problem [Gaw., 1]. Another problem which arises is called the *placement problem* [Gaw., 1] and it is defined as attempting to find a subset which has controllability/observability properties close to the original requirements for a given set of sensors and actuators. It should be noted however that not every controllability and observability property can be obtained with a given set of actuator and sensor locations.

For control system purposes it is advantageous to have a tool for modifying or shaping the controllability and observability properties of a system. This can be achieved in two ways. One way is to determine proper sensor and/or actuator configurations. The other way is by modifying the system properties (such as introducing a feedback loop) and it is this that will be considered in the remainder of the thesis.

This chapter forms a basis for a wider study of controllability and observability, which will follow in the next two chapters. Chapter 4 will look at existing and newly developed measures of controllability and observability. In Chapter 5 there will be a study of how energy and the singular values of the output controllability grammian can be linked. Chapters 6, 7 and 8 will cover the area of how feedback can be used to modify the system properties.

4 MEASURES OF CONTROLLABILITY AND OBSERVABILITY

4.1 INTRODUCTION

The basic concepts of controllability and observability were reviewed in Chapter 3. The definitions in Section 3.2 provide merely a binary concept of controllability/observability. Either something is controllable/observable or not. But any uncontrollable system is in a certain sense arbitrarily close to some controllable system, and on the other hand a controllable system may or may not be close to an uncontrollable one [Pai., 1]. It may be possible to alter the structural properties of an uncontrollable system (e.g. selection of inputs) in order to make it controllable. But just how uncontrollable or controllable is a system? Merely to know whether something is controllable or not is not enough, and therefore a measure of controllability would be hugely advantageous in control design. The degree to which a system can be controlled/observed (degrees of controllability/observability) is a useful and sensible tool in analysing systems of physical variables, and are invariant as functions of coordinate transformations. Thus a measure of controllability/observability is vital to the satisfactory placement of sensors and actuators on a structure. This chapter will examine the notions of measuring controllability and observability with a view to sensor and actuator placement. The next section will comprise of a review of existing measures and a study of some new ones, which will be proceeded by a comparison of some of the techniques in this field. A new measure based on Markov parameters is also considered, followed by a section dealing with open and closed loop degrees of controllability. A study of how exterior algebra and characteristics of Plücker matrices can be used to develop measures is also presented. This leads conveniently to a documentation of a new method of input selection based on minimising the condition number of the controllability matrix, which concludes the chapter.

In this chapter the measure of strength of the presence of properties is explored for use as tools which enable the selection of sensors and actuators. System properties are characterised by values of property indicators which very frequently are expressed in terms of the rank of system matrices. In fact if the rank is full, it is deemed that the property is present, and if the rank is less than full then the system lacks that property. This exact characterisation is not often satisfactory. In fact a matrix can be nearly singular (i.e. due to small singular values) which indicates an "almost" lack of a particular property.

Given that there is some form of continuity in the presence or absence of properties in a system, it is important to introduce measures of strength of such properties (i.e. of controllability and observability). The selection of the input-output structure and more so the coordinate frame for state space models, strongly affects the measure of strength of such properties.

In this thesis, properties such as state controllability, observability, input and output controllability, and minimality are studied. Defining the strength of such properties in a formal way involves measuring the distance of the given system (having a fixed set of actuators and sensors, i.e. a specific input-output structure) from the family of systems which lack these types of properties. Most of the research done so far implies this but tackles it in an indirect way by considering norm properties of the appropriate property indicator.

4.2 Review of Controllability Measures

4.2.1 Measures Based on General Distance Functions

There are several mathematically equivalent approaches to determining controllability or observability of a linear time invariant system of the form

$$\begin{aligned} \dot{x} &= \underset{n\times n}{A}\, x + \underset{n\times m}{B}\, u \\ y &= \underset{p\times n}{C}\, x \end{aligned} \tag{4.1}$$

where A has eigenvalues $\lambda_1, \ \ldots, \ \lambda_n$. Yet these different approaches lead to computational methods that give significantly differing results which may stem from rounding errors. Although determining the binary aspect of controllability of (4.1) is a simple enough problem mathematically using the conditions of Theorem 3.1, computing the degree of controllability is by no means a trivial task.

It has been shown that the traditional methods of computing controllability are not satisfactory in the sense that they may lead to inaccurate conclusions, i.e. a system deemed to be controllable may in some way be very close to an uncontrollable one. An initial attempt to finding the distance of the closest uncontrollable system from the given one is made [Pai., 1]. The approach taken is derived from an analysis of the numerical rank of a matrix, perturbation analysis and sensitivity issues of eigenvalues. The measure is taken as

$$\mu(A, B) = \min\|\delta A, \delta B\|_2 \tag{4.2}$$

where the system described by

$$(A + \delta A, \quad B + \delta B) \tag{4.3}$$

is uncontrollable. It is argued [Pai., 1] that the measure of (4.2) is invariant under orthogonal transformations, yet may be altered under nonunitary transformations and scaling. The measure $\mu(A, B)$ is obtained by allowing for all possible perturbations, yet due to the modelling uncertainties in the system matrices A and B, the control engineer may lack confidence in using such a measure due to the issues discussed in Section 2.4.6 which highlighted the problems of computing nongeneric values of poorly modelled systems. It is for this reason that scaling is required [Don., et al, 1] so that the uncertainties of the elements in A and B are all of the same magnitude. The crux of the investigation by Paige [Pai., 1] is to draw attention to the importance of finding a reliable and efficient numerical algorithm for determining the distance of the closest uncontrollable system from the given one. However, the problem posed is an open computational one, yet it has stimulated further work.

The concept of a controllable subspace has been explored [Moore, 1] with the application to principal component analysis. Moore classifies the controllable and unobservable subspaces as two important subspaces of the state space, $\mathbb{R}^n$. The controllable subspace is defined as the smallest subspace which contains the state response (when the initial conditions are zero, i.e. $x(0) = 0$) to every piecewise continuous vector signal injected into the model input terminals ($u(t)$). Similarly the unobservable subspace is characterised as the largest subspace in which arbitrary piecewise signals can be injected with no output response. Moore presents an analysis by which these subspaces are computed approximately but is not extended to measuring the distance between them and the uncontrollable/unobservable subspaces. Yet the realisation of computing this distance has been explored by Eising [Eis., 1]. This is defined as the distance between a system (A, B) and the set of all uncontrollable systems, denoted by $\text{UNCO}_{n,m}$, and it is argued that the minimum of the smallest singular value, $\sigma_{\min}$, of $[\lambda I - A, \quad B]$ with respect to λ, is a measure of this distance, i.e.

$$d\langle (A,B),\ \text{UNCO}_{n,m} \rangle = \min_{\lambda \in \mathbb{C}} \sigma_{\min}[\lambda I - A, \quad B] \tag{4.4}$$

$\sigma_{\min}[\lambda I - A, \quad B]$ is the smallest singular value of $[\lambda I - A, \quad B]$, and is a continuous function of λ. It is explained that the measure requires the determination of the singular value of a polynomial matrix and minimising a function of a complex variable. Although the algorithm is computationally strenuous, the result has been extended to define a decentralised eigenvalue assignability measure [Vaz, & Dav., 1].

It has been the goal by certain authors merely to explore some of the properties of a controllable system which is near to an uncontrollable pair [Bol., & Lu, 1]. Here once more the distance between a controllable system and the nearest uncontrollable pair is considered. The method developed by Eising [Eis., 1] is extended to give a characterisation of when a system is "hard to control" in the presence of perturbations in the system matrices for either the complex or real case. It is argued that for physical systems, it is not necessary to take into consideration complex perturbations, hence making the measure of (4.4) [Eis., 1] conservative. It is suggested that it is sufficient to restrict the analysis to real perturbations. Due to the high level of computations involved, certain bounds are proposed in an attempt to reduce the complexities. The definitions of the distances given are significant for certain situations, especially when the data defining the coefficient matrices are not known to a great level of accuracy, or when computer simulations involving round off errors are carried out. The author also obtains a relation between the developed distance and the feedback gain required to shift a pole of the system. But this relationship is limited in the sense that it holds only for significantly small feedback gains, but the scope is there for further analysis. It is shown also that for small distances, there are correspondences to certain properties of the singular values of the controllability matrix and the energy considerations of the controllability grammian. The results presented provide a solid foundation on which to interpret the distance measures and provide grounds for an extension into investigating the connection between the controllability grammian and energy consumption.

4.2.2 MEASURES BASED ON ANGLES

Taking a different route to computing such indices using the distance criterion, Hamdan and Nayfeh [Ham., & Nay., 1] proposed measures of modal controllability/observability by considering angles between vectors. The angles between the left eigenvectors of A and the columns of the matrix B are used to propose modal controllability measures. Likewise the angles between the rows of the C matrix and the right eigenvectors of A are employed to put forward modal observability measures. The measures are an extension of the modal controllability and observability tests examined by Kailath [Kai., 1]. The controllability test is stated as the *i*th mode is not controllable from the *j*th input if and only if

$$q_i^T\{\lambda_i I - A \vdots b_j\} = 0^T \tag{4.5}$$

where q_i corresponds to the set of left eigenvectors, b_j is the *j*th column of B and 0 is a zero column. Similarly the *i*th mode is not observable at the *k*th output if and only if

$$\begin{Bmatrix} c_b^T \\ \lambda_i I - A \end{Bmatrix} p_i = 0 \tag{4.6}$$

where p_i are the respective right eigenvectors and c_k is the *k*th row of C. Whilst it has been suggested that the magnitudes of $q_i^T b_j$ and $c_k^T p_i$ can be used as measures of modal controllability and observability respectively, due to the inappropriateness of the scaling of the eigenvectors involved, the authors propose new algorithms. The magnitude of $q_i^T b_j$ can be rewritten as

$$|q_i^T b_j| = \|q_i\|\|b_j\|\cos\theta_{ij} \tag{4.7}$$

where the angle θ_{ij} is taken to be of an acute nature. Thus it is proposed that a measure of controllability of the *i*th mode from the *j*th input of a system model described by (4.1) is $\cos\theta_{ij}$, where θ_{ij} is the angle between b_j and q_i. Thus if the two subspaces are orthogonal (cos 90° = 0) then it is argued that the *i*th mode is uncontrollable from the *j*th input. Similarly, the magnitude of $c_k^T p_i$ can be defined as

$$|c_k^T p_i| = \|c_k\|\|p_i\|\cos\phi_{ki} \tag{4.8}$$

and so it is proposed that a measure of observability of the *i*th mode at the *k*th output is $\cos\phi_{ki}$, where ϕ_{ki} is the angle between c_k and p_i. Again if the two subspaces concerned in (4.8) are orthogonal, it is said that the *i*th mode will not be observable at the *k*th output. The measure of controllability outlined here is inversely proportional to the angle between the subspaces spanned by b_j and q_i. When they are orthogonal, the distance is a maximum and the degree of controllability is zero. A parallel argument can be given for the measure of observability algorithm derived from (4.8). yet despite the genuineness of the approach, the authors stop short of extending their measures to the generalised eigenvector case.

The above notions have been extended to form a relationship with the residues of the transfer function [Lind., et al, 1] of system (4.1) described by

$$y(s) = G(s)u(s) = \left[\sum_{i=1}^{n} \frac{R_i}{(s-\lambda_i)}\right]u(s) \tag{4.9}$$

where the poles of $G(s)$ are distinct and the $l \times m$ residue matrix R_i is given by

$$R_i = Cp_i q_i^T B \tag{4.10}$$

It is shown that the magnitude of the residues can be bounded by the product of the measures of controllability and observability of (4.7) and (4.8) respectively. It is implied that when the residue is exactly zero then the corresponding mode is either uncontrollable or unobservable (but not necessarily both). But a state space transformation can change the measure of observability or controllability of a particular mode. However the residue of the mode concerned is invariant to state space transformations and so the norm of the residue provides a lower bound on the product of the observability and controllability measures. The authors [Lind., et al, 1] point out that as a mode becomes less observable (through a state space transformation) it becomes more controllable. This leads to the assumption that the interaction of the controllability and observability measures works to preserve the input-output properties of such a system described by (4.1) under state transformations.

4.2.3 MEASURES BASED ON NORMS

The measures proposed above [Lind., et al, 1], [Ham., & Nay., 1] are only applicable to systems with distinct eigenvalues and poles. To solve this problem, Tarokh [Tar., 1] proposed simple controllability and observability measures for systems with distinct or repeated eigenvalues. It is shown that the proposed measures are directly related to important time and frequency domain characteristics of a system. For the system described by (4.1), the input-output transfer function is described by

$$y(s) = C(sI - A)^{-1} Bu(s) = \frac{\Phi(s)}{\Delta(s)} u(s) \tag{4.11}$$

where $\Phi(s) = C\text{adj}(sI - A)B$ is the numerator transfer function matrix and $\Delta(s) = |sI - A|$ is the characteristic polynomial. In addition to $\Phi(s)$, two more numerator matrices are defined

$$\begin{aligned} \Phi_B(s) &= \text{adj}(sI - A)B \\ \Phi_C(s) &= C\text{adj}(sI - A) \end{aligned} \tag{4.12}$$

It is proposed [Tar., 1] that if the system of (4.1) has distinct eigenvalues, then the mode λ_i is uncontrollable if and only if $\Phi_B(\lambda_i) = 0$. Similarly it is unobservable if and only if $\Phi_C(\lambda_i) = 0$. This is verified by the fact that in a system with distinct eigenvalues, common pole-zero cancellations in the transfer function matrix (4.11) result in uncontrollability or unobservability of the system [Tar., 2]. However, this only provides an addition to the already established binary tests for controllability and observability. So as an extension the following measures for the mode λ_i are introduced for controllability, m_{ci}, and observability m_{oi}

$$\begin{aligned} m_{ci} &= \left\|\Phi_B(\lambda_i)\right\|_F \\ m_{oi} &= \left\|\Phi_C(\lambda_i)\right\|_F \end{aligned} \tag{4.13a}$$

where $\|\bullet\|_F$ is the Frobenius norm. The controllability and observability measures for the system (A, B, C) are defined respectively as

$$\begin{aligned} m_c &= \min_i\{m_{ci}\} \\ m_o &= \min_i\{m_{oi}\} \end{aligned} \tag{4.13b}$$

The proposed measures are extended to systems with repeated eigenvalues, and the derivation of an algorithm becomes more complicated because consideration has to be made of the zero polynomials of the square subsystems within the system described by (4.1). The numerator transfer functions of (4.12) are in fact zero polynomials of single-input single-output systems and are not adequate for developing a measure for systems with repeated eigenvalues. So the concept of transmission zeros [Mac., & Kar., 1] is employed to define measures of such systems. The measure of controllability is thus defined by

$$\tilde{m}_{ci} = \left[\sum_{r=1}^{m}\sum_{\alpha=1}^{\sigma_r}\left|\psi_\alpha^r(\lambda_i)\right|^2\right]^{1/2} \tag{4.14}$$

λ_i is the mode whose controllability is to be measured, m is the number of rows of B, r is the size of the square subsystem and

$$\psi_\alpha^r(s) = \det\begin{bmatrix} sI_n - A & B_\beta^r \\ C_\gamma^r & 0 \end{bmatrix} \tag{4.15}$$

where B_β^r, $\beta = 1, \ 2, \ \ldots, \ \eta_r$, are the set of $n{\times}r$ submatrices formed from r columns of the matrix B. C_γ^r, $\gamma = 1, \ 2, \ \ldots, \ \rho_r$, are the set of $r{\times}n$ submatrices formed from r rows of the matrix C. The number of $n{\times}r$ submatrices of B is

$\eta_r = (m!/(m-r)!r!)$. The number of $r \times n$ submatrices of C is $\rho_r = (l!/(l-r)!r!)$, where l is the number of rows of C. Thus the number of r-dimensional subsystems is defined as $\sigma_r = \eta_r \rho_r$ and $\alpha = \{1, 2, \ldots, \sigma_r\}$. Similarly the observability measure of the mode λ_i is

$$\tilde{m}_{oi} = \left[\sum_{r=1}^{l} \sum_{\alpha=1}^{\sigma_r} \left| \psi_{\alpha}^{r}(\lambda_i) \right|^2 \right]^{1/2} \tag{4.16}$$

The measures defined by (4.15) and (4.16) are equal to the sum of distances of the eigenvalue λ_i to the transmission zeros of all square subsystems of (A, B, I_n) for controllability and (A, I_n, C) for observability. When all square subsystems have a transmission zero at λ_i all the distances are zero and the system becomes uncontrollable (or unobservable) [Tar., 1]. The thoroughness of these algorithms provides a useful and in depth study into the development of measures of controllability and observability. The author has taken into consideration the often neglected problem of repeated eigenvalues, yet the question of whether such measures are invariant under coordinate transformations remains debateable.

4.2.4 USE OF MEASURES OF CONTROLLABILITY, OBSERVABILITY IN THE SENSOR/ACTUATOR SELECTION PROBLEM: DYNAMIC CONSIDERATIONS

So far the measures reviewed up to this point have been based on a purely mathematical level, without the authors stressing any applications to practical problems. This section of the review concentrates on procedures which indicate ways of selecting inputs/outputs in order to maximise their relative degrees of controllability and observability. The importance of a definition of an effective input-output structure has been highlighted [Kar., et al, 1] in order to guarantee basic structural properties such as controllability and observability. Assuming that a system model exists, various values of performance can be addressed, such as the degrees of controllability and observability at this design stage. The problem here is thus to retain the achieved structural features of the system, and to implement some additional properties for the input/output structure using tuning parameters. A variety of performance tests and criteria such as energy requirements for control and observation [Lev., et al, 1] as well as properties as maximising the degree of controllability and observability can be used to help reach the design goals.

An initial study into the area of sensor/actuator selection was made by Wu, Rice and Juang [Wu., et al, 1]. They considered the controllability and observability of a reduced order system of a flexible structure in order to determine the minimum number of actuators and sensors required. But the results presented are only tests and not measures. They are based on the rank tests of the controllability and observability matrices which have been tailored for a flexible structure system. The concept of the degree of controllability of a control system has been developed starting from physical considerations [Vis., et al, 1]. A link is made with the presented measure and the question of how to choose the number and locations of the control system actuators. The results obtained allow the control system designer to rank the effectiveness of alternative actuator distributions, and hence to choose the locations based on a rational selection process. The degree of controllability is shown to take a simple form when the dynamic equations of a particular system (the example given in the literature is of a satellite) are in second order modal form. It is argued that the degree of controllability concept has fundamental uses in that it allows the system structural relations between the various inputs and outputs of a linear system to be studied. The analysis starts from a set of actuator locations which produce an uncontrollable system, but for which the number of actuators is sufficient to produce controllability. It is suggested that by moving one of the actuators by a distance $\varepsilon > 0$, a controllable system can be produced, regardless of the size of ε. But for small ε, even though the system may technically be controllable, in some sense it will not be "very controllable". From this, the authors set out five conditions that their measure of controllability must meet [Vis., et al, 1]:

- The degree of controllability is zero when the system is uncontrollable
- The system stability properties must somehow be represented
- It must be dependent on the total time T
- It must standardise or restrict the control effort in some way

- The control objective must be reflected in the definition

These conditions provide a sensible platform for any potential development of a measure of degree of controllability. The control objective is to return the state variable x to zero after a disturbance [Vis., et al, 1] as this is the most common attitude and shape control objective for flexible structures. Controllability requires the existence of a control function which can transfer any initial state to any final state in finite time. In an uncontrollable system there will be at least one direction in the state space for which initial conditions in this direction cannot be returned to the origin. For a controllable system whose parameters are such that it is almost uncontrollable, then only initial conditions very close to $x = 0$ along the aforementioned direction could be returned to the origin in time T. Thus a definition of the degree controllability is generated based on the minimum distance from the origin to a normalised state that cannot be brought to the origin in time T [Vis., et al, 1]. So the degree of controllability in time T for the case when $x = 0$ (initial condition) of a normalised version of the system described by (4.1) is

$$\rho = \inf\left\|x(0)\right\| \tag{4.17}$$

where $\left\|\bullet\right\|$ is the Euclidean norm.

The degree of controllability defined in (4.17) is a scalar measure which exists within a region where all of the initial conditions (or disturbed states) that can be returned to the origin in time T can be identified. This is called the recovery region [Vis., et al, 1]. This measure may be applicable to flexible structure systems, but fails to take into consideration the case when the system exhibits repeated eigenvalues. Although the measure is independent of using eigenvalues in the calculation, they do play an important role in the system response.

Various definitions of the degree of controllability and observability have been used in guiding the search for optimal actuator and sensor locations. Among these, the degree of controllability defined by scalar measures has been addressed [Vis., et al, 1], [Vis., et al, 2], [Long., et al, 1]. A second approach uses the projection magnitudes of eigenvectors into the input and output matrices to define gross measures of controllability and observability [Ham., & Nay., 1]. The problem of defining and obtaining the *optimal* actuator and sensor locations has been addressed by Lim, [Lim, 1]. The method is based on the controllability and observability of an actuator/sensor pair. The selection is based on the *effectiveness* (controllability and observability of a particular mode) and *versatility* (controllability and observability of all modes) of pairs of actuators and sensors. The method introduced has the advantage of not requiring a specification of the number of actuators and sensors and it is centred around the orthogonal projection of structural modes into the intersection subspace of the controllable and observable subspaces corresponding to an actuator/sensor pair. The controllability and observability grammians are then used to weight the projections to reflect the degrees of controllability and observability. This method produces a three-dimensional design space within which sets of optimal actuators and sensors may be selected based on the criteria set out by the designer without the need for elaborate nonlinear programming strategies. The method also allows for the comparison of many actuator and sensor candidate locations since the computational effort depends only on the product of the number of actuator and sensor location candidates rather than laborious computational search methods. The main drawback of such a method is the time consumed to test for all the possible sensor and actuator locations, thus potentially turning the algorithm into one based on a trial and error methodology.

Keeping to the theme of actuator placement, yet another measure has been developed [Kim, & Jun., 1] in order to aid this delicate design problem. This measure is computed as follows

$$\alpha = \sum_{i=1}^{n} \frac{V_i}{V} \rho_i^2 \tag{4.18}$$

where α is the measure of controllability, V_i the ith mode component cost in the cost function V, and ρ_i is the gross measure of modal controllability from all inputs of the ith mode. ρ_i has already been proposed [Ham., and Nay., 1] and V is computed from

$$V = \mathrm{tr}\left\{Q_v C_d X C_d^T\right\} \tag{4.19}$$

where Q_v is a weighting matrix, C_d is the output matrix, X is the controllability grammian that satisfies the Lyapunov equation

$$XA^T + AX + BB^T = 0 \tag{4.20}$$

where (A, B) are the system matrices. Qualitatively this index represents a measure of output controllability (Chapter 5) that measures both modal controllability and modal participation of all modes in the physically important cost function. However, despite its ease of use, this measure fails to incorporate an optimal solution for the placement of actuators, and the authors merely make use of a trail and error basis to arrive at an index that improves their placement.

4.2.5 MEASURES BASED ON ENERGY CONSIDERATIONS

A measure based on energy consumption has been considered [V Vel., & Car., 1] to give a quantitative indication of how well a system can be controlled with a given set of actuators (a measure of controllability). Similarly, a measure of observability is defined which is a quantitative indication of how well a system can be observed with a given set of sensors. The measure of controllability formulated results from a four step procedure. The first step is to find the minimum control energy strategy for driving the system from a given initial state to the origin in a prescribed time using

$$E = \frac{1}{2}\int_0^T u^T R u \, dt \tag{4.21}$$

where R is a positive definite weighting matrix and u is a control input. The second step is to establish the region of initial states which can be driven to the origin with constrained control energy and time using an optimal control technique. This region is bounded by an ellipsoidal surface in the state space. The third step is to equalise the priority of each unit displacement in every direction within this region so that they are all equally important to control. Finally the degree of controllability is taken as

$$DC = \left[V_S + \frac{V_S}{V_E}(V_E - V_S) \right]^{1/n} \tag{4.22}$$

where V_E is the n-dimensional volume of the ellipsoid in what is described as an "equicontrol" space, set out by the third step in this procedure and is defined as

$$V_E = \prod_{i=1}^{n} \sqrt{v_i} \tag{4.23}$$

where v_i are the eigenvalues of DV_0D, where D is the matrix that transforms the state vector to an equicontrol space $z = Dx$ and V_0 is an initial condition of the weighted volume of the ellipsoid. V_S is the spherical volume within the ellipsoid and is also the shortest distance to the surface, and is given by

$$V_S = 1 \Big/ \sqrt{\lambda_{\max}} \tag{4.24}$$

where λ_i are the eigenvalues of $(DV_0D)^{-1}$. The method presented [V Vel., & Car., 1], whilst radical and complex in approach, has a few loose ends that need to be tied up. For example, no mention is made of the type of eigenvalues (i.e. distinct, real) that can be used in the algorithm, however the authors do go on to extend the measure for optimal actuator placement.

The problem of deriving a measure of controllability has been a strongly disputed issue in the control engineering community. But a novel concept is proposed which uses a systematic approach to select optimal candidate sets for actuator placement [Roh, & Par., 1]. The method proposed relies on a new quantitative measure of controllability and is related to the minimum control energy input needed to regulate the system from initial modal disturbances. The modal degree of controllability presented offers the control system designer a tool with which the ranking of the effectiveness of a specific distribution of actuators leading to a rational based choice for their locations. The proposed method represents the relative performance of a specific set of a predetermined number of actuators compared to the performance achievable with a full set of actuators. The performance metric used in the definition is the control energy required to regulate the system from a disturbance of a specific structural mode. The cost function, which is the weighted sum of the modal degrees of controllability (MDOCs [Roh, & Par., 1]) corresponding to the modes of interest, and is used to find optimal actuator locations. It is argued that by placing a predetermined number of actuators at the set which optimises the cost function, the control energy required to regulate the system from disturbances can be minimised. The control objective is defined as regulating a system from a set of initial disturbances x_0 to zero within a given time interval of t_1-t_0 , with the minimum control energy $E(t_0,t_1;x_0)$ defined by

$$E(t_0,t_1;x_0) = \min_u \int_{t_0}^{t_1} \|u(\tau)\|^2 \, d\tau \tag{4.25}$$

subject to $x(t_0) = x_0$, $x(t_1) = 0$, and $[t_0,t_1]$ fixed. If a linear time invariant flexible structure described in Section 3.3 is controllable, the optimal solution to (4.25) is [Roh, & Par., 1]

$$E(t_0,t_1;x_0) = x_0^T W_C^{-1}(t_1,t_0)x_0 \tag{4.26}$$

where $W_C(t_1,t_0)$ is the controllability grammian matrix described by equation (3.22). $E(t_0,t_1;x_0)$ can provide a quantitative measure of controllability, but it is dependent on the initial condition x_0. If $E(t_0,t_1;x_0)$ of system S_1 is smaller than that of system S_2, then system S_1 is said to be more controllable than S_2 [Roh, & Par., 1]. The derivation of the modal degree of controllability using this energy minimisation concept, involves the selection of and initial condition which disturbs the system with unit energy, ξ_i. This is then used in the following definition for an MDOC of a particular mode i

$$\mathrm{MDOC}_i = \frac{\overline{E}(t_0,t_1;\xi_i)}{E(t_0,t_1;\xi_i)} \tag{4.27}$$

$\overline{E}(t_0,t_1;\xi_i)$ represents the maximum of the set of minimum input energies when the actuators are located at all of the possible locations on the flexible structure. $E(t_0,t_1;\xi_i)$ is defined as the maximum of the set of minimum control energies required to regulate the system from its modal displacement with unit initial energy. This is indicative of the relative ratio of the minimum energies required to regulate a system from a modal disturbance when a chosen specific set and a full set of actuators are used respectively. The bound on the MDOC is given as

$$0 < \mathrm{MDOC}_i \leq 1 \tag{4.28}$$

It is argued that if the MDOC for a specific number of actuators is small, it implies that an increase in the number of actuators can improve the controllability of the mode. This method adds another condition to developing measures of controllability. Not only are measures vital to input and output instrumentation selection, but it is shown here how gauging controllability can be used to accommodate energy utilisation concerns. This will be extensively dealt with in Chapter 5.

4.2.6 REVIEW SUMMARY

The review of the measures of controllability and observability has demonstrated the various criteria used to formulate methodologies. The principal motivation for such research is the optimal placement of actuators and sensors on large scale systems like flexible space structures. It has been shown that the majority of methods devised are based on eigenvector/eigenvalue solutions. However, little consideration is paid to the case of repeated eigenvalues. Several authors have adopted the use of norms to solve this problem, but such methodologies remain incomplete. What is provided, however, are certain objectives that potential measures should be based upon. Most of the existing methods handle the measure of controllability and observability by considering the relaxation of the exact conditions which lead to the absence of the corresponding property. The relaxation of exact mathematical conditions is important, but it is difficult to relate it to the physical implications of the weak presence of such properties. The energy study of Chapter 5 is an attempt to reintroduce the physical dimension to this degree of presence of properties. The methods reviewed in this section form a foundation for which to carry the study forwards. In the next section, a comparison will be made of newly developed measures and existing ones which have been slightly modified.

4.3 COMPARISON OF EXISTING MEASURES

4.3.1 MEASURES TO BE INVESTIGATED

Here, four measures of controllability and observability will be presented and compared. Three measures are based on finding the minimum singular value of certain matrices that are linked to determining controllability. A fourth measure is based on the calculation of the norm of a transfer function matrix. Singular values and norms have been extensively used in the literature in determining measures of controllability and observability. For the measures to be studied in this section, it must be pointed out that the comparison will only be carried out on a mathematical basis, and not with a view for the placement of actuators and sensors, which would require an extension to the current research by way of optimisation techniques.

The four measures of controllability (MoC) and observability (MoO) are presented as follows, where A, B and C are the state, input and output matrices of equation (4.1)

MoC1

The minimum singular value of the controllability matrix $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$.

MoO1

The minimum singular value of the observability matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

MoC2

The minimum singular value of the matrix pencil $\begin{bmatrix} \lambda_i I - A, & B \end{bmatrix} \forall \lambda_i$.

MoO2

The minimum singular value of the matrix pencil $\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} \forall \lambda_i$.

MoC3

The frobenius norm of the numerator transfer function matrix $\operatorname{adj}(sI - A)B$.

MoO3

The frobenius norm of the numerator transfer function matrix $C\operatorname{adj}(sI - A)$.

MoC4

The minimum singular value of the toeplitz matrix [Ant., & Mic., 1] *of* $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$.

MoO4

The minimum singular value of the toeplitz matrix [Ant., & Mic., 1] *of* $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

A study of the progression of development of these measures can be found in Section 4.6. Measures MoC4 and MoO4 are purely speculative, and require further elaboration. Toeplitz matrices [Ant., & Mic., 1] have dynamical implications, and the analysis of links with measuring the degrees of controllability and observability is an avenue for further investigations.

4.3.2 COMPARISON

The comparison of these four measures is carried out using MATLAB. The corresponding code can be found in the Appendix. The following system model of a flexible rocket [Med., et al, 1] will be used for the analysis

$$A = \begin{bmatrix} -0.2105 & -0.1056 & -0.0007 & 0 & -0.0706 & 0 \\ 1 & -0.0354 & -0.0001 & 0 & -0.0004 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -605.1 & -4.92 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3906.3 & -12.5 \end{bmatrix}, \quad B = \begin{bmatrix} -7.211 \\ -0.0523 \\ 0 \\ 794.7 \\ 0 \\ -448.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0.0003 & 0 & -0.0077 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is stable in the sense that all eigenvalues are distinct and have negative real parts. The system is also controllable and observable, according to the full rank tests of the controllability and observability matrices respectively of Theorems 3.1 and 3.2. A second uncontrollable and unobservable, yet stable system with matrices

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

is also used to compare measures. The table below summarises the measures for both the rocket and uncontrollable systems.

	Flexible Rocket	Uncontrollable System
MoC1	6.4961	0
MoC2	0.4720	8.8×10^{-16}
MoC3		
λ_1	1.79×10^{7}	11.3137
λ_2	1.79×10^{7}	1.38×10^{-14}
λ_3	3.89×10^{10}	6.92
λ_4	3.89×10^{10}	n/a
λ_5	3.60×10^{11}	n/a
λ_6	3.60×10^{11}	n/a
MoC4	4.29×10^{9}	4.02

Table 4.1 Comparison of controllability measures

	Flexible Rocket	Unobservable System
MoO1	0.1572	0
MoO2	1.23×10^{-5}	1.87×10^{-15}
MoO3		
λ_1	2.61×10^{6}	3.01×10^{-14}
λ_2	2.61×10^{6}	19.59
λ_3	1.49×10^{4}	6.92
λ_4	1.49×10^{4}	n/a
λ_5	6.16×10^{6}	n/a
λ_6	6.16×10^{6}	n/a
MoO4	1.73×10^{6}	10.0948

Table 4.2 Comparison of observability measures

Table 4.1 compares the measures of controllability between a controllable model of a rocket and an uncontrollable system. As can be seen, the higher values in the *Rocket* column indicate that it is more controllable for all the measures used than the *uncontrollable system*, for which there are lower values. The implication here is that the pair of matrices (A, B) describing the flexible rocket contain a higher proportion of controllable modes than the matrix pair describing the uncontrollable system.

Table 4.2 compares the observability measures of the rocket model to those of an unobservable one. Once again, the higher values of the measures for the rocket model satisfy the implication that it is more observable than the second system. Once again, the matrix pair (A, C) of the flexible rocket has a lower proportion of unobservable modes in comparison with the system described in the adjacent column of Table 4.2.

4.4 DEGREE OF MINIMALITY OF STATE SPACE DESCRIPTIONS

4.4.1 PROBLEM DEFINITION

For a given state space description there exist many measures for evaluating distance from uncontrollability, unobservability or alternatively measuring the strength of such properties. It is worth pointing out that these measures are functions of coordinate transformations. These functions may change as the coordinate transformations are varied. In this section a new measure is introduced that estimates the aggregate distance from minimality of the description (controllability and observability) without differentiating between the two important constituent properties. Furthermore this new measure is based on Markov [Ant., & Mic., 1] parameters and thus it is invariant under state coordinate transformations.

4.4.2 DEGREE OF MINIMALITY

Consider in the following that $S(A, B, C, D)$ is a state space model and that the transfer function $H(s)$ is used for evaluating the McMillan degree, which here is done based on the standard Hankel matrix characterisation of this property [Ants. & Mich., 1], [Kai., 1].

Firstly

Lemma 4.1: Let $H(s)$ be a transfer function and $S(A, B, C, D)$ be a realisation of $H(s)$. $S(A, B, C, D)$ is a minimal realisation of $H(s)$ if the McMillan degree of $H(s)$ is $\delta_M(H) = \partial\{|sI - A|\}$.

□

The above result seems to be of an algebraic nature, but it may take a numerical form allowing the introduction of some notion of distance from minimality by using the state space characterisation of the McMillan degree which is established as shown below. Consider the Laurent series expression of $H(s)$ [Ants. & Mic., 1], i.e.

$$H(s) = H_0 + \hat{H}(s) = H_0 + H_1 s^{-1} + H_2 s^{-2} + H_3 s^{-1} + \ldots \tag{4.29}$$

where $\hat{H}(s)$ is the strictly proper part and the $q \times r$ real matrices H_0, H_1,... are the Markov parameters where [Ant., & Mic., 1]

$$\begin{aligned} H_0 &= D \\ H_i &= CA^{i-1}B, \quad i = 1, 2, \ldots \end{aligned} \tag{4.30}$$

The Hankel matrix $M_H(i, j)$ corresponding to the Markov parameter sequence H_0, H_1,... is defined as the $iq \times jr$ matrix given by [Ant., & Mic., 1]

$$M_H(i,j) \equiv \begin{bmatrix} H_1 & H_2 & \cdots & H_j \\ H_2 & H_3 & \cdots & H_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_i & H_{i+1} & \cdots & H_{i+j-1} \end{bmatrix} \tag{4.31}$$

Lemma 4.2: [Ants. & Mich., 1] The McMillan degree of the transfer function $H(s)$ is the rank of $M_H(v, v)$, where v is the degree of the least common denominator of the entries of $H(s)$.

□

By computing the least common multiple (lcm) of the entries of $H(s)$, i.e. $d_H(s)$, then $v = \partial\{d_H(s)\}$. Using the Markov parameters (CB, CAB, ...) the matrix referred to as the <u>principal Markov matrix</u> may be defined as

$$M_H(v,v) \equiv M_H^v = \begin{bmatrix} CB & CAB & \dots & CA^{v-1}B \\ CAB & CA^2B & \dots & CA^vB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{v-1}B & CA^vB & \dots & CA^{2v-1}B \end{bmatrix} \tag{4.32}$$

Evaluating the rank of M_H^v provides the means to estimate the degree of minimality of the model and this follows from the lemmas above and is summarised below.

Proposition 4.1: [Ant., & Mic., 1] Let $S(A, B, C, D)$ be a state space description with dimension n, $H(s)$ the corresponding transfer function matrix and M_H^v the principal Markov matrix. Then, the system is minimal if and only if

$$n = \text{rank}\{M_H^v\} \tag{4.33}$$

□

The above characterisation allows the translation of the McMillan degree result in a sense that accepts measuring uncertainty and provides a measure of the aggregate distance from minimality. For the matrix M_H^v which has dimensions $mv \times pv$ the ordered set of singular values in descending order is denoted as

$$\bar{\gamma} = \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n = \underline{\gamma} \tag{4.34}$$

Remark 4.1: Since there is a state space model with state dimension n, the rank of M_H^v is at most n and thus all singular values γ_i for $i = n+1, n+2, \dots$ are zero.

□

The following definition may now be given

Definition 4.1: Let $S(A, B, C, D)$ be a state space model with n states, $H(s)$ be the corresponding transfer function and M_H^v be the principal Hankel matrix. If $\in$ is some tolerance, $\in>0$, then the following can be defined:

1. If $\gamma_n \geq \in$, then the system is <u>$\in$-strongly minimal</u> and $|\gamma_n - \in|$ provides a measure of distance from loss of minimality.
2. If $\gamma_1 \geq \cdots \geq \gamma_k \geq \in \geq \gamma_{k+1} \geq \cdots \geq \gamma_n > 0$, then the system is called <u>$\in$-weakly minimal</u> and $|\in - \gamma_n|$ also provides a measure of distance from $\in$-strongly minimality.

□

A better measure of distance from loss of minimality which is independent of scaling is provided by the condition number

$$\mu_H = \gamma_1 / \gamma_n \tag{4.35}$$

which is referred to as the <u>Hankel condition number</u>. Clearly the following properties hold true.

Remark 4.2: The condition number $\mu_H < \infty$ if the system is minimal. This number provides a measure of distance from loss of minimality, which occurs when $\mu_H \to \infty$.

□

The property of non-minimality, or loss of minimality does not differentiate between loss of controllability and/or loss of observability, but expresses an aggregation of the two. The importance of μ_H as the measure of distance from loss of minimality is due to its independence from state space coordinate transformations, since it is based on Markov parameters.

4.5 OPEN AND CLOSED LOOP DEGREES OF CONTROLLABILITY

4.5.1 PROBLEM DEFINITION

It has been documented that controllability is invariant under state feedback. However what will be investigated here is how the degree of controllability varies (if it does vary) with state feedback. Also, the effect of the structure of the state feedback matrix on the degree of controllability will be examined, particularly its rank and orthogonality. It will be demonstrated through examples that although the property is invariant under feedback its degree varies according to the structure of feedback matrix used. The degree of controllability used for this task is closely related to that developed by Hamdan and Nayfeh [Ham., & Nay., 1].

4.5.2 THEORY

As has been investigated already in Section 4.2, the measure of controllability devised by Hamdan and Nayfeh [Ham., & Nay., 1] is based on the test

$$q_i^T\left\{\lambda_i I - A \vdots b_j\right\} = 0^T \tag{4.29}$$

The condition

$$\underline{q}_i^T B = 0 \tag{4.30}$$

is used here to investigate if the structure of the state feedback affects the measure of controllability. Each column of the left eigenvector matrix q^T is multiplied by the corresponding row of the input matrix B, i.e.

$$\underline{q}_i^T b_j = \underline{\varepsilon}_i^T \tag{4.31}$$

The measure of controllability, ϕ, is taken as

$$\min\left\{\|\varepsilon_i\|,\ i \in n\right\} = \phi \tag{4.32}$$

Small values of ϕ indicate uncontrollable modes. When state feedback is applied, the test of controllability becomes

$$q_i^T\left\{\lambda_i I - A \ - BL \vdots \ b_i\right\} = 0^T \tag{4.33}$$

and thus

$$\underline{\tilde{q}}_i^T B = 0 \tag{4.34}$$

The corresponding measure of controllability, when state feedback is applied, is now

$$\min\left\{\|\tilde{\varepsilon}_i\|, \ i \in n\right\} = \tilde{\phi} \tag{4.35}$$

4.5.3 SOFTWARE AND TESTS

A MATLAB routine was written and compiled in order to carry out the measure of controllability described above, and to compare the two indices for the open loop and state feedback (closed loop) cases. The code can be found in the Appendix, and is named *moc5.m*. The program has been written as a function, and thus by simply typing `[phi,phi_f]=moc5(a,b,l)` where *a*, *b*, and *l* are the predefined state, input and feedback matrices, the controllability measures for the open loop and feedback cases, *phi* and *phi_f* respectively, are returned. Small values of *phi* and *phi_f* indicate less controllable states.

The first test to be carried out examined whether the implementation of feedback has any affect on the degree of controllability. This was done using the matrices [Bod., & Gro., 1]

$$A = \begin{bmatrix} -0.7685 & 1.0137 & -0.0185 & 0.0019 & 0.0018 \\ -4.3448 & -1.9816 & 0.4991 & 0.0598 & 0.0788 \\ 0.2155 & -0.1958 & -0.0636 & 0.0585 & -0.9273 \\ -1.8760 & -0.4775 & -20.3609 & -1.3178 & 1.9133 \\ -0.0432 & 0.0018 & 4.9747 & -0.0017 & -0.4948 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0096 & -0.0193 & -0.0457 \\ -6.8978 & -0.3138 & -0.0961 \\ -0.2652 & -0.1649 & -0.0714 \\ -0.0131 & 18.7269 & -2.0886 \\ 0.0556 & 1.4760 & -2.6271 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The pair (A, B) is both stable and controllable both in the open loop and closed loop. It was found that for the above state feedback matrix, L, that the degree of closed loop controllability varied from the open loop case. For the same system, different feedback matrices were also tested, and each time the degree of controllability differed from that of the open loop system.

The second test was to investigate how the rank of the state feedback matrix L affected the degree of controllability. The same matrices [Bod., & Gro., 1] were used for the investigation, with the exception of the L state feedback matrix, whose rank was altered to three, two and one. When this was done, there was a noticeable change in the degree of controllability for different ranks of L.

The third and final test investigated the effect of skewness of the feedback matrix L on the degree of controllability. The standard test for orthogonality, later to be detailed in Section 7.3.1, using grammians was used as part of this procedure. The same set of matrices as those in task one, were used in conjunction with different types of state feedback matrices (L) exhibiting differing degrees of skewness. It was deduced that skewness/orthogonality of L does have a bearing on the measure of controllability.

