# On the Hochschild cohomology of blocks of finite group algebras

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# Declaration

I, William Murphy confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract. This thesis sets out to investigate the first Hochschild cohomology of finite group algebras and their blocks, as well as of twisted group algebras. We employ a range of methods and techniques to calculate the structure of some explicit examples, as well as develop the general theory. Our first main result, found in Chapter 3, is actually an alternative proof of our own pre-existing result that a certain 9-dimensional algebra does not arise as the basic algebra of a block; this time round we use Hochschild cohomology as a key ingredient. Our second main result, in Chapter 4, concerns the Lie algebra structure of the first Hochschild cohomology of the Mathieu groups, and in particular we show that these Lie algebras are nontrivial for blocks with a nontrivial defect group. Our third and final main result in Chapter 5 concerns the first Hochschild cohomology of twisted group algebras, and we show that their Lie algebras are also nontrivial for the twisted group algebras of the finite simple groups.

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# Chapter 1

# Introduction

Given a finite group $G$, a lot can be said in general about the ordinary representation theory of $G$, that is, the theory over a field such as $\mathbb{C}$. In some sense, a field of prime characteristic $p$, such as $\mathbb{F}_p$, is much easier to understand than $\mathbb{C}$, and arithmetic, calculations and computations in $\mathbb{F}_p$ are hugely simplified in comparison. In contrast to this, if $p$ happens to divide the order of $G$, then we are thrown into the realm of the modular representation theory of $G$ which is undeniably far more complex than its ordinary counterpart. It is not without reason that the theory of algebras with *wild representation type* is named so.

Within the staggeringly vast subspace of mathematics that makes up modular representation theory, there is a corner occupied by those who, like the author, are interested in Hochschild cohomology. This cohomology theory for associative, unital algebras plays a crucial role in the classification of different algebraic structures, not least the group algebras. It is used to determine relationships between the group algebras, the underlying groups themselves, and the *block structure* of the group algebra: its decomposition into indecomposable 2-sided ideals.

A guiding philosophy of this thesis is the desire to learn more about that interaction between the Hochschild cohomology of a group algebra and that of its blocks. To that end, in this work we make thorough investigation of the Hochschild cohomology of a number of group algebras and their blocks, including group algebras over explicit groups such as the Mathieu groups and some small order semi-direct products. In addition, we delve into the Hochschild cohomology of the (twisted) group algebras over the finite simple groups.

This thesis is structured as follows. In Chapter 2 we provide the necessary background material to dive into the theory behind the Hochschild cohomology of blocks of finite groups. We give a high-paced overview of some of the basic results on the representation theory of group algebras and their blocks, a detour into the required category theory and homological algebra, and then, of course, define our main object of study, Hochschild cohomology. We take the time to look in detail at this, ensuring the reader is armed with an expository example (this continues throughout the entire thesis).

In Chapter 3 we showcase some of the power of Hochschild cohomology. We do this by providing an alternative proof of the work done by Linckelmann and the author in [54], this time using Hochschild cohomology as a key ingredient. The work done there rules out the possibility of a certain 9-dimensional algebra arising as a basic algebra of a block of some finite group.

The work done in Chapter 4 is the author's own work, building on the paper [60]. There, we look at the first Hochschild cohomology of the blocks of the Mathieu groups, providing a thorough description of them as Lie algebras. This work also showcases the theory of blocks with a cyclic defect group, develops many results that aid with finding the structure of Hochschild cohomology, and brings in a powerful tool to compute the dimension of the Hochschild cohomology of a particularly tricky block.

Finally, in Chapter 5, we make a detailed investigation of how the first Hochschild cohomology applies to twisted group algebras. In particular, we embellish the author's work done in [61], showing the non-triviality of the first Hochschild cohomology of the twisted finite simple group algebras. Also in that chapter, we conclude our collection of expository examples on the first Hochschild cohomology of some explicit groups and blocks, this time with a twisted group algebra flavour.

# Chapter 2

# Background definitions and preliminary results

## 2.1 Finite group algebras and their blocks

Throughout this thesis, $k$ will denote a field, though we remark that some results hold more generally on replacing $k$ with a ring. We will also fix throughout that $G$ and $H$ denote finite groups, unless otherwise stated. Our sources for the background results of this chapter are primarily the works of Linckelmann [52, 53], and further details may be found there.

A *$k$-algebra* $A$ is a $k$-vector space with a notion of multiplication, that is, a $k$-bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, that is, unital, associative and distributes over the vector space addition. Explicitly, there is a unit element $1_A \in A$ such that $1_A a = a = a1_A$ for all $a \in A$, and for all $a, b, c \in A$ we have that $(ab)c = a(bc)$, $a(b+c) = ab+ac$ and $(b+c)a = ba+bc$. We will sometimes simply call $A$ an algebra, if $k$ is clear from the context.

An algebra $A$ is called *finite-dimensional* if it is a finite-dimensional vector space over $k$, and *infinite-dimensional* otherwise. We remark that $A$ will "most often" be a finite-dimensional $k$-algebra, though we will also encounter infinite-dimensional algebras: this will always be made clear.

**Definition 2.1.1.** The *group algebra $kG$* is the $k$-algebra whose $k$-basis is indexed by $G$ itself so that all elements of $kG$ are formal linear combinations of elements of $G$ with scalars from $k$, of the form $\sum_{g\in G} \lambda_g g$ for some $\lambda_g \in k$. The sum and the scalar multiplication in $kG$ are performed component-wise, and for $x = \sum_{g\in G} \lambda_g g, y = \sum_{h\in G} \mu_h h \in kG$, the product is

$$xy = \sum_{l\in G} \left( \sum_{gh=l} \lambda_g \mu_h \right) l.$$

Since we are only interested in $G$ being a finite group, one sees that the group algebra $kG$ is finite-dimensional. We will also see a generalisation of group algebras, similarly defined but with multiplication *twisted by a scalar*, and will encounter more examples of $k$-algebras throughout.

**Example 2.1.2.** The field $k$ and more generally the set $M_n(k)$ of $n \times n$ matrices with entries in $k$, is a $k$-algebra. Here, the addition and scalar multiplication are given entry-wise, the multiplication is matrix multiplication, and we note that if given a vector space $V$ over $k$ of dimension $n$, we have $M_n(k) \cong \mathrm{End}_k(V)$, the $k$-algebra of all $k$-linear maps on $V$ with point-wise addition and multiplication given by composition of maps.

The *polynomial ring* $k[x_1, x_2, \ldots, x_n]$ in $n$ commuting indeterminates is an infinite-dimensional $k$-algebra. The *truncated polynomial ring*

$$k[x_1, x_2, \ldots, x_n] \Big/ (f_1, f_2, \ldots, f_m),$$

also forms a $k$-algebra, where the $f_i$ are polynomials in $k[x_1, x_2, \ldots, x_n]$ that we identify with 0. This algebra may be finite- or infinite-dimensional, depending on the $f_i$.

Let $A, B$ be algebras. An *algebra homomorphism* is a $k$-linear map $A \to B$, that respects algebra multiplication and preserves the unit. A *k-subalgebra* of $A$ is a $k$-subspace of $A$ that is also an algebra, and in particular when we have a subgroup $H \subseteq G$ then it is the case that $kH$ is a $k$-subalgebra of $kG$. A *(left/right) ideal* of $A$ is a $k$-subspace of $A$ that is also a (left/right) ideal of $A$ as a ring. The three main isomorphism theorems for algebraic structures of importance also hold for $k$-algebras: we do not repeat them here.

The *opposite* algebra of $A$, denoted $A^{\mathrm{op}}$, is the $k$-algebra with underlying vector space equal to that of $A$, but with an opposite multiplication defined: if "$\cdot_{\mathrm{op}}$" the multiplication in $A^{\mathrm{op}}$, then $a \cdot_{\mathrm{op}} b = ba$ for all $a, b \in A^{\mathrm{op}}$. The *direct product* of $A$ and $B$, $A \times B$, is the $k$-algebra given by the vector space equal to Cartesian product of $A$ and $B$, with component-wise multiplication. The centre, $Z(A)$, consists of the elements of $A$ that commute with every element of $A$.

A *left A-module* $M$ is a $k$-vector space equipped with a $k$-bilinear action $A \times M \to M$, $(a, m) \mapsto a \cdot m$ such that $1_A \cdot m = m$ and $(ab) \cdot m = a \cdot (b \cdot m)$ for all $a, b \in A$ and $m \in M$. The action of $A$ on $M$ will always be denoted by "$\cdot$" unless it is clear from the context. A *right A-module* is defined analogously, with action on the right, and an *(A, B)-bimodule* is a $k$-vector space $M$ which is simultaneously a left $A$-module and a right $B$-module, that satisfies $(a \cdot_A m) \cdot_B b = a \cdot_A (m \cdot_B b)$ for $\cdot_A, \cdot_B$ the actions of $A$ and $B$ respectively, and for all $a \in A, b \in B$ and $m \in M$. Note that a left $A$-module is equivalent to a right $A^{\mathrm{op}}$-module. Throughout this thesis, unless otherwise stated, all modules will be left modules.

Let $V$ be an $A$-module, $U$ be a right $A$-module, and define $M$ to be the $k$-vector space with basis indexed by the elements of $U$ and $V$, denoted $\{u \otimes v \mid u \in U, v \in V\}$. Let $I \subseteq M$ be the subspace generated by all linear combinations of symbols of the form $u \cdot a \otimes v - u \otimes a \cdot v$, $(u + u') \otimes v - u \otimes v - u' \otimes v$, $u \otimes (v + v') - u \otimes v - u \otimes v'$, $\lambda(u \otimes v) - (\lambda u) \otimes v$ and $\lambda(u \otimes v) - u \otimes (\lambda v)$, for all $u, u' \in U$, $v, v' \in V$, $a \in A$ and $\lambda \in k$. Then we define the *tensor product of A-modules* $U$ and $V$ as the quotient space $U \otimes_A V := M/I$. The tensor product is unique up to unique isomorphism (hence we refer to it as "the" tensor product) and is generated as a vector space by the *elementary tensors*, that is, those of form $u \otimes v$ as $u, v$ run over $U, V$ respectively. Given $k$-algebras $B$ and $C$, if $U$ is an $(A, B)$-bimodule and $V$ is a $(B, C)$-bimodule, then the tensor product $U \otimes_B V$ has a unique $(A, C)$-bimodule structure given by $a \cdot (u \otimes v) \cdot c = (a \cdot_A u) \otimes (v \cdot_C c)$ for all $a \in A, c \in C, u \in U$ and $v \in V$.

Given $k$-algebras $A$ and $B$, there is a unique $k$-algebra structure defined on $A \otimes_k B$, given by $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$ for all $a, a' \in A$ and all $b, b' \in B$. What is more, if $U$ is an $A$-module and $V$ is a $B$-module, then there is a unique $A \otimes_k B$-module structure defined on $U \otimes_k V$ given by $(a \otimes b) \cdot (u \otimes v) = (a \cdot_A u) \otimes (b \cdot_B v)$ for all $a, b, u, v$ in $A, B, U, V$ respectively. Note that by this, an $(A, B)$-bimodule is equivalent to an $A \otimes_k B^{\mathrm{op}}$-module.

In the case that $A = kG$ and $B = kH$ for groups $H$ and $G$, we have a $k$-algebra isomorphism $kG \otimes_k kH \cong k(G \times H)$, the group algebra over the direct product of groups $G \times H$, given by the map sending $g \otimes h \mapsto (g, h)$ for all $g \in G$ and $h \in H$.

Suppose now that $B \subseteq A$ is a subalgebra, and that $U$ is a $B$-module. Then we define the *induced module* of $U$ from $B$ to $A$ as the $A$-module $\mathrm{Ind}_B^A(U) = A \otimes_B U$, with action given by $a \cdot (a' \otimes u) = aa' \otimes u$ for all $a, a' \in A$ and $u \in U$. When $A = kG$ and $B = kH$ for some subgroup $H$ of $G$, we write $\mathrm{Ind}_H^G(U)$.

Dually, if $B \subseteq A$ as above and $V$ is an $A$-module, we have the *restricted module*, that is the $B$-module $\mathrm{Res}_B^A(V) = {}_BA \otimes_A V$, where ${}_BA$ is simply $A$ viewed as a $(B, A)$-bimodule. Note that this is module-isomorphic to the $B$-module equal to $V$ as a vector space and with action from $B$ given by the restricted from the action of $A$. As before, if $A = kG$ and $B = kH$ for some subgroup $H$ of $G$ then we write $\mathrm{Res}_H^G(V)$.

We find at this point it is appropriate to remind ourselves a *representation* of $A$ is a $k$-algebra homomorphism $\varphi : A \to \mathrm{End}_k(M)$ for some vector space $M$; whence, defining an action of $A$ on $M$ by $a \cdot m = \varphi(a)(m)$ one observes the 1-to-1 correspondence between representations of $A$ and modules of $A$. In particular, there is a bijective correspondence

$$\{kG\text{-modules}\} \longleftrightarrow \{\text{representations of } kG\} \longleftrightarrow \{\text{representations of } G \text{ over } k\}.$$

Let $M$ be an $A$-module. Then an $A$-submodule of $M$ is a module $N \subseteq M$ such that $a \cdot N \subseteq N$ for all $a \in A$. Given $A$-modules $M$ and $N$, an *$A$-module homomorphism* is an $A$-linear map $M \to N$, and as before we have the three main isomorphism theorems for modules, that we do not state here. Note that a (left/right) ideal of $A$ is the same as a (left/right) $A$-submodule of the *regular (left/right) module* $A$, that is, the $A$-module $A$ with action given by multiplication in $A$.

An $A$-module $M$ is called *finitely generated* if there is some finite subset $S \subseteq M$ such that for all $m \in M$, $m = \sum_{s \in S} a_s \cdot s$ for some $a_s \in A$. A module $M$ is called *free* if it is isomorphic as an $A$-module to a direct sum of (possibly infinitely many) copies of $A$, and *free of rank $n$* for some integer $n$, if $M \cong A^{\oplus n}$ as $A$-modules. Note that we are restricting our attention to the case that $k$ is a field, hence any $A$-module is a $k$-vector space, so that if $A = k$ then any $A$-module is automatically free. On the other hand for more general $A$, not all modules will necessarily be free.

**Remark 2.1.3.** Recall that in practice we will be most interested in finite-dimensional $k$-algebras. If $A$ is such an algebra and $M$ an $A$-module, then $M$ is finite-dimensional as a $k$-vector space if and only if it is finitely generated as an $A$-module. One direction of this equivalence is immediate, for a $k$-basis of $M$ is automatically a generating set for $M$ as an $A$-module. To see that the converse holds, note that a finitely generated module is always (isomorphic to) a quotient of a free module of finite rank.

As we will see, the following definition will be called upon multiple times throughout this thesis, and is an important object of study in the representation theory of algebras, whence its formal definition status.

**Definition 2.1.4.** Let $A$ be a $k$-algebra. An $A$-module $P$ is called *projective* if there is some free $A$-module $M$ such that $M = P \oplus Q$ for some $A$-module $Q$.

Note that in the definition, $Q$ is necessarily also a projective $A$-module, and that every free module is projective (by setting $Q = 0$). We remark again that since $k$ is a field, $k$-modules are simply $k$-vector spaces, and therefore every $k$-module is free; in particular, every $k$-module is projective. When $k$ is more general - such as a commutative ring - a projective $k$-module need not necessarily be free.

We have a notion dual to that of a projective module, an *injective* $A$-module: this is a module, $I$, such that for any injective $A$-module homomorphism $\iota : U \to V$ and any $A$-module homomorphism $\varphi : U \to I$, there exists an $A$-module homomorphism $\psi : V \to I$ such that $\psi \circ \iota = \varphi$. Taken at face value it is not clear that this definition is dual (in the categorical sense) to that of a projective module. On the other hand, once we have encountered some more category theory this will become clear: in particular, referring to Definition 2.2.2 (iv) and Proposition 2.2.3 in the sequel will make the connection explicit.

A non-zero $A$-module $M$ is called *indecomposable* if it cannot be written as a direct sum of $A$-modules, $M = U \oplus V$, such that neither $U$ nor $V$ are zero. A non-zero $A$-module $S$ is called *simple* if $S$ contains no $A$-submodules other than $\{0\}$ and $S$. Note that all simple modules are by default indecomposable, though the converse does not hold. A non-zero $A$-module $U$ is called *semisimple* if $U$ may be written as a direct sum of the simple $A$-submodules of $U$.

We say that $A$ itself is *simple* if its only ideals are $\{0\}$ and $A$, and *semisimple* if it is semisimple as an $A$-module. We say that $A$ is *indecomposable as a $k$-algebra* if it cannot be written as a direct product of two $k$-algebras, and we note that this is precisely the case when $A$ is indecomposable as an $(A, A)$-bimodule. Simplicity, semi-simplicity and indecomposability are properties of $A$ and its modules that we are highly interested in: they provide us with the basic components that form the more complicated structure of an arbitrary algebra and its modules.

Our first formal result concerns a situation whereby indecomposable algebras arise.

**Lemma 2.1.5** ([52], Corollary 1.11.4)**.** *Suppose that* $\mathrm{char}(k) = p > 0$ *and that $G$ is a $p$-group. Then the group algebra $kG$ is indecomposable.*

The *Grothendieck group* of a $k$-algebra $A$ is the abelian group generated by the set of isomorphism classes $[U]$ of finitely generated $A$-modules $U$, subject to the relations $[U] + [W] - [V] = 0$ precisely when there is a short exact sequence of $A$-modules $0 \to U \to V \to W \to 0$, and we denote this group by $\mathcal{R}(A)$. If $A$ is finite-dimensional then $\mathcal{R}(A)$ is a finitely generated abelian group with basis equal to the set of isomorphism classes of simple $A$-modules. Thus, if $U$ is a finitely generated $A$-module for $A$ finite-dimensional, then $[U] = \sum_S d(U, S) \cdot [S]$ where $S$ runs over a set of representatives of the isomorphism classes of simple $A$-modules and where $d(U, S)$ is the number of composition factors isomorphic to $S$ in a composition series of $U$. Whence the images in $\mathcal{R}(A)$ of two finitely generated $A$-modules coincide if and only if they have equivalent composition series.

Our next few definitions and results concern a certain ideal of $A$, the *Jacobson radical of $A$*, which is a hugely important tool in the classification of algebras, as it points us in the direction of where to look for simple $A$-modules. Let $M$ be an $A$-module. The *annihilator of $M$* is the ideal in $A$, $\mathrm{Ann}_A(M) = \{a \in A \mid a \cdot m = 0 \text{ for all } m \in M\}$. Given this, we define the *Jacobson radical of $A$* as the ideal

$$J(A) = \bigcap_{\substack{S \text{ a simple} \\ A\text{-module}}} \mathrm{Ann}_A(S) = \{a \in A \mid a \cdot S = 0 \text{ for all simple } A\text{-modules } S\}.$$

If $A$ is finite-dimensional, then the Jacobson radical has an alternative characterisation as the unique maximal *nilpotent ideal* of $A$, that is, the unique ideal of $A$ that is maximal with respect to the property that $J(A)^n = \{0\}$ for some integer $n$. In addition, it is characterised as the intersection of all maximal left ideals of $A$, or alternatively the intersection of all maximal right ideals of $A$.

Let $A$ be finite-dimensional, $I$ be an ideal of $A$ and $M$ an $A$-module. Denote by $I \cdot M$ the $A$-module consisting of all finite sums of elements of the form $a \cdot m$ for all $a \in I$ and all $m \in M$. Then $M$ is semisimple if and only if the product $J(A) \cdot M = \{0\}$, whence $A/J(A)$ is always semisimple, and in fact the largest semisimple quotient of $A$. From this, we see that $J(A) = \{0\}$ if and only if $A$ itself is semisimple.

We have a complete description of finite-dimensional semisimple algebras thanks to Wedderburn.

**Theorem 2.1.6** (Wedderburn, [73]). *Suppose that $k$ is algebraically closed and let $A$ be a finite-dimensional semisimple $k$-algebra. Let $\{S_i \mid 1 \leq i \leq m\}$ a complete set of representatives of the isomorphism classes of simple $A$-modules and let $n_i$ be the unique positive integer such that $A \cong \bigoplus_{i=1}^m S_i^{\oplus n_i}$ as an $A$-module. Then there is an isomorphism of $k$-algebras*

$$A \cong \prod_{i=1}^{m} M_{n_i}(k).$$

Thus, a semisimple algebra can always be written as a product of matrix algebras, which themselves are easy to describe: a matrix algebra is itself a simple algebra. As we are particularly interested in the structure of group algebras, the following result of Maschke cannot be understated.

**Theorem 2.1.7** (Maschke). *The group algebra $kG$ is a semisimple algebra if and only if* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) = p$ *does not divide the order of $G$.*

Thus the representation theory of $G$ over a field $k$ as in Maschke's theorem is completely understood: this is the *ordinary representation theory* of $G$. We are interested, therefore, in the case when $\mathrm{char}(k) = p$ divides $|G|$, the *modular representation theory* of $G$. As we cannot hope for $kG$ to always be semisimple, we broaden our scope somewhat, to consider indecomposability.

This brings us, at last, to our main objects of study: the *blocks* of a $k$-algebra, which we reward with a definition.

**Definition 2.1.8.** Let $A$ be a $k$-algebra. A *block* of $A$ is an indecomposable direct factor of $A$, or, equivalently, an indecomposable, 2-sided ideal, that is a direct summand of $A$ as an $(A, A)$-module.

When $k$ is of prime characteristic $p$, we sometimes refer to the *$p$-blocks of $A$*. When $A = kG$ for some group $G$, and $k$ is clear from the context, we may also refer to these as the *blocks of $G$*.

There is another, equivalent definition of a block of $A$, for which we require a little more terminology. An *idempotent* of $A$ is a non-zero element $b \in A$ such that $b^2 = b$. Two elements $a, b \in A$ are called *orthogonal* if $ab = ba = 0$, and a *primitive idempotent* is one which cannot be written as the sum of two orthogonal idempotents. A *block idempotent* is a primitive idempotent of the centre of $A$, $Z(A)$, which allows us to redefine a block of $A$.

**Proposition 2.1.9** ([19, 2.1]). *Let $A$ and $B$ be a $k$-algebras. Then the following are equivalent.*

*(i) The algebra $B$ is a block of $A$.*

*(ii) The algebra $B = Ab = \{ab \mid a \in A\}$ for some idempotent $b$, primitive in $Z(A)$.*

*Proof.* First assume (i) holds, and write $A = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ where each $B_i$ is an indecomposable 2-sided ideal of $A$, the blocks of $A$, so that $B = B_j$ for some $j$. Now consider the decomposition of $1_A$ in this direct sum, $1 = b_1 + b_2 + \cdots + b_n$ where each $b_i \in B_i$. Since each $B_i$ is a 2-sided ideal, we have that $b_i b_j \in B_i \cap B_j$ for all $i, j$, and in particular this intersection is zero when $i \neq j$. Thus $b_i b_j = 0$ if $i \neq j$, and the $b_i$ are all orthogonal. Whence

$$\sum_{i=1}^{n} b_i = 1 = 1^2 = \left(\sum_{i=1}^{n} b_i\right)^2 = \sum_{i=1}^{n} b_i^2,$$

which shows that $b_i^2 = b_i$; the $b_i$ are idempotents for all $i$. What is more, for $a \in A$, we have

$$a \cdot \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (ab_i) = \sum_{i=1}^{n} (b_i a) = \sum_{i=1}^{n} b_i \cdot a,$$

since $a = a \cdot 1 = 1 \cdot a$. Noting that each $B_i$ is a 2-sided ideal, so $ab_i, b_i a \in B_i$, this shows that $ab_i$ and $b_i a$ are both the projection of $a$ into $B_i$. Whence $ab_i = b_i a$ and the $b_i \in Z(A)$ for all $i$. Next, we have

$$A = A \cdot 1 = A \cdot \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} Ab_i,$$

however each $Ab_i \subseteq B_i$, so that in fact $A = \bigoplus_{i=1}^{n} Ab_i$ and $Ab_i = B_i$. Finally choose a $b_i$ and suppose that $b_i = a + a'$ for some orthogonal central idempotents $a$ and $a'$. This gives $B_i = Ab_i = Aa + Aa'$. If $x \in Aa \cap Aa'$, then $x = ca = da'$ for some $c, d \in A$ however $xa = ca^2 = ca = x$ whereas $xa = da'a = 0$ so that $x = 0$ and $B_i = Aa \oplus Aa'$. Since the $B_i$ are indecomposable one of $Aa$ or $Aa'$ is zero, and consequently one of $a$ or $a'$ is zero, proving that the $b_i$ are primitive. Thus we have shown that any indecomposable 2-sided ideal of $A$ is of the form $Ab$ for $b$ a primitive central idempotent in $A$, whence (i) implies Definition (ii).

Now assume that (ii) holds. It is clear that $Ab$ is a 2-sided ideal of $A$: $a(cb) = (ac)b \in Ab$ and $(cb)a = c(ba) = c(ab) = (ca)b \in Ab$ for all $a, c \in A$. We now show that $1 - b$ is a central idempotent in $A$, orthogonal to $b$:

$$(1-b)^2 = 1 - 2b + b^2 = 1 - b,$$

showing that $1 - b$ is an idempotent,

$$a(1-b) = a - ab = a - ba = (1-b)a,$$

showing that $1 - b \in Z(A)$, and

$$b(1-b) = b - b^2 = b - b = 0,$$

showing that $b$ and $1 - b$ are orthogonal. Since $1 = b + (1 - b)$ we have $A = Ab \oplus A(1-b)$ as $(A, A)$-bimodules, and $A = Ab \times A(1-b)$ as $k$-algebras, so that $Ab$ is both a direct factor of the $k$-algebra $A$ and 2-sided ideal summand of the regular $A$-module $A$. It remains to show that $Ab$ is indecomposable though this results directly from the primitivity of $b$: for otherwise, writing $Ab = U \oplus V$, the projection of $b$ onto each of $U$ leads to $b \in U$, whence $U = A$ and $V = \{0\}$. This completes the proof. □

In the proof, the decomposition of $1_A$ into a sum of pairwise orthogonal block idempotents is known as a *primitive decomposition* of $1_A$.

The next result is a good indication of the utility of the Jacobson radical, and highlights how important the primitive idempotents of $A$ are when determining the structure of its modules.

**Theorem 2.1.10** ([52, Corollary 4.6.9]). *Let $A$ be finite-dimensional. Then every simple $A$-module $S$ is isomorphic to $Ai/J(A)i$ for some primitive idempotent $i \in A$, unique up to conjugation by an invertible element in $A$, and every finitely generated projective indecomposable $A$-module $P$ is isomorphic to $Aj$ for some primitive idempotent $j \in A$. To be precise, the maps*

$$i \mapsto Ai \mapsto Ai/J(A)i$$

*where $i$ runs over the primitive idempotents in $A$, induce bijections between the conjugacy classes of primitive idempotents in $A$, the isomorphism classes of finitely generated projective indecomposable $A$-modules, and the isomorphism classes of simple $A$-modules.*

Given an $A$-module $M$, the *radical* of $M$, $\mathrm{rad}(M)$ is defined as the intersection of all maximal submodules of $M$. Observe that if $A$ is finite-dimensional, we have $\mathrm{rad}(M) = J(A) \cdot M$. We therefore have that for a finitely generated projective indecomposable $A$-module $P_i \cong Ai$,

$$\mathrm{rad}(P_i) = J(A) \cdot P_i \cong J(A)Ai = J(A)i,$$

and that for $S$ a simple $A$-module, $S = S_i \cong P_i/\mathrm{rad}(P_i) \cong Ai/J(A)i$ as expected. We define the *socle* of $M$ is the sum (not necessarily direct) of all simple submodules of $M$: this plays a somewhat dual role to the radical, as we will see.

We can push our understanding of the structure of the projective indecomposable $A$-modules further, with some significant results that can help us gain a better understanding of the algebra $A$ itself. This is of course all highly useful when one wishes to classify certain types of algebras. Let $M$ be an $A$-module, then the *radical* or *Loewy series* of $M$ is defined inductively:

$$\mathrm{rad}^0(M) = M, \mathrm{rad}^j(M) = \mathrm{rad}(\mathrm{rad}^{j-1}(M)), j \geq 1.$$

The *socle series* of $M$ is also defined inductively:

$$\mathrm{soc}^0(M) = 0, \mathrm{soc}^j(M)/\mathrm{soc}^{j-1}(M) = \mathrm{soc}(M/\mathrm{soc}^{j-1}(M)), j \geq 1.$$

The $j$'th *radical* or *Loewy layer* of $M$ is $\mathrm{rad}^{j-1}(M)/\mathrm{rad}^j(M)$, $j \geq 1$, and the $j$'th socle layer is $\mathrm{soc}^j(M)/\mathrm{soc}^{j-1}(M)$. The *socle length* of $M$, is the positive integer $n$ such that $\mathrm{soc}^n(M) = M$ but $\mathrm{soc}^{n-1}(M) \neq M$. The socle length of $M$ is $n$ if and only if the *radical* or *Loewy length* of $M$ is $n$, defined by $\mathrm{rad}^n(M) = 0$ but $\mathrm{rad}^{n-1}(M) \neq 0$.

Returning our attention to the blocks of an algebra, we are particularly interested in classifying the blocks of group algebras, and us such would like to know what they look like. For this, we have some useful results.

**Proposition 2.1.11** ([52, 1.1.2]). *Let $H$ be a subgroup of $G$. Suppose that $|H|$ is invertible in $k$. Then the element*

$$e_H = \frac{1}{|H|} \sum_{h \in H} h$$

*is an idempotent in $kG$, and we have an isomorphism of $kG$-modules*

$$kGe_H \cong k(G/H),$$

*where $k(G/H)$ is the permutation module on the set of left cosets $G/H$. Moreover, if $H$ is normal in $G$, then $e_H$ is in the centre $Z(kG)$ of $kG$.*

*Proof.* Let $g \in H$. Then

$$g \cdot \sum_{h\in H} h = \sum_{h\in H} gh = \sum_{gh\in H} h = \sum_{h\in H} h,$$

so that

$$\left(\sum_{h\in H} h\right)^2 = \sum_{g\in H} g \cdot \sum_{h\in H} h = \sum_{g\in H}\sum_{h\in H} h = |H| \sum_{h\in H} h.$$

Whence

$$e_H = \frac{1}{|H|}\sum_{h\in H} h = \frac{1}{|H|^2}\left(\sum_{h\in H} h\right)^2 = e_H^2.$$

Define a map $\varphi : k(G/H) \to kGe_H$ by $gH \mapsto ge_H$, extended linearly. We verify that this is an isomorphism of $kG$-modules: first let $g, a \in G$, then

$$\varphi(a \cdot gH) = \varphi(agH) = age_H = a \cdot \varphi(gH).$$

We have

$$\mathrm{im}(\varphi) = \{xe_H \mid x \in kG\} = kGe_H,$$

and, choosing $aH, bH \in G/H$ then $\varphi(aH) = \varphi(bH)$ if and only if $ae_H = be_H$ if and only if $b^{-1}ae_H = e_H$ which tells us that $b^{-1}a \in H$, for only elements of $H$ will permute the sum in $e_H$. Whence $aH = bH$ and (extending this argument linearly) $\varphi$ is injective, thus an isomorphism of $kG$-modules. Finally, if $H$ is normal in $G$ then for all $g \in G$ then

$$ge_Hg^{-1} = g \cdot \frac{1}{|H|}\sum_{h\in H} h \cdot g^{-1} = \frac{1}{|H|}\sum_{h\in H} ghg^{-1} = \frac{1}{|H|}\sum_{ghg^{-1}\in H} h = \frac{1}{|H|}\sum_{h\in H} h = e_H,$$

so that $e_H \in Z(kG)$, completing the proof. $\square$

**Corollary 2.1.12.** *Let $H$ be a subgroup of $G$. Suppose that $|H|$ is invertible in $k$ and that $\chi : H \to k^\times$ is a group homomorphism. Then the element*

$$e_{H,\chi} = \frac{1}{|H|}\sum_{h\in H} \chi(h)^{-1}h$$

*is an idempotent in $kG$, and we have an isomorphism of $kG$-modules*

$$kGe_{H,\chi} \cong \mathrm{Ind}_H^G(k_\chi),$$

*where $k_\chi$ is the one-dimensional $kH$-module defined by $\chi$; that is, with action given by $h \cdot x := \chi(h)x$ for all $h \in H$ and $x \in k_\chi$.*

To prove the corollary, we need an extra result.

**Lemma 2.1.13.** *Let $A$ and $B$ be a $k$-algebras, $b \in A$ an idempotent, and $\alpha : A \to B$ an algebra homomorphism. Then $\alpha(b)$ is either an idempotent in $B$, or else $b \in \ker(\alpha)$.*

*Proof.* We have $\alpha(b)^2 = \alpha(b^2) = \alpha(b)$, and the result follows. □

*Proof of Corollary 2.1.12.* Define a map, $\alpha : kH \to kH$, by $\alpha : h \mapsto \chi(h)^{-1}h$ for all $h \in H$, extended linearly. We first verify that $\alpha$ is an algebra endomorphism. It is clear that $\alpha$ sends $\alpha(1_{kH}) = \chi(1_H)^{-1}1_H = 1_H = 1_{kH}$. Now, for all $x = \sum_{g\in H} \lambda_g g, y = \sum_{h\in H} \mu_h h \in kH$,

$$\begin{aligned}\alpha(x)\alpha(y) &= \sum_{g\in H} \lambda_g \chi(g)^{-1} g \cdot \sum_{h\in H} \mu_h \chi(h)^{-1} h \\ &= \sum_{l\in H} \left( \sum_{gh=l} \lambda_g \mu_h \chi(h^{-1})\chi(g^{-1}) \right) l \\ &= \sum_{l\in H} \left( \sum_{gh=l} \lambda_g \mu_h \chi(l^{-1}) \right) l \\ &= \alpha(xy).\end{aligned}$$

What is more, for $e_H$ as defined in Proposition 2.1.11, we have

$$\alpha(e_H) = \frac{1}{|H|} \sum_{h\in H} \chi(h)^{-1} h = e_{H,\chi}.$$

By Lemma 2.1.13 it is clear that $e_{H,\chi}$ is an idempotent in $kH \subseteq kG$. Finally, the map $ge_{H,\chi} \mapsto g \otimes 1_{k_\chi}$ gives a $kG$-module isomorphism $kGe_{H,\chi} \cong \mathrm{Ind}_H^G(k_\chi) = kG \otimes_{kH} k_\chi$, checking in the same manner as in the proof of Proposition 2.1.11: we do not repeat this here. □

We begin here the first in a series of linked examples, designed to exposit the concepts which we will encounter. This series of examples may be thought of as one overarching "case study", focusing on a specifically chosen group algebra and its blocks. This choice has been made for a number of reasons, not least because the structure is particularly elegant to describe. Moreover, it is an object of interest for reasons we will detail in the sequel, once we have encountered some more of the theory.

**Example 2.1.14** (The blocks of $k(C_3^2 \rtimes Q_8)$).

Let $k$ be algebraically closed of prime characteristic $p = 3$. Let $C_n$ denote the cyclic group of order $n$, written multiplicatively, and fix $E = C_2 \times C_2$ and $Z = C_2$. Consider a central extension

$$1 \to Z \xrightarrow{\iota} L \xrightarrow{\pi} E \to 1,$$

that is, a sequence of groups such that $\iota$ is injective, $\pi$ is surjective, $\mathrm{im}(\iota) = \ker(\pi)$, and $\iota(Z) = Z \subseteq Z(L)$. This setup will be formalised in the sequel, for now we satisfy ourselves that this implies

$L/Z \cong E$. Up to isomorphism there are 4 possible choices for $L$, namely $C_2^3$, $C_4 \times C_2$, $D_8$ or $Q_8$. We make the choice that $L \cong Q_8$ and note that $Z(L) = Z$.

Let $P = C_3 \times C_3$, and consider the faithful action of $E$ on $P$ where a generator of the first copy of $C_2$ in $E$ inverts a generator of the first copy of $C_3$ in $P$, and similarly for the second copies. We can lift this action to an action of $L$ on $P$, defining $Z$ to act trivially on $P$, and we have an induced central extension

$$1 \to Z \to P \rtimes L \to P \rtimes E \to 1. \tag{2.1}$$

Throughout these examples $G$ and $H$ will denote $P \rtimes L$ and $P \rtimes E$ respectively, and we remark that $H \cong S_3 \times S_3$.