From the three tests conducted and described above, there is evidence to point to the assertion that the structure of the state feedback matrix does affect the closed loop degree of controllability. The first test provided evidence that the open loop degree of controllability is variant once feedback is applied. The implication of the second test is that the rank of the state feedback matrix, L, affects the degree of controllability. Finally the third test showed that the degree of controllability was dependent on the skewness or orthogonality of L. It is evident that state feedback affects the degree of controllability and thus there is need to define such measures of properties in a way that is independent of feedback. An effort to produce such tests is made in the following section.

4.6 Exterior algebra based criteria for controllability and observability and the notion of almost decoupling zeros

4.6.1 INTRODUCTION

This section comprises of a brief review of the notions of controllability and observability that have been developed within the exterior algebra framework. Tests for controllability and observability which are based on the Plücker matrices are also presented here. These results allow the characterisation of input-output decoupling zeros (uncontrollable/unobservable modes) [Ant., & Mic., 1] in an "almost" sense and provide new ways for measuring the distance from uncontrollability/ unobservability based on the singular value analysis of the controllability and observability Plücker matrices [Kar. & Gia., 1,2]. The "almost decoupling zeros" characterisation is based on the "almost" zero characterisation of a set of polynomials [Kar., et al, 2] and for this there is a computational framework that is based on optimisation techniques. The use of the controllability, observability and restricted controllability/observability pencils for the characterisation of distance from the loss of the corresponding properties has the advantage of permitting the definition of a distance from uncontrollability/unobservability in a way that is independent from the effects of state feedback (output injection). The latter provides the means to characterise distance from uncontrollability/unobservability based on the properties of the Plücker matrices of the restricted controllability and observability pencils [Kar., 3].

Throughout this section the standard notation for the exterior algebra framework [Mar., & Min., 1] is used. Thus the range space of a map H is denoted by $\mathcal{R}(H)$ and its right, left null spaces by $\mathcal{N}_r(H)$, $\mathcal{N}_l(H)$ respectively. If $\mathcal{V}$ is a vector space, then a vector is denoted by $\underline{v}$, V a basis and V a basis matrix of $\mathcal{V}$. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from 1, 2, ..., n. If $\{\underline{x}_{i_1}, \ \ldots, \ \underline{x}_{i_k}\}$ is a set of vectors of $\mathcal{V}$, $\omega = (i_1, \ \ldots, \ i_k) \in Q_{k,n}$, then $\underline{x}_{i_1} \wedge \ \ldots \wedge \ \underline{x}_{i_k} = \underline{x}_{\omega} \wedge$ denotes their exterior product and by $\wedge^r \mathcal{V}$ the r-th exterior power of $\mathcal{V}$ is denoted. If $H \in \mathcal{F}^{m \times n}$ and $r \leq \min\{m,n\}$, then by $C_r(H)$ the r-th compound matrix of H is denoted. Finally if a property is said to be true for $i \in \tilde{n}$, this means that it is true for all $1 \leq i \leq n$.

4.6.2 THE DETERMINANTAL ASSIGNMENT PROBLEM FOR STATE SPACE MODELS [Kar., et al, 2]

Consider the linear system described by

$$S(A,B,C,D): \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u}, \ A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m} \\ \underline{y} = C\underline{x} + D\underline{u}, \ C \in \mathbb{R}^{p \times n}, \ D \in \mathbb{R}^{p \times m} \end{array} \tag{4.36}$$

The classical state space design problems of pole assignment by state feedback and design of observers may be formulated in the following way.

<u>1. Pole Assignment by State Feedback</u>: Consider $L \in \mathbb{R}^{n \times m}$, where L is a state feedback applied to the system of (4.36). The closed loop characteristic polynomial is given by

$$p_L(s) = \det\{sI - A - BL\} = \det\{B(s)\tilde{L}\} \tag{4.37}$$

where $B(s) = [sI - A, -B]$ and $\widetilde{L} = \left[I_n, L'\right]'$.

2. Design of an n-state Observer: Consider the problem of designing an n-state observer for the system of (4.36). The characteristic polynomial of the observer is then defined by

$$p_T(s) = \det\{sI - A - TC\} = \det\{\widetilde{T}C(s)\} \tag{4.38}$$

where $T \in \mathbb{R}^{n \times p}$ is a feedback matrix, $\widetilde{T} = \left[I_n, T\right]$ and $C(s) = [sI - A', -C']'$.

The common formulation of these problems clearly suggest that they are special cases of a more general problem, which is defined below [Kar., & Gia., 1]:

The Determinantal Assignment Problem: Let $M(s) \in \mathbb{R}^{q \times r}[s]$, $r \leq q$, where $\text{rank}_{\mathbb{R}(s)}\{M(s)\} = r$ and let $H = \{\mathrm{H} : \mathrm{H} \in \mathbb{R}^{r \times q},\ \text{rank}\{\mathrm{H}\} = r\}$. Finding $\mathrm{H} \in H$ such that the polynomial

$$f_M(s, \mathrm{H}) = \det\{\mathrm{H}M(s)\} \tag{4.39}$$

has assigned zeros is defined as the *determinantal assignment problem* (DAP). If $\underline{h}_i'$, $\underline{m}_i(s)$, $i \in \underline{r}$ denote the rows of H and columns of $M(s)$ respectively, then $C_r(\mathrm{H}) = \underline{h}_1' \wedge \ \ldots \wedge \ \underline{h}_r' = \underline{h}' \wedge \in \mathbb{R}^{1 \times \sigma}$ and $C_r(M(s)) = \underline{m}_1(s) \wedge \ \ldots \wedge \ \underline{m}_r(s) = \underline{m}(s) \wedge$ $\in \mathbb{R}^{\sigma}[s]$, $\sigma = \begin{pmatrix} q \\ r \end{pmatrix}$ and by the Binet-Cauchy theorem [Mar., & Min., 1]

$$f_M(s, \mathrm{H}) = C_r(\mathrm{H}) C_r(M(s)) = \langle \underline{h} \wedge, \underline{m}(s) \wedge \rangle = \sum_{\omega \in Q_{r,p}} h_\omega m_\omega(s) \tag{4.40}$$

where $\langle \bullet, \bullet \rangle$ denotes the inner product, $\omega = (i_1, \ \ldots, \ i_r) \in Q_{r,p}$, and h_ω, $m_\omega(s)$ are the coordinates of $\underline{h} \wedge$, $\underline{m}(s) \wedge$ respectively. Note that h_ω is the $r \times r$ minor of H which corresponds to the ω set of columns of H and thus h_ω is a multilinear alternating function of the entries h_{ij} of H. The multilinear, skew symmetric nature of DAP suggests that the natural framework for its study is that of exterior algebra [Mar., 1]. The study of the zero structure of the multilinear function $f_M(s,\mathrm{H})$ may thus be reduced to a linear subproblem and a standard multilinear algebra problem as shown below.

1. Linear Subproblem of DAP: Set $\underline{m}(s)^\wedge = \underline{p}(s) \in \mathbb{R}^{\sigma}[s]$. Determine whether there exists a $\underline{k} \in \mathbb{R}^{\sigma}$, $\underline{k} \neq \underline{0}$, such that

$$\begin{aligned} &f_M(s,\underline{k}) = \underline{k}'\,\underline{p}(s) = \sum k_i p_i(s) = f(s) \\ &i \in \sigma,\ f(s) \in \mathbb{R}[s] \end{aligned} \tag{4.41}$$

2. Multilinear Subproblem of DAP: Assume that K is the family of solution vectors $\underline{k}$ of (4.41). Determine whether there exists $\mathrm{H}' = [\underline{h}_1, \ \ldots, \ \underline{h}_r]$, where $\mathrm{H}' \in \mathbb{R}^{p\times r}$, such that

$$\underline{h}_1{}^\wedge \ \ldots \ \underline{h}_r{}^\wedge = \underline{h}^\wedge = \underline{k}, \ \ \underline{k} \in K \tag{4.42}$$

Polynomials defined by equation (4.41) are called *polynomial combinants* [Kar., et al, 2] and the zero assignability of them provides necessary conditions for the solution of the DAP. The solution of the exterior equation (4.42) is a standard problem of exterior algebra and it is known as decomposability of multivectors [Mar., 1]. Note that notions and tools from exterior algebra also play an important role in the linear subproblem, since $f_M(s,\underline{k})$ is generated by the decomposable multivector $\underline{m}(s)^\wedge$. This introduces some new system invariants (independent from the standard ones defined by the Kronecker theory) and they are considered next.

4.6.3 GRASSMAN VECTORS AND PLÜCKER MATRICES [Kar., & Gia., 1]

Let $T(s)\in \mathbb{R}_{o}^{q\times r}(s)$, $T(s)=\left[\underline{t}_1(s),\ \ldots,\ \underline{t}_r(s)\right]$, $q\geq r$, $\operatorname{rank}_{\mathbb{R}(s)}\{T(s)\}=r$ and let $\mathcal{X}_t=\mathcal{R}_{\mathbb{R}(s)}(T(s))$. If $T(s)=M(s)D(s)^{-1}$ is a RCMFD of $T(s)$, then $M(s)$ is a polynomial basis for $\mathcal{X}_t$. If $Q(s)$ is a greatest right divisor of $M(s)$ [Kai., 1] then $T(s)=\widetilde{M}(s)Q(s)D(s)^{-1}$, where $\widetilde{M}(s)$ is a least degree polynomial basis for $\mathcal{X}_t$ [For., 1]. A *Grassman representative* (GR) for $\mathcal{X}_t$ is defined by

$$\underline{t}(s)^\wedge=\underline{t}_1(s)^\wedge\ \ldots{}^\wedge\ \underline{t}_r(s)=\underline{\widetilde{m}}_1(s)^\wedge\ \ldots{}^\wedge\ \underline{\widetilde{m}}_r(s)\cdot z_t(s)/p_t(s) \qquad (4.43)$$

where $z_t(s)=\det\{Q(s)\}$, $p_t(s)=\det\{D(s)\}$ are the zero, pole polynomials of $T(s)$ and $\underline{\widetilde{m}}_1(s)^\wedge=\underline{\widetilde{m}}_1(s)^\wedge\ \ldots{}^\wedge\ \underline{\widetilde{m}}_r(s)\in\mathbb{R}_{o}^{\sigma}[s]$, $\sigma=\begin{pmatrix}p\\ r\end{pmatrix}$ is also a GR of $\mathcal{X}_t$. Since $\widetilde{M}(s)$ is a least degree polynomial basis of $\mathcal{X}_t$, the polynomials of $\underline{\widetilde{m}}(s)^\wedge$ are coprime and $\underline{\widetilde{m}}(s)^\wedge$ will be referred to as a *reduced polynomial GR* $(R-\mathbb{R}[s]-GR)$ of $\mathcal{X}_t$. If $\delta=\deg(\underline{\widetilde{m}}(s)^\wedge)$, then δ is the Forney dynamical order [For., 1] of $\mathcal{X}_t$. $\underline{\widetilde{m}}(s)^\wedge$ may always be expressed as

$$\begin{aligned}&\underline{\widetilde{m}}(s)^\wedge=\underline{p}(s)=\underline{p}_0+s\underline{p}_1+\ldots+s^{\delta}\underline{p}_{\delta}=P_{\delta}\underline{e}_{\delta}\\ &P_{\delta}\in\mathbb{R}_{o}^{\sigma\times(\delta+1)}\end{aligned} \qquad (4.44)$$

where P_δ is a basis matrix for $\underline{\widetilde{m}}(s)^\wedge$ and $\underline{e}_{\delta}(s)=\left[1,\ \ s,\ \ \ldots,\ \ s^{\delta}\right]'$. It can be readily shown that all $R-\mathbb{R}[s]-GR$s of $\mathcal{X}_t$ differ by only a nonzero scalar factor $a\in\mathbb{R}$. By choosing an $\underline{\widetilde{m}}(s)^\wedge$ for which $\left\|\underline{p}_{\delta}\right\|=1$, a monic $R-\mathbb{R}[s]-GR$ is defined. Such a GR of $\mathcal{X}_t$ is defined as the *canonical polynomial Grassman representative* $(C-\mathbb{R}[s]-GR)$ of $\mathcal{X}_t$ [Kar., & Gia., 1] and shall be denoted by $\underline{g}(\mathcal{X}_t)$. The basis matrix P_δ of $\underline{g}(\mathcal{X}_t)$ is defined as the *Plücker matrix* of $\mathcal{X}_t$ [Kar., & Gia., 1]. The significance of these new types of invariants is emphasised by the following result [Kar., & Gia., 1].

Theorem 4.1: [Kar., & Gia., 1] $\underline{g}(\mathcal{X}_t)$, or the associated Plücker matrix P_δ is a complete (basis free) invariant of $\mathcal{X}_t$.

□

If $M(s) \in \mathbb{R}^{q\times r}[s]$, $q \geq r$, $\text{rank}_{\mathbb{R}(s)}\{M(s)\} = r$, then $M(s) = \widetilde{M}(s)Q(s)$, where $\widetilde{M}(s)$ is a least degree basis and $Q(s)$ is a greatest right divisor of the rows of $M(s)$ and thus

$$\begin{aligned}\underline{m}(s)\wedge &= \underline{\widetilde{m}}(s)\wedge \cdot \det\{Q(s)\} = \underline{p}(s)\cdot z_m(s)\\ &= P_\delta \underline{e}_\delta(s)\cdot z_m(s)\end{aligned} \tag{4.45}$$

The linear part of the DAP is thus reduced to

$$f_M(s,\underline{k}) = \underline{k}'\,\underline{p}(s)z_m(s) = \underline{k}'P_\delta \underline{e}(s)z_m(s) \tag{4.46}$$

Proposition 4.2: The zeros of $M(s)$ are fixed zeros of all combinants of $\underline{m}(s)\wedge$.

□

The zeros of $f_M(s,\underline{k})$ which may be freely assigned are those of the combinant $f_{\widetilde{M}}(s,\underline{k}) = \underline{k}'\,\underline{\widetilde{m}}(s)\wedge$ where $\underline{\widetilde{m}}(s)\wedge$ is reduced. Given that the zeros of $f_{\widetilde{M}}(s,\underline{k})$ are not affected by scaling with constraints, it is assumed that $\underline{\widetilde{m}}(s)\wedge = P_\delta \underline{e}_\delta(s)$.

For the control problems discussed earlier the matrix $M(s)$ has a special structure. Thus the matrix coefficient of $\underline{m}(s)\wedge$ has important properties which stem from the properties of the corresponding control problem. A number of Plücker type matrices associated with a linear system are defined below.

(i) For the pair (A, B), $\underline{b}(s)'\wedge$ denotes the exterior product of the rows of $B(s)$ = $[sI - A, -B]$ and $P(A, B)$ is the $(n+1)\times\binom{n+m}{n}$ basis matrix of $\underline{b}(s)'\wedge$. $P(A, B)$ will be called the *controllability Plücker matrix*.

(ii) For the pair (A, C), $\underline{c}(s)^{\wedge}$ denotes the exterior product of the columns of $C(s) = [sI - A^t, -C^t]^t$ and $P(A, C)$ is the $\binom{n+p}{n} \times (n+1)$ basis matrix of $\underline{c}(s)^{\wedge}$. $P(A, C)$ will be called the *observability Plücker matrix.*

(iii) For the pair (A, B), the restricted input state pencil $R(s) = sN - NA \in \mathbb{R}^{(n-p)\times n}[s]$ may be introduced. If $\underline{r}(s)^{t\wedge}$ denotes the exterior product of the rows of $R(s)$, and $P_R(A, B)$ is the $(n-m+1) \times \binom{n}{n-m}$ basis matrix of $\underline{r}(s)^{t\wedge}$, then $P_R(A, B)$ may be called the *restricted controllability Plücker matrix.*

(iv) For the pair (A, C) the introduction of the restricted state-output pencil $Q(s) = sM - AM \in \mathbb{R}^{n\times(n-p)}[s]$ may be made. If $\underline{q}(s)^{\wedge}$ denotes the exterior product of the columns of $Q(s)$ and $P_Q(A, C)$ is the $\binom{n}{n-p} \times (n-p+1)$ basis matrix of $\underline{q}(s)^{\wedge}$, $P_Q(A, C)$ will be called the *restricted observability Plücker matrix.*

The restricted pencils $sN - NA$ and $sM - AM$ provide alternative means for testing controllability and observability respectively which has the advantage of being independent from state feedback and output injection correspondingly [Kar., 3, 4]. The general properties of the Plücker matrices for linear systems have been addressed [Kar., & Gia., 2]. Specifically the link of $P(A, B)$ and $P(A, C)$ characterise the pairs (A, B) and (A, C) respectively and it is expected to be linked to controllability and observability. This is established in the following result.

Theorem 4.2: [Kar., & Lev., 1] Let $P(A, B)$ and $P(A, C)$ be the Plücker matrices associated with the pairs (A, B) and (A, C) respectively.

(i) (A, B) is controllable if and only if $\text{rank}\{P(A, B)\} = n + 1$

(ii) (A, C) is observable if and only if $\text{rank}\{P(A, C)\} = n + 1$

□

The proof of the above result is based on the Kronecker structure of the pencils $B(s)$ and $C(s)$ [Kar., & Gia., 2]. This allows the development of a corresponding result based on the restricted pencils $R(s) = sN - NA$ and $Q(s) = sM - AM$ which is stated below.

Corollary 4.1: [Kar., & Gia., 2] Let $P_R(A, B)$ and $P_Q(A, C)$ be the restricted controllability, observability Plücker matrices respectively of the linear system $S(A,B,C)$. The following properties hold true:

(i) $P_R(A, B)$ is invariant under state feedback. Furthermore the pair (A, B) is controllable if and only if

$$\text{rank}\left\{P_R\left(A,B\right)\right\} = n - m + 1 \tag{4.47}$$

(ii) $P_Q(A, C)$ is invariant under output injection. Furthermore the pair (A, C) is observable if and only if

$$\text{rank}\left\{P_Q\left(A,C\right)\right\} = n - p + 1 \tag{4.48}$$

(iii) The gcd of $\underline{r}(s)'^{\wedge}$ defines the polynomial of input-decoupling zeros of (A, B).

(iv) The gcd of $\underline{q}(s)^{\wedge}$ defines the polynomial of output-decoupling zeros of (A, C).

□

The above result readily follows from the results describing the link between Kronecker invariants and Grassman invariants [Kar., & Lev., 1] and the properties of the restricted pencils [Kar., 3].

4.6.4 PROJECTIVE MEASURES FOR DISTANCE FROM UNCONTROLLABILITY/UNOBSERVABILITY

The previous subsection provides the motivation for the introduction of some new measures for evaluating the distance from uncontrollability and unobservability which are invariant under compensation and are defined below.

Definition 4.2: [Kar., & Gia., 2] For the linear system $S(A,B,C)$, consider the Plücker matrices $P(A, B)$ and $P(A, C)$ and the restricted versions $P_R(A, B)$ and $P_Q(A, C)$. The smallest of the singular values of the corresponding matrices are denoted by $\underline{\sigma}(A,B)$, $\underline{\sigma}(A,C)$, $\underline{\sigma}_R(A,B)$ and $\underline{\sigma}_Q(A,C)$. The respective condition numbers are denoted by $\mu(A,B)$, $\mu(A,C)$, $\mu_R(A,B)$ and $\mu_Q(A,C)$. Then the following definitions can be made

(i) $\underline{\sigma}(A,B)$, $\mu(A,B)$ are <u>projective open-loop measures for controllability</u>.

(ii) $\underline{\sigma}(A,C)$, $\mu(A,C)$ are <u>projective open-loop measures for observability</u>.

(iii) $\underline{\sigma}_R(A,B)$, $\mu_R(A,B)$ are <u>projective measures for controllability</u>.

(iv) $\underline{\sigma}_Q(A,C)$, $\mu_Q(A,C)$ are <u>projective measures for observability</u>.

□

In the above definition the term projective is used because the measures are defined by projective invariants, i.e. the Plücker matrices [Kar., & Gia., 1]. The properties of these new measures stem from their definition (Plücker matrices corresponding to matrix pencils [Kar., & Lev., 1]) and they are summarised below.

Corollary 4.2: For the projective measures the following properties hold true:

(i) $\underline{\sigma}(A,B)$,$\mu(A,B)$ are dependent on state-coordinate and state feedback transformations and $\underline{\sigma}(A,C)$, $\mu(A,C)$ are dependent on state-coordinate and output injection.

(ii) $\underline{\sigma}_R(A,B)$, $\mu_R(A,B)$ are dependent on state-coordinate transformations, but are invariant under state feedback.

(iii) $\underline{\sigma}_Q(A,C)$, $\mu_Q(A,C)$ are dependent on state-coordinate transformations, but are invariant under output injection.

<u>Proof</u>

The result is readily established for controllability pencils and the analysis carries over to observability. Thus under the input, state coordinate transformations and state feedback

$$Q^{-1}[sI-A,-B]\begin{bmatrix} Q & 0 \\ L & R \end{bmatrix} = Q^{-1}B(s)T = B'(s) \tag{4.49}$$

Using the Binet-Cauchy Theorem [Mar. & Min., 1]

$$\begin{aligned} G_n\left(B'(s)\right) &= G_n\left(Q^{-1}\right)G_n\left(B(s)\right)G_n(T) \\ &= \left|Q^{-1}\right| \cdot G_n\left(B(s)\right)G_n(T) \end{aligned}$$

and thus

$$\underline{e}_n(s)' P(A', B') = \underline{e}_n(s)' \cdot \left|Q^{-1}\right| \cdot P(A, B)G_n(T)$$

or equivalently, if $c = \left|Q^{-1}\right|$, then

$$P(A', B') = c \cdot P(A, B)G_n(T) \qquad (4.50)$$

The above condition implies that $P(A', B')$, $P(A, B)$ are left equivalent under state-coordinates and state feedback, and in general it implies variability of singular values and the condition numbers.

If state coordinate transformations and any state feedback are taken into consideration, then the restriction pencils for (A, B) and $(A + BL, B)$ are the same and the effect of state coordinate transformation is expressed by

$$R'(s) = (sN - NA)Q = R(s)Q \qquad (4.51)$$

and from the Binet-Cauchy theorem

$$G_{n-m}\left(R'(s)\right) = G_{n-m}(sN - NA)G_{n-m}(Q)$$

from which

$$\underline{e}'_{n-m}(s)P_R(A', B') = \underline{e}'_{n-m}(s)P_R(A, B)G_{n-m}(Q)$$

or

$$P_R(A',B') = P_R(A,B)G_{n-m}(Q) \tag{4.52}$$

The invariance of $\underline{\sigma}_R(A,B)$, $\mu_R(A,B)$ under state feedback and their variability under state coordinate transformations is obvious.

□

The above new indicators provide some estimates of distance from uncontrollability, unobservability and indicate in a non ambiguous way that some measures are affected by feedback ($\underline{\sigma}(A,B)$, $\mu(A,B)$, $\underline{\sigma}(A,C)$, $\mu(A,C)$). but there are measures which are feedback invariant such as $\underline{\sigma}_R(A,B)$, $\mu_R(A,B)$, $\underline{\sigma}_Q(A,C)$, $\mu_Q(A,C)$. These measures are linked with the notions of "almost decoupling zeros" which have already been introduced [Kar., & Gia., 1] and will be developed further next.

4.6.5 THE NOTION OF "ALMOST ZERO" OF A SET OF POLYNOMIALS

Let $\mathcal{P}\left\{p_i(s)\text{: } p_i(s) \in \mathbb{R}[s],\ i \in \underline{m},\ d_i = \deg p_i(s)\right\}$ be a set of polynomials and let $d = \max\left\{d_i,\ i \in \underline{m}\right\}$. A polynomial vector $\underline{p}(s)$, $\underline{p}(s) \in \mathbb{R}^m[s]$, may always be associated with $\mathcal{P}$, where

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ \vdots \\ p_k(s) \\ \vdots \\ p_m(s) \end{bmatrix} = \begin{bmatrix} p_0^1 & p_1^1 & \cdots & p_{d_1}^1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \vdots \\ p_0^k & p_1^k & \cdots & \cdots & \cdots & \cdots & p_{d_k}^k \\ \vdots & \vdots & & & & & \\ p_0^m & p_1^m & \cdots & p_{d_m}^m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^d \end{bmatrix} \tag{4.43}$$
$$= \left[\underline{p}_0,\ \ldots,\ \underline{p}_d\right]\underline{e}_d(s) = P_d\,\underline{e}_d(s)$$

where $P_d \in \mathbb{R}^{m\times(d+1)}$ and $\underline{e}_d(s) \in \mathbb{R}^{d+1}[s]$. The polynomial vector $\underline{p}(s)$ is defined as a *vector representative* of $\mathcal{P}$ and $d = \deg \underline{p}(s)$ will be referred to as the *degree* of $\mathcal{P}$.

The matrix P_d characterises the properties of $\mathcal{P}$ and it is defined as a *basis matrix* of $\mathcal{P}$. The set $\mathcal{P}$ will be called reduced if the polynomials $p_i(s)$ are coprime; otherwise it will be called *nonreduced*. Finally $\mathcal{P}$ will be called monic if $\left\|\underline{p}_d\right\|=1$ (Euclidean norm).

The coprimeness of a set of polynomials $\mathcal{P}$ may be investigated by using one of the standard resultant tests [Bar., 1]. The notion of an "almost zero" or an "almost common divisor" of $\mathcal{P}$ is discussed next. When $s \in \mathbb{C}$, the vector representative $\underline{p}(s)$ of $\mathcal{P}$ defines a vector analytic function with domain $\mathbb{C}$ and codomain $\mathbb{C}^m$. The norm of $\underline{p}(s)$ (or the norm of $\mathcal{P}$) is defined as

$$\left\|\underline{p}(s)\right\| = \phi(\sigma,\omega) = \sqrt{\underline{p}(s^*)'\,\underline{p}(s)} = \sqrt{\underline{e}_d(s^*)'\,P_d'\,P_d\,\underline{e}_d(s)} \qquad (4.44)$$

where s^* is the complex conjugate of s $(s=\sigma \pm j\omega)$. Note that if $q(s)=s+\alpha$ is a divisor of $\mathcal{P}$, then $p(-\alpha)=\underline{0}$ and thus $\left\|p(-\alpha)\right\|=0$. This observation leads to the following definition.

Definition 4.3: [Kar., & Gia., 1] Let $\mathcal{P}$ be a reduced set of polynomials. If $s = z$, $s \in \mathbb{C}$, is a local minimum of $\left\|\underline{p}(s)\right\|$, then z will be called an *almost zero* (AZ) of $\mathcal{P}$ and the value of $\left\|\underline{p}(z)\right\| = \varepsilon$ will be referred to as the *order* of the AZ. If $s=\tilde{z}$ is the global minimum of $\left\|\underline{p}(s)\right\|$, then $\tilde{z}$ will be called the *prime almost zero* (PAZ) of the set $\mathcal{P}$.

□

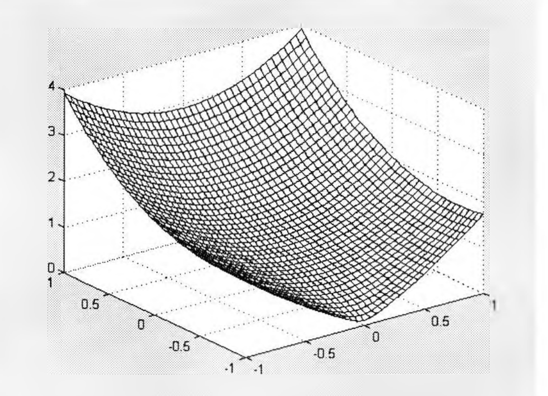
Clearly if $\mathcal{P}$ is not reduced, then the set of AZs, which have order $\varepsilon=0$, defines the zeros of $\mathcal{P}$. Thus the above definition unifies the notions of exact and "approximate" zeros, since both emerge as minima of a norm function of $\mathcal{P}$. The order ε of an AZ indicates how well z may be considered as an "approximate" zero of $\mathcal{P}$. It should be noted however that scaling of the polynomials of $\mathcal{P}$ by a $c \in \mathbb{R}$, $c \neq 0$, affects the size of the error and thus a better, standardised definition of the almost zero may be given by assuming all polynomials in the set are monic.

Whenever this convention is used, the almost zero takes a unique form and will be referred to as the *monic almost zero*. The computation of AZs is made by using numerical optimisation procedures and an algorithm has been given [Kar., et al, 2].

Example 4.1: Let $\mathcal{P}$ be defined by

$$\underline{p}(s)=\begin{bmatrix} s+1.1 \\ s^2+s \end{bmatrix}=\begin{bmatrix} 1.1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}=P_2\underline{e}_2(s)$$

A plot of $\phi(\sigma,\omega)$ surfaces in the region of the prime AZ is shown in Figure 4.1 and the corresponding MATLAB code, *paz.m* can be found in the appendix. The numerical search reveals that the PAZ is at z = -1.046 and that its order is ε = 0.005.



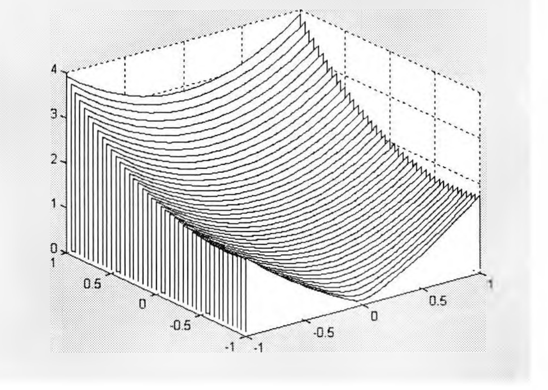


Figure 4.1 Plot of the surface $\left\|\underline{p}(s)\right\|$ in the prime region

□

The properties of the set of AZs of $\mathcal{P}$ are summarised next [Kar., et al, 2]. It will be shown next that the similarities between exact and almost zeros extend beyond their common definition.

Let $\underline{p}(s) = P_d\underline{e}_d(s)$ be a vector representative of the set of polynomials $\mathcal{P}\{p_i(s)\text{: } p_i(s) \in \mathbb{R}[s],\ i \in \underline{m}\}$ where $P_d \in \mathbb{R}^{m\times(d+1)}$ and let $\underline{k} \in \mathbb{R}^m$. The polynomial function of the parameter vector $\underline{k}$ defined by [Kar., & Gia., 1]

$$f(s,\mathcal{P},\underline{k}) = \underline{k}' P_d \underline{e}_d(s) = \sum_{i=1}^{m} k_i p_i(s) \tag{4.45}$$

is called a *$\underline{k}$-polynomial combinant* of $\mathcal{P}$ and shall be denoted in short by $f(s, \underline{k})$. For a set of polynomials $\mathcal{P}$ represented by a basis matrix P_d, the zero assignment problem for polynomial combinants can be defined by finding a $\underline{k} \in \mathbb{R}^m$ such that $f(s, \underline{k}) = \underline{k}'P_d\underline{e}_d(s) = a(s)$, where $a(s) \in \mathbb{R}[s]$ is arbitrary. It is clear that the maximum degree of $a(s)$ has to be equal to the degree d of $\mathcal{P}$, and if $a(s) = a_0 + a_1s + \ldots + a_ds^d = [a_0,\ a_1,\ \ldots,\ a_d] \bullet \underline{e}_d(s) = \underline{a}'\underline{e}_d(s)$, then the problem is reduced to the solution of the equation

$$\widetilde{P}_d\underline{k} = \underline{a},\ \ \widetilde{P}_d = P_d' \in \mathbb{R}^{(d+l)\times m},\ \ \underline{a} \in \mathbb{R}^{(d+l)} \tag{4.46}$$

A set $\mathcal{P}$ for which equation (4.45) has a solution for all $\underline{a} \in \mathbb{R}^{(d+l)}$ will be called *completely assignable* (CA). Otherwise $\mathcal{P}$ will be referred to as *nonassignable* (NA). An important family of nonassignable sets are those for which there is no $\underline{k} \in \mathbb{R}^m$ such that $f(s, \underline{k}) = c,\ c \in \mathbb{R}$. Such sets will be called *strongly nonassignable* (SNA) and they have the additional property that there is no combinant with all its zeros at $s = \infty$.

Proposition 4.3: Let $P_d = \left[\underline{p}_0, \ \underline{p}_1, \ \ldots, \ \underline{p}_d\right] = \left[\underline{p}_0, \overline{P}_d\right] \in \mathbb{R}^{m\times(d+l)}$, with π $= \mathrm{rank}\{P_d\}$ and $\overline{\pi} = \mathrm{rank}\{\overline{P}_d\}$, where P_d is a basis matrix of a set $\mathcal{P}$ [Kar., & Gia., 1].

1. $\mathcal{P}$ is completely assignable if and only if $\pi = d + l$.
2. $\mathcal{P}$ is strongly nonassignable if and only if $\overline{\pi} = m$.

□

The significance of almost zeros in the distribution of zeros of polynomial combinants is described by the following result [Kar., et al, 2].

Corollary 4.3: Let $\mathcal{P}$ be a strongly nonassignable set and z be the prime AZ of $\mathcal{P}$. For all $\underline{k} \in \mathbb{R}^m$ there is always a disk centred at z with a radius R, $D[z, R]$ such that it contains at least a zero of $f(s, \underline{k})$.

□

The above result demonstrates that as far as distribution of zeros of $f(s, \underline{k})$, the prime AZ acts in a similar way to that of the exact zero. In fact an exact zero implies a fixed zero for $f(s, \underline{k})$, whereas an AZ implies an extension of a fixed point to that of a disk. Results for computing the radii of these disks have been given [Kar., et al, 2]. The above property is demonstrated by the following example.

Example 4.2: For the polynomial set of Example 4.1 the set of combinants is defined by

$$f(s,\underline{k}) = \begin{bmatrix} k & 1 \end{bmatrix} \begin{bmatrix} s+1.1 \\ s^2+s \end{bmatrix} = k(s+1.1) + (s^2+s) \qquad (4.47)$$

For this special case, the zero assignment is a standard root locus problem. The root locus for equation (4.47) is shown in Figure 4.2. The polynomial set has an AZ at $z = -1.046$, and from Figure 4.2, the radius of the minimal disk, which is centred at $z = -1.046$, is found to be $\widetilde{R}_m(z) = 0.43$. The predicted radius has a value $R_{\text{pred}} = 0.79$ [Kar., et al, 2].

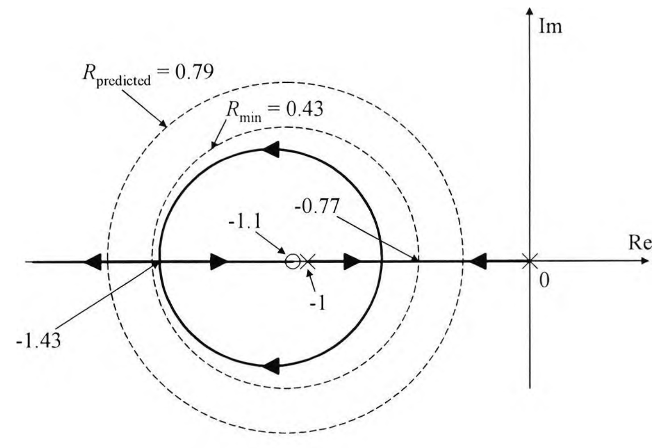


Figure 4.2 Root Locus

□

4.6.6 ALMOST DECOUPLING ZEROS AND INVARIANT ALMOST DECOUPLING ZEROS [Kar., & Gia., 1]

The general results on polynomial vectors will now be used to define notions of almost decoupling zeros. For the system *S*(A, B, C, D) the following polynomial vectors are defined

$$\underline{b}(s)^t \wedge = G_n([sI - A, -B]) = \underline{e}_n^t(s)P(A,B) = \underline{g}_{A,B}(s)^t \tag{4.48}$$

$$\underline{c}(s) \wedge = G_n\left(\begin{bmatrix} sI - A \\ -C \end{bmatrix}\right) = P(A,C)\underline{e}_n(s) = \underline{g}_{A,C}(s) \tag{4.49}$$

$$\underline{r}(s)^t \wedge = G_{n-m}([sN - NA]) = \underline{e}_{n-m}^t(s)P_R(A,B) = \underline{g}_{A,B}^r(s)^t \tag{4.50}$$

$$\underline{q}(s) \wedge = G_{n-p}(sM - AM) = P_Q(A,C)\underline{e}_{n-p}(s) = \underline{g}_{A,C}^q(s) \tag{4.51}$$

where $\underline{g}_{A,B}(s)$, $\underline{g}_{A,B}^r(s)$ are referred to as the *controllability Grassman representative* (C-GR) and the *restricted controllability Grassman representative* (RC-GR). $\underline{g}_{A,C}(s)$, $\underline{g}_{A,C}^q(s)$ are called the *observability Grassman representative* (O-

GR) and the *restricted observability Grassman representative* (RO-GR). The polynomial vectors defined above have the following properties.

Proposition 4.4: For the linear system $S(A, B, C)$ the following properties hold true [Kar., & Gia., 1]:

(i) The polynomial vectors $\underline{g}_{A,B}(s)$, $\underline{g}^{r}_{A,B}(s)$ have the same gcd, the roots of which define the system input decoupling zeros.

(ii) The polynomial vectors $\underline{g}_{A,C}(s)$, $\underline{g}^{q}_{A,C}(s)$ have the same gcd, the roots of which define the system output decoupling zeros.

Proof:

If $(N, B^{\dagger})$ is a pair of a left annihilator and a left inverse of B then [Kar., 3]

$$\begin{gathered}
\begin{bmatrix} N \\ \hline B^{\dagger} \end{bmatrix}[sI-A, \quad -B] = Q[sI-A, \quad -B] = \begin{bmatrix} sN-NA & 0 \\ (sB^{\dagger}-B^{\dagger}A) & -I_m \end{bmatrix} \\
|Q| \neq 0 \Rightarrow \\
\begin{bmatrix} N \\ \hline B^{\dagger} \end{bmatrix}[sI-A, \quad -B]\begin{bmatrix} I_n & 0 \\ (sB^{\dagger}-B^{\dagger}A) & -I_p \end{bmatrix} = \begin{bmatrix} sN-NA & 0 \\ 0 & -I_m \end{bmatrix}
\end{gathered} \tag{4.52}$$

Condition (4.52) implies that $[sI - A, -B]$ and $sN - NA$ have the same set of nontrivial invariant polynomials and thus the same gcd. The proof for part (ii) follows along similar lines.

□

Definition 4.4: [Kar., & Gia., 2] For the linear system $S(A, B, C)$ the following can be defined:

(i) The almost zeros of $\underline{g}_{A,B}(s)$, $\underline{g}^{r}_{A,B}(s)$ are the *system almost input decoupling zeros* (SA-IDZ) and *invariant almost input decoupling zeros* (IA-IDZ) correspondingly.

(ii) The almost zeros of $\underline{g}_{A,C}(s)$, $\underline{g}^{q}_{A,C}(s)$ are the *system almost output decoupling zeros* (SA-ODZ), and *invariant almost output decoupling zeros* (IA-ODZ) correspondingly.

□

The location of the almost zeros depends on the properties of the corresponding Plücker matrices. The term invariant above refers to the property of invariance under state feedback, $\underline{g}^r_{A,B}(s)$, and invariance under output injection, $\underline{g}^q_{A,C}(s)$, respectively. The above definition clarifies the important notion that although decoupling zeros are invariant under feedback, their almost versions are not always invariant and invariant almost zeros may be introduced through the restricted Grassman representatives $\underline{g}^r_{A,B}(s)$ and $\underline{g}^q_{A,C}(s)$.

4.7 SELECTION OF CONTROL INPUT DIRECTION

4.7.1 PROBLEM DEFINITION

Some state feedback design techniques are based on dyadic feedback which is equivalent to making the system controllable from one input. This problem may be seen within the overall framework considered here. In fact it is desired to make the system controllable from one input, and the aim is to select such an input which also produces the best degree of controllability. The approach considered here is based on an input selection that leads to the well conditioning of the controllability matrix, and is independent of the system eigenvalues and initial conditions, yet dependent on the singular values. The required selection of appropriate input directions leads to the minimisation of the condition number of the controllability matrix.

Given the system

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{4.53}$$

where the eigenvalues of A are distinct, the mode $(\lambda_i, \ \underline{u}_i, \ \underline{v}_i^t)$ (eigenvalue, right eigenvector and left eigenvector respectively) is <u>uncontrollable</u> if and only if

$$\begin{aligned} \underline{\beta}_i' &= \underline{v}_i' B = 0 \\ \underline{v}_i'(\lambda I - A) &= 0 \end{aligned} \tag{4.54}$$

or in matrix form

$$\underline{v}_i'\left[\lambda_i I - A, \quad -B\right] = 0 \tag{4.55}$$

where $\left[\lambda_i I - A, \quad -B\right]$ is the *input-state pencil*. If the system described by (4.53) is subjected to the transformation

$$\underline{x} = U\underline{\tilde{x}} \tag{4.56}$$

where $U = V^1$ is the matrix of right eigenvectors and V the matrix of left eigenvectors, then the system can be redefined as

$$\begin{aligned} \dot{\underline{\tilde{x}}} &= \Lambda\underline{\tilde{x}} + \beta\underline{u} \\ \mathrm{B} &= VB \end{aligned} \tag{4.57}$$

and the system modes are controllable if and only if $\underline{\beta}_i' \neq 0$. The equivalent modal controllability matrix is thus defined as

$$\begin{aligned} \tilde{Q} &= \left[\mathrm{B}, \quad \Lambda\mathrm{B}, \quad \ldots, \quad \Lambda^{n-1}\mathrm{B}\right] \\ &= \left[\begin{array}{c|c|c|c} \underline{\beta}_1' & \lambda_1\underline{\beta}_1' & \cdots & \lambda_1^{n-1}\underline{\beta}_1' \\ \underline{\beta}_2' & \lambda_2\underline{\beta}_2' & \cdots & \lambda_2^{n-1}\underline{\beta}_2' \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\beta}_n' & \lambda_n\underline{\beta}_n' & \cdots & \lambda_n^{n-1}\underline{\beta}_n' \end{array}\right] \end{aligned} \tag{4.58}$$

$\tilde{Q}$ is non-singular if and only if for all i $\underline{\beta}_i' \neq 0$ (controllable). Hence, in order for the system modes to be controllable, the controllability matrix $\tilde{Q}$ has to be non-singular (full rank for multi input case). The controllability matrix of the system described by the untransformed set of (4.58) is defined as

$$\tilde{Q}V^{-1} = Q = \begin{bmatrix} B, & AB, & \ldots, & A^{n-1}B \end{bmatrix} \tag{4.59}$$

and is a function of $\tilde{Q}$. For non-singular matrices, the condition number is finite (small) and so for matrices that are nearly non-singular, their condition numbers are fairly small. For matrices that are nearly singular, their condition numbers tend to infinity. The implication made here is that if the controllability matrix is non-singular (i.e. its condition number is a small finite number), then all the modes are controllable. Finally if $\tilde{Q}$ is almost singular (i.e. it has a very high condition number), then the modes are uncontrollable. Hence, if a set of inputs can be chosen to minimise the condition number of the controllability matrix, and hence influence the overall controllability of the modes, the following problem can be formulated.

Problem 4.1: Given the controllability matrix $Q(A,B) = \left[B, AB, \ldots, A^{n-1}B\right]$, find a set of control input directions $\underline{u}$ such that the condition number of

$$\left[B\underline{u}, AB\underline{u}, \ldots, A^{n-1}B\underline{u}\right] = \left[\underline{b}_u, A\underline{b}_u, \ldots, A^{n-1}\underline{b}_u\right] \tag{4.60}$$

is minimised.