As mentioned, one reason for this choice of group $G$ is that it affords a nice description of its blocks: by Corollary 2.1.12 they are in one-to-one correspondence with the characters of $Z$. Now, $Z$ has two irreducible characters, the trivial character $\chi_0$ and the sign character $\chi_1$ which sends $z \mapsto -1$ in $k^\times$. These correspond to two block idempotents of $kG$: writing $Z = \langle z \rangle$ we have

$$\begin{aligned} b_0 &= \frac{1}{|Z|}(\chi_0(1)1 + \chi_0(z)z) = \frac{1}{2}(1+z), \\ b_1 &= \frac{1}{|Z|}(\chi_1(1)1 + \chi_1(z)z) = \frac{1}{2}(1-z). \end{aligned}$$

We proceed to check that these are indeed block idempotents. First we double-check:

$$b_0^2 = \frac{1}{4}(1+z)^2 = \frac{1}{4}(1+2z+z^2) = \frac{1}{4}(2+2z) = \frac{1}{2}(1+z) = b_0,$$

and

$$b_1^2 = \frac{1}{4}(1-z)^2 = \frac{1}{4}(1-2z+z^2) = \frac{1}{4}(2-2z) = \frac{1}{2}(1-z) = b_1,$$

so that $b_1$ and $b_2$ are indeed idempotents. Next, let $g \in G$, then $gb_0 = \frac{1}{2}(g+gz) = \frac{1}{2}(g+zg) = b_0 g$ and similarly for $b_1$, so that $b_0, b_1 \in Z(kG)$. The idempotents are orthogonal, for $b_0 b_1 = \frac{1}{4}(1-z^2) = 0$, and that they sum to 1 is easily seen.

Finally, suppose that $b_0 = e + e'$ for two orthogonal idempotents $e, e' \in kG$, and write $e = \sum_{g\in G} \lambda_g g$, $e' = \sum_{g \in G} \mu_g g$. Then we must have

$$\lambda_1 + \mu_1 = 2, \tag{2.2}$$
$$\lambda_z + \mu_z = 2, \tag{2.3}$$

and $\lambda_g + \mu_g = 0$ for all $g \neq 1, z$. Rearranging and squaring Eq.(2.2) and Eq.(2.3), one sees that $\lambda_1^2 = 1 + 2\mu_1 + \mu_1^2$ and $\lambda_z^2 = 1 + 2\mu_z + \mu_z^2$. Whence

$$\lambda_1^2 + \lambda_z^2 = 2 + 2\mu_1 + 2\mu_z + \mu_1^2 + \mu_z^2. \tag{2.4}$$

Since $e^2 = e$ we obtain $\sum_{g\in G}\left(\sum_{h^2=g} \lambda_h^2\right) g = \sum_{g\in G} \lambda_g g$, which, on equating coefficients gives $\sum_{h^2=1} \lambda_h^2 = \lambda_1$. On the other hand, since $Z$ is the unique subgroup of order 2 in $G$, this simplifies to $\lambda_1^2 + \lambda_z^2 = \lambda_1$. The analogous argument shows that $\mu_1^2 + \mu_z^2 = \mu_1$. Substituting these into either side of Eq.(2.4) gives

$$\lambda_1 = 2 + 2\mu_z. \tag{2.5}$$

However, by rearranging Eq.(2.2) one sees that $\lambda_1 = 2 + 2\mu_1$, and so combining with Eq.(2.5) we have that $\mu_1$ and $\mu_z$ must be equal. Thus,

$$\begin{aligned}\mu_1 &= \mu_1^2 + \mu_z^2 \\ &= 2\mu_1^2,\end{aligned}$$

forcing $\mu_1 + \mu_1^2 = 0$, so that $\mu_1(1+\mu_1) = 0$. This gives two cases: if $\mu_1 = 0$ then (as $\mu_1 = \mu_z$) we must have $e' = 0$ and $b_0 = e$. On the other hand, if $\mu_1 = \mu_z = 2$ then by Eq.(2.2) and Eq.(2.3) we have that $e = 0$ and $b_0 = e$. In either case, we conclude that $b_0$ cannot be written as a sum of orthogonal idempotents, and is therefore primitive. The analogous argument shows that $b_1$ is primitive, and we do not reproduce that here.

Writing $B_i = kGb_i$ for $i = 0, 1$, we have a block decomposition $kG = B_0 \oplus B_1$. By Proposition 2.1.11 we have an isomorphism of $kG$-modules, $B_0 = kGb_0 \cong k(G/Z) \cong kH$, and by Corollary 2.1.12 we have $B_1 \cong \mathrm{Ind}_Z^G(k_{\chi_1})$. This of course means that $kG \cong kH \times B_1$, so that $B_0$ is itself a group algebra. On the other hand we will see in the sequel that $B_1$ when considered as an algebra, forms a *twisted group algebra*.

As mentioned, we will return to this specific group algebra $kG$ and the structure of its two blocks as we develop the theory, providing a toy example to highlight key concepts. Our next encounter with this example may be found at Example 2.1.19.

We return to our preliminary definitions and theory. Given an arbitrary group algebra $kG$, there is a trivial $kG$-module structure on $k$, defined by $g \cdot \lambda = \lambda$ for all $g \in G, \lambda \in k$, extended linearly. The *augmentation homomorphism* of $kG$ is the unique $k$-algebra homomorphism determined by this module structure, that is, $\eta : kG \to k$, $\sum_{g\in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g$. We define the *augmentation ideal* as the kernel of $\eta$, denoted $I(kG)$. It is simple to check that $I(kG)$ has a $k$-basis given by $\{g - 1 \mid g \in G \backslash \{1\}\}$, for $\eta(g-1) = 0$. If $b$ is a block of $kG$, then by Lemma 2.1.13 the element $\eta(b)$ is an idempotent of $k$ or else equal to zero. However $1_k$ is the unique idempotent of $k$, which allows us to define *the principal block* of $kG$ as the unique block, $B_0 = kGb_0$, whose block idempotent $b_0 \notin I(kG)$.

We would like to be able to classify and categorise different types of blocks of group algebras. One step in this direction is the notion of a *defect group*, a certain $p$-group which we can assign to a given block, for a fixed prime $p$. To define this formally, we need *the Brauer map*: given a $p$-subgroup $P$ of $G$, let $C_G(P) = \{g \in G \mid gx = xg \text{ for all } x \in P\}$, then this map is defined as the canonical projection $\mathrm{Br}_P : kG \to kC_G(P)$,

$$\mathrm{Br}_P : \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in C_G(P)} \lambda_g g.$$

For $H$ a subgroup of $G$, we write $(kG)^H = \{x \in kG \mid axa^{-1} = x \text{ for all } a \in H\}$, the $H$-fixed points of $kG$, and note that $(kG)^G = Z(kG)$. We remark here that on restriction to $(kG)^P$, the Brauer map becomes an algebra homomorphism $(kG)^P \to kC_G(P)$.

We are now ready to define a defect group of a block.

**Definition 2.1.15.** Let $p$ be a prime. Let $P$ be a subgroup of $G$, $b$ be a block idempotent of $kG$ and $B = kGb$ the corresponding block. Then $P$ is a *defect group* of $B$ (or of $b$) if $P$ is a maximal $p$-subgroup with respect to the property that $\mathrm{Br}_P(b) \neq 0$.

If $B$ has a defect group $P$ of order $p^r$, then we say that the *defect* of $B$ is the integer $r$.

There are a number of equivalent characterisations of defect groups, and we note one more for our purposes. Let $H$ be a subgroup of $G$, and choose a set of coset representatives of $H$ in $G$, denoted $[G/H]$, then we define the *relative trace map from $H$ to $G$* as the $k$-linear map $\mathrm{Tr}_H^G : (kG)^H \to (kG)^G = Z(kG)$,

$$\mathrm{Tr}_H^G(a) = \sum_{x \in [G/H]} xax^{-1}.$$

We will write $(kG)_H^G = \mathrm{im}(\mathrm{Tr}_H^G)$.

**Proposition 2.1.16** ([53, 6.2.1]). *Let $P$ be a subgroup of $G$, $b$ be a block idempotent of $kG$ and $B = kGb$ the corresponding block. Then $P$ is a defect group of $B$ (or of $b$) if and only if $P$ is a minimal subgroup with respect to the property that $b \in (kG)_P^G$.*

The proof of Proposition 2.1.16 requires some technical machinery that we do not reproduce here. Note, however, that Proposition 2.1.16 does not require $P$ to be a $p$-group a-priori. We state, without proof, some useful results on defect groups.

**Theorem 2.1.17** ([53, 6.1.2, 6.1.5]). *Let $P$ be a defect group of a block $B = kGb$. Then the following hold.*

*(i) For any subgroup $H$ of $G$ such that $b \in (kG)_H^G$, there is some $x \in G$ such that $P \subseteq xHx^{-1}$.*

*(ii) The defect group $P$ is unique up to conjugation in $G$.*

*(iii) If $B$ is the principal block of $kG$, then $P$ is a Sylow $p$-subgroup of $G$.*

Thus the principal block has maximal defect. One might ask, what do blocks with trivial defect groups look like?

**Theorem 2.1.18** ([14, Corollary 3]). *Let $B$ be a block with defect zero, or, equivalently with trivial defect group. Then $B$ is isomorphic as a $k$-algebra to a matrix algebra $M_n(k)$ for some $n$.*

We see that the modular representation theory of blocks with defect zero is equivalent to the ordinary representation theory: blocks with trivial defect are completely understood (in the context of Wedderburn's and Maschke's theorems). The defect groups of a block may therefore be thought of as a measure of complexity of the structure of the block, when considered as a $k$-algebra: in particular, a measure of how "far" the block is from a matrix algebra. The next step in complexity are blocks with defect one, those which have a cyclic defect group. These are also well-understood, and we will return to them in Chapter 4, when we develop the theory further in order to apply it to the blocks of some finite simple groups.

Let $B = kGb$ be a block of a group algebra $kG$, and $P$ a $p$-subgroup of $G$. Suppose that $H$ is a subgroup of $G$ containing $N_G(P) = \{g \in G \mid gP = Pg\}$. Then the *Brauer correspondence* tells us that if $P$ is a defect group of $B$, then there is a unique block $C = kHc$ of $kH$ with $P$ as a defect group, such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$. This gives us a bijection between the set of blocks of $kG$ with defect group $P$ and the set of blocks of $kH$ with defect group $P$. In the case that $H = N_G(P)$, we call the unique block $C$ of $kN_G(P)$ with $P$ as a defect group and such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$ the *Brauer correspondent* of $B$.

If $B$ is a block with defect group $P$ and Brauer correspondent $C = kN_G(P)c$ then we define the *inertial quotient* of $B$ to be the group $E = N_G(P, f)/PC_G(P)$ where $N_G(P, f) = N_G(P) \cap \{g \in G \mid gfg^{-1} = f\}$ and $f$ is a block of $kC_G(P)$ satisfying $fc = f$. The inertial quotient is unique up to conjugacy, and so is independent of choice of $f$. If $B = B_0$ is the principal block, then $\mathrm{Br}_P(b)$ is the principal block of $kC_G(P)$, so that $f = c$ and $E = N_G(P)/PC_G(P)$. Note also that if $P$ is abelian, $PC_G(P) = C_G(P)$.

**Example 2.1.19** (The defect groups of the blocks of $k(C_3^2 \rtimes Q_8)$)**.**

We continue with Example 2.1.14: all notation is defined as in that example. Our next step is to find a defect group of the blocks $B_0$ and $B_1$ of $kG$. Since a defect group of $B_0$ is a Sylow 3-subgroup of $G$, it is clear that $P = C_3 \times C_3$ is the required defect group, and $B_0$ is a block of defect 2: we would expect the defect to be non-trivial for, as we have seen, $B_0 \cong kH$ as an algebra, and since $|H|$ is divisible by 3 we are far from $B_0$ being a simple algebra. Since $P$ is normal in $G$, it is the unique Sylow 3-subgroup and consequently the only defect group of $B_0$. We will proceed to verify explicitly from the definitions that $P$ is the defect group of $B_0$.

First note that since $P$ is a Sylow subgroup and so is maximal among all 3-subgroups, it remains only to verify that $\mathrm{Br}_P(b_0) \neq 0$ when using Definition 2.1.15. To check this, we need a description of $C_G(P)$: this is readily verified as equal to $P \times Z$. Whence $\mathrm{Br}_P(b_0) = b_0$ and we are done.

Next we want to find a defect group of $B_1$. Again, using Definition 2.1.15, we look first at the 3-subgroups of $G$, namely $P$ and its subgroups which are isomorphic to $C_3$ and the trivial group. where $C_3$ occurs as a representative of three distinct $G$-conjugacy classes. As it happens, we need only concern ourselves with $P$, for $\mathrm{Br}_P(b_1) = b_1$ and we see again that $P$ is the unique defect group of $B_1$.

Since $P = C_3 \times C_3$ is normal in $G$ we have $N_G(P) = G$, whilst $C_G(P) = P \times Z$. Thus the Brauer correspondent of the principal block $B_0$ of $kG$ is simply $B_0$, and the inertial quotient $(C_3^2 \rtimes Q_8)/(C_3^2 \times C_2) \cong C_2 \times C_2$, isomorphic to the group $E$ as defined in Example 2.1.14 earlier. It is no coincidence that we have encountered the group $H = P \rtimes E$ already in the central extension (2.1) when constructing the group $G$, and indeed that $B_0 \cong k(P \rtimes E)$; we will see similar constructions appearing many times throughout this thesis, in particular we will focus on groups and blocks of this type in Section 4.3.

We will next return to this case study in Example 2.3.22, where we will look at the dimensions of the first Hochschild cohomology of $kG$ and its blocks.

Returning to our more general setting of an arbitrary $k$-algebra $A$, for a given $(A, A)$-bimodule $M$, let the *dual of* $M$ be the $(A, A)$-bimodule $M^* = \mathrm{Hom}_k(M, k)$, with action given by $(a{\cdot}\varphi{\cdot}b)(m) = \varphi(bma)$ for all $a, b \in A$ and $\varphi \in M^*$. There is a particular class of algebras that we are interested in for their properties: these are the *symmetric* algebras, namely, those finite-dimensional algebras $A$ that are isomorphic as $(A, A)$-bimodules to their duals $A^*$. An element $s \in A^*$ is called a *symmetrising form* if there is an $(A, A)$-bimodule isomorphism $A \cong A^*$ sending $1_A$ to $s$.

For a symmetric algebra $A$, we obtain some further information from the radical and Loewy series of the projective indecomposable modules of $A$ than from an arbitrary algebra. Let $A$ be symmetric and let $I = \{i_1, \ldots, i_r\}$ be a set of conjugacy class representatives of the primitive idempotents in $A$, $\mathcal{S} = \{S_{i_1}, \ldots, S_{i_r}\}$ be a set of representatives of the isomorphism classes of simple $A$-modules, and fix $S \in \mathcal{S}$ so that $S = S_i \cong P_i/\mathrm{rad}(P_i) \cong Ai/J(A)i$ for some finitely generated projective indecomposable $A$-module $P = P_i \cong Ai$ and some primitive idempotent $i$ in

$I$. Then the $j$'th radical layer of $P_i$ is the semi-simple $A$-module,

$$\mathrm{rad}^{j-1}(P_i)/\mathrm{rad}^j(P_i) \cong J(A)^{j-1}i/J(A)^j i \cong \bigoplus_{i_s \in I} S_{i_s}^{\oplus n_s}$$

for some $n_s \in \mathbb{N}$, and we can describe the radical series of $P_i$ diagrammatically:

$$\begin{aligned}
\text{1st radical layer: } & P_i/\mathrm{rad}(P_i) = S_i \\
\text{2nd radical layer: } & \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i) = S_{i_1}^{\oplus n_{1,i_1}} \oplus S_{i_2}^{\oplus n_{1,i_2}} \oplus \cdots \oplus S_{i_r}^{\oplus n_{1,i_r}} \\
& \vdots \\
j\text{'th radical layer: } & \mathrm{rad}^{j-1}(P_i)/\mathrm{rad}^j(P_i) = S_{i_1}^{\oplus n_{j,i_1}} \oplus S_{i_2}^{\oplus n_{j,i_2}} \oplus \cdots \oplus S_{i_r}^{\oplus n_{j,i_r}} \\
& \vdots \\
n\text{'th radical layer: } & \mathrm{rad}^{n-1}(P_i) = S_i,
\end{aligned}$$

and typically we drop the $\oplus$ symbols for clarity.

**Proposition 2.1.20** ([52, Theorem 2.11.2])**.** *The group algebra $kG$ is a symmetric algebra, with symmetrising form given by*

$$s : kG \to k, \;\; s(\sum_{g \in G} \lambda_g g\;) = \lambda_1.$$

*Proof.* We define the isomorphism $\Phi : (kG)^* \cong kG$ as follows: a map $f \in (kG)^*$ is sent to the element $\Phi(f) = \sum_{g \in G} f(g^{-1})g$. One checks that this is a $(kG, kG)$-bimodule isomorphism:

(i) For all $f_1, f_2 \in (kG)^*$,

$$\begin{aligned}
\Phi(f_1 + f_2) &= \sum_{g \in G}(f_1 + f_2)(g^{-1})g \\
&= \sum_{g \in G}(f_1(g^{-1}) + f_2(g^{-1}))g \\
&= \sum_{g \in G}(f_1(g^{-1})g + f_2(g^{-1})g) \\
&= \sum_{g \in G} f_1(g^{-1})g + \sum_{g \in G}(f_2(g^{-1})g \\
&= \Phi(f_1) + \Phi(f_2).
\end{aligned}$$

(ii) For all $a = \sum_{h \in G} \lambda_h h, b = \sum_{l \in G} \mu_l l \in kG$ and $g \in G$,

$$\begin{aligned} bg^{-1}a &= \left( \sum_{l \in G} \mu_l l \right) g^{-1} \left( \sum_{h \in G} \lambda_h h \right) \\ &= \sum_{lg^{-1} \in G} \mu_l l \sum_{h \in G} \lambda_h h \\ &= \sum_{m \in G} \left( \sum_{lg^{-1}h = m} \mu_l \lambda_h \right) m, \end{aligned}$$

so that for all $f \in (kG)^*$,

$$\begin{aligned} \Phi(a \cdot f \cdot b) &= \sum_{g \in G} (a \cdot f \cdot b)(g^{-1}) g \\ &= \sum_{g \in G} f(bg^{-1}a) g \\ &= \sum_{g \in G} f \left( \sum_{m \in G} \left( \sum_{lg^{-1}h = m} \mu_l \lambda_h \right) m \right) g \\ &= \sum_{g \in G} \sum_{m \in G} \left( \sum_{lg^{-1}h = m} \mu_l \lambda_h \right) f(m) g \\ &= \sum_{g \in G} \sum_{m \in G} \left( \sum_{hgl = m} \mu_l \lambda_h \right) f(g^{-1}) m, \end{aligned}$$

whilst on the other hand

$$\begin{aligned} a \cdot \Phi(f) \cdot b &= a \sum_{g \in G} f(g^{-1}) g b \\ &= \sum_{g \in G} agbf(g)^{-1} \\ &= \sum_{g \in G} \left( \sum_{h \in G} \lambda_h h \right) g \left( \sum_{l \in G} \mu_l l \right) f(g^{-1}) \\ &= \sum_{g \in G} \left( \sum_{h \in G} \lambda_{hg} h \right) \left( \sum_{l \in G} \mu_l l \right) f(g^{-1}) \\ &= \sum_{g \in G} \sum_{m \in G} \left( \sum_{hgl = m} \mu_l \lambda_h \right) f(g^{-1}) m, \end{aligned}$$

showing that $\Phi(a \cdot f \cdot b) = a \cdot \Phi(f) \cdot b$.

(iii) The ideal

$$\operatorname{im}(\Phi) = \{\sum_{g \in G} f(g^{-1})g \mid f \in (kG)^*\}$$

is easily seen to be equal to $kG$, and the ideal

$$\ker(\Phi) = \{f \in (kG)^* \mid \sum_{g \in G} f(g^{-1})g = 0\}$$

is easily seen to equal $\{0\}$.

Thus (i), (ii) and (iii) show that $\Phi$ is indeed a $(kG, kG)$-bimodule isomorphism between $(kG)^*$ and $kG$. Finally, $\Phi(s) = \sum_{g \in G} s(g^{-1})g = s(1_G) = 1_{kG}$, completing the proof. □

**Corollary 2.1.21.** *Let $B$ be a block of a group algebra $kG$. Then $B$ is symmetric, with symmetrising form given by the restriction to $B$ of the symmetrising form on $kG$.*

Let $A$ be a symmetric algebra with symmetrising form $s$ and basis $X = \{a_i \mid i = 1, \ldots, n\}$. We define the *dual basis* of $A$ with respect to $s$ and $X$ as the basis $X' = \{a'_i \mid i = 1, \ldots, n\}$, such that $s(a_i a'_j) = \delta_{ij} 1_k$. One verifies that if $A = kG$ with symmetrising form as in Proposition 2.1.20 then the dual basis of $kG$ with respect to $s$ and the basis $G$ is the basis $G^{-1} = \{g^{-1} \mid g \in G\}$.

Let $A$ be symmetric with symmetrising form $s$, basis $X$ and dual basis $X'$ with respect to $s$. Another way of putting this is that sending $x$ to $x'$ is a bijection between $X$ and $X'$ such that $s(xx') = 1$ and $s(xy') = 0$ for all $x, y \in X$, $x \neq y$. Define a map $T : A \to Z(A)$ by $T(a) = \sum_{x \in X} x'ax$, and note that this depends on choice of $s$ but not of $X$. Then we define the *projective ideal* in $Z(A)$ as the the image $Z^{\mathrm{pr}}(A) = \operatorname{im}(T)$, and note that this does not depend on $s$. In the case $A = kG$, one sees that $T = \mathrm{Tr}_1^G$, the relative trace map from the trivial subgroup $\{1\}$ to $G$ as defined earlier.

Returning to the general case, the *stable center* of a symmetric algebra $A$ is defined as the quotient $\underline{Z}(A) = Z(A)/Z^{\mathrm{pr}}(A)$. As we will see in Chapter 3, the notion of stable centre is useful when comparing algebras, in particular since a *stable equivalence of Morita type preserves the stable centre*. We will encounter stable equivalences in Definition 2.2.20.

We close this subsection by mentioning briefly the definition of a *selfinjective algebra*: this is a $k$-algebra $A$ that is injective as a left $A$-module. We will need this definition when looking at how Hochschild cohomology is preserved by certain categorical equivalences, and so state without proof the following.

**Lemma 2.1.22** ([52, Theorem 4.12.1])**.** *Let $A$ be a symmetric $k$-algebra. Then $A$ is selfinjective. In particular, the group algebra $kG$ is selfinjective.*

## 2.2 Category theory and homological algebra

Category theory provides a framework within which we can consider algebras, their modules and the maps between their modules. It can be thought of as a large universe of algebraic terminology, in which the corner containing algebras and their modules is but a fraction, and where high-level and generally abstract comparisons between all manner of algebraic objects can be made.

Homological algebra, on the other hand, gives us a collection of tools and techniques with which we can compute important invariants of groups and algebras, allowing us to make the comparisons between different algebraic objects explicit. The two areas of category theory and homological algebra go hand in hand, with category theory providing the vocabulary with which to describe homological algebra. As we will make great use of homological algebra throughout this thesis in order to better understand the blocks of group algebras, not least when computing the titular *Hochschild cohomology*, we detail the preliminary concepts and theory here.

**Definition 2.2.1.** A *category*, $\mathcal{C}$, consists of the following data:

(i) A class $\mathrm{Ob}(\mathcal{C})$, called the *objects of* $\mathcal{C}$.

(ii) For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, called the *morphisms from* $X$ *to* $Y$ *in* $\mathcal{C}$.

(iii) For all $X, Y, Z \in \mathcal{C}$ a map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, $(f, g) \to g \circ f$, called the *composition map*.

Additionally, the data is subject to the following properties:

(a) For all $X, Y \in \mathcal{C}$, the classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are pairwise disjoint.

(b) For all $X \in \mathrm{Ob}(\mathcal{C})$, there is a morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, called the *identity morphism* of $X$, such that for all $Y \in \mathcal{C}, f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, we have $f \circ \mathrm{Id}_X = f$ and $\mathrm{Id}_X \circ g = g$.

(c) For all $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$ and any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z), h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$, an equality of morphisms in $\mathrm{Hom}_{\mathcal{C}}(X, W)$.

We will often write $f : X \to Y$ or $X \xrightarrow{f} Y$ to denote the situation that, given a category $\mathcal{C}$ and objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we have $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

A *subcategory* of $\mathcal{C}$ is a category $\mathcal{D}$ such that $\mathrm{Ob}(\mathcal{D})$ is a sub-class of $\mathrm{Ob}(\mathcal{C})$, such that for any objects $X, Y \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ is a sub-class of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and such that composition in $\mathcal{D}$ is the restriction of composition in $\mathcal{C}$. We say that $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{D})$.

We will be most interested in the category of $A$-modules, for a given $k$-algebra $A$. We denote this category by $\mathrm{Mod}(A)$: here, the objects are the $A$-modules and the morphisms are $A$-module homomorphisms, with the categorical composition map given by the standard composition of homomorphisms. We will write $\mathrm{Hom}_A(U, V)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(U, V)$ for $U, V \in \mathrm{Ob}(\mathrm{Mod}(A))$. We will be typically most interested in the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated $A$-modules, denoted $\mathrm{mod}(A)$.

**Definition 2.2.2.** Let $\mathcal{C}$ be a category, $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f : X \to Y$.

(i) The morphism $f$ is called an *epimorphism* if for any two morphisms $g$, $h$ from $Y$ to any other object $Z$ satisfying $g \circ f = h \circ f$, we have $g = h$.

(ii) The morphism $f$ is called a *monomorphism* if for any two morphisms $g$, $h$ from any other object $Z$ to $X$ satisfying $f \circ g = f \circ h$, we have $g = h$.

(iii) The morphism $f$ is called an *isomorphism* if there exists a morphism $g : Y \to X$ satisfying $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

(iv) An object $P$ in $\mathcal{C}$ is called *projective* if for any epimorphism $h : X \to Y$ and any morphism $g : P \to Y$, there is a morphism $\varphi : P \to X$ such that $h \circ \varphi = g$.

As the names suggest, epimorphisms in the category $\mathrm{Mod}(A)$ are simply surjective $A$-module homomorphisms, monomorphisms in $\mathrm{Mod}(A)$ are injective $A$-module homomorphisms, isomorphisms in $\mathrm{Mod}(A)$ are $A$-module isomorphisms and projective objects in $\mathrm{Mod}(A)$ are projective modules. We show the latter.

**Proposition 2.2.3** ([52, Theorem 1.12.3])**.** *Let $P \in \mathrm{Ob}(\mathrm{Mod}(A))$. Then $P$ is projective as an object in* $\mathrm{Mod}(A)$ *if and only if $P$ is projective as an $A$-module, as in Definition 2.1.4.*

*Proof.* Let $P$ be a projective object in $\mathrm{Mod}(A)$. Then, for any $A$-modules $X$ and $Y$, any surjective $A$-module homomorphism $h : X \to Y$ and any $A$-module homomorphism $g : P \to Y$, there exists an $A$-module homomorphism $f : P \to X$ such that $g = h \circ f$.

Let $S \subseteq P$ be a generating set for $P$ as an $A$-module, that is, a set such that for all $z \in P$, $z$ may be written as a (not necessarily unique) sum $z = \sum_{s \in S} a_s \cdot s$, for all but finitely many $a_s \in A$ non-zero. Define a free $A$-module $F$ with basis indexed by $S$, $E = \{e_s \mid s \in S\}$ so that $F \cong \oplus_{s \in S} A \cdot e_s$. Since $F$ is free, there is a (unique) surjective map $h : F \to P$ sending $e_s \mapsto s$. We have the map $\mathrm{Id}_P : P \to P$, so that by the property above, there is some $f : P \to F$ such that $\mathrm{Id}_P = h \circ f$.

We proceed to show that $F \cong \ker(h) \oplus P$ as $A$-modules. Let $y \in F$. Then $h(y - fh(y)) = h(y) - hf(h(y)) = h(y) - h(y) = 0$, so that $y - fh(y) \in \ker(h)$. Now, $y = y - fh(y) + fh(y) \in \ker(h) + \mathrm{im}(f)$. Since $\ker(h), \mathrm{im}(f) \subseteq F$, we obtain $F = \ker(h) + \mathrm{im}(f)$. Now let $y \in \ker(h) \cap \mathrm{im}(f)$, so that $h(y) = 0$ and $y = f(a)$ for some $a \in P$. Then $0 = h(y) = hf(a) = a$, so that $y = f(a) = f(0) = 0$. Whence $F = \ker(h) \oplus \mathrm{im}(f)$. Finally, it is evident that $f$ is an injective $A$-module homomorphism: pick $z, z' \in P$, $z \neq z'$, and suppose that $f(z) = f(z')$, then $hf(z) = hf(z')$ contradicting our choice of $z \neq z'$. So $\mathrm{im}(f) \cong P$, and one direction of the result follows: $F \cong \ker(h) \oplus P$, that is, $P$ is a direct summand of a free $A$-module.

Now suppose that $P$ is a projective $A$-module, as defined in Definition 2.1.4, and write $F = P \oplus P'$ for some free $A$-module $F$ with a basis $S$, and some $A$-module $P'$. Let $X, Y \in \mathrm{Ob}(\mathrm{Mod}(A))$, $h : X \to Y$ be an epimorphism and $g : P \to Y$ a morphism. Define a morphism $\hat{g} : F \to Y$ by $\hat{g}(z + z') = g(z)$ for all $z \in P, z' \in P'$. For all $s \in S$, let $x_s \in X$ denote the element such that $h(x_s) = \hat{g}(s)$. Since $F$ is free there is an epimorphism $\varphi : F \to X$ such that $\varphi(s) = x_s$ for all $s \in S$. Then $h\varphi(s) = \hat{g}(s)$ for all $s \in S$ and so $h \circ \varphi = \hat{g}$. Now let $s \in P$. Then $h\varphi(s) = \hat{g}(s) = g(s)$, so that the restriction of $\varphi$ to $P$ is the required morphism giving us the property that $P$ is a projective object in the category $\mathrm{Mod}(A)$. This completes the proof. $\square$

**Definition 2.2.4.** Let $\mathcal{C}, \mathcal{D}$ be categories. A *covariant functor* $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ is a map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with a family of maps, abusively all denoted by the same letter $\mathcal{F}$, from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, with the following properties.

(i) For all $X \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$.

(ii) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$, we have $\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\psi) \circ \mathcal{F}(\varphi)$.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with a family of maps, $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, with the following properties.

(i) For all $X \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$.

(ii) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$, we have $\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi)$.

Throughout, we will take the term *functor* to implicitly mean *covariant functor*, unless otherwise specified.

On every category $\mathcal{C}$ is defined the *identity functor*, $\mathrm{Id}_{\mathcal{C}}$, given by identity map on $\mathrm{Ob}(\mathcal{C})$ together with the family of identity maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$. Functors may be associatively composed in the obvious way, by composing the maps on objects and on morphisms.

There is one important example of a functor that we will need presently.

**Example 2.2.5.** Let $A, B$ be $k$-algebras, $M$ be an $(A, B)$-bimodule, and set $\mathcal{C} = \mathrm{Mod}(A), \mathcal{D} = \mathrm{Mod}(B)$. Then we can define a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ by the assignment

$$\mathcal{F}(U) = \mathrm{Hom}_A(M, U)$$

for all $U \in \mathrm{Ob}(\mathcal{C})$, and by $\mathcal{F}(\varphi) : \mathcal{F}(U) \to \mathcal{F}(V)$,

$$\mathcal{F}(\varphi)(f) = \varphi \circ f$$

for all $U, V \in \mathrm{Ob}(\mathcal{C})$, all $\varphi : U \to V$ and all $f \in \mathcal{F}(U)$. Note that $\mathcal{F}(U)$ is indeed in $\mathrm{Ob}(\mathcal{D})$ for all $U \in \mathrm{Ob}(\mathcal{C})$, defining the action by $(b \cdot f)(m) = f(mb)$ for all $f \in \mathcal{F}(U)$, all $b \in B$ and all $m \in M$.

We proceed to check that $\mathcal{F}$ is a functor. Let $U \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_A(M, U)$ and $m \in M$, then

$$\mathcal{F}(\mathrm{Id}_U)(f)(m) = \mathrm{Id}_U \circ f(m) = f(m) = (1 \cdot f)(m) = \mathrm{Id}_{\mathrm{Hom}_A(M,U)}(f)(m) = \mathrm{Id}_{\mathcal{F}(U)}(f)(m).$$

Now let $U, V, W \in \mathrm{Ob}(\mathcal{C})$, $\varphi : U \to V$, $\psi : V \to W$, and $f : M \to U$ and $m \in M$, then

$$\mathcal{F}(\psi \circ \varphi)(f)(m) = (\psi \circ \varphi) \circ f(m) = \psi\varphi f(m) = \mathcal{F}(\psi)[\mathcal{F}(\varphi)f](m) = [\mathcal{F}(\psi) \circ \mathcal{F}(\varphi)](f)(m).$$

Exchanging the roles of the arguments in the covariant functor $\mathcal{F}$ given above, one also obtains a contravariant functor $\mathcal{G} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G}(U) = \mathrm{Hom}_A(U, M)$ for all $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{G}(\psi) : \mathcal{G}(V) \to \mathcal{G}(U)$, $\mathcal{G}(\psi)(f) = f \circ \psi$ for all $\psi : U \to V$, all $U, V \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathcal{G}(V)$.

In other words, doing away with the general categorical notation, for each $A$-module $M$ we have two functors $\mathcal{F}, \mathcal{G} : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$, dependent on $M$, defined by

$$\mathcal{F}(-) = \mathrm{Hom}_A(M, -),$$
$$\mathcal{G}(-) = \mathrm{Hom}_A(-, M),$$

with the obvious induced maps $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$. These functors will play an important role in the definitions of group and Hochschild cohomology.

We will see plenty more examples of functors between module categories when we consider the different types of module category *equivalences*, that is, different ways to compare the module categories of two algebras. Additionally, we have already encountered some examples of functors.

For an $A$-module $V$ and right $A$-module $U$, the assignment $U \mapsto U \otimes_A V$ is a covariant functor from $\mathrm{Mod}(A)$ to $\mathrm{Vect}(k)$, the category of $k$-vector spaces, which sends an $A$-module homomorphism $f : U \to X$ to the linear map $f \otimes 1 : U \otimes_A V \to X \otimes_A V$. Similarly, $V \mapsto U \otimes_A V$ is a covariant functor $\mathrm{Mod}(A^{\mathrm{op}}) \to \mathrm{Vect}(k)$.

**Definition 2.2.6.** Let $\mathcal{C}, \mathcal{D}$ be categories, and let $\mathcal{F}, \mathcal{G}$ be functors from $\mathcal{C} \to \mathcal{D}$. A *natural transformation from $\mathcal{F}$ to $\mathcal{G}$* is a family $\varphi = (\varphi(X))_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms $\varphi(X) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{G}(X))$ such that for any morphism $f : X \to Y$ in $\mathcal{C}$ we have $\mathcal{G}(f) \circ \varphi(X) = \varphi(Y) \circ \mathcal{F}(f)$; that is, we have a commutative diagram of morphisms in the category $\mathcal{D}$ of the form

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\varphi(X)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(X) \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(Y) \arrow[r, "\varphi(Y)"'] & \mathcal{G}(Y)
\end{tikzcd}$$

Every functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ defines a natural transformation, the *identity transformation* $\mathrm{Id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$, given by the family $\mathrm{Id}_{\mathcal{F}} = (\mathrm{Id}_{\mathcal{F}}(X))_{X \in \mathrm{Ob}(\mathcal{C})} = (\mathrm{Id}_{\mathcal{F}(X)})_{X \in \mathrm{Ob}(\mathcal{C})}$ of identity morphisms on $\mathcal{F}(X)$. Natural transformations may be associatively composed in the obvious way, which allows us to define *isomorphic functors*: these are functors $\mathcal{F}, \mathcal{G}$ from $\mathcal{C}$ to $\mathcal{D}$ such that there exists some natural transformations $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{F}$ with $\psi \circ \varphi = \mathrm{Id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{Id}_{\mathcal{G}}$. In this case, we write $\mathcal{F} \cong \mathcal{G}$.

**Definition 2.2.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if there are functors $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ and $\mathcal{G}$ from $\mathcal{D}$ to $\mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$, and in this case we write $\mathcal{C} \cong \mathcal{D}$.

For the remainder of this chapter, we will focus on categories of modules over a $k$-algebra. We restrict our attention to this case since these are the categories which we will be most interested throughout this thesis. In addition, these categories have certain properties that allow us to simplify the exposition: in particular they are both *$k$-linear* and *abelian categories*, though we do not define those here. We refer the interested reader instead to the all-encompassing opus of Weibel [75] for further details and greater generality.

Let $A$, $B$ be $k$-algebras. One way to compare $A$ and $B$ and their representation theories is to look at their module categories. We say that $A$ and $B$ are *Morita equivalent* if there is an equivalence of abelian categories $\mathrm{Mod}(A) \cong \mathrm{Mod}(B)$. Thanks to Morita, there is an explicit characterisation of Morita equivalent $k$-algebras.

**Theorem 2.2.8** (Morita's theorem)**.** *Let $A$ and $B$ be $k$-algebras, then $A$ and $B$ are Morita equivalent if and only if there is an $(A, B)$-bimodule $M$ and a $(B, A)$-bimodule $N$ such that $M$, $N$ are finitely generated and projective as left/right modules, with $M \otimes_B N \cong A$ as $(A, A)$-bimodules and $N \otimes_A M \cong B$ as $(B, B)$-bimodules. In this case, the functors $M \otimes_B -$ and $N \otimes_A -$ induce an equivalence* $\mathrm{Mod}(A) \cong \mathrm{Mod}(B)$.

In practice, we are most concerned with $A = kG$ for $G$ finite, hence $A$ is finite-dimensional. Recall also that we have fixed $k$ to be a field. Consequently (and by Remark 2.1.3) we are most often interested in $\mathrm{mod}(A)$. It is important to note, therefore, that Morita's theorem implies that an equivalence $\mathrm{Mod}(A) \cong \mathrm{Mod}(B)$ yields an equivalence $\mathrm{mod}(A) \cong \mathrm{mod}(B)$, and that the statements, in the sequel, concerning equivalences of algebras therefore all hold for finitely generated modules.

Given a finite-dimensional $k$-algebra $A$, one can begin to search for algebras Morita equivalent to $A$ by looking at the *basic algebras of* $A$. We say that $A$ is *basic* if, in a primitive decomposition $I$ of $1_A$, the elements of $I$ are pairwise non-conjugate - for example, if $A$ is commutative. Equivalently, in the module decomposition $A = \bigoplus_{i \in I} Ai$, the summands $Ai$ are pairwise non-isomorphic.

If $A$ is finite-dimensional, we can take a primitive decomposition $I$ of $1_A$, and from that choose a set of conjugacy class representatives $J$. Setting $e$ to be the idempotent $e = \sum_{i \in J} i$, then the algebra $eAe$ is basic and Morita equivalent to $A$,

$$eA \otimes_A - : \mathrm{Mod}(A) \cong \mathrm{Mod}(eAe),$$

with inverse functor $Ae \otimes_{eAe} -$. Since $e$ is unique up to conjugation, the algebra $eAe$ is unique up to isomorphism, and we call the algebras that are basic and Morita equivalent to $A$ the *basic algebras of* $A$.

**Remark 2.2.9.** Morita equivalence is a strong condition between algebras. One might ask, how can this condition be relaxed? The answer gives rise to a number of different equivalences between algebras, all intimately linked. In particular, given $k$-algebras $A$ and $B$, if $\mathrm{Mod}(A) \ncong \mathrm{Mod}(B)$, all is far from lost: we have plenty of other ways to compare the modules of $A$ and $B$, as we shall see shortly.

Let $A$ be a $k$-algebra and $\mathcal{C}$ be a category of $A$-modules. A *graded object over* $\mathcal{C}$ is a family $X = (X_n)_{n \in \mathbb{Z}}$ of objects $X_n \in \mathrm{Ob}(\mathcal{C})$. Given two graded objects $X = (X_n)_{n \in \mathbb{Z}}$ and $Y = (Y_n)_{n \in \mathbb{Z}}$ over $\mathcal{C}$, a *graded morphism of degree* $m$ is a family $f = (f_n)_{n \in \mathbb{Z}}$ of morphisms $f_n : X_n \to Y_{n+m}$ in $\mathcal{C}$. The category of graded objects over $\mathcal{C}$ with graded morphisms of degree zero is denoted $\mathrm{Gr}(\mathcal{C})$. A *zero object* in $\mathcal{C}$, denoted by 0, is an object such that for all objects $X$ in $\mathcal{C}$, there is a unique morphism $X \to 0$ and a unique morphism $0 \to X$ (this is equivalent to 0 being both *terminal* and *initial*). The *zero morphism* from $X$ to $Y$ in $\mathcal{C}$, is the unique morphism that factors through the zero object.

A *chain complex* over $\mathcal{C}$ is a pair $(X, \delta)$ consisting of a graded object $X = (X_n)_{n \in \mathbb{Z}}$ and a graded endomorphism $\delta = (\delta_n)_{n \in \mathbb{Z}}$ of degree $-1$, such that $\delta_{n-1} \circ \delta_n = 0$, the zero morphism, for all $n$. A *cochain complex* over $\mathcal{C}$ is a pair $(X, \delta)$ consisting of a graded object $X = (X^n)_{n \in \mathbb{Z}}$ and a graded endomorphism $\delta = (\delta^n)_{n \in \mathbb{Z}}$ of degree $+1$, such that $\delta^{n+1} \circ \delta^n = 0$ for all $n$. We will often write $X$ for a (co)chain complex, when it is clear from the context. The graded endomorphism $\delta$ is known as the *differential* of the (co)chain complex.

A *chain map* between two chain complexes $(X, \delta)$ and $(Y, \varepsilon)$ is a graded morphism $f : X \to Y$ of degree 0, such that $f_{n-1} \circ \delta_n = \varepsilon_n \circ f_n$. A *cochain map* between two cochain complexes $(X, \delta)$ and $(Y, \varepsilon)$ is a graded morphism $f : X \to Y$ of degree 0, such that $f^{n+1} \circ \delta^n = \varepsilon^n \circ f^n$. The category of chain complexes over $\mathcal{C}$ with morphisms given by chain maps, is denoted $\mathrm{Ch}(\mathcal{C})$ and the category of cochain complexes with cochain maps is denoted $\mathrm{coCh}(\mathcal{C})$.

Let $(X, \delta)$ be a chain complex of $A$-modules. The condition that $\delta^2 = 0$ implies that $\mathrm{im}(\delta_{n+1}) \subseteq \ker(\delta_n)$ as $A$-modules: let $x \in \mathrm{im}(\delta_{n+1})$ for some $x \in X_n$, so that $x = \delta_{n+1}(y)$ for some $y \in X_{n+1}$. Then $\delta_n(x) = \delta_n \delta_{n+1}(y) = 0$. Thus, the following definition makes sense.

**Definition 2.2.10.** Let $A$ be a $k$-algebra, and $(X, \delta)$ a chain complex of $A$-modules. Then the *homology* of $(X, \delta)$ is the graded $A$-module $H_n(X, \delta) = \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$. If $(Y, \varepsilon)$ is a cochain complex of $A$-modules, then the *cohomology* of $(Y, \varepsilon)$ is the graded $A$-module $H^n(Y, \varepsilon) = \ker(\varepsilon^n)/\mathrm{im}(\varepsilon^{n-1})$. If $\delta$ is clear from the context, we will write $H^n(X) = H^n(X, \delta)$, and similarly for cohomology. We will write $H_*(X) = (H_n(X))_{n\in\mathbb{Z}}$ and $H^*(Y) = (H^n(Y))_{n\in\mathbb{Z}}$.

**Proposition 2.2.11.** *Let $A$ be a $k$-algebra, $\mathcal{C} = \mathrm{coCh}(\mathrm{Mod}(A))$ and $\mathcal{D} = \mathrm{Gr}(\mathrm{Mod}(A))$. For $X \in \mathrm{Ob}(\mathcal{C})$, let $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ be the assignment*

$$\mathcal{F}(X) = H^*(X),$$

*and for all $(X, \delta), (Y, \varepsilon) \in \mathrm{Ob}(\mathcal{C})$ define $\mathcal{F} = H^* : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ by*

$$H^n(\varphi)(x + \mathrm{im}(\delta^{n-1})) = \varphi^n(x) + \mathrm{im}(\varepsilon^{n-1})$$

*for all cochain maps $\varphi : X \to Y$ and for all $n \in \mathbb{Z}$. Then $\mathcal{F}$ is a functor. In other words, taking cohomology is a functor from the cochain complexes of $A$-modules to the graded objects of $A$-modules.*

*Proof.* We will make use of the following diagram visualising the chain complex $X$ in the top row and $Y$ in the bottom:

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "\delta^{n-1}"] \arrow[d, "\varphi^{n-1}"] & X^n \arrow[r, "\delta^n"] \arrow[d, "\varphi^n"] & X^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "\varepsilon^{n-1}"] & Y^n \arrow[r, "\varepsilon^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.6}$$

We first show that $\mathcal{F}$ is well-defined. Let $x, x' \in \ker(\delta^n)$ and suppose that $x + \mathrm{im}(\delta^{n-1}) = x' + \mathrm{im}(\delta^{n-1})$. Then $x - x' \in \mathrm{im}(\delta^{n-1})$, say, $\delta^{n-1}(z) = x - x'$ for some $z \in X^{n-1}$. By commutativity of the squares in (2.6) we get that

$$\varphi^n(x) - \varphi^n(x') = \varphi^n(x - x') = \varphi^n\delta^{n-1}(z) = \varepsilon^{n-1}\varphi^{n-1}(z),$$

so that $\varphi^n(x) - \varphi^n(x') \in \mathrm{im}(\varepsilon^{n-1})$, and in particular $\varphi^n(x) + \mathrm{im}(\varepsilon^{n-1}) = \varphi^n(x') + \mathrm{im}(\varepsilon^{n-1})$. Whence $H^n(\varphi)(x + \mathrm{im}(\delta^{n-1})) = H^n(x' + \mathrm{im}(\delta^{n-1}))$.

A similar argument shows that $\varphi$ sends $\ker(\delta)$ to $\ker(\varepsilon)$ and $\mathrm{im}(\delta)$ to $\mathrm{im}(\varepsilon)$: first let $x \in \ker(\delta^n)$, then

$$\varepsilon^n\varphi^n(x) = \varphi^{n+1}\delta^n(x) = 0,$$

so $\varphi^n(x) \in \ker(\varepsilon^n)$. Now let $x \in \mathrm{im}(\delta^{n-1})$, say $x = \delta^{n-1}(z)$ for some $z \in X^{n-1}$, then

$$\varphi^n(x) = \varphi^n\delta^{n-1}(z) = \varepsilon^{n-1}\varphi^{n-1}(x),$$

so $\varphi^n(x) \in \mathrm{im}(\varepsilon^{n-1})$. Thus, $\mathcal{F}$ is a well-defined map from $\mathcal{C}$ to $\mathcal{D}$.

Since $H^*(\varphi)$ is induced by $\varphi$ a routine verification shows that given two composable chain maps $\varphi : X \to Y, \psi : Y \to Z$, we have $H^n(\psi\circ\varphi)(x+\mathrm{im}(\delta^{n-1})) = \psi\varphi(x+\mathrm{im}(\delta^{n-1})) = H^n(\psi)\circ H^n(\varphi)(x+\mathrm{im}(\delta^{n-1}))$, so that $\mathcal{F}(\psi\circ\varphi) = \mathcal{F}(\psi)\circ\mathcal{F}(\varphi)$. Finally, $\mathcal{F}(\mathrm{Id}_X)(x+\mathrm{im}(\delta^{n-1})) = \mathrm{Id}_{X^n}(x+\mathrm{im}(\delta^{n-1})) = x + \mathrm{im}(\delta^{n-1}) = \mathrm{Id}_{H^n(X)}(x + \mathrm{im}(\delta^{n-1})) = \mathrm{Id}_{\mathcal{F}(X)}(x + \mathrm{im}(\delta^{n-1}))$. This completes the proof. $\square$

We have the analogous statement for homology, that we provide for convenience (though we do not prove it here).

**Proposition 2.2.12.** *Let $A$ be a $k$-algebra. Then taking homology of chain complexes of $A$ is functorial, and gives a functor from* $\mathrm{Ch}(\mathrm{Mod}(A))$ *to* $\mathrm{Gr}(\mathrm{Mod}(A))$.

A chain complex $(X, \delta)$ is called *exact* or *acyclic* if $H_*(X) = 0$ or, equivalently, if $\ker(\delta_n) = \mathrm{im}(\delta_{n+1})$ for all $n \in \mathbb{Z}$, and we have the analogous case occuring for cochain complexes. A *quasi-isomorphism* of chain complexes $X$ and $Y$ is a chain map $f : X \to Y$ such that the induced map $H_*(f) : H_*(X) \to H_*(Y)$ is an isomorphism of graded $A$-modules, and we have the analogous definition for a quasi-isomorphism of cochain complexes.

We have at last encountered (co)homology, and are well on our way to defining both the eponymous Hochschild cohomology and the cohomology of groups, which we will need for considering the Hochschild cohomology of group algebras. Before we do, we need some more concepts. Given an $A$-module $U$, a *projective resolution of* $U$ is a pair $(P, \pi)$ consisting of a chain complex $P$ of projective $A$-modules, such that $P_i = 0$ for all $i < 0$, and a quasi-isomorphism $\pi : P \to U$, viewing $U$ as a chain complex concentrated in degree zero. If $P$ has differential $\delta$ we have the following visualisation:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\delta_2"] \arrow[d] & P_1 \arrow[r, "\delta_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\pi_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & U \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} \tag{2.7}$$

where $\pi_n = 0$ for all $n \neq 0$. Taking homology of both complexes and considering that $\pi$ is a quasi-isomorphism forces exactness of the following chain complex

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi_0} U \longrightarrow 0 \ . \tag{2.8}$$

To see this, first note that $\ker(\delta_n)/\mathrm{im}(\delta_{n+1}) = 0$ for all $n > 0$, where the left hand side of the equality is the homology of the top row of (2.7), and the right hand side is the homology of the bottom row. Next, in degree 0 we have $P_0/\mathrm{im}(\delta_1) \cong U$ (again by considering homology of each row). Finally, by commutativity of each square in (2.7), we have that $\pi_0 \circ \delta_1 = 0$, whence $\mathrm{im}(\delta_1) \subseteq \ker(\pi_0)$ and we obtain $U \cong P_0/\mathrm{im}(\delta_1) \cong P_0/\ker(\pi_0)$, so that $\pi_0$ is surjective and consequently $\mathrm{im}(\delta_1) = \ker(\pi_0)$. If it is clear from the context, we write $\pi = \pi_0$.

**Remark 2.2.13.** Given an $A$-module $U$, one can always construct a projective resolution of $U$. In fact, one can always construct a *free* resolution of $U$, and therefore projective, in the following manner. Let $S \subseteq U$ be a generating set for $U$ as an $A$-module, and let $P_0$ be the free $A$-module with basis indexed by $S$, $\{e_s \mid s \in S\}$ so that $P_0 \cong \bigoplus_{s \in S} A \cdot e_s$. Since $P_0$ is free, there is a (unique) surjective map $P_0 \to U$ sending $e_s \mapsto s$: define $\pi_0$ to be this surjection. Now let $P_1$ be the free $A$-module with basis indexed by a generating set of $\ker(\pi_0)$, and define $\delta_1$ to be the composition of the projection and inclusion maps $P_1 \to \ker(\pi_0) \to P_0$. Then it is clear that by construction, $\mathrm{im}(\delta_1) = \ker(\pi_0)$ and the sequence is exact in degree 0. Continuing in this manner, we obtain a free resolution as desired.

Let $(X, \delta)$, $(Y, \varepsilon)$ be chain complexes of $A$-modules. A *chain homotopy from* $X$ *to* $Y$ is a graded map of degree 1, $h : X \to Y$. Two chain maps $f, g : X \to Y$ are called *homotopic*, denoted $f \sim g$, if there is a homotopy $h : X \to Y$ such that $f - g = h \circ \delta + \varepsilon \circ h$. Explicitly, at the level of $A$-modules, we have $f_n, g_n : X_n \to Y_n$ are such that $f_n - g_n = h_{n-1} \circ \delta_n + \varepsilon_{n+1} \circ h_n$ for all $n \in \mathbb{Z}$.

A chain map $f : X \to Y$ is called a *homotopy equivalence* if there is a chain map $g : Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$, and in this case we write $X \simeq Y$. A simple check of the definitions confirms that we have $X \simeq 0$ as complexes, if and only if $\mathrm{Id}_X \sim 0$ as chain maps. In this case, we say that $X$ is *contractible*.

We remark that the analogous definitions hold for $X$ and $Y$ cochain complexes.

**Proposition 2.2.14.** *Let $A$ be a $k$-algebra, $(X, \delta), (Y, \varepsilon) \in \mathrm{Ch}(\mathrm{Mod}(A))$ and suppose that $h : X \to Y$ is a homotopy. Then the map $f : X \to Y$ defined by $f = h \circ \delta + \varepsilon \circ h$ is a chain map. What is more, the induced map on homology, $H_*(f) : H_*(X) \to H_*(Y)$, is the zero map.*

*Proof.* The following diagram may be used to verify indices throughout the proof:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[dl] & X_n \arrow[r, "\delta_n"] \arrow[dl, "h_n"'] & X_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "\varepsilon_{n+1}"'] & Y_n \arrow[r, "\varepsilon_n"'] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

At the level of $A$-modules, we have, for all $n \in \mathbb{Z}$, $f_n : X_n \to Y_n$, $f_n = h_{n-1} \circ \delta_n + \varepsilon_{n+1} \circ h_n$. One verifies that

$$\varepsilon_n \circ f_n = \varepsilon_n \circ (h_{n-1} \circ \delta_n + \varepsilon_{n+1} \circ h_n) = \varepsilon_n \circ h_{n-1} \circ \delta_n + \varepsilon_n \circ \varepsilon_{n+1} \circ h_n = \varepsilon_n \circ h_{n-1} \circ \delta_n.$$

On the other hand we have

$$f_{n-1} \circ \delta_n = (h_{n-2} \circ \delta_{n-1} + \varepsilon_n \circ h_{n-1}) \circ \delta_n = h_{n-2} \circ \delta_{n-1} \circ \delta_n + \varepsilon_n \circ h_{n-1} \circ \delta_n = \varepsilon_n \circ h_{n-1} \circ \delta_n,$$

so that $\varepsilon_n \circ f_n = f_{n-1} \circ \delta_n$ and $f$ is indeed a chain map as required.

Now, for all $n \in \mathbb{Z}$, let $x \in \ker(\delta_n)$ so that $\delta_n(x) = 0$. Then

$$f_n(x) = h_{n-1}\delta_n(x) + \varepsilon_{n+1}h_n(x) = \varepsilon_{n+1}h_n(x)$$

and we see that $f_n(x) \in \mathrm{im}(\varepsilon_{n+1})$. Thus the induced map on homology, $H_n(f_n) : H_n(X_n) \to H_n(Y_n)$ sends $H_n(f_n)(x + \mathrm{im}(\delta_{n+1})) = f_n(x) + \mathrm{im}(\varepsilon_{n+1}) = 0 + \mathrm{im}(\varepsilon_{n+1})$, completing the proof. $\square$

The next result provides a shortcut to showing that a given chain complex is in fact exact, crucial to finding projective resolutions of $A$-modules.

**Proposition 2.2.15.** *Let $A$ be a $k$-algebra and $X$ be a chain complex of $A$-modules. If $X \simeq 0$ then $X$ is exact.*

*Proof.* Let $X \simeq 0$ as in the statement. Then as we have already seen, this means $\mathrm{Id}_X \sim 0$. In other words, there is a homotopy $h : X \to X$ such that $\mathrm{Id}_{X_n} = h_{n-1} \circ \delta_n + \delta_{n+1} \circ h_n$ for all $n \in \mathbb{Z}$. By Proposition 2.2.14, we have therefore that $H_n(\mathrm{Id}_{X_n}) = 0$, the zero map on homology. On the other hand, since by Proposition 2.2.12 taking homology is functorial, $\mathrm{Id}_{H_n(X_n)} = H_n(\mathrm{Id}_{X_n})$, so that the identity map on $H_n(X_n)$ is zero for all $n \in \mathbb{Z}$. This can only be the case, of course, if $H_n(X_n) = 0$ for all $n$, completing the proof. $\square$

**Definition 2.2.16.** Let $A$ be a $k$-algebra, $U, V$ be $A$-modules and $n \in \mathbb{Z}_{\geq 0}$. Then we define the $k$-vector space $\mathrm{Ext}^n_A(U, V)$ in the following way. First, choose a projective resolution of $U$ as in (2.7). Next, apply the contravariant functor $\mathrm{Hom}_A(-, V)$ (see Example 2.2.5) to $P$, explicitly to the top row of (2.7), to obtain a cochain complex:

$$\mathrm{Hom}_A(P_0, V) \xrightarrow{\delta^0} \mathrm{Hom}_A(P_1, V) \xrightarrow{\delta^1} \mathrm{Hom}_A(P_2, V) \longrightarrow \cdots, \tag{2.9}$$

with differential $\delta^n : \mathrm{Hom}(P_n, V) \to \mathrm{Hom}(P_{n+1}, V)$ given by $\delta^n(\varphi) = \varphi \circ \delta_{n+1}$ for all $\varphi : P_n \to V$. Finally, take cohomology of the cochain complex (2.9), to obtain

$$\mathrm{Ext}^n_A(U, V) = H^n(\mathrm{Hom}_A(P, V)).$$

**Remark 2.2.17.** We mention, without proof, that the definition of Ext is independent of the choice of projective resolution $P$ - as it should be in order to be well-defined. As ever, the interested reader is directed to Weibel's treatise on homological algebra [75, Chapter 2].

The following result, known as the *Eckmann-Shapiro Lemma* will be used in the sequel to find an additive decomposition of Hochschild cohomology (see Theorem 2.3.16).

**Proposition 2.2.18** ([52, Proposition 2.20.7])**.** *Let $G$ be a group and $H$ a subgroup of $G$. Let $U$ be a $kG$-module and $V$ a $kH$-module. For any $n \geq 0$ we have natural isomorphisms of $k$-vector spaces*

$$\mathrm{Ext}^n_{kG}(\mathrm{Ind}^G_H(V), U) \cong \mathrm{Ext}^n_{kH}(V, \mathrm{Res}^G_H(U)),$$
$$\mathrm{Ext}^n_{kG}(U, \mathrm{Ind}^G_H(V)) \cong \mathrm{Ext}^n_{kH}(\mathrm{Res}^G_H(U), V).$$

For $U$ and $A$-module, we have a graded $k$-algebra product defined on $\mathrm{Ext}^*_A(U, U)$; though we do not define it formally here, we include the following result so as to be able to refer to "the $k$-algebra $\mathrm{Ext}^*_A(U, U)$".

**Proposition 2.2.19.** *Let $A$ be a $k$-algebra and $U$ be an $A$-module. Then the graded $k$-vector space $\mathrm{Ext}^*_A(U, U) := \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(U, U)$ is a unital associative $k$-algebra.*

Returning to the question posed in Remark 2.2.9, we are now equipped to explore equivalences between algebras, weaker than a Morita equivalence. Let $A$ be a $k$-algebra, and consider $\mathrm{Ch}(\mathrm{Mod}(A))$. We define the *homotopy category of complexes over* $\mathrm{Mod}(A)$ to be the category whose objects are those of $\mathrm{Ch}(\mathrm{Mod}(A))$, but whose morphisms are equivalence classes of morphisms in $\mathrm{Ch}(\mathrm{Mod}(A))$ under homotopy equivalence. That is, $\mathrm{Hom}_{K(\mathrm{Mod}(A))}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathrm{Mod}(A))}(X, Y)/\sim$ where $f \sim g$ if and only if $f$ and $g$ are homotopic.

Let $\mathrm{Proj}(A)$ be the subcategory of $\mathrm{Mod}(A)$ consisting of projective $A$-modules. The *bounded derived category* of $\mathrm{Proj}(A)$, denoted $D^b(\mathrm{Mod}(A))$ is then defined as the additive subcategory of $K(\mathrm{Proj}(A))$ whose objects are those of $\mathrm{Ch}(\mathrm{Proj}(A))$ that are bounded below and have nonzero homology in finitely many degrees).

A *derived equivalence* between two $k$-algebras $A$ and $B$ is an equivalence of bounded derived categories $D^b(\mathrm{Mod}(A)) \cong D^b(\mathrm{Mod}(B))$, and it is trivial from the definition that a Morita equivalence implies a derived equivalence.

The *stable module category* $\underline{\mathrm{Mod}}(A)$ is the $k$-linear category that has the same objects as $\mathrm{Mod}(A)$ and identifies with zero all morphisms that factor through a projective module. A *stable equivalence* between $A$ and $B$ is an equivalence of $k$-linear categories $\underline{\mathrm{Mod}}(A) \cong \underline{\mathrm{Mod}}(B)$. In practice, "most" stable equivalences that occur are of the following form.

**Definition 2.2.20** (Broué)**.** There is a *stable equivalence of Morita type* between $A$ and $B$ if there is an $(A, B)$-bimodule $M$ and a $(B, A)$-bimodule $N$, such that $M$, $N$ are finitely generated and projective as left/right modules, with $M \otimes_B N \cong A$ in $\underline{\mathrm{Mod}}(A \otimes_k A^{\mathrm{op}})$ and $N \otimes_A M \cong B$ in $\underline{\mathrm{Mod}}(B \otimes_k B^{\mathrm{op}})$. In this case, the functors $M \otimes_B -, N \otimes_A -$ induce an equivalence $\underline{\mathrm{Mod}}(A) \cong \underline{\mathrm{Mod}}(B)$.

**Remark 2.2.21.** The definition of a stable equivalence of Morita type above is equivalent to the situation that $M \otimes_B N \cong A \oplus X$ in $\mathrm{Mod}(A \otimes_k A^{\mathrm{op}})$ for some projective $(A, A)$-bimodule $X$, and $N \otimes_A M \cong B \oplus Y$ in $\mathrm{Mod}(B \otimes_k B^{\mathrm{op}})$ for some $(B, B)$-bimodule $Y$. Setting $X$ and $Y$ to 0 immediately shows that a Morita equivalence between $A$ and $B$ implies a stable equivalence of Morita type between $A$ and $B$.

As in the remark above, it is trivial to show that a Morita equivalence is a specific instance of a stable equivalence of Morita type. Much less trivial, a result of Rickard [64, Corollary 5.5] and of Broué [12, Proposition 5.2] shows that for self-injective algebras, a derived equivalence implies a stable equivalence of Morita type. Thus we have the following chain of implications, for self-injective algebras $A$ and $B$:

$A$ and $B$ are Morita equivalent $\implies$ $A$ and $B$ are derived equivalent $\implies$ there is a stable equivalence of Morita type between $A$ and $B$ $\implies$ $A$ and $B$ are stably equivalent.

## 2.3 Hochschild cohomology

There are numerous cohomology theories within algebra and mathematics more generally. Arguably the most important and certainly the most pervasive in group representation theory, are the theories of the cohomology of groups, of associative algebras, and of Lie algebras, the second item on this list being the eponymous Hochschild cohomology. Given a group $G$, one of our main guiding principles for this thesis is the question: what does the first Hochschild cohomology of the group algebra $kG$ and its blocks actually look like? As it happens, the Hochschild cohomology of $kG$ and its blocks is (perhaps unsurprisingly) intimately linked to the group cohomology of $G$, and so we dedicate this section to defining these cohomology theories and looking at some of their elementary properties.

**Definition 2.3.1.** Let $A$ be a $k$-algebra and $M$ be an $(A, A)$-bimodule, viewed as an $A \otimes_k A^{\mathrm{op}}$-module. Then the *Hochschild cohomology with coefficients in $M$ of $A$* is the graded $k$-vector space

$$HH^*(A; M) := \mathrm{Ext}^*_{A \otimes_k A^{\mathrm{op}}}(A, M),$$

and the *Hochschild cohomology of $A$* is the graded $k$-algebra

$$HH^*(A) := HH^*(A; A).$$

For the majority of this thesis, we will be interested only in $HH^0(A)$ and $HH^1(A)$. We would like to be able to describe these explicitly and therefore will dedicate the next few pages to doing so. As a précis of this, we state here the following well-known results: there is an isomorphism of $k$-algebras

$$HH^0(A) \cong Z(A), \tag{2.10}$$

and an isomorphism of $k$-vector spaces

$$HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A), \tag{2.11}$$

where $\mathrm{Der}(A) = \{f \in \mathrm{Hom}_k(A, A) \mid f(ab) = af(b) + f(a)b \text{ for all } a, b \in A\}$ and $\mathrm{IDer}(A)$ is the $k$-subspace of $\mathrm{Der}(A)$ consisting of those $f$ such that there is some $a \in A$ with the property that $f(b) = ba - ab$ for all $b \in A$.

Our first step to showing this is, of course, determining an explicit projective resolution, $(P, \pi)$, of $A$ as an $A \otimes_k A^{\mathrm{op}}$-module, and so $(P, \pi)$ takes the following form,

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi_0} A \longrightarrow 0\ . \tag{2.12}$$

Let $n \in \mathbb{Z}$. First, for $n > 0$, let $A^{\otimes n}$ be the $n$-fold tensor product of $A$ with itself and define $A^{\otimes 0} = k$. Then, we define $P$ in (2.12) as follows: let $P_n = A^{\otimes(n+2)}$ for $n \geq 0$ and, of course, $P_n = 0$ for $n < 0$. We wish to show that the $P_n$ are projective $A \otimes_k A^{\mathrm{op}}$-modules, and remark that in the more general case of $k$ a commutative ring, we would need the additional condition that $A$ is a projective $k$-module for this to be the case; restricting to our case of $k$ a field simplifies this. This is because $A$ is a vector space over $k$, or equivalently "$A$ is free as a $k$-module", whence $A$ is projective as a $k$-vector space (as previously mentioned this is the case for *all* finite-dimensional vector spaces over a field).

The action of $A \otimes_k A^{\mathrm{op}}$ on $P_n$ is given by $(a \otimes b) \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = aa_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}b$ for all $a, b, a_i \in A$, $i = 0, \ldots, n+1$. Let $\mathcal{B}$ be a basis for $A$ as a $k$-vector space, of cardinality $m$. Then $P_n$ is a free $A \otimes_k A^{\mathrm{op}}$-module with basis

$$\{1 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m \otimes 1 \mid a_i \in \mathcal{B}, i = 1, \ldots, m\},$$

so that

$$P_n = \bigoplus_{\substack{a_i \in \mathcal{B} \\ i=1,\ldots,m}} (A \otimes_k A^{\mathrm{op}}) \cdot (1 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m \otimes 1) \cong (A \otimes_k A^{\mathrm{op}})^{\oplus m^n}$$

as $A \otimes_k A^{\mathrm{op}}$-modules. Thus the $P_n$ are projective $A \otimes_k A^{\mathrm{op}}$-modules.

We define the differential $\delta$ on $P$ as follows: for all $n \geq 1$, $\delta_n : P_n \to P_{n-1}$ is given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \tag{2.13}$$

Finally, define the quasi-isomorphism $\pi : P_0 \to A$ to simply be multiplication in $A$.

We check that this does indeed define a projective resolution of $A$ as an $A \otimes_k A^{\mathrm{op}}$-module. We have already seen that the $P_n$ are projective $A \otimes_k A^{\mathrm{op}}$-modules for all $n \geq 0$ . Next we will show that $\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$ is a chain complex. For all $n \geq 0$, let $\delta_{n,i} : P_n \to P_{n-1}$ be the map sending $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \mapsto a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$, extended linearly, so that $\delta_n = \sum_{i=0}^{n} (-1)^i \delta_{n,i}$. The maps $\delta_{n,i}$ are $A \otimes_k A^{\mathrm{op}}$-bimodule homomorphisms in the obvious way, and therefore extend to bimodule homomorphisms $\delta_n$.

For all $n \geq 1$, consider

$$\delta_{n-1} \circ \delta_n = \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \delta_{n-1,j} \circ \delta_{n,i}. \tag{2.14}$$

One checks that for $j \geq i$, $\delta_{n-1,j} \circ \delta_{n,i} = \delta_{n-1,i} \circ \delta_{n,j+1}$, and that for $j < i$, $\delta_{n-1,j} \circ \delta_{n,i} = \delta_{n-1,i-1} \circ \delta_{n,j}$. Thus in (2.14), if $j \geq i$, the summand indexed by $(i,j)$ and the summand indexed by $(j+1,i)$ will be equal but with opposite sign, and if $j < i$, the summand indexed by $(i,j)$ and the summand indexed by $(j,i-1)$ will be equal but with opposite sign. We see that by appropriately pairing these summands, $\delta_{n-1} \circ \delta_n = 0$, and it is trivial to check that $\pi \circ \delta_1 = 0$. Thus $\cdots \to P_2 \to P_1 \to P_0 \to 0$ is a chain complex of projective $A \otimes_k A^{\mathrm{op}}$-modules, and (2.12) is a chain complex of $A \otimes_k A^{\mathrm{op}}$-modules.

Now we show that the complex (2.12) is exact. To do so, we will in fact show that it is contractible as a complex of $A \otimes_k A^{\mathrm{op}}$-modules, by finding a suitable homotopy $h$, and then applying Proposition 2.2.15. Thus we need to find $h$ such that $h_n : P_n \to P_{n+1}$, and such that $\mathrm{Id}_{P_n} = h_{n-1} \circ \delta_n + \delta_{n+1} \circ h_n$. We do this in two parts, considering the complex as one of right $A$-modules, and then one of left $A$-modules. First define $h$ as follows:

$$h_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1},$$

extended linearly, is evidently a right $A$-module homomorphism. Now,

$$\begin{aligned} h_{n-1} \circ \delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) &= h_{n-1}\left(\sum_{i=0}^{n}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}\right) \\ &= \sum_{i=0}^{n}(-1)^i h_{n-1}(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &= \sum_{i=0}^{n}(-1)^i 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}, \end{aligned} \tag{2.15}$$

whilst

$$\begin{aligned} \delta_{n+1} \circ h_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) &= \delta_{n+1}(1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\ &= \sum_{\substack{i=-1 \\ a_{-1}=1}}^{n} (-1)^{i+1} 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\ &= a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} + \\ &+ \sum_{i=0}^{n}(-1)^{i+1} 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \end{aligned} \tag{2.16}$$

For all $i = 0, \ldots, n$, the second term in the sum (2.16) is equal and with opposite sign to the term (2.15), whence

$$(h_{n-1} \circ \delta_n + \delta_{n+1} \circ h_n)(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1},$$

and so $h$ is indeed a homotopy of (2.12) considered as right $A$-modules, and (2.12) is indeed contractible as a complex of right $A$-modules.

Now, defining

$$h_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1,$$

and extending linearly, $h$ is a left $A$-module homomorphism, and the analogous computations (we do not repeat this here) show that $h$ defined in this way is a homotopy of (2.12) considered as left $A$-modules, which is again contractible as a complex of left $A$-modules. Whence (2.12) is a contractible chain complex of $A \otimes_k A^{\mathrm{op}}$-modules, and therefore by Proposition 2.2.15 is an exact chain complex of $A \otimes_k A^{\mathrm{op}}$-modules.

Whence we have constructed a projective resolution of $A$ as an $A \otimes_k A^{\mathrm{op}}$-module, and we will fix the notation of $(P, \pi)$ and $\delta$ for the remainder of this section.

We are now in a position to describe, explicitly, the low-dimensional Hochschild cohomology groups. By direct use of the definitions above, we have that $HH^0(A) = \ker(\delta^0)$, that is, the abelian group consisting of all $f \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes 2}, A)$ such that $f(ab \otimes c) = f(a \otimes bc)$ for all $a, b, c \in A$. We also have that $HH^1(A) = \ker(\delta^1)/\mathrm{im}(\delta^0)$, where $\ker(\delta^1)$ consists of all $f \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes 3}, A)$ such that $f(a \otimes bc \otimes d) = f(ab \otimes c \otimes d) + f(a \otimes b \otimes cd)$ for all $a, b, c, d \in A$, and $\mathrm{im}(\delta^0)$ consists of all $f \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes 3}, A)$ such that there exists some $g \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes 2}, A)$ with the property that $f(a \otimes b \otimes c) = g(ab \otimes c) - g(a \otimes bc)$ for all $a, b, c \in A$.

As explicit as this is, it is still not immediately intuitive how to find or construct elements in $HH^0(A)$ and $HH^1(A)$, and though it bears resemblance to the isomorphisms in (2.10) and (2.11), it is not exactly the same. Luckily, we can simplify things further.

**Lemma 2.3.2.** *Let $A$ be a $k$-algebra. Then, for $n \geq 0$ we have an isomorphism of abelian groups*

$$\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes(n+2)}, A) \cong \mathrm{Hom}_k(A^{\otimes n}, A).$$

*Proof.* Define the isomorphism $\Phi : \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes(n+2)}, A) \to \mathrm{Hom}_k(A^{\otimes n}, A)$ by

$$\Phi(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = f(1 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1)$$

for all $f \in \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(A^{\otimes(n+2)}, A)$, $a_1, \cdots, a_n \in A$, if $n > 0$, and by $\Phi(f)(1) = 1 \otimes 1$ if $n = 0$ (recall that we define $A^{\otimes 0} = k$). It is clear that $\Phi(f + f') = \Phi(f) + \Phi(f')$ and so a group homomorphism. We define

$$\Psi(g)(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \cdot g(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}.$$

A trivial verfication shows that $\Phi^{-1} = \Psi$, completing the proof. ☐

Perhaps unsurprisingly, the isomorphism in Lemma 2.3.2 induces a simplified route to calculating Hochschild cohomology.