□

The condition number of a matrix is given by

$$cond(G) = \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)} \tag{4.61}$$

where $\overline{\sigma}(G)$ and $\underline{\sigma}(G)$ are the maximum and minimum singular values of a matrix respectively. The maximum singular value of a matrix is also the same as the l_2 norm, i.e.

$$\overline{\sigma}(G) = \|A\|_2 \tag{4.62}$$

In order to facilitate the analysis, the reciprocal of the condition number will be used. This changes the optimisation problem from a minimisation approach to a maximisation approach. So now the reciprocal of the condition number ranges from one (for well conditioned, non-singular matrices) to zero (for badly conditioned, singular matrices).

4.7.2 INVESTIGATION OF THE TWO INPUT CASE

Let

$$
\begin{aligned}
&Q = \left[B, AB, A^2B, \ldots, A^{n-1}B\right] \\
&\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad B = \begin{bmatrix} \underline{b}_1 & \underline{b}_2 \end{bmatrix} \quad A \in \mathbb{R}^{n \times n}
\end{aligned}
\tag{4.63}
$$

where Q is the controllability matrix, A and B are the system matrices and $\underline{u}$ represents two control input directions. Multiplying Q by the input vector $\underline{u}$ gives

$$
\begin{aligned}
&Q\underline{u} = \left[B\underline{u}, AB\underline{u}, \ldots, A^{n-1}B\underline{u}\right] = \\
&\left[(b_1u_1 + b_2u_2), (Ab_1u_1 + Ab_2u_2), \ldots, (A^{n-1}b_1u_1 + A^{n-1}b_2u_2)\right] = \\
&\left[\underline{b}_1, A\underline{b}_1, \ldots, A^{n-1}\underline{b}_1\right]u_1 + \left[\underline{b}_2, A\underline{b}_2, \ldots, A^{n-1}\underline{b}_2\right]u_2 = \\
&\left[Q_1u_1 + Q_2u_2\right]
\end{aligned}
$$

This problem is first solved for two input directions, u_1 and u_2 only. This is because the condition number has to be optimised. Since this is being done graphically, using MATLAB, the condition number cannot be plotted with more than two other axes, and in this case, the axes will be the inputs u_1 and u_2. Hence the condition number of the equation below is to be plotted against two input directions,

$$
Q(A, B\underline{u}) = u_1Q_1(A, \underline{b}_1) + u_2Q_2(A, \underline{b}_2) \tag{4.64}
$$

So the task is to investigate how the condition number of Q varies for different values of the control inputs u_1 and u_2.

4.7.3 SIMULATION

The condition number is to be plotted against the two gains of the general input directions, $\underline{u}_1$ and $\underline{u}_2$ by making use of equation (4.64) above. The task is to investigate how the condition number of Q varies for different values of the control inputs $\underline{u}_1$ and $\underline{u}_2$. MATLAB code was used to perform this operation which can be found in the Appendix. The system matrices (A, B) of a C5A transport aircraft were used to demonstrate the task. Conveniently, the B matrix consists of two columns, and hence two input directions.

$$A = \begin{bmatrix} -0.1192 & 0.58060 & 4.75800 & -1.4640 & 2.0600 & 1.6400 \\ -0.4412 & -0.04412 & -0.10140 & 1.3430 & -0.4941 & -0.5637 \\ -5.3660 & 0.50390 & -0.93810 & -2.1740 & 4.6320 & 3.2380 \\ 0.7003 & -0.88560 & 0.14910 & -1.2320 & 4.4520 & 5.5330 \\ -0.9315 & -0.39540 & -0.15980 & -0.4563 & -6.5790 & -2.5920 \\ 0.0298 & -0.26970 & 0.02673 & -0.4245 & -0.4385 & -7.3640 \end{bmatrix}$$

$$B = 10^{7}\begin{bmatrix} -25.77000 & 18.64999 & -2.4910 & -1.8750 & -1.1390 & -0.3218 \\ 29.85001 & 23.45000 & -8.58700 & -2.8170 & -1.8510 & -0.2683 \end{bmatrix}$$

The above model is a two-input, sixth-order system [Enns, 1], with stable left hand plane eigenvalues. The resulting plot is shown below in Figure 4.3.

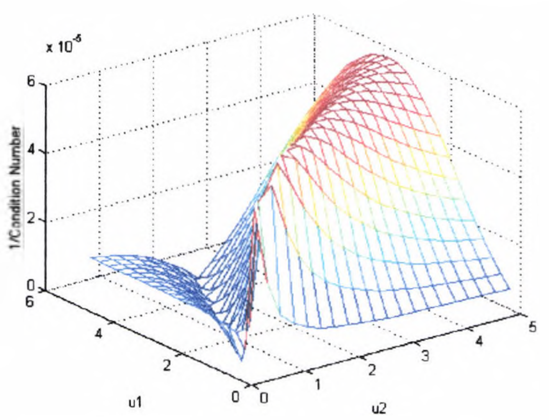


Figure 4.3 3D Plot of the reciprocal condition number v the inputs

The maximum ridge of the plot indicates the input directions at which the reciprocal of the condition number of the $Q(A,B\underline{u}) = u_1Q_1(A,\underline{b}_1) + u_2Q_2(A,\underline{b}_2)$ matrix is at its maximum, i.e. when it is well conditioned.

This procedure could be extended to the case where there are more than two inputs to deal with, but this would extend the computational method to that of optimising several unknowns. The case presented here is for a system with two inputs, and this is fairly trivial to solve using the graphical approach.

4.8 Summary

The review of measures of controllability and observability have shown how important this area of control design is, especially when developing control laws for large scale systems like flexible space structures. Most of the measures reviewed were derived especially for this purpose. The latter part of the chapter concentrated on comparing measures of controllability/observability and examining how different indicative properties can be used. This was followed by a section examining how the structure of the state feedback matrix affects the degree of controllability. It was found that not only does the application of state feedback affect the degree of controllability, but the rank and skewness of the feedback matrix is also a factor in the measure. This was proceeded by a section detailing how controllability and observability properties can be determined from Plücker matrices of transfer function matrices. This was linked to the notion of "almost" zeros. Finally, a new method of measuring controllability was introduced. This was based on selecting inputs in order to improve the conditioning of the controllability matrix. However, the method was limited to systems with two inputs, and an extension of this work is required to extend the algorithm to a multi-input case. In the next chapter, a link will be made between energy consumed and the singular values of the output controllability grammians.

5 INPUT-OUTPUT CONTROLLABILITY AND ENERGY

5.1 INTRODUCTION

Energy conservation (or energy usage minimisation) linking measures of controllability has already been considered [Roh, & Par., 1], [Bol., & Lu, 1], [V Vel., & Car., 1]. Together with the energy of the control signals, the *quantitative* side of the system properties can be studied. Such an area is the output controllability, which is the principal topic of this chapter. Quantitative controllability can be characterised in terms of the singular values of the controllability grammian in relation to the minimal energy needed in control. The singular values of the controllability grammian are the indicators of quantitative controllability. This concept will be demonstrated later in this chapter. The next section deals with the mathematical analysis of output controllability, particularly the output controllability grammian, and the link between its singular values and the energy required to transfer the output from one position to another. This is followed by a practical analysis concerning the importance of energy use in large scale systems, and ends with a few examples demonstrating the algorithms developed.

5.2 INPUT-OUTPUT CONTROLLABILITY

5.2.1 DEFINING CONDITION FOR INPUT-OUTPUT CONTROLLABILITY

The observability, controllability and output disturbability of a system are known as *qualitative properties*. In real life applications, a system is usually disturbed by certain unavoidable sources of noise. For such cases it is desirable to eliminate the effect that this unwanted noise has on the output. These disturbances are denoted as $\underline{\omega}(t)$ in equation (5.1) below. Output disturbability defines the degree to which the response to a disturbance can be eliminated either totally or partially. Qualitative properties are important in the sense that they reveal the capabilities and limitations of the system. For example, if a system is state controllable, then there will exist a certain input signal which will enable any given initial state of a system to be brought to zero in a finite time interval. On the other hand, if a system is not state controllable, then it will not be possible for the system initial conditions to be brought to zero with any control signal in a finite time.

The following set of equations describe a state space model *S(A, B, C, D, H, J)*:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) + H\underline{\omega}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) + J\underline{\omega}(t) \end{aligned} \tag{5.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times l}$, $H \in \mathbb{R}^{n\times p}$, $J \in \mathbb{R}^{m\times p}$, $\underline{x} \in \mathbb{R}^{n}$, $\underline{u} \in \mathbb{R}^{l}$, $\underline{y} \in \mathbb{R}^{m}$ and $\underline{\omega} \in \mathbb{R}^{p}$. $\underline{\omega}(t)$ describes the disturbances corresponding to the state (*H*) and output (*J*).

If such a system is output controllable, then there will always exist an input $\underline{u}(t)$ such that an arbitrarily specified final output state $\underline{y}(t_f)$ can be reached from an arbitrary starting position $\underline{y}(t_0)$. The study conducted in this chapter is aimed at finding among all the possible inputs that enables such a transition to be made, a particular input that utilises the least amount of energy and subsequently investigate the relationship between the minimum energy and the output of the controllability grammian, which has already been discussed in Chapter 3. The following definition for output controllability applies.

Definition 5.1: A system is said to be completely output-controllable over the time interval $[t_0, t_f]$ if for a given t_0 and t_f, any final output can be achieved from arbitrary initial conditions of the system at $t = t_0$.

□

The concept of *output-function controllability* has been established by Rosenbrock [Ros., 1]. In many industrial processes, control laws are implemented in order to make the output vector y of a plant take a certain form as a function of time. Let a system have the $(r + m) \times (r + m)$ polynomial system matrix [Ros., 1]

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{5.2}$$

so that the $m \times m$ transfer function matrix is defined by [Ros., 1]

$$G(s) = V(s)T^{-1}(s)U(s) + W(s) \tag{5.3}$$

The system is called *functionally controllable* if it satisfies the following condition. Let the McMillan degree $\delta(G)$ of $G(s)$ be p. Then given any y which is initially at zero for $t < 0$, has its pth derivative $D_p y \in D_R$ (D_R is a set of physical variables that are continuous [Ros., 1]), and satisfies the boundary condition $\|D_p y(t)\| < Me^{ct}$ for some M, α, and all t, there exists a control input u such that with $x(0) = 0$, u generates y.

Theorem 5.1: [Ros., 1] The system described by (5.2) is functionally controllable if and only if one of the following holds

(i) $|G(s)| \neq 0$

(ii) $|P(s)| \neq 0$

□

Theorem 5.1 implies that when $G(s)$ and $P(s)$ have full rank, the system that they describe is functionally controllable [Ros., 1].

5.2.2 CONSIDERATION OF STRICTLY PROPER TIME INVARIANT LINEAR SYSTEMS

For linear time variant systems, such as those described by (5.1), the system output is defined as [Ske., 1]

$$\begin{aligned} Y(t_f,t_0) = D(t_f)D^T(t_f)\delta + D(t_f)B^T(t_f)C^T(t_f) \\ + C(t_f)B(t_f)D^T(t_f) + G_{OC}(t_0,t_f) \end{aligned} \tag{5.4}$$

where the output controllability grammian is [Ske., 1]

$$G_{OC}(t_0,t_f) = \int_{t_0}^{t_f} C(t_f)\Phi(t_f,\sigma)B(\sigma)B^T(\sigma)\Phi^T(t_f,\sigma)C^T(t_f)d\sigma \tag{5.5}$$

where δ is the Dirac delta function, and the following theorem is of relevance.

Theorem 5.2: [Ske., 1] The system described by (5.1) is output controllable at t_0 if equation (5.4) has the property $Y(t_0,\ t_f) > 0$ for some $t_f > t_0$.

□

However, for strictly proper linear time invariant systems, which are only considered in this thesis, the matrices are constants and the output controllability grammian of the system becomes [Ske., 1]

$$G_{OC}(t_0,t_f) = C\left[\int_{t_0}^{t_f} e^{A(t_f-\epsilon)}BBe^{A^T(t_f-\epsilon)}d\in\right]C^T = C\left[\int_0^{t_f-t_0} e^{A\sigma}BB^Te^{A^T\sigma}d\sigma\right]C^T \tag{5.6}$$

and the control $\underline{u}(\sigma)$ is given as [Ske., 1]

$$\begin{aligned} \underline{u}(\sigma) &= B^T \Phi^T(t_f,\sigma) C^T \left[\int_{t_0}^{t_f} C\Phi(t_f,\sigma) BB^T \Phi^T(t_f,\sigma) C^T d\sigma \right]^{-1} \underline{y}(t_f) \\ &= B^T e^{A^T(t_f-\sigma)} C^T \left[C\left(\int_0^{t_f-t_0} e^{A\sigma} BB^T e^{A^T\sigma} d\sigma \right) C^T \right] \underline{y}(t_f) \\ &= G^T(\sigma) G_{OC}^{-1}(t_f,t_0) \underline{y}(t_f) \end{aligned} \tag{5.7}$$

The minimum energy of the control signal which drives the system from $\underline{y}(t_0)$ at $t = t_0$ to $\underline{y}(t_f)$ at $t = t_f$ is given by the following [Ske., 1]

$$\begin{aligned} E &= \int_{t_0}^{t_f} \underline{u}^T(\sigma) \underline{u}(\sigma) d\sigma \\ &= \int_{t_0}^{t_f} \left\{ \underline{y}^T(t_f) \left[\int_{t_0}^{t_f} G(\in) G^T(\in) d\in \right]^{-1} G(\sigma) G^T(\sigma) \left[\int_{t_0}^{t_f} G(\in) G^T(\in) d\in \right]^{-1} \underline{y}(t_f) \right\} d\sigma \\ &= \underline{y}^T(t_f) G_{OC}^{-1}(t_f,t_0) \underline{y}(t_f) \end{aligned}$$

(5.8)

Because the system is output controllable, the output controllability grammian, $G_{OC}(t_f,t_0)$, is a symmetric positive definite real matrix. For such a matrix, the following theorem states:

Theorem 5.3: [Gan.,1] Given a real, symmetric and positive definite matrix G, there always exists a set of orthonormal eigenvectors $\underline{u}_1$, $\underline{u}_2$, …, $\underline{u}_n$ with corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Setting $Q = [\underline{u}_1, \ \underline{u}_2, \ \ldots, \ \underline{u}_n]$, then

$$Q^T G Q = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \ or \ G = Q diag(\lambda_1, \lambda_2, \ldots, \lambda_n) Q^T \tag{5.9}$$

where the transformation matrix Q satisfies the following

$$QQ^T = I$$
$$Q^{-1} = Q^T$$
$$|Q| = \pm 1$$

□

Furthermore, the quadratic form defined by the matrix A has the following property:

Theorem 5.4: [Gan.,1] If a matrix A is a real symmetric and positive definite, then the quadratic defined by the matrix satisfies

$$\underline{x}^T A \underline{x} \le \lambda_1 \underline{x}^T \underline{x} \tag{5.10}$$

where λ_1 is the largest of the eigenvalues.

□

Assuming that the output controllability grammian of the system has as a set of singular values $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n$, then there exists an orthonormal transformation U such that

$$G_{OC}(t_f,\ t_0) = U * \mathrm{diag}\{\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n\} U^T \tag{5.11}$$

and the inverse of which is

$$G_{OC}^{-1}(t_f,\ t_0) = U * \mathrm{diag}\{\sigma_1^{-1},\ \sigma_2^{-1},\ \ldots,\ \sigma_n^{-1}\} U^T \tag{5.12}$$

so equation (5.8) satisfies

$$E = \int_{t_0}^{t_f} \underline{u}^T(\sigma)\underline{u}(\sigma)d\sigma \le (1/\sigma_n)\underline{y}^T(t_f)\underline{y}(t_f) \tag{5.13}$$

So it can be concluded that the singular values of the output controllability grammian of a system are very important measures. When the smallest singular value is large, then the energy consumed in transferring the outputs from $\underline{y}(t_0)$ to

$\underline{y}(t_f)$ will be small. It is shown that the shorter the time available for control action, then the greater the energy required to steer the output from $\underline{y}(t_0)$ to $\underline{y}(t_f)$, [Sei.,1]. This can also be demonstrated by looking at the singular values of the finite time output controllability grammian. Indeed, it will be shown later in the MATLAB examples that the shorter the time available (i.e. for a small t_f), the smaller the singular values of the output controllability grammian. Summarising, the following proposition can be made.

Remark 5.1: The singular values of the output controllability Grammian of a linear time invariant system are important indicators for the energy needed to transform one output to another. In particular, the minimum energy required to transfer from one output to the other is reciprocal to the minimal singular value.

□

5.2.3 CALCULATION OF THE OUTPUT CONTROLLABILITY MATRIX IN THE TIME INVARIANT CASE

The calculation of the output controllability grammian involves at least $m \times m$ integrations, and as defined it is the integral of an $m \times m$ matrix. This may computationally be very expensive when the system is of large dimensions. However in the time invariant case, the grammian

$$G_{oc}(0,t) = C\left(\int_0^t e^{A\sigma} BB' e^{A'\sigma} d\sigma\right)C' \tag{5.14}$$

can be found in closed form in terms of the solution to a single Lyapunov equation. In fact if Z is of the form

$$\begin{aligned} Z &= CYC^T \\ Y &= X - e^{At} X e^{A^T t} \\ AX + XA^T &= -BB^T \end{aligned} \tag{5.15}$$

then

$$\begin{aligned}\frac{dY}{dt} &= -Ae^{At}Xe^{A^Tt} - e^{At}Xe^{A^Tt}A^T \\ &= e^{At}\left(-AX - XA^T\right)e^{A^Tt} \\ &= e^{At}BB^Te^{A^Tt}\end{aligned} \tag{5.16}$$

therefore

$$Y(t) = \int_0^t e^{At}BB^Te^{A^Tt}dt \tag{5.17}$$

which proves that the Z matrix defined above is the same as the output controllability grammian $G_{oc}(0,t)$. This suggests the following computation scheme, which only involves the solution of a Lyapunov equation and of a matrix exponential:

a) Solve the Lyapunov equation: $AX + XA^T = -BB^T$

b) Substitute X into $G_{oc}(0,t) = C\left(X - e^{At}Xe^{A^Tt}\right)C^T$

Furthermore, the above closed form solution has the following implications:

a) If A is stable, then $\lim_{t\to\infty} G_{oc}(0,t) = CXC^T$

b) $\left.\dfrac{dG_{oc}(0,t)}{dt}\right|_{t=0} = CBB^TC^T$

Therefore the trajectories of the graphs produced in the next section of the singular values of $G_{oc}(0,t)$ as t varies are determined by two factors. The first of these is the singular values of CBB^TC^T and the second is the singular values of CXC^T. Rough estimates of the graphs of the singular values $\sigma_i(t)$ of the grammian are given by

$$\sigma_i(t) = x_i\left(1 - e^{-\frac{b_i}{x_i}t}\right) \tag{5.18}$$

[Ske., 1] where x_i, b_i are the i-th singular values of CXC^T and CBB^TC^T respectively.

5.2.4 COMPUTATIONAL PROCEDURE

Therefore to summarise the computations of the previous section:

1. The output controllability grammian $G_{oc}(t_0, t_f)$ is the operator involved in the relationship between the input $\underline{u}(t)$ and the necessity of it to transfer $\underline{y}(t_0)$ to $\underline{y}(t_f)$ and $\underline{y}^0(t_f)$. This is described by the following equality

$$\underline{u}(\sigma) = G^T(\sigma) G_{oc}(t_0, t_f)^{-1} \underline{y}(t_f) \tag{5.19}$$

2. The relation between the energy of $\underline{u}(t)$ and $\underline{y}^0(t_f)$ can be described by

$$\|\underline{u}(\sigma)\|^2_{[0,t_f]} = \underline{y}(t_f)^T G_{oc}(t_0, t_f)^{-1} \underline{y}(t_f) \tag{5.20}$$

This suggests that the singular values of the grammian are important indicators for the energy needed to transfer the output from one state to another.

3. The energy of the input $\underline{u}(t)$ that transfers $\underline{y}(t_0)$ to $\underline{y}(t_f)$ is bounded according to the following inequality

$$\|\underline{u}\|^2 \le \frac{1}{\sigma_n} \|\underline{y}(t_f)\|^2 \tag{5.21}$$

where σ_n is the smallest singular value of $G_{oc}(t_0, t_f)$.

4. The output controllability grammian $G_{oc}(0, t)$ can be calculated as follows

a) Solve the Lyapunov equation: $AX + XA^T = -BB^T$

b) Substitute X into $G_{oc}(0,t) = C\left(X - e^{At} X e^{A^T t}\right)C^T$

The above steps form a basis from which a suitable structure of MATLAB code can be created in order to convert this theory onto a practical arena. This will be dealt with in the following section. The Lyapunov approach allows a relaxation in the computational burden, and this will clearly be demonstrated once the problem of creating the code is tackled.

5.3 MATLAB EXAMPLES

5.3.1 COMPUTATIONAL DESCRIPTION

As discussed in Section 5.2, the singular values of the output controllability grammian can be used as indicators of the energy used to transfer the output of a system from one state at time t_0 to another state at time t_f. Section 5.2.5 explains the link between the solution to a single Lyapunov equation and the computation of the grammian (from conditions (5.14) to (5.17)). The code written to undertake the computation of the singular values of the output controllability grammian thus approaches the problem from a Lyapunov viewpoint. Addressing the issue using calculus techniques would involve additional programming and increase computational time. The Lyapunov method is a computationally quick and simple way of achieving the objective.

The file *outcon.m* was written to calculate the output controllability grammian of a system, and can be seen in the Appendix. The program first asks the user to define the system matrices A, B, and C and for the sample time to be specified. The `lyap` command is then used to calculate the Lyapunov solution for the equation $AX + XA^T = -BB^T$. The first `for` loop creates two arrays. The first consists of the singular values of the output controllability grammian in the form of $G_{oc}(0,t) = C\left(X - e^{At} X e^{A^T t}\right)C^T$ over the interval from 0 to a specified time t. The second array is of the corresponding set of condition numbers (the ratio of the largest singular value to the smallest). The remainder of the routine enables the singular values and condition numbers to be plotted with respect to time t on two separate plots.

If a routine were to be written in order to solve and plot the same problem described above using a straight integration approach, it would have to be rather lengthy in order to accommodate the integration steps. This would also increase the computation time and greater manipulation of the system matrices would be required. This is why it was decided to go with the Lyapunov method. There now follows some examples to demonstrate the *outcon.m* code.

5.3.2 EXAMPLES

The following linear time invariant system is described by the state-space model as:

$$\underline{\dot{x}} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}$$

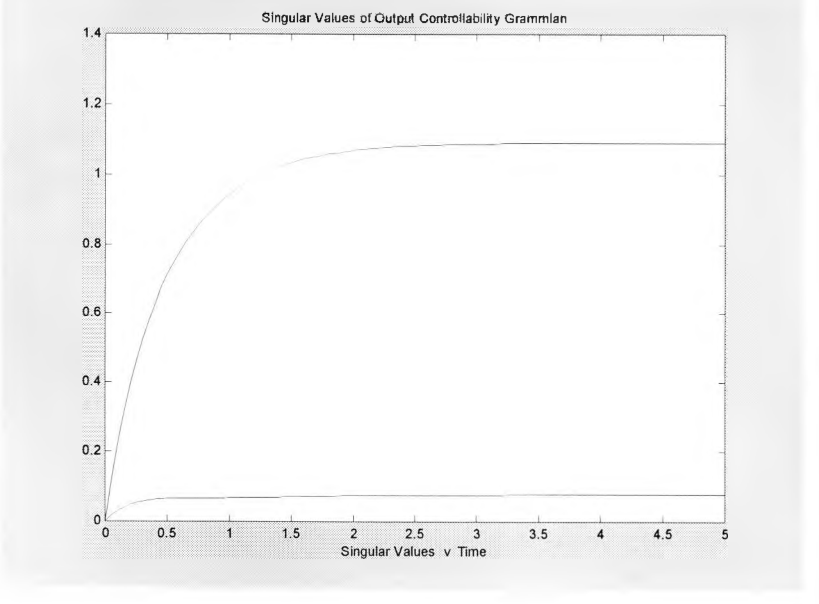


***Figure 5.1** Corresponding singular value plot*

the system is output controllable, because the rank of the matrix [CB CAB CA^2B] is two, which is the same as the number of inputs, i.e. the system has full rank. The singular values of the output controllability grammian are plotted against t in Figure (5.1) above. The singular values are functions of the final time t_f. When t_f is small, the smallest singular value is also small. So by using equation (5.13), the energy needed to transfer the output from one state to another is large.

The corresponding condition number plot is shown in Figure 5.2.

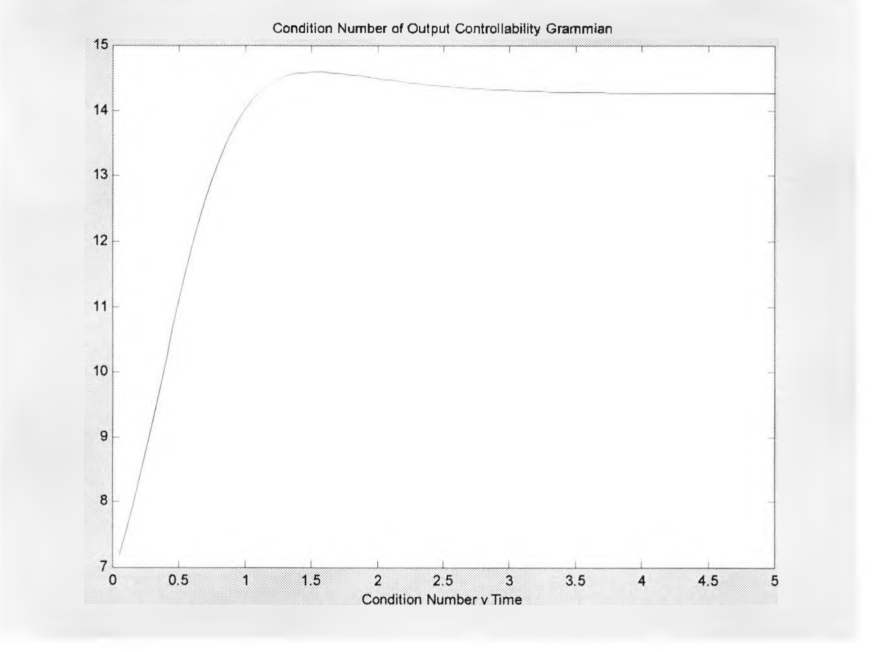


Figure 5.2 *Corresponding condition number plot*

Now consider the same system but with a slightly different input matrix, B, to the previous example.

$$\underline{\dot{x}} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0.5 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}$$

The singular values of the output controllability of the above system are shown in Figure 5.3. This system is also output controllable. However, the control will not be as easy as in the previous system because more energy is required to transfer the same initial state. This is deduced by examining the smallest singular value of the output controllability Grammian. It is smaller compared with that of the first system.

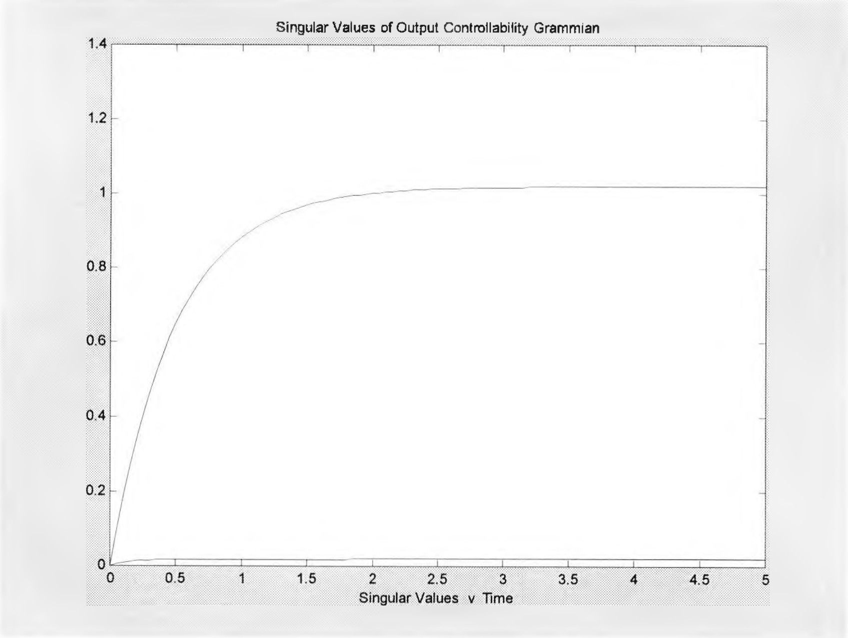


Figure 5.3 *Singular value plot with new B matrix*

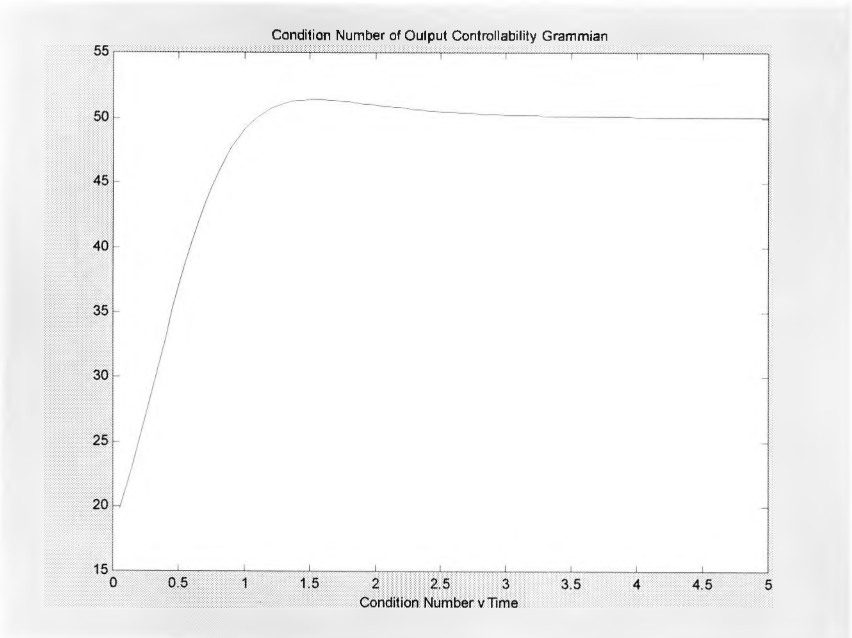


Figure 5.4 *Condition number plot with new B matrix*

Figures 5.2 and 5.4 are plots of the corresponding condition numbers of the two systems. It has already been demonstrated that because the second system has the smallest minimum singular value, then the energy expended to transfer the state from one point to the other greater than for the first system. The condition number can also provide a useful indication. For the first system, the plot of Figure 5.2 shows a final condition number of about 14.3. Yet for the second system, the final condition number is about 50. This shows that the output controllability grammian of the first system is better conditioned than that of the second.

5.4 SELECTION OF CONTROLLER PLACEMENT BASED ON ENERGY CONSIDERATIONS

Consider a large scale linear time invariant system $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ is a state vector and A is a matrix of large dimension. In order to improve the dynamical behaviour of this system, it will be necessary to apply some form of linear state feedback control scheme. But this will in turn raise some important issues, such as [Chi., et al, 1]

- How many controllers are needed
- Where to place these controllers
- How to design these controllers

Such criteria may be subject to certain desired objective functions and constraints that need to be minimised. One of the problems considered in this thesis is how to find a feedback gain matrix K such that the closed loop system has a desired eigenstructure, and will be dealt with in the following chapters.

The criteria sited above are part of the *controller placement problem.* Since the choices of controller locations for a large scale system are enormous, this problem is by no means a trivial one. In recognition of the difficulty in attempting to solve this problem analytically, a couple of methodologies will be cited. From Chang and Soong [Cha., & Soo., 1], the optimal locations are chosen combinatorially such that an energy function is minimised for modal control. This scheme is numerically impractical for a system with large number of controller locations to choose from. In an approach from Chiang et al., [Chi., et al., 1], the controller placement problem is formulated as an optimisation problem with a single objective function. However, in the design of control systems, a designer is usually confronted with multiple design objectives and these design objectives are, in general, in conflict with each other. There is no existing design method which is optimal with respect to all the specified design objectives. Several researchers [Khar., & Rot., 1], [Kri., & Stei., 1], [Kyr, & Buch., 1] have formulated the design of control systems as a multi-objective optimisation problem, yet they are computationally demanding.

Later, it will be shown that through the application of linear feedback (either state or output), the dynamical behaviour of a system will change. So the ensuing problem is how to find an appropriate feedback gain matrix such that the resulting closed loop system has the desired eigenstructure, subject to the following objective functions

- Minimisation of (or a bound on) control effort
- Minimisation of sensitivity relative to system perturbation caused by the closed loop eigenframe

The problem of eigenstructure assignment will be considered and reviewed in subsequent chapters, for both the state feedback and output feedback cases. Eigenstructure assignment in a linear multivariable system is of vital importance in control theory and applications. The specified effect of the controller is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed loop system. In general terms the speed of the response is determined by the eigenvalues, whereas the shape of the response is defined by the assigned eigenvectors.

Having a computable measure of how well a large scale system can be controlled (observed) with any given set of actuators (sensors), with the expected effect of component degradation or failures during its operational lifespan reflected in the measure, it becomes possible to optimise the choice of component locations so as to maximise the performance measure (or minimise detrimental effects). This task may be computationally burdensome, yet conceptually it is quite straightforward. A constraint that applies in most applications is that component placement will be restricted to a discrete set of permissible locations. Structural considerations may require that certain components may not be placed in sensitive, unshielded areas, which in turn increases the limitations of the placement algorithm. Having the optimum set of component locations and the corresponding maximum degree of controllability (observability) for a given number of components, it is possible to compute the maximum performance measure for several choices of component numbers. The choice of how many actuators and sensors to use in the system cannot be resolved as an optimisation problem unless additional factors are incorporated in the criterion. The degree of controllability or observability will always improve with additional components if the best locations are used in each case. However, it would be beneficial to observe the trend of the performance measure with the number of components. Some locations are more advantageous than others. With a realistic restriction that only one component can be placed at any one of the allowable locations, it is expected to see diminishing returns in performance with increasing number as the more favourable locations are occupied. This information should be helpful to the designer in making the trade-off between improved performance and increased cost, power required, energy consumption, etc.

5.5 SUMMARY

This chapter concludes the part of the thesis that deals with controllability and observability issues. In this chapter, the theory of grammians was developed and extended in order to provide a link between the singular values of the output controllability grammians and the energy consumed in changing the output of a system from one position to another. This is essentially a measure of the energy needed for control action, thus output controllability is a *quantitative* measure. The quantitative measure can be further developed to be an interaction measure between the inputs and outputs, but that would take this research down a different path. The solution to a Lyapunov equation was used to solve the output controllability grammian at regular time intervals. The resulting singular values and condition numbers could then be plotted and systems with different input parameters could be compared in order to aid in the selection of input signals where the minimum expendable energy is a criteria of the control problem. The method described in this chapter has potential for use in applications where the conservation of energy is of paramount importance. Such applications are space stations where the rationing of energy is always a prioritised concern.

The following chapters of this thesis will concentrate on the role that eigenvalues and eigenvectors play in shaping system responses and as indicators of system performance.

6 EIGENSTRUCTURE ASSIGNMENT: BASIC CONCEPTS AND BACKGROUND RESULTS

6.1 INTRODUCTION

There has been a substantial amount of work performed in the field of control theory over the past three decades that examines the control of systems through the restructuring of the eigenvalues and eigenvectors, namely eigenstructure assignment. More recently, these techniques have been successfully applied to the control of flexible structures, especially in the area of enhancing modern flight control systems where existing systems are often hampered by the limitations exhibited by the classical control methods. The eigenstructure assignment problem therefore has a very important role to play in order to guarantee successful controller design in the sense of stability and robustness. It must be stated however, that eigenstructure assignment can only be carried out if the system is described by state space equations, which are made up from physical variables. In this case, it makes sense to impose conditions on the eigenframe which is linked to variables with a physical significance.

Firstly, the countenance of this thesis exhibits issues concerning controllability and observability, so the link between these two qualitative properties and eigenvectors has to be established. If the left eigenvector $\underline{v}_i'$ is in the left null space of the input matrix B, i.e. if $\underline{v}_i' B = 0$, then the corresponding mode λ_i is deemed to be uncontrollable. Likewise, if the right eigenvector $\underline{u}_i'$ is contained in the right null space of C, $C\underline{u}_i' = 0$, then the mode λ_i that it is associated with is said to be unobservable. Therefore, if it is desired to reassign open loop eigenvalues, in order to ensure controllability and observability of the respective modes, the designer has to take into consideration the above criteria, in the sense of how assignment affects the "degrees" of controllability/observability which thus have to be defined in an appropriate way.

Another issue of importance is that of robustness. A desired effect of a closed loop system is that its response is impervious to modelling errors and external disturbances. Close attention has to be paid to sensitivity minimisation and control system robustness. Therefore it is necessary to devise an algorithm that reduces the sensitivity of the closed loop eigenvalues to such undesired features.

In view of the problems of stability, robustness, controllability and observability that arise in an open loop configuration, it may be necessary to reassign, or shift, certain modes and reshape the eigenframe of a system by implementing some kind of feedback, so as to improve the dynamical response and properties of the system. This chapter will start off by examining the background on eigenvalues and eigenvectors, especially the relationship with rectilinear motions. The theoretical analysis will then go on to examine the notion of transmission subspaces, and the association of closed loop eigenvalues with feedback. Finally there will be a review of the results in the literature concerned with methods of assigning the eigenstructure of a system.

6.2 BACKGROUND ON EIGENVALUES AND EIGENVECTORS

6.2.1 RECTILINEAR MOTIONS

To begin with, it will be necessary to examine the theory related to rectilinear motions in the state space for free motions, which is primarily concerned with the internal workings of a linear system. Subspaces of the state space that are of a one dimensional nature which have the property of retaining any free motion for every $t \geq 0$ are in fact the eigenvectors of the dynamic map A. The corresponding motions are of the exponential type $e^{\lambda t}\underline{x}(0)$, where λ is the eigenvalue related to the corresponding eigenvector. Such motions are called *rectilinear*. The ensuing problem is thus to restrict the free motion in a one-dimensional subspace with a view to finding the pairs of a vector and a frequency satisfying the eigenvalue-eigenvector relationship already described in Chapter 2.

The problem of keeping the state trajectory of a linear system within a given subspace of the state space is of great importance in a number of control problems. This section will concentrate on the restriction of the free motion in a given subspace, and will begin with by stating the following theorem.

Theorem 6.1: [Won., 1], [Kar. & Kou., 1] Let $S(A, B, C, D)$ be a linear system and $\mathcal{V}$ an r-dimensional subspace of the state space $\mathcal{X}$. A necessary and sufficient condition for the free motion part of the state trajectory $\underline{x}(t)$ to be kept within $\mathcal{V}$ $\forall t \geq 0$ whenever the state is released from any initial condition $\underline{x}(0) \in \mathcal{V}$ is

(i) For every trajectory $\underline{x}(t) \in \mathcal{V}$ there exists another trajectory $\underline{x}'(t) \in \mathcal{V}$ such that

$$A\underline{x}(t) = \underline{x}'(t) \quad \forall t \geq 0 \tag{6.1a}$$

(ii)

$$A\mathcal{V} \in \mathcal{V} \tag{6.1b}$$

The subspace $\mathcal{V}$ satisfying the above conditions is called an A-invariant subspace.

□

The above theorem provides links with the fundamental notion of rectilinear motions [Kar., 5]. In fact, any point $\underline{x}_1 \in \mathcal{V}$ may be considered as lying on some $\underline{x}(t) \in \mathcal{V}$ for some $t = t_0$. Therefore condition (6.1) implies that $\forall \underline{x}_1 \in \mathcal{V}$ there exists $\underline{x}'_1 \in \mathcal{V}$ such that

$$A\underline{x}_1 = \underline{x}'_1, \ \forall \underline{x}_1 \in \mathcal{V}, \ \underline{x}'_1 \in \mathcal{V}$$

or that

$$A\mathcal{V} \subset \mathcal{V} \tag{6.2}$$

For one-dimensional subspaces $\mathcal{V}$, the Laplace Transform of the state trajectory can be expressed by $\underline{x}(s) = \phi(s)\underline{x}_0$. If this is substituted into (6.1a), then

$$s\phi(s)\underline{x}_0 = \phi(s)A\underline{x}_0 + \underline{x}_0$$

which can be rewritten as

$$\frac{-1 + s\phi(s)}{\phi(s)}\underline{x}_0 = A\underline{x}_0 \tag{6.3}$$

It is desired for the left hand side of (6.3) to be independent of s, therefore the following representation is necessary

$$\frac{-1 + s\phi(s)}{\phi(s)} = s_0 \Rightarrow \phi(s) = \frac{1}{s - s_0} \tag{6.4}$$

Condition (6.3) implies that

$$A\underline{x}_0 = s_0\underline{x}_0 \tag{6.5}$$

where $\underline{x}_0$ is an eigenvector of A and s_0 is the eigenvalue corresponding to $\underline{x}_0$. The free motion of the system starting from $\underline{x}_0$ is thus

$$\underline{x}(t) = 1(t)\underline{x}_0 e^{s_0 t} \tag{6.6}$$

which is referred to as a *simple rectilinear motion*, or a motion along an eigenvector. From the above, the following result can be deduced:

Remark 6.1: One-dimensional A-invariant subspaces of $\mathcal{X}$ are simply the eigenspaces of the state matrix A. Each eigenspace is characterised by a unique frequency s_0 which is the corresponding eigenvalue.

□

For the case where the subspaces are of higher dimensions, the above results have to be slightly modified. Let $\mathcal{V}$ be an r-dimensional A-invariant subspace and $\{\underline{v}_i\}$, $i = 1, \ldots, r$ be a basis for $\mathcal{V}$. It is possible using condition (6.1b) to find certain vectors $\underline{w}_i \in \mathcal{V}$, $\{\underline{v}_i\}$, $i = 1, \ldots, r$ such that

$$\begin{aligned} &A\underline{v}_i = \underline{w}_i, \quad i = 1, \ldots, r \\ &\Rightarrow AV = W \end{aligned} \tag{6.7}$$

where $V = [\underline{v}_1 \vdots \ \ldots \ \vdots \underline{v}_r]$, $W = [\underline{w}_1 \vdots \ \ldots \ \vdots \underline{w}_r]$. Because $\{\underline{v}_i\}$ is a basis for $\mathcal{V}$, it is possible to say that $W = V\overline{A}$, or

$$AV = V\overline{A} \tag{6.8}$$

where $\overline{A}$ is a restriction matrix of the $r \times r$ matrix A having $Q\Lambda Q^{-1}$ as a characteristic decomposition. If a new basis is defined by the transformation $U = VQ$, then

$$AU = U\Lambda \tag{6.9}$$

The matrix Λ may have a simple or nonsimple structure including Jordan blocks. Therefore, if $\{\underline{u}_i\}$ is defined as the characteristic basis of $\mathcal{V}$, then the following conditions hold

$$A\underline{u}_i = \lambda_i \underline{u}_i \quad i = 1, \ldots, r \tag{6.10}$$

or if $\overline{A}$ has one Jordan block

$$\begin{aligned} &A\underline{u}_j = \lambda_j \underline{u}_j \quad j = 1, \ldots, \mu \\ &A\underline{u}_i = \lambda_\mu \underline{u}_i + \underline{u}_{i-1} \quad i = \mu + 1, \ldots, r \end{aligned} \tag{6.11}$$

The basis $\{\underline{u}_i\}$ is unique (unless there are repeated eigenvalues λ_i), and is spanned by eigenvectors and generalised eigenvectors of the matrix A. The subspaces corresponding to the Jordan blocks are called Jordan eigenspaces. The above may be summarised by the following result:

Remark 6.2: [Kar., 5] If $\mathcal{V}$ is an r-dimensional A-invariant subspace, there exists a uniquely defined decomposition of $\mathcal{V}$ into a direct sum of Jordan eigenspaces, and to each of the eigenspaces there corresponds a uniquely defined frequency.