**Proposition 2.3.3.** *Let $A$ be a $k$-algebra, $n \geq 0$ and fix the notation $C^n(A) = \mathrm{Hom}_k(A^{\otimes n}, A)$. Define maps $\varepsilon^n : C^n(A) \to C^{n+1}(A)$ by*

$$\begin{aligned}\varepsilon^n(f)(a_0 \otimes \cdots \otimes a_n) &= a_0 \cdot f(a_1 \otimes \cdots \otimes a_n) + \\ &+ \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n) + (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) \cdot a_n\end{aligned}$$

*for all $f \in C^n(A)$. Then $(C(A), \varepsilon)$ is a cochain complex, isomorphic to $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P, A)$ where $P$ is as defined in (2.12). In particular, $HH^*(A)$ is isomorphic (as cohomology groups) to the cohomology of the complex $(C(A), \varepsilon)$.*

*Proof.* By Lemma 2.3.2 we have that $C^n(A)$ is isomorphic to $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_n, A)$ in (2.12). Let $f \in C^n(A)$. We will show that $\varepsilon^n(f)$ agrees with the map that results from chasing $f$ from $C^n(A)$ to $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_n, A)$ via $\Psi$, from $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_n, A)$ to $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_{n+1}, A)$ via $\delta^n$, and from $\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_{n+1}, A)$ to $C^{n+1}(A)$ via $\Phi$, where $\Phi$ and $\Psi$ are as in Lemma 2.3.2. In other words, we have the following picture

$$\begin{tikzcd}
\mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_n, A) \arrow[r, "\delta^n"] & \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(P_{n+1}, A) \arrow[d, "\Phi"] \\
C^n(A) \arrow[u, "\Psi"] \arrow[r, "\varepsilon^n"'] & C^{n+1}(A)
\end{tikzcd},$$

and we will show that $\varepsilon^n = \Phi \circ \delta^n \circ \Psi$. Let $a_0, a_1, \ldots, a_n \in A$, and first note that

$$\begin{aligned}
\delta_{n+1}(1 \otimes a_0 \otimes \cdots \otimes a_n \otimes 1) &= \sum_{\substack{i=-1 \\ a_{-1}=a_{n+1}=1}}^{n} (-1)^{i+1} 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes 1 \\
&= \sum_{\substack{i=0 \\ a_{-1}=a_{n+1}=1}}^{n+1} (-1)^{i} 1 \otimes a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n \otimes 1 \\
&= a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1 + \\
&\quad + \sum_{i=1}^{n} (-1)^i 1 \otimes a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n \otimes 1 + \\
&\quad\quad + (-1)^{n+1} (1 \otimes a_0 \otimes \ldots \otimes a_n).
\end{aligned}$$

Then

$$\begin{aligned}
\Phi(\delta^n(\Psi(f))(a_0 \otimes \cdots \otimes a_n) &= \Phi(\Psi(f) \circ \delta_{n+1})(a_0 \otimes \cdots \otimes a_n) \\
&= (\Psi(f) \circ \delta_{n+1})(1 \otimes a_0 \otimes \cdots \otimes a_n \otimes 1) \\
&= \Psi(f)(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1) + \\
&\quad + \sum_{i=1}^{n} (-1)^i \Psi(f)(1 \otimes a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n \otimes 1) + \\
&\quad\quad + (-1)^{n+1} \Psi(f)(1 \otimes a_0 \otimes \ldots \otimes a_n) \\
&= a_0 \cdot f(a_1 \otimes \cdots \otimes a_n) + \\
&\quad + \sum_{i=1}^{n} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n) + \\
&\quad\quad + (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) \cdot a_n \\
&= \varepsilon^n(a_0 \otimes \cdots \otimes a_n).
\end{aligned}$$

This completes the proof. □

We are ready to compute $HH^0(A)$ and $HH^1(A)$ explicitly, for the second time, via the isomorphism of Lemma 2.3.2. First note that $C^0(A) = \mathrm{Hom}_k(A^{\otimes 0}, A) = \mathrm{Hom}_k(k, A) \cong A$, for any map

$f \in \mathrm{Hom}_k(k, A)$ is uniquely determined by its value at $f(1_k)$. Thus, we are taking cohomology of the following cochain complex

$$A \xrightarrow{\varepsilon^0} C^1(A) \xrightarrow{\varepsilon^1} C^2(A) \to \cdots,$$

where $\varepsilon^0(a)(b) = ab - ba$ and $\varepsilon^1(f)(a \otimes b) = af(b) - f(ab) + bf(a)$ for all $a, b \in A$. We obtain

$$HH^0(A) \cong \ker(\varepsilon^0) = \{a \in A \mid ab = ba \text{ for all } b \in A\} = Z(A),$$

and we have $HH^1(A) \cong \ker(\varepsilon^1)/\mathrm{im}(\varepsilon^1)$ where

$$\begin{aligned}\ker(\varepsilon^1) &= \{f \in C^1(A) \mid \varepsilon^1(f)(a \otimes b) = 0 \text{ for all } a, b \in A\} \\ &= \{f \in C^1(A) \mid f(ab) = af(b) + f(a)b \text{ for all } a, b \in A\},\end{aligned}$$

and

$$\begin{aligned}\mathrm{im}(\varepsilon^0) &= \{\varepsilon^0(a) \mid a \in A\} \\ &= \{f \in C^1(A) \mid \text{ there is some } a \in A \text{ such that } f(b) = ba - ab \text{ for all } b \in A\}.\end{aligned}$$

**Definition 2.3.4.** Let $A$ be a $k$-algebra. We say that a $k$-linear map $f : A \to A$ is a *derivation* of $A$ if $f(ab) = af(b) + f(a)b$. A derivation $f$ in $A$ is called *inner* if there is some $a \in A$ such that $f(b) = ba - ab$ for all $b \in A$. We write $\mathrm{Der}(A)$ for the $k$-vector space of derivations on $A$, and $\mathrm{IDer}(A)$ for the $k$-subspace of $\mathrm{Der}(A)$, of inner derivations on $A$.

Thus we have shown that the isomorphisms of (2.10) and (2.11) hold, in particular that $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$, providing us with a convenient and explicit description of elements of $HH^1(A)$ that we will use throughout this thesis. Often when we write $HH^1(A)$, we are really thinking about a set of class representatives of derivations on $A$, modulo inner derivations: this will be our point of view throughout.

**Remark 2.3.5.** Given a group $G$ and a block decomposition $kG = B_0 \oplus B_1 \oplus \cdots \oplus B_n$, taking Hochschild cohomology on both sides gives an isomorphism of $k$-vector spaces,

$$HH^1(kG) \cong HH^1(B_0) \oplus HH^1(B_1) \oplus \cdots \oplus HH^1(B_n).$$

This may be seen directly from the definitions of Ext: when viewed as a functor, Ext is additive in both arguments [75, Proposition 3.3.4] and vanishes when applied to modules belonging to different direct factors (see [19, Section 3.4]).

We will now see one of the most compelling arguments for the importance of the Hochschild cohomology of algebras: it is invariant under certain categorical equivalences, whence allows explicit comparisons between algebras.

**Theorem 2.3.6** ([52, Proposition 2.20.10])**.** *Let $A$ and $B$ be $k$-algebras and suppose that $A$ and $B$ are Morita equivalent. Then there is an isomorphism of graded $k$-algebras $HH^*(A) \cong HH^*(B)$.*

One might ask, what about more relaxed equivalences, what can be said of the Hochschild cohomology there? The answers are known for all equivalences described in Section 2.2. Originally due to Happel and formalised by Rickard, we have the following.

**Theorem 2.3.7** ([64, Proposition 2.5]). *Let $A$ and $B$ be $k$-algebras and suppose that $A$ and $B$ are derived equivalent. Then there is an isomorphism of graded $k$-algebras $HH^*(A) \cong HH^*(B)$.*

**Remark 2.3.8.** Theorems 2.3.6 and 2.3.7 above hold more generally for $k$ any commutative ring.

As we relax the conditions on the equivalence that holds, we require a more strict hypothesis on the algebras in question.

**Theorem 2.3.9** ([50, Remark 2.13]). *Let $A$ and $B$ be symmetric $k$-algebras and suppose that there is a stable equivalence of Morita type between $A$ and $B$. Then there is an isomorphism of graded $k$-algebras $HH^*(A) \cong HH^*(B)$ in degree $n > 0$.*

As mentioned, the Hochschild cohomology of a group algebra $kG$, and the group cohomology of $G$ are intimately linked: we proceed now to describe the latter.

**Definition 2.3.10.** Let $G$ be a group, $M$ be a $kG$-module and $K$ an abelian group. Then the *cohomology of $G$ with coefficients in $M$* is the graded $k$-vector space

$$H^*(G; M) := \mathrm{Ext}^*_{kG}(k, M),$$

and the *cohomology of $G$ with coefficients in $K$* is the graded $k$-vector space

$$H^*(G; K) := \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, K).$$

We will most often be interested in the case where $M = k$, the trivial $kG$-module where elements of $G$ act trivially on elements of $k$, extended linearly, and with structural homomorphism equal to the augmentation homomorphism $\eta$. As in the case of Hochschild cohomology, we would like to be able to describe explicitly what the spaces $H^0(G; k)$ and $H^1(G; k)$ look like.

**Remark 2.3.11.** There is another case that we are interested in, namely when $k$ is algebraically closed, and $K = k^\times = k\backslash\{0\}$ in Definition 2.3.10. This will be discussed in the sequel, in reference to *twisted group algebras*, where we will be interested in *2-cocycles*, that is, elements of $H^2(G; k^\times)$.

Thus we want to know what a projective resolution of $k$ as a $kG$-module looks like, and use it to calculate $\mathrm{Ext}^*_{kG}(k, k)$, in a manner similar to how we presented it for Hochschild cohomology. Fortunately we can sidestep a lot of the work, using Hochschild cohomology of $kG$ to help us, and arrive quickly at the analogue to Proposition 2.3.3 for group cohomology. We state this without proof, however the interested reader may, as ever, consult Weibel [75] for details.

**Theorem 2.3.12.** *Let $G$ be a group, $n \in \mathbb{Z}_{\geq 0}$ and write $G^n$ for the direct product of $n$ copies of $G$, defining $G^0 = \{1\}$. Let $M$ be a $kG$-module, and set $C^n(G; M) = \{f : G^n \to M \mid f \text{ is a function}\}$, so that $C^0(G; M) \cong M$. For $n > 0$ define $k$-linear maps $d^n : C^n(G; M) \to C^{n+1}(G; M)$ by*

$$\begin{aligned} d^n(f)(g_0, g_1, \ldots, g_n) = g_0 \cdot f(g_1, \ldots, g_n) + \\ + \sum_{i=1}^{n} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_n) + (-1)^{n+1} f(g_0, \ldots, g_{n-1}), \end{aligned}$$

*for all $g_i \in G, i = 0, \ldots, n$, and $d^0 : M \to C^1(G; M)$, $d^0(m)(g) = g \cdot m - m$ for all $g \in G$ and all $m \in M$. Then $(C(G; M), d)$ is a cochain complex whose cohomology is isomorphic to $H^*(G; M)$.*

We remark that the proof boils down to taking the projective resolution $(P, \delta)$ of $A$ as an $A \otimes_k A^{\mathrm{op}}$-module as defined in (2.12), setting $A = kG$, and computing $(P \otimes_{kG} k, \delta \otimes_{kG} \mathrm{Id}_k)$: this yields a projective resolution of $k$ as a $kG$-module, which, after applying $\mathrm{Hom}_{kG}(-, k)$ to obtain $\mathrm{Hom}_{kG}(P \otimes_{kG} k, k)$ gives a cochain complex equal to that of Theorem 2.3.12.

Using the maps provided in Theorem 2.3.12, we are able to compute $H^0(G; M)$ and $H^1(G; M)$ explicitly. We are taking cohomology of the cochain complex

$$M \xrightarrow{d^0} C^1(G; M) \xrightarrow{d^1} C^2(G; M) \to \cdots,$$

where $d^0(m)(g) = g \cdot m - m$ and $d^1(f)(g, h) = g \cdot f(h) - f(gh) + f(g)$ for all $g, h \in G$ and all $m \in M$. Thus

$$H^0(G; M) \cong \ker(d^0) = \{m \in M \mid g \cdot m = m \text{ for all } g \in G\} = M^G,$$

and we have $H^1(G; M) \cong \ker(d^1)/\mathrm{im}(d^1)$ where

$$\begin{aligned}\ker(d^1) &= \{f : G \to M \mid d^1(f)(g, h) = 0 \text{ for all } g, h \in G\} \\ &= \{f : G \to M \mid f(gh) = f(g) + g \cdot f(h) \text{ for all } g, h \in G\},\end{aligned}$$

and

$$\begin{aligned}\mathrm{im}(d^0) &= \{d^0(g) \mid g \in G\} \\ &= \{f : G \to M \mid \text{ there is some } m \in M \text{ such that } f(g) = g \cdot m - m \text{ for all } g \in G\}.\end{aligned}$$

In particular, when $M = k$ on which $G$ acts trivially, we see that $\ker(d^0) = k$, $\ker(d^1) = \mathrm{Hom}(G, k)$ the space of all homomorphisms from $G$ to $k$, and $\mathrm{im}(d^0) = \{0\}$, so that

$$H^0(G; k) \cong k, \;\; H^1(G; k) \cong \mathrm{Hom}(G, k).$$

Returning to the general case, now consider $H^2(G; M) = \ker(d^2)/\mathrm{im}(d^1)$, where

$$\begin{aligned}\ker(d^2) &= \{f : G \times G \to M \mid d^2(f)(a, b, c) = 0, \text{ for all } a, b, c \in G\} \\ &= \{f : G \times G \to M \mid f(a, b) + f(ab, c) = f(a, bc) + a \cdot f(b, c)\},\end{aligned}$$

and $\mathrm{im}(d^1) = \{d^1(g) \mid g : G \to M\}$, that is, the set of all $f : G \times G \to M$ such that there is some $g : G \to M$ with the property that $f(a, b) = a \cdot g(b) - g(ab) + g(a)$ for all $a, b \in G$. Let $M$ now be an abelian group written multiplicatively, on which $G$ acts on the left, writing ${}^g m$ for the action of $G$ on $M$, for all $g \in G$ and $m \in M$. Then $\ker(d^2) = \{f : G \times G \mid f(a, b)f(ab, c) = f(a, bc)\,({}^a f(b, c))$ for all $a, b, c \in G\}$. Such an $f \in \ker(d^2)$ is called a 2-*cocycle*, and the defining property of $f$ in $\ker(d^2)$ is a called the 2-*cocycle identity*. Importantly, when $G$ acts trivially on $M$, for example if $M = k^\times = k \backslash \{0\}$, then $H^2(G; k^\times)$ parametrises *central extensions of* $G$; we have encountered central extensions in Example 2.1.14, and will look at them in more detail when we consider *twisted group algebras* in Chapter 5. The following definition gives us an alternative point of view of first and second cohomology groups, defined explicitly via cocycles.

**Definition 2.3.13.** Let $G$ be a group and $Z$ be an abelian group, written multiplicatively, on which $G$ acts (on the left). A 1-*cocycle of* $G$ *with coefficients in* $Z$ is a map $f : G \to Z$ satisfying

$f(xy) = f(x)({}^x f(y))$ for all $x, y \in G$. Denote by $Z^1(G; Z)$ the set of all 1-cocycles of $G$ with coefficients in $Z$. Let $B^1(G; Z) \subseteq Z^1(G; Z)$ be the set of maps $f : G \to Z$ for which there exists some $z \in Z$ such that $f(x) = ({}^x z)z^{-1}$ for all $x \in G$. The elements in $B^1(G; Z)$ are called the *1-coboundaries of $G$ with coefficients in $Z$*. The set $Z^1(G; Z)$ forms an abelian group under pointwise multiplication, with $B^1(G; Z)$ a subgroup, and the quotient group

$$H^1(G; Z) = Z^1(G; Z)/B^1(G; Z)$$

is called the *first cohomology group of $G$ with coefficients in $Z$*.

A *2-cocycle of $G$ with coefficients in $Z$* is a map $f : G \times G \to Z$ satisfying $f(x, y)f(xy, z) = f(x, yz)({}^x f(y, z))$ for all $x, y, z \in G$. Denote by $Z^2(G; Z)$ the abelian group of all 2-cocycles of $G$ with coefficients in $Z$. Let $B^2(G; Z) \subseteq Z^1(G; Z)$ be the subgroup of maps $f : G \times G \to Z$ for which there exists some $g : G \to Z$ such that $f(x, y) = g(x)({}^x g(y))g(xy)^{-1}$ for all $x, y \in G$. The elements in $B^2(G; Z)$ are called the *2-coboundaries of $G$ with coefficients in $Z$*, and the quotient group

$$H^2(G; Z) = Z^2(G; Z)/B^2(G; Z)$$

is called the *second cohomology group of $G$ with coefficients in $Z$*.

It is trivial to see that these definitions are equivalent to those derived "from the ground up", that is, by using projective resolutions of $Z$ as a $\mathbb{Z}G$-module or by applying Theorem 2.3.12. The cocycle point of view has been studied historically in its own right, before the unifying themes of (co)homology and category theory were formalised.

We will need some results from low degree group cohomology. Recall that an *exact sequence* of groups is a sequence of groups connected by homomorphisms such that the image of one homomorphism is equal to the kernel of the next.

**Theorem 2.3.14** ([26, Corollary 7.2.3])**.** *Let $M$ be a $kG$-module and $N$ be a normal subgroup of $G$. Then there is an exact sequence of abelian groups*

$$0 \to H^1(G/N; M^N) \to H^1(G; M) \to H^1(N; M)^{G/N} \to H^2(G/N; M^N) \to H^2(G; M). \quad (2.17)$$

The maps between the abelian groups appearing in Eq.(2.17) are given, in that order, by the following.

(i) The zero map.

(ii) The inflation map on 1-cocycles, $\inf_N^G(f)(g) = f(gN)$ for all $g \in G$ and $f \in Z^1(G/N; M^N)$.

(iii) The restriction map on 1-cocycles, $\mathrm{res}_N^G(f)$ for all $f \in Z^1(G; M)$, simply the restriction of $f$ to $N$.

(iv) The transgression map: the reader is invited to consult [40, Theorems 1.1.11 and 1.1.12] for full details of this map, whose level of technicality is not required here.

(v) The inflation map on 2-cocycles, defined analagously to (ii): $\inf_N^G(\alpha)(g, h) = \alpha(gN, hN)$ for all $g, h \in G$ and $\alpha \in Z^2(G/N; M^N)$.

Note that the maps (i) to (v) above all take coboundaries to coboundaries, and so are well-defined on the sequence (2.17). We will write inf and res when the context is clear, and we will refer to this sequence as the *fundamental exact sequence*.

We will need one more important map which we will see in the sequel (Proposition 2.3.30) allows us to find a basis of derivations of $HH^1(kG)$.

**Definition 2.3.15.** Let $M$ be a $kG$-module and $H$ a subgroup of $G$. We define the *corestriction map*,

$$\mathrm{cor}_H^G : Z^1(H;M) \to Z^1(G;M),$$

as follows. Take a set $X$ of left coset representatives of $H$ in $G$, and for each $x \in X$ and $g \in G$, let $y \in X$ be the unique coset representative such that $gyH = xH$. Then

$$\mathrm{cor}_H^G(f)(g) = \sum_{x \in X} x \cdot f(x^{-1}gy)$$

for all $f \in Z^1(H;M)$ and all $g \in G$. Note that $\mathrm{cor}_H^G(h) \in B^1(G;M)$ for all $h \in B^1(H;M)$, so that the corestriction map induces a well defined map $H^1(H;M) \to H^1(G;M)$.

### 2.3.1 The centraliser decomposition

Recall that given a group $G$ with blocks $B_0, B_1, \ldots$, one of our main guiding principles for this thesis is the question: what do $HH^1(kG)$ and $HH^1(B_i)$ for $i = 0, 1, \ldots$, actually look like? A first step in this direction would be to determine what the dimensions of these spaces are as $k$-vector spaces. For this, we have a shortcut known as the *centraliser decomposition*, a powerful link between the Hochschild cohomology of $kG$ and the group cohomology of $G$. This shortcut is of huge importance and sees numerous applications throughout our work.

**Theorem 2.3.16** ([5, Theorem 2.11.2(ii)])**.** *As a graded $k$-vector spaces, we have a canonical isomorphism*

$$HH^*(kG) \cong \bigoplus_x H^*(C_G(x);k), \tag{2.18}$$

*where $x$ runs over a complete set of conjugacy class representatives of $G$.*

*Proof.* First recall that there is a $k$-algebra isomorphism $kG \otimes_k (kG)^{\mathrm{op}} \cong k(G \times G)$ induced by the map sending $g \otimes h$ to $(g, h^{-1})$. Let $H = G \times G$, then $kH$ has a $k$-subalgebra $k(\triangle G)$, where $\triangle G = \{(g,g) \mid g \in G\} \leq H$ the *diagonal subgroup*. There is then an isomorphism of $k$-algebras $kG \cong k(\triangle G)$ induced by the diagonal map sending $g$ to $(g,g)$, and so we may embed $kG$ in $kG \otimes_k (kG)^{\mathrm{op}}$ by viewing the embedding of $k(\triangle G)$ in $kH$.

The group algebra $k(\triangle G)$ may be viewed as an $kH$-module with action given by $(a,b) \cdot (g,g) = (agb^{-1}, agb^{-1})$ for all $a,b,g \in G$. There is then an isomorphism of $kH$-modules $k(\triangle G) \cong \mathrm{Ind}_{\triangle G}^H(k)$, where $k$ is the trivial $k(\triangle G)$-module, given by the map sending $(g,g)$ to $(g,1) \otimes 1$ which has inverse sending $(g,h) \otimes 1$ to $(gh^{-1}, gh^{-1})$. The action of $kH$ on $\mathrm{Ind}_{\triangle G}^H(k) = kH \otimes_{k(\triangle G)} k$ is given by $(a,b) \cdot ((c,d) \otimes 1) = ((ac,bd) \otimes 1)$ for all $a,b,c,d \in G$.

By definition, for all $n \geq 0$, $HH^n(kG) = \mathrm{Ext}^n_{kG \otimes_k (kG)^{\mathrm{op}}}(kG, kG)$, where $kG$ is identified with its isomorphic image $k(\triangle G)$ and considered as a $kH$-module with action given as above. This is isomorphic as a vector space to $\mathrm{Ext}^n_{kH}(\mathrm{Ind}_{\triangle G}^H(k), k(\triangle G))$, where the first argument is the induced module with action also considered above. By the Eckmann-Shapiro Lemma (Proposition 2.2.18), we have a vector space isomorphism

$$\mathrm{Ext}^n_{kH}(\mathrm{Ind}^H_{\triangle G}(k), k(\triangle G)) \cong \mathrm{Ext}^n_{k(\triangle G)}(k, \mathrm{Res}^H_{\triangle G}(k(\triangle G))),$$

where $k$ is still the trivial $k(\triangle G)$-module, but $\mathrm{Res}^H_{\triangle G}(k(\triangle G)) = k(\triangle G)$ as a vector space, with $k(\triangle G)$-module action restricted from the $kH$-action. In particular, $k(\triangle G)$ is viewed as a $k(\triangle G)$-module via the action $(a, a) \cdot (g, g) = (aga^{-1}, aga^{-1})$. Identifying $k(\triangle G)$ with $kG$, this is now simply the action of $G$ on $kG$ by conjugation. So

$$\mathrm{Ext}^n_{k(\triangle G)}(k, \mathrm{Res}^H_{\triangle G}(k(\triangle G))) \cong \mathrm{Ext}^n_{kG}(k, \mathrm{Res}^H_G(kG)) = \mathrm{Ext}^n_{kG}(k, kG),$$

and by definition this is equal to $H^n(G; kG)$ with the conjugation action of $G$ on $kG$. To summarise what we have so far,

$$HH^*(kG) = H^*(G; kG).$$

Now we look more closely at the action of $G$ on the $kG$-module $kG$ in $H^*(G; kG)$. As this is via conjugation we may $kG$ decompose into a direct sum of $kG$-submodules corresponding to the conjugacy classes of $G$. Letting $\mathcal{C}_x$ be the conjugacy class of $x \in G$ on which $G$ acts by conjugation, $k\mathcal{C}_x$ the vector space with basis $\mathcal{C}_x$ of dimension $[G : C_G(x)]$ (and so a permutation module of $kG$), and $G/\sim$ be a set of conjugacy class representatives in $G$, then $G = \bigcup_{x \in G/\sim} \mathcal{C}_x$ is a disjoint union, and

$$kG = \bigoplus_{x \in G/\sim} k\mathcal{C}_x$$

as a direct sum of $kG$-submodules of $kG$ under the conjugation action. Since cohomology commutes with direct sums, one sees that

$$H^*(G; kG) \cong \bigoplus_{x \in G/\sim} H^*(G; k\mathcal{C}_x).$$

For each $x \in G/\sim$ the conjugacy class $\mathcal{C}_x$ is a transitive $G$-set, with stabiliser the centraliser of $x$ in $G$, $C_G(x) = \{g \in G \mid gxg^{-1} = x\}$. This gives an isomorphism of $kG$-modules $k\mathcal{C}_x \cong k(G/C_G(x))$, where $G$ acts on $G/C_G(x)$ by left multiplication. One sees therefore that we have an isomorphism of vector spaces

$$H^*(G; k\mathcal{C}_x) \cong H^*(G; k(G/C_G(x))).$$

On the other hand there is an isomorphism of $kG$-modules $k(G/C_G(x)) \cong \mathrm{Ind}^G_{C_G(x)}(k)$ viewing $k$ as the trivial $kC_G(x)$-module, induced by the map sending $gC_G(x)$ to $g \otimes 1_k$. So for each $x \in G/\sim$ we have

$$H^*(G; k\mathcal{C}_x) \cong H^*(G; \mathrm{Ind}^G_{C_G(x)}(k))$$

as $k$-vector spaces. By definition this right hand side is equal to $\mathrm{Ext}^*_{kG}(k, \mathrm{Ind}^G_{C_G(x)}(k))$ which by a further application of the Eckmann-Shapiro Lemma is isomorphic to $\mathrm{Ext}^*_{kC_G(x)}(\mathrm{Res}^G_{C_G(x)}(k), k) = \mathrm{Ext}^*_{kC_G(x)}(k, k)$. By definition, this is $H^*(C_G(x); k)$ and so we obtain our result,

$$HH^*(kG) \cong \bigoplus_{x \in G/\sim} H^*(C_G(x); k).$$

□

Restricting to degree one, keeping in mind the low-degree descriptions of group cohomology that we have encountered, in particular with $k$ as the trivial $kG$-module, we therefore see that

$$HH^1(kG) \cong \bigoplus_x \mathrm{Hom}(C_G(x), k), \tag{2.19}$$

where $x$ runs over a set of conjugacy class representatives of $kG$. As initially stated, we intend to use (2.19) to find $\dim_k(HH^1(kG))$. It therefore suffices to determine the homomorphisms from $C_G(x)$ to $k$, for

$$\dim_k\left(HH^1(kG)\right) = \sum_x \dim_k\left(\mathrm{Hom}(C_G(x), k)\right).$$

To do this, we need to develop some more group theory.

**Lemma 2.3.17.** *Let $k$ have prime characteristic $p$. Then the $k$-vector space $\mathrm{Hom}(G, k)$ is non-zero if and only if $G$ has a quotient isomorphic to a non-trivial $p$-group.*

*Proof.* Recall that $O^p(G)$ denotes the normal subgroup of $G$ generated by all elements in $G$ of $p'$-order (order coprime to $p$). Equivalently, $G/O^p(G)$ is the largest quotient group of $G$ that is also a $p$-group. In addition, let $\Phi(G)$ denote the Frattini subgroup of $G$, that is the intersection of all maximal subgroups of $G$ (or if no such subgroups exist, then $\Phi(G)$ is defined to be $G$).

If $\mathrm{Hom}(G, k) \neq \{0\}$, take a non-zero homomorphism $f : G \to k$. This implies $G/\ker(f)$ is isomorphic to a non-trivial (additive) subgroup of $k$, and all such subgroups of $k$ will have additive exponent $p$.

In the other direction let $R = G/O^p(G)$ be the largest non-trivial $p$-group quotient of $G$. Then $P := R/\Phi(R) \cong C_p \times C_p \times \ldots \times C_p$ is an elementary abelian $p$-group, and so there exists a non-zero homomorphism $P \to k$ given by the map sending a generator of each copy of $C_p$ to $1_k$. The homomorphism composed of the following maps

$$G \twoheadrightarrow G/O^p(G) = R \twoheadrightarrow R/\Phi(R) = P \to k,$$

is then a non-zero homomorphism $G \to k$. □

The proof of Lemma 2.3.17 highlights the following useful fact.

**Corollary 2.3.18.** *Let $R = G/O^p(G)$, and let $P$ be the elementary abelian $p$-group of rank $r$, $P = R/\Phi(R)$. Then we have an equality of dimensions*

$$\dim_k(\mathrm{Hom}(G, k)) = \dim_k(\mathrm{Hom}(P, k)) = r.$$

Applying this to the centraliser decomposition, Lemma 2.3.17 and Corollary 2.3.18 tell us that to find the dimensions $\dim_k(\mathrm{Hom}(C_G(x), k))$ we need to know the group structure of $C_G(x)$. With that in hand, one can then find $R_x := C_G(x)/O^p(C_G(x))$ and determine the rank of the elementary abelian $p$-group $R_x/\Phi(R_x)$: this rank will be equal to the desired dimension.

**Remark 2.3.19.** We will see in Appendix A.1 a basic `GAP` code that, for a group $G$ stored in the `GAP` library, uses the two results above to compute the right hand side of the isomorphism seen in (2.19) in order to efficiently calculate the dimension of $HH^1(kG)$.

As a second useful corollary to Lemma 2.3.17 that we will repeatedly use, we have the following.

**Corollary 2.3.20.** *Let $k$ have characteristic $p$ and let $E$ be a group of $p'$ order. Then there are no non-zero homomorphisms from $E$ to $k$, that is,* $\mathrm{Hom}(E, k) = \{0\}$.

The following theorem, known as the Künneth formula, tells us how direct products of groups behave under cohomology; this will also see plenty of use when it comes to applying the centraliser decomposition to explicit groups.

**Theorem 2.3.21** ([52, Proposition 2.21.7])**.** *Let $G, H$ be groups and $M$ be an abelian group on which $G$ and $H$ act. Then we have an isomorphism of cohomology groups*

$$H^n(G \times H; M) \cong \bigoplus_{i+j=n} H^i(G; M) \otimes_k H^j(H; M).$$

*In particular,* $\mathrm{Hom}(G \times H, k) \cong \mathrm{Hom}(G, k) \oplus \mathrm{Hom}(H, k)$.

We will now proceed to tie together some of the fundamental results that we have seen so far, to get a flavour of what the Hochschild cohomology of blocks looks like, with a concrete example.

**Example 2.3.22** (The dimensions of $HH^1\left(k(C_3^2 \rtimes Q_8)\right)$ and its blocks)**.**

We continue with the case study of Examples 2.1.14 and 2.1.19: in particular, all the notation used here is as defined in those examples. Our next aim is to describe explicitly the centraliser decomposition theorem as applied to the group $G$.

Recall that this group $G$ has been carefully chosen as a case study for a number of reasons: we have now encountered enough theory to give those details.

(i) The structure of $G$ is well understood and appears (more generally) in plenty of block theory situations. For example, Rouquier showed in [65] that for a block $B$ of a group $G$ with an elementary abelian defect group $P$ of rank 2, there is a *stable equivalence of Morita type* between $B$ and a unique block $B'$ of $kN_G(P)$: as we will see, this induces an isomorphism of Lie algebras $HH^1(B) \cong HH^1(B')$. In our case, the defect group $P = C_3 \times C_3$ of $B_0$ and $B_1$ is normal in $G$, so that $N_G(P) = G$ and we don't learn a lot from Rouquier's result: this result is, of course, a very powerful simplification in general.

(ii) The actual block structure of $kG$ is particularly simple to describe and has also been studied (more generally) elsewhere, in particular in the context of Hochschild cohomology, allowing verifications (see [7]). What is more, as we have already seen, the principal block of $kG$ is itself a group algebra, $B_0 = kH$, which makes for a more explicit description of the blocks of $kG$ and even simpler calculations to find $HH^1(B_0)$.

(iii) The non-principal block $B_1$ arises as a *twisted group algebra*, $k_\alpha H$, the group algebra over $H$ with *multiplication twisted by a 2-cocycle* $\alpha$. In Chapter 5 we will return to these and in particular return to this example, to describe the structure of $B_1$ and its first Hochschild cohomology in more detail: see the example in Section 5.4.

Let us turn to calculating the dimensions of the first Hochschild cohomology of $kG$, $B_0$ and $B_1$, via the centraliser decomposition. Recall the setup: $k$ is algebraically closed, $p = \mathrm{char}(k) = 3$, $E = C_2 \times C_2$ and $Z = C_2$. We have $P = C_3 \times C_3$, and a faithful action of $E$ on $P$ where a generator of the first copy of $C_2$ in $E$ inverts a generator of the first copy of $C_3$ in $P$, and similarly for the second copies. Consider the central extension

$$1 \to Z \to L \to E \to 1,$$

such that $L \cong Q_8$ and $Z(L) = Z$. With $Z$ acting trivially on $P$, we have the induced central extension

$$1 \to Z \to P \rtimes L \to P \rtimes E \to 1,$$

and we write $G$ and $H$ for $P \rtimes L$ and $P \rtimes E$ respectively. As some of the following calculations are verified in `GAP` [29], we note that there, $H$ is given by `SmallGroup(36,10)` and $G$ is given by `SmallGroup(72,24)`.

The group $Z$ has two irreducible characters, the trivial character $\chi_0$ and the sign character $\chi_1$ which sends $z \mapsto -1$ in $k^\times$, and we saw that these correspond to two block idempotents of $kG$: writing $Z = \langle z \rangle$ we have

$$b_0 = \frac{1}{2}(1+z),\ b_1 = \frac{1}{2}(1-z),$$

and writing $B_i = kGb_i$ for $i = 0, 1$, we have $kG = B_0 \oplus B_1$. In Example 2.1.19 we saw that $P$ was the unique defect group for both $B_0$ and $B_1$, and that $B_0 \cong kH$ as a $k$-algebra.

By Remark 2.3.5, we have $HH^1(kG) = HH^1(B_0) \oplus HH^1(B_1)$, so that

$$\dim_k(HH^1(kG)) = \dim_k(HH^1(B_0)) + \dim_k(HH^1(B_1)).$$

We will now use the centraliser decomposition, Theorem 2.3.16, to calculate $\dim_k(HH^1(kG))$. We will proceed to explicitly write the group structure of $G$ and its centralisers to highlight the processes taking place, though we remark that all of what follows may be computed and verified using the `GAP` code in Appendix A.1.

Let $\mathrm{Dic}_3$ denote the non-trivial split extension of $C_4$ by $C_3$, $\mathrm{Dic}_3 \cong C_3 \rtimes C_4$. Then, up to isomorphism, the centralisers $C_G(x)$ of a set of conjugacy class representatives $x$ of $G$, in descending order of cardinality, are given by: two copies of $G$, four copies of $C_3 \times \mathrm{Dic}_3$, two copies of $C_3^2 \times C_2$, six copies of $C_{12}$ and one copy of $C_4$. Using the notation as in the proof of Lemma 2.3.17, in particular that $R_x = C_G(x)/O^p(C_G(x))$ for each conjugacy class representative $x$, the groups $R_x$ corresponding to the centralisers above are isomorphic to, in that order,

$$\{1\}, \{1\}, C_3, C_3, C_3, C_3, C_3^2, C_3^2, C_3, C_3, C_3, C_3, C_3, C_3, \{1\}.$$

Whence as $k$-vector spaces we have

$$\begin{aligned} HH^1(kG) &\cong \bigoplus_x \mathrm{Hom}(C_G(x), k) \\ &= \mathrm{Hom}(G,k)^{\oplus 2} \oplus \mathrm{Hom}(C_3 \times \mathrm{Dic}_3, k)^{\oplus 4} \oplus \mathrm{Hom}(C_3^2 \times C_2, k)^{\oplus 2} \\ &\qquad \oplus \mathrm{Hom}(C_{12}, k)^{\oplus 6} \oplus \mathrm{Hom}(C_4, k) \\ &\cong \{0\}^{\oplus 2} \oplus \mathrm{Hom}(C_3, k)^{\oplus 4} \oplus \mathrm{Hom}(C_3^2, k)^{\oplus 2} \oplus \mathrm{Hom}(C_3, k)^{\oplus 6} \oplus \{0\}, \end{aligned}$$

where we have invoked Corollary 2.3.20. The $k$-vector space dimensions of these homomorphism spaces is now given by the $p$-rank of the elementary abelian groups $R_x/\Phi(R_x)$ which are easily found, and one verifies that

$$\begin{aligned}\dim_k(HH^1(kG)) &= \sum_x \dim_k(\mathrm{Hom}(C_G(x),k))\\ &= 4\times 1 + 2\times 2 + 6\times 1\\ &= 14.\end{aligned}$$

We now wish to understand how the dimension 14 splits as a sum of dimensions of $HH^1(B_0)$ and $HH^1(B_1)$. By Proposition 2.1.11 we have an isomorphism of $kG$-modules, $B_0 = kGb_0 \cong k(G/Z) \cong kH$. Thus if we can find $\dim_k(HH^1(kH))$ then we are done; we proceed in the same manner as above. The centralisers $C_H(x)$ of conjugacy class representatives $x$ of $H$, in descending order of cardinality, are isomorphic to one copy of $H$, two copies of $C_3\times S_3$, two copies of $D_{12}$, one copy of $C_3\times C_3$, two copies of $C_6$ and one copy of $C_2\times C_2$. The corresponding $p$-quotient groups $R_x$ are isomorphic to, in that order,

$$\{1\}, C_3, C_3, \{1\}, \{1\}, C_3\times C_3, C_3, C_3, \{1\}.$$

Whence as $k$-vector spaces we have

$$\begin{aligned}HH^1(B_0) &\cong HH^1(kH)\\ &\cong \bigoplus_x \mathrm{Hom}(C_H(x),k),\\ &= \mathrm{Hom}(H,k)\oplus \mathrm{Hom}(C_3\times S_3,k)^{\oplus 2}\oplus \mathrm{Hom}(D_{12},k)^{\oplus 2}\\ &\qquad \oplus \mathrm{Hom}(C_3\times C_3,k)\oplus \mathrm{Hom}(C_6,k)^{\oplus 2}\oplus \mathrm{Hom}(C_2\times C_2,k), \qquad (2.20)\\ &\cong \{0\}\oplus \mathrm{Hom}(C_3,k)^{\oplus 2}\oplus \{0\}^{\oplus 2}\oplus \mathrm{Hom}(C_3\times C_3,k)\oplus \mathrm{Hom}(C_3,k)^{\oplus 2}\oplus \{0\},\end{aligned}$$

where we have again invoked Corollary 2.3.20. The $p$-rank of the elementary abelian groups $R_x/\Phi(R_x)$ are easily found, giving the dimensions of these homomorphism spaces and one checks that

$$\begin{aligned}\dim_k(HH^1(B_0)) &= \dim_k(HH^1(kH))\\ &= \sum_x \dim_k(\mathrm{Hom}(C_H(x),k))\\ &= 2\times 1 + 1\times 2 + 2\times 1\\ &= 6.\end{aligned}$$

We can now use $\dim_k(HH^1(kG)) = \dim_k(HH^1(B_0)) + \dim_k(HH^1(B_1))$ to see that the dimension of $HH^1(B_1)$ is equal to 8.