□

The set of frequencies $\{\lambda_i\}$, taking into account their multiplicity, as this is expressed by the dimensions of the Jordan blocks, is called the spectrum of $\mathcal{V}$. If the characteristic basis of $\mathcal{V}$ is denoted by $\{\underline{x}_0^1, \ \ldots, \ \underline{x}_0^r\}$, and if it is assumed that A is of a simple structure, then the transform of the state trajectory $\underline{x}(s) \in \mathcal{V}$ can be expressed as

$$\underline{x}(s) = \sum_{i=1}^{r} \phi_i(s)\underline{x}_0^i \tag{6.12}$$

For an initial condition

$$\underline{x}_0 = \sum_{i=1}^{r} a_i \underline{x}_0^i \tag{6.13}$$

condition (6.1b) yields

$$\sum_{i=1}^{r} \{-a_i + s\phi_i(s)\}\underline{x}_0^i = \sum_{i=1}^{r} \phi_i(s) A \underline{x}_0^i \tag{6.14}$$

It is clear that the concept of rectilinear motions is strongly related to the notion of A-invariance. A-invariance is strongly linked to the study of the problem of restricting the free motion of a system inside a subspace $\mathcal{V}$ for any initial condition $\underline{x}_0 \in \mathcal{V}$. It has been shown that that the concept of a simple rectilinear motion is connected with the motion along a simple eigenvector.

So it can be concluded that A-invariant subspaces are associated with the free motion behaviour of the system. Such subspaces are also linked to the zero input problem whilst the state and output trajectories are rectilinear. This can be illustrated in the following diagram

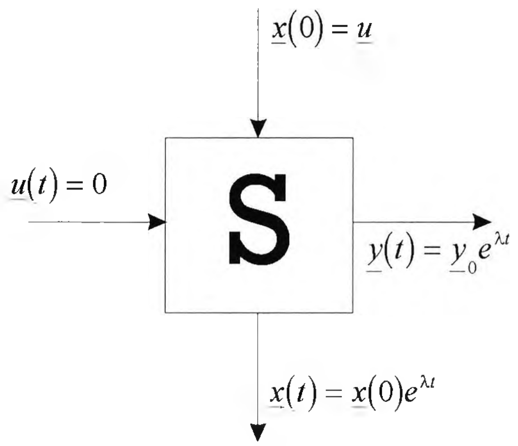


Figure 6.1 *Zero input problem*

Both $\underline{x}(t)$ and $\underline{x}(0)$ exist within the subspace $\mathcal{V}$. But what happens to the frequencies and their associated rectilinear motions when $\underline{u}(t) \neq 0$? This is where A-invariance is extended to (A, B)-invariance, and will be dealt with later in this chapter.

The above notions have shown that the free motion of a system starting from an initial condition is called rectilinear, which is in fact a motion along an eigenvector. The frequency corresponding to this motion is called an eigenvalue. A-invariance is a condition for the free motion part of the trajectory to be kept within the boundaries of a certain subspace when released from an initial point. The definition of A-invariance is given by equation (6.1).

6.2.2 SUMMARY OF SPECTRAL CHARACTERISATION

As a recollection from earlier, an eigenvector $\underline{u}_i$ that corresponds to an eigenvalue λ_i is a nontrivial solution of

$$[\lambda_i I_n - A]\underline{u}_i = 0 \tag{6.18}$$

The *spectral decomposition* of A in the case of distinct eigenvalues is of the form

$$\begin{aligned} A &= U\Lambda V \\ VA &= \Lambda V \end{aligned} \tag{6.19}$$

where U is the matrix of eigenvectors and $V = U^{-1}$ is the matrix of dual eigenvectors and $\Lambda = \mathrm{diag}(\lambda_i)$. If $\mathcal{B}$ and $\mathcal{B}'$ represent the eigenbasis and dual eigenbasis described by $\{\underline{u}_1, \;\ldots,\; \underline{u}_n\}$ and $\{\underline{v}_1, \;\ldots,\; \underline{v}_n\}$ respectively, then

$$\begin{aligned} VU &= \begin{bmatrix} \underline{v}_1^t \\ \vdots \\ \underline{v}_n^t \end{bmatrix} [\underline{u}_1, \;\ldots,\; \underline{u}_n] = I_n \\ &\Rightarrow \underline{v}_i^t \underline{u}_j = \delta_{ij} \end{aligned} \tag{6.20}$$

Consider an $n \times n$ linear multivariable system, described by the following state space model

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned} \tag{6.21}$$

The system transfer function matrix is given by

$$G(s) = C(sI_n - A)^{-1} B \tag{6.22}$$

If U and V satisfy condition (6.19), and A is of simple structure and $\Lambda = \mathrm{diag}(\lambda_i)$, the transfer function matrix can be expressed in the *dyadic* form below

$$G(s) = \sum_{i=1}^{n} \frac{C\underline{u}_i \underline{v}_i' B}{s - \lambda_i} \tag{6.23}$$

As can be seen from equation (6.23), eigenvalues, eigenvectors and dual eigenvectors have an important role to play in the formulation of the system transfer function.

6.2.3 CONTROLLABILITY AND OBSERVABILITY ISSUES

[Gil., 1] The countenance of this thesis exhibits issues concerning controllability and observability, so a link between these two qualitative properties and the eigenstructure of a system has to be established. Take the system described by equation (6.21), where A has distinct eigenvalues, and the modes of interest are λ_i, $\underline{u}_i$ and $\underline{v}_i'$. The complete mode $(\lambda_i,\ \underline{u}_i,\ \underline{v}_i')$ is uncontrollable if

$$\begin{aligned} \underline{\beta}_i' &= \underline{v}_i' B = 0 \\ \underline{v}_i'(\lambda_i I_n - A) &= 0 \end{aligned} \tag{6.24}$$

The mode $(\lambda_i,\ \underline{u}_i,\ \underline{v}_i')$ is unobservable if

$$\begin{aligned} \underline{\gamma}_i &= C\underline{u}_i = 0 \\ (\lambda_i I_n - A)\underline{u}_i &= 0 \end{aligned} \tag{6.25}$$

A mode $(\lambda_i,\ \underline{u}_i,\ \underline{v}_i')$ is said to be:

- Controllable and observable if $\underline{\beta}_i' \neq 0$ and $\underline{\gamma}_i \neq 0$
- Controllable and unobservable if $\underline{\beta}_i' \neq 0$ and $\underline{\gamma}_i' = 0$
- Uncontrollable and observable if $\underline{\beta}_i' = 0$ and $\underline{\gamma}_i \neq 0$

- Uncontrollable and unobservable if $\underline{\beta}_i' = 0$ and $\underline{\gamma}_i' = 0$

The conditions $\underline{\beta}_i' = \underline{v}_i' B = 0$ and $\underline{\gamma}_i = C\underline{u}_i = 0$ provide the basis for a geometric characterisation of uncontrollability and unobservability. In fact condition (6.24) implies that the left eigenvector $\underline{v}_i$ satisfies the geometric condition

$$\underline{v}_i \in \mathcal{N}_l(B) \equiv \mathcal{N} \tag{6.26}$$

Likewise, equation (6.25) implies that the right eigenvector satisfies the geometric condition

$$\underline{u}_j \in \mathcal{N}_r(C) \equiv \mathcal{M} \tag{6.27}$$

The above geometric conditions are expressed as conditions on spaces and thus they may be used to provide measures of the "degree" on controllability and of observability by measuring the proximity of the left eigenvector to the $\mathcal{N}$ space and the proximity of the right eigenvector to the $\mathcal{M}$ space. Although controllability is invariant under state feedback and observability invariant under output injection [Won., 1], [Kai., 1], [Kar., 5], their respective degrees are not. Thus in shaping the closed loop eigenframe by feedback, the degree of controllability and observability due to positioning of the resulting closed loop eigenframes is an important indicator that can be considered as a design parameter.

6.3 FORCED RECTILINEAR MOTIONS AND CLOSED LOOP EIGENSTRUCTURE

6.3.1 PHYSICAL PROBLEM

In Subsection 6.2.1, the problem of rectilinear motions for zero input conditions was examined. An extension of this problem can be stated as follows:

Problem 6.1: [Kar., 5] Given the system $S(A, B, C, D)$ and a subspace of $\mathcal{X}$, $\mathcal{V}$, find under what conditions, for any $\underline{x}_0 \in \mathcal{V}$ there exists a control input which restricts the state trajectory in $\mathcal{V}$, $\forall t \geq 0$.

□

Here, the case when $\underline{u}(t) \neq 0$ will be looked at. So the question that must be posed is if the rectilinear motion problem can be extended to forced systems, i.e. when $\underline{u}(t) \neq 0$. In order to examine this, it is necessary to make use of the input-state pencil [Kar., 5]

$$[sI_n - A, \quad -B]\begin{bmatrix} \hat{\underline{x}} \\ \hat{\underline{u}} \end{bmatrix} = \underline{x}_0 \tag{6.28}$$

From equation (6.28), the input-state pencil is given by $C(s) = [sI_n - A, \quad -B]$, and is used to help describe the coupling between the input and the state. Taking into consideration the initial condition $\underline{x}(0) = \underline{x}_0$, and the system description of equation (6.21), the problem of forced rectilinear motions can be formulated as follows

Problem 6.2: [Kar., 5] Is it possible to find a specific $\underline{x}_0$ and $\underline{u}(t)$ such that $\underline{x}(t) = e^{\lambda_i t}\underline{x}_0$, $\forall t \geq 0$, for some $\lambda_i \in \mathbb{C}$?

□

In order to tackle this problem, it is necessary to look back at Section 6.2, where the study of A-invariant subspaces and rectilinear motions within them was introduced. For the case of forced systems a more general situation arises. Apart from the internal mechanism characterised by the A matrix, and the way it is coupled to the environment via the output map C, the way in which the outside is coupled to the system via the input map B is taken into consideration. Thus the initial concept of A-invariance is now extended to (A, B)-invariance. This can best be explained by considering the system $S(A, B)$ and the r-dimensional subspace of $\mathcal{X}$, $\mathcal{V}$, then a necessary and sufficient condition for the trajectory $\underline{x}(t)$ to remain in $\mathcal{V}$ for all time $t \geq 0$, for an appropriate input vector $\underline{u}(t)$, when released from a general initial condition $\underline{x}(0) = \underline{x}_0$ is

(i) the vector $\underline{x}'(t)$ is defined by

$$\underline{x}'(t) = A\underline{x}(t) + B\underline{u}(t) \quad \forall t \geq 0 \tag{6.29}$$

remains in $\mathbb{V}$, or equivalently

(ii)
$$A\mathbb{V} \subset \mathbb{V} + \mathcal{B} \tag{6.30}$$

where $\mathcal{B}$ is the range space of the matrix B.

For a 1-dimensional subspace $\{\underline{x}_0\}$, the following theorem can be stated:

Theorem 6.2: [Kar., 5] Given the system $S(A,B)$ and the subspace $\{\underline{x}_0\}$ of $\mathcal{X}$, then for a release condition $\underline{x}_0 \in \{\underline{x}_0\}$, the trajectory $\underline{x}(t)$ remains in $\{\underline{x}_0\}$ for some appropriate control vector $\underline{u}(t)$ if and only if $\underline{x}_0$ is an (A, B)-invariant direction given by

$$Ax_0 = s_0 x_0 - Bu_0 \tag{6.31}$$

and the control input $\underline{u}(t)$ is of the rectilinear type defined by the pair $(s_0, \ \underline{u}_0)$ as

$$\underline{u}(t) = 1(t)\exp(s_0 t)\underline{u}_0 \tag{6.32}$$

The ensuing motion in $\mathcal{X}$ will be rectilinear according to the frequency s_0

$$\underline{x}(t) = 1(t)\exp(s_0 t)\underline{x}_0 \tag{6.33}$$

□

The above theorem implies an arbitrary s_0 when the subspace $\{\underline{x}_0\}$ has an intersection with $\mathcal{B}$, i.e. $\{\underline{x}_0\} \cap \mathcal{B} \neq \{0\}$. For the case when $\{\underline{x}_0\} \cap \mathcal{B} = \{0\}$, the 1-dimensional (A, B)-invariant subspaces have a uniquely defined frequency, s_0. The implication of this is that given a specific $\{\underline{x}_0\}$, a pair $(s_0, \underline{u}_0)$ can be found. This is also conversely true. Therefore, unlike the case of A-invariant subspaces, (A, B)-invariant subspaces may not be described in terms of a frequency only, and this is associated with a *generalised* eigenvalue-eigenvector problem.

6.3.2 CHARACTERISATION OF TRANSMISSION SUBSPACE

The difference between the frequency and vector correspondence for the two cases of A and (A, B)-invariance can be summarised in the following way. The spectrum $\sigma[\{\underline{x}_0\}] = s_0$ can uniquely characterise a 1-dimensional A-invariant subspace $\{\underline{x}_0\}$. Each spectral frequency s_0 has a unique characteristic vector $\underline{w}[s_0] = \underline{x}_0$. For ($A$, B)-invariant cases, each subspace $\{\underline{x}_0\}$ (for $\{\underline{x}_0\} \cap \mathscr{B} \neq \{0\}$) is uniquely characterised by a generalised spectral frequency s_0, but unlike A-invariance, there is no unique corresponding characteristic vector $\underline{x}_0$. Any vector $\underline{x}_0$ satisfying

$$\begin{aligned} N(sI - A)x_0 &= 0 \\ NB &= 0 \end{aligned} \tag{6.34}$$

where N is a basis matrix for $\mathcal{N}_l(B)$, is (A, B)-invariant and is uniquely characterised by s_0. However, equation (6.34) has more than one solution for $\underline{x}_0$. In order to be able to distinguish between the correspondence of frequencies and characteristic subspaces for the two cases of A and (A, B)-invariance it is necessary to introduce concepts relating to the frequency transmission through forced systems.

The first concept is the *transmission subspace* of s_0, $\mathcal{T}(s_0)$ [Kar. & Kou., 1], to be the subspace spanned by the totality of the solutions to equation (6.31) in $\underline{x}_0$ for the same frequency s_0. The second concept is that the frequency s_0 corresponding to $\mathcal{T}(s_0)$ is called the *frequency content* of the frequency subspace.

The concept of $\mathcal{T}(s_0)$ is quite an important one. In order for the successful transmission of a particular frequency s_0, the initial condition x_0 and the associated trajectory $\underline{x}(t)$ must remain within $\mathcal{T}(s_0)$. Furthermore, because the transmission subspace is uniquely characterised by a frequency, rectilinear motions sustained in any subspace of $\mathcal{T}(s_0)$ will enable the transmission of the frequency s_0 only. It must be noted that these statements only hold true for (A, B)-invariant subspaces that do not intersect $\mathcal{B}$.

Remark 6.3: [Kar. & Kou., 1] An (A, B)-invariant subspace that intersects with $\mathcal{B}$ has part of its spectrum arbitrarily assignable and contains a controllability subspace.

□

Proposition 6.1: [Kar. & Kou., 1] All transmission subspaces of a system $S(A, B)$, where A and B are of sizes $n \times n$ and $n \times l$ respectively, for which $l > n/2$ have an intersection with $\mathcal{B}$. Otherwise, when $l \leq n/2$, then such an intersection generally does not exist.

□

Before looking at a way to compute the transmission subspace, it is necessary to look at its physical significance with respect to frequency propagation. The transmission of the frequency s_0 only takes place in subspaces of the transmission subspace $\mathcal{T}(s_0)$. Conversely, every subspace of $\mathcal{T}(s_0)$ only allows the transmission of the frequency s_0. This focuses attention solely on the behaviour of the state vector without taking into consideration the type of input vector required to initialise a frequency transmission. This can be justified by looking at equation (6.31), where the existence of a solution for $\underline{x}_0$ immediately implies a solution for $\underline{u}_0$, which is indicated by

$$\underline{u}_0 = B^{+}(s_0 I - A)\underline{x}_0 \tag{6.35}$$

where $(B^{+}B = I_l)$. An interesting exercise would be to identify the particular subspace in the input space $\mathcal{U}$ from which $\mathcal{T}(s_0)$ in $\mathcal{X}$ may be reached. The subspace in $\mathcal{U}$ is defined as the *input transmission subspace*, and is denoted by $\mathcal{T}_u(s_0)$. With this in mind, the following proposition can be made

Proposition 6.2: [Kar., & Kou., 1] $\mathcal{T}_u(s_0)$ coincides with the whole input space $\mathcal{U}$ for all frequencies if $\mathcal{B}$ does not intersect with any $\mathcal{T}(s_0)$. If there is an intersection, then the same applies to all frequencies s_0 again, except for those that belong to the controllable part of the spectrum of A.

□

The proof of this once again lies with equation (6.31), from which all vectors in $\mathcal{T}(s_0)$ for any frequency s_0 which is not an eigenvalue of A are given as

$$\underline{x}_0 = (s_0 I - A)^{-1} B\underline{u}_0 \tag{6.36}$$

From the above condition, any vector $\underline{u}_0$ leads to a vector $\underline{x}_0 \in \mathcal{T}(s_0)$. But if $s_0 = \lambda_i$, $\lambda_i \in \sigma(A)$, equation (6.31) becomes

$$A\underline{x}_0 = \lambda_i \underline{x}_0 - B\underline{u}_0 \tag{6.37}$$

when this is projected onto the eigenframe of A, the following condition arises

$$\underline{v}_i' B\underline{u}_0 = 0 \tag{6.38}$$

where $\underline{v}_i'$ is the left eigenvector of A corresponding to the eigenvalue λ_i. Thus it is still possible to use any vector as so long as $\underline{v}_i' B = 0$, that is $\mathcal{T}_u(\lambda_i) = \mathcal{U}$ if λ_i is an uncontrollable mode. If $\underline{v}_i' B \neq 0$, then $\underline{u}_0$ may not assume values for which $B\underline{u}_0 \in \{\underline{w}_i\}$, where $\underline{w}_i$ is the eigenvector of A that corresponds to λ_i.

Equation (6.36) gives the totality of vector solutions for $\underline{x}_0$, where $\underline{x}_0 \in \mathcal{T}(s_0)$ for any frequency s_0 such that s_0 is not in the spectrum of A, i.e. $s_0 \notin \sigma(A)$. Therefore $\mathcal{T}(s_0)$ can be expressed as

$$\mathcal{T}(s_0) = \text{range}\left\{(s_0 I - A)^{-1} B\right\} \tag{6.39}$$

Remark 6.4: [Kar. & Kou., 1] For the general case, $s_0 \in \mathbb{C}$, where $s_0 \in \sigma(A)$, the transmission subspace is defined as the $\underline{x}_0$ vector solutions of $(s_0 N - NA)\underline{x}_0 = 0$.

□

6.3.3 FEEDBACK AND CLOSED LOOP EIGENVALUES

The transmission of the frequency s_0 is generally affected from any input $\underline{u}_0$ in the input space $\mathcal{U}$. However it may only be propagated along a direction belonging to a given subspace of the output space $\mathcal{Y}$. It is required that such transmissions are only possible if the state vector is restricted to the transmission subspace $\mathcal{T}(s_0)$, and that the ensuing trajectories in the input, state and output spaces are all of the rectilinear type. The rectilinear motions in $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ all share the same frequency s_0. The need for an external excitation in the form of a controlled input $\underline{u}(t)$ could be eliminated by deploying suitable feedback connections from either the states or the outputs back to the inputs. Therefore applying an appropriate state feedback operator K_s, or output feedback operator K_o such that

$$K_s \underline{x}_0 = \underline{u}_0 \tag{6.40}$$

or

$$K_o \underline{y}_0 = \underline{u}_0 \tag{6.41}$$

it is possible to generate the control input $\underline{u}(t)$ needed to sustain the rectilinear motions by closing the loops around the system S in the manner indicated in Figure 6.2.

The top right of the diagram shows a state feedback configuration, and the bottom part shows an output feedback one. The actual physical interpretations of these diagrams do not need an external input, and can be considered as free responding systems. The associated problem can be linked to restricting the state vector (and output vector) of an autonomous system. So now the vector $\underline{x}_0$ which originally was seen as a member of a transmission subspace becomes a member of a closed-loop eigenspace. If equations (6.40) and (6.41) are substituted into (6.31), then

$$(s_0 I - A - BK_s)\underline{x}_0 = 0 \tag{6.42}$$

$$(s_0 - A - BK_o C)\underline{x}_0 = 0 \tag{6.43}$$

are obtained. $(A + BK_s)$ and $(A + BK_o C)$ are the closed-loop state matrices under state and output feedback respectively. These play a huge part in the problem of the placement of closed-loop poles and eigenvectors.

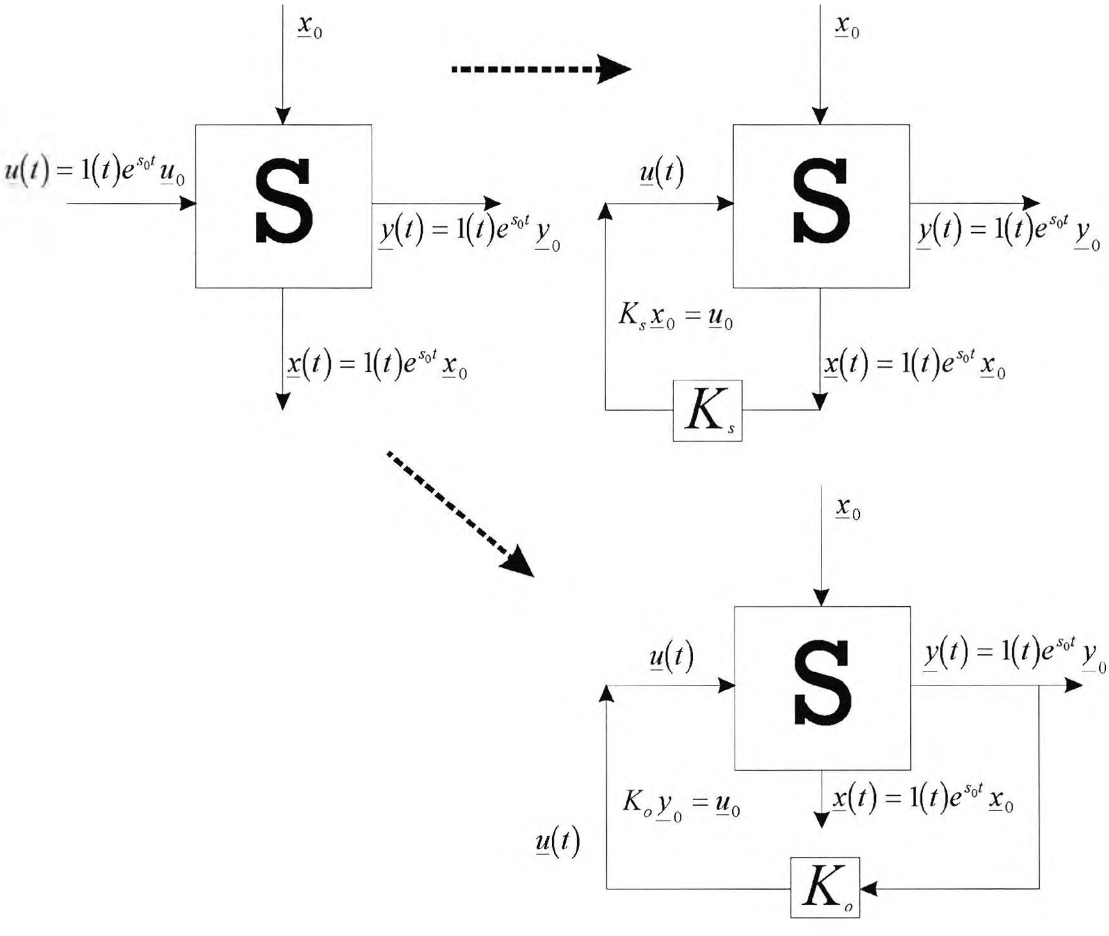


Figure 6.2 *Feedback systems*

6.3.4 THE PROBLEM OF EIGENSPACE ASSIGNMENT

An adequate way of summarising the above subsection would be to say that frequency transmissions along (A, B)-invariant directions could be self generated by the utilisation of an appropriate family of feedback operators connecting the states (or outputs) back to the inputs. Therefore rectilinear motions of the type $\exp(s_0 t)\underline{x}_0$ stimulated by an input $\exp(s_0 t)\underline{u}_0$ could be made to be self perpetuating if the input signal were generated by a combination of the state (or output) variables and the action of a feedback operator. To keep things simple, only state feedback will be considered, such that $K_s \underline{x}_0 = \underline{u}_0$.

It has been documented that any motion in a general r-dimensional (A, B)-invariant subspace may be broken into a number of simple and higher order rectilinear motions, each linked to a specific spectral frequency s_{0_i} that take place along the generalised eigenspace defined by the vectors $\left[\underline{x}_{0_i}^{0},\ \underline{x}_{0_i}^{1},\ \ldots,\ \underline{x}_{0_i}^{\mu}\right]$. Such motions are sustained by inputs that consist of rectilinear type components, of which each are associated with one particular frequency s_{0_i}. These frequencies take place along the input characteristic vectors, defined by $\left[\underline{u}_{0_i}^{0},\ \underline{u}_{0_i}^{1},\ \ldots,\ \underline{u}_{0_i}^{\mu}\right]$. Therefore the state feedback law of (6.40) may be restated in order to satisfy the conditions of r-dimensional (A, B)-invariant subspaces as follows

$$K_s \underline{x}_0^j = \underline{u}_{0_0}^j \tag{6.44}$$

and in matrix form

$$K_s V_0 = U_0 \tag{6.45}$$

The action of the matrix K_s as a feedback operator has already been illustrated in Figure 6.2. The diagram shows how the restriction of the state trajectory $\underline{x}(t) \in \mathcal{V}^*$ can be achieved by a closed-loop system without the necessity of a control input $\underline{u}(t)$. Thus the concept of (A, B)-invariance can be extended to (A + BK_s)-invariance, which leads onto the problem of eigenspace assignment. The equivalence between the two can be investigated by first considering a derivation of equation (6.31)

$$AV_0 = V_0 J_R - BU_0 \tag{6.46}$$

where J_R is the Jordan block diagonal canonical form of $\Lambda_R = \text{diag}\{s_{0_i}\}$. If U_0 from (6.45) is substituted into (6.46), then

$$(A + BK_s)V_0 = V_0 J_R \tag{6.47}$$

which in turn can be expressed in vector space notation by

$$(A + BK_s)\mathbb{V} \subset \mathbb{V} \tag{6.48}$$

The following theorem states under what circumstances the assignment of an eigenspace can be considered:

Theorem 6.3: [Kar., 5] The sufficient and necessary condition for the assignability of a given vector as a closed-loop eigenvector is that it belongs to a transmission subspace, of which the frequency content designates the corresponding closed loop eigenvalue.

□

With this in mind, the general form of the eigenstructure assignment problem can be formulated as follows:

Problem 6.3: Given the system $S(A, B)$, find a set of independent vectors associated with the frequencies $\{\lambda_i\}$, $i = 1, \ldots, r$, find an appropriate feedback operator (either K_s for state feedback or K_o for output feedback) that makes the frequencies λ_i closed-loop eigenvalues, and the corresponding closed loop eigenvectors while at the same time the resulting eigenframe satisfies some given properties.

□

Basically the point of eigenstructure assignment is to shift certain undesirable characteristic frequencies to new locations and to exercise some control over the resulting eigenvectors. The latter, in tandem with the input and output maps B and C respectively, are vital for the problem of well conditioning controllability and observability properties. It is well known that the controllability and observability properties have certain invariance properties under feedback/output injection as stated below.

Theorem 6.4: [Kai., 1] [Kar., 5] Given the system $S(A, B, C, D)$, the following hold true:

(i) The controllability properties are invariant under state feedback.

(ii) The observability properties are invariant under output injection.

□

The above suggests that state feedback cannot make a controllable system uncontrollable, but it can affect the degrees of controllability as demonstrated in Chapter 4, when these are suitably defined. However, state feedback can make the system unobservable, if the system has zeros and a suitable feedback is selected [Kar., 5] [Sha., & Kar., 1]. Similar arguments can be made for the output injection. Thus the general eigenstructure assignment involves a simultaneous selection of a suitable closed loop set of frequencies and a suitable eigenframe that can guarantee some additional properties.

In the next section, a review of some of the literature dealing with some of the methods formulated to tackle the problem of eigenstructure assignment is made.

6.4 REVIEW OF RESULTS ON EIGENSTRUCTURE ASSIGNMENT

6.4.1 EARLY RESULTS

The progression of work done in formulating methods that attempt to solve the eigenstructure assignment problem will now be reviewed. The response of a control system is largely dependent on its eigenvalues and eigenvectors, namely its eigenstructure. The eigenvalue assignment problem was first addressed by Wonham [Won., 2] in 1967. The author proved that a system was controllable if and only if state feedback could be applied and calculated so as to make the newly formed closed loop system have an arbitrary set of self-conjugate scalars as its poles. Since this paper, there have been hundreds of publications dedicated to the subject of pole placement and its applications, which go on to discuss the assignment of eigenvectors as well. A handful have been selected in order to give an insight into some of the methodologies that have been developed for both output and state feedback cases.

The problem of using eigenvectors and assigning them was first considered by Karcanias [Kar., 6], and was used by Shaked and Karcanias [Sha. & Kar., 1] as part of the wider issues of model reduction of linear systems. The aim of their work was to find a state feedback matrix such that the closed loop system had the maximum number of eigenvectors possible in the kernel of the output matrix C. An algorithm was developed whereby the maximum number of newly assigned eigenvectors, which corresponded to stable modes, lay in the kernel of C. This meant that the maximum possible number of stable modes became unobservable. This took advantage of the fact that the observability properties of a system are not invariant under state feedback. At around the same time, Moore [Moore, 1] established the fact that state feedback could be used to assign the closed loop system and desired self conjugate set of eigenvalues, if and only if the open loop system was controllable. The purpose of his paper was to identify the freedom (other than the choice of eigenvalues to be assigned) offered by state feedback. It was shown that the freedom available was a choice of one particular set from the class of "allowable" sets of closed loop eigenvectors. Porter and D'Azzo [Por. & D'Az., 1] presented a set of results for closed loop eigenstructure assignment by state feedback in multivariable linear systems which took advantage of the freedom available due to the pole placement method by Moore [Moore, 1]. The results provided a method for the direct computation of the state feedback matrix which can be used to assign prescribed Jordan canonical forms, eigenvectors and generalised eigenvectors to the plant matrices of closed loop systems. Also it is pointed out that even in the case of systems for which the pair (A, B) is uncontrollable, certain prescribed eigenvectors of the feedback system $(A + BK_s)$

can be assigned by state feedback. In the case of systems with asymptotically stable but uncontrollable modes, they state that it is often possible to achieve significant improvements in the dynamical behaviour of such systems by the introduction of appropriate state feedback controllers. The results from this paper led to a further development by Porter and D'Azzo [Por. & D'Az., 2]. The algorithm presented is based along solving

$$\left[A + BK_s - \lambda_i I\right]\underline{u}_i = 0 \tag{6.49}$$

for K_s by arbitrarily assigning a vector ω_i to find the set of eigenvectors $\underline{u}_i$ which satisfy the relationship $K_S\underline{u}_i = \omega_i$. The nature of the computations is simple due to the case of the elementary column operations involved.

The early results of eigenstructure assignment described here pioneered further investigations into this novel control problem. These early studies opened a new channel in control design that steered away from standard classical techniques (second order PID controllers) to allow more complex feedback controllers to be designed and implemented.

6.4.2 STATE FEEDBACK RESULTS

The poles of a system are also the roots of the characteristic equation that gives rise to the eigenvalues of a system. Therefore the term "pole-shifting" means the same as relocating the eigenvalues of a system to obtain improved behavioural patterns. In view of this, Retallack and MacFarlane [Ret. & MacF., 1] derived a straightforward state feedback pole-shifting algorithm, which relates the open and closed loop characteristic frequencies of multivariable feedback systems to the Bode return difference of the system. The useful algorithm developed provided an interesting link between state-space and transfer function matrix representations in the treatment of pole shifting. Although many algorithms exist for the solution of the pole placement problem using state feedback, it can generally be concluded that most of them are numerically unstable, yet the paper by Minimis and Paige [Min. & Pai., 1] attempted to prove that their algorithm was numerically stable. They suggested a direct algorithm for the computation of the linear state feedback matrix for multi-input systems such that the resultant closed-loop system matrix has specified eigenvalues. This method has the added advantage of an extra degree of freedom which can be used in different ways, for example to decrease some norm of the feedback matrix and hence the control effort or to improve the condition of some eigenvalues of the closed loop matrix. The algorithm devised uses unitary transformations for numerical reliability, and its stability results from the use of explicit shifting for the allocation of each eigenvalue. Another numerically stable and efficient computational algorithm for pole assignment of linear multi-input systems was proposed by Petkov et al [Pet. et al., 1]. The preliminary stage of the algorithm involves the reduction of the state matrices into an orthogonal transformation of the closed loop system matrix into an upper quasi-triangular form whose diagonal blocks correspond to the desired poles. The computed gain matrix, due to its numerical stability, is also exact for a system with slightly perturbed matrices. It works equally well with real and complex, distinct and multiple poles and is also applicable to ill-conditioned and high order problems.

The problem with using state feedback is that the states of a system are not always readily available. This creates the problem of the inability of the designer to shift all the states of a system. This is where output feedback has an advantage, where the states can be fed back as functions of the output.

6.4.3 OUTPUT FEEDBACK RESULTS

In 1978, Porter and Bradshaw [Por. & Brad., 1] derived a method for entire eigenstructure assignment which was applicable to the design of multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers. The method was illustrated by designing an error-actuated dynamic controller which caused the output of a second order continuous time plant to track a constant command input in the presence of an unmeasurable constant disturbance input. The feedback matrix K_o is solved using the eigenvalue-eigenvector relationship

$$\left[A + BK_oC - \lambda_i I\right]\underline{u}_i = 0 \qquad (6.50)$$

where A, B and C are the state, input and output matrices respectively. λ_i represents the eigenvalues to be assigned, and $\underline{u}_i$ is the corresponding eigenvector set of the new system. A new approach was developed by Alexandridis and Parakevopoulos [Ale. & Par., 1], which identifies the eigenspaces for the desired set of all the closed loop eigenvalues. In order for the desired set of eigenvalues to be successfully assigned, necessary and sufficient conditions are established and met. The proposed approach is based on the idea of breaking down the problem of the output feedback pole assignment in the following two steps. In the first step, an expression for K_o is derived which relates the output feedback gain matrix to the eigenstructure assignment for the set Λ_1 of the closed loop eigenvectors. In the second step, the remaining closed loop eigenvectors are assigned to be in the set Λ_2 without affecting the assignment of the first set of Λ_1 eigenvalues. The problem of determining the free parameters in K_o either to a bilinear system of real algebraic equations in the general case or to a linear system is achieved by algebraic manipulations. Sobel et al [Sob. et al., 1] also presented a comprehensive use of eigenstructure assignment design methodology using output feedback. The implementation of their technique is applicable to the design of advanced flight control systems. Their method enables the designer to satisfy damping, settling time and mode decoupling specifications by directly choosing the eigenvalues and eigenvectors. They also tackle the problem of eigenvalue sensitivity, which arises due to the incremental change in the eigenvalues as a result of incremental changes in the stability of the aircraft and control derivatives. Duan [Duan, 1] proposed a simple and effective algorithm for robust pole assignment in multivariable linear systems via output feedback. The presented method gives a robust solution in the sense that the closed loop eigenvalues are as insensitive as possible to perturbations in the system coefficient matrices. The solution to the problem involves three steps, the first of which is aimed at trying to find a proper eigenvalue sensitivity index. The second step involves stating the freedom of the control system and in the final step, the freedom of the system is optimised by minimising the proposed eigenvalue sensitivity index. The eigenvalue sensitivity index can be described appropriately by the condition number of the eigenvector matrix of the closed loop system. The algorithm conveniently includes closed loop eigenvalues as optimising parameters and it also possesses stable numerical properties, as well as being fairly simple to implement. Kabamba and Longman [Kab. & Long., 1] produced a note addressing the problem of the assignability of the eigenvalues of the matrix $A + BK_oC$ by the choice of the feedback matrix K_o. This mathematical problem corresponds to pole assignment in the direct output feedback problem, and by proper changes of variables it also represents the pole assignment problem with dynamic feedback controllers. The key to the solution presented by the authors is the introduction of the concept of local assignability which in loose terms is the arbitrary perturbability of the eigenvalues of $A + BK_oC$ by the perturbations of K_o. If n is the order of the system, they show that if $A + BK_oC$ has distinct eigenvalues, a necessary and sufficient condition for local complete assignability at K_o is that the matrices $C[A + BK_oC]^{i-1}$ be linearly independent for $1 \leq i \leq n$. In special cases, this condition can be reduced to known criteria for controllability and observability. Although such properties are necessary conditions for assignability, the paper also addresses the question of assignability of uncontrollable and unobservable systems, both by direct output feedback and dynamic compensation. Fletcher et al [Flet. et al., 1] presented a set of necessary and sufficient conditions for closed loop eigenvector assignment by output feedback in time invariant linear multivariable control systems. The basis of the paper is a simple condition on a square matrix, which is necessary and sufficiently adequate for it to be the closed loop plant matrix of a given system. It is employed to obtain a condition concerning the assignment of an eigenstructure consisting of the eigenvalues with a mixture of left and right eigenvectors. Thus their arguments suggest that the analysis of the closed loop eigenstructure should be carried out in terms of a mixture of left and right eigenvectors.

The disadvantage of the output feedback approach is that it is limited by a lack of degree of freedom. The output feedback matrix is restricted by the size of the output matrix, C, whereas state feedback is not. The nature of the control problem dictates whether state or output feedback is used.

6.4.4 COMBINED STATE AND OUTPUT FEEDBACK APPROACH

An interesting result was produced by Lovass-Nagy et al [Lov.-N. et al., 1] where the output feedback matrix can be calculated from knowledge of the state feedback matrix. A method using matrix generalised inverses is developed for the computation of the matrix K_s (state feedback) such that the matrix $A + BK_s$ has prescribed eigenvalues which need satisfy only the condition that a certain number of them are distinct and real. A feedback law of the form $\underline{u} = \underline{v} + K_s\underline{x}$ is used to achieve the desired eigenvalue placement. The method does not require the solution of sets of non-linear equations or manipulation of polynomial matrices, and no knowledge of the eigenvalues and/or the eigenvectors of A is necessary. If the computed matrix K_s and the given matrix C satisfy a consistency condition, then the output feedback matrix K_o can be found from the relationship $K_oC = K_s$, and the desired eigenvalue placement can be realised by the output feedback law $\underline{u} = \underline{v} + K_o\underline{y}$.

This interesting result allows direct information of the state space to be used to calculate an output feedback controller. It is worth further investigation in order to check system responses that indicate just how valid the approach is.

6.4.5 APPROACH THAT REDUCES CONTROLLER COMPLEXITY

A note dealing with the use of feedback to approximate the closed loop eigenstructure of a system to a prescribed set of values was proposed by Calvo-Ramon [Cal.-R., 1] in order to reduce the controller complexity based on eigenvalue sensitivity concepts. Output feedback is used to approximate the closed loop eigenstructure of the system to a desired set of values. The method is quite systematic and the design of a constrained output feedback system from a prescribed eigenstructure is well established. Residue analysis (based on left and right eigenvectors) is used to estimate the effect on the eigenvalues of constraints in the feedback gains. The numerical results show that some eigenvectors can be approximately preserved, although eigenvector sensitivities have not been considered. The main drawback of this method is that the eigenvector sensitivities are estimated, which may lead to inaccurate controller designs as stronger poles may be mistakenly overlooked.

6.4.6 RESULTS OBTAINED FROM A SUBSPACE THEME

The problem with Wonham's [Won., 2] fundamental state feedback result is that in most practical situations the state is not available directly. Kwon and Youn [Kwon & Youn, 1] attempted to find a condition under which the system is eigenvalue assignable despite the system having incomplete state observation. They presented a generalisation of an entire eigenstructure assignment method for linear time-invariant multivariable systems, without using assumptions and with the eigenvalues of the closed-loop system being distinct or different from any of the eigenvalues of the open-loop system. The presented method has sufficient conditions that show that the closed loop eigenstructure assignment by output feedback is constrained by the requirement that the generalised right and left eigenvectors lie in certain subspaces. Following on from the subspace theme, Søgaard, Trostmann and Conrad [S-And., et al., 1] presented a method whereby all the residuals assignable by state feedback must be characterised geometrically in terms of subspaces. These subspaces are defined by the freely selectable closed loop eigenvalues. Any desired residual may be selected from these subspaces. The applicability of this result is complimented by the fact that basic control design objectives like I/O response and robustness can be expressed in terms of the residuals.

The approach here stimulates further analysis into the assignable spectra of controllability subspaces, and will be studied in greater detail in Chapter 8.

6.4.7 PARAMETRIC STATE FEEDBACK RESULTS

Roppenecker [Rop., 1] derived an explicit parametric expression for the controller gain matrix of a linear state-variable feedback system. It is based on a modal analysis of the input control vector $\underline{u}(t)$ under linear state-variable feedback conditions. The parameterisation of the class of all state feedback controllers that assign a prescribed set of distinct eigenvalues was found in terms of certain parameter vectors which are functions of the gain matrix and the new eigenvectors to be derived. The same algorithm, provided the prescribed eigenvalues are distinct and that the system is completely controllable, can always calculate the controller gain matrix. The method for deriving the controller parameters is also applicable to the case where all the open-loop eigenvalues are required to be shifted by an appropriate control action. Fahmy and O'Reilly [Fah. & O'Re., 1] devised another parametric solution for closed-loop eigenstructure assignment via state feedback in a linear multivariable system with n states and r control inputs. This was achieved by introducing a lemma on the differentiation of the determinant of the matrix $\left[I_r - K_s(\lambda_i I_n - A)^{-1} B\right]$, the class of assignable eigenvectors and generalised eigenvectors associated with the assigned eigenvalues can be explicitly described by a set of free parameter vectors. Fahmy and O'Reilly followed this up in another paper [Fah. & O'Re., 2], where a general eigenstructure assignment (EA) problem for linear multivariable systems was formulated and solved within the framework of the parametric eigenstructure assignment methodology derived earlier [Fah. & O'Re., 1]. It was shown that EA control is achievable by means of a family of classes of state feedback controllers. The number of classes is equal to the number of admissible Jordan forms of the closed loop system. Each class is characterised by a specific minimum number of free parameters (degrees of freedom) in the parametric form of the feedback gain matrix. The class of EA controllers with the greatest value of free parameters is used for the assignment of the eigenstructure. A significant advantage of this method occurs when not all of the eigenvalues need to be shifted, thus releasing extra free parameters for other design purposes.