**Remark 2.3.23.** In the work of Benson, Kessar and Linckelmann [7] they compute the first Hochschild cohomology of the *quantum complete intersection* algebra

$$A = k\langle x, y \mid x^p = y^p = xy - qyx = 0\rangle,$$

for $p = \mathrm{char}(k)$ an odd prime and $q \in k^\times$ an element of order dividing $p-1$. As we will see, with $p=3$ and $q=-1$ this algebra is a basic algebra for the block $B_1$ in our example, and so

$HH^1(A) \cong HH^1(B_1)$ as $k$-vector spaces. Thus we may verify directly that the dimension we have just calculated, $\dim_k(HH^1(B_1)) = 8$, agrees with that shown more generally by Benson, Kessar and Linckelmann in [7, Theorem 1.1(i)].

We will return to this example in Section 2.4, where we will examine more carefully the structure of $HH^1(B_0)$, in particular how it behaves as a *Lie algebra*.

### 2.3.2 The Lie algebra structure of Hochschild cohomology

We have seen that given a group $G$, we can find the dimension of $HH^1(kG)$ as a $k$-vector space, and, depending on certain restrictions that we place on $G$, we can find the dimensions of the $HH^1$ of (some of) the blocks of $kG$. This is of course very useful, and can immediately tell us whether we have an isomorphism of vector spaces $HH^1(A) \cong HH^1(B)$ for two $k$-algebras $A$ and $B$. One might ask whether that is enough to tell us that $A \cong B$ as $k$-algebras, or as vector spaces, or whether they are Morita equivalent, or even if there is a stable equivalence of Morita type between $A$ and $B$. In general this is not the case based on the dimensions of the first Hochschild cohomology spaces alone; on the other hand, Hochschild cohomology has a rich structure that is not just restricted to the additive structure of a vector space. In particular, the first Hochschild cohomology of $A$ has a *Lie algebra* structure, and the graded $k$-algebra $HH^*(A) := \bigoplus_{n\geq 0} HH^n(A)$ has a *graded Lie algebra* structure. Empirical evidence is building to suggest that given blocks $B$ and $B'$ with a common defect group and $HH^1(B) \cong HH^1(B')$ as Lie algebras, then it may well be the case that $B$ and $B'$ are stably equivalent. This motivates our study of the Lie algebra structure of Hochschild cohomology.

**Definition 2.3.24.** A *Lie algebra* over $k$ is a pair $(\mathcal{L}, [-,-])$, consisting of a $k$-vector space $\mathcal{L}$, together with a bilinear map called the *Lie bracket*

$$\mathcal{L} \times \mathcal{L} \to \mathcal{L}, \ (x, y) \mapsto [x, y],$$

satisfying the properties that $[x, x] = 0$ for all $x \in \mathcal{L}$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathcal{L}$.

In the definition, the second condition on the Lie bracket is referred to as the *Jacobi identity.* We call the Lie bracket $[x, y]$ the *commutator* of $x$ and $y$. Note that by the bilinearity of the Lie bracket, and the condition that $[x, x] = 0$ for all $x \in \mathcal{L}$, together imply that $[x, y] = -[y, x]$ for all $x, y \in \mathcal{L}$, for we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$$

for all $x, y \in \mathcal{L}$.

Given a Lie algebra $(\mathcal{L}, [-,-])$, a *Lie subalgebra* of $\mathcal{L}$ is a $k$-vector subspace $\mathcal{K} \subseteq \mathcal{L}$ such that $[x, y] \in \mathcal{K}$ for all $x, y \in \mathcal{K}$, hence a Lie algebra in its own right. A *Lie ideal* of $\mathcal{L}$ is a subspace $\mathcal{I}$ of $\mathcal{L}$ such that $[x, y] \in \mathcal{L}$ for all $x \in \mathcal{L}$ and all $y \in \mathcal{I}$. The *centre* of $\mathcal{L}$ is the ideal $Z(\mathcal{L}) = \{x \in \mathcal{L} \mid [x, y] = 0 \text{ for all } y \in \mathcal{L}\}$, and we say that $\mathcal{L}$ is *abelian* if $\mathcal{L} = Z(\mathcal{L})$. We say that $\mathcal{L}$ is *simple* if its only ideals are $\{0\}$ and $\mathcal{L}$ itself.

Fixing an ideal $\mathcal{I}$ of $\mathcal{L}$, let $\mathcal{L}/\mathcal{I}$ be the quotient space of cosets $\{x + \mathcal{I} \mid x \in \mathcal{L}\}$, then, under the bracket

$$[x + \mathcal{I}, y + \mathcal{I}] := [x, y] + \mathcal{I},$$

where on the left the bracket is defined on $\mathcal{L}/\mathcal{I}$ and on the right it is as defined on $\mathcal{L}$, we obtain a Lie algebra $(\mathcal{L}/\mathcal{I}, [-,-])$ called a *quotient Lie algebra.*

**Example 2.3.25.** Let $A$ be a $k$-algebra, and $f, g \in \mathrm{Der}(A)$. Define $[-,-] : \mathrm{Der}(A) \times \mathrm{Der}(A) \to \mathrm{Der}(A)$, $[f,g] = f \circ g - g \circ f$. Then $(\mathrm{Der}(A), [-,-])$ is a Lie algebra.

To see this we will in fact show that $(\mathrm{Der}(A), [-,-])$ is a Lie subalgebra of $(\mathrm{End}_k(A), [-,-])$, the Lie algebra of linear endomorphisms on $A$, with the definition of the Lie bracket as above, extended from $\mathrm{Der}(A)$ to $\mathrm{End}_k(A)$.

It is evident that $[f,f] = 0$ for all $f \in \mathrm{End}_k(A)$. We will show that the Jacobi identity holds for $(\mathrm{End}_k(A), [-,-])$. Let $f, g, h \in \mathrm{End}_k(A)$ and $a \in A$. Then

$$\begin{aligned}
[f,[g,h]](a) &= [f, g \circ h - h \circ g](a) \\
&= (f \circ (g \circ h - h \circ g))(a) - ((g \circ h - h \circ g) \circ f) \\
&= (f \circ g \circ h - f \circ h \circ g)(a) - (g \circ h \circ f - h \circ g \circ f)(a) \\
&= fgh(a) - fhg(a) - ghf(a) + hgf(a).
\end{aligned}$$

Thus

$$\begin{aligned}
[f,[g,h]] + [g,[h,f]](a) &= fgh(a) - fhg(a) - ghf(a) + hgf(a) + ghf(a) - gfh(a) - hfg(a) + fhg(a) \\
&= fgh(a) + hgf(a) - gfh(a) - hfg(a) \\
&= -(hfg(a) - hgf(a) - fgh(a) + gfh(a)) \\
&= -[h,[f,g]](a),
\end{aligned}$$

and the Jacobi identity holds. Whence $(\mathrm{End}_k(A), [-,-])$ is a Lie algebra as stated.

Next, for arbitrary $f, g \in \mathrm{Der}(A)$, observe that $[f,g]$ is indeed a derivation on $A$:

$$\begin{aligned}
[f,g](ab) &= fg(ab) - gf(ab) \\
&= f(g(a)b + ag(b)) - g(f(a)b + af(b)) \\
&= f(g(a)b) + f(ag(b)) - g(f(a)b) - g(af(b)) \\
&= fg(a)b + g(a)f(b) + f(a)g(b) + afg(b) - gf(a)b - f(a)g(b) - g(a)f(b) - agf(b) \\
&= fg(a)b - gf(a)b + afg(b) - agf(b) \\
&= [f,g](a)b + a[f,g](b),
\end{aligned}$$

for all $a, b \in A$. Whence $[f,g] \in \mathrm{Der}(A)$ for all $f, g \in \mathrm{Der}(A)$, and $\mathrm{Der}(A) \subseteq \mathrm{End}_k(A)$ is indeed a Lie subalgebra.

We will now show that $(\mathrm{IDer}(A), [-,-])$ is a Lie ideal of $(\mathrm{Der}(A), [-,-])$. It is clear that $\mathrm{IDer}(A)$ is a subspace of $\mathrm{Der}(A)$. Let $f \in \mathrm{Der}(A)$, $g \in \mathrm{IDer}(A)$ and $a \in A$. Thus, by definition there is some $b \in A$ such that $g(a) = g_b(a) = ab - ba$. If we let $h = h_{f(b)} \in \mathrm{IDer}(A)$ be the map that sends $a \mapsto h(a) = af(b) - f(b)a$, then

$$\begin{aligned}[f,g](a) &= fg(a) - gf(a)\\ &= f(ab-ba) - f(a)b + bf(a)\\ &= f(a)b + af(b) - f(b)a - bf(a) - f(a)b + bf(a)\\ &= af(b) - f(b)a\\ &= h(a) \in \mathrm{IDer}(A)\end{aligned}$$

and we are done.

Thus, importantly, $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$ has the structure of a quotient Lie algebra.

Let $(\mathcal{L}, [-,-])$ be a Lie algebra, and let $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ be the $k$-subspace of $\mathcal{L}$ with basis given by the set of all commutators in $\mathcal{L}$: it is evident that this is a Lie subalgebra of $\mathcal{L}$ and we call $\mathcal{L}'$ the *derived Lie subalgebra of* $\mathcal{L}$. The derived Lie subalgebra of $\mathcal{L}' =: \mathcal{L}^{(1)}$ is denoted $\mathcal{L}^{(2)}$, and iterating this gives rise to the *derived series* of $\mathcal{L}$, namely, the series with terms

$$\mathcal{L}^{(1)} = \mathcal{L}',\ \mathcal{L}^{(m)} = \left(\mathcal{L}^{(m-1)}\right)', m \geq 2.$$

We say that $\mathcal{L}$ is *solvable* if for some $m \geq 1$ we have $\mathcal{L}^{(m)} = \{0\}$, and in this case say that $\mathcal{L}$ has *derived length equal to* $m$.

We define the *lower central series* of $\mathcal{L}$ to be the series with terms

$$\mathcal{L}^1 = \mathcal{L}',\ \mathcal{L}^m = [\mathcal{L}, \mathcal{L}^{m-1}], m \geq 2,$$

and say that $\mathcal{L}$ is *nilpotent of class* $m$ if $\mathcal{L}^m = \{0\}$ for some $m \geq 1$.

A *homomorphism of Lie algebras* $\varphi : \mathcal{L} \to \mathcal{K}$ is a $k$-linear map that respects the Lie bracket, that is $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathcal{L}$; as before, the bracket on the left is the bracket for $\mathcal{L}$, and on the right is the bracket for $\mathcal{K}$. We have the three isomorphism theorems for Lie algebras, that we do not repeat here. We designate special status to the map known as the *adjoint representation* of $\mathcal{L}$, that is, the map $\mathrm{Ad} : \mathcal{L} \to \mathrm{End}_k(\mathcal{L})$ which sends $x$ to the map $\mathrm{Ad}(x) : \mathcal{L} \to \mathcal{L}, \mathrm{Ad}(x)(y) = [x,y]$.

Now that we have seen that the Hochschild cohomology has the structure of a Lie algebra, we are able to make comparisons of algebras via the following result of Koenig, Liu and Zhou.

**Theorem 2.3.26** ([47, Theorem 10.7])**.** *Let $k$ be algebraically closed, let $A$ and $B$ be symmetric $k$-algebras and suppose that there is a stable equivalence of Morita type between $A$ and $B$. Then there is an isomorphism of graded Lie algebras $HH^*(A) \cong HH^*(B)$ in degree $n > 0$.*

**Definition 2.3.27.** Suppose that $k$ has prime characteristic $p$. We say that $\mathcal{L}$ is a *restricted* or *$p$-restricted* Lie algebra if there exists a map, called a *$p$-mapping,* $(-)^{[p]} : \mathcal{L} \to \mathcal{L}$, $x \mapsto x^{[p]}$, such that $\mathrm{Ad}(x^{[p]})(y) = \mathrm{Ad}(x)^p(y)$, such that $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ and such that $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$, for all $x, y \in \mathcal{L}$, all $\lambda \in k$, and where $is_i(x,y)$ is the coefficient of $t^{i-1}$ in the evaluation of $\mathrm{Ad}^{p-1}(ta+b)(a)$. In this case we write that the pair $(\mathcal{L}, [p])$ is a *restricted Lie algebra.*

**Remark 2.3.28.** As technical as this definition is, we need only the following example: let $\mathrm{char}(k) = p > 0$, let $A$ be a $k$-algebra and let $\mathcal{L} = HH^1(A)$. Define the $p$-mapping $(-)^{[p]}$ : $\mathcal{L} \to \mathcal{L}, (f)^{[p]} = f^p$, that is the composition of $f$ with itself $p$ times, for all $f \in \mathrm{Der}(A)$. Then $(\mathcal{L}, [p])$ is a restricted Lie algebra.

Let $(\mathcal{L}, [p])$ be a restricted Lie algebra over a field $k$ of prime characteristic $p$. We say that $x \in \mathcal{L}$ is *p-toral* if $x^{[p]} = x$, and *p-nilpotent* if $x^{[p]} = 0$. A *p-toral Lie subalgebra* of $\mathcal{L}$ is an abelian Lie subalgebra of $\mathcal{L}$ with a basis given by $p$-toral elements of $\mathcal{L}$. We define the *p-toral rank of* $\mathcal{L}$ as the dimension of a maximal $p$-toral subalgebra of $\mathcal{L}$.

Given two $p$-restricted Lie algebras $(\mathcal{L}, [p]_1)$ and $(\mathcal{K}, [p]_2)$, we say that a Lie algebra homomorphism $f : \mathcal{L} \to \mathcal{K}$ is $p$-restricted if $f(x^{[p]_1}) = f(x)^{[p]_2}$ for all $x \in \mathcal{L}$.

Since $HH^1(A)$ is a $p$-restricted Lie algebra, these definitions add to the rich structure of Hochschild cohomology. Importantly, we can use the (restricted) Lie algebra structure to make comparisons between algebras, via the following theorem of Briggs and Rubio y Degrassi.

**Theorem 2.3.29** ([15, Theorem 1])**.** *Suppose that* $\mathrm{char}(k) = p > 0$*, let* $A$ *and* $B$ *be finite-dimensional, self-injective* $k$*-algebras, and suppose that there is a stable equivalence of Morita type between* $A$ *and* $B$*. Then there is an isomorphism of* $p$*-restricted graded Lie algebras* $HH^*(A) \cong HH^*(B)$ *in degree* $n > 0$*.*

Let us recapitulate. Given a group $G$ with blocks $B_0, B_1, \dots$, we know how to find the dimensions of $HH^1(kG)$ using the centraliser decomposition, and, depending on the structure of $G$, we may also be able to find the dimensions of the first Hochschild cohomology of (some of) the blocks of $kG$. We now know that if $k$ has characteristic $p$ then $HH^1(kG)$ and the $HH^1(B_i)$ form $p$-restricted Lie algebras. Whence a next step might be to find a basis for $HH^1(kG)$, and its blocks, as Lie algebras. For that we have the following theorem, adapted from [69], that allows us to take the centraliser decomposition as a starting point to construct derivations in $HH^1(kG)$. Recall from Theorem 2.3.16 and the argument that follows it that we have $HH^1(kG) \cong H^1(G; kG)$, where $kG$ on the right is viewed as a $kG$-module on which $G$ acts via conjugation. Recall also from Definition 2.3.13 that $H^1(G; kG) = Z^1(G; kG)/B^1(G; kG)$: we will use this point of view in the proof of the following proposition. We remark that this proposition is the reader's first encounter with novel work of the author.

**Proposition 2.3.30** ([60], Proposition 2.2)**.** *Let* $\{g_1 = 1, g_2, \dots, g_\ell\}$ *be a complete set of conjugacy class representatives of* $G$*, let* $G_i = C_G(g_i)$*, and let* $f_i \in \mathrm{Hom}(G_i, k)$ *for* $i = 1, \dots, \ell$*. Then there is a derivation* $d_i \in \mathrm{Der}(kG)$*, defined by*

$$d_i(g) = \mathrm{cor}_{G_i}^{G}(\hat{f}_i)(g)g,$$

*for all* $g \in G$ *and extended linearly to* $kG$*, where* $\hat{f}_i \in Z^1(G_i; kG)$ *is given by* $\hat{f}_i(h) = f_i(h)g_i$ *for all* $h \in G_i$*.*

*Proof.* The centraliser decomposition tells us that there is an isomorphism of $k$-vector spaces $\Phi : \bigoplus_{i=1}^{\ell} \mathrm{Hom}(G_i, k) \to H^1(G; kG)$. Siegel and Witherspoon give this isomorphism explicitly in [69, Lemma 4.2]: a homomorphism $f_i \in \mathrm{Hom}(G_i, k)$ is mapped to

$$\Phi(f_i) = [\mathrm{cor}_{G_i}^{G}(\hat{f}_i)] \in H^1(G; kG). \tag{2.21}$$

This has inverse given by the map $\Psi : H^1(G; kG) \to \bigoplus_{i=1}^{\ell} \mathrm{Hom}(G_i, k)$ that sends $[f] \in H^1(G; kG)$ to $\Psi([f]) = \pi_i \circ \mathrm{res}_{G_i}^{G}(f) \in \mathrm{Hom}(G_i, k)$, where $\pi_i : kG \to k, \pi_i(\sum_{x \in G} \lambda_x x) = \lambda_{g_i}$ (see [69, Sections 3 and 4] for more details on this).

One checks that for arbitrary $\alpha \in Z^1(G; kG)$ there is a derivation $d_\alpha \in \mathrm{Der}(kG)$, given by $d_\alpha(g) = \alpha(g)g$: let $a, b \in G$ then

$$\begin{aligned} d_\alpha(ab) &= \alpha(ab)ab \\ &= (\alpha(a) + a \cdot \alpha(b))ab \\ &= (\alpha(a) + a\alpha(b)a^{-1})ab \\ &= \alpha(a)ab + a\alpha(b)b \\ &= d_\alpha(a)b + ad_\alpha(b). \end{aligned}$$

so that $d_\alpha$ is indeed a derivation on $kG$. Now let $\varphi$ be the map sending $\alpha$ to $d_\alpha$ and define $\psi : \mathrm{Der}(kG) \to Z^1(G; kG)$, $\psi(D)(g) = D(g)g^{-1}$ for all $D \in \mathrm{Der}(kG)$ and all $g \in G$. First observe that this is indeed a 1-cocycle, for

$$\begin{aligned} \psi(D)(ab) &= D(ab)b^{-1}a^{-1} \\ &= (D(a)b + aD(b))\, b^{-1}a^{-1} \\ &= D(a)a^{-1} + a\left(D(b)b^{-1}\right)a^{-1} \\ &= \psi(D)(a) + a \cdot \psi D(b). \end{aligned}$$

for all $a, b \in G$. Next we verify that $\psi$ maps $\mathrm{IDer}(kG)$ to $B^1(G; kG)$: let $f \in \mathrm{IDer}(kG)$ be such that there is some $a \in kG$ with the property that $f(b) = ba - ab$ for all $b \in kG$. Then, for all $g \in G$,

$$\begin{aligned} \psi(f)(g) &= f(g)g^{-1} \\ &= (ga - ag)g^{-1} \\ &= gag^{-1} - a \\ &= g \cdot a - a, \end{aligned}$$

whence $\psi(f) \in B^1(G; kG)$. Finally, one checks that for all $g \in G$ and $D \in \mathrm{Der}(kG)$,

$$\varphi\psi(D)(g) = d_{\psi(D)}(g) = \psi(D)(g)g = D(g)g^{-1}g = D(g)$$

and for all $g \in G$ and $\alpha \in Z^1(G; kG)$, that

$$\psi\varphi(\alpha)(g) = \psi(d_\alpha)(g) = d_\alpha(g)g^{-1} = \alpha(g)gg^{-1} = \alpha(g),$$

showing that $\psi$ and $\varphi$ are mutually inverse linear maps inducing the isomorphisms $\mathrm{Der}(kG) \cong Z^1(G; kG)$, $\mathrm{IDer}(kG) \cong B^1(G; kG)$ and $HH^1(kG) \cong H^1(G; kG)$.

Pre-composing the map $\Phi$ in (2.21) with the map $\varphi$ and applying the result to a homomorphism $f_i \in \mathrm{Hom}(G_i, k)$ for all $i = 1, \dots, \ell$ gives the desired derivation as in the the statement of the proposition, completing the proof. □

**Remark 2.3.31.** Note that the proposition above tells us nothing about inner derivations (or from another point of view, Hochschild 1-coboundaries). Rather, it starts with a vector space basis and directly constructs a collection of Hochschild cohomology class representatives that form a basis for $HH^1(kG)$. This is rarely a problem, however one does need to be careful when inspecting the Lie algebra structure: we will see that calculating the Lie bracket on derivations leads to some cases whereby inner derivations do indeed need to be considered, in Section 5.4.

## 2.4 Example. (The Lie algebra structure of the $HH^1$ of the principal block of $k(C_3^2 \rtimes Q_8)$)

We continue with our case study, concluding this chapter by providing an example of Proposition 2.3.30 above, applying it to the case study we have seen in Examples 2.1.14, 2.1.19 and 2.3.22. We adopt all notation used there.

We state here that our main aim of this section is to calculate the 3-restricted Lie algebra structure of $HH^1(B_0)$, where $B_0$ denotes the principal block of $kG$. Since $B_0 \cong kH$ we are able to use Proposition 2.3.30 above to do so; for $B_1$ we will need to invoke some generalisations to twisted group algebras - see the example in Section 5.4 for the next steps in this direction.

A summary of the upcoming results is found in Tables 2.1 and 2.2 below. Moreover, we will state now the ultimate aim of this particular example: to show that $HH^1(B_0)$ is a solvable, non-nilpotent Lie algebra of derived length 2, and has 3-toral rank also equal to 2.

We will first need some more notation. In $H = P \rtimes E$, denote by $[g, h]$ the multiplicative group commutator $[g, h] = ghg^{-1}h^{-1}$ for all $g, h \in H$. We fix the following presentation of $H$,

$$H = \langle a, b, r, s \mid a^3 = b^3 = r^2 = s^2 = 1, [a, b] = [r, s] = [a, s] = [b, r] = 1, rar^{-1} = a^2, sbs^{-1} = b^2\rangle.$$

Then a complete set of conjugacy class representatives of $H$ can given by $\{1, a, b, r, s, ab, rs, as, br\}$. The centralisers in $H$ of these representatives correspond, of course, to the 9 groups that arise in the centraliser decomposition of $HH^1(kH)$ in Example 2.3.22, given explicitly at (2.20). In particular, we are interested in $C_H(ab) = \langle a, b\rangle \cong C_3 \times C_3$, $C_H(a) = \langle a, b, s\rangle \cong C_3 \times S_3$, $C_H(b) = \langle a, b, r\rangle \cong C_3 \times S_3$, $C_H(as) = \langle a, s\rangle \cong C_6$ and $C_H(br) = \langle b, r\rangle \cong C_6$, for these are the centralisers with non-trivial 3-group quotient, and thus by Lemma 2.3.17 the centralisers with a non-trivial contribution to the overall dimension of $HH^1(kH)$. As before, though more explicitly and after applying Corollary 2.3.20 to identify with $\{0\}$ the trivial homomorphism groups, we have

$$\begin{aligned} HH^1(B_0) &\cong HH^1(kH) \\ &\cong \mathrm{Hom}(C_H(a), k) \oplus \mathrm{Hom}(C_H(b), k) \\ &\qquad \oplus \mathrm{Hom}(C_H(ab), k) \oplus \mathrm{Hom}(C_H(as), k) \oplus \mathrm{Hom}(C_H(br), k) \\ &\cong \mathrm{Hom}(C_3 \times S_3, k)^{\oplus 2} \oplus \mathrm{Hom}(C_3 \times C_3, k) \oplus \mathrm{Hom}(C_6, k)^{\oplus 2} \\ &\cong \mathrm{Hom}(C_3, k)^{\oplus 2} \oplus \mathrm{Hom}(C_3 \times C_3, k) \oplus \mathrm{Hom}(C_3, k)^{\oplus 2}. \end{aligned} \tag{2.22}$$

To see that this final isomorphism holds, we use the Künneth formula, Theorem 2.3.21: more generally for some abelian group $M$ on which $C_3 \times S_3$ acts, we would obtain an isomorphism of $k$-vector spaces $H^1(C_3 \times S_3; M) \cong H^1(C_3; M) \otimes_k H^0(S_3; M) \oplus H^0(C_3; M) \otimes_k H^1(S_3; M)$. Concentrating on our case with $M = k$ on which $H$ and its subgroups act trivially, we obtain that $\mathrm{Hom}(C_3 \times S_3, k)$ is isomorphic as a $k$-vector space to $\mathrm{Hom}(C_3, k) \otimes_k k \oplus k \otimes_k \mathrm{Hom}(S_3, k)$ which itself is isomorphic to $\mathrm{Hom}(C_3, k)$ by applying Corollary 2.3.20 to the second summand, and noting that $V \otimes_k k \cong k \otimes_k V \cong V$ for any $k$-vector space $V$. Similarly, $\mathrm{Hom}(C_H(as), k) \cong \mathrm{Hom}(C_H(br), k) \cong \mathrm{Hom}(C_6, k)$, via the Künneth formula is isomorphic to $\mathrm{Hom}(C_3, k) \otimes k \oplus k \otimes \mathrm{Hom}(C_2, k)$ which in turn is isomorphic to $\mathrm{Hom}(C_3, k)$ by Corollary 2.3.20.

We will now use the decomposition of $HH^1(kH)$ given in (2.22), in conjunction with Proposition 2.3.30, to find a basis for $HH^1(kH)$ as a Lie algebra, and consequently recover the Lie algebra structure of $HH^1(B_0)$. By linearity and the fundamental derivation property $f(yz) = f(y)z+yf(z)$ for some $f \in \mathrm{Der}(kH)$ and all $y, z \in kH$, to define where $f$ sends an arbitrary element of $kH$, it suffices to define where $f$ sends a set of generators of $H$.

We first inspect $\mathrm{Hom}(C_H(a), k) = \mathrm{Hom}(\langle a, b, s\rangle, k) \cong \mathrm{Hom}(C_3 \times S_3, k) \cong \mathrm{Hom}(C_3, k)$, and therefore is 1-dimensional as a $k$-vector space, by Corollary 2.3.18. Choose a basis $\{f_a\}$ of $\mathrm{Hom}(C_H(a), k)$ where $f_a$ sends $a \mapsto 1$ and $b, s \mapsto 0$. Using the notation of Proposition 2.3.30, in particular relabelling $H_a = C_H(a)$ this gives a 1-cocycle $\hat{f}_a \in Z^1(H_a; kH)$ which sends $a \mapsto a$ and $b, s \mapsto 0$. For ease of notation, we will write the conjugation action of $H_a$ on $kH$ as $x \cdot y$ for all $x \in H_a$ and $y \in kH$. We will do the same for the conjugation taking place in $H$ itself, so that $r \cdot a = a^2$ and $s \cdot b = b^2$. To calculate $\mathrm{cor}_{H_a}^H(\hat{f}_a)$ we first find a set of left coset representatives of $H_a$ in $H$ to be $\{1, r\}$. Then one verifies that

$$\begin{aligned}
\mathrm{cor}_{H_a}^H(\hat{f}_a)(a) &= \hat{f}_a(a) + r \cdot \hat{f}_a(r \cdot a) \\
&= a + r \cdot \hat{f}_a(a^2) \\
&= a + r \cdot (2a) \\
&= a + 2a^2.
\end{aligned}$$

It is routine to check that $\mathrm{cor}_{H_a}^H(\hat{f}_a)$ maps $b, r, s \mapsto 0$. Finally, this gives us a derivation in $\mathrm{Der}(kH)$, $d_a$, where $d_a(a) = \mathrm{cor}_{H_a}^H(\hat{f}_a)(a)a = a^2 + 2$, and $d_a(b) = d_a(r) = d_a(s) = 0$.

By the symmetry of the action of $E$ on $P$, there is an isomorphism of $k$-vector spaces

$$\mathrm{Hom}(C_H(a), k) \cong \mathrm{Hom}(C_H(b), k)$$

given by mapping $a \mapsto b$, $b \mapsto a$ and $s \mapsto r$. With this in mind, it is almost immediate to see that a second, linearly independent derivation in $\mathrm{Der}(kH)$ is given by the derivation we label $d_b$, which maps $b \mapsto b^2 + 2$ and $a, r, s \mapsto 0$: we do not go through the details here but remark that the analogous construction to $d_a$, after swapping the roles of $a$ with $b$, and $r$ with $s$, holds.

Next we turn to $\mathrm{Hom}(C_H(as), k) = \mathrm{Hom}(\langle a, s\rangle, k) \cong \mathrm{Hom}(C_6, k) \cong \mathrm{Hom}(C_3, k)$. We choose a basis $\{f_{as}\}$ of $\mathrm{Hom}(C_H(as), k)$ where $f_{as}$ sends $a \mapsto 1$ and $s \mapsto 0$. Labelling $H_{as} = C_H(as)$ this gives a 1-cocycle $\hat{f}_{as} \in H^1(H_{as}; kH)$ which sends $a \mapsto as$ and $s \mapsto 0$. Fix a set of left coset representatives of $H_{as}$ in $H$ as $\{1, r, b, rb, b^2, rb^2\}$. Then one verifies that

$$\begin{aligned}
\mathrm{cor}_{H_{as}}^H(\hat{f}_{as})(a) &= \hat{f}_{as}(a) + b \cdot \hat{f}_{as}(b^2 \cdot a) + r \cdot \hat{f}_{as}(r \cdot a) + b^2 \cdot \hat{f}_{as}(b \cdot a) + \\
&\qquad + rb \cdot \hat{f}_{as}(rb^2 \cdot a) + rb^2 \cdot \hat{f}_{as}(rb \cdot a) \\
&= \hat{f}_{as}(a) + b \cdot \hat{f}_{as}(a) + r \cdot \hat{f}_{as}(a^2) + b^2 \cdot \hat{f}_{as}(a) + \\
&\qquad + rb \cdot \hat{f}_{as}(a^2) + rb^2 \cdot \hat{f}_{as}(a^2) \\
&= as + b \cdot (as) + r \cdot (2as) + b^2 \cdot (as) + rb \cdot (2as) + rb^2 \cdot (2as) \\
&= as + ab^2 s + 2a^2 s + abs + 2a^2 b^2 s + 2a^2 bs \\
&= (a + 2a^2) \left( \sum_{i=0}^{2} b^i \right) s.
\end{aligned}$$

It is routine to check that $\mathrm{cor}_{H_{as}}^H(\hat{f}_{as})$ maps $b, r, s \mapsto 0$. Finally, this gives us a derivation $d_{as}$, where $d_{as}(a) = \mathrm{cor}_{H_{as}}^H(\hat{f}_{as})(a)a = (a^2 + 2)(\sum_{i=0}^{2} b^i)s$, and $d_{as}(b) = d_{as}(r) = d_{as}(s) = 0$.

As before, by the symmetry of the action of $E$ on $P$, we have an isomorphism of vector spaces

$$\mathrm{Hom}(C_H(as), k) \cong \mathrm{Hom}(C_H(br), k),$$

and it follows immediately (on swapping $a$ with $b$ and $r$ with $s$) that another, linearly independent derivation in $\mathrm{Der}(kH)$ is given by $d_{br}$ which maps $b \mapsto (\sum_{i=0}^{2} a^i)(b^2+2)r$ and $a, r, s \mapsto 0$.

It remains to deal with $\mathrm{Hom}(C_H(ab), k) = \mathrm{Hom}(\langle a, b\rangle, k) \cong \mathrm{Hom}(C_3 \times C_3, k)$. For this, choose a basis $\{f_1, f_2\}$ of $\mathrm{Hom}(C_H(ab), k)$ where $f_1$ sends $a \mapsto 1$ and $b \mapsto 0$ whilst $f_2$ sends $a \mapsto 0$ and $b \mapsto 1$. Labelling $H_{ab} = C_H(ab)$ this gives 1-cocycles $\hat{f}_1, \hat{f}_2 \in H^1(H_{ab}; kH)$ where $\hat{f}_1$ sends $a \mapsto ab$ and $b \mapsto 0$ whilst $\hat{f}_2$ sends $a \mapsto 0$ and $b \mapsto ab$. Fix a set of left coset representatives of $H_{ab}$ in $H$ as $\{1, r, s, rs\}$. Then one verifies that

$$\begin{aligned}
\mathrm{cor}_{H_{ab}}^{H}(\hat{f}_1)(a) &= \hat{f}_1(a) + r \cdot \hat{f}_1(r \cdot a) + s \cdot \hat{f}_1(s \cdot a) + rs \cdot \hat{f}_1(rs \cdot a) \\
&= \hat{f}_1(a) + r \cdot \hat{f}_1(a^2) + s \cdot \hat{f}_1(a) + rs \cdot \hat{f}_1(a^2) \\
&= ab + r \cdot (2ab) + s \cdot (ab) + rs \cdot (2ab) \\
&= ab + 2a^2b + ab^2 + 2a^2b^2,
\end{aligned}$$

and that $\mathrm{cor}_{H_{ab}}^{H}(\hat{f}_1)$ maps $b, r, s \mapsto 0$. This gives us a derivation $d_{ab,1}$, where

$$d_{ab,1}(a) = \mathrm{cor}_{H_{ab}}^{H}(\hat{f}_1)(a)a = a^2b + 2b + a^2b^2 + 2b^2 = (a^2+2)(b+b^2),$$

and $d_{ab,1}(b) = d_{ab,1}(r) = d_{ab,1}(s) = 0$.

As in all previous cases, the symmetry of the action of $E$ on $P$ (on swapping $a$ with $b$ and $r$ with $s$) allows the reader to quickly verify that a sixth and final linearly independent derivation in $\mathrm{Der}(kH)$, induced by the map $\hat{f}_2$ is given by $d_{ab,2}$ which maps $b \mapsto (a+a^2)(b^2+2)$ and $a, r, s \mapsto 0$.

We summarise this data in Table 2.1 below: the first row are the six maps in $\mathrm{Der}(kH)$ defined above, which are the representatives of the classes that form a basis for $HH^1(kH)$. The first column are the given generators of $H$, and the entries of the table describe where these generators are sent under the corresponding class representatives. A "." signifies that this element is sent to 0 under the corresponding map.

Table 2.1: Class representatives of a basis for $HH^1(kH)$ as a Lie algebra

| | $d_a$ | $d_b$ | $d_{as}$ | $d_{br}$ | $d_{ab,1}$ | $d_{ab,2}$ |
|---|---|---|---|---|---|---|
| $a$ | $a^2+2$ | . | $(a^2+2)\left(\sum_{i=0}^{2} b^i\right)s$ | . | $(a^2+2)(b+b^2)$ | . |
| $b$ | . | $b^2+2$ | . | $\left(\sum_{i=0}^{2} a^i\right)(b^2+2)r$ | . | $(a+a^2)(b^2+2)$ |
| $r$ | . | . | . | . | . | . |
| $s$ | . | . | . | . | . | . |

Now that we have a good handle on the (representatives of the non-inner) derivations of $kH$ we can proceed to compute the Lie bracket on $HH^1(kH)$. Doing so gives Table 2.2 below: as before, the first row are the six maps in $\mathrm{Der}(kH)$, which are the representatives of the classes that form a basis for $HH^1(kH)$. This is also the case for the first column, and the entries of the table describe the derivation that results from applying the Lie bracket, with first argument an entry in the first column, and second argument an entry in the first row. We only compute this for the upper right half entries of this table, as the anticommutative property of the Lie bracket may be used to quickly find the remaining entries. As before, a "." denotes that the Lie bracket evaluates to 0.

Table 2.2: The Lie bracket relations on the class representatives of a basis of $HH^1(kH)$

| $[-,-]$ | $d_a$ | $d_b$ | $d_{as}$ | $d_{br}$ | $d_{ab,1}$ | $d_{ab,2}$ |
|---|---|---|---|---|---|---|
| $d_a$ | . | . | . | $d_{br}$ | . | $d_b + d_{ab,2}$ |
| $d_b$ | | . | $d_{as}$ | . | $d_a + d_{ab,1}$ | . |
| $d_{as}$ | | | . | . | . | $d_{as}$ |
| $d_{br}$ | | | | . | $d_{br}$ | . |
| $d_{ab,1}$ | | | | | . | $d_a+d_{ab,1}+2(d_b+d_{ab,2})$ |
| $d_{ab,2}$ | | | | | | . |

From this, writing $\mathcal{L} = HH^1(kH)$, we obtain that a basis for $\mathcal{L}'$ may be given by

$$\{d_{as}, d_{br}, d_a + d_{ab,1}, d_b + d_{ab,2}\},$$

and that $\mathcal{L}'' = \{0\}$, so that $HH^1(kH)$ has derived length of 2, as stated. On the other hand, it is trivial to check that $\mathcal{L}^2 = \mathcal{L}'$, so that $\mathcal{L}$ is not nilpotent. We also remark that $Z(\mathcal{L}) = \{0\}$.

One also sees that for $x \in \{a, b\}$,

$$d_x^3(x) = d_x^2(x^2 + 2) = d_x(2xd_x(x)) = d_x(2 + x) = d_x(x),$$

so that $d_a$ and $d_b$ are both 3-toral elements. Since $[d_a, d_b] = 0$, this gives us a maximal abelian 3-toral Lie subalgebra of $\mathcal{L}$ with basis $\{d_a, d_b\}$, and we see that the 3-toral rank of $\mathcal{L}$ is 2, as stated. As an additional computation, it is trivial to verify that the elements $d_{as}$ and $d_{br}$ are 3-nilpotent.

We will return to and conclude this example in Section 5.4, when will perform an examination similar to that which we have just seen, this time looking at the Lie algebra $HH^1(B_1)$.

# Chapter 3

# A 9-dimensional algebra which is not a block of a finite group algebra

This chapter takes the work of [54] as a starting point, developing it further within the specific context of Hochschild cohomology of finite dimensional algebras. In this chapter we will show the utility and versatility of computing the Hochschild cohomology of an algebra as a method to compare and distinguish different algebras. In particular, we will show that one can use Hochschild cohomology in order to prove some strong statements regarding the existence of blocks of group algebras of a specific structure.

Throughout, we will fix $k$ to be an algebraically closed field of prime characteristic $p$.

## 3.1 Introduction

A conjecture of Donovan (see [2, Conjecture M]) states that for a fixed $p$-group $P$, there are only finitely many Morita equivalence classes of blocks with defect group isomorphic to $P$. Motivated by this conjecture, Linckelmann classified in 2016 "almost all" algebras of dimension less than 13, that can arise as the basic algebra of a block of a finite group [51].

Here, "almost all" algebras of dimension less than 13 refers specifically to the exception of a single $k$-algebra of dimension 9 for which it was unknown at the time of publication whether or not it arose as the basic algebra of a block. This final exception remained until 2020 when work by Linckelmann and the author [54] was done to rule it out as a possibility.

Following this more recent development, Sambale in [67] extended this classification to algebras of dimension less than 15, and just this year Benson and Sambale [9] classified dimensions 15 and 16, along the way highlighting an open case in dimension 14 as a result of Macgregor's work on tame blocks [58].

The work done by Linckelmann and the author in [54] roughly takes the following form: this remaining exceptional case of dimension 9, a $k$-algebra which we will denote by $A$ throughout this chapter, were it indeed to be the basic algebra of a block would necessarily exist in characteristic 3. Moreover, it would be isomorphic (as $k$-algebras) to the group algebra over the group $H := (C_3 \times C_3) \rtimes C_2$, with $C_2$ acting by inversion on a generator of each copy of $C_3$. It is then shown that this is not possible, by calculating the stable centre of both $A$ and $kH$, and then showing that the dimensions of $J(\underline{Z}(A))^2$ and $J(\underline{Z}(kH))^2$ are not equal: on the other hand, a stable equivalence of Morita type preserves the stable centre, whence these dimensions should be the same if such an equivalence were to exist.

The purpose of this chapter is therefore to provide an alternative proof that there is no stable equivalence of Morita type between $A$ and $kH$. This is intended as an illuminating example that neatly ties together some far-reaching aspects of modular representation theory, including the structure theory of finite-dimensional algebras, the motivating conjecture of Donovan, Hochschild cohomology and its Lie algebra structure, and the centraliser decomposition theorem.

We remark at this point that Benson and Sambale's work [9] significantly simplifies certain proofs that rule out possible algebras found in [51]. Moreover, the same is true in the case of the algebra $A$, as Benson and Sambale's methods quickly rule out the possibility of $A$ arising. By the end of this chapter there will therefore exist three proofs ruling out the possibility that $A$ can arise as the basic algebra of a block, all of varying lengths and different flavours.

This chapter, and our proof will proceed as follows. First we will discuss some background material on finite dimensional algebras that will be key to our proof. Next we will apply these concepts and results to the $k$-algebra $A$, to determine as much of its structure as possible. Once we have established the structure of $A$, in particular its behaviour when considered as an Ext-*quiver with relations*, we will use results on the Hochschild cohomology of algebras to determine the Lie algebra structure of $HH^1(A)$, along the way computing its dimension. We will then use the centraliser decomposition to find the dimension of $HH^1(kH)$, as well as results from the author's work ([60], see for example Proposition 2.3.30) to determine the Lie algebra structure of $HH^1(kH)$. Were there indeed a stable equivalence of Morita type between $A$ and $kH$, we would expect to see that $HH^1(A) \cong HH^1(kH)$ as 3-restricted Lie algebras: this will not be the case, of course, completing the proof.

To summarise, in this chapter we will provide a new proof of the following result, and that proof uses the first Hochschild cohomology as a key ingredient, in contrast to previous proofs.

**Theorem 3.1.1.** *There is no* 9*-dimensional non-local* $k$*-algebra that arises as a basic algebra of a block with a non-cyclic defect group.*

## 3.2 Background results on algebras

Building off of the background results in Chapter 2, we detail some additional concepts that will be necessary to establish the context and complete the proof of Theorem 3.1.1. Further details and proofs of the results in this section may be found in any standard textbook on representation theory, for example Benson's volumes [4] and [5].

**Remark 3.2.1.** For Section 3.2 only, $A$ will denote an arbitrary finite-dimensional $k$-algebra, in contrast to the remainder of Chapter 3 where $A$ will denote the $k$-algebra that is the object of focus in the work of Linckelmann and the author in [54]. One might query why then is Section 3.2 included here in Chapter 3 and not in Chapter 2? Our choice to keep this general theory here has been made in order to highlight how it ties in specifically to the main results of this chapter, whilst being less relevant elsewhere.

Let $I$ be a set of conjugacy class representatives of the primitive idempotents of $A$, and let $S_i$, $S_j$ be simple $A$-modules for some $i, j \in I$. It is a standard result that $\mathrm{Ext}^1_A(S_i, S_j) \cong \mathrm{Hom}_A(J(A)i/J(A)^2i, S_j)$ as $A$-modules, so that if $k$ is algebraically closed, $\dim_k(\mathrm{Ext}^1_A(S_i, S_j))$ equals the number of summands, isomorphic to $S_j$, in a direct sum decomposition of the 2nd radical layer $J(A)i/J(A)^2i$. Whence we can read the (vector space) dimensions of the Ext-spaces for the simple $A$-modules from the 2nd layer of the radical series.

What is more, if $A$ is a symmetric algebra (for example a group algebra or a block), and so $A$ is isomorphic to its dual, we have that for any integer $n$ there is an isomorphism of $A$-modules given by $\mathrm{soc}^n(A)^* \cong A/J(A)^n$, $\mathrm{soc}^n(A) \cong (A/J(A)^n)^*$, and more generally there are isomorphisms of $A$-modules $\mathrm{soc}^n(Ai)^* \cong Ai/J(A)^n i$ and $\mathrm{soc}^n(Ai) \cong (Ai/J(A)^n i)^*$ (see, for example [52, 2.9.2(vii)]). This duality therefore allows us to read the dimensions of the Ext-spaces for the simple $A$-modules directly from the penultimate layer of the socle series.

The classification of low-dimensional algebras by Linckelmann in [51] begins by looking at restrictions on the different types of algebras that can arise at each dimension. In particular, the following definition is key to determining how many algebras can arise, at a given dimension.

**Definition 3.2.2.** Recall that we have fixed $A$ to be finite-dimensional for this section. Let $I$ be a set of representatives of the conjugacy classes of primitive idempotents in $A$. For each $i \in I$, set $P_i = Ai$ and $S_i = P_i/\mathrm{rad}(P_i) = Ai/J(A)i$. For any $i, j \in I$, denote by $c_{i,j}$ the number of composition factors isomorphic to $S_i$ in a composition series of $P_j$. The square matrix $C = C_A = (c_{i,j})_{i,j \in I}$ is called the *Cartan matrix* of $A$.

The Cartan matrix can be computed for $A$, via the following proposition.

**Proposition 3.2.3** ([52, Theorem 4.10.2])**.** *Let $I$ be a set of representatives of the conjugacy classes of primitive idempotents in $A$. Then for any $i, j \in I$ we have $c_{i,j} = \dim_k(iAj) = \dim_k(\mathrm{Hom}_A(Ai, Aj))$.*

Recall the definition of a basic algebra from Section 2.2. We remark that there is a significant implication of the previous proposition: if $A$ is basic then and we have a direct sum decomposition $A = \bigoplus_{i,j \in I} iAj$, then $\dim_k(A) = \sum_{i,j \in I} \dim_k(iAj) = \sum_{i,j \in I} c_{i,j}$. Using the Cartan matrix to place restrictions on possible algebras of a given dimension is therefore a good starting point. As the aim is to determine whether algebras of a given dimension are in fact block algebras, one might ask what does the Cartan matrix of a block algebra actually look like? Recall that we have fixed $k$ to be algebraically closed of prime characteristic $p$ for this chapter.

**Proposition 3.2.4** ([53, Corollary 6.5.15])**.** *Given a group $G$ and a $p$-block $B$ of $kG$ with defect group $P$, then the elementary divisors of the Cartan matrix $C_B$ of $B$ must divide $|P|$. In particular, $\det(C_B) = p^m$ for some $m \in \mathbb{Z}_{\geq 0}$.*

One can introduce a number of sub-cases of types of algebras to look at, when classifying them by dimension. The algebra $A$ is called *local* if the quotient $A/J(A)$ is a *division algebra*, that is, every element of $A/J(A)$ is invertible. As we are restricting to the case where $k$ is an algebraically closed field, this is equivalent to the condition that $A/J(A) \cong k$ as $k$-algebras.

Linckelmann's classification is in fact a list of basic algebras, containing all basic algebras of dimension at most 12 of block algebras of finite groups. Note two equivalent definitions to the ones given in Section 2.2: the algebra $A$ is called *basic* if every simple $A$-module is 1-dimensional, or equivalently if $A/J(A)$ is algebra isomorphic to the $k$-algebra given by a direct product of a finite number of copies of the $k$-algebra $k$. The reason we want to restrict our attention to basic algebras is given in the following result, reproduced from Section 2.2.

**Theorem 3.2.5.** *Given a $k$-algebra $A$, there is some idempotent $i \in A$ such that $iAi$ is basic and Morita equivalent to $A$.*

Thus every algebra has a less complex algebra attached to it, and we can simplify the "version" of the algebra in question, by looking at its basic algebras, without altering its representation theory. In particular, in our context of the first Hochschild cohomology of an algebra, it is clear that since Morita equivalence preserves Hochschild cohomology, if $A$ is a basic algebra for some other algebra $B$ then $HH^1(A) \cong HH^1(B)$ as Lie algebras.

Our next two results provide us with a quick way to find the Jacobson radical of a $k$-algebra $A$, provided other information is already known, such as a basis for $A$. Recall that a *separable* $k$-algebra $A$ is one that is projective as an $(A, A)$-bimodule.

**Theorem 3.2.6** (Wedderburn-Mal'cev theorem, [59, 73, 74])**.** *Let $A$ be a $k$-algebra and suppose that $A/J(A)$ is separable. Then there exists a subalgebra $E \subset A$ such that $E \cong A/J(A)$ and $A = E \oplus J(A)$.*

**Corollary 3.2.7.** *Suppose that $A$ is a basic $k$-algebra with $n$ simple modules up to isomorphism. Then $E$ as in the decomposition of $A$ in Theorem 3.2.6 is algebra isomorphic to a direct product of $n$ copies of $k$.*

The next tool in our toolkit of classifying the low-dimensional basic algebras worthy of mention is the Ext-*quiver* of an algebra.

**Definition 3.2.8.** Let $A$ be a basic $k$-algebra and let $I$ be a primitive decomposition of $1_A$ in $A$. Then the Ext-*quiver of $A$* is the finite directed graph whose vertices are indexed by $I$ (or equivalently, by the isomorphism classes of simple/projective indecomposable $A$-modules). In addition, the number of arrows from the vertex represented by $i \in I$ to the vertex represented by $j \in I$ is equal to $\dim_k(\mathrm{Ext}^1_A(S_i, S_j)) = \dim_k(\mathrm{Hom}_A(P_j, \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)))$, where $S_i \cong Ai/J(A)i$ and $P_i \cong Ai$ for all $i \in I$.

Shifting focus to Hochschild cohomology, our final result in this section will help us to find derivations in an arbitrary algebra. This result will only be used in this chapter whence its appearance here rather than Chapter 2. Recall that a derivation on $A$ is a $k$-linear map $f : A \to A$ such that $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. We will use the isomorphism of Lie algebras $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$ as detailed in Chapter 2 to view $HH^1(A)$ as the quotient of the Lie algebra of derivations on $A$ modulo the inner derivations; those that may be written as a commutator map (refer to Chapter 2 for more details).

The following result, by Linckelmann and Rubio-y-Degrassi, provides an excellent starting point when one wishes to compute explicitly the derivations of an algebra. We remark that these results hold for $k$ *any* algebraically closed field and $A$ *any* finite dimensional $k$-algebra.

**Proposition 3.2.9** ([56, Lemmas 2.4 and 2.5])**.** *Let $E$ be a separable subalgebra of $A$, complementary to $J(A)$ in $A$, as in the Wedderburn-Mal'cev theorem (Theorem 3.2.6). Then the following hold.*

*(i) Every class in $HH^1(A)$ has a representative $f \in \mathrm{Der}(A)$ satisfying $E \subseteq \ker(f)$.*

*(ii) Let $f$ be as in* (i)*. Then for any two idempotents $e_1, e_2 \in E$, we have $f(e_1Ae_2) \subseteq e_1Ae_2$.*

## 3.3 Quoted results on the 9-dimensional algebra $A$

The classification of algebras that arise as basic algebras of blocks completed in [51] employs a case-wise method, with each dimension being a separate case. Fixing some integer $n \leq 12$, the possible Cartan matrices of an $n$-dimensional algebra are computed - a finite list restricted by the various properties of that Cartan matrix, the most important of which are that its entries will sum to $n$, and its determinant will be power of $p$. It follows naturally to then partition the $n$-dimensional classification into the local and non-local sub-cases, for the local algebras will have a $1 \times 1$ Cartan matrix, significantly simplifying the list of possible algebras that may arise.

We first describe the structure of the 9-dimensional algebras that arise as basic algebras of blocks, as was shown by Linckelmann in [51]. Let $C_n$ denote the cyclic group of order $n$.

**Theorem 3.3.1** ([51, 2.9])**.** *Let $A'$ be the basic $k$-algebra of a block of a finite group algebra. Suppose further that $A'$ is 9-dimensional. Then $p = 3$ and either $A' \cong A$ is the algebra as described in Proposition 3.3.2 below, or $A'$ is isomorphic to one of the following four algebras.*

*(i) The group algebra $kC_9$.*

*(ii) The group algebra $k(C_3 \times C_3)$.*

*(iii) The quantum complete intersection $k\langle x, y\rangle/(x^3, y^3, xy + yx)$.*

*(iv) A Brauer tree algebra of a tree with two edges, exceptional multiplicity 4, and exceptional vertex at one end of the tree.*

The first three cases complete the classification of such basic algebras that are local. We have encountered case (iii) as a basic algebra of the non-principal block of the group algebra found in Examples 2.1.14, 2.1.19, 2.3.22 and 2.4. That is, case (iii) arises as a basic algebra of the twisted group algebra $k_\alpha((C_3 \times C_3) \rtimes (C_2 \times C_2))$ for some 2-cocycle $\alpha \in H^2(C_2 \times C_2; k^\times)$, where the two copies of $C_2$ act by inversion on the two copies of $C_3$. This is itself (isomorphic to) the non-principal block of the group algebra over the group $C_2.(S_3 \times S_3) = (C_3 \times C_3) \rtimes Q_8$; in the sequel we will see more details of this non-principal block, its Hochschild cohomology, and twisted group algebras in general (see Chapter 5).

In the non-local situation, case (iv) occurs as the basic algebra of a block with cyclic defect group (isomorphic to $C_9$), for example the principal block of $PSL_2(8)$ (see [67, Table 1]). For a block with non-cyclic defect group, we arrive at the algebra $A$.

**Proposition 3.3.2** ([51, 2.9])**.** *Let $A$ be a non-local, 9-dimensional, basic $k$-algebra of a block $B$, with non-cyclic defect group $P$. Then $p = 3$ and the following statements hold.*

*(i) The defect group* $P \cong C_3 \times C_3$.

*(ii) The Cartan matrix of* $A$ *is given by*

$$C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

*whence* $A$ *has 2 simple modules up to isomorphism.*

*(iii) The dimension* $\dim_k(Z(A)) = |\mathrm{Irr}(B)| = 6$.

*(iv) The algebra* $A$ *is uniquely determined by the endomorphism algebras of the two projective indecomposable* $A$*-modules, and these endomorphism algebras are isomorphic to*

$$k[x,y]/(xy, x^2 - y^3) \text{ and } k[x]/(x^2).$$

The next result was determined by Eaton at some point in the 4 years following the publication of [51], and stated without proof at the "Block Library" [23], an online database of blocks with the specific intention of keeping track of the status of Donovan's conjecture. This was then verified in [54], as Linckelmann and the author worked to establish whether $A$ did indeed arise as the basic algebra of a block.

**Theorem 3.3.3** ([23], [54, Theorem 2.1])**.** *Let* $A$ *be as in Proposition 3.3.2. Then* $A$ *is isomorphic to the algebra given by the quiver*

$$\begin{tikzcd} i \arrow[loop left, "\delta"] \arrow[loop, out=120, in=240, distance=4em, "\gamma"'] \arrow[r, shift left, "\alpha"] & j \arrow[l, shift left, "\beta"] \end{tikzcd}$$

*with relations* $\delta^2 = \gamma^3 = \alpha\beta$, $\delta\gamma = \gamma\delta = 0$, $\delta\alpha = \gamma\alpha = 0$, *and* $\beta\delta = \beta\gamma = 0$.

The next quoted result on the structure of $A$ is determined by a routine calculation directly from the quiver and relations of $A$. For the remainder of the chapter we will keep the notation of Theorem 3.3.3, in particular identifying the generators $i$, $j$, $\alpha$, $\beta$, $\gamma$ and $\delta$ with their images in $A$.

**Lemma 3.3.4** ([54, Lemma 3.1])**.**

(i) *The set* $\{i, j, \alpha, \beta, \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\}$ *is a* $k$*-basis of* $A$.

(ii) *The set* $\{\alpha, \beta, \delta^2 - \beta\alpha\}$ *is a* $k$*-basis of* $[A, A]$.

(iii) *The set* $\{1, \gamma, \gamma^2, \delta, \delta^2, \beta\alpha\}$ *is a* $k$*-basis of* $Z(A)$.

(iv) *The set* $\{\delta^2, \beta\alpha\}$ *is a* $k$*-basis of* $\mathrm{soc}(A)$.

**Remark 3.3.5.** Recall that $\delta^2 = \alpha\beta$; the "symmetry" of the commutator subspace $[A, A]$ and the socle $\mathrm{soc}(A)$ becomes more apparent. See Remark 3.4.2 below for more on this and the basis of $\mathrm{soc}(A)$.

The algebra $A$ is symmetric, as seen in the following lemma, which may also be verified by a routine calculation.

**Lemma 3.3.6** ([54, Lemma 3.2]). *There is a unique symmetrising form $s : A \to k$ such that*

$$s(\alpha\beta) = s(\beta\alpha) = 1$$

*and such that*

$$s(i) = s(j) = s(\alpha) = s(\beta) = s(\gamma) = s(\gamma^2) = s(\delta) = 0$$

*The dual basis with respect to the form $s$ of the basis*

$$\{i, j, \alpha, \beta, \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\}$$

*is, in this order, the basis*

$$\{\alpha\beta, \beta\alpha, \beta, \alpha, j, \gamma^2, \gamma, \delta, i\}$$

**Lemma 3.3.7** ([54, Lemma 3.3]). *Let $\mathrm{char}(k) = 3$. The projective ideal $Z^{\mathrm{pr}}(A)$ is one-dimensional, with basis $\{\alpha\beta - \beta\alpha\}$, we have an isomorphism of $k$-algebras*

$$\underline{Z}(A) \cong k[x, y]/(x^3 - y^2, xy, y^3)$$

*induced by the map sending $x$ to $\gamma$ and $y$ to $\delta$, and after identifying $x$ and $y$ with their images in the quotient, the following statements hold:*

(i) *The set $\{1, x, x^2, y, y^2\}$ is a $k$-basis of $\underline{Z}(A)$, and in particular $\dim_k(\underline{Z}(A)) = 5$.*

(ii) *The set $\{x, x^2, y, y^2\}$ is a $k$-basis of $J(\underline{Z}(A))$.*

(iii) *The set $\{x^2, y^2\}$ is a $k$-basis of $J(\underline{Z}(A))^2$.*

(iv) *The set $\{y^2\}$ is a $k$-basis of $\mathrm{soc}(\underline{Z}(A))$, and $J(\underline{Z}(A))^3 = \mathrm{soc}(\underline{Z}(A))$.*

(v) *The $k$-algebra $\underline{Z}(A)$ is a symmetric algebra.*

*Proof.* It follows from Lemma 3.3.6 that the relative trace map $\mathrm{Tr}_1^A$ from $A$ to $Z(A)$ is given by

$$\mathrm{Tr}_1^A(u) = iu\alpha\beta + ju\beta\alpha + \alpha u\beta + \beta u\alpha + \beta\alpha uj + \gamma u\gamma^2 + \gamma^2 u\gamma + \delta u\delta + \delta^2 ui$$

for all $u \in A$. One checks, using $\mathrm{char}(k) = 3$, that

$$\mathrm{Tr}_1^A(i) = -\mathrm{Tr}_1^A(j) = \beta\alpha - \alpha\beta$$

and that $\mathrm{Tr}_1^A$ vanishes on all basis elements different from $i$, $j$. Statement (i) then follows from the relations in the quiver of $A$ and Lemma 3.3.4. The algebra $\underline{Z}(A)$ is split local, proving statement (ii), whilst a straightforward computation shows both statement (iii) and (iv). Finally, a simple verification proves that the map $s : \underline{Z}(A) \to k$ such that

$$s(y^2) = 1$$

and such that

$$s(1) = s(x) = s(x^2) = s(y) = 0$$

is a symmetrising form on $\underline{Z}(A)$. One verifies also that the dual basis with respect to the form $s$ of the basis

$$\{1, x, y, x^2, y^2\}$$

is, in this order, the basis

$$\{y^2, x^2, y, x, 1\}.$$

This completes the proof. □

**Remark 3.3.8.** Note that by a result of Erdmann [25, I.10.8(i)], $A$ is of wild representation type.

## 3.4 Further structural results on $A$

We now arrive at unpublished work, novel and computed explicitly for the purposes of this chapter. Before determining $HH^1(A)$, we want (and need) to determine a little more structure on $A$ itself.

**Lemma 3.4.1.** *Let $A = E \oplus J(A)$ as in the Wedderburn-Mal'cev theorem. Then a $E$ can be chosen with a $k$-basis $\{i, j\}$, and a $k$-basis for $J(A)$ is then given by $\{\alpha, \beta, \beta\alpha, \delta, \delta^2, \gamma, \gamma^2\}$.*

*Proof.* Recall the Wedderburn-Mal'cev theorem, Theorem 3.2.6. The two primitive idempotents in $A$, $i$ and $j$ correspond to the two (up to isomorphism) simple $A$-modules which we label $S_i$ and $S_j$ respectively. Since $A$ is basic, by Corollary 3.2.7 we have that $E$ is algebra isomorphic to $k \times k$, hence as a left $A$-module to $S_i \oplus S_j$: $\dim_k(E) = 2$. Since $i, j \notin J(A)$ by definition, the result follows. □

**Remark 3.4.2.** Recall that $\mathrm{soc}(A)$ has a basis given by $\{\delta^2, \beta\alpha\}$. By definition of the socle and by comparing dimensions this means that $\mathrm{soc}(A) = S_i \oplus S_j$ is a direct sum. Using the quiver of $A$ we now have explicit the $k$-bases for $S_i$ and $S_j$ as $A$-submodules of the regular module $A$ as $\{\delta^2\}$ and $\{\beta\alpha\}$ respectively - as opposed to via the isomorphisms $S_i \cong Ai/J(A)i$, $S_j \cong Aj/J(A)j$ which of course gives $\{i + J(A)i\}$ and $\{j + J(A)j\}$ as respective bases.

Using these bases $\{\delta^2\}$ and $\{\beta\alpha\}$ for $S_i$ and $S_j$, and the basis of $J(A)$ as computed in Lemma 3.4.1 above, one now verifies that the definition of $J(A)$ as the intersection of all annihilators of all simple $A$-modules, is indeed satisfied in this case.

Now that we have a basis for $J(A)$ we are able to determine, explicitly and with ease, more structural information regarding $A$. In particular, we wish to know more about the simple and projective indecomposable $A$-modules, and so will be using techniques from Chapter 2 and Section 3.2 to calculate their radical series' and composition series'.

We fix the notation of $S_i$ and $S_j$ as in the previous lemma, and write $P_i \cong Ai$ and $P_j \cong Aj$ for the two projective indecomposable $A$-modules. The following proposition is easily verified using the quiver and relations for $A$ given in Theorem 3.3.3, and the bases given in Lemmas 3.3.4 and 3.4.1.

**Proposition 3.4.3.**

*(i) The projective indecomposable $A$-module $P_i$ has a $k$-basis given by $\{i, \beta, \delta, \delta^2, \gamma, \gamma^2\}$.*

*(ii) The projective indecomposable A-module $P_j$ has a k-basis given by $\{j, \alpha, \beta\alpha\}$.*

*(iii) The endomorphism algebras $iAi$ and $jAj$ of the two projective indecomposable A-modules, have k-bases given by $\{i, \delta, \delta^2, \gamma, \gamma^2\}$ and $\{j, \beta\alpha\}$ respectively.*

*(iv) The $(A, A)$-bimodules $iAj$ and $jAi$ have k-bases given by $\{\alpha\}$ and $\{\beta\}$ respectively.*

*Proof.* For $e_1, e_2 \in \{1, i, j\}$, we calculate $e_1Ae_2$ by computing $e_1ae_2$ for all elements $a$ in the $k$-basis for $A$ given in Lemma 3.3.4, using also the quiver and relations given in Theorem 3.3.3; we do not reproduce these calculations here as they are routine and easily verified. This gives the bases as stated. □

**Remark 3.4.4.** The endomorphism algebras of (iii) in the proposition above match the results given in Proposition 3.3.2(iv) as the truncated polynomial algebras $k[x, y]/(xy, x^2 - y^3)$ and $k[x]/(x^2)$. This may be seen via the isomorphisms sending $x \mapsto \delta$, $y \mapsto \gamma$, $1 \mapsto i$, and $x \mapsto \beta\alpha$, $1 \mapsto j$ respectively.

We now have a basis for the Jacobson radical of $A$, the socle of $A$, the projective indecomposable $A$-modules and of course for the simple $A$-modules. The next step is to determine how they are all connected structurally.

**Proposition 3.4.5.** *We have that $J(A)^4 = \{0\}$ or equivalently $\mathrm{soc}^4(A) = A$. What is more, $J(A)^4 i = J(A)^3 j = \{0\}$, and the following equalities hold in the Grothendieck group of A.*

*(i)* $[P_i] = 5[S_i] + [S_j]$, $[P_j] = [S_i] + 2[S_j]$.

*(ii)* $[J(A)i] = 4[S_i] + [S_j]$, $[J(A)j] = [S_i] + [S_j]$.

*(iii)* $[J(A)^2 i] = 2[S_i]$, $[J(A)^2 j] = [S_j]$.

*(iv)* $[J(A)^3 i] = [S_i]$.

*Proof.* Using the bases for $P_i$ and $P_j$ in Proposition 3.4.3, the quiver and relations in Theorem 3.3.3 prove that the equalities in (i) hold. Recalling that for an ideal $I$ and integer $n$ the ideal $I^n$ is generated by all linear combinations of products of $n$ elements from $I$, one can easily find bases for $J(A)^n e$, $e \in \{i, j\}$ (we do not reproduce this here). Following again from Theorem 3.3.3, these bases give the equalities of (ii), (iii) and (iv) as stated. □

**Proposition 3.4.6.** *The following equalities hold in the Grothendieck group of A,*

*(i)* $[\mathrm{soc}^2(Ai)] = 3[S_i] + [S_j]$.

*(ii)* $[\mathrm{soc}^3(Ai)] = 4[S_i] + [S_j]$.

*(iii)* $[\mathrm{soc}^4(Ai)] = [P_i]$.

*(iv)* $[\mathrm{soc}^2(Aj)] = [S_i] + [S_j]$.

*(v)* $[\mathrm{soc}^3(Aj)] = [P_j]$.

*Proof.* To compute the socle and its "powers" we recall the fact from Section 3.2 that there is an isomorphism of $A$-modules given by $\mathrm{soc}^n(Ae) \cong (P_e/J(A)^n e)^*$: the results are now routinely verified. □

We are now ready to complete our ultimate aim of Section 3.4, namely to determine a radical and composition series for both $P_i$ and $P_j$.

**Proposition 3.4.7.** *The Loewy lengths of the projective indecomposable modules of $A$, $P_i$ and $P_j$, are 4 and 3 respectively. What is more, the following hold.*

*(i) The radical series of the projective indecomposable modules of $A$ are:*

$$\begin{array}{ccc} & P_i & \\ & & \\ & S_i & \\ S_i & S_i & S_j \\ & S_i & \\ & S_i & \end{array} \qquad\qquad\qquad \begin{array}{c} P_j \\ \\ S_j \\ S_i \\ S_j \end{array}$$

*(ii) A composition series for $P_i$ is given by*

$$P_i \supset J(A)i \supset U_1 \supset U_2 \supset J(A)^2 i \supset J(A)^3 i \supset J(A)^4 i = \{0\},$$

*with composition factors $S_i, S_j$ and 4 more copies of $S_i$ respectively, where $U_1$ and $U_2$ are indecomposable $A$-submodules of $P_i$ of dimension 4 and 3 respectively.*

*(iii) A composition series for $P_j$ is given by*

$$P_j \supset J(A)j \supset J(A)^2 j \supset J(A)^3 j = \{0\},$$

*with composition factors $S_i, S_j$ and $S_i$ respectively.*

**Remark 3.4.8.** The radical series' of $P_i$ and $P_j$ match those given (without proof) in the "Block Library" [23], and all details agree, of course, with the entries of the Cartan matrix.

## 3.5 A proof of Theorem 3.1.1

With a good description of $A$ in hand, we are now ready to compute $HH^1(A)$. Once we have found a basis and therefore a dimension for $A$, we will use the centraliser decomposition to quickly find the first Hochschild cohomology of $kH$ as a vector space, where $H = (C_3 \times C_3) \rtimes C_2$ and $\mathrm{char}(k) = 3$ as in Section 3.1. This will, of course, allow us to re-prove Theorem 3.1.1 - our ultimate goal in this chapter - by comparing the dimensions of $HH^1(A)$ and $HH^1(kH)$, in particular showing that they are different.

Let us first define the following 6 derivations of $A$. Let $a \in A$. Then, in Table 3.1 below, we use the notation $f_a$ or $d_a$ to denote an element of $\mathrm{Der}(A)$. The entries in the table denote where the derivation of $\mathrm{Der}(A)$ in the first column sends the basis element of $A$ in the first row, where a "." denotes that element is sent to 0.

Table 3.1: A basis for $\mathrm{Der}(A)$ as a Lie algebra

| | $i$ | $j$ | $\alpha$ | $\beta$ | $\beta\alpha$ | $\delta$ | $\delta^2$ | $\gamma$ | $\gamma^2$ |
|---|---|---|---|---|---|---|---|---|---|
| $f_i$ | . | . | $-\alpha$ | $\beta$ | . | . | . | . | . |
| $f_\alpha$ | $\alpha$ | $-\alpha$ | . | $\beta\alpha - \delta^2$ | . | . | . | . | . |
| $f_\beta$ | $-\beta$ | $\beta$ | $\delta^2 - \beta\alpha$ | . | . | . | . | . | . |
| $d_\gamma$ | . | . | . | . | . | . | . | $\gamma$ | $-\gamma^2$ |
| $d_{\gamma^2}$ | . | . | . | . | . | . | . | $\gamma^2$ | $-\delta^2$ |
| $d_{\delta^2}$ | . | . | . | . | . | . | . | $\delta^2$ | . |

**Theorem 3.5.1.** *For $a \in A$, let $f_a$ and $d_a$ be as in Table 3.1. Then the set $\{f_i, f_\alpha, f_\beta, d_\gamma, d_{\gamma^2}, d_{\delta^2}\}$ is a $k$-basis for* $\mathrm{Der}(A)$ *as a Lie algebra, and the set $\{f_i, f_\alpha, f_\beta\}$ is a $k$-basis for* $\mathrm{IDer}(A)$ *as a Lie ideal of* $\mathrm{Der}(A)$*. In particular,* $\dim_k(HH^1(A)) = 3$.

*Proof.* Recall that we have a $k$-linear map $A \to \mathrm{IDer}(A)$ that sends $a \in A$ to the inner derivation $f_a = [-, a]$ in $\mathrm{IDer}(A)$, that is, the derivation such that $f_a(b) = [b, a] = ba - ab$ for all $b \in A$. This $k$-linear map from $A$ to $\mathrm{IDer}(A)$ has kernel $Z(A)$ and induces an isomorphism of vector spaces $A/Z(A) \cong \mathrm{IDer}(A)$. Whence $\dim_k(\mathrm{IDer}(A)) = \dim_k(A) - \dim_k(Z(A)) = 3$. Evidently, $f_a = 0$ for all $a \in Z(A)$, leaving $f_b \neq 0$ for $b$ one of $i, j, \alpha$ or $\beta$. A routine verification shows that $f_i$, $f_\alpha$ and $f_\beta$ are as given in Table 3.1, and that $f_i = -f_j$. This proves that $\{f_i, f_\alpha, f_\beta\}$ forms a $k$-basis for $\mathrm{IDer}(A)$ as a Lie ideal in $\mathrm{Der}(A)$, as in the theorem.

Now let $d + \mathrm{IDer}(A) \in HH^1(A)$. By Proposition 3.2.9(i) and Lemma 3.4.1 we may assume that $d(i) = d(j) = 0$. Since $\mathrm{char}(k) = 3$ and $\gamma \in Z(A)$ one also sees that $d(\gamma^3) = 3\gamma d(\gamma) = 0$. As $\gamma^3 = \delta^2$ and $\delta \in Z(A)$, this tells us that $0 = d(\gamma^3) = d(\delta^2) = 2\delta d(\delta)$, whence $d(\delta) = 0$. Additionally, as $\delta^2 = \alpha\beta$, we have $0 = d(\delta^2) = d(\alpha\beta) = \alpha d(\beta) + d(\alpha)\beta$.

By Proposition 3.2.9(ii), $d(iAj) \subseteq iAj$ and $d(jAi) \subseteq jAi$, and bases for these bimodules have been given in Proposition 3.4.3(iv) as $\{\alpha\}$ and $\{\beta\}$ respectively. This tells us that $d(\alpha) = \lambda\alpha$ and $d(\beta) = \mu\beta$ for some $\lambda, \mu \in k$. Consequently, we have that $0 = \alpha d(\beta) + d(\alpha)\beta = \mu\alpha\beta + \lambda\alpha\beta$, so that $\mu = -\lambda$. This also forces $d(\beta\alpha) = 0$. To summarise so far: if $d + \mathrm{IDer}(A)$ is in a basis of $HH^1(A)$, then without loss of generality $d$ may be chosen such that it sends $i, j, \beta\alpha, \delta$ and $\delta^2$ to 0.