6.4.8 PARAMETRIC OUTPUT FEEDBACK RESULTS

There have also been methodologies for the output feedback case that follow the parametric approaches devised under state feedback conditions. Fahmy and O'Reilly [Fah. & O'Re., 3] proposed the development of an effective multistage parametric approach for eigenstructure assignment in linear multivariable systems by output feedback control. The sets of closed loop eigenvalues and associated eigenvectors are suitably divided into subsets and the entire eigenstructure is constructed by parts in two (or more) consecutive stages. The eigenvalue-vector subset assigned in a certain stage is intermediately protected, i.e. made invariant under output feedback, so that another subset can be assigned in a subsequent stage without disturbing the former subset. This allows the subsets of right and left eigenvectors to be assigned in separate stages, which relaxes the computational algorithm from the orthogonality conditions. The number of effective free parameters beyond the eigenvalue assignment is also determined, and the notion of redistributing these parameters among the assignable right and left eigenvectors is introduced. The approach as a whole is remarkably simple and systematic, and it provides much insight into the mechanism of eigenstructure assignment by output feedback control. Duan [Duan, 2] introduced another complete parametric approach for eigenstructure assignment by decentralised output feedback. By using a complete parametric solution of a generalised Sylvester matrix equation, parametric representations of both the left and right closed loop eigenvectors and generalised eigenvectors and two series of partially free parameter vectors are established. The whole problem is therefore divided into two subproblems. The first is concerned with the solution of two generalised Sylvester matrix equations, and solved by using a complete parametric solution to the generalised Sylvester matrix equation. The second subproblem is concerned with the solution of a series of real matrices satisfying two sets of linear matrix equations. The obtained algorithm does not require any conditions on the closed loop eigenvalues, and provides a high number of degrees of design freedom for the eigenstructure assignment problem.

From the studies of parametric methods for both state and output feedback cases, it is evident that such approaches allow greater flexibility in the eventual controller design. Such an advantage reduces computational complexity and is indeed used in the new methods devised in Chapter 8.

6.4.9 OTHER APPROACHES

To conclude the review, a couple of unconventional assignment methods will be looked at. Datta [Dat., 1] proposed a conceptually simple algorithm to assign eigenvalues in a Hessenberg matrix. The method is based on the evaluation of a simple recursive relation. A matrix $H=(h_{ij})$ is an upper Hessenberg matrix if $h_{ij}=0$ whenever $i>j+1$. Such a matrix is unreduced if $h_{i,i-1}\neq 0$. Datta considered the problem of replacing the first row of a given unreduced upper Hessenberg matrix such that the resulting matrix has the desired spectrum of eigenvalues. Murdoch and Shriba considered the same problem [Mur. & Shr., 1], however one disadvantage of their method is that the case of the assignment of repeated eigenvalues cannot be considered without considerable alterations to the algorithm. Yet it does have a couple of advantages, the first of those being that the required first row elements are yielded by the solution of a set of linear equations for which reliable algorithms exist in program libraries. The second advantage is that the effect of each assigned eigenvalue on the solution is easily identified, as each is associated with one row of respective equations. Olbrot [Olb., 1] considered arbitrary robust eigenvalue placement by static state feedback. The author demonstrated that robust eigenvalue placement in the disk of an arbitrary radius r centred at $-2r$, can be achieved by a static state feedback controller for systems with so called matched perturbations of uncertain parameters in the state coefficient matrix A (i.e. with perturbations of A in the range of the input matrix B). This implies that such systems can be robustly stabilised with an arbitrarily fixed degree of exponential decay.

The next chapter of will deal with the significance of eigenvectors with a view to robust eigenvector assignment. It is well known that due to the presence of uncertainty or the variation of parameters, and that a mathematical model of a control system is at best an approximation of its corresponding physical problem. The analysis of stability robustness or performance robustness has been very important for control systems under perturbations. From a practical point of view, the analysis of robustness is one of the most important problems that attempts to obtain a quantitative measure of the perturbations under which the systems still maintain the desired performance. A condition for robustness is the orthogonality of the eigenframe, which was examined primarily by Wilkinson [Wil., 1] in 1965. Since then, several papers have been dedicated to the issue of assigning the eigenstructure to satisfy robustness criteria. Juang, Hong and Wang [Jua., et al., 1] based their robust pole assignment method upon the Lyapunov approach [Lan. & Tis., 1], where the upper bounds of the perturbations are obtained to retain the system eigenvalues located within an arbitrarily chosen region in the complex plane. The bounds derived by the proposed method provide useful quantitative measures in consideration of both the stability robustness and performance robustness of uncertain systems. However Wang and Lin [Wang & Lin, 1] argued that the robustness bounds for eigenvalue assignment could be obtained without the need to solve the Lyapunov equation. The analysis of the problem of eigenvalue assignment is based on some essential properties of the induced norms and certain matrix measures, which eliminate the heavy computational burden of the Lyapunov approach. However the Lyapunov approach was taken a step further by Wilson, Cloutier and Yedavalli [Wil., et al., 1]. They presented a generalised eigenstructure assignment procedure for designing a controller which has the best eigenstructure achievable while simultaneously maintaining stability robustness to time varying parametric variations. The problem was approached by constraining the minimisation of the difference between the actual and desired eigenstructure. This minimisation is made subject to the constraints of the eigenstructure equation and the closed loop Lyapunov equation. A more detailed examination of system robustness will be made in Chapter 7.

6.4.10 SUMMARY OF REVIEW

Eigenstructure assignment has attracted a lot of attention but it has focussed on a standard parameterisation of possible eigenstructures and has addressed mainly the robustness of performance using as a test the orthogonality of the eigenframe. Other features and implications of the eigenstructure have not been considered with the exception of the effect of the eigenstructure on the degrees of controllability and observability. In this thesis the above robustness criteria are extended by introducing a new property that demonstrates the effect of the eigenstructure on the state overshoots of corresponding systems.

Most of the techniques on eigenstructure assignment deal with ways to maximise the orthogonality of the eigenframe, which is one particular problem and is indeed only one issue within the eigenstructure design problem family. Issues such as the best selection of closed loop spectrum that guarantees the most orthogonal solution are not sufficiently addressed. In this thesis, this problem is touched on through examples (Chapter 7), but overall it remains open.

Eigenstructure assignment algorithms which can handle a multitude of performance criteria require more flexible parameterisations. Specifically, what is required, are parameterisations tuned to the needs of the specific criteria. The new algebraic criterion to be introduced in Chapter 8 seems to be the most flexible since it provides an explicit description of the structure of the eigenframe based on the properties of the closed loop spectrum. This new form has the potential to study problems such as specification of closed loop spectra that can guarantee the most orthogonal closed loop eigenstructure as well as selection of eigenstructures with the best degrees of controllability and observability. The alternative test based on open loop and closed loop spectra is also important since it permits the linking of state feedback design to pole mobility using energy considerations or norm of the feedback matrix used.

6.5 SUMMARY AND OPEN ISSUES

In light of the literature review that examined numerous methodologies for the application of procedures that assign the eigenframe of a system to a new predetermined state so as to enhance its performance, it is evident that such techniques can be split into the following categories.

- Effect of the eigenstructure on system performance
- Eigenstructure assignment using a *state feedback* approach
- Eigenstructure assignment using an *output feedback* approach
- Eigenstructure assignment by *parameterising* the eigenvectors

Before examining the way the eigenstructure can be changed by certain forms of compensation, it is important to examine the role of the eigenstructure on different aspects of system performance. The issues that are fundamental to this are:

(i) Eigenstructure and system properties such as controllability, observability, robustness, stability, etc.

(ii) Measuring the degree of orthogonality of the eigenframe and its effects on system properties.

(iii) The selection of desirable spectra and its effect on resulting orthogonality.

(iv) Alternative forms for parameterising eigenframes.

Such properties are very important and have not been paid the appropriate attention in the study of eigenstructure assignment problems.

The state feedback approach is centred on the solutions for $\underline{u}_i$ and K_s of equation (6.49). Pivotal to the method that uses output feedback is equation (6.50), which is used to find solutions for $\underline{u}_i$ and K_o. The third procedure is the parametric approach, whereby either of the relationships for state or output feedback are used to formulate methods that make use of parametric equations to determine solutions for the respective feedback matrices and corresponding eigenvectors. Generally, feedback has an effect on the closed-loop characteristic polynomial of a system, and thus affects stability and system performance. The advantage of state feedback is that it presents the designer with extra freedom with which multivariable control systems can be successfully applied. However, there are systems in which the states are not measurable, and so the use of full state feedback is impractical. Therefore eigenstructure assignment by output feedback is used.

It is essential that the solutions obtained are such that the sensitivity of the assigned eigenvalues to system modelling discrepancies and external disturbances is minimised. In the next chapter it will be shown that a degree of closed loop system robustness can be achieved by ensuring that the eigenvector matrix is as orthogonal as possible. This presents another hurdle with respect to measuring the orthogonality of a matrix, or a frame. The problem of overshoots in the free response of a system also appears to have been neglected by those addressing the eigenstructure assignment problem. The question of the existence of overshoots despite system stability will be dealt with in the next chapter, where a link to the nature of the structure of the eigenvector matrix will be proposed.

Another criterion central to the theme of the work carried out in this thesis is the requirement to accommodate system controllability and observability. It is desired to maintain these two properties when assigning the eigenstructure of a system. As discussed earlier, this is achieved by ensuring that the eigenvectors are in the left null space of the input matrix B and the right null space of the output matrix C for controllability and observability respectively. Therefore the fundamental problem to be considered is that given the system matrices A and B and a set

$\Lambda = \text{diag}\{\lambda_1, \ \lambda_2, \ \ldots, \ \lambda_n\}$ of stable, controllable eigenvalues, find an appropriate feedback matrix F, and an eigenvector matrix U such that a measure of the conditioning, or robustness, is minimised. With regards to feedback, because open and closed loop systems have the same restricted input-state pencil $(sN - NA)$, the controllability properties of a system are invariant under state feedback, yet the observability properties change. Note that when considering the effect of eigenframe properties on controllability and observability, the pivotal issue is not the exact notion of controllability and observability, but instead it is their "degrees," which can be measured in an appropriate way. In the next chapter, issues regarding the desired properties of the eigenframes are considered such as robustness, orthogonality, skewness of eigenframes and free response overshoots will be looked at. To conclude the thesis, a new method for eigenstructure assignment involving the parameterisation of minimal bases will be considered, developed and tested in Chapter 8.

7 SIGNIFICANCE OF EIGENVECTORS

7.1 EIGENVECTORS AND ROBUSTNESS

7.1.1 INTRODUCTION

It has been well documented, as was seen in the review of Section 6.4, that eigenstructure assignment (i.e. the reallocation of eigenvalues and eigenvectors) is a powerful tool that can be used to shape the dynamic response of a linear time-invariant system as desired. The application of either state or output feedback is a popular technique for altering the shape of the system response. Such design algorithms are usually used to assign the eigenstructure of a closed-loop system to a different frame under the assumption of complete controllability. However, the eigenstructure should only be assigned by taking into consideration certain performance criteria.

One of the most crucial performance measures is that of the closed-loop system robustness to parameter variations, external disturbances and system modelling errors. A major cause for concern is eigenvalue sensitivity to such perturbations, and it is part of the design process to minimise its detrimental effect. Closed-loop system robustness is a big concern of control designers, because knowledge of system parameters is often limited and rarely match those that occur during normal operations. Component ageing is a primary cause of variations in the system, which could lead to performance deterioration and possibly even stability concerns. Therefore it is important that the eigenstructure to be assigned is formed with a view that the resulting system is as robust as possible. Thus it is necessary to examine whether there is a link between the frame of eigenvectors and system robustness. As mentioned earlier, eigenvectors are not strictly vectors, but are directions represented by vectors. The directions form a set, which is referred to as a frame (called an *eigenframe*). Followed by a short literature review on some of the papers that have dealt with robustness and measuring it, the objectives of this chapter will be to firstly examine the link between the orthogonality of the eigenframe and system robustness. Next comes an analysis and demonstration using MATLAB of the presence of overshoots in the state response for asymptotically stable systems, and how the response is affected by changes in the nature of the eigenvector matrix. The presence of overshoots leads to the new notion of strong stability, and this will also be examined in Section 7.2. To conclude the chapter, ways to efficiently measure the orthogonality of matrices will be looked at.

7.1.2. BACKGROUND RESULTS

One of the earliest attempts at examining the significance of orthogonality of the eigenframe for robustness was made by Wilkinson [Wil.,1] in 1965. Wilkinson examined certain properties of Hermitian matrices. A matrix A is defined as Hermitian if

$$\overline{A}^T = A \tag{7.1}$$

where $\overline{A}^T$ denotes the complex conjugate transpose of A. $\overline{A}^T$ is frequently denoted by A^H. Similarly a column vector x is denoted by $\underline{x}^H$ after performing the Hermitian operation. The following standard result provides the basis of the analysis:

***Result** 7.1:* [Horn & Jon., 1] If a Hermitian matrix has distinct eigenvalues, then its eigenvectors satisfy the relationship

$$\underline{x}_i^H \underline{x}_j = 0, \quad i \neq j \tag{7.2}$$

If $\underline{x}_i$ is normalised so that $x_i^H x_i = 1$, then the matrix X formed by the columns of eigenvectors satisfies

$$X^H X = I$$
$$X^H = X^{-1} \tag{7.3}$$

□

A matrix that meets the above conditions is called a unitary matrix. A real unitary matrix is called an orthogonal matrix. Wilkinson goes on to say that there is a link between the condition number of the eigenvector matrix and the eigenvalue sensitivity problem with a view to maintaining a high degree of robustness. The condition number is equal to the inverse of the sensitivity of the perturbations of the eigenvalues, which is defined by

$$s_i = y_i^T x_i \tag{7.4}$$

where s_i is the associated sensitivity of the λ_i eigenvalue and (y_i, x_i) are the corresponding left and right eigenvectors. This leads to the following result.

***Result** 7.2:* [Wil., 1] The sensitivity of the eigenvalue λ_i to perturbations in the components of the state matrix A, is largely dependent on the magnitude of the condition number c_i given by

$$c_i = 1/s_i = \|y_i\|_2 \|x_i\|_2 / |y_i^T x_i| \geq 1 \tag{7.5}$$

where s_i is the sensitivity which is the cosine of the angle between the right and left eigenvectors (x_i and y_i) corresponding to λ_i. This is subject to the bound on the sensitivities given by

$$\max_i c_i \leq \text{cond}(X) \equiv \|X\|_2 \|X^{-1}\|_2 \tag{7.6}$$

where $X = [x_1, \ x_2, \ \ldots, \ x_n]$ is the matrix of eigenvectors. The aim is to compute X such that it is well conditioned, i.e. cond(X) = 1.

□

Kautsky, Nichols and Van Dooren [Kau., et al, 1] used the above result to formulate the robust pole assignment problem as follows

Problem 7.1: Given (A, B) and $\Lambda = \{\lambda_1, \ \lambda_2, \ \ldots, \ \lambda_n\}$, find a real feedback matrix, F, and a non-singular matrix X of eigenvectors satisfying

$$(A + BF)X = X\Lambda \tag{7.7}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \ \lambda_2, \ \ldots, \ \lambda_n\}$, and such that some measure v of robustness of the eigenproblem is optimised.

□

The authors remark that the measure v could be chosen to be $v_1 = \|c\|_\infty$, where $c^T = [c_1, \ c_2, \ \ldots, \ c_n]$ is the vector of condition numbers corresponding to the selected matrix X of eigenvectors. Alternatively, the measure v_2 could be taken to be the overall condition number of the matrix X, described by (7.6). The third measure introduced was defined by v_3 as $n^{-1/2}\|X^{-1}\|_F$, where $\|\bullet\|_F$ denotes the Frobenius norm, and n is the size of the state matrix A. The final measure reaffirms the link between orthogonality of the eigenvector matrix and robustness. The conditioning, v_2, of the eigenproblem is said to be optimal if and only if the matrix X of normalised eigenvectors x_i is unitary. Therefore the aim is to select eigenvectors such that $\|x_i\|_2 = 1$ and that the vectors x_i are as orthogonal as possible to each other. v_4 involves the angle between a unitary eigenvector $\hat{x}_i$ and the selected eigenvector x_i, and is denoted by θ_i. Thus v_4 is given by $\left(\sum_j \sin^2\theta_i\right)^{1/2} \Big/ n^{1/2}$. So the four measures can be summarised in the following result:

Definition 7.1: [Kau., Nic., & Van D., 1] The following four measures of robustness may be defined as

$$\begin{aligned} v_1 &= \|c\|_\infty \\ v_2 &= \mathrm{cond}(X) \\ v_3 &= n^{-1/2}\|X^{-1}\|_F \\ v_4 &= \left(\sum_j \sin^2\theta_i\right)^{1/2} \Big/ n^{1/2} \end{aligned} \tag{7.8}$$

□

Ibbini and Alawneh [Ibb. & Ala., 1] split the robustness issue into two problems. The first was the type of robustness associated with parameter variations (i.e. modelling errors, ageing components). They stated that one of the following two conditions had to be satisfied in order for the system to be minimally impervious to such variations. The first was that the condition number of the closed loop eigenvectors matrix has to be minimised. The other condition stated was that the closed loop eigenvectors are as close as possible to being individually orthogonal. When the eigenvectors are orthonormal, a condition number of one is obtained, which indicates that one can either minimise the overall system condition number or adjust the closed loop eigenvectors to become as close as possible to being individually orthogonal. The second type of robustness problem stated was that linked to external disturbances, where a norm constraint has to be satisfied in order for this case to be successfully dealt with. Mudge and Patton, [Mud. & Pat., 1], through their technique of robust eigenstructure assignment, also point to the fact that in order to maximise robustness, it is necessary to have an eigenvector matrix whose condition number is small, and with a high degree of orthogonality.

In the next section another aspect of the eigenframe is considered. This is the case of linking skewness of the eigenvectors to overshoots in the state trajectory. This problem will be examined and demonstrated using MATLAB.

7.2 EIGENVECTORS AND OVERSHOOTS: THE NOTION OF STRONG STABILITY

7.2.1 INTRODUCTION

In this section, the link between overshoots in the free response of the state vector and the orthogonality of the eigenvector matrix is to be examined for systems that are asymptotically stable. It will be demonstrated using MATLAB that skewed frames produce undesired overshoots in the trajectory of the state vectors for some initial conditions. This leads to the need for achieving orthogonality of the frames, since orthogonal frames do not exhibit overshoots for any initial conditions. The latter will also be demonstrated by some examples.

7.2.2 STRONG STABILTY: OVERSHOOTS IN THE FREE RESPONSE

Firstly the problem will be explained in greater detail. The question that arises is that assuming the system is stable and overdamped, is it possible to have overshoots in the free response of the state trajectory, even for one initial condition? If so, then is it possible to characterise the type of state matrices A for which such a property holds true?

The free response of a system is given by

$$\underline{x}(t) = e^{At}\,\underline{x}(0) \tag{7.9}$$

With this in mind, the characterisation of state space overshoots can be given by the following definition:

Definition *7.2:* The autonomous system $S(A)$ defined by $\dot{\underline{x}} = A\underline{x},\ A \in \mathbb{R}^{n\times n}$, exhibits state space overshoots if, for at least one initial condition from the sphere $\mathrm{S_p}(0,r)$ (centred at the origin with radius r), the resulting trajectory $\underline{x}(t)$ satisfies

$$\left\|\underline{x}(t)\right\| > r \tag{7.10}$$

for a specific time interval of $[t_0, t_1]$. It can be said that the system will exhibit no overshoots if for all $\underline{x}_0 \in S_p(0,r)$ and all $t_1 > 0$, $\|\underline{x}(t)\| \leq r$.

□

A system that exhibits no overshoots for all initial conditions and for all spheres $S_p(0,r)$ will be called *strongly stable.* It will be shown later on the property of strong stability, if it holds, is independent of the radius *r*. For the sake of simplicity, initial conditions within the unit sphere can be considered.

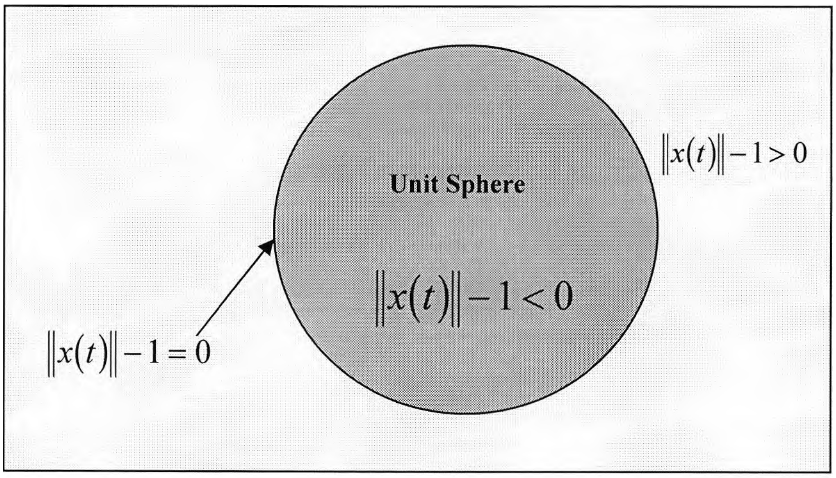


Figure 7.1 *Overshoot conditions*

Note that

- $\|x(t)\| - 1 < 0$ for $\forall t > 0$, indicates that the free response is contained within the unit sphere for all $t > 0$
- $\|x(t)\| - 1 \leq 0$ for $\forall t > 0$, indicates that the free response may touch the circumference of the unit sphere for some time
- $\|x(t)\| - 1 > 0$ for some $t > t_e$ indicates that the free response exceeds the boundaries of the unit sphere after some time t_e and thus the system exhibits overshoots. This may imply instability, or stability with overshoots.

The following remarks can be made:

Remark 7.1: A system that exhibits no overshoots in the sense of the above definition is also stable. Instability clearly implies the existence of overshoots.

□

***Remark** 7.2:* For linear systems, the radius of the sphere $\mathrm{S_p}(0,r)$ does not affect the overshooting property, and it can always be assumed that $r = 1$.

□

7.2.3 OVERSHOOTS AND SIGN DEFINITENESS

For the system $S(A)$, the following properties hold true which characterise the absence of overshoots,

***Theorem** 7.1:* The system $S(A)$ has no overshoot for initial conditions originating within $\mathrm{S_p}(0,1)$, if and only if the quadratic $\underline{x}'A\underline{x}$ is negative definite. The system exhibits overshoots for initial conditions in $\mathrm{S_p}(0,1)$ if $\underline{x}'A\underline{x}$ is positive, or sign indefinite.

□

***Proof** 7.1:* Consider an initial condition $\underline{x}(0)$ and the resulting trajectory $\underline{x}(t)$, which behaves with respect to the surface function $V(\underline{x}) = x_1^2+\ldots+x_n^2 - r^2$ in the following way

(i) If the trajectory starts on the surface defined by $V(\underline{x})$, then it remains beneath the surface for all time, or moves beneath the surface of $V(\underline{x})$, if and only if

$$\cos\left(\underline{\dot{x}},\ \mathrm{grad}V(\underline{x})\right) \le 0 \tag{7.11}$$

(ii) If the trajectory starts on the surface defined by $V(\underline{x})$, it transgresses the surface when

$$\cos\left(\underline{\dot{x}},\ \mathrm{grad}V(\underline{x})\right) > 0 \tag{7.12}$$

Note that $\mathrm{grad}V(\underline{x}) = \left[\dfrac{\partial V(\underline{x})}{\partial x_1},\ \ldots,\ \dfrac{\partial V(\underline{x})}{\partial x_n}\right]$, thus

$$\begin{aligned}\cos(\underline{\dot{x}},\ \text{grad}\ V(\underline{x})) &= \frac{2[x_1,\ \ldots,\ x_n]\cdot\underline{\dot{x}}}{\|\underline{\dot{x}}\|\|\text{grad}\ V(\underline{x})\|} \\ &= \frac{2\underline{x}'A\underline{x}}{\|\underline{\dot{x}}\|\|\text{grad}\ V(\underline{x})\|}\end{aligned} \tag{7.13}$$

The sign of $\cos(\underline{\dot{x}},\ \text{grad}V(\underline{x}))$ is clearly defined by the sign of the quadratic $\underline{x}'A\underline{x}$, which establishes the proof of the result.

□

The above result provides the necessary and sufficient conditions for testing the problem of state trajectory overshoots. Before proceeding, the following lemma needs to be stated, that clarifies the nature of the associated quadratic.

Lemma 7.1: [Gant., 1] The quadratic $\underline{x}'A\underline{x}$ is generated by the symmetric part of A, where

$$A = \frac{1}{2}(A + A') \tag{7.14}$$

and thus

$$Q(\underline{x}, A) = \underline{x}'A\underline{x} = \underline{x}'\overline{A}\underline{x} \tag{7.15}$$

□

The above is readily established since if

$$\overline{A} = \frac{1}{2}(A + A'),\ \ \widetilde{A} = \frac{1}{2}(A - A')$$

where $\overline{A}$ is symmetric, and $\widetilde{A}$ is antisymmetric, then

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

and

$$Q(\underline{x}, A) = \underline{x}' A \underline{x} = \underline{x}' \overline{A} \underline{x} + \underline{x}' \tilde{A} \underline{x}$$

However,

$$\underline{x}' \tilde{A} \underline{x} = \frac{1}{2} \underline{x}' A \underline{x} - \frac{1}{2} \underline{x}' A' \underline{x} = \frac{1}{2} \underline{x}' A \underline{x} - \frac{1}{2} \underline{x}' A \underline{x} = 0$$

which establishes the result.

***Lemma** 7.2:* [Horn & Jon., 1] The quadratic $Q(\underline{x}, A) = \underline{x}' A \underline{x}$ is negative definite if and only if $\overline{A} = \frac{1}{2}\left(A + A'\right)$ satisfies either of the following conditions:

1. $\overline{A}$ is negative definite.
2. $\overline{A}$ has eigenvalues which are all negative.

□

The above result is standard, and no proof is required. However, what must be addressed is the characterisation of the properties of A. Such properties may guarantee the negative definiteness of $\overline{A}$, or equivalently may lead to conditions where the negative definiteness is violated. Hence, the following proposition:

***Proposition** 7.1:* If A is unstable, then $\overline{A}$ is either sign indefinite or positive definite.

□

This can be verified by saying that if A is unstable, then there exists initial conditions for which the state trajectory $\underline{x}(t) = e^{At} \underline{x}(0)$ leaves the sphere defined by $\mathrm{S_p}(0,r)$. This means that the cosine of the angle of $\langle \dot{\underline{x}}, \mathrm{grad} V(\underline{x}) \rangle$ is positive for some $\underline{x}(0)$ on the sphere. Therefore the quadratic $Q(\underline{x}, A) = \underline{x}' A \underline{x}$ is positive in some regions at least, which proves the result.

The following proposition can be made in view of the above:

Proposition 7.2: A necessary condition for $\bar{A}$ to be negative definite is that A has to be stable.

□

The above follows directly from Proposition 7.1. The next problem to be taken into consideration is the characterisation of the special conditions on stable matrices, which either guarantee the negative definiteness of $\bar{A}$, or violate this condition. Firstly it has to be noted that not every stable matrix $A \in \mathbb{R}^{n \times n}$ has a symmetric part that is negative definite. To prove this, it is necessary to demonstrate this property with an example.

Example 7.1: Consider the stable matrix

$$A = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}$$

with eigenvalues at -1 and -3. The symmetric part of A is

$$\begin{aligned} \bar{A} = \frac{1}{2}(A + A') &= \frac{1}{2}\left\{ \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \right\} \\ &= \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \end{aligned}$$

whose eigenvalues are 0.2 and -4.2. Therefore A is *sign indefinite*.

□

The above example shows that special conditions are needed to classify the set of stable matrices that have a negative definite symmetric part, and those not possessing such a property. The following two problems can be formulated from the ones presented in this subsection.

Problem 7.2: Determine the conditions which A must satisfy in order that the symmetric part $\bar{A}$ is negative definite.

□

***Problem** 7.3:* Determine the degree of the skewness of the eigenframe of A that lead to violations of the negative definiteness if $\overline{A}$, for asymptotically stable state matrices, A.

□

The conditions for A that enable it to be defined as a negative definite quadratic may be derived by using Sylvester's Theorem for sign definiteness and this can be illustrated using the following example.

***Example** 7.2:* Consider a general 2×2 case where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow A' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

and the symmetric part is

$$\overline{A} = \begin{bmatrix} a_{11} & \frac{1}{2}(a_{12} + a_{21}) \\ \frac{1}{2}(a_{12} + a_{21}) & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

For the above matrix, the Sylvester Theorem conditions [Gant., 1] imply that

$$\alpha_{11} < 0, \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} > 0$$

are conditions for negative definiteness i.e.

$$a_{11} < 0 \text{ and}$$
$$a_{11}a_{22} - \frac{1}{4}(a_{12} + a_{21})^2 > 0$$

or that

$$\boxed{\begin{array}{l} a_{11} < 0 \\ (a_{12} + a_{21})^2 > 4a_{11}a_{22} \end{array}} \tag{7.16}$$

which are nonlinear inequalities.

□

Clearly, these types of conditions can be generalised for the case where n is anything.

7.2.4 STRONG STABILITY: QUADRATIC INEQUALITIES

A matrix $A \in \mathbb{R}^{n \times n}$ that has a symmetric part $\overline{A}$ which is negative definite will be called *strongly stable*. The notion of strong stability can therefore be related to the lack of overshoots in the free response by way of Definition 7.1. Examples 7.1 and 7.2 show that a natural way to parameterise the family of strongly stable matrices is to use the Sylvester conditions on $\overline{A}$. But this becomes a computational burden for dimensions higher than two. The following examples lead to some important points.

Example 7.3: Consider the case of a symmetric 3 × 3 matrix

$$\overline{A} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

In order for $\overline{A}$ to be negative definite, the Sylvester conditions [Gant., 1] are

$$\boxed{\begin{array}{l} \alpha_{11} < 0, \quad \alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0, \\ \alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}^2 - \alpha_{33}\alpha_{12}^2 - \alpha_{22}\alpha_{13}^2 + 2\alpha_{12}\alpha_{23}\alpha_{13} < 0 \end{array}} \tag{7.17}$$

The conditions of (7.17) are clearly quadratic inequalities. These may be generated in an algorithmic way from the parameter vector $\underline{\alpha}$, where

$$\underline{\alpha}' = [\alpha_{11} \mid \alpha_{12} \quad \alpha_{22} \mid \alpha_{13} \quad \alpha_{23} \quad \alpha_{33}] \tag{7.18}$$

□

The first column of $\underline{\alpha}'$ is based on the $[\alpha_{11}]$ vector and leads to

$$\alpha_{11} < 0 \tag{7.19}$$

Columns two and three are based on $[\alpha_{11} \mid \alpha_{12} \quad \alpha_{22}]$ and corresponds to

$$\alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0 \tag{7.20}$$

The last three columns are based on the overall vector, and is linked to

$$\alpha_{11}\alpha_{22}\alpha_{33} + 2\alpha_{12}\alpha_{13}\alpha_{23} - \alpha_{11}\alpha_{23}^2 - \alpha_{22}\alpha_{13}^2 - \alpha_{33}\alpha_{12}^2 < 0 \tag{7.21}$$

From the above, it can be seen that there is a clear rule emerging. The question that arises is whether a rule can be defined that generates these inequalities.

There are a number of problems that need to be tackled in this area. The first of which is as follows

Problem 7.4: Is it possible to derive a systematic algorithmic procedure based on operations on the parameter vector $\underline{\alpha}'$, such that the complete set of quadratic inequalities may be generated?

□

Take the example below

Example7.4: Consider a matrix A in the companion form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \in \mathbb{R}^{3\times 3} \text{ and } a_0,\ a_1,\ a_2 > 0$$

The symmetric part $\overline{A}$ is

$$\begin{bmatrix} 0 & 1 & -a_0 \\ 1 & 0 & 1-a_1 \\ -a_0 & 1-a_1 & -2a_2 \end{bmatrix}$$

Clearly $\Delta_1 = 0$, $\Delta_2 = -1$ and $\Delta_3 = 2a_2 - 2a_0$. Since $\Delta_1 = 0$, $\Delta_2 < 0$ and $\Delta_3 > 0$ if $a_2 > a_0$, therefore $\underline{x}' A \underline{x}$ is sign indefinite.

□

The above example demonstrates that for certain types of matrices, strong stability is not possible. In fact, it can be stated that

Proposition 7.3: If $A \in \mathbb{R}^{n \times n}$ and is in companion form, then it cannot be negative definite for any of the values of the nonunity coefficients.

□

This leads to the conclusion that for certain families of matrices, strong stability is not possible. Special types of matrices (i.e. Upper Triangular, Circulant, Toeplitz, Hankel, Hessenberg and Tridiagonal) are linked with the selection of special co-ordinate systems. In certain cases, such matrices may arise naturally. With respect to the problem of overshoots considered here, it can only be considered when the state variables are natural. Therefore it makes sense to impose constraints on their behaviour in order to ensure that the modelled elements of the state matrix do not violate certain mathematical conditions that will lead to overshoots, and thus will not guarantee strong stability. Carrying out arbitrary co-ordinate transformations for the sake of studying strong stability makes little sense, since it is evident that strong stability is a property of the original co-ordinate frame and thus the specific, natural description of A.

7.2.5 INVESTIGATIONS USING MATLAB

Problem 7.2 can best be tackled by studying the effect that a skewed eigenvector matrix has on the trajectory of the free state response. It has already been suggested above that closed-loop robustness is directly related to the orthogonality of the eigenframe. To conclude this section, a MATLAB routine will be implemented that plots the response of the state trajectory over a finite time length, for a given eigenframe. Laios [Lai., 1] has already investigated the significance of the eigenframe in relation to the appearance of overshoots, but no explanation for the causes of such phenomenon was given. It was stated that if the eigenvector matrix was normal (i.e. $AA^T = A^TA$) or almost normal, then overshoots are avoided. This will be further investigated using MATLAB and in order to begin, it is necessary to take a look at the equation that dictates the shape of the free response of the state vector

$$\underline{x}(t) = e^{At}\underline{x}(0) = \sum_{i=1}^{n} \underline{u}_i e^{\lambda_i t} \underline{v}_i^t \underline{x}(0) \tag{7.22}$$

where $\underline{x}(0)$ represents the initial conditions and for this case are assumed to be on a sphere of unity radius. $\underline{u}$ and $\underline{v}$ are the right and left eigenvectors, and A is the state matrix, with λ_i representing the eigenvalues. It also has to be assumed that the eigenvectors are of unit length and that the eigenvalues are stable and real. The following assertion was made, based on the above assumptions:

Assertion 7.1: *If the eigenframe is orthogonal or almost orthogonal, and the closed-loop eigenvalues are in the left half plane (i.e. stable), then the norm of* $\underline{x}(t)$ *should be less than unity, i.e.* $\|\underline{x}(t)\| < 1,\ \forall t$ *. And if the frame is skewed, then the norm exceeds unity, i.e.* $\|\underline{x}(t)\| > 1,\ \forall t$ *for some appropriate initial conditions.*

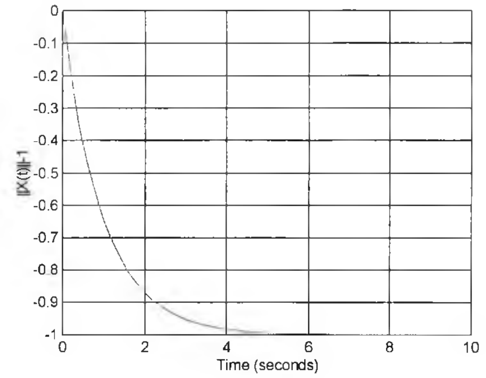
□

A proper study of this assertion is undertaken later on. But first it will demonstrated through some examples. The MATLAB routine *newframe.m* was used to test whether the free response of the state vector in equation (7.22) $\underline{x}(t)$ crossed the circumference of a circle with unit radius. The three conditions to look out for are

- $\|x(t)\| - 1 < 0$ for all $t > 0$ which would indicate that the free response was contained within the unit circle
- $\|x(t)\| - 1 = 0$ for all $t > 0$ which would indicate that the free response lay on the circumference of the unit circle

➤ $\|x(t)\| - 1 > 0$ for some $t > t_e$ which would indicate that the free response exceeded the boundaries of the unit circle and thus the presence of overshoots

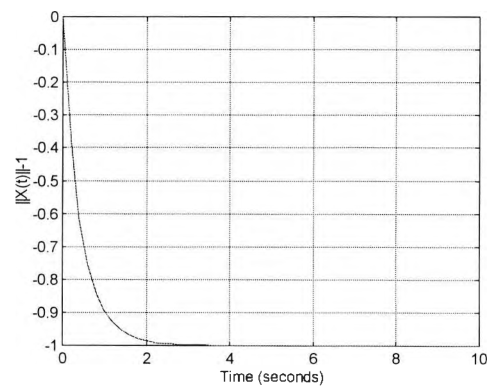
Example 7.5: In this first example the eigenframe, represented by U for the orthogonal set of eigenvectors and Λ for the eigenvalues, was tested for various initial conditions, $\underline{x}(0)$.



$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ -0.0463 & 0.7167 & 0.6959 \end{bmatrix}$$

$\Lambda = [-1 \quad -2 \quad -3]$ $\underline{x}(0) = [0 \quad 1 \quad 0]$

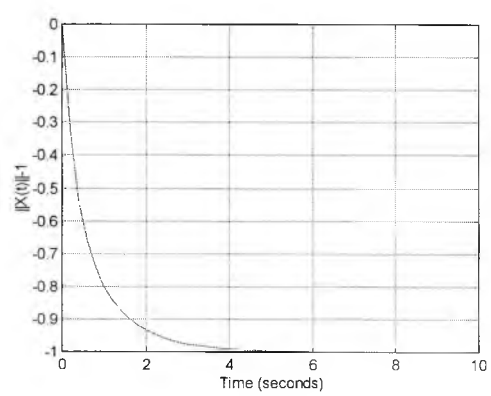
U is orthogonal



$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ -0.0463 & 0.7167 & 0.6959 \end{bmatrix}$$

$\Lambda = [-1 \quad -2 \quad -3]$ $\underline{x}(0) = [0 \quad 0 \quad 1]$

U is orthogonal

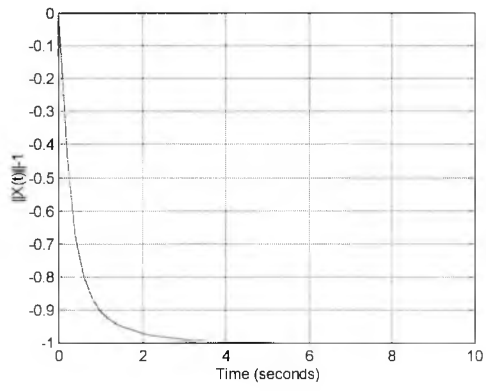


$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ -0.0463 & 0.7167 & 0.6959 \end{bmatrix}$$

$\Lambda = [-1 \quad -2 \quad -3]$

$\underline{x}(0) = [\sqrt{0.5} \quad \sqrt{0.5} \quad 0]$

U is orthogonal

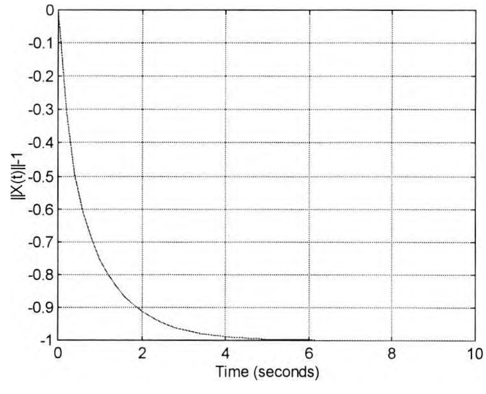


$$U=\begin{bmatrix}-0.2691 & -0.6798 & 0.6822\\ 0.9620 & -0.1557 & 0.2243\\ -0.0463 & 0.7167 & 0.6959\end{bmatrix}$$

$$\Lambda=\begin{bmatrix}-1 & -2 & -3\end{bmatrix}$$

$$\underline{x}(0)=\begin{bmatrix}\sqrt{0.5} & 0 & \sqrt{0.5}\end{bmatrix}$$

U is orthogonal

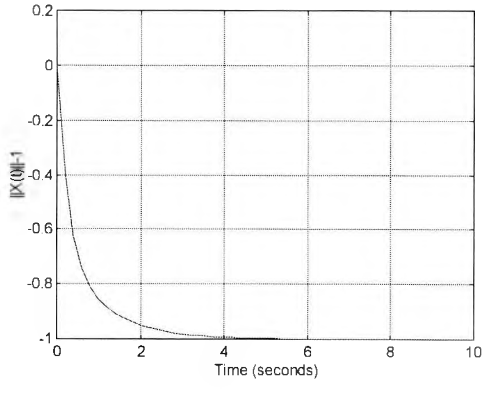


$$U=\begin{bmatrix}-0.2691 & -0.6798 & 0.6822\\ 0.9620 & -0.1557 & 0.2243\\ -0.0463 & 0.7167 & 0.6959\end{bmatrix}$$

$$\Lambda=\begin{bmatrix}-1 & -2 & -3\end{bmatrix}$$

$$\underline{x}(0)=\begin{bmatrix}0 & \sqrt{0.5} & \sqrt{0.5}\end{bmatrix}$$

U is orthogonal



$$U=\begin{bmatrix}-0.2691 & -0.6798 & 0.6822\\ 0.9620 & -0.1557 & 0.2243\\ -0.0463 & 0.7167 & 0.6959\end{bmatrix}$$

$$\Lambda=\begin{bmatrix}-1 & -2 & -3\end{bmatrix}$$

$$\underline{x}(0)=\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix}$$

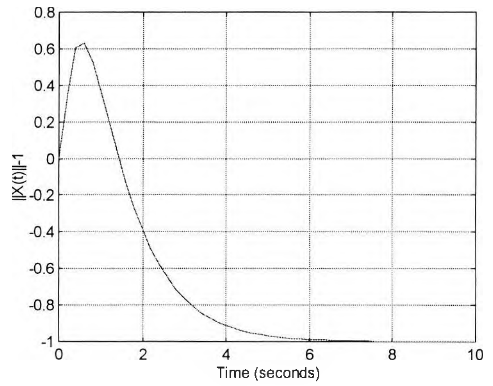
U is orthogonal

Using a standard grammian test for orthogonality, $\left|"U*"U'\right|$, which will be discussed in Section 7.3, it was deduced that the eigenvector matrix was orthogonal as the aforementioned index was equal to 1. The condition number of u was also equal to 1.

As can be seen from the response above, the norm of the state vector for all of the initial conditions does not transgress 0, i.e. it remains within the unit circle. Therefore the response in the time domain of a system with this type of orthogonal eigenframe does not contain overshoots. To explore this further, an eigenvector matrix that is skewed is considered.

□

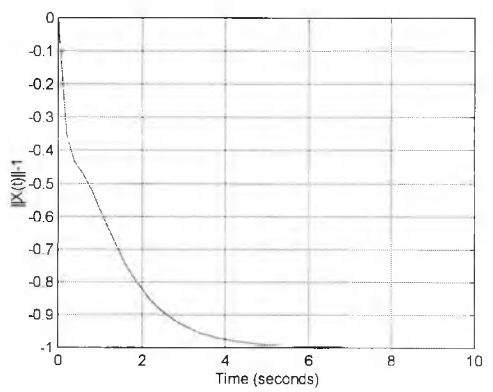
Example 7.6: These responses were obtained for an eigenframe with a skewed set of eigenvectors



$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix} \ \underline{x}(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

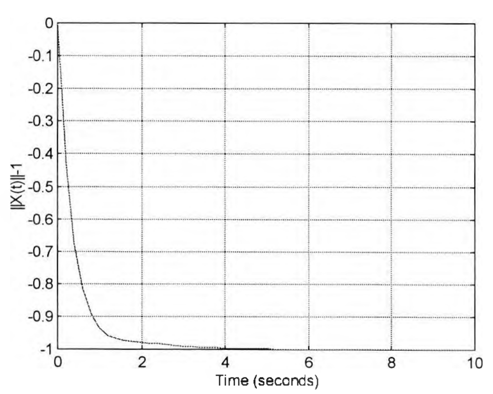
U is skewed



$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix} \ \underline{x}(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

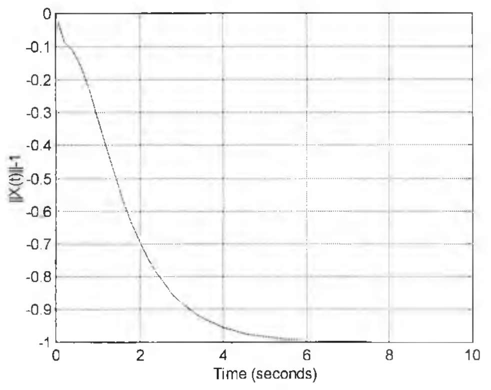
U is skewed



$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix} \ \underline{x}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

U is skewed

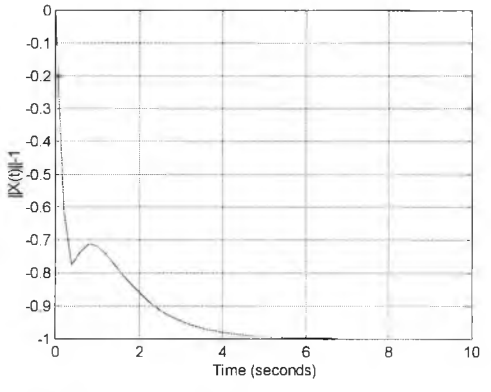


$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} \sqrt{0.5} & \sqrt{0.5} & 0 \end{bmatrix}$$

U is skewed

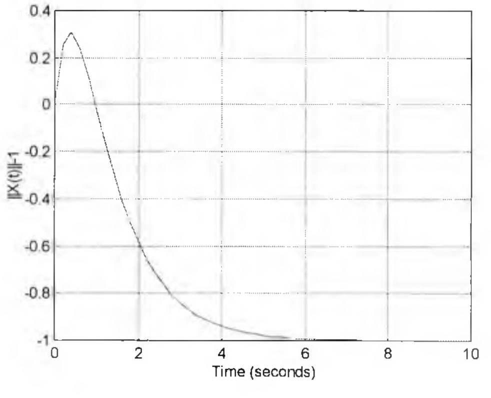


$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} \sqrt{0.5} & 0 & \sqrt{0.5} \end{bmatrix}$$

U is skewed

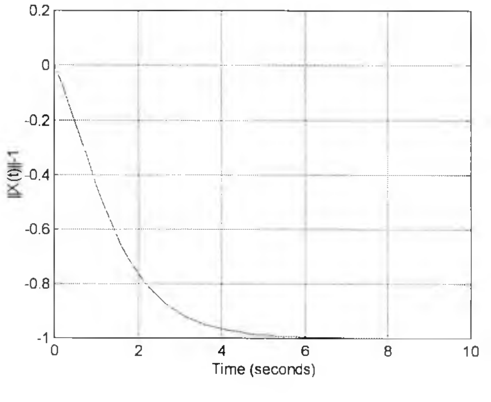


$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 0 & \sqrt{0.5} & \sqrt{0.5} \end{bmatrix}$$

U is skewed



$$U = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

U is skewed

From Proposition 7.1, the resulting A matrix is sign indefinite. In this case U is skewed, which was indicated by the fact that $|"U*"U'|$ was close to 0, and that the condition number of u was 22. As can be seen from the above responses, the state trajectory does contain overshoots for some of the initial conditions. For the initial conditions that do not have overshoots, the settling time for the responses are longer than those of Example 7.3.

□

***Example** 7.7:* Now take a practical example [Mud. & Pat., 1]. The state matrix of a model of a remotely piloted aircraft flying at a constant airspeed of 33 ms^{-1} is

$$A = \begin{bmatrix} -0.277 & 0 & 32.9 & 9.81 & 0 \\ -0.1033 & -8.525 & 3.75 & 0 & 0 \\ 0.3649 & 0 & -6.639 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

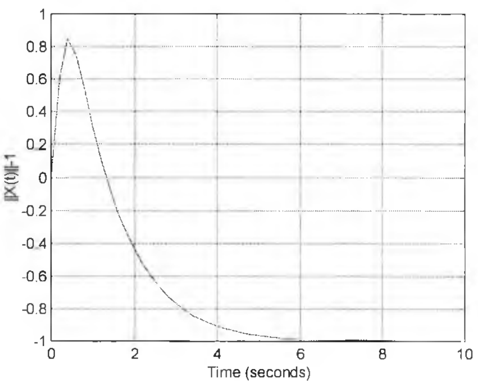
where the state vector is

$$\begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \text{Sideslip} \\ \text{Roll rate} \\ \text{Yaw rate} \\ \text{Bank angle} \\ \text{Yaw angle} \end{bmatrix}$$

Sideslip is the angle between the plane of symmetry and the direction of motion of the craft. The roll rate is the rate at which the aircraft rotates around its longitudinal axis. The yaw rate and angle are concerned with rotational or oscillatory movement of the aircraft about a vertical axis. The bank angle is the angle about the longitudinal axis for the purpose of turning. The corresponding matrix of eigenvectors is

$$U = \begin{bmatrix} 0 & 0.7957 & -0.3486 & 0.4708 & 0.9983 \\ 0 & 0.0099 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 0.0588 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 0.1099 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 0.0356 \end{bmatrix}$$

The above eigenvector matrix is highly skewed, indeed the index of orthogonality, $\left|"U*"U'\right|$, is 8.541×10^{-4}. The condition number is 16.43, and thus the matrix is ill-conditioned. The following trajectories was obtained with the closed-loop eigenvalues given as $\Lambda = [-4, \quad -1.5, \quad -0.5, \quad -1.75, \quad -1]$

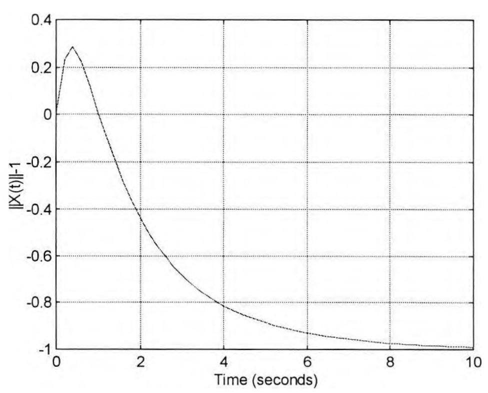


$$U = \begin{bmatrix} 0 & 0.7957 & -0.3486 & 0.4708 & 0.9983 \\ 0 & 0.0099 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 0.0588 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 0.1099 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 0.0356 \end{bmatrix}$$

$$\Lambda = [-4 \quad -1.5 \quad -0.5 \quad -1.75 \quad -1]$$

$$\underline{x}(0) = [0 \quad 0 \quad 0 \quad 1 \quad 0]$$

U is highly skewed



$$U = \begin{bmatrix} 0 & 0.7957 & -0.3486 & 0.4708 & 0.9983 \\ 0 & 0.0099 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 0.0588 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 0.1099 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 0.0356 \end{bmatrix}$$

$$\Lambda = [-4 \quad -1.5 \quad -0.5 \quad -1.75 \quad -1]$$

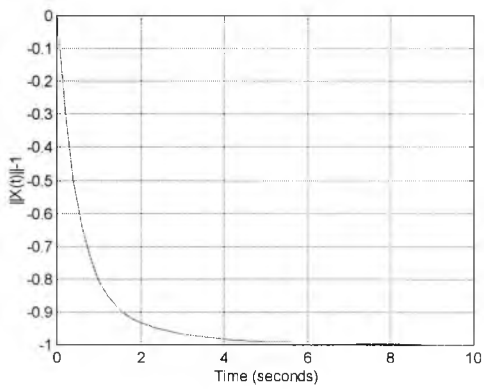
$$\underline{x}(0) = \left[1/\sqrt{5} \quad 1/\sqrt{5} \quad 1/\sqrt{5} \quad 1/\sqrt{5} \quad 1/\sqrt{5} \right]$$

U is highly skewed

As can be seen, due to the highly skewed nature of the eigenvectors, the trajectories for both initial conditions exhibit overshoots. In an attempt to eliminate these undesired overshoots, it will be necessary for the designer to implement a feedback controller in order to reassign the eigenvector matrix to a more orthogonal state. If the resulting feedback system were to have as its eigenvector frame

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 1 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 1 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 1 \end{bmatrix}$$

where the orthogonality index, $|''U^*''U'|$ is 0.0997 and the condition number is 4.48, the following trajectories for the same initial conditions used above

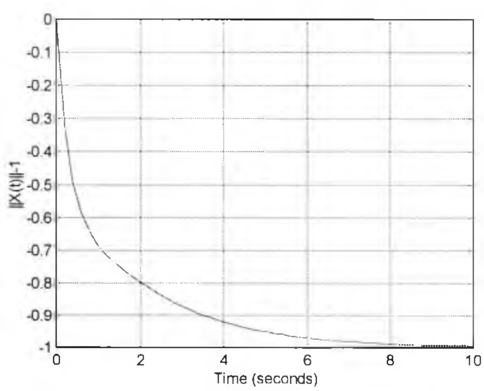


$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 1 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 1 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 1 \end{bmatrix}$$

$\Lambda = [-4, \ -1.5, \ -0.5, \ -1.75, \ -1]$

$\underline{x}(0) = [0 \ \ 0 \ \ 0 \ \ 1 \ \ 0]$

U is less skewed



$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 1 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 1 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 1 \end{bmatrix}$$

$\Lambda = [-4, \ -1.5, \ -0.5, \ -1.75, \ -1]$

$\underline{x}(0) = \left[1/\sqrt{5} \ \ 1/\sqrt{5} \ \ 1/\sqrt{5} \ \ 1/\sqrt{5} \ \ 1/\sqrt{5} \right]$

U is less skewed

Clearly, the higher index of orthogonality has reduced the overshoot problem at the initial part of both of the responses.

□

From all of the examples above, it can be deduced that the orthogonality of the eigenvector matrix is crucial to the free response of a system. In fact, the above examples demonstrate that there is a strong link between skewness of the eigenframes and the presence of overshoots in the state vector of overdamped systems. If the aim of the designer is to eliminate overshoots in the state vector, then a criteria for this to be achieved is clearly that the eigenframe has to be orthogonal, or as close to orthogonality as possible. Therefore it is highly necessary for there to be tests for orthogonality, not simply as a binary concept (i.e. orthogonal or not), but as a measure of the distance of a frame from being orthogonal. Such tests will be examined in the next section.

7.3 ORTHOGONALITY OF EIGENFRAMES

7.3.1 STANDARD MEASURES

If the inner product of two vectors is 0, then they are orthogonal. For real matrices, the standard binary test for orthogonality is the condition $UU^T = I$. So if the product of a matrix and its transpose is an identity matrix, then it is said to be orthogonal. Another indication of whether a matrix is orthogonal or not is the condition number, which is the ratio of the largest to the smallest of the singular values. If the matrix is well conditioned (i.e. the condition number is 1), then the matrix is also said to be orthogonal. However, despite these measures, simply saying whether a frame is orthogonal or not, is not enough. Therefore what is needed is a measure or an index that indicates how far a frame is from being orthogonal.

A standard way of achieving this is to use the grammian approach. To recap from earlier in the thesis, let $\left[\underline{x}_1, \quad \underline{x}_2, \quad \ldots, \quad \underline{x}_n\right] \in \mathbb{R}^{n\times n}$ be a set of vectors. The matrix

$$G = \begin{bmatrix} (\underline{x}_1 \cdot \underline{x}_1) & (\underline{x}_1 \cdot \underline{x}_2) & \cdots & (\underline{x}_1 \cdot \underline{x}_n) \\ (\underline{x}_2 \cdot \underline{x}_1) & (\underline{x}_2 \cdot \underline{x}_2) & \cdots & (\underline{x}_2 \cdot \underline{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ (\underline{x}_m \cdot \underline{x}_1) & (\underline{x}_m \cdot \underline{x}_2) & \cdots & (\underline{x}_m \cdot \underline{x}_m) \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{7.23}$$

is called the *gram matrix* [Gant., 1] of the vectors $[\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n]$ and the determinant $|G| = G(\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n)$ is called the *grammian* of the vector matrix. Grammians have several characteristic properties. The grammian of linearly independent vectors is positive, and that of linearly dependent vectors is zero. Negative grammians do not exist, so for arbitrary vectors $[\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n]$

$$|G| \geq 0 \tag{7.24}$$

If the vectors $[\underline{x}_1, \ \underline{x}_2, \ \ldots, \ \underline{x}_n]$ are of unit length, the following inequality always holds true

$$|G| \leq 1 \tag{7.25}$$

Grammians also provide a useful way of indicating the distance between complete orthogonality and any degree of skewness. The simple steps that make up this procedure are as follows

- Define the matrix, X
- Normalise the columns of the matrix
- Multiply the resultant matrix by its transpose
- Find the determinant
- As the determinant tends to one, then the original matrix X becomes more orthogonal
- As the determinant approaches zero, then X becomes more skewed.

The MATLAB routine that executes this measure is called *orthgtst.m*, and will be symbolised by $\left|{}''u{*}''u'\right|$. Later in this section, the measures mentioned above will be compared, not only with each other, but to a new method for identifying the distance from orthogonality, which follows.

7.3.2 NEW ORTHOGONALITY INDEX BASED ON DISTANCE

In this section, a new measure for determining the distance that a matrix is from orthogonality will be developed. This is closely related to the distances of matrices from being symmetric and normal. So in tandem with finding the distance from orthogonality, distances from the set of symmetric and normal matrices will also be found.

Given a matrix $X \in \mathbb{R}^{n\times n}$ with normalised columns, i.e. if $X = [\underline{x}_1, \quad \underline{x}_2, \quad \ldots, \quad \underline{x}_n]$ and $\|\underline{x}_i\| = 1$, the aim is to investigate a fundamental distance problem which plays a key role in this study.

Problem 7.5: Find the distance of X from the set of $n \times n$ orthogonal matrices.

□

$A \in \mathbb{R}^{n\times n}$ is symmetric if $A = A'$, orthogonal if $AA' = A'A = I_n$, and normal if $AA^* = A^*A$. The above problems will be considered here with the objective to developing computational tests for such distances, where it will be necessary to use the Euclidean norm for a matrix $A \in \mathbb{R}^{n\times n}$, defined by

$$\|A\|_F \equiv \left\{ \sum_{i,j=1}^{n} |a_{ij}|^2 \right\}^{1/2} = \left\{ \text{trace}(AA') \right\}^{1/2} \tag{7.26}$$

Consider first Problem 7.5. If U is any orthogonal matrix, i.e. $UU' = I_n$, then finding the distance of X from the set of U orthogonal matrices implies solving a minimisation problem of the form

$$\min\|X - U\|_F = \min\left[\left[tr(X^T - U^T)(X - U) \right]^{1/2} \right] \tag{7.27}$$

or equivalently minimising

$$\begin{aligned}\|X-U\|_F^2 &= tr\{(X'-U')(X-U)\} \\ &= tr\{X'X+U'U-2trX'U\}\end{aligned} \tag{7.28}$$

Given that $UU' = I_n$ and the columns of X are of unit length, then $tr(X'X) = tr(U'U) = n$. So equation (7.28) leads to

$$\begin{aligned}\|X-U\|_F^2 &= \left[tr(X^T X) + tr(U^T U) - 2tr(XU^T)\right] \\ &= 2n - 2tr(XU^T)\end{aligned} \tag{7.29}$$

The function $\|X-U\|$ is minimised when the function $tr(X'U)$ is maximised, subject to $UU' = I_n$. If

$$\begin{aligned}X &= [\underline{x}_1, \quad \underline{x}_2, \quad \ldots, \quad \underline{x}_n] \\ U &= [\underline{u}_1, \quad \underline{u}_2, \quad \ldots, \quad \underline{u}_n]\end{aligned} \tag{7.30}$$

then

$$tr(X'U) = tr(U'X) = \underline{u}_1'\underline{x}_1 + \underline{u}_2'\underline{x}_2 + \ldots + \underline{u}_n'\underline{x}_n \tag{7.31}$$

subject to the following conditions [Horn & Jon., 1]

$$\underline{u}_i'\underline{u}_i = 1, \quad \underline{u}_i'\underline{u}_j = 0, \quad i \neq j \tag{7.32}$$

From the above, it is clear that the following optimisation problem has to be solved:

Problem 7.6: Maximise the function

$$f_X(U) = \underline{u}_1'\underline{x}_1 + \underline{u}_2'\underline{x}_2 + \ldots + \underline{u}_n'\underline{x}_n \tag{7.33}$$

subject to the constraints of (7.32).

□

Firstly the Lagrangian operator has to be defined

$$
\begin{aligned}
g_X(U,\underline{\lambda}) = {} & \underline{u}_1'\underline{x}_1 + \underline{u}_2'\underline{x}_2 + \ldots + \underline{u}_n'\underline{x}_n \\
& + \lambda_{11}\left(\underline{u}_1'\underline{u}_1 - 1\right) + \lambda_{22}\left(\underline{u}_2'\underline{u}_2 - 1\right) + \ldots + \lambda_{nn}\left(\underline{u}_n'\underline{u}_n - 1\right) \\
& + \sum_{j \neq i} \lambda_{ij}\underline{u}_i'\underline{u}_j
\end{aligned}
\tag{7.34}
$$

In light of the fact that $\underline{u}_i'\underline{u}_j = 0$ also implies that $\underline{u}_j'\underline{u}_i = 0$, it can be deduced that $\lambda_{ij} = \lambda_{ji}$, because corresponding Lagrangian coefficients satisfy a symmetry condition. The conditions that must be satisfied in order for an extremal to exist are derived from

$$
\begin{aligned}
& \frac{\partial g_x}{\partial \underline{u}_i'} = 0, \quad i = 1,\ 2,\ \ldots,\ n \\
& \frac{\partial g_x}{\partial \lambda_{ii}} = 0 \rightarrow \underline{u}_i'\underline{u}_i = 1, \ i = 1,\ 2,\ \ldots,\ n \\
& \frac{\partial g_x}{\partial \lambda_{ij}} = 0 \rightarrow \underline{u}_i'\underline{u}_j = 0, \ \forall i \neq j
\end{aligned}
\tag{7.35}
$$

These conditions are a precursor to the following set

$$
\begin{aligned}
& \frac{\partial g_x}{\partial \underline{u}_1'} = \underline{x}_1 + 2\lambda_{11}\underline{u}_1 + \sum_{j \neq i} \lambda_{1j}\underline{u}_j = 0 \\
& \quad \vdots \\
& \frac{\partial g_x}{\partial \underline{u}_n'} = \underline{x}_n + 2\lambda_{nn}\underline{u}_n + \sum_{j \neq i} \lambda_{nj}\underline{u}_j = 0
\end{aligned}
\tag{7.36}
$$

The assembly of conditions (7.35) and (7.36) leads to

$$X+\begin{bmatrix} 2\lambda_{11} & \lambda_{12} & & \lambda_{1n} \\ \lambda_{21} & 2\lambda_{22} & & \lambda_{2n} \\ & & \ddots & \\ \lambda_{n1} & \lambda_{n2} & & 2\lambda_{nn} \end{bmatrix} U=0 \tag{7.37}$$

and so it can be stated that:

Proposition 7.4: The extremals of the function $f_X(U)=tr(X'U)$, where $UU' = I_n$, are defined by those U corresponding to the factorisation of X as

$$X=-\Lambda U \tag{7.38}$$

where Λ is a symmetric matrix and U is an orthogonal matrix.

□

The fact that the extremals correspond to a maximum can be defended by the following arguments. Let the singular value decomposition of X be

$$X=V\Sigma W',\ \Sigma=\mathrm{diag}\{\sigma_1,\quad \sigma_2,\quad \ldots,\quad \sigma_n\} \tag{7.39}$$

Then

$$tr(X'U)=tr\{(V\Sigma W')'U\} \tag{7.40}$$

or

$$tr(X'U)=tr\{W\Sigma V'U\}=tr\{\Sigma V'UW\} \tag{7.41}$$

The matrix $Q=V'UW$ is orthogonal since each of the matrices V. U and W are also orthogonal. Thus each of the main diagonal entries of these matrices has a modulus of 1, say $q_{tt}=r_t$, $|r_t|=1$, t = 1, 2, ..., n. If k = rank(X), then

$$tr(\Sigma Q)=\sigma_1 r_1+\sigma_2 r_2+\ldots+\sigma_k r_k \tag{7.42}$$

The largest value of $tr(\Sigma Q)$ is $\sigma_1+\sigma_2+...+\sigma_k = tr(\Sigma)$. So U can be chosen such that

$$\begin{aligned} Q &= V'UW = I_n \\ U &= VW' \end{aligned} \tag{7.43}$$

The above analysis leads to

Proposition 7.5: The function $f_X(U) = tr(X'U)$, where $UU' = I_n$, always has a maximum. If $X = V\Sigma W'$ is the singular value decomposition of X, then the maximum is obtained for $U = VW'$.

□

It is now possible to return to (7.37), which in fact is the polar decomposition of the given matrix X. In fact,

Lemma 7.3: [Horn & Jon., 1] Let $A \in \mathbb{C}^{m\times n}$, $m \le n$, then A may be factorised as

$$A = PU \tag{7.44}$$

where $P \in \mathbb{C}^{m\times m}$ is positive semidefinite, rank P = rank A, and $U \in \mathbb{C}^{m\times n}$ has orthonormal rows (i.e. $UU^* = I$). P is always uniquely defined as

$$P = \left(AA^*\right)^{1/2} \tag{7.45}$$

and U is uniquely determined when A has rank m. For the case when $m = n$, and A is nonsingular, then U can be determined by

$$U = P^{-1}A \tag{7.46}$$

□

using the above result, the polar decomposition of $X \in \mathbb{R}^{n\times n}$ can be expressed as

$$X = (XX')^{1/2}(XX')^{-1/2}X \\ = -\Lambda U \tag{7.47}$$

where

$$\Lambda = -(XX')^{1/2}, \quad U = (XX')^{-1/2}X \tag{7.48}$$

and it can be readily verified that

(i) Λ is symmetric

(ii) U is orthogonal

Using the above expressions, the distance may be computed as follows

$$\begin{aligned} \min\|X - U\|_2^2 &= 2n - 2tr\{XU'\} \\ &= 2n - 2tr\left\{XX'(XX')^{-1/2}\right\} \\ &= 2n - 2tr\left\{(XX')^{1/2}\right\} \end{aligned} \tag{7.49}$$

Given that if σ_i are the singular values of X, then

$$tr\{(XX')\} = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 = n \tag{7.50}$$

the following theorem can be stated.

Theorem 7.2: Let $X \in \mathbb{R}^{n \times n}$ be a unit normalised matrix (i.e. $\|\underline{x}_i\| = 1$, i = 1, 2, ..., n) and let $U \in \mathbb{R}^{n \times n}$ be an arbitrary orthogonal matrix $(U'U = I_n)$. If the singular decomposition of X is as in (7.39)

$$X = V\Sigma W', \ \Sigma = \text{diag}\{\sigma_1, \quad \sigma_2, \quad \ldots, \quad \sigma_n\}$$

then

$$\min_{UU'=I} \|X - U\|_2^2 = 2n - 2(\sigma_1 + \sigma_2 + \ldots + \sigma_n) \tag{7.51}$$

and the orthogonal matrix which is closest to X is defined by

$$U = VW' = \left(XX'\right)^{-1/2} X \tag{7.52}$$

□

Note that the matrix $X = [\underline{x}_1, \ \ldots, \ \underline{x}_n]$ is assumed to have columns with unit length i.e. $\underline{x}_i' \underline{x}_i = 1$, i = 1, ..., n. Thus

$$X'X = \begin{bmatrix} \underline{x}_1'\underline{x}_1 & \underline{x}_1'\underline{x}_2 & \cdots & \underline{x}_1'\underline{x}_n \\ \underline{x}_2'\underline{x}_1 & \underline{x}_2'\underline{x}_2 & \cdots & \underline{x}_2'\underline{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ \underline{x}_n'\underline{x}_1 & \underline{x}_n'\underline{x}_2 & \cdots & \underline{x}_n'\underline{x}_n \end{bmatrix} \tag{7.53}$$

Clearly

$$tr\left(X'X\right) = \underline{x}_1'\underline{x}_1 + \underline{x}_2'\underline{x}_2 + \ldots + \underline{x}_n'\underline{x}_n = n \tag{7.54}$$

However, it has also been shown that

$$tr\left(X'X\right) = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 \tag{7.55}$$

and thus

$$\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 = n \tag{7.56}$$

It is also known from the Hadamard inequality that for the normalised (column) matrix X

$$0 \le \left|X'X\right| = G(X) \le 1 \tag{7.57}$$

where $G(X)$ is the grammian of X. Given that [Gant., 1]

$$\left|X^{*}X\right| = G(X) = \sigma_1^2 \cdot \sigma_2^2 \cdot \ldots \cdot \sigma_n^2 \tag{7.58}$$

then the following corollary can be validated:

Corollary 7.1: [Gant., 1] If $X = [\underline{x}_1, \ \ldots, \ \underline{x}_n] \in \mathbb{R}^{n \times n}$ is column normalised, i.e. $\|\underline{x}_i\| = 1$, i = 1, 2, ..., n, then its singular values satisfy the conditions

$$\begin{aligned} &0 \leq \sigma_1 \sigma_2 \ldots \sigma_n \leq 1 \\ &\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 = n \end{aligned} \tag{7.59}$$

□

The analysis presented here for the case of unit normalised matrices may be extended to the general case where normalisation is not used. In this case the result may be expressed in the following form

Theorem 7.3: Let $X \in \mathbb{R}^{n \times n}$ and assume that its singular value decomposition is

$$X = V \Sigma W', \ \ \Sigma = \operatorname{diag}\{\sigma_1, \ \ldots, \ \sigma_n\}$$

If $U \in \mathbb{R}^{n \times n}$ is an arbitrary orthogonal matrix, $U'U = I_n$, then the minimum distance problem $\min\limits_{UU' = I_n} \|X - U\|_2$ has a solution for the matrix

$$U = VW' = \left(XX'\right)^{-1/2} X \tag{7.60}$$

and the minimal distance d is

$$
\begin{aligned}
d^2 = \min_{U'U=I_n} \|X-U\|_2^2 &= \|X\|_2^2 + \|I_n\|_2^2 - 2\sum_{i=1}^{n}\sigma_i \\
&= \sum_{i=1}^{n}\sigma_i^2 + n - 2\sum_{i=1}^{n}\sigma_i \\
&= \sum_{i=1}^{n}(\sigma_i - 1)^2
\end{aligned}
\tag{7.61}
$$

□

The above result reduces to the previous condition of Theorem 7.2 when unit normalisation is introduced.

To conclude, these results have been presented for the case of real matrices, X. However, they can be generalised to the case when $X = \mathbb{C}^{n\times n}$ along similar lines. For such complex cases, the distance from unitary matrices is considered. So by assuming that U is unitary and with X complex, Theorem 7.3 is also valid, but with the necessary changes, i.e. orthogonal to unitary.

7.3.3 SOFTWARE AND CALCULATION OF DEGREES

The MATLAB routine to compute the standard grammian test of the distance from orthogonality from a matrix is called *orthgtst.m* and is denoted by $|"U*"U'|$, and can be seen in the Appendix. The first line of the routine signifies that it can be implemented as a function, i.e. to run the program, it is sufficient to simply type `orthgtst(u)` where u is the matrix under test. The `for` loop normalises the matrix, and the last two lines compute the eventual index of orthogonality.

The second, new index based on the singular values of the matrix is computed by the MATLAB routine *orthtest.m* and is symbolised by the notation $\min_{UU'=I}\|X-U\|$ and can also be seen in the Appendix. Once again, the program is set to be a function. The first part normalises the stated matrix. The remaining section computes the singular values, which are then summed. The last line computes the index.

To finish, these two methods will be tested and compared for eigenvector matrices, along with the condition numbers. Firstly, take the eigenvector matrix of Example 7.3.

$$U_1 = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ -0.0463 & 0.7167 & 0.6959 \end{bmatrix}$$

On execution of the standard grammian test routine, *orthgtst.m*, $\left|"u*"u'\right|$, the orthogonality index was computed to be 1. On execution of the new test, *orthtest.m*, the distance $\min\limits_{UU^T=I}\|X-U\|$ was found to be 0. The condition number of U_1 was also 1. Hence, it can be concluded that U_1 is orthogonal.

Taking the eigenvector matrix of Example 7.4

$$U_2 = \begin{bmatrix} -0.2691 & -0.6798 & 0.6822 \\ 0.9620 & -0.1557 & 0.2243 \\ 4 & 0.7167 & 0.6959 \end{bmatrix}$$

the indices $\left|"u*"u'\right|$ and $\min\limits_{UU^T=I}\|X-U\|$ were 0.0389 and 0.9053 respectively. The condition number was 22.0775. Thus, U_2 is said to exhibit a high degree of skewness.

Moving onto a practical example, consider the aerospace application of Example 7.5

$$U_3 = \begin{bmatrix} 0 & 0.7957 & -0.3486 & 0.4708 & 0.9983 \\ 0 & 0.0099 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 0.0588 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 0.1099 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 0.0356 \end{bmatrix}$$

The standard grammian measure $\left|"U*"U'\right|$ was 8.5426×10-4, and the new measure $\min_{UU^T=I}\|X-U\|$ was 1.847. The condition number was found to be 16.4166. If the less skewed matrix of the same example is considered, i.e.

$$U_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.9294 & -0.8637 & 0.0070 \\ 0 & 0.0442 & 1 & -0.1412 & 0.0459 \\ 0 & -0.1320 & 0.1056 & 1 & 0.0055 \\ 1 & -0.5894 & -0.0067 & 0.0180 & 1 \end{bmatrix}$$

$\left|"U*"U'\right|$ and $\min_{UU^T=I}\|X-U\|$ are 0.0997 and 0.9887, whereas the condition number was 4.48, hence implying that U_4 is closer to orthogonality than U_3.

7.4 SUMMARY

Having earlier discussed the theoretical purpose of the eigenframe, and the concept of eigenstructure assignment, this chapter took a more practical route in the analysis of eigenvectors. Firstly, some of the methods examining the significance of the orthogonality of the eigenframe to closed loop system robustness were examined. It was shown that robustness was linked mainly to the nature of the eigenvector matrix. The degree of robustness could be identified from the condition number and the norm of the eigenframe. Another way to quantify the robustness of a system was to examine the shape of the eigenvector matrix, i.e. whether it is skewed or orthogonal. It was shown using MATLAB that the response of the state vector of asymptotically stable systems contained overshoots for certain initial conditions when the eigenvectors were not orthogonal to each other. This stimulated research into the notion of strong stability, which is related to the lack of overshoots in the free response. It was shown that the natural way to parameterise the family of strongly stable matrices was by way of Sylvester's Theorem. In order to help combat state overshoots, i.e. to make the system more robust, a new measure for the distance of a matrix from orthogonality was proposed and demonstrated, based on the singular values. This new measure was compared to existing techniques and proved to be just as reliable for matrices with real elements. So it was proposed that there is a link between overshoots in the free response of the state vector and the orthogonality of the eigenvector matrix. Software developed using MATLAB showed that this proposition was well founded.

In the next chapter, a new robust method for assigning the eigenstructure of a system will be proposed and developed.

8 MINIMAL BASES: BEHAVIOURS, ZEROS AND EIGENSTRUCTURE ASSIGNMENT

8.1 INTRODUCTION

In this chapter the problem of eigenvector frame parameterisation using two distinct methods and the characterisation of system properties using algebraic means are both considered. An algebraic description of the total system behaviour is presented which in turn allows the study of closed loop eigenvectors in a systematic way by providing parameterisations. An algebraic characterisation of the total input, state and output behaviour in an implicit formulation is given based on properties of MFD descriptions. This allows a novel unifying characterisation of poles and zeros based on input and output zeroing problems [MacF. & Kar., 1]. The analysis also provides explicit algebraic means for characterising the zero structure and providing a new algebraic characterisation of the family of closed loop eigenvectors and related input and output directions. This enables the derivation of a new method of eigenstructure assignment via state feedback, using minimal basis theory, and is demonstrated via an example. Also presented is a way to optimise the eigenframe, which contains the closed loop eigenvalues in order to guarantee maximum system robustness by making it as close to orthogonality as possible. To begin with, a parametric expression is derived for the total behaviour vector of the system. This analysis is based on the computation of the input-state generator pair $\left(\overline{N}(s),\ \overline{D}(s)\right)$ and it is followed by a study of closed loop eigenvectors, frequency transmission and eigenvalue/eigenvector assignment by output feedback. The duality of the pair $\left(\overline{N}(s),\ \overline{D}(s)\right)$ to the zero pencil and zero polynomial case is then discussed. Finally, using the mathematical theory of minimal bases, a new method of eigenstructure parameterisation and assignment is presented, that has the potential for improved robust solutions based on the new parameterisation.

8.2 ASSIGNABILITY OF THE SPECTRUM OF A CONTROLLABILITY SUBSPACE

8.2.1 PROBLEM STATEMENT

The family of controllability subspaces (c.s.) [Won., 1] are special types of (A, B)-invariant subspaces that intersect with the range space $\mathcal{B}$. In fact controllability subspaces are (A, B)-invariant subspaces with the property that any two points may be connected by some appropriate trajectory generated by a control input with the property that the trajectory always remains in the given space [Won.,1]. Their spectra are not fixed, and so the question arises as to whether or not such subspaces may assume any given spectrum. An alternative to the solution already established [Won., & Mor., 1] based on an eigenvector approach is proposed here and involves the construction of characteristic bases having as a spectrum the set of assignable frequencies. This section provides an alternative parameterisation of eigenframes based on the property that such frames are arbitrarily assignable spectra that are characteristic bases of controllability subspaces [Kar., 6]. The results in this section provide an eigenvalue assignment algorithm that conveniently follows the approach mentioned above.

8.2.2 ASSIGNING THE SPECTRUM OF A CONTROLLABILITY SUBSPACE

An alternative establishment of the classical result of the geometric theory is considered here [Won., & Mor., 1]. Consider first the following lemma [Kar., 6].

Lemma 8.1: Let $\mathcal{R}$ be a c.s. of the pair (A, B) and $\{\underline{u}_i\}$ a characteristic basis for $\mathcal{R}$. A vector control input $\underline{u} \in \mathcal{R} \cap \mathcal{B}$ can always be found such that

$$\begin{gathered} \underline{u} = \sum_{j=1}^{r} a_j \underline{u}_j = B\underline{m} \\ a_j \neq 0 \ \forall j, j = 1, \ldots, r, \ r = \dim \mathcal{R} \end{gathered} \tag{8.1}$$

<u>Proof:</u> [Kar., 6] With respect to the basis $\{\underline{u}_i\}$, the vector $\underline{u}$ may be written as

$$\underline{u} = \left[\underline{u}_1 \mid \underline{u}_2 \mid \dots \mid \underline{u}_r\right]\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix} = BG\underline{r} = \hat{B}\underline{r} \tag{8.2}$$

where G ($l \times r$) ($l \leq r$) is the input transformation gain matrix such that the space $\mathscr{R}$ is generated by vectors in the range of $\hat{B}$, $\hat{\mathscr{B}}$, i.e. $\mathscr{R} = \left\{(A + BL)/\hat{\mathscr{B}}\right\}$. For some state feedback matrix L the vectors of the basis $\{\underline{u}_j\}, j = 1, \dots, r$ become a subset of the eigenvectors of the matrix $(A + BL)$ defined by the columns of the matrix U. If V defines the dual eigenvector frame to U and if $\mathscr{R}$ is the controllable subspace of the pair $\left(A + BL, \quad \hat{B}\right)$, then the matrix $V\hat{B}$ has no row that contain all zero elements. Multiplying (8.2) on the left by V gives

$$a_i = v_i^t \hat{B}\underline{r} \tag{8.3}$$

where v_i^t denotes the rows of V. Since none of the $\underline{v}_i^t \hat{B}$ rows are zero, $\underline{r}$ can always be chosen such that $a_i \neq 0$. Then $\underline{m} = G\underline{r}$.

□

Having established this lemma the main results of this section will now be stated, which is the assignment to $\mathscr{R}$ of a characteristic basis having any given spectrum.

Theorem 8.1: [Kar., 6] Let $\mathscr{R}$ be a c.s. of the pair (A, B) and $\{\underline{u}_j\}, j = 1, \dots, r$ a characteristic basis of $\mathscr{R}$, $r = \dim\mathscr{R}$. A new characteristic basis $\left\{\underline{u}_{\mu_i}\right\}$ of $\mathscr{R}$ can always be found such that the spectrum associated with $\left\{\underline{u}_{\mu_i}\right\}$ is any given $\left\{\mu_i\right\}$, $i = 1, \dots, r$.

<u>Proof:</u> For the sake of simplicity it is assumed that $\{\underline{u}_i\}$ is a characteristic basis of $\mathscr{R}$ and has a simple structure that corresponds to eigenvalues with a diagonalisable Jordan form. Then

$$A\underline{u}_i = \lambda_i \underline{u}_i + B\underline{k}_i \tag{8.4}$$

Making the further assumption that the assignable spectrum $\{\mu_i\}$, $i = 1, \ldots, r$ consists of distinct frequencies, then [Kar., 6]

(i) Assume that $\{\mu_i\} \cap \{\lambda_j\} = \mathit{0} \ \forall \ i, j, i, j = 1, \ldots, r$, where $\mathit{0}$ is the zero space. Making use of Lemma 8.1, a vector $\underline{u} = \sum_{j=1}^{r} a_j \underline{u}_j = B\underline{m}$ with $a_i \neq 0$ and vectors $\underline{u}_{\mu_i}$, $\underline{k}_{\mu_i}$ can be found such that

$$A\underline{u}_{\mu_i} = \mu_i \underline{u}_{\mu_i} + B\underline{k}_{\mu_i} \tag{8.5}$$

where

$$\underline{u}_{\mu_i} = [\underline{u}_1 \mid \underline{u}_2 \mid \ldots \mid \underline{u}_r] \begin{bmatrix} a_1(\mu_i) \\ a_2(\mu_i) \\ \vdots \\ a_r(\mu_i) \end{bmatrix} = [\underline{u}_1 \mid \underline{u}_2 \mid \ldots \mid \underline{u}_r] \mathrm{diag}\{a_i\} \begin{bmatrix} \dfrac{1}{\mu_i - \lambda_1} \\ \dfrac{1}{\mu_i - \lambda_2} \\ \vdots \\ \dfrac{1}{\mu_i - \lambda_r} \end{bmatrix} \tag{8.6}$$

The set of r vectors defined this way can be written in a matrix form as follows

$$U_\mu = UD_a M_{\mu,\lambda} \tag{8.7}$$

where U_μ designates the matrix having as columns the vectors $\underline{u}_{\mu_i}$, U is the matrix having as columns the vectors $\underline{u}_i$, D_a the diagonal matrix of the a_i elements, and finally by $M_{\mu,\lambda}$ the matrix with its entries defined by $\delta_{i,j}(\mu,\lambda) = 1/(\mu_j - \lambda_i)$. Because the elements of D_a are nonzero, it always has full rank. Furthermore the matrices $M_{\mu,\lambda}$ always have full rank whenever the sets $\{\mu_i\}$, $\{\lambda_i\}$ have no common element between them. Thus the matrix U_{μ} has full column rank and the vectors $\{\underline{u}_{\mu_i}\}$ form a basis for $\mathcal{R}$ with the desirable spectrum.

(ii) Now assume that the $\{\mu_i\}$, $\{\lambda_i\}$ sets have some common elements. In that case a new distinct spectrum, $\{\xi_i\}$, may be defined such that $\{\xi_i\}\cap\{\lambda_j\}=\emptyset$ and $\{\xi_i\}\cap\{\mu_k\}=\emptyset$ $\forall\, i, j, k$. To the spectrum $\{\xi_i\}$, there will correspond a new basis $\{\underline{u}_{\xi_i}\}$ which according to condition (8.7) can be derived from

$$U_{\xi}=UD_a M_{\xi,\lambda} \tag{8.8}$$

The vector $\underline{u}\in\mathcal{R}\cap\mathcal{B}$ is now expressed with respect to the new basis $\{\underline{u}_{\xi_i}\}$ as

$$\underline{u}=U\underline{a}=U_{\xi}M_{\xi,\lambda}^{-1}D_a^{-1}\underline{a}=U_{\xi}M_{\xi,\lambda}^{-1}\underline{e}=\sum_{i=1}^{r}a_{i\xi}\underline{u}_{i\xi} \tag{8.9}$$

with

$$\begin{aligned}\underline{a}' &=[a_1,\ a_2,\ \ldots,\ a_r]\\ \underline{e}' &=[1,\ 1,\ \ldots,\ 1]\\ \underline{a}_{\xi} &= M_{\xi,\lambda}^{-1}\underline{e}\end{aligned} \tag{8.10}$$

By Lemma (8.1) it is evident that $a_{i\xi}\neq 0\ \forall\, i$. The new basis $\{\underline{u}_{\xi_i}\}$ with the desired spectrum $\{\mu_i\}$ can be easily determined using (8.7) with the assumption that $\{\xi_i\}\cap\{\mu_i\}=\emptyset$.

□

The above theorem implies that, given the characteristic basis $\{\underline{u}_i\}$ for a c.s., $\mathcal{R}$, all that is needed to generate a new characteristic basis $\{\underline{u}_{\mu_i}\}$ which will have as its spectrum the prescribed set of frequencies $\{\mu_i\}$ is a vector $\underline{u} = B\underline{m} = \sum_{j=1}^{r} a_j \underline{u}_j$. It thus appears appropriate to refer to the vector $\underline{u}$ as the "generator" of the c.s. $\mathcal{R}$. It is worth noting that due to the minimal property of a c.s. $\mathcal{R}$, that the generator $\underline{u}$ can be chosen to be any vector $\underline{u} \in \mathcal{R} \cap \mathcal{B}$. The characterisation of the basis $\{\underline{u}_{\mu_i}\}$ in terms of its spectrum is given in matrix form by the following condition,

$$AU_\mu = U_\mu \Lambda_\mu + BK_\mu \tag{8.11}$$

where for generality the matrix Λ_μ is assumed to have a Jordan block structure. Since U_μ has full column rank, a state feedback matrix L can always be found such that

$$LU_\mu = -K_\mu \tag{8.12}$$

Then (8.11) and (8.12) yield

$$(A + BL)U_\mu = U_\mu \Lambda_\mu \tag{8.13}$$

These results may be summarised in the following corollary.