Now we use Proposition 3.2.9(ii) again, this time looking at the endomorphism algebra $iAi$. That $d(iAi) \subseteq iAi$ means $d(\gamma) = \lambda_0 i + \lambda_1\gamma + \lambda_2\gamma^2 + \lambda_3\delta + \lambda_4\delta^2$, for some $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in k$, which also forces $d(\gamma^2) = 2\gamma d(\gamma) = -(\lambda_0\gamma + \lambda_1\gamma^2 + \lambda_2\delta^2)$. To complete the proof we use the derivation property of $d$ on the defining relations of $A$: we do not reproduce this in its entirety here as in fact the only other relation that provides any new information is $\delta\gamma = 0$. From this, one obtains $0 = d(0) = d(\delta\gamma) = d(\delta)\gamma + \delta d(\gamma) = \delta d(\gamma) = \lambda_0\delta + \lambda_3\delta^2$, whence $\lambda_0 = \lambda_3 = 0$.

As all possible restrictions on $d$ have been placed by the defining relations $A$, it remains only to set one of each of $\lambda, \mu, \lambda_1, \lambda_2, \lambda_4$ equal to 1 in turn, and the remaining to 0, and determine which derivations arise. Doing so gives us $f_i$, $d_\gamma$, $d_{\gamma^2}$ and $d_{\delta^2}$ as in Table 3.1. This completes the proof. □

We now turn to using the centraliser decomposition to find the dimension of the first Hochschild cohomology of $kH$. The reader should by now be comfortable with the centraliser decomposition as made explicit in Example 2.3.22; the methods employed here will follow in the same manner.

**Theorem 3.5.2.** *Let $H$ be the group $(C_3 \times C_3) \rtimes C_2$, where the generator of $C_2$ acts by inversion on a generator of each copy of $C_3$. Then $\dim_k(HH^1(kH)) = 8$.*

*Proof.* Recall that in $H$, the action of $C_2$ inverts a generator in each copy of $C_3$: $H$ is the non-trivial split extension of $S_3$ by $C_3$ and in `GAP` is given by `SmallGroup(18,4)`. The centralisers $C_H(x)$ of conjugacy class representatives $x$ of $H$, in descending order of cardinality, are isomorphic to one copy of $H$, one copy of $C_2$ and four copies of $C_3 \times C_3$. The corresponding $p$-quotient groups $R_x$ are isomorphic to, in that order,

$$\{1\}, \{1\}, C_3 \times C_3, C_3 \times C_3, C_3 \times C_3 \text{ and } C_3 \times C_3.$$

Whence as $k$-vector spaces we have

$$\begin{aligned} HH^1(kH) &\cong \bigoplus_x \operatorname{Hom}(C_H(x), k) \\ &= \operatorname{Hom}(H, k) \oplus \operatorname{Hom}(C_2, k) \oplus \operatorname{Hom}(C_3 \times C_3, k)^{\oplus 4} \\ &\cong \{0\}^{\oplus 2} \oplus \operatorname{Hom}(C_3 \times C_3, k)^{\oplus 4}, \end{aligned}$$

where we have invoked Corollary 2.3.20 to obtain the trivial homomorphism spaces. The $p$-rank of the elementary abelian groups $R_x/\Phi(R_x)$ are easily found, giving the dimensions of the non-trivial homomorphism spaces, and one checks that

$$\begin{aligned} \dim_k(HH^1(kH)) &= \sum_x \dim_k(\operatorname{Hom}(C_H(x), k)) \\ &= 4 \times 2 \\ &= 8. \end{aligned}$$

□

**Remark 3.5.3.** In the sequel, we will see that the structure of $H$ and its collection of centralisers of conjugacy class representatives in fact generalises to a situation from which we are able to obtain the dimension $\dim_k(HH^1(kH))$ immediately: see Proposition 4.3.3 for a formula for this dimension in terms of the order of certain subgroups of $H$. In that case, $P = C_3 \times C_3$ is of order 9 and rank $r = 2$, $E = C_2$ is of order 2, and $\dim_k(HH^1(kH)) = 2 \cdot \left(\frac{9-1}{2}\right) = 8$.

*Proof of Theorem 3.1.1.* Arguing by contradiction, suppose $A'$ is a 9-dimensional, basic non-local $k$-algebra of a block $B$ of a finite group with a non-cyclic defect group. Then $A'$ is isomorphic as a $k$-algebra to the algebra $A$ as described in Proposition 3.3.2 and Theorem 3.3.3 with defect group $P \cong C_3 \times C_3$, and there is a Morita equivalence between $A$ and $B$.

Recall that Lemma 3.3.7 computes the stable centre $\underline{Z}(A)$ of $A$, and Lemma 3.3.6 shows that $\underline{Z}(A)$ is a symmetric $k$-algebra. It then follows from [42, Lemma 3.8(iii)] that we have an algebra isomorphism

$$\underline{Z}(A) \cong (kP)^E$$

where $E$ is the inertial quotient of the block $B$. By Lemma 3.3.7, we have $\dim_k((kP)^E) = 5$, or equivalently, $E$ has five orbits in $P$. The list of possible inertial quotients in Kiyota's paper [43] shows that $E$ is isomorphic to one of 1, $C_2$, $C_2 \times C_2$, $C_4$, $C_8$, $D_8$, $Q_8$, $SD_{16}$. In all cases except for $E \cong C_2$ is the action of $E$ on $P$ determined, up to equivalence, by the isomorphism class of $E$. Thus if $E$ contains a cyclic subgroup of order 4, then $E$ has at most 3 orbits, and if $E$ is the Klein four group, then $E$ has 4 orbits. Therefore we have $E \cong C_2$. If the non-trivial element $r$ of $E$ has a non-trivial fixed point in $P$ (or equivalently, if $r$ centralises one of the factors $C_3$ of $P$ and acts as inversion on the other), then $E$ has 6 orbits. Thus $r$ has no non-trivial fixed point in $P$, and the group $H = P \rtimes E$ is the Frobenius group as defined in Theorem 3.5.2. By a result of Puig [63, 6.8] (also described in [53, Theorem 10.5.1]), there is a stable equivalence of Morita type between $B$ and $kH$, hence between $A$ and $kH$. As we have seen, stable equivalences of Morita type preserve Hochschild cohomology in positive degree, whence there is a Lie algebra isomorphism $HH^1(A) \cong HH^1(kH)$. On the other hand, by comparing dimensions in Theorem 3.5.1 and Theorem 3.5.2 one sees that this is not the case, completing the proof. □

## 3.6 The Lie algebra structure of $HH^1(A)$ and $HH^1(kH)$

Though it is not required to prove Theorem 3.1.1, we provide additional details on the Lie algebra structure of the first Hochschild cohomology groups calculated in this chapter. We do so for the benefit of the reader as it makes explicit certain constructions: of the Lie algebra nature of spaces of derivations and of the construction that results from the author's Proposition 2.3.30.

We begin by computing the Lie bracket on the basis of $HH^1(A)$ given in Theorem 3.5.1. In Table 3.2 below, the entries denote what the Lie bracket $[-,-]$ looks like in $\mathrm{Der}(A)$, where the first argument of $[-,-]$ is taken from corresponding entry in the first column and the second argument is taken from the corresponding entry in the first row, and where a "." denotes that $[-,-]=0$.

Table 3.2: The Lie bracket relations on $\mathrm{Der}(A)$

| $[-,-]$ | $f_i$ | $f_\alpha$ | $f_\beta$ | $d_\gamma$ | $d_{\gamma^2}$ | $d_{\delta^2}$ |
|---|---|---|---|---|---|---|
| $f_i$ | . | $-f_\alpha$ | $f_\beta$ | . | . | . |
| $f_\alpha$ | $f_\alpha$ | . | . | . | . | . |
| $f_\beta$ | $-f_\beta$ | . | . | . | . | . |
| $d_\gamma$ | . | . | . | . | $d_{\gamma^2}$ | $-d_{\delta^2}$ |
| $d_{\gamma^2}$ | . | . | . | $-d_{\gamma^2}$ | . | . |
| $d_{\delta^2}$ | . | . | . | $d_{\delta^2}$ | . | . |

**Theorem 3.6.1.** *The Lie algebra $HH^1(A)$ is a solvable Lie algebra of derived length 2, has a trivial centre, and is not nilpotent.*

*Proof.* These statements are immediate from Table 3.2: first note that no element in the basis given for $\mathrm{Der}(A)$ commutes with all others. Next, letting $\mathcal{L} = HH^1(A)$, we have a $k$-basis for $\mathcal{L}'$ given by $\{d_{\gamma^2}, d_{\delta^2}\}$, so that $\mathcal{L}'' = \{0\}$ and $HH^1(A)$ is solvable. On the other hand $\mathcal{L}^2 = \mathcal{L}'$ so that $HH^1(A)$ is not nilpotent. □

**Lemma 3.6.2.** *The derivations $d_\gamma$ and $f_i$ are 3-toral, and $d_{\gamma^2}$ and $d_{\delta^2}$ are 3-nilpotent. In particular, $\mathrm{Der}(A)$ has 3-toral rank of 2 and $HH^1(A)$ has 3-toral rank of 1.*

*Proof.* A routine verification shows that this is the case. □

We now move on to using Proposition 2.3.30 to calculate the Lie algebra structure of $HH^1(kH)$ which we will see differs from that of $HH^1(A)$ in more ways than just the dimension. As was the case for Theorem 3.5.2 - using Example 2.3.22 as a guiding example - the reader should also be comfortable with the constructions here: a similar situation is provided, more explicitly, in Example 2.4; the methods employed here will follow in the same manner.

With the aid of Proposition 2.3.30 we can explicitly find the class representatives of a basis of derivations of $\mathrm{Der}(kH)/\mathrm{IDer}(kH)$. To do so, we need to fix some notation. Let $P = C_3 \times C_3$ have as generators $a$ and $b$ and let $E = C_2$ be generated by $r$. Set $H = P \rtimes E$ and let $[-,-]$ denote the group commutator bracket, a presentation for $H$ is given by

$$H = \langle a, b, r \mid a^3 = b^3 = r^2 = [a,b] = 1, rar^{-1} = a^2, rbr^{-1} = b^2 \rangle.$$

Fix a set of conjugacy class representatives of $H$, denoted $H/\sim\, = \{1, r, a, b, ab, a^2b\}$, and write $\mathcal{H} = (H/\sim)\backslash\{1, r\}$. Then, in Table 3.3 below, we use the notation $f_x$ or $d_x$ to denote a coset representative in $\mathrm{Der}(kH)/\mathrm{IDer}(kH)$ for some $x \in \mathcal{H}$. The entries in the table denote where the derivation in $\mathrm{Der}(kH)$ in the first row sends a generator of $H$ given in the first column, where a "." denotes that element is sent to 0. Note that if $f \in \mathrm{Der}(kH)$ then it suffices to define where $f$ sends a set of generators of $H$; for remaining elements of $kH$ one can then construct $f$ explicitly using the fundamental derivation property $f(yz) = f(y)z + yf(z)$ for all $y, z \in kH$ and by extending linearly.

Table 3.3: A basis for $HH^1(kH)$ as a Lie algebra

| | $d_a$ | $f_a$ | $d_b$ | $f_b$ | $d_{ab}$ | $f_{ab}$ | $d_{a^2b}$ | $f_{a^2b}$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $a^2+2$ | . | $ab + 2ab^2$ | . | $a^2b + 2b^2$ | . | $b + 2a^2b^2$ | . |
| $b$ | . | $ab + 2a^2b$ | . | $b^2+2$ | . | $ab^2 + 2a^2$ | . | $a^2b^2 + 2a$ |
| $r$ | . | . | . | . | . | . | . | . |

Let for $x \in \mathcal{H}$ and a coset representative $f_x$ or $d_x$, let $[f_x]$ and $[d_x]$ denote their respective coset classes.

**Theorem 3.6.3.** *For $x \in \mathcal{H}$, let $f_x$ and $d_x$ be as in Table 3.3. Then the set $\{[f_x], [d_x] \mid x \in \mathcal{H}\}$ is a $k$-basis for $HH^1(kH)$ as a Lie algebra.*

*Proof.* By the centraliser decomposition as seen in the proof of Theorem 3.5.2, we have that

$$\begin{aligned} HH^1(kH) &\cong \mathrm{Hom}(H,k) \oplus \mathrm{Hom}(E,k) \oplus \mathrm{Hom}(P,k)^{\oplus 4} \\ &\cong \mathrm{Hom}(P,k)^{\oplus 4}. \end{aligned}$$

For ease of notation, let $C$ denote the set of generators of $H$ given in the presentation of $H$ above, $C = \{a, b, r\}$. In order to use Proposition 2.3.30 we need to distinguish between the four copies of $P$ in the collection of centralisers of conjugacy class representatives of $H$. To that end, for $x \in \mathcal{H}$, let $P_x = C_H(x)$, so that $HH^1(kH) \cong \oplus_{x\in\mathcal{H}}\mathrm{Hom}(P_x, k)$. As we will see, each $d_x$ and $f_x$ in Table 3.3 is induced by a homomorphism in a (2-dimensional) $k$-basis of $\mathrm{Hom}(P_x, k)$, $x \in \mathcal{H}$, hence the labelling given there in the table.

For each $x \in \mathcal{H}$, $\mathrm{Hom}(P_x, k)$ has a $k$-basis given by $\{\varphi_x, \rho_x\}$, where $\varphi_x$ maps $a \mapsto 1$ and $C\backslash\{a\}$ to 0 whilst $\rho_x$ maps $b \mapsto 1$ and $C\backslash\{b\}$ to 0. Using the notation of Proposition 2.3.30, $H^1(P_x; kH)$

has a $k$-basis with class representatives $\hat{\varphi}_x$ and $\hat{\rho}_x$, where $\hat{\varphi}_x$ maps $a \mapsto x$ and $C\backslash\{a\}$ to 0 whilst $\hat{\rho}_x$ maps $b \mapsto x$ and $C\backslash\{b\}$ to 0.

Now let $\psi \in \{\hat{\varphi}_x, \hat{\rho}_x\}$ and $P = P_x$ for some $x \in \mathcal{H}$. Recall that for all $g \in H$,

$$\mathrm{cor}_P^H(\psi)(g) = \sum_{z \in E} y\psi(y^{-1}gz)y^{-1},$$

where $y \in E$ is the unique element such that $gzP = yP$. In other words,

$$\begin{aligned}\mathrm{cor}_P^H(\psi)(a) &= \psi(a) + r\psi(a^2)r\\ \mathrm{cor}_P^H(\psi)(b) &= \psi(b) + r\psi(b^2)r\\ \mathrm{cor}_P^H(\psi)(r) &= 0.\end{aligned}$$

Finally, by Proposition 2.3.30 we have a derivation $D = D_\psi \in \mathrm{Der}(kH)$ given by

$$\begin{aligned}D(a) &= \psi(a)a + r\psi(a^2)ra\\ D(b) &= \psi(b)b + r\psi(b^2)rb\\ D(r) &= 0.\end{aligned}$$

Replacing $\psi$ with each of $\hat{\varphi}_x$ and $\hat{\rho}_x$ gives the eight derivations as in Table 3.3, where for each $x \in \mathcal{H}$ we have labelled $d_x = D_{\hat{\varphi}_x}$ and $f_x = D_{\hat{\rho}_x}$. This completes the proof. □

Now that we have a basis for $HH^1(kH)$ we are interested in the Lie algebra structure that is defined on $A$ by this basis. In Table 3.4 below, the entries denote how the Lie bracket $[-,-]$ behaves in $HH^1(kH)$, where the first argument of $[-,-]$ is taken from corresponding entry in the first column and the second argument is taken from the corresponding entry in the first row, and where a "." denotes that $[-,-] = 0$. For ease of notation, we have relabelled the basis for $HH^1(kH)$ given in Theorem 3.5.2 and Table 3.3,

$$\{d_a, f_a, d_b, f_b, d_{ab}, f_{ab}, d_{a^2b}, f_{a^2b}\},$$

in that order, as

$$\{D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8\}.$$

For clarity, we have only included the upper triangular entries of the table, as the lower triangular entries are easily found by the antisymmetric property of the Lie bracket, $[D_m, D_n] = -[D_n, D_m]$ for all $m, n = 1, \ldots, 8$.

Table 3.4: The Lie bracket relations on $HH^1(kH)$

| $[-,-]$ | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ |
|---|---|---|---|---|---|---|---|---|
| $D_1$ | . | $-D_2$ | $-D_5-D_7$ | . | $D_3$ | $D_8-D_4$ | $D_3$ | $D_6-D_4$ |
| $D_2$ | | . | $D_5-D_6-D_7-D_8$ | $D_6-D_8$ | $D_7-D_3-D_4-D_8$ | $D_8-D_4$ | $D_3-D_5-D_4-D_6$ | $D_4-D_6$ |
| $D_3$ | | | . | $D_3$ | $-D_1-D_7$ | $D_7-D_1-D_8-D_2$ | $D_5-D_1$ | $D_1+D_5-D_2+D_6$ |
| $D_4$ | | | | . | $-D_1-D_7$ | $D_2$ | $D_1-D_3$ | $-D_2$ |
| $D_5$ | | | | | . | $D_5-D_6$ | $-D_3$ | $D_3-D_1+D_2+D_4$ |
| $D_6$ | | | | | | . | $D_4-D_1-D_2-D_3$ | $D_2$ |
| $D_7$ | | | | | | | . | $D_7+D_8$ |
| $D_8$ | | | | | | | | . |

**Theorem 3.6.4.** *Let $\{D_i \mid i = 1,\dots,8\}$ be the $k$-basis for $HH^1(kH)$ as a Lie algebra as above. Then the Lie bracket relations on $HH^1(kH)$ are as given in Table 3.4.*

*Proof.* Routine verification. □

**Proposition 3.6.5.** *Let $\mathcal{L} = HH^1(kH)$ and $\mathcal{L}'$ the derived Lie subalgebra of $\mathcal{L}$. Then $\mathcal{L}' = \mathcal{L}$, and consequently the Lie algebra $HH^1(kH)$ is not solvable.*

*Proof.* Using the Lie bracket relations in Table 3.4 one is able to reproduce the entire basis $\{D_i \mid i = 1,\dots,8\}$. One verifies that:

(i) $[D_4, D_7] + [D_1, D_5] = D_1$,

(ii) $[D_2, D_1] = D_2$,

(iii) $[D_1, D_5] = D_3$,

(iv) $[D_4, D_7] + D_3 + [D_3, D_7] + [D_6, D_5] + [D_8, D_1] = D_4$,

(v) $[D_3, D_7] + D_1 = D_5$,

(vi) $[D_6, D_5] + D_5 = D_6$,

(vii) $[D_3, D_1] - D_5 = D_7$,

(viii) $[D_7, D_8] - D_7 = D_8$.

This completes the proof. □

**Lemma 3.6.6.** *The derivations $D_1, D_4, D_5, D_6, D_7$ and $D_8$ are 3-toral, the derivations $D_2$ and $D_3$ are 3-nilpotent, and $HH^1(kH)$ has 3-toral rank of 2.*

*Proof.* Recalling that $D_1, D_4, D_5, D_6, D_7$ and $D_8$ are given by $d_a, f_b, d_{ab}, f_{ab}, d_{a^2b}$ and $f_{a^2b}$ respectively in Table 3.3, and that there we also have $D_2 = f_a$, $D_3 = d_b$, a routine verification shows that these properties hold. In addition, $[D_1, D_4] = 0$, so that $\{D_1, D_4\}$ is a $k$-basis for a maximal abelian 3-toral Lie subalgebra of $HH^1(kH)$. □

**Remark 3.6.7.** To make the comparison more explicit between the situation involving $kH$ and its first Hochschild cohomology in the section above, and the situation in Examples 2.3.22 and 2.4, we give a little bit more information on the structure of the group algebra $kH$, in particular how it arises as a block of a larger group.

Let $E = C_2$ as before, and set $Z = C_2$. Consider the central extension

$$1 \to Z \to L \to E \to 1,$$

such that $L \cong C_4$. With $P = C_3 \times C_3$ also as before, and consider the free action of $E$ on $P$ where a generator of $E$ inverts a generator of each copy of $C_3$ in $P$. If $u$ is a generator of $L$, then $Z = \langle u^2 \rangle$ acts trivially on $P$, we have an induced central extension

$$1 \to Z \to P \rtimes L \to P \rtimes E \to 1.$$

Write $G = P \rtimes L$ and we recover $H = P \rtimes E$.

As in Example 2.3.22, the blocks of $kG$ are in one-to-one correspondence with the characters of $Z$. $Z$ has two irreducible characters, the trivial character $\chi_0$ and the sign character $\chi_1$ which sends $u^2 \mapsto -1$ in $k^\times$. By Corollary 2.1.12 these correspond to two block idempotents of $kG$:

$$\begin{aligned} b_0 &= \frac{1}{|Z|}(\chi_0(1)1 + \chi_0(u^2)u^2) = \frac{1}{2}(1 + u^2), \\ b_1 &= \frac{1}{|Z|}(\chi_1(1)1 + \chi_1(u^2)u^2) = \frac{1}{2}(1 - u^2). \end{aligned}$$

Write $B_i = kGb_i$ for $i = 0, 1$, so that $kG = B_0 \oplus B_1$, $HH^1(kG) = HH^1(B_0) \oplus HH^1(B_1)$, and

$$\dim_k(HH^1(kG)) = \dim_k(HH^1(B_0)) + \dim_k(HH^1(B_1)).$$

By Proposition 2.1.11, $kH \cong B_0$ is the principal block of $kG$ in this situation. In `GAP`, $G$ is given by `SmallGroup(36,7)` and a computer verification using the `GAP` code found in Appendix A.1 shows that $\dim_k(HH^1(kG)) = 16$, meaning that $\dim_k(HH^1(B_1)) = 8$. As in Examples 2.3.22, 2.4 above and 5.4 below, $B_1$ arises as a twisted group algebra over $H$, $B_1 \cong k_\alpha H$ for some 2-cocycle in $Z^2(E; Z)$, inflated (trivially, since $Z$ injects into $k^\times$ and acts trivially on $P$) to a 2-cocycle $\alpha \in Z^2(H; k^\times)$.

# Chapter 4

# The first Hochschild cohomology of the blocks of the Mathieu groups

This chapter is based on the author's paper [60]. In what follows we will calculate the Lie algebra structure of the first Hochschild cohomology of the $p$-blocks of the sporadic simple Mathieu groups. To be more precise, these Lie algebras will be computed over an algebraically closed field $k$ of characteristic $p$, as $p$ varies over all the prime divisors of all the orders of all the sporadic simple Mathieu groups. Our most note-worthy result that we will see is that for the blocks with non-trivial defect group, this Lie algebra is non-trivial, adding to our understanding of how the structure of a block is intimately tied to the structure of its Hochschild cohomology algebra.

Throughout this chapter, $k$ will be an algebraically closed field of prime characteristic $p$.

## 4.1 Introduction

As we have seen, for a given (finite) group $G$, the first Hochschild cohomology group of the group algebra $kG$ forms a Lie algebra over $k$. By a seemingly innocuous result of Fleischmann, Janiszczak and Lempken [28], it turns out that providing that $G$ is not the trivial group, then this Lie algebra has non-zero dimension over $k$ (we will return to this in more detail in Chapter 5). One might ask whether the same is true for the blocks of a group algebra? The answer is still unknown; the problem of determining whether $HH^1(B) \neq \{0\}$ for a $p$-block $B$ of $kG$ with non-trivial defect groups is still open in general.

In this chapter we settle this problem for the blocks of the sporadic simple Mathieu groups by showing the following.

**Theorem 4.1.1.** *Let $G$ be one of the sporadic simple Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ or $M_{24}$. Then for all primes $p$ dividing $|G|$ and for all $p$-blocks $B$ of $kG$ with a non-trivial defect group, $HH^1(B) \neq \{0\}$. Moreover, for each such $p$-block $B$ the dimensions $\dim_k(HH^1(B))$ are as given in Table 4.5, Table 4.6, Table 4.7, Table 4.8, Table 4.9 for $G = M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ respectively.*

Aside from generalising the non-triviality of the first Hochschild cohomology from the case of group algebras to the case of blocks, one other motivation to study this problem comes from a wider investigation into the links between the Lie algebra structure of $HH^1(B)$ and the $k$-algebra structure of $B$, examining how each influences the other. For example, Linckelmann and Rubio y Degrassi have shown in [55] that a block with a unique isomorphism class of simple modules has degree one Hochschild cohomology a simple Lie algebra if and only if the block is Morita equivalent to $kP$, where $P$ is elementary abelian of order at least 3: this result establishes such a link between structures. See also Chapter 2 as well as the publications [8, 15, 17, 24, 55, 66], for more results on the Lie algebra structure of Hochschild cohomology.

In addition, the Lie algebra structure is expected to provide information useful to the Auslander-Reiten conjecture, which predicts that two finite-dimensional and stably-equivalent $k$-algebras will have the same number of non-projective simple modules up to isomorphism [3, Conjecture (5)]. As we have seen, for a symmetric $k$-algebra $A$, the graded Lie algebra structure on $HH^*(A)$ is preserved in degree $n > 0$ by stable equivalences of Morita type. Conversely, empirical evidence suggests that if $B$ and $B'$ are blocks with the same defect group, and if there is a Lie algebra isomorphism between their first Hochschild cohomology groups, then they are stably equivalent. Studying the links between the structure of $B$ and $HH^1(B)$ can therefore add to the growing wealth of information that supports the Auslander-Reiten conjecture.

We use elementary dimension-counting arguments to prove most of Theorem 4.1.1 and a wide array of methods to determine the Lie algebra properties. In particular, we will make use of the centraliser decomposition, Theorem 2.3.16, and Proposition 2.3.30 to find the dimensions and the Lie algebra structure.

In the process of proving the theorem we collect interesting information on the $p$-blocks of the sporadic simple Mathieu groups. Out of the collection of 40 such $p$-blocks with non-zero defect, we determine that 37 have degree one Hochschild cohomology that is a solvable Lie algebra, and 1 has degree one Hochschild cohomology that is a simple Lie algebra. Of the 18 $p$-blocks whose degree one Hochschild cohomology is a 2-dimensional Lie algebra, we determine that 15 are non-abelian and 1 is abelian. Moreover, we have that for $G = M_{11}$ or $M_{22}$, $HH^1(kG)$ is a solvable Lie algebra for all primes $p = \mathrm{char}(k)$ dividing $|G|$.

This chapter is structured as follows. In Section 4.2 we give the definitions, notation and quoted results used throughout. In Section 4.3 we determine in general the Lie algebra structure of $HH^1(B)$ for $B$ a block with a non-trivial cyclic defect group, giving an explicit formula for its dimension (though this is already folklore) and a characterisation of a basis for $HH^1(B)$ as a Lie algebra. This is a solvable Lie algebra in general (see [56, Example 5.7]), and we note that for some blocks $B$ with a non-trivial cyclic defect group, $HH^1(B)$ has dimension as small as 1. In this case we are also able to determine that there is a $p$-toral basis of $HH^1(B)$.

In Section 4.4 we determine the dimension of the first Hochschild cohomology group of the non-principal 2-block of the Mathieu group $M_{12}$ via an alternative method than that which is used in the rest of the proof of Theorem 4.1.1. When $G = M_{12}$ and $p = 2$, the non-principal block of $kG$ has a non-cyclic defect group; this is the only such case over all primes dividing the orders of all the simple sporadic Mathieu groups $G$, and so the only case where the centraliser decomposition alone is insufficient to compute the desired dimensions. In particular we use a result of Külshammer and Robinson [48] to verify this dimension. This result gives an alternating sum formula for the dimension of the Tate-Hochschild cohomology, equivalent to the Hochschild cohomology in positive degree, and to our knowledge the work done by the author in [60] marks the first explicit use of this formula. Whilst a shorter proof exists to determine these dimensions (see Theorem 4.5.2(i)), our verification showcases the power of this formula and its potential application to other finite groups that do not have such "nice" $p$-block structure as the Mathieu groups (see Sections 4.2.1 and 4.2.2 for more details on the block structure of the Mathieu groups).

Aside from this exceptional case, to determine the dimensions of the first Hochschild cohomology groups of the blocks $B$ of $kG$ in general, the dimension of $HH^1(kG)$ is first calculated, and given in Table 4.3. In Section 4.5 we prove our result using the centraliser decomposition and the $p$-block structure of each sporadic simple Mathieu group.

## 4.2 Background results

Throughout this chapter we will write $C_n$, $S_n$, $A_n$ and $D_{2n}$ the cyclic, symmetric, alternating and dihedral groups of orders $n$, $n!$, $n!/2$ and $2n$ respectively, with $Q_{2^n}$ and $SD_{2^n}$ denoting the quaternion and semi-dihedral groups of order $2^n$. The projective special linear group of degree $n$ over the finite field with $q$ elements is denoted $L_n(q)$, and for $G$ a group acting on a set $X$, we will denote by $X^G$ the fixed points of $X$ under this action. For $x \in G$, $C_G(x)$ will denote the centraliser of $x$ in $G$. Finally, for a block $B$, let $\mathrm{IBr}_k(B)$ denote the set of irreducible Brauer characters belonging to $B$.

Let $A$ be an associative $k$-algebra and let $M$ be an $(A, A)$-bimodule. Denote by $A^e$ the tensor product of $k$-algebras $A \otimes_k A^{\mathrm{op}}$ and view $M$ as an $A^e$-module. Recall the definition of Hochschild cohomology from section 2.3: for all $n \geq 0$ the $n$'th Hochschild cohomology group of $A$ with coefficients in $M$ is the $k$-module

$$HH^n(A; M) = \mathrm{Ext}^n_{A^e}(A; M),$$

and the $n$'th Hochschild cohomology group of $A$ is $HH^n(A; A) = HH^n(A)$. Recall also that the first Hochschild cohomology admits an alternative characterisation as the Lie algebra of $k$-linear derivations on $A$, modulo inner derivations: $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$. As remarked, this Lie algebra structure extends to a graded Lie algebra structure on $HH^*(A)$, though in this chapter we are only interested in degree one.

Now fix $G$ such that the characteristic $p$ of $k$ divides the order $|G|$. We restate the centraliser decomposition for convenience.

**Theorem** (Theorem 2.3.16)**.** As graded $k$-vector spaces, we have a canonical isomorphism

$$HH^*(kG) \cong \bigoplus_x H^*(C_G(x); k),$$

where $x$ runs over a complete set of conjugacy class representatives of $G$.

The first cohomology group of a group $H$ with coefficients in $k$, $H^1(H; k)$, will always be considered with trivial action of $H$ on $k$. In particular, we have a $k$-linear isomorphism $HH^1(kG) \cong \bigoplus_x \mathrm{Hom}(C_G(x), k)$, and hence when viewed as $k$-vector spaces we have a sum of dimensions

$$\dim_k(HH^1(kG)) = \sum_x \dim_k(\mathrm{Hom}(C_G(x), k)).$$

Since we are interested in determining not just the dimensions of the first Hochschild cohomology group but its Lie algebra structure wherever possible, we restate for convenience our earlier result on constructing derivations using the centraliser decomposition.

**Proposition** (Proposition 2.3.30)**.** Let $\{g_1 = 1, g_2, \dots, g_\ell\}$ be a complete set of conjugacy class representatives of $G$, let $G_i = C_G(g_i)$, and let $f_i \in \mathrm{Hom}(G_i, k)$ for $i = 1, \dots, \ell$. Then there is a derivation $d_i \in \mathrm{Der}(kG)$, defined by

$$d_i(g) = \mathrm{cor}_{G_i}^{G}(\hat{f}_i)(g)g,$$

for all $g \in G$ and extended linearly to $kG$, where $\hat{f}_i \in Z^1(G_i; kG)$ is given by $\hat{f}_i(h) = f_i(h)g_i$ for all $h \in G_i$.

In addition, the following theorem will be used to determine the solvability of the first Hochschild cohomology of some of the blocks we will encounter.

**Theorem 4.2.1** ([56, 66])**.** *Let $A$ be a finite dimensional $k$-algebra such that $\dim_k(\mathrm{Ext}_A^1(S,S)) = 0$ and $\dim_k(\mathrm{Ext}_A^1(S,T)) \leq 1$, for all non-isomorphic simple $A$-modules $S$ and $T$. Then $HH^1(A)$ is a solvable Lie algebra.*

Finally, we remark that we are only concerned with the first Hochschild cohomology groups of the blocks of $kG$ that have a non-trivial defect group. In the case of a trivial defect group the Hochschild cohomology vanishes, for any such block is isomorphic to a matrix algebra over $k$ (see [14]), and the Hochschild cohomology group $HH^n(A)$ of a separable algebra $A$ is equal to zero in degree $n > 0$ [30, Theorem 4.1].

### 4.2.1 The Külshammer-Robinson dimension formula

We have an additional method to compute dimensions of the first Hochschild cohomology groups of blocks, detailed here, and used in the proof of Proposition 4.4.1. In [46] Knörr and Robinson consider several simplicial complexes related to the $p$-local structure of $G$, on which $G$ acts naturally by conjugation. Let $\mathcal{P}_G$ denote the simplicial complex with $m$-simplices the (possibly empty) chains of non-trivial $p$-subgroups of $G$ of the form $Q_1 < Q_2 < \cdots < Q_m$ (with strict inclusion) for some integer $m$. The empty chain is then viewed as a $-1$-simplex, and so for explicit calculations (and also for notational convenience) all chains will in fact be viewed as chains of the form $Q_0 < Q_1 < \cdots < Q_m$, where it is always the case that $Q_0 = \{1_G\}$ (c.f [46, Section 2, Remark 1]).

We denote by $\sigma$ an element of $\mathcal{P}_G$ (also known as a $p$-chain), that is $\sigma = Q_0 < \cdots < Q_m$ for some integer $m \geq 0$. Given a $p$-chain $\sigma = Q_0 < \cdots < Q_m$, we define the subgroups $C_G(\sigma) = C_G(Q_m)$ and $N_G(\sigma) = \cap_{i=0}^m N_G(Q_i)$. The *length* of $\sigma$, denoted $|\sigma|$, is equal to the number of non-trivial subgroups in the chain: in our case it will always be equal to $m$ as our chains always start with the trivial subgroup.

In the course of the proof of Proposition 4.4.1, we will restrict our attention to the subcomplex $\mathcal{U}_G \subset \mathcal{P}_G$ of chains of the form $Q_0 < Q_1 < \cdots < Q_m$ such that $Q_i = O_p(N_G(Q_i))$ for all $i = 0, \dots, m$. We call such subgroups $Q_i$ *radical $p$-subgroups*. We say that $\sigma = Q_0 < \cdots < Q_m \in \mathcal{P}_G$ is a *radical* or *normal* chain if $Q_0 = O_p(G)$ and $Q_i = O_p(N_G(\sigma_i))$, where $\sigma_i = Q_0 < \cdots < Q_i$ for all $0 \leq i \leq m$. Denote by $\mathcal{R}_G \subset \mathcal{P}_G$ the set of all radical $p$-chains of $G$ (note this is not a simplicial complex, see the remarks following [46, 3.4]), which will also be considered throughout the course of the proof of Proposition 4.4.1.

Since $G$ acts by conjugation on $\mathcal{P}_G$ and all its subsets we may consider $\mathcal{P}_G/G$, a set of $G$-conjugacy class representatives of $p$-chains, with the analogous notation defined on the subsets of $\mathcal{P}_G$ considered above.

For a $p$-subgroup $Q$ of $G$, denote by $\mathrm{Br}_Q : (kG)^Q \to kC_G(Q)$ the Brauer homomorphism (see Chapter 2), sending $\sum_{g\in G} \lambda_g g$ to $\sum_{g\in C_G(Q)} \lambda_g g$. With this notation, we have the following result (restricted to positive degrees for Hochschild cohomology, c.f [53, 10.7.11]), in which we may replace $\mathcal{P}_G/G$ with either $\mathcal{U}_G/G$ or $\mathcal{R}_G/G$, without altering the result.

**Theorem 4.2.2** ([48], Theorem 1)**.** *Let $b$ a block idempotent of $kG$. For any chain $\sigma = Q_0 < Q_1 < \cdots < Q_m$ in $\mathcal{P}_G$ set $B_\sigma = kN_G(\sigma)\mathrm{Br}_{Q_m}(b)$. Then for any integer $n > 0$ we have*

$$\sum_{\sigma\in\mathcal{P}_G/G} (-1)^{|\sigma|}\dim_k(HH^n(B_\sigma)) = 0.$$

In this alternating sum, the chain $\sigma = Q_0$ recovers the dimension $\dim_k(HH^1(kGb))$ which we may isolate on one side of the equality, and the other terms require only calculations of the dimensions of the degree one Hochschild cohomology of smaller group algebras, or blocks of smaller group algebras, for which we can use the centraliser decomposition or Lemma 4.4.2 below.