Corollary 8.1: [Kar., 6] Given a c.s. $\mathcal{R}$ and a set of frequencies $\{\mu_i\}$, $i = 1, \ldots, r$, $r = \dim\mathcal{R}$, there always exists a state feedback matrix L such that the restriction $(A + BL)/\mathcal{R}$ has the set $\{\mu_i\}$ as its spectrum.

□

If the pair (A, B) is controllable, then the whole state space $\mathcal{X}$ is a c.s. since $\mathcal{X} \cap \mathcal{B} = \mathcal{B}$ and $\mathcal{X} = \{A/\mathcal{X} \cap \mathcal{B}\} = \{A/\mathcal{B}\}$. Thus the theorem for the assignability of the poles by state feedback stated [Won., & Mor., 1] follows immediately if Corollary (8.1) is used. This theorem is stated as follows.

Theorem 8.2: Let (A, B) be a controllable pair and let $\{\mu_i\}$, $i = 1, \ldots, n$, be a set of complex numbers symmetrically distributed along the real axis. There always exists a state feedback matrix L which assigns the frequencies μ_i's as closed loop eigenvalues of the dynamic map $A_c = A - BL$.

□

The above theorem provides a closed loop eigenvector based alternative proof to the assignability of the spectrum of a c.s. Unlike the original approach [Won., & Mor., 1] which was based on the definition of characteristic polynomials of cyclic subspaces, the treatment given in this section constitutes an eigenvector approach in as far as it is based on the construction of characteristic bases. Next, a pole assignment algorithm is proposed which is based on the concepts outlined.

8.2.3 EIGENVALUE PLACEMENT ALGORITHM BASED ON MOBILITY OF OPEN TO CLOSED LOOP SPECTRA

The above eigenvector approach to the fundamental theorem of assignability of the closed loop eigenvalues yields an algorithm for eigenvalue placement that involves the following fundamental steps [Kar., 6].

(i) Given A, the set of eigenvalues and the corresponding eigenvectors $\{\underline{u}_i,\ \lambda_i\}$ are first found. The vectors $\underline{u}_i$ form a basis for the c.s. $\mathcal{R} = \mathcal{X}$ with the corresponding input directions $\underline{k}_i = 0\ \forall\ i = 1, \ldots, n$.

(ii) If $\{\mu_i\}$ is the assignable spectrum it is safe to always assume that $\{\mu_i\} \cap \{\lambda_i\} = \emptyset$. This is admissible since if $\{\mu_i\} \cap \{\lambda_i\} \neq \emptyset$ then it may necessary to resort to the technique suggested by equations (8.8) and (8.9) and thus define a new basis with spectrum $\{\xi_i\}$ for which $\{\xi_i\} \cap \{\mu_i\} = \emptyset$. Alternatively it is possible to initially apply an arbitrary state feedback which without changing the controllability properties of the pair (A, B) that scatters the closed loop poles to a new spectrum $\{\lambda_i'\}$ such that $\{\lambda_i'\} \cap \{\mu_i\} = \emptyset$.

(iii) A generator $\underline{u}$ of the c.s. is in the form $\underline{u} = \sum_{i=1}^{n} a_i \underline{u}_i = B\underline{m}$. If $\underline{v}_i'$ denotes the eigenvectors dual to $\underline{u}_i$, then the set a_i is given by

$$a_i = \underline{v}_i^t B \underline{m} \quad i = 1,\ldots,n \tag{8.14}$$

Since the pair (A, B) is controllable, none of the $\underline{v}_i^t B$ vectors are zero and the vector $\underline{m}$ may be chosen such that each a_i is non zero.

(iv) Given the sets of the frequencies $\{\mu_i\}$, $\{\lambda_i\}$ such that $\{\mu_i\} \cap \{\lambda_i\} = \emptyset$ and having found the coefficients of a_i the basis $\{\underline{u}_{\mu_i}\}$ may be defined by using the following conditions

$$\begin{aligned} U_\mu &= UD_a M_{\mu,\lambda} \\ \underline{u}_p(\mu_i) &= \sum_{j=1}^{r} a_j^{(p)}(\mu_i)\underline{u}_j \\ \underline{k}_p(\mu_i) &= -\underline{m} + \sum_{j=1}^{r} a_j^{(p)}(\mu_i)\underline{k}_j \end{aligned} \tag{8.15}$$

where the coefficients $a_j^{(p)}(\mu_i)$ are defined by the following expressions

$$\begin{aligned} a_1^{(p)} &= \sum_{j=1}^{v} a_j \delta_{j,p}(\mu, \lambda_1), \\ a_2^{(p)} &= \sum_{j=2}^{v} a_j \delta_{j-1,p}(\mu, \lambda_1), \ldots \\ a_{v-1}^{(p)} &= \sum_{j=v-1}^{v} a_j \delta_{j-v+2,p}(\mu, \lambda_1), \\ a_v^{(p)} &= a_v \delta_{1,p} 0(\mu_1, \lambda_1) \\ a_i^{(p)} &= a_i \delta_{1,p}(\mu, \lambda_1), \\ i &= v+1,\ldots,r \end{aligned} \tag{8.16}$$

where the functions $\delta_{\tau,p}(\mu, \lambda_i)$ are given by

$$\delta_{\tau,p}(\mu, \lambda_i) = \frac{s_\tau(1)}{(\mu - \lambda_i)^\tau} - \frac{s_\tau(2)}{(\mu - \lambda_i)^{\tau+1}} + \cdots + (-1)^{p-1} \frac{s_\tau(p)}{(\mu - \lambda_i)^{\tau+p-1}} \tag{8.17}$$

where $s_\tau(v),\ \tau,\ v = 1,\ldots$ denote the elements of the sequence

$$s_\tau(v+1) - s_\tau(v) = s_{\tau-1}(v+1) \tag{8.18}$$

(v) The input directions corresponding to the vectors $\{\underline{u}_{\mu_i}\}$ are then

$$\underline{k}_{\mu_i} = -\underline{m} + \sum_{j=1}^{n} a_j(\mu_i)\underline{k}_j = -\underline{m} \tag{8.19}$$

Since every $\underline{k}_j = 0\ \forall\ j = 1, \ldots, n$ every A-invariant subspace is also (A, B)-invariant with zero input directions.

(vi) The state feedback matrix is now defined to be

$$L\left[\underline{u}_{\mu_1} \mid \underline{u}_{\mu_2} \mid \ldots \mid \underline{u}_{\mu_n}\right] = -\left[\underline{m} \mid \underline{m} \mid \ldots \mid \underline{m}\right] \tag{8.20}$$

Because $\underline{u}_{\mu_i}$ is linearly independent the matrix $V_\mu = U_\mu^{-1}$ exists and

$$L = -\left[\underline{m} \mid \underline{m} \mid \ldots \mid \underline{m}\right]V_\mu \tag{8.21}$$

(vii) The closed loop dynamic map A_c is then given by

$$A_c = A - BL = U_\mu \operatorname{diag}\{\mu_i\} V_\mu \tag{8.22}$$

and thus can be computed without needing to work out the state feedback matrix L.

It is worth noting that the eigenvalue assignment algorithm presented here yields a unity rank state feedback matrix L. This is due to the fact that the matrix of the input directions corresponding to the closed loop eigenvectors is of unity rank. An alternative approach leading to a full rank state feedback matrix can be formulated as follows.

Given A, an arbitrarily state feedback with a matrix L_0 having full rank may be applied. For the new matrix $A_0 = A - BL_0$, the previously described algorithm may be applied, yielding a unity rank state feedback matrix L_u assigning the poles of A_0 at the desirable locations. The controller $L = L_0 + L_u$ is in general in full rank and assigns the eigenvalues of A at the desired locations.

The essence of the proposed modification is that instead of using the eigenframe of A as a characteristic basis of $\mathcal{X}$ with an associated set of input directions zero, any other characteristic basis of $\mathcal{X}$ may be used with a full rank set of corresponding input directions. Such a basis may be defined as the eigenframe U_0 of some closed loop matrix $A - BL_0$, where L_0 is a state feedback matrix having full rank. The input directions corresponding to this new eigenframe are given by $L_0 \underline{u}_{i_0}$ and the resulting matrix formed, K_0 is of full rank. The successive application of the steps detailed above yield a full rank state feedback matrix L in general. The eigenvalue assignment algorithm is illustrated using the following examples.

8.2.4 EXAMPLES OF EIGENVALUE PLACEMENT ALGORITHM

Example 8.1: Let A and B be given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

The sets of eigenvalues of A and their corresponding eigenvectors are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and

$$\underline{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \quad \underline{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \quad \underline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system is controllable and therefore pole assignable.

1. Assume the desired closed loop frequencies to be $\mu_1 = -1$, $\mu_2 = -2$, $\mu_3 = -3$. Then $\{\mu_i\} \cap \{\lambda_i\} = \emptyset$ and hence no modification to the A matrix is needed.

2. Choosing a $\underline{u}$ vector as generator

$$\underline{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \underline{m} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and it is seen that

$$\underline{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \sum_{i=1}^{3} a_i \underline{u}_i = (1)\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + (4)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which gives $a_1 = 1$, $a_2 = 1$, $a_3 = 4$.

3. From the conditions of (8.15)

$$\underline{u}_{\mu_i} = \frac{1}{\mu_i - 1}\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{\mu_i - 2}\begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + \frac{4}{\mu_i - 3}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

4. Thus the vectors $\underline{u}_{\mu_i}$ are

$$\underline{u}_{\mu_1} = \frac{1}{60}\begin{bmatrix} 10 \\ -10 \\ -10 \end{bmatrix},\ \underline{u}_{\mu_2} = \frac{1}{60}\begin{bmatrix} 5 \\ -10 \\ -3 \end{bmatrix},\ \underline{u}_{\mu_3} = \frac{1}{60}\begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$$

5. The input directions then are given as

$$\underline{k}_{\mu_1} = \underline{k}_{\mu_2} = \underline{k}_{\mu_3} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

6. The state feedback matrix L can be derived from

$$L = 60\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 10 & 5 & 3 \\ -10 & -10 & -9 \\ -10 & -3 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 21 & 12 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

7. Finally the closed loop matrix $A_c = A - BL$ is

$$A_c = A - BL = \begin{bmatrix} 0 & 1 & 0 \\ -23 & -9 & -15 \\ 5 & 1 & 3 \end{bmatrix}$$

which have $\mu_1 = -1$, $\mu_2 = -2$, $\mu_3 = -3$ as eigenvalues and the $\underline{u}_{\mu_i}$'s as eigenvectors.

□

Example 8.2: Let A and B be

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ -4 & 0 & 0 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c|c} 1 & -1 & 1 \\ 2 & -1 & -1 \\ 4 & 0 & 1 \end{array}\right]}_{\equiv U}\underbrace{\left[\begin{array}{c|c|c} -1 & 1 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -4 \end{array}\right]}_{\equiv \Lambda}\underbrace{\left[\begin{array}{ccc} -1 & 1 & 2 \\ \hline -6 & -3 & 3 \\ \hline 4 & -4 & 1 \end{array}\right]}_{\equiv V} \cdot \tfrac{1}{9}$$

and let the assignable spectrum be $\mu_1 = \mu_2 = \mu_3 = -5$. The above pair is controllable and $\{\mu_i\} \cap \{\lambda_i\} = \emptyset$.

1. Choosing the generator vector $\underline{u}$ as

$$\underline{u} = \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 9 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + (-4)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{m}^t$ = [0 9] and $a_1 = 1$, $a_2 = -3$, $a_3 = -4$.

2. The closed loop eigenvectors are given by

$$\begin{aligned}\underline{u}_\mu^{(1)} &= a_1^{(1)}\underline{u}_1 + a_2^{(1)}\underline{u}_2 + a_3^{(1)}\underline{u}_3 \\ \underline{u}_\mu^{(2)} &= a_1^{(2)}\underline{u}_1 + a_2^{(2)}\underline{u}_2 + a_3^{(2)}\underline{u}_3 \\ \underline{u}_\mu^{(3)} &= a_1^{(3)}\underline{u}_1 + a_2^{(3)}\underline{u}_2 + a_3^{(3)}\underline{u}_3\end{aligned}$$

where

$$a_1^{(1)} = \frac{a_1}{(\mu-\lambda_1)} + \frac{a_2}{(\mu-\lambda_1)^2} = -\frac{7}{16},$$

$$a_2^{(1)} = \frac{a_2}{(\mu-\lambda_1)} = \frac{12}{16}, \quad a_3^{(1)} = \frac{a_3}{(\mu-\lambda_3)} = \frac{64}{16}$$

$$a_1^{(2)} = a_1\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2}\right\} + a_2\left\{\frac{1}{(\mu-\lambda_1)^2} - \frac{2}{(\mu-\lambda_1)^3}\right\} = \frac{-19}{32}$$

$$a_2^{(2)} = a_2\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2}\right\} = \frac{30}{32}, \quad a_3^{(2)} = a_3\left\{\frac{1}{\mu-\lambda_3} - \frac{1}{(\mu-\lambda_3)^2}\right\} = \frac{256}{32}$$

$$\begin{aligned}a_1^{(3)} = a_1&\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2} + \frac{1}{(\mu-\lambda_1)^3}\right\} + \\ a_2&\left\{\frac{1}{(\mu-\lambda_1)^2} - \frac{2}{(\mu-\lambda_1)^3} + \frac{3}{(\mu-\lambda_1)^4}\right\} = \frac{-165}{256}\end{aligned}$$

$$a_2^{(3)} = a_2\left\{\frac{1}{\mu-\lambda_1} - \frac{1}{(\mu-\lambda_1)^2} + \frac{1}{(\mu-\lambda_1)^3}\right\} = \frac{252}{256}$$

$$a_3^{(3)} = a_3\left\{\frac{1}{\mu-\lambda_3} - \frac{1}{(\mu-\lambda_3)^2} + \frac{1}{(\mu-\lambda_3)^3}\right\} = \frac{3072}{256}$$

and

$$\underline{u}_{\mu}^{(1)} = \frac{1}{16}\begin{bmatrix} 45 \\ -90 \\ 36 \end{bmatrix}, \quad \underline{u}_{\mu}^{(2)} = \frac{1}{32}\begin{bmatrix} 207 \\ -324 \\ 180 \end{bmatrix}, \quad \underline{u}_{\mu}^{(3)} = \frac{1}{256}\begin{bmatrix} 2655 \\ -3654 \\ 2412 \end{bmatrix}$$

3. The input directions corresponding to the vectors $\underline{u}_{\mu}^{(i)}$ are

$$\underline{k}_{\mu}^{(1)} = \underline{k}_{\mu}^{(2)} = \underline{k}_{\mu}^{(3)} = \begin{bmatrix} 0 \\ -9 \end{bmatrix}$$

4. The state feedback matrix is given by

$$L = 256\begin{bmatrix} 0 & 0 & 0 \\ -9 & -9 & -9 \end{bmatrix}\begin{bmatrix} 720 & 1656 & 2655 \\ -1440 & -2592 & -3654 \\ 576 & 1440 & 2412 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 39 & 9 & -30.25 \end{bmatrix}$$

5. For the matrix L the closed loop matrix $A_c = A - BL$ becomes

$$A_c = \begin{bmatrix} -3 & 1 & 0 \\ -39 & -12 & 31.25 \\ -4 & 0 & 0 \end{bmatrix}$$

The eigenvalues of A_c are at $\mu_1 = \mu_2 = \mu_3 = -5$ while $\underline{u}_{\mu}^{(1)}$ is an eigenvector and $\underline{u}_{\mu}^{(2)}$ and $\underline{u}_{\mu}^{(3)}$ are pseudo-eigenvectors if A_c.

□

The final example given here is intended to illustrate the modified algorithm which yields a full rank matrix L.

Example 8.3: Let A and B be

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 5 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

By applying an arbitrary state feedback by the matrix L_0

$$L_0 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

a closed loop matrix $A_c = A - BL_0$ is obtained having the following set of eigenvalues and eigenvectors

$$\lambda_1 = 3,\ \underline{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \lambda_2 = 3,\ \underline{u}_2 = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix},\ \lambda_3 = 6,\ \underline{u}_3 = \begin{bmatrix} 3 \\ 18 \\ 17 \end{bmatrix}$$

The input directions corresponding to the $\underline{u}_i$ set are defined by $\underline{k}_i = L_0\underline{u}_i$ or

$$\underline{k}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \underline{k}_2 = \begin{bmatrix} -6 \\ 0 \end{bmatrix},\ \underline{k}_3 = \begin{bmatrix} -6 \\ -18 \end{bmatrix}$$

The sets $\underline{u}_i$ and $\underline{k}_i$ define the new characteristic basis of the controllable state space of $\mathcal{X}$. Choosing the generator $\underline{u}_m$ as $\underline{u}_m = B\underline{m}$ with $\underline{m}^t = [18\ 0]$ yields

$$\underline{u}_m = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix} = (-22)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 18 \\ 17 \end{bmatrix}$$

such that a_1 = -22, a_2 = -1, a_3 = 1. If the desired closed loop spectrum is $\mu_1 = -1$, $\mu_2 = -2$, $\mu_3 = -3$ then a new characteristic basis of $\mathcal{X}$ having the set μ_i as a spectrum is defined by using the equations of (8.15) as follows

$$\underline{u}_{\mu_1} = \begin{bmatrix} 36 \\ -36 \\ -27 \end{bmatrix},\ \underline{k}_{\mu_1} = \begin{bmatrix} -324 \\ 36 \end{bmatrix},\ \underline{u}_{\mu_2} = \begin{bmatrix} 45 \\ -90 \\ -9 \end{bmatrix},\ \underline{k}_{\mu_2} = \begin{bmatrix} -810 \\ 90 \end{bmatrix},\ \underline{u}_{\mu_3} = \begin{bmatrix} 6 \\ -18 \\ 1 \end{bmatrix},\ \underline{k}_{\mu_3} = \begin{bmatrix} -174 \\ 18 \end{bmatrix}$$

The state feedback matrix L assigning the set of frequencies $\{\mu_i\}$ as eigenvalues of the closed loop matrix $A_c = A - BL$ is

$$L = \begin{bmatrix} -324 & -810 & -174 \\ 36 & 90 & 18 \end{bmatrix} \begin{bmatrix} 36 & 45 & 6 \\ -36 & -90 & -18 \\ -27 & -9 & 1 \end{bmatrix}^{-1}$$

which has full rank.

□

8.2.5 SUMMARY

The results in this section provide a new parameterisation of closed loop eigenvectors based on the properties of the existence of characteristic bases of controllability subspaces with arbitrarily assignable spectra. The derived expressions are parameterised by appropriate input directions and the differences between open and closed loop eigenvalues and they provide an alternative approach to the existing parameterisations and related feedbacks.

8.3 PARAMETRIC EXPRESSION OF TOTAL BEHAVIOUR, CLOSED-LOOP EIGENVECTORS AND FEEDBACK

8.3.1 INTRODUCTION

An alternative algebraic approach that can provide a characterisation of the closed loop eigenvectors will be considered here, as well as introducing a new way of characterising the system properties based on an algebraic characterisation of the behaviour for linear systems.

8.3.2 IMPLICIT SYSTEM DESCRIPTIONS AND BEHAVIOUR

For the system $S(A,B,C,E)$, which will be assumed to be minimal, i.e. controllable and observable, the total behaviour solution of system equations under zero initial conditions is expressed by [Ros., 1]

$$\begin{bmatrix} sI-A & -B \\ -C & -E \end{bmatrix}\begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -\underline{y}(s) \end{bmatrix} \tag{8.23a}$$

or equivalently [Kar. & Hay., 1]

$$\begin{bmatrix} sI-A & -B & 0 \\ -C & -E & -I \end{bmatrix}\begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \\ \underline{y}(s) \end{bmatrix} = 0 \tag{8.23b}$$

$\underline{\xi}(s) = \left[\underline{x}(s)', \quad \underline{u}(s)', \quad \underline{y}(s)'\right]'$ will be referred to as the *total behaviour vector* of the system. Also it should be noted that if $s=\lambda$ and $\underline{x}(\lambda)$, $\underline{u}(\lambda)$, $\underline{y}(\lambda)$ denote corresponding constant vectors, then (8.23a) or (8.23b) denote vector solutions of the rectilinear motion problem discussed in Chapter 6 for the given λ.

The initial problem is to define the solution of (8.23) in parametric form using the system model structure. The system equations are

$$\begin{aligned} (sI-A)\underline{x}(s) &= B\underline{u}(s) \\ \underline{y}(s) &= C\underline{x}(s) + E\underline{u}(s) \end{aligned} \tag{8.24}$$

Consider the relationship

$$\underline{x}(s) = (sI-A)^{-1}B\underline{u}(s) \tag{8.25a}$$

and let

$$(sI-A)^{-1}B = \overline{N}(s)\overline{D}(s)^{-1} \tag{8.26}$$

be a right coprime MFD of the input state transfer function. Then

$$\underline{x}(s) = (sI-A)^{-1}B\underline{u}(s) = \overline{N}(s)\overline{D}(s)^{-1}\underline{u}(s) \tag{8.25b}$$

By defining

$$\underline{h}(s) = \overline{D}(s)^{-1}\underline{u}(s) \text{ or } \underline{u}(s) = \overline{D}(s)\underline{h}(s) \tag{8.27a}$$

the output relationship may be expressed as

$$\begin{aligned} \underline{y}(s) &= C\underline{x}(s) + E\underline{u}(s) \\ &= \left\{C(sI-A)^{-1}B + E\right\}\underline{u}(s) \\ &= \left\{C\overline{N}(s)\overline{D}(s)^{-1} + E\right\}\underline{u}(s) \\ &= \left\{C\overline{N}(s) + E\overline{D}(s)\right\}\overline{D}(s)^{-1}\underline{u}(s) \end{aligned} \tag{8.27b}$$

Proposition 8.1: If the system $S(A,B,C,E)$ is controllable and observable and the state input factorisation in (8.26) is coprime, then a right MFD is defined by

$$G(s) = C(sI-A)^{-1}B + E = N(s)D(s)^{-1} \tag{8.28a}$$

where

$$N(s) = C\overline{N}(s) + E\overline{D}(s), \quad D(s) = \overline{D}(s) \tag{8.28b}$$

and is a right coprime MFD.

Proof: If the system is minimal, then $n = \deg\left|\overline{D}(s)\right|$. From equation (8.27b), it is obvious that $\left\{C\overline{N}(s) + E\overline{D}(s),\ \overline{D}(s)\right\}$ defines an MFD since $\deg\left|\overline{D}(s)\right| = n = \delta_M(G(s))$, the factorisation is minimal.

□

Substituting (8.27a) into (8.27b)

$$\underline{y}(s) = \left\{C\overline{N}(s) + E\overline{D}(s)\right\}\underline{h}(s) \tag{8.27c}$$

and assembling (8.27a), (8.27b) and (8.27c), the following result is obtained.

Proposition 8.2: The total behaviour vector of the system is defined in parametric form as

$$\begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \\ \underline{y}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} \underline{h}(s) = Q_r(s)\underline{h}(s) \tag{8.29}$$

where $\left(\overline{N}(s),\ \overline{D}(s)\right)$ is a coprime right MFD pair of the input state transfer function and $\underline{h}(s) \in \mathbb{R}[s]$ is an arbitrary vector parameter for the rational behaviour.

□

The matrix $Q_r(s) \in \mathbb{R}[s]^{(r+m+p)\times p}$ is referred to as the <u>behavioural representation</u>, and contains as a submatrix the input-output behavioural representation $T_r(s)$ which is defined below as

$$Q_r(s) = \begin{bmatrix} \overline{N}(s) \\ \hline \overline{D}(s) \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \hline T_r(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ D(s) \\ N(s) \end{bmatrix} \tag{8.30a}$$

The rational vector space

$$\mathcal{Q} \equiv \mathrm{colsp}_{\mathbb{R}(s)}\{Q_r(s)\} \tag{8.30b}$$

characterises the total behaviour and has as a complete invariant a corresponding Plücker matrix, or the Grassman Representative of $\mathcal{Q}$ [Kar. & Gia., 1], as defined in Chapter 4.

Remark 8.1: The expression of the total behaviour as in (8.29) suggests that the whole theory of transformations and invariants may be expressed in terms of properties of the $Q_r(s)$ matrix. Furthermore, for minimal $S(A,B,C,E)$ systems all aspects of behavioural structures are generated by the input-state factorisation, i.e.

$$Q_r(s)=\begin{bmatrix}\overline{N}(s)\\ \overline{D}(s)\\ C\overline{N}(s)+E\overline{D}(s)\end{bmatrix}=\left[\begin{array}{c|c|c} I & & \\ \hline & I & 0\\ \hline & E & C\end{array}\right]\bullet\begin{bmatrix}\overline{N}(s)\\ \hline \overline{D}(s)\\ \hline \overline{N}(s)\end{bmatrix}=\begin{bmatrix}\overline{N}(s)\\ \hline D(s)\\ N(s)\end{bmatrix} \tag{8.31a}$$

which clearly denotes how MFDs are generated from the input-state transfer function, which has implications for their computation.

□

Remark 8.2: Given that the Smith structure of $N(s)$ defines the zeros, the zero structure formation may be considered as a model projection problem [Kar., 7] defined in polynomial terms by

$$N(s)=[E,\quad C]\begin{bmatrix}\overline{D}(s)\\ \hline \overline{N}(s)\end{bmatrix}=E\overline{D}(s)+C\overline{N}(s) \tag{8.31b}$$

The "squaring down" [Kar. & Gia., 1] is thus a special case of the above problem of selecting (E, C) to assign the structure of $N(s)$. The important issue here is the problem of transformation of the controllability indices (Forney indices of $\left[\overline{D}(s)', \overline{N}(s)'\right]'$ to those of $N(s)$. Note that "squaring down" corresponds to the boundary case where all Forney indices of $N(s)$ are zero.

□

The framework already developed on zero assignment [Kar. & Gia., 1] may be extended to model projection using the above formulation. This, however, is now a more complex problem since now controllability indices are transformed to Forney dynamical orders and possible zeros. This is a topic for future research.

The MFD pair $\left(\overline{D}(s), \overline{N}(s)\right)$ emerges as a crucial element for the overall analysis and shall be referred to as an input-state generator pair. Such pairs will always be assumed to be coprime.

8.3.3 DUALITY ISSUES AND BEHAVIOURS

Consider the solutions of

$$\left[\underline{z}(s)', \quad \underline{v}'(s)\right]\begin{bmatrix} sI - A \\ -C \end{bmatrix} = 0 \tag{8.32a}$$

which in a sense are dual to those of (8.25a). From (8.32a)

$$\underline{z}(s)' = \underline{v}'(s)C(sI - A)^{-1} \tag{8.32b}$$

If the coprime factorisation of $C(sI - A)^{-1}$ is considered i.e.

$$C(sI - A)^{-1} = \widetilde{D}(s)^{-1}\widetilde{N}(s) \tag{8.33a}$$

then

$$\underline{z}(s)' = \underline{v}(s)'\widetilde{D}(s)^{-1}\widetilde{N}(s) \tag{8.32c}$$

and by defining $\underline{f}(s)' = \underline{v}(s)'\widetilde{D}(s)^{-1}$, this leads to

$$\begin{aligned} \underline{v}(s)' &= \underline{f}(s)'\widetilde{D}(s) \\ \underline{z}(s)' &= \underline{f}(s)'\widetilde{N}(s) \end{aligned} \tag{8.33b}$$

or

$$\left[\underline{z}(s)', \quad \underline{v}(s)'\right] = \underline{f}(s)'\left[\widetilde{N}(s), \quad \widetilde{D}(s)\right] \tag{8.33c}$$

From the above, the left coprime MFDs of the transfer function can be obtained as shown below

$$\begin{aligned} G(s) &= C(sI - A)^{-1} B + E = \widetilde{D}(s)^{-1} \widetilde{N}(s) B + E \\ &= \widetilde{D}(s)^{-1} \left\{ \widetilde{N}(s) B + \widetilde{D}(s) E \right\} = D'(s)^{-1} N'(s) \end{aligned} \tag{8.34a}$$

Proposition 8.3: If the system is minimal and $\widetilde{D}(s)$, $\widetilde{N}(s)$ are left coprime MFDs of $C(sI - A)^{-1}$, then $D'(s)$, $N'(s)$ where

$$D'(s) = \widetilde{D}(s),\ N'(s) = \widetilde{N}(s) B + \widetilde{D}(s) E \tag{8.34b}$$

are left coprime MFDs of $G(s)$.

□

$\left(\widetilde{D}(s),\ \widetilde{N}(s)\right)$ is the state-output generator pair, and the generation of the left coprime MFDs is described by

$$\left[\widetilde{N}(s),\quad \widetilde{D}(s),\quad \widetilde{N}(s)\right] \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & E \\ \hline 0 & 0 & B \end{array}\right] = \left[\widetilde{N}(s),\quad D'(s),\quad N'(s)\right] \tag{8.35}$$

The above is the dual of the relationship of (8.31), and it should be noted that

$$G(s) = D'(s)^{-1} N'(s) = N(s) D(s)^{-1} \tag{8.36a}$$

and thus some interesting relationships between the input-state and state-output generator pairs are derived below. In fact (8.36a) implies that

$$N'(s) D(s) - D'(s) N(s) = 0 \tag{8.36b}$$

and by substituting from (8.28) and (8.34)

$$\left\{ \widetilde{N}(s) B + \widetilde{D}(s) E \right\} \overline{D}(s) - \widetilde{D}(s) \left\{ C \overline{N}(s) + E \overline{D}(s) \right\} = 0 \tag{8.37a}$$

or

$$\tilde{N}(s)B\overline{D}(s) - \tilde{D}(s)C\overline{N}(s) = 0 \tag{8.37b}$$

which leads to the following result:

Proposition 8.4: If $\left(\overline{N}(s),\ \overline{D}(s)\right)$ is an input-state and $\left(\tilde{N}(s),\ \tilde{D}(s)\right)$ a state-output generator pair, then the following relationship holds true

$$\Rightarrow \qquad \left[\tilde{N}(s),\quad \tilde{D}(s)\right]\begin{bmatrix} B & 0 \\ 0 & -C \end{bmatrix}\begin{bmatrix} \overline{D}(s) \\ \overline{N}(s) \end{bmatrix} = 0 \tag{8.37c}$$

□

It should be noted that $\left(\tilde{N}(s),\ \tilde{D}(s)\right)$ contain information on observability indices and $\left(\overline{D}(s),\ \overline{N}(s)\right)$ on controllability indices. Condition (8.37c) expresses constraints on their values.

The computation of state output generator pairs is based on the fact that (8.33) implies

$$\left[\tilde{N}(s) \quad \tilde{D}(s)\right]\begin{bmatrix} sI - A \\ -C \end{bmatrix} = 0 \tag{8.38a}$$

and if M and C^{+} are right annihilators and inverses of the full rank output matrix C, then by multiplying on the right by the full rank matrix $\left[M \mid C^{+}\right]$, the following result is obtained

$$\left[\tilde{N}(s) \quad \tilde{D}(s)\right]\begin{bmatrix} sI - A \\ -C \end{bmatrix}\left[M \mid C^{+}\right] = 0$$

or

$$\begin{bmatrix} \widetilde{N}(s) & \widetilde{D}(s) \end{bmatrix} \begin{bmatrix} sM - AM & sC^{+} - AC^{+} \\ 0 & -I_m \end{bmatrix} = 0$$

and thus

Proposition 8.5: $\widetilde{N}(s)$ is constructed as a minimal basis on the left kernel of $sM - AM$ i.e.

$$\widetilde{N}(s)(sM - AM) = 0 \tag{8.38b}$$

and

$$\widetilde{D}(s) = \widetilde{N}(s)(sC^{+} - AC^{+}) \tag{8.38c}$$

□

The above expressions together with (8.37c) may be used to work out more detailed relationships between the controllability and observability indices of the system. Starting from (8.37c) and using (8.38c) and (8.33b), it is readily shown that:

Remark 8.3: The numerators $\widetilde{N}(s)$ and $\overline{N}(s)$ of the output-state and input-state generator pairs are related as

$$\widetilde{N}(s)\left\{s\left(BB^{+} - C^{+}C\right) - \left(BB^{+}A - AC^{+}C\right)\right\}\overline{N}(s) = 0 \tag{8.39}$$

where $\widetilde{N}(s)$ is a minimal basis of $\mathcal{N}_l\{sM - AM\}$ and $\overline{N}(s)$ is a minimal basis of $\mathcal{N}_r\{sN - NA\}$.

□

8.3.4 COMPUTATION OF INPUT-STATE GENERATOR PAIRS

By definiton

$$(sI - A)^{-1} B = \overline{N}(s)\overline{D}(s)^{-1} \tag{8.26}$$

and this implies that

$$\begin{aligned} &B = (sI - A)\overline{N}(s)\overline{D}(s)^{-1} \\ &B\overline{D}(s) = (sI - A)\overline{N}(s) \Leftrightarrow \\ &[sI - A, \quad -B]\begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \end{bmatrix} = 0 \end{aligned} \tag{8.40}$$

Remark 8.4: The computation of a pair $\left(\overline{D}(s),\ \overline{N}(s)\right)$ is equivalent to computing a minimal basis for the right kernel of $[sI - A, \quad -B]$.

□

Reduced complexity computations may be achieved by using the pair $(N,\ B^{+})$ for the B matrix where N is a left annihilator and B^{+} a left inverse, i.e.

$$\rho(B) = p,\ NB = 0,\ B \in \mathbb{R}^{(n-p)\times n},\ \rho(N) = n - p,\ B^{+}B = I$$

Using $(N,\ B^{+})$, (8.40) is equivalent [Kar., 7] to

$$\begin{aligned} &(sN - NA)\overline{N}(s) = 0 \\ &\overline{D}(s) = B^{+}(sI - A)\overline{N}(s) \end{aligned} \tag{8.41}$$

Remark 8.5: The results developed later in this chapter on minimal bases of matrix pencils are used for computing $\overline{N}(s)$. Then $\overline{D}(s)$ is defined by (8.41) and the denominator of the MFD of $G(s)$ by

$$N(s) = C\overline{N}(s) + E\overline{D}(s),\ \ D(s) = \overline{D}(s) \tag{8.42}$$

□

The above may form the basis for a numerical method for computing MFDs that are different to those already existing. The approach is algebraic in nature and it provides links with fundamental aspects of the underlying system structure.

8.3.5 CLOSED LOOP EIGENVECTORS AND FREQUENCY TRANSMISSION

The algebraic analysis given before is now used to characterise the structure of closed loop eigenvectors and to produce a new characterisation of them. The solution to the frequency transmission problem [Kar. & Kou., 1] is defined by

$$\begin{bmatrix} \lambda_n I - A & -B & 0 \\ -C & -E & -I \end{bmatrix} \begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = 0 \qquad (8.43a)$$

and thus from Proposition 8.2 and condition (8.29), it can be shown that:

Proposition 8.6: The solution of the input, state and output rectilinear motion problem is given by

$$\begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} \underline{x}(\lambda_i) \\ \underline{u}(\lambda_i) \\ \underline{y}(\lambda_i) \end{bmatrix} = \begin{bmatrix} \overline{N}(\lambda_i) \\ \overline{D}(\lambda_i) \\ E\overline{D}(\lambda_i) + C\overline{N}(\lambda_i) \end{bmatrix} \underline{h}_i = Q_r(\lambda_i)\underline{h}_i \qquad (8.43b)$$

□

The above generates all solutions of the frequency transmission problem in parametric form. In fact, $Q_r(\lambda_i)$ is a basis for the total composite transmission space [Kar. & Kou., 1]. This framework will be subsequently used to derive an eigenstructure assignment method.

Remark 8.6: [Kar. & Kou., 1] The solutions of the frequency transmission problem are given by the fact that the generation of any general frequency requires that the state $x(t)$ be restricted in an (A, B)-invariant subspace. This condition may be ensured by selecting an appropriate release condition x_0 that lies in this particular subspace and some appropriate rectilinear input trajectory with the same frequency. The resulting output trajectory will then be the sum of the rectilinear motions whose frequency components are defined by the same exponential.

□

8.3.6 POLE ASSIGNMENT BY OUTPUT FEEDBACK AND CLOSED LOOP EIGENVECTORS

Subsection 8.3.8 will examine the role that state feedback plays in the design of controllers that assign the eigenstructure of a system. First, an examination of output feedback is considered. For the sake of simplicity the strictly proper case is first considered, i.e. when $E = 0$. If K_O is the output feedback matrix, then the closed loop eigenvectors and eigenvalues are defined by

$$(\lambda_i I - A - BK_O C)\underline{x}_i = 0 \tag{8.44a}$$

where $\{\lambda_i \in \mathbb{C}\}$ is a complex conjugate set and the set of corresponding eigenvectors $\{\underline{x}_i,\ i \in n\}$ is linearly independent. For this case, (8.43b) takes the form

$$\begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} \overline{N}(\lambda_i) \\ \overline{D}(\lambda_i) \\ C\overline{N}(\lambda_i) \end{bmatrix} \underline{h}_i \tag{8.44b}$$

and

$$\underline{y}_i = C\underline{x}_i,\ \underline{u}_i = K_O \underline{y}_i,\ \forall i \in n \tag{8.44c}$$

From (8.44b)

$$\underline{u}_i = \underline{u}(\lambda_i) = \overline{D}(\lambda_i)\underline{h}_i$$
$$\underline{y}_i = \underline{y}(\lambda_i) = C\overline{N}(\lambda_i)\underline{h}_i$$

and thus (8.44c) leads to

$$\boxed{\left[\overline{D}(\lambda_i) - K_O C\overline{N}(\lambda_i)\right]\underline{h}_i = 0} \tag{8.45}$$

Remark 8.7: $D_K(s) = \overline{D}(s) - K_O C\overline{N}(s)$ is the denominator of the closed loop transfer function under output feedback and thus $\underline{h}_i$ are the vectors associated with the loss of rank of the $D_K(s)$ denominator (closed loop poles). The selection of $\underline{h}_i$ has to be such that the eigenvectors of (8.44a) defined by

$$\underline{x}_i = \underline{x}(\lambda_i) = \overline{N}(\lambda_i)\underline{h}_i,\ \forall i \in n \tag{8.44d}$$

have to be linearly independent. If $\underline{h}_i$ are treated as free parameters then a design problem may be posed as that aiming at maximising the orthogonality of the $\left[N(\lambda_1)\underline{h}_1,\ \ldots,\ N(\lambda_n)\underline{h}_n\right]$ frame.

□

Remark 8.8: Given that $(sI - A)\overline{N}(s) = B\overline{D}(s)$, then for an eigenvalue $\lambda \in \sigma(A)$, $\overline{D}(\lambda)$ is rank deficient and $\exists\ \underline{h}_\lambda(\lambda)$, then

$$\overline{D}(\lambda)\underline{h}_\lambda = 0$$
$$\underline{x}_\lambda = \overline{N}(\lambda)\underline{h}_\lambda$$

where $\underline{x}_\lambda$ is the λ-closed loop eigenvector since $(\lambda I - A)\overline{N}(\lambda)\underline{h}_\lambda = 0$.

□

The selection of parameter $\underline{h}_i$ is dependent on the input vector $\underline{u}_i$ and the denominator of the input-state transfer function $\overline{D}(s)$ defined by equation (8.26). It is also dependent on the eigenvectors determined by (8.44a) and the corresponding condition (8.44d). The resulting selection problem is thus a crucial one because issues such as linear independence and orthogonality are involved. This approach is independent of the feedback used and can be employed for procedures that lead to eigenstructure assignment.

8.3.7 POLES AND ZEROS

The simplest case of an autonomous system is one that has no physical inputs, i.e. $\underline{u}(t) = 0$. This reduces the state space description to merely $\dot{\underline{x}} = A\underline{x}$ and $\underline{y} = C\underline{x}$.

Because of the absence of inputs, the term *frequency transmission* discussed in Chapter 6 needs to be interpreted differently. As a recap, for a forced system (i.e. a system with physical inputs) a frequency s_0 is said to be transmitted through the system when the application of a signal with this same frequency is applied to the inputs. The system then yields an output response of the same frequency. However, when $\underline{u}(t) = 0$, a frequency cannot be transmitted in this fashion. This does not imply that the system itself, which is free responding under zero input conditions, is not capable of exciting a response of an exponential type. The notion of the zeros of a system is strongly related to the physical situation whereby the system has an identically zero output whilst the states and inputs are not themselves identically zero. It has been shown [Mac., & Kar., 1] that given a transfer function matrix $G(s)$ there are certain specific values of the complex frequency s associated with certain specific non-zero input transform vectors $\underline{u}(s)$ in the input space that transform the output vector $\underline{y}(s)$ to zero. The matrix $G(s)$ corresponds to an external description of the system behaviour in terms of how sets of exponential signals are propagated through it. The internal structural aspects for the case in which a system can have a zero output for non-zero inputs or states have already been examined [Mac., & Kar., 1].

In the following part of the analysis, the case of selecting the parameter $\underline{h}_i$ for both cases of input and output zeroing will be examined. The behaviour form provides an ideal characterisation of the poles and zeros and corresponding directions, because from equation (8.29)

$$\begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \\ \underline{y}(s) \end{bmatrix} = \begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \\ C\overline{N}(s) + E\overline{D}(s) \end{bmatrix} \underline{h}(s) = Q_r(s)\underline{h}(s) \tag{8.29}$$

the following results can be readily deduced from the above description:

Corollary 8.2: **<u>Characterisation of Poles</u>** Consider the zero input problem with $u(t) = 0$. Then $\forall \lambda : |D(\lambda)| = 0 \ \exists \ \underline{h}_p$ such that

$$D(\lambda)\underline{h}_p = 0 \text{ and } \underline{u}_p = 0 \tag{8.46a}$$

then λ_i is a pole of the system and

$$\begin{aligned} \underline{x}_p &= \overline{N}(\lambda)\underline{h}_p \\ \underline{y}_p &= C\overline{N}(\lambda)\underline{h}_p \end{aligned} \tag{8.46b}$$

are the corresponding eigenvectors and output pole directions.

□

Corollary 8.3: **Characterisation of Zeros** Consider the output zeroing problem i.e. $y(t) = 0$. Then $\forall z{:}\mathcal{N}_r\{C\overline{N}(z)+E\overline{D}(z)\} \neq \{0\}\ \exists\ \underline{h}_z$ such that

$$\left[C\overline{N}(z)+E\overline{D}(z)\right]\underline{h}_z = 0 = \underline{y}_z \tag{8.47a}$$

then z is a zero and

$$\begin{aligned} \underline{x}_z &= \overline{N}(z)\underline{h}_z \\ \underline{u}_z &= \overline{D}(z)\underline{h}_z \end{aligned} \tag{8.47b}$$

are the corresponding state and input zero directions of the system.

□

From the behaviour viewpoint, poles and zeros are distinct frequency solutions of zero input and zero output problems. Although $\underline{u}(s)$ cannot by definition become identically zero in a well defined system (although this may happen in implicit autoregressive descriptions), $\underline{y}(s)$ may become identically zero. Thus

$$\underline{y}(s) = 0 \Leftrightarrow \exists \underline{h}_z(s){:}\ \left[C\overline{N}(s)+E\overline{D}(s)\right]\underline{h}_z = 0 \tag{8.47a}$$

The polynomial solution $\underline{h}_z(s)$ then defines the vectors

$$\begin{aligned} \underline{x}_z(s) &= \overline{N}(s)\underline{h}_z(s) \\ \underline{u}_z(s) &= \overline{D}(s)\underline{h}_z(s) \end{aligned} \tag{8.47b}$$

which in turn defines the output nulling controllability spaces for the system [Won., 1].

Remark 8.9: The zeros are those frequencies associated with the further expansion of the kernel of $\left[C\overline{N}(s)+E\overline{D}(s)\right]$ and the corresponding $\underline{x}_z$ are independent from those of colsp.$\{\underline{x}_z\}$.

□

8.3.8 DESIGN OF STATE FEEDBACK CONTROLLERS USING EIGENVECTOR PARAMETERISATION

The general analysis on the solution of the system equations in an algebraic-behavioural sense leads to a parameterisation of closed loop eigenvectors and an explicit design of state feedback that assigns the eigenstructure, and is presented here. The problem of state feedback is stated as follows:

Problem 8.1: Given a complex symmetric set $\Lambda = \{\lambda_i,\ i \in n\}$, find an independent set of closed loop eigenvectors $\{\underline{x}(\lambda_i) = \underline{x}_i,\ i \in n\}$ with corresponding input directions $\{\underline{u}(\lambda_i) = \underline{u}_i,\ i \in n\}$ such that

$$K_S \underline{x}_i = \underline{u}_i,\ \forall i \in n \tag{8.49a}$$

or equivalently

$$\begin{aligned} &K_S[\underline{x}_1,\ \underline{x}_2,\ \ldots,\ \underline{x}_n] = [\underline{u}_1,\ \underline{u}_2,\ \ldots,\ \underline{u}_n] \\ &\Rightarrow K_S X(\Lambda) = U(\Lambda) \end{aligned} \tag{8.49b}$$

□

The above problem can be solved if the frame $X(\Lambda)$ has full rank. Furthermore, it is necessary for the frame $X(\Lambda)$ to be as close to orthogonality as possible, [Wil., 1], since this is related to robustness. Clearly, if for the given Λ a frame $X(\Lambda)$ which has full rank may be found, the solution of (8.49b) is not unique and for the selected frame $X(\Lambda)$ it is shown that

$$K_S = U(\Lambda)X(\Lambda)^{-1} \tag{8.50}$$

The two important issues that emerge here are:

(i) Selection of an independent set of eigenvectors for any given Λ.

(ii) Selection of the most orthogonal frame, if a procedure for selection of independent vectors is found.

Considering the first of the interrelated problems, condition (8.43b) is used to characterise the solution of the rectilinear motion problem, i.e.

$$\begin{bmatrix} \underline{x}_i \\ \underline{u}_i \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} \overline{N}(\lambda_i) \\ \overline{D}(\lambda_i) \\ N(\lambda) \end{bmatrix} \underline{h}_i,\ i = 1,\ 2,\ \ldots,\ n \tag{8.51}$$

Assume that in the factorisation

$$(sI - A)^{-1} B = \overline{N}(s)\overline{D}(s)^{-1} \tag{8.52a}$$

$\overline{N}(s)$ is an ordered minimal basis and is expressed as

$$\overline{N}(s) = \left[\underline{\bar{n}}_1(s),\quad \underline{\bar{n}}_2(s),\quad \ldots,\quad \underline{\bar{n}}_p(s)\right] \tag{8.52b}$$

where $\delta\left[\underline{\bar{n}}_i(s)\right] = \varepsilon_i$, $i \in p$ and $\varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_p$, and

$$\underline{n}_i(s) = \overline{N}_i \underline{e}_{\varepsilon_i}(s),\ \underline{e}_{\varepsilon_i}(s) = \left[1,\quad s,\quad \ldots,\quad s^{\varepsilon_i}\right] \tag{8.52c}$$

where $\overline{N}_i \in \mathbb{R}^{n\times(\varepsilon_i+1)}$ and from the properties of minimal bases of matrix pencils which have already been characterised from the mathematical background detailed in Chapter 2, $rank(\overline{N}_i) = \varepsilon_i + 1$ [Mit., & Kar., 1]. The above properties suggest a simple procedure for selection of an independent eigenframe, and this will be demonstrated in the next section. The selection of independent closed loop eigenvectors is considered next.

8.3.9 SELECTION OF AN INDEPENDENT EIGENFRAME AND RESULTING STATE FEEDBACK

The selection of an eigenframe that corresponds to a given closed loop spectrum is based on the following steps.

STEP (1): For every $\Lambda = \{\lambda_i,\ i \in n\}$ symmetric, it is possible to partition it into the following subsets $\Lambda_{\varepsilon_1} = \{\lambda_1^{\varepsilon_1}, \lambda_{21}^{\varepsilon_1}, \ldots, \lambda_{\varepsilon_1+1}^{\varepsilon_1}\}, \ldots, \Lambda_{\varepsilon_{p1}} = \{\lambda_1^{\varepsilon_p}, \lambda_{21}^{\varepsilon_p}, \ldots, \lambda_{\varepsilon_1+1}^{\varepsilon_p}\}$. It is assumed that each of the Λ_{ε_i} subsets with $\varepsilon_i + 1 = \sigma_i$ eigenvalues is also symmetric. The partitioning corresponds to the dimensions of the controllability subspaces defined by the $\varepsilon_i + 1$ indices. Clearly

$$\Lambda = \{\lambda_i,\ i \in n\} = \Lambda_{\varepsilon_1} \cup \Lambda_{\varepsilon_2} \cup \ldots \cup \Lambda_{\varepsilon_p} \tag{8.53}$$

Definition 8.1: For a given set of Λ and a system with controllability indices $\{\sigma_i = \varepsilon_i + 1,\ i \in p\}$, the ability to split Λ into symmetric subsets Λ_{ε_i} such that (8.53) holds true characterises a property referred to as *compatibility* of the Λ, with respect to the $\{\sigma_i,\ i \in p\}$ sets.

□

In the following, compatibility of the Λ, $\{\sigma_i,\ i \in p\}$ sets will be assumed.

Remark 8.10: Compatibility of the Λ, $\{\sigma_i,\ i \in p\}$ sets implies that the minimal decomposition of the state space implied by the minimal basis can lead to a real state feedback matrix. If compatibility is not valid, nonminimal decompositions will have to be dealt with, i.e. controllability subspaces of higher dimensions. This may be readily overcome but requires additional work going through the results characterising the possible dimensions of controllability subspaces [War., & Ech., 1], [Kar., 4].

□

STEP (2): Having assumed compatibility, the free parameters in the selection of eigenvectors are defined as

$$\begin{aligned} &h_1 = \ldots = h_{\sigma_1} = [1, \quad 0, \quad \ldots, \quad 0]' = \underline{e}_1 \\ &h_{\sigma_1+1} = \ldots = h_{\sigma_1+\sigma_2} = [0, \quad 1, \quad 0, \quad \ldots, \quad 0]' = \underline{e}_2 \\ &\vdots \\ &h_{\sigma_1+\ldots+\sigma_{p-1}+1} = \ldots = h_n = [0, \quad \ldots, \quad 0, \quad 1]' = \underline{e}_p \end{aligned}$$

STEP (3): For every ε_i, the $\sigma_i = \varepsilon_i + 1$ vectors are defined based on the common $h = [0, \ \ldots, \ 0, \ \underset{\leftarrow \ i \ \rightarrow}{1}, \ 0, \ \ldots, \ 0]'$ and the selected spectrum Λ_{ε_i} as

$$\underline{x}_j^{\varepsilon_i} = \overline{N}_{\varepsilon_i} \begin{bmatrix} 1 \\ \lambda_j^i \\ \vdots \\ \lambda_j^{i\varepsilon_i} \end{bmatrix}, j = 1, \ \ldots, \ \varepsilon_i + 1 \tag{8.54a}$$

and thus a set of vectors

$$\left[\underline{x}_1^{\varepsilon_i}, \ \ldots, \ \underline{x}_{\sigma_i}^{\varepsilon_i}\right] = \overline{N}_{\varepsilon_i} \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1^i & & \lambda_{\sigma_i}^i \\ \vdots & & \vdots \\ \lambda_1^{i\varepsilon_i} & \cdots & \lambda_{\sigma_i}^{i\varepsilon_i} \end{bmatrix} = \overline{N}_{\varepsilon_i} V(\Lambda_{\varepsilon_i}) \tag{8.54b}$$

can be derived, where $\overline{N}_{\varepsilon_i} \in \mathbb{R}^{n \times \sigma_i}$, and $V(\Lambda_{\varepsilon_i}) \in \mathbb{R}^{\sigma_i \times \sigma_i}$, and since it is assumed that the eigenvalues are distinct (for the sake of simplicity), the Vandermonde matrix has full rank. For the case of repeated eigenvalues, corresponding Jordan vectors can be defined by using derivatives of the $\underline{e}_{\varepsilon_i}(s) = [1, \quad s, \quad \ldots, \quad s^{\varepsilon_i}]$ vector evaluated at λ_i.

Proposition 8.7: For any given symmetric set Λ_{ε_i}, the set of vectors

$$X(\Lambda_{\varepsilon_i}) = \overline{N}_{\varepsilon_i} V(\Lambda_{\varepsilon_i}) = \overline{N}_{\varepsilon_i} V_{\varepsilon_i} \tag{8.55a}$$

is linearly independent. Furthermore, if the original set is a compatibly partitioning set as in (8.53), then the set of vectors

$$\begin{aligned} X(\Lambda) &= \left[X(\Lambda_{\varepsilon_1}), \quad \ldots, \quad X(\Lambda_{\varepsilon_p})\right] \\ &= \left[\overline{N}_{\varepsilon_1}, \quad \overline{N}_{\varepsilon_1}, \quad \ldots, \quad \overline{N}_{\varepsilon_p}\right] \begin{bmatrix} V_{\varepsilon_1} & & & 0 \\ & V_{\varepsilon_2} & & \\ & & \ddots & \\ 0 & & & V_{\varepsilon_p} \end{bmatrix} \end{aligned} \tag{8.55b}$$

is symmetric (pairwise complex conjugate) within each of the σ_i subsets and it is linearly independent.

STEP (4): For every Λ_{ε_i} set and with $\underline{h} = \underline{e}_i$ vector, it may be possible to define the input vectors $\underline{u}_j^{\varepsilon_i}$, $j = 1, \ldots, \varepsilon_1 + 1$ as follows. Firstly, it is necessary to express $\overline{D}(s)$ as

$$\overline{D}(s) = [\underline{d}_1(s), \quad \underline{d}_2(s), \quad \ldots, \quad \underline{d}_p(s)] \tag{8.56a}$$

where $\partial[\underline{d}_i(s)] = \varepsilon_i + 1 = \sigma_i$, $i \in p$. Then

$$\underline{u}_j^{\varepsilon_i} = \underline{d}_i(\lambda_j^i), \, j = 1, \; 2, \; \ldots, \; \sigma_i \tag{8.56b}$$

and for the set Λ_{ε_i}, a new set is defined

$$U\left(\Lambda_{\varepsilon_i}\right)=\left[\underline{u}_1^{\varepsilon_i}, \quad \ldots, \quad \underline{u}_{\sigma_i}^{\varepsilon_1}\right]=\left[\underline{d}_i\left(\lambda_1^i\right), \quad \ldots, \quad \underline{d}_i\left(\lambda_{\sigma_i}^i\right)\right] \tag{8.56c}$$

□

STEP (5): The state feedback matrix that assigns Λ as closed loop eigenvalues with $X(\Lambda)$ as the corresponding closed loop eigenvectors is then defined by

$$\begin{aligned} K_S &= \left[U\left(\Lambda_{\varepsilon_1}\right), \quad \ldots, \quad U\left(\Lambda_{\varepsilon_p}\right)\right]\left[X\left(\Lambda_{\varepsilon_1}\right), \quad \ldots, \quad X\left(\Lambda_{\varepsilon_p}\right)\right]^{-1} \\ &= U(\Lambda)X(\Lambda)^{-1} \end{aligned} \tag{8.57}$$

Remark 8.11: The construction of the frame $X(\Lambda)$ is based on the properties of minimal bases of matrix pencils and thus this theory is instrumental in defining all such families of eigenframes. The advantage of this construction is that it leads to maximal rank feedback and provides constructive means for shaping the properties of the eigenframe $X(\Lambda)$. Furthermore the selection of the $\underline{h}_i$ vectors for each of the subspaces of the decomposition is arbitrary and this expresses the p degrees of freedom in the eigenstructure assignment, which may be further explored to achieve additional properties of the eigenframe beyond the linear independence.

□

8.4 EXAMPLES

8.4.1 EXAMPLE 1

The equations of motion of a satellite of mass m in earth orbit are given by

$$\dot{\underline{x}}=\underline{f}\left(\underline{x}, \underline{u}\right)=\begin{bmatrix} \dot{r} \\ r\dot{\theta}\cos^2\phi+r\dot{\phi}^2-k/r^2+u_r m \\ \dot{\theta} \\ -2\dot{r}\,\dot{\theta}/r+2\dot{\theta}\dot{\phi}\sin\phi/\cos\phi+u_0/mr\cos\theta \\ \dot{\phi} \\ -\dot{\theta}^2\cos\phi\sin\phi-2\dot{r}\,\dot{\phi}/r+u_\phi/mr \end{bmatrix} \tag{8.58}$$

where the state vector $\underline{x} = [r, \ \dot{r}, \ \theta, \ \dot{\theta}, \ \phi, \ \dot{\phi}]^t$ represents the position and velocity polar co-ordinates. The control vector $\underline{u} = [u_r, \ u_\theta, \ u_\phi]$ represents the forces which may be applied by small rocket thrusters to position and control the satellite. The linearised sixth order set of equations defined in (8.58) can be split into two uncoupled subsets, one involving only the state variables $(r, \ \dot{r}, \ \theta, \ \dot{\theta})$ and control variables $(u_r, \ u_\theta)$ which describe the motion in the equatorial $\{r, \ \theta\}$ plane. The second subset describes the azimuthal variables $(\phi, \ \dot{\phi})$ and control u_ϕ. If the radius of the orbit, $r_0 = m = 1$, and the angular velocity ω_0 is constant, then the linearised equations describing the motion on the circular equatorial orbit are given by

$$\begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \tag{8.59}$$

The system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and the desired closed loop eigenvalues $\Lambda = [\lambda_1, \ \lambda_2, \ \lambda_3, \ \lambda_4]$. The first stage is to compute N and $B^\dagger$ in order to satisfy equation (8.41).

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B^\dagger = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From these, $sN - NA$ and $B^\dagger(sI - A)$ can be derived

$$sN - NA = \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & 0 & s & -1 \end{bmatrix}$$

$$B^{\dagger}(sI - A) = \begin{bmatrix} -3\omega_0^2 & s & 0 & -2\omega_0 \\ 0 & 2\omega_0 & 0 & s \end{bmatrix}$$

The computation of the pair $(\overline{N}(s), \overline{D}(s))$ is the same as computing a minimal basis for the right kernel of $[sI - A, \ -B]$. Thus from (8.41)

$$\overline{N}(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}, \quad \overline{D}(s) = \begin{bmatrix} -3\omega_0^2 + s^2 & -2\omega_0 s \\ 2\omega_0 s & s^2 \end{bmatrix}$$

So from (8.40)

$$[sI - A, \ \ -B]\begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \end{bmatrix} = 0 \Rightarrow$$

$$\left[\begin{array}{cccc|cc} s & -1 & 0 & 0 & 0 & 0 \\ -3\omega_0^2 & s & 0 & -2\omega_0 & -1 & 0 \\ 0 & 0 & s & -1 & 0 & 0 \\ 0 & 2\omega_0 & 0 & s & 0 & -1 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \\ \hline -3\omega_0^2 + s^2 & -2\omega_0 s \\ 2\omega_0 s & s^2 \end{array}\right] = 0$$

$\overline{N}(s)$ can be split into two columns, i.e. $[\underline{\overline{n}}_1(s), \ \underline{\overline{n}}_2(s)]$, which have degrees of $\varepsilon_1 = \varepsilon_2 = 1$ respectively. This leads to the formation of the $X(\Lambda)$ matrix, which is formed from $\overline{N}(s)$. The individual column degrees of $\overline{N}(s)$ contribute $\varepsilon_i + 1$ columns in $X(\Lambda)$, where s is substituted by the closed loop eigenvalues. For this example

$$X(\Lambda)=\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{array}\right]$$

$$\lambda_1 \neq \lambda_2 \quad \lambda_3 \neq \lambda_4$$

Similarly, $U(\Lambda)$ is derived from $\overline{D}(s)=\left[\underline{d}_1(s),\ \underline{d}_2(s)\right]$. The degree of each column of $\overline{D}(s)$ is $\sigma_i=\varepsilon_i+1$ respectively. Thus, for each column in $\overline{D}(s)$, σ_i columns are inputted into $U(\Lambda)$. In this case, the degrees of each of the columns of $\overline{D}(s)$ are both 2, thus

$$U(\Lambda)=\left[\begin{array}{cc|cc} -3\omega_0^2+\lambda_1^2 & -3\omega_0^2+\lambda_2^2 & -2\omega_0\lambda_3 & -2\omega_0\lambda_4 \\ 2\omega_0\lambda_1 & 2\omega_0\lambda_2 & \lambda_3^2 & \lambda_4^2 \end{array}\right]$$

Finally, from equation (8.57), the state feedback matrix K_S is

$$U(\Lambda)X(\Lambda)^{-1}=\left[\begin{array}{cc|cc} -3\omega_0^2+\lambda_1^2 & -3\omega_0^2+\lambda_2^2 & -2\omega_0\lambda_3 & -2\omega_0\lambda_4 \\ 2\omega_0\lambda_1 & 2\omega_0\lambda_2 & \lambda_3^2 & \lambda_4^2 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{array}\right]^{-1}$$

where the specified closed loop eigenvalues are $\Lambda=\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\}$. As discussed earlier in this thesis, it is desirable that the matrix of eigenvectors, $X(\Lambda)$, is as close to being orthogonal as possible. Therefore an optimisation routine had to be created in order to achieve this. Firstly one of the orthogonality indices described in Chapter 7 had to be chosen. For simplicity and ease of computation, the grammian method from Section 7.3.1 was used. For a matrix to be orthogonal, its grammian must be equal to 1, provided the original matrix has been normalised. If the grammian is 0, then the matrix is dependent and cannot be an eigenframe. This leads to the following problem definition

Problem 8.2: From the grammian, G, of the given matrix of eigenvectors, $X(\Lambda)$ as computed from equation (8.55), determine the closed loop eigenvalues so as to maximise G.

□

The MATLAB routine `eigopt1.m` was created to calculate the grammian of $X(\Lambda)$ for the above example. This was found to be

$$G=\left(\lambda_2^2-2\lambda_1\lambda_2+\lambda_1^2\right)*\left(\lambda_4^2-2\lambda_3\lambda_4+\lambda_3^2\right)/\left(\lambda_4^2+1\right)/\left(\lambda_3^2+1\right)/\left(1+\lambda_1^2\right)/\left(1+\lambda_2^2\right)$$

There are no commands in MATLAB that maximise functions, so in order to carry out the optimisation, it was necessary to find the minimum of $1/G$, which is equivalent to finding the maximum of G. This meant that for the function to be orthogonal, $1/G$ still had to tend to one, but now skewness was portrayed by values approaching infinity. The functions `dan1.m` and `funone.m` were created to carry out the optimisation, using the `fmincon.m` command. On execution, the optimum closed loop eigenvalues were found to be –20.027, -0.1, -30.0116 and –0.1. The corresponding value of $1/G$ was 1.0410.

The explicit formulation of the eigenframe used here allows the study of the optimal location of eigenvalues (in the stable region) which permits achievement of maximal orthogonality of the eigenframe.

8.4.2 EXAMPLE 2

For the second example, consider the pair

$$A=\left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which yield

$$sN - NA = \left[\begin{array}{cc|ccc} s & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & s & -1 \end{array}\right]$$

$$\overline{N}(s) = \left[\begin{array}{c|c} 1 & 0 \\ s & 0 \\ \hline 0 & 1 \\ 0 & s \\ 0 & s^2 \end{array}\right]$$

where $\varepsilon_1 = 1$ and $\varepsilon_2 = 2$, the matrix of eigenvectors is

$$X(\Lambda) = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda_3 & \lambda_4 & \lambda_5 \\ 0 & 0 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 \end{array}\right]$$

On execution of the altered MATLAB functions, the minimum value of $1/G$ was found to be 3.8052 for the closed loop eigenvalues –0.1, -65.8873, -192.6109, -1.0756 and -0.1000.

It is worth noting that both the examples above indicate how an optimal spectrum to achieve maximisation of orthogonality may be achieved for a given selection of minimal basis. The reverse problem where a spectrum is given, and it is desired to find the basis that guarantees the maximal orthogonality, is an open one.

8.5 Summary

This chapter introduced a framework for discussing frequency transmission and blocking problems, as well two new methods for assigning desired closed loop eigenvalues to a system, given the state matrix A and the input matrix B. The second method described makes use of minimal bases theory and matrix pencils in order to compute a state feedback matrix K_S. The methodology starts off by deriving the total behaviour under zero initial conditions of a minimal system. This is followed by the derivation of the right coprime matrix fraction description of the input-state transfer function. The computation of the $\left(\overline{N}(s),\quad \overline{D}(s)\right)$ pair is the same as computing a minimal basis for the right kernel of the input-state pencil $\left[sI-A,\quad -B\right]$. This equivalence eases the computational burden. From this MFD pair, the column degrees determine the number of allowable columns for the frames $X(\Lambda)$ and $U(\Lambda)$, from which the feedback matrix K_S is derived. The problem of optimal distribution of eigenvalues to guarantee stability and maximal orthogonality of the eigenframe has also been addressed using Grammian based criteria. The first method is based on the properties of characteristic bases of controllability subspaces and has the advantage that expresses the desirable closed loop eigenframe in terms of differences of the open and closed loop spectra. This permits the linking of robustness criteria (orthogonality of the frame) to pole mobility.

The general behaviour framework introduced here provides the means to also examine problems of the creation of Forney dynamical indices from controllability indices, or observability indices, in terms of problems of general model projection involving selection of the matrix B, or C, respectively. Such problems are generalisations of the squaring down problem.

9 CONCLUSIONS AND OPEN ISSUES

In this thesis, two problems have been addressed. Specifically, the problems of measuring the degree of controllability and observability with a view to use them eventually in optimising the placement of sensors and actuators on a system and that of eigenstructure assignment to guarantee robustness and a desired response were examined.

In Chapter 2, general theory related to control systems was introduced, together with their governing equations and their characteristics. The section dealing with the fundamentals of eigenvalues and eigenvectors paved the way for the thorough study of the eigenstructure assignment problem in Chapters 6, 7 and 8. The material covered in Chapter 5 is based on the solution to linear time invariant systems which was described in Section 2.3. There are several measures of controllability and observability studied in Chapter 4, and are all related to the necessary geometric and computational issues in the solution of linear systems of Section 2.4. Section 2.5 prepared the foundation for the new method of eigenstructure assignment presented in Chapter 8 by dealing with the relevant mathematical material.

Chapters 3, 4, and 5 dealt with the concept of controllability and observability, and determining ways to measure these important behavioural properties. Chapter 3 started off with a description of how control problems are formulated and considered the implications of how components (sensors and actuators) play a pivotal role in the evolution of the system from its design stage to its eventual construction. The notions of controllability and observability are applicable to all kinds of systems, but only large scale systems (particularly flexible structures) were considered in this thesis to keep within an engineering perspective. Thus the representation and properties of flexible structures were described in Section 3.4.

Here it was shown how state space representations of such systems could be derived from a set of differential equations describing a simple flexible structure. It was then shown how different state space modal descriptions could be obtained. It is from such descriptions that controllability (observability) and more significantly the degree of controllability (observability) can be determined. As an initial study into this, the mathematical means of examining controllability and observability using grammians was discussed in Section 3.5. This study showed how the participation of a state variable in a system can be gauged. The controllability and observability grammians presented however merely allow a binary determination of these properties, i.e. something is either controllable (observable) or not. Indeed if the solutions of equation (3.24) are positive definite, then a typical system described by (3.1) is both completely controllable and observable. But the question posed is just how close to uncontrollability and unobservability is the system, and it is a problem that grammians cannot address.

A great deal of emphasis is put on developing criteria and tools for possible sensor and actuator locations when testing and analysing the control behaviour of large scale systems. The locations of such components affect the dynamic response and closed loop behaviour. From this stems the problems of grammian assignment and the placement problem. The first involves finding the locations of the sensors/actuators of an open loop system in order to meet the specified observability/controllability requirements using the respective grammians. The second problem addresses the attempt to find a subset which has certain controllability and observability properties close to the original requirements of a given set of actuators and sensors. However, the drawback of such problems is that not every controllability and observability property can be obtained with a given set of actuator and sensor locations.

For control system purposes it is advantageous to have a tool for modifying or shaping the controllability and observability properties of a system. This can be achieved in two ways. One way is to determine proper sensor and/or actuator configurations. The other way is by modifying the system properties (such as introducing a feedback loop) which has been tackled in this thesis.

The study of existing measures of controllability in Chapter 4 showed the importance of this area of control design, especially in the development of control laws of large scale systems. Any uncontrollable system is in a way arbitrarily close to some controllable system, and conversely a controllable system may or may not be close to an uncontrollable one. It is possible to change the structural properties of an uncontrollable system (i.e. input-output structure selection) in order to make it controllable. But the question that has to be asked that despite impending changes to the system, just how far is it from being controllable or uncontrollable, and thus a measure of this "distance" is the crux of the investigations of Chapter 4. Section 4.3 consisted of a mathematical comparison between a selection of existing measures. It was demonstrated how the results of such measures differed for the two systems compared, one of which was controllable and the other uncontrollable.

In Section 4.4 a new measure was introduced that estimates the aggregate distance from minimality of a given state space description. The reason for this measure was to counter the fact that existing measures of controllability are functions of coordinate transformations, and may change as these are varied. Thus the new measure, based on Markov parameters, was shown to be invariant under state coordinate transformations. It was also shown how the degree of controllability varies with state feedback in Section 4.5, despite it being documented that controllability properties are invariant on undergoing the same control configuration. Also examined was how the structure of the state feedback matrix affected the degree of controllability. It was shown that with changes in rank and orthogonality, although controllability is maintained, the degree of controllability did vary according to the structure of the state feedback matrix used. Section 4.6 dealt with how controllability and observability properties can be determined from Plücker matrices of transfer function matrices. A new method of measuring controllability was presented at the end of Chapter 4 and was based on selecting an input structure in order to improve the conditioning of the controllability matrix. However, the method was limited to systems with two inputs, and an extension of this work is required to develop the algorithm for use on multi-input cases.

Chapter 5 looked at the link between input-output controllability and energy. This was investigated by studying the link between the singular values of the output controllability grammian and the energy required to transfer the output of a system from one position to another. The quantitative measure developed was for the energy needed for control action. It could have been developed further into an interaction measure between the inputs and the outputs, but that would have taken this research down a different route. The solution to a Lyapunov equation was used to solve the output controllability grammian at regular time intervals. The resulting singular values and condition numbers were then plotted and systems with different input parameters were compared in order to aid in the selection of input signals where the minimum expendable energy was a criteria of the control problem. The method described in Chapter 5 has the potential for use in applications where the conservation of energy is amongst the primary control objectives. Such applications are space stations where the rationing of energy is always a prioritised concern.

Chapters 6, 7 and 8 examined the area of eigenstructure assignment, which can only be carried out if the system is described by state space equations stemming from a set of physical variables. The link between controllability (observability) and the eigenstructure of a system was established in Section 6.2. Another issue connected with eigenstructure assignment is that of robustness to modelling errors and external disturbances and is a necessary property of a closed loop system. In view of the problems of stability, robustness, controllability and observability that arise in an open loop configuration, the studies carried out in the second half of the thesis examined the necessity to reassign, or shift, certain modes resulting in the reshaping of the eigenframe of a system under the implementation of some kind of feedback, leading to an improvement in the dynamical response and properties of the system.

As a result of the literature review into several methods of eigenstructure assignment in Section 6.4, it was evident that this area of control design could be split into four categories. Some papers made little or no attempt in actually assigning the eigenstructure, and merely studied the affect that closed loop eigenvalues and eigenvectors have on system performance. There has been a substantial effort by several authors in designing feedback schemes (both output and state feedback architectures) that assign a given set of open loop eigenvalues to a new set of closed loop modes. There have also been attempts in tackling this area of control design by parameterising a set of eigenvectors. There have been several issues that have however been neglected in the literature. The way in which the eigenstructure affects system properties such as controllability, observability, robustness and stability has not been sufficiently addressed. Also, the effect on system properties by the orthogonality (or skewness) of the matrix of eigenvectors, and how this can be measured has not been properly addressed.

The state feedback approach is based on the solutions of equation (6.49). Central to the method that uses output feedback is equation (6.50). The third procedure is the parametric approach, whereby either of the relationships for state or output feedback are used to formulate methods that make use of parametric equations to determine solutions for the respective feedback matrices and corresponding eigenvectors. In general, feedback affects the closed-loop characteristic polynomial of a system, and thus influences stability and system performance. The advantage of state feedback is that is presents the designer with extra freedom with which multivariable control system designs can be successfully applied. Yet, there are systems in where the states are not all measurable, and so the use of full state feedback is impractical. Therefore eigenstructure assignment via an output feedback scheme is preferred.

Fundamental to this research are the properties of controllability and observability. It is desirable to maintain these two properties when assigning the eigenstructure of a system. As discussed earlier, this is achieved by ensuring that the eigenvectors are in the left null space of the input matrix B and the right null space of the output matrix C for controllability and observability respectively. Therefore the elementary problem considered was that given the system matrices A and B and a set $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of stable, controllable eigenvalues, find an appropriate feedback matrix F, and an eigenvector matrix $\underline{u}$ such that a measure of the conditioning, or robustness, is minimised.

A prime design concern of eigenstructure assignment is that the solutions obtained take into account the minimisation of the sensitivity of the assigned eigenvalues to system modelling discrepancies and external disturbances. An additional problem of overshoots in the free response of a system also appears to have been overlooked by those addressing the eigenstructure assignment issue. The results of Chapter 7 showed that a degree of closed loop system robustness can be achieved by setting the eigenvector matrix to be as close to orthogonality as possible. Thus, an effective method of measuring orthogonality had to be used. The chapter took a more practical approach in the analysis of eigenvectors, departing from the theoretical background of Chapter 6. Methods examining the significance of the orthogonality of the eigenframe to closed loop system robustness were examined. It was shown that robustness was linked mainly to the nature of the eigenvector matrix. It was found that the degree of robustness could be obtained from the condition number and the norm of the eigenframe. Another way to quantify the robustness of a system was to examine the shape of the skewness or orthogonality of the eigenvector matrix. Through MATLAB demonstrations it was shown that the response of the state vector of asymptotically stable systems contained overshoots for certain initial conditions when the eigenvectors were not orthogonal to each other. This prompted research into the notion of strong stability, which is related to the lack of overshoots in the free response. It was shown that the natural way to parameterise the family of strongly stable matrices was by way of Sylvester's Theorem. In order to help combat state overshoots, i.e. to increase the level of robustness, a new measure for the distance of a matrix from orthogonality was proposed and demonstrated, based on singular values. This new measure was compared to existing techniques and proved to be just as reliable for matrices with real elements. So it was proposed that there is a link between overshoots in the free response of the state vector and the orthogonality of the eigenvector matrix. Software developed using MATLAB showed that this proposition was well founded.

In Chapter 8 an algebraic description of the total system behaviour was formulated which in turn led to the study of closed loop eigenvectors in a systematic way by using a parametric approach. An algebraic characterisation of the total input, state and output behaviour in an implicit formulation was stated based on properties of matrix fraction descriptions (MFDs), which led to a novel unification of the characterisation of poles and zeros based on input and output zeroing problems.

Explicit algebraic means for characterising the zero structure were provided as well as a new algebraic characterisation of the family of closed loop eigenvectors and related input and output directions. The derivation of a new method of eigenstructure assignment, given the state and input matrices, via state feedback ensued, using minimal basis theory, and was demonstrated via an example. The method described made use of minimal base theory and matrix pencils in order to compute a state feedback matrix. The derivation of the total behaviour under zero initial conditions of a minimal system was central to the methodology formulated, followed by expressing the right coprime matrix fraction description of the input-state transfer function. From this MFD expression, the column degrees were used to determine the number of allowable columns for the frames $X(\Lambda)$ and $U(\Lambda)$, from which the feedback matrix was derived. The problem of optimal distribution of eigenvalues to guarantee stability and maximal orthogonality of the eigenframe was also addressed using Grammian based criteria.

The two problems considered in this thesis, measures of controllability and observability and eigenstructure assignment, are invariably linked. These problems created subproblems, which were also addressed. Although not a measure of controllability, the input structure selection problem for a system with two inputs considered in Chapter 4, could be used to choose actuator locations to guarantee a certain level of controllability. As an extension to this subproblem, there is scope for developing the methodology for a multiple input system. The problem of energy utilisation discussed in Chapter 5 showed how such a performance criteria can be linked to output controllability. This could be developed further into an algorithm where the maximisation of controllability can be derived through the minimisation of energy utilisation. Controllability and observability properties are inherent in the eigenvector matrix of system state space equations. In addition to these properties, the question of closed loop robustness was also tackled.

As an extension to the work carried in this thesis, the problems and subproblems could all be linked in one unifying methodology. It may be possible to assign a set of open loop eigenvalues, which guarantee a set of eigenvectors that meet specific degrees of controllability, observability and robustness for a desired positioning of a prescribed number of actuators and sensors with a view to minimising the energy utilisation of the system. It is hoped that the methodologies covered in this thesis can be combined to develop one single algorithm that helps to meet a number of control performance criteria. What would be needed first is an optimisation methodology that can explore measures of controllability and observability, eigenstructure properties and for large scale problems, the structure of the underlying graph. Such a methodology has to use in an explicit form optimisation which has to be multiobjective. In the area of eigenstructure assignment the selection of the best stable spectrum as far as the maximisation of the conditioning of the eigenframe orthogonality remains open. The classical result that orthogonality of the eigenframe implies improvement in robustness has been established under the assumption of real eigenvalues. Developing state feedback algorithms for eigenvector assignment using criteria which are not only based on the orthogonality of the eigenframe, but also on the balancing of the degrees of controllability and observability is a new problem which is also left for further work. The new test for the degree of controllability and observability based on optimisation and the notion of almost zeros needs further expansion and linking with other measures (distances).

APPENDIX

Programs from Section 4.3.2

```
function mc1 = degcon1(a,b)
%DEGCON1 Calculates the minimum singular value
%       of the matrix [B AB A^2B ...]
%
%       Copyright (c) 1997 by CEC.

t = ctrb(a,b);
p = svd(t);
mc1 = min(p);

function mc2 = degcon2(a,b)
%DEGCON2 Calulates the minimum singular value
%       of the matrix [(lamda)I - A, B] for all
%       lamda.
%
%       Copyright (c) 1997 by CEC.

l = eig(a);
[m,n] = size(l);
q=[];
for i=1:m
    p=svd([l(i)*eye(m)-a,b]);
    q=[q,p];
end
t=min(q);
mc2=min(t);

function mc3 = degcon3(a,b)
% DEGCON3       This measure of controllability is
%          according to Tarokh.
%
%       Copyright (c) 1997 by CEC.

modes=eig(a)
[m,n]=size(a);
l=input('Which mode do you want to control:');
t=l*eye(m);
r=adjoint(t);
phi=r*b;
mc3=norm(phi, 'fro');

function mc4 = degcon4(a,b)
% DEGCON4       Calculates the minimum singular value of
%       the matrix toeplitz([B AB A^2B ...])
%
%       Copyright (c) 1997 by CEC.

cbm=ctrb(a,b);
toe=toeplitz(cbm);
s=svd(toe);
mc4=min(s);

function mo1 = degobs1(a,c)
%DEGOBS1 Calculates the minimum singular value
%       of the matrix [C; CA; CA^2 ...]
%
```

```
%     Copyright (c) 1997 by CEC.

u = obsv(a,c);
q = svd(u);
mo1 = min(q);

function mo2 = degobs2(a,c)
%DEGOBS2 Calulates the minimum singular value
%      of the matrix [(lamda)I - A; C] for all
%      lamda.
%
%     Copyright (c) 1997 by CEC.

l = eig(a);
[m,n] = size(l);
q=[];
for i=1:m
    p=svd([l(i)*eye(m)-a; c]);
    q=[q,p];
end
t=min(q);
mo2=min(t);

function mo3 = degobs3(a,c)
% DEGOBS3       This measure of observability is
%         according to Tarokh.
%
%     Copyright (c) 1997 by CEC.

modes=eig(a)
[m,n]=size(a);
l=input('Which mode do you want to control:');
t=l*eye(m);
r=adjoint(t);
phi=c*r;
mo3=norm(phi, 'fro');

function mo4 = degobs4(a,c)
% DEGOBS4       Calculates the minimum singular value of
%      the matrix toeplitz([C; CA; CA^2 ...])
%
%     Copyright (c) 1997 by CEC.

obm=obsv(a,c);
toe=toeplitz(obm);
t=svd(toe);
mo4=min(t);
```

Program from Section 4.5.3

```
% function [phi,phi_f]=moc5(a,b,l);
% phi returns the minimum of the norms of the rows
% of the V*B matrix, where V is the inverse eigenvetor
% matrix and B is the input matrix.
% phi_f returns the minimum of the norms of the rows
% of the V_F*B matrix, where V_F is the inverse eigenvector
% matrix of the state feedback system [A-BL] and L is the state
% feedback matrix.

function [phi,phi_f]=moc5(a,b,l)
```

```
% a is the state matrix (nxn)
% b is the input matrix (nxm)
% l is the state feedback matrix (mxn)

[u,d]=eig(a);
v=inv(u);
beta=v*b;
[m,n]=size(b);

q=[];
for i=1:m;
    t=norm(beta(i,:));
    q=[q;t];
end

f=[a-(b*l)];
[u_f,d_f]=eig(f);
v_f=inv(u_f);
beta_f=v_f*b;

q_f=[];
for i=1:m;
    t_f=norm(beta_f(i,:));
    q_f=[q_f;t_f];
end

phi=min(q);
phi_f=min(q_f);
```

Program for Section 4.6.5

```
clear all;
close all;

range=-1:0.05:1;
[re,im]=meshgrid(range);

[m,n]=size(re);
Z=zeros(n);

k=1;
for sig=range
    l=1;
    for ome=range
        %p=[sig    +    ome*i    +    1.1;sig*(sig+1)    -    ome.^(2)    +
ome*((2*sig)+1)*i];
        s=sig+ome*i;
        p=[s+1.1; s^2+s];
        q=conj(p);
        qt=transpose(q);
        phi = sqrt(qt*p);
        Z(l,k)=phi;
        l=l+1;
    end
    k=k+1;
end
map=[0 0 0];
figure
mesh(im,re,Z)
colormap(map);
```

```
figure
waterfall(im',re',Z')
colormap(map);
```

Program from Section 4.7.3

```
clear
load c5a;
b1=b(:,1);
b2=b(:,2);

range=0.1:0.25:5.0;

q1=ctrb(a,b1);
q2=ctrb(a,b2);

[u1,u2]=meshgrid(range);
[m,n]=size(u1);
Q=zeros(n);

i=1;
for ii=range
    j=1;
        for k=range
            y=1/cond((ii*q1)+(k*q2));
            Q(j,i)=y;
            j=j+1;
        end
    i=i+1;
end

for i=1:n
    for j=1:n
        if Q(i,j)==inf %| Q(i,j)>1000
           Q(i,j)=NaN;
        end
    end
end

mesh(u1,u2,Q)
grid
ylabel('u1')
xlabel('u2')
zlabel('Condition Number')
```

Program from Section 5.3.1

```
% outcon.m calculates the output controllability Grammian of a
system.
% The singular values of the output controllability Grammian of
the
% system are indicators of the output assignability of the system.
%
stem1=0;
stem2=0;
sou=input('Will the data be entered via the keyboard or from an m-
file, ''k'' or ''m''?','s');
if sou=='k'
   A=input('The system matrix A is:')
   B=input('The input matrix B is:')
```

```
   C=input('The output matrix C is:')
   tff=input('The final time tff is:')
else
   disp('please specify the m-file which contains the data')
   inn=input('','s');
   inn=['load ',inn];
   eval(inn)
   A
   B
   C
   tff
end


X= lyap(A, B*B')
sing=[];

for t=0:(tff/100):tff
sing=[sing,svd(C*X*C'-C*expm(t*A)*X*expm(t*A')*C')];
end

%[r,c]=size(B);

t=0:(tff/100):tff;

ran=[0,0,1];

while (ran(3)>0.5)

ran=input('Input range of singular values: ');
for i=ran(1):ran(2)
plot(t,1+sing(i,:))
hold on
end
title('singular values of output controllability Grammian')
xlabel('Singular Values  v  Time')
hold off

end
```

Program for Section 7.2.2

```
u=input('Please enter eigenvector matrix:')
v=inv(u);
l=input('Please enter eigenvalues in column matrix form:')
l=diag(l);
L=sym(l);
t=input('Specify time interval:')
%
x0=input('Please enter initial conditions:')
[n,m]=size(u); %symbollic size of A
%
E=zeros(n,n);
for i=1:n
   E=sym(E,i,i, 'exp(eval(sym(L,j,j)))')
end
El=zeros(n,n)
for j=1:n
   El(j,j)=eval(sym(E,j,j))
end
```

```
nxt=[];
for t=0:0.2:10;
   xt=u*(El^t)*v*x0;
   q=(norm(xt, 'fro'))-1;
   nxt=[nxt,q];
end

t=input('Respcify time interval:')
plot(t,nxt)
xlabel('Time (seconds)')
ylabel('||X(t)||-1')
```

Programs for Section 7.4.3

```
function x = orthgtst(u)

[m,n]=size(u);

q=[];
for i = 1:n;
   v=norm(u(:,i));
   q=[q,v];
end

[s,t]=size(q);

k=[];
for j=1:t;
   l=u(:,j)/q(j);
   k=[k,l];
end

z=k*transpose(k);
x=det(z);

function x = orthtest(u)

[m,n]=size(u);

q=[];
for i = 1:n;
   v=norm(u(:,i));
   q=[q,v];
end
[s,t]=size(q);

k=[];
for j=1:t;
   l=u(:,j)/q(j);
   k=[k,l];
end

s=svd(k);
x=(2*n)-2*(sum(s))
```

Program for Section 8.4.1

```
clear all;
%X=sym('[0,0,c,d;1,1,0,0;a,b,0,0;0,0,-1,-1]')
X=sym('[1,1,0,0;a,b,0,0;0,0,1,1;0,0,c,d]')
%
c1=X(:,1)
dpc1=transpose(c1)*c1
sqdpc1=sqrt(dpc1)
nc1=c1/sqdpc1
%
c2=X(:,2)
dpc2=transpose(c2)*c2
sqdpc2=sqrt(dpc2)
nc2=c2/sqdpc2
%
c3=X(:,3)
dpc3=transpose(c3)*c3
sqdpc3=sqrt(dpc3)
nc3=c3/sqdpc3
%
c4=X(:,4)
dpc4=transpose(c4)*c4
sqdpc4=sqrt(dpc4)
nc4=c4/sqdpc4
%
NX=[nc1,nc2,nc3,nc4]
TNX=transpose(NX)
GX=NX*TNX
G=det(GX)
```

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