Since Proposition 4.4.1 concerns the principal block of the $M_{12}$, we will make use of the following when evaluating the Brauer homomorphism involved in the structure of the $B_\sigma$ above.

**Theorem 4.2.3** ([53], Theorem 6.3.14, Brauer's Third Main Theorem)**.** *Let $b$ be the principal block idempotent of $kG$. Then, for any $p$-subgroup $Q$ of $G$, $\mathrm{Br}_Q(b)$ is the principal block idempotent of both $kC_G(Q)$ and $kN_G(Q)$.*

Similarly, the following result from Chapter 2 will often be used to show that the $\mathrm{Br}_{Q_m}(b)$ in the structure of the $B_\sigma$ above is simply equal to the identity of $kG$, whence we repeat it here.

**Lemma** (Lemma 2.1.5)**.** If $G$ is a $p$-group then $kG$ is indecomposable.

Recall also from Chapter 2 that if $B = kGb$ is a block of $kG$ with block idempotent $b$ and a defect group $P$, and if $C = kN_G(P)c$ is the Brauer correspondent of $B$ in $N_G(P)$ for some idempotent $c$, then we form the inertial quotient of $B$, the group $E = N_G(P, f)/PC_G(P)$ where $f$ is a block of $kC_G(P)$ satisfying $fc = f$. The inertial quotient is unique up to conjugacy, and so independent of choice of $f$. If $B = B_0$ is the principal block, then $\mathrm{Br}_P(b)$ is the principal block of $kC_G(P)$, so that $f = c$ and $E = N_G(P)/PC_G(P)$. Note also that if $P$ is abelian, $PC_G(P) = C_G(P)$.

### 4.2.2 The dimension calculations.

As we will see in Section 4.5, the structure of the Mathieu groups lends itself to calculating the dimensions of the first Hochschild cohomology of their blocks. For each sporadic simple Mathieu group $G$ and each prime $p$ dividing $|G|$, all non-principal $p$-blocks have a cyclic defect group aside from the aforementioned exception of $G = M_{12}$ and $p = 2$ (see the the last remark of §1 in [1]). The blocks of the Mathieu groups with a non-trivial cyclic defect group have structure that is well understood and the dimensions of the degree one Hochschild cohomology are easily calculated in this case (see Section 4.3). As stated earlier the degree one Hochschild cohomology vanishes for the blocks with a trivial defect group.

The dimensions of the degree one Hochschild cohomology of the principal block $B_0$ are then readily calculated (aside from the noted exception) by comparing the dimensions of $HH^1(kG)$ - which are easily computed using the centraliser decomposition and the `GAP` code in Appendix A.1 - and the dimensions of $HH^1(B)$ for the non-principal blocks $B$ with cyclic defect groups:

$$\dim_k(HH^1(B_0)) = \dim_k(HH^1(kG)) \quad - \sum_{\substack{B\neq B_0,\\ B \text{ has cyclic} \\ \text{defect group}}} \dim_k(HH^1(B)).$$

In the exceptional case, $G = M_{12}$ has a non-principal 2-block with Klein four defect group and so to calculate the dimensions and show solvability of the degree one Hochschild cohomology of this block we use results of Holm [33], Erdmann [25] and Linckelmann and Rubio y Degrassi [56].

## 4.3 The $HH^1$ of blocks with cyclic defect groups

The theory of blocks with cyclic defect groups is well understood (see Chapter 2 and, for example [20, 53, 68]) and the first Hochschild cohomology groups of such blocks has a Lie algebra structure that is particularly simple to describe. In this section, unless otherwise stated $P$ will denote a non-trivial cyclic group of order $p$ and $E$ a $p'$-subgroup of $\mathrm{Aut}(P)$, of order $e$. Let $x$ and $y$ be generators of $P$ and $E$ respectively, then $E$ acts on $P$ and we will write the action as $y \cdot x = x^s$ for some $s = 2, \dots, p-1$, such that $s^e \equiv 1 (\mathrm{mod}\ p)$. Let $H = P \rtimes E$ denote the semidirect product.

The group algebra $kH$ is a Nakayama (also selfinjective serial) algebra (see [53, §11.3]), and in particular, if $E$ is non-trivial, then by Linckelmann and Rubio y Degrassi ([56, Example 5.7]) $HH^1(kH)$ is a solvable Lie algebra with nilpotent derived Lie subalgebra.

**Theorem 4.3.1** ([53], 11.1.1/3/11)**.** *Let $G$ be a finite group and $B$ be a block of $kG$ with a non-trivial cyclic defect group $P$ and inertial quotient $E$, and let $H = P \rtimes E$. Then the following hold.*

*(i) The inertial quotient $E$ is cyclic of order dividing $p-1$, and acts freely on $P$.*

*(ii) There is an equality $|E| = |\mathrm{IBr}_k(B)|$.*

*(iii) There is a derived equivalence between $B$ and $kH$.*

Derived equivalences preserve Hochschild cohomology so that in particular, with $B$ and $H$ as above, we have an isomorphism of restricted Lie algebras $HH^1(B) \cong HH^1(kH)$. The group $H$ is a Frobenius group by virtue of the free action of $E$ on $P$.

The next result is well-known (see [68, §2]) and provides a characterisation of the degree one Hochschild cohomology of blocks with a cyclic defect group in terms of that of the group algebra $k(P \rtimes E)$ and the derivations of $HH^1(kP)$ that are fixed under the action of $E$ induced by the action on $P$. We point out that this theorem holds more generally for $P$ an elementary abelian $p$-group.

**Theorem 4.3.2.** *Let $P$ be an elementary abelian $p$-group, $E \leq \mathrm{Aut}(P)$ and write $G = P \rtimes E$. Then the action of $E$ on $P$ induces an action on $HH^1(kP)$, and there is an inclusion of Lie algebras*

$$HH^1(kP)^E \hookrightarrow HH^1(kG).$$

*Moreover if the action of $E$ on $P$ is a Frobenius action then this inclusion is an isomorphism.*

*Proof.* We show something slightly more general. Let $G$ be a finite group with $N$ a normal subgroup of $p'$-index, and recall from the proof of Theorem 2.3.16 that $HH^*(kG) \cong H^*(G; kG)$, where the action of $G \cong \Delta G$ on $kG$ is restricted from the action of $kG \otimes_k (kG)^{\mathrm{op}} \cong k(G \times G)$ on $kG$, whence $G$ acts on $kG$ by conjugation. By [16, III.10.4], $H^*(G; kG) \cong H^*(N; kG)^{G/N}$, which by definition equals $\mathrm{Ext}^*_{kN}(k, kG)^{G/N}$.

Since $N$ is of index coprime to $p$ we have a decomposition under the conjugation action, $kG = kN \oplus k(G\backslash N)$, such that each summand is stable under the action of $G$. One sees that

$$\begin{aligned} HH^*(kG) &\cong \mathrm{Ext}^*_{kN}(k, kN)^{G/N} \oplus \mathrm{Ext}^*_{kN}(k, k(G\backslash N))^{G/N} \\ &= HH^*(kN)^{G/N} \oplus \mathrm{Ext}^*_{kN}(k, k(G\backslash N))^{G/N}. \end{aligned}$$

Now if $N = P$ is an elementary abelian $p$-group and $G = P \rtimes E$, we have $HH^1(kP)^{G/P} \cong HH^1(kP)^E \subseteq HH^1(kG)$. Recall that $x, y$ generate $P, E$ respectively, with $y \cdot x = x^s$ for some $s = 2, \ldots, p-1$. One checks that for $d \in HH^1(kP)$ a derivation, then the action of $E$ on $HH^1(kP)$ is given by ${}^y d \in HH^1(kP)$, the derivation sending $x \mapsto y \cdot d(y^{-1} \cdot x)$.

We also have that if the action of $E$ on $P$ is Frobenius then $k(G\backslash P) = \bigoplus_{y \in E\backslash\{1\}} kPy$ is a direct sum of projective $kP$-modules under the conjugation action of $P$ on $Py$, so that $\mathrm{Ext}^1_{kP}(k, k(G\backslash P)) = \{0\}$, and the second statement follows. $\square$

**Proposition 4.3.3.** *Let $P$ be a finite abelian $p$-group, $r$ the rank of $P/\Phi(P)$, and $E$ be an abelian $p'$-subgroup of $\mathrm{Aut}(P)$ acting freely on $P\backslash\{1\}$. Set $H = P \rtimes E$. Then*

$$\dim_k(HH^1(kH)) = r \cdot \left(\frac{|P|-1}{|E|}\right).$$

*Proof.* Since $E$ acts freely and is abelian, we have the $\frac{|P|-1}{|E|} + |E|$ conjugacy classes of $H$ as described in [68, §2], generalised to the finite abelian $p$-group case. Noting that $\mathrm{Hom}(H, k) = \mathrm{Hom}(E, k) = \{0\}$, we obtain an isomorphism of $k$-modules $HH^1(kH) \cong \mathrm{Hom}(P, k)^{\oplus \ell}$ where $\ell = (|P|-1)/|E|$ and the result then follows from the observation that $\dim_k(\mathrm{Hom}(P, k)) = r$. $\square$

**Remark 4.3.4.** The dimensions calculated when using this formula agree, of course, with the dimensions calculated using the centraliser decomposition in Example 2.3.22 as well as the group $H$ as defined in Chapter 3.

From this proposition we obtain the following simple corollary via the derived equivalence of Theorem 4.3.1.

**Corollary 4.3.5.** *Let $B$ be a block of $kG$ with a non-trivial cyclic defect group $P$ and non-trivial inertial quotient $E$. Then*

$$\dim_k(HH^1(B)) = \frac{|P|-1}{|E|}.$$

The following theorem provides us with a simple recipe to determine a Lie algebra basis of the degree one Hochschild cohomology of blocks with cyclic defect group.

**Theorem 4.3.6.** *Let $P$ be a cyclic group of order $q = p^t$ with a generator $x$, and $E$ a $p'$-subgroup of $\mathrm{Aut}(P)$ of order $e$, with a generator sending $x$ to $x^s$ and set $H = P \rtimes E$. Let $\ell = (q-1)/e$ and $\{x^{s_i} \mid i = 0, \dots, \ell\}$ be a complete set of $E$-orbit representatives of $P$. For $i = 0, \dots, \ell$, let $d_i$ denote the map from $kH \to kH$ that sends*

$$x \longmapsto \sum_{j=0}^{e-1} s^j x^{s_i s^{-j}+1},$$

*and sends $y \mapsto 0$. Then we have the following.*

*(i) The set $\{d_1, \dots, d_\ell\}$ forms a $k$-basis for $HH^1(kH)$ as a Lie algebra.*

*(ii) If $E$ is non-trivial, then for all $i$, $d_i(x) \in J(kP)$.*

*(iii) Let $p > 2$, $t = 1$ and let $E$ have order $p-1$. Then $HH^1(kH)$ has a (1-dimensional) $p$-toral basis.*

*Proof.* The automorphism group $\mathrm{Aut}(P)$ is isomorphic to the direct product of an abelian group of order $p^{t-1}$ and a cyclic group of order $p-1$. Whence $E$ is cyclic acting freely on $P$. Let $E$ have a generator $y$, so that $y \cdot x = x^s$. As in Proposition 4.3.3 we have an isomorphism $HH^1(kH) \cong \mathrm{Hom}(P,k)^{\oplus \ell}$. Let $x_i = x^{s_i}$ be the $\ell$ distinct non-trivial $E$-orbit representatives of $P$ and write $P_i = C_H(x_i) = P$ so that $HH^1(kH) \cong \bigoplus_{i=1}^{\ell} \mathrm{Hom}(P_i, k)$. For each $P_i$, let $f_i \in \mathrm{Hom}(P_i, k)$ be the homomorphism sending $x \mapsto 1$ so that as a $k$-module $\mathrm{Hom}(P_i, k)$ has a basis given by $\{f_i\}$. We note that $f_i$ sends $x^j \mapsto j$ for all $i = 0, \dots, \ell$ and $j = 0, \dots, p-1$. By Proposition 2.3.30 we see that

$$d_i(x) = \sum_{z \in E} f_i(z \cdot x)(z^{-1} \cdot x_i)x,$$

for $H/P_i \cong E$ for all $i$ and $z_1^{-1} x z_2 \in P_i$ if and only if $z_1 = z_2$ for all $z_1, z_2 \in E$ since $H$ is Frobenius. One verifies that the formula for $d_i(x)$ is as given in the statement, and a similar calculation shows that $d_i(y) = 0$ for all $i$, proving (i).

Recall that $s$ is such that $s^e \equiv 1 \pmod p$. One sees that

$$(1-s)\left(\sum_{i=0}^{e-1} s^i\right) = 1 - s^e \equiv 0,$$

and if $E$ is non-trivial, then the free action of $E$ on $P$ implies $s > 1$, whence $\sum_{i=0}^{e-1} s^i \equiv 0$, from which it is clear that $d_0 \equiv 0$ and

$$d_i(x) = \sum_{j=0}^{e-1} s^j x^{s_i s^{-j}+1} \in I(kP) = J(kP),$$

proving (ii).

Now suppose that $p \neq 2$, $t = 1$ so that $P$ is cyclic of order $p$, and that $E$ has order $p-1$. Fix an element $w = \sum_{i=1}^{p-1} i^{-1}x^i \in kP$ (cf. [68, Lemma 2]). Then $w \in I(kP) = J(kP)$, as $\sum_{i=1}^{p-1} i^{-1} = \sum_{i=1}^{p-1} i = 0$. On the other hand $w \notin J(kP)^2$. Indeed $J(kP)^2$ is generated as $k$-vector space by all elements of the form $x^i(x-1)^2$ as $i$ ranges from 0 to $p-1$, and if we set $w = \sum_{i=0}^{p-1} \lambda_i x^i(x-1)^2$ then on comparing coefficients we get a system of linear equations in the variables $\lambda_i$ whose coefficient matrix has rank $p-2$ but whose augmented matrix has rank $p-1$, whence no such $\lambda_i \in k$ exist. By Nakayama's Lemma $w$ generates $J(kP)$ as a $kP$-module, and noting that

$$w^p = \left(\sum_{i=1}^{p-1} i^{-1}x^i\right)^p = \sum_{i=1}^{p-1} i^{-p}x^{pi} = \sum_{i=1}^{p-1} i^{-1} = 0,$$

one sees that $\{1, w, w^2, \dots, w^{p-1}\}$ is a basis for $kP$ as a $k$-vector space.

One checks that $w$ is sent to $sw$ under the action of $y$, and that $d_i(x) = s_iwx$ for all $i$. By Proposition 4.3.5, $HH^1(kH)$ has dimension 1 in this case so we may choose $i$ such that $s_i = -1$, that is we choose $x^{-1}$ as the representative of the non-trivial $E$-orbit of $P$, without loss of generality. Write $d_i = d$ so that $HH^1(kH)$ has a basis given by $\{d\}$. It is now simple to check that $d(w) = -\sum_{j=1}^{p-1} x^jw$, and we will show that in fact $d(w) = w$, proving (iii).

We have

$$\sum_{j=1}^{p-1} x^jw = \sum_{j=1}^{p-1} x^{-j}w = \sum_{j=1}^{p-1}\sum_{m=1}^{p-1} m^{-1}x^{m-j} = \sum_{\substack{j=1 \\ j\neq n}}^{p-1}\sum_{n=0}^{p-1} (n-j)^{-1}x^n,$$

where the final equality is seen explicitly by expanding the sum indexed by $m$ followed by collecting terms with common powers of $x$. Fixing $n \neq 0$, one checks that

$$\sum_{j=1, j\neq n}^{p-1} (n-j)^{-1} = \sum_{j=1-n, j\neq 0}^{p-n-1} j^{-1} = -n^{-1},$$

and when $n = 0$ we recover $\sum_{j=1}^{p-1}(-j)^{-1} = 0$. Combining this with the sum indexed by $n$, we have $d(w) = \sum_{n=1}^{p-1} n^{-1}x^n = w$. This completes the proof.

□

**Remark 4.3.7.** In part (iii) of the previous result it is possible to show that for $P$ cyclic of order $q = p^t$, $t > 1$, there is still a basis of $HH^1(kP)^E$ consisting of maps $d_i$ such that $d_i^p = d_i$. Indeed, fixing $w = \sum_{j=1, p\nmid j}^{q-1} j^{-1}x^j$ one verifies by an analogous proof that $\{1, w, w^2, \dots, w^{q-1}\}$ is a basis for $kP$ as a $k$-vector space, and that $d_i^p(w) = d_i(w) = w$ for all $i = 0, \dots, (q-1)/e$, $i \neq 1$. On the other hand a simple verification shows that in general $[d_i, d_j] \neq 0$ for $i \neq j$ and so this basis is not $p$-toral.

We note also that it is possible to arrive at the same formula for the $d_i$ by finding the fixed points of $HH^1(kP)$ under the induced action of $E$, then using the isomorphism $HH^1(kH) \cong HH^1(kP)^E$. One basis for $HH^1(kP)$ is given by the $q$ distinct maps which send $x$ to $x^i$ for $i = 0, \dots, q-1$, from which the fixed points are easy to find.

## 4.4 The Külshammer-Robinson dimension formula for $M_{12}$

Using the result of Külshammer and Robinson (Theorem 4.2.2) we verify the following statement which we will see again in the sequel (Theorem 4.5.2(i)), that ultimately forms part of the proof of Theorem 4.1.1. As mentioned before we use this alternative method for verification in order to demonstrate its use in applications to other groups and more general cases, and its power as a computational tool.

**Proposition 4.4.1.** *Let $G = M_{12}$, let $k$ be an algebraically closed field of characteristic 2 and let $B_0$ be the principal block of $kG$. Then $\dim_k(HH^1(B_0)) = 13$.*

From Table 4.4 below we see that $M_{12}$ is the only sporadic simple Mathieu group with a non-principal block that has defect greater than 1, occuring when $p = 2$. The results of Section 4.3 no longer apply here and so the methods used to find the dimensions of the first Hochschild cohomology group of the principal block are different for this case only.

For the remainder of this section we fix $G = M_{12}$, $\mathrm{char}(k) = 2$, and $B_0 = kGb$ the principal 2-block of $kG$, with block idempotent $b$. In order to prove Proposition 4.4.1 we will calculate some auxiliary dimensions that appear in the alternating sum. In particular, $G$ has a unique conjugacy class of subgroups isomorphic to $C_2 \times S_5$, to $S_3 \times A_4$ and to $C_2^2 \times S_3$, and the principal block of these groups arises in the alternating sum as some $kN_G(\sigma)\mathrm{Br}_Q(b)$ for some chain $\sigma$ and 2-subgroup $Q$ of $G$.

**Lemma 4.4.2.** *Let $H$ be one of $C_2 \times S_5$, $S_3 \times A_4$ or $C_2^2 \times S_3$. Then $kH$ has two blocks $B_0'$ and $B_1'$ and we have the following dimensions.*

*(i) Let $H = C_2 \times S_5$ Then $\dim_k(HH^1(B_0')) = 22$ and $\dim_k(HH^1(B_1')) = 8$.*

*(ii) Let $H = S_3 \times A_4$. Then $\dim_k(HH^1(B_0')) = 12$ and $\dim_k(HH^1(B_1')) = 2$.*

*(iii) Let $H = C_2^2 \times S_3$. Then $\dim_k(HH^1(B_0')) = 24$ and $\dim_k(HH^1(B_1')) = 8$.*

*Proof.* First let $H = C_2 \times S_5$. Blocks of $H$ are in bijection with blocks of $kC_2 \otimes_k kS_5$, and $kS_5$ decomposes into two blocks $B_0(kS_5)$ and $B_1(kS_5)$ with defect groups isomorphic to $D_8$ and $C_2$ respectively. The block $B_0(kS_5)$ is then a tame block of type $D(2\mathcal{A})$, derived equivalent to $kS_4$ by Holm [32], which by the centraliser decomposition has $\dim_k(HH^1(kS_4)) = 6$ and has $\dim_k(HH^0(kS_4)) = \dim_k(Z(kS_4)) = 5$. One sees therefore that $HH^1(B_0') \cong HH^1(k(C_2 \times S_4))$ and the latter has an additive decomposition isomorphic to $kC_2 \otimes_k Z(kS_4) \oplus kC_2 \otimes_k HH^1(kS_4)$ by the Künneth formula (Theorem 2.3.21), whence the dimension of $HH^1(B_0')$ is 22.

Similarly, the block $B_1(kS_5)$ is derived equivalent to $kC_2$ by Theorem 4.3.1(iii) so that $HH^1(B_1') \cong HH^1(k(C_2 \times C_2))$ which has dimension 8. By this block decomposition of $kH$, we may add the dimensions of the respective Hochschild cohomology rings to see that $\dim_k(HH^1(kH)) = 30$: this may also be verified using the centraliser decomposition.

Now let $H = S_3 \times A_4$. The proof follows in the same vein as (i). The group algebra $kA_4$ is indecomposable whereas $kS_3$ has 2 blocks $B_0(kS_3)$ and $B_1(kS_3)$ with defect 1 and 0 respectively, isomorphic to $kC_2$ and $M_n(k)$ respectively, for some $n$. Whence $B_0' \cong B_0(kS_3 \otimes_k kA_4) \cong k(C_2 \times A_4)$ which has degree one Hochschild cohomology of dimension 12 by the centraliser decomposition. Similarly $B_1' \cong kA_4$ which has first Hochschild cohomology group of dimension 2, and using the centraliser decomposition one verifies that $HH^1(kH)$ has dimension 14.

Finally let $H = C_2^2 \times S_3$. One sees that $B_0' \cong B_0(kC_2^2 \otimes_k kS_3) \cong kC_2^2 \otimes_k B_0(kS_3) \cong kC_2^3$ which has degree one Hochschild cohomology of dimension 24, $B_1' \cong kC_2^2$ which has first Hochschild cohomology of dimension 8, and using the centraliser decomposition one verifies that $HH^1(kH)$ has dimension 32. $\square$

*Proof of Proposition 4.4.1.* We will apply Külshammer and Robinson's result using $\mathcal{U}_G$, the complex of radical 2-subgroups of $G$. A defect group of $B_0$ is given by $P \cong C_4^2 \rtimes C_2^2$ (see `SmallGroup(64,134)` in the `GAP` library), and using [1, Table 2D] we deduce the information displayed in Table 4.1, which gives us the radical 2-subgroups of $G$, up to $G$-conjugacy (the details of the actions of the semidirect product may be found using `GAP` and the database [22]).

Table 4.1: The radical 2-subgroups of $M_{12}$

| Label, $Q$ | Isomorphic to | $N_G(Q)$ | $C_G(Q)$ |
|---|---|---|---|
| $P_0$ | $\{1_G\}$ | $G$ | $G$ |
| $P_1$ | $C_2$ | $C_2 \times S_5$ | $C_2 \times S_5$ |
| $P_2$ | $C_2 \times C_2$ | $S_3 \times A_4$ | $C_2^2 \times S_3$ |
| $P_3$ | $C_4 \rtimes D_8$ | $C_4^2 \rtimes D_{12}$ | $C_2 \times C_2$ |
| $P_4$ | $2_+^{1+4}$ | $Q_8 \rtimes S_4$ | $C_2$ |
| $P$ | $C_4^2 \rtimes C_2^2$ | $P$ | $C_2$ |

The $G$-conjugacy classes of subgroups of $P$ and their lattice are given in Figure 4.1, where reading left to right a connecting line denotes strict inclusion. We therefore have that $\mathcal{U}_G/G$

Figure 4.1: Lattice of radical 2-subgroups of $M_{12}$

$$\begin{tikzcd}
P_0 \arrow[r, dash] & P_1 \arrow[rr, dash] \arrow[dr, dash] & & P_3 \arrow[r, dash] & P \\
& & P_2 \arrow[r, dash] & P_4 \arrow[ur, dash] &
\end{tikzcd}$$

consists of 20 chains of radical 2-subgroups (all starting at $P_0 = \{1_G\}$); one of length 0, five of length 1, eight of length 2, 5 of length 3 and one of length 4. From these 20 chains, we obtain the information displayed in Table 4.2. We note that the $B_\chi$ are found using Theorem 4.2.3, Lemma 2.1.5 and `GAP`, the dimensions $\dim_k(HH^1(B_\chi))$ for $\chi \in \{\beta, \gamma, \eta\}$ are calculated in Lemma 4.4.2, and all other dimensions are calculated using the centraliser decomposition.

Table 4.2: Representatives of the classes of $\mathcal{U}_G/G$

| $\chi$ | Representative | $B_\chi$ | $\dim_k(HH^1(B_\chi))$ |
|---|---|---|---|
| $\alpha$ | $P_0$ | $B_0(kG) = kGb$ | 13 |
| $\beta$ | $P_0 < P_1$ | $B_0(k(C_2 \times S_5))$ | 22 |
| $\gamma$ | $P_0 < P_2$ | $B_0(k(S_3 \times A_4))$ | 12 |
| $\delta$ | $P_0 < P_3$ | $k(C_4^2 \rtimes D_{12})$ | 30 |
| $\epsilon$ | $P_0 < P_4$ | $k(Q_8 \rtimes S_4)$ | 23 |
| $\zeta$ | $P_0 < P$ | $kP$ | 40 |
| $\eta$ | $P_0 < P_1 < P_2$ | $B_0(k(C_2^2 \times S_3))$ | 24 |
| $\theta$ | $P_0 < P_1 < P_4$ | $k(C_2 \times D_8)$ | 28 |
| $\iota$ | $P_0 < P_1 < P$ | $k(C_2 \times D_8)$ | 28 |
| $\kappa$ | $P_0 < P_1 < P_3$ | $k(C_2 \times S_4)$ | 22 |
| $\lambda$ | $P_0 < P_2 < P_4$ | $k(C_2 \times A_4)$ | 12 |
| $\mu$ | $P_0 < P_2 < P$ | $kC_2^3$ | 24 |
| $\nu$ | $P_0 < P_4 < P$ | $kP$ | 40 |
| $\xi$ | $P_0 < P_3 < P$ | $kP$ | 40 |
| $\pi$ | $P_0 < P_1 < P_2 < P_4$ | $kC_2^3$ | 24 |
| $\rho$ | $P_0 < P_1 < P_2 < P$ | $kC_2^3$ | 24 |
| $\sigma$ | $P_0 < P_1 < P_4 < P$ | $k(C_2 \times D_8)$ | 28 |
| $\tau$ | $P_0 < P_1 < P_3 < P$ | $k(C_2 \times D_8)$ | 28 |
| $\phi$ | $P_0 < P_2 < P_4 < P$ | $kC_2^3$ | 24 |
| $\psi$ | $P_0 < P_1 < P_2 < P_4 < P$ | $kC_2^3$ | 24 |

Let $\Lambda = \{\beta, \gamma, \ldots, \psi\}$, that is, the set of all chain labels $\chi$ in Table 4.2, excluding $\chi = \alpha$. Then the alternating sum formula of Theorem 4.2.2 tells us that

$$\dim_k(HH^1(B_\alpha)) = \dim_k(HH^1(B_0(kG))) = \sum_{\chi \in \Lambda} (-1)^{|\chi|+1} \dim_k(HH^1(B_\chi)) = 13.$$

This completes the proof.

□

## 4.5 The proof of Theorem 4.1.1

Let $G$ be one of the sporadic simple Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ or $M_{24}$, and let $k$ be an algebraically closed field of prime characteristic $p$ dividing $|G|$. Throughout this section we use `GAP` [29], the ATLAS [18], the online group database found at [22], the Modular Atlas project [76] and its print counterpart [37] as our sources of data on the groups $G$, their subgroups and their blocks.

To prove Theorem 4.1.1 we first calculate the dimensions of $HH^1(kG)$ using the centraliser decomposition. To efficiently do so, we use the `GAP` code in Appendix A.1, which provides the following data on the dimensions of $HH^1(kG)$ at the different primes $p$.

Table 4.3: The dimensions of $HH^1(kG)$ for $p = \mathrm{char}(k)$ dividing $|G|$.

| $G$ | Order | $p = 2$ | $p = 3$ | $p = 5$ | $p = 7$ | $p = 11$ | $p = 23$ |
|---|---|---|---|---|---|---|---|
| $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 6 | 2 | 1 | – | 2 | – |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 15 | 4 | 2 | – | 2 | – |
| $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 9 | 3 | 1 | 2 | 2 | – |
| $M_{23}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ | 9 | 5 | 3 | 4 | 2 | 2 |
| $M_{24}$ | $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ | 22 | 10 | 4 | 6 | 1 | 2 |

We need to describe how the dimensions of $HH^1(kG)$ given in Table 4.3 decompose into the dimensions of the degree one Hochschild cohomology of the $p$-blocks of $kG$, in view of the structure provided in Table 4.4 below (see Section 4.2.2). This table - the data of which one may verify in `GAP` - describes the $p$-block structure of $kG$ for the prime divisors $p$ of $|G|$, in particular the number of blocks and their defect. The notation $[n_0, n_1, \dots, n_m]$ denotes that $G$ has $p$-block structure $kG = B_0 \oplus B_1 \oplus \cdots \oplus B_m$ where $B_0$ is the principal block of $kG$, $B_i$ has defect $n_i$, and the blocks are ordered according to the decreasing size of their defect. The notation $1^n$ (resp. $0^n$) denotes $n$ successive $p$-blocks whose defect groups are non-trivial cyclic (resp. trivial).

Table 4.4: The defects of the $p$-blocks of $kG$ for $p = \mathrm{char}(k)$ dividing $|G|$.

| $G$ | $p = 2$ | $p = 3$ | $p = 5$ | $p = 7$ | $p = 11$ | $p = 23$ |
|---|---|---|---|---|---|---|
| $M_{11}$ | $[4, 0^2]$ | $[2, 0]$ | $[1, 0^5]$ | – | $[1, 0^3]$ | – |
| $M_{12}$ | $[6, 2]$ | $[3, 1, 0]$ | $[1^2, 0^5]$ | – | $[1, 0^8]$ | – |
| $M_{22}$ | $[7]$ | $[2, 1, 0^3]$ | $[1, 0^7]$ | $[1, 0^7]$ | $[1, 0^5]$ | – |
| $M_{23}$ | $[7, 0^2]$ | $[2, 1, 0^5]$ | $[1^2, 0^8]$ | $[1^2, 0^7]$ | $[1, 0^{10}]$ | $[1, 0^4]$ |
| $M_{24}$ | $[10]$ | $[3, 1^4, 0]$ | $[1^3, 0^{12}]$ | $[1^3, 0^{11}]$ | $[1, 0^{15}]$ | $[1, 0^{13}]$ |

We divide Theorem 4.1.1 into the following 5 results, organised with regards to which Mathieu group they relate to. In each theorem, we will write $B_0$ for the principal $p$-block of $kG$, and $B_i$ for the non-principal $p$-blocks of $kG$, $i = 1, 2, \dots$, if they exist, ordered by descending defect. We adopt the convention that the "zero Lie algebra" $\mathcal{L} = \{0\}$ with Lie bracket $[0, 0] = 0$ is trivially solvable.

**Theorem 4.5.1.** *Let $G = M_{11}$. Then $HH^1(kG)$ is a solvable Lie algebra isomorphic to $HH^1(B_0)$, every non-principal block of $kG$ has defect 0, and the following isomorphisms of Lie algebras with given dimensions hold.*

*(i) Let $p = 2$ and $B_0(kL_3(q))$ be the principal 2-block of $L_3(q)$, $q \equiv 3 \pmod 4$. Then there are isomorphisms of 6-dimensional solvable Lie algebras,*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(B_0(kL_3(q))).$$

*(ii) Let $p = 3$. Then there are isomorphisms of 2-dimensional Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(k(C_3^2 \rtimes SD_{16})).$$

*(iii) Let $p = 5$. Then there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_5)^{C_4} \cong k.$$

*(iv) Let $p = 11$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_{11})^{C_5}.$$

Table 4.5: Data on the $p$-blocks of $kM_{11}$.

| $G = M_{11}$ | $p = 2$ | $p = 3$ | $p = 5$ | $p = 11$ |
|---|---|---|---|---|
| No. of blocks, $B_i$, of $kG$ | 3 | 2 | 6 | 4 |
| Defects of the $B_i$ | $[4, 0^2]$ | $[2, 0]$ | $[1, 0^5]$ | $[1, 0^3]$ |
| Dimensions $\dim_k(HH^1(B_i))$ | $[6, 0^2]$ | $[2, 0]$ | $[1, 0^5]$ | $[2, 0^3]$ |
| Sylow $p$-subgroup $P$ of $G$ | $SD_{16}$ | $C_3^2$ | $C_5$ | $C_{11}$ |
| $E = N_G(P)/PC_G(P)$ | $\{1\}$ | $SD_{16}$ | $C_4$ | $C_5$ |

In Table 4.5 above, and Tables 4.6, 4.7, 4.8, and 4.9 below, the notation of the second row (the defects of the blocks) is the same notation as in Table 4.4. The notation of the third row is similar: $[n_0, n_1, \ldots, n_m]$ denotes that $G$ has $p$-block structure $kG = B_0 \oplus B_1 \oplus \cdots \oplus B_m$ where $HH^1(B_i)$ has dimension $n_i$, whilst $n^m$ denotes $m$ successive $p$-blocks whose first Hochschild cohomology group has dimension $n$.

*Proof of Theorem 4.5.1.* By the data in Table 4.5 one sees that for all primes dividing $|G|$, the block structure of $kG$ provides the isomorphism $HH^1(kG) \cong HH^1(B_0)$. The dimensions $\dim_k(HH^1(B_0))$ are therefore simply the same as the dimensions of $HH^1(kG)$, which are calculated using the centraliser decomposition.

Let $p = 2$. Benson and Carlson showed in [6, (14.1)] that there is a Morita equivalence between $G$ and $L_3(3)$, whence $HH^1(kG) \cong HH^1(B_0(L_3(3))$. One uses the decomposition matrix of $B_0$, available on `GAP`, to see that $B_0$ has 3 irreducible Brauer characters, and so 3 simple modules up to isomorphism. Observing that $B_0$ is a tame block, by Erdmann's classification [25, pp.300] one verifies that the decomposition and Cartan matrices of $B_0$ match those of $SD(3\mathcal{B})_1$. This may be further verified in [33, 3.3/3.4] which gives us explicitly a derived equivalence between $B_0$ and $B_0(L_3(q)), q \equiv 3 \pmod 4$. The quiver and relations of $B_0$ given by Erdmann enable us to determine that it is not a local algebra, which by a result of Eisele and Raedschelders [24, 4.7] provides the solvability of $HH^1(B_0)$. This proves (i).

Now let $p = 3$. A Sylow 3-subgroup $P$ is isomorphic to $C_3 \times C_3$, with inertial quotient $E = SD_{16}$ as in Table 4.5. By a result of Rouquier [65] there is a stable equivalence of Morita type between $B_0$ and its Brauer correspondent $kN_G(P) = k(P \rtimes E)$, (in fact there is derived equivalence between $B_0$ and $kN_G(P)$ constructed explicitly by Okuyama in an unpublished result [62]). Whence $HH^1(B_0) \cong HH^1(k(P \rtimes E))$ proving (ii). For (iii) and (iv), Theorems 4.3.1(iii) and 4.3.2 provide the second isomorphism, and using Corollary 4.3.5 one may verify the dimensions of both match those found by the centraliser decomposition, completing the proof. □

**Theorem 4.5.2.** *Let $G = M_{12}$. Then the following isomorphisms of Lie algebras with given dimensions hold.*

*(i) Let $p = 2$. Then $HH^1(B_0)$ is a 13-dimensional Lie algebra and there is an isomorphism of 2-dimensional abelian Lie algebras*

$$HH^1(B_1) \cong HH^1(kA_4).$$

*(ii) Let $p = 3$. Then $HH^1(B_0)$ is a 3-dimensional solvable Lie algebra and there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(B_1) \cong HH^1(kS_3) \cong k.$$

*(iii) Let $p = 5$. Then there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(B_0) \cong HH^1(B_1) \cong HH^1(C_5)^{C_4} \cong k.$$

*(iv) Let $p = 11$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_{11})^{C_5}.$$

**Remark 4.5.3.** Let $p = 3$. By Usami [71, Proposition 4] there is a perfect isometry between the principal blocks of $kM_{12}$ and $L_3(3)$, and we note that the first Hochschild cohomology group of both is a 3-dimensional solvable Lie algebra. The dimension of the degree one Hochschild cohomology in the $L_3(3)$ case is computed by the centraliser decomposition, and to see solvability in this case we apply a result of Koshitani [44, Theorem 1] to Theorem 4.2.1. On the other hand we are not aware of the existence of a derived equivalence in the literature and so we are unable to determine whether $HH^1(B_0)$ and $HH^1(kL_3(3))$ are isomorphic as Lie algebras.

Table 4.6: Data on the $p$-blocks of $kM_{12}$

| $G = M_{12}$ | $p = 2$ | $p = 3$ | $p = 5$ | $p = 11$ |
|---|---|---|---|---|
| No. of blocks, $B_i$ of $kG$ | 2 | 3 | 7 | 9 |
| Defects of the $B_i$ | $[6, 2]$ | $[3, 1, 0]$ | $[1^2, 0^5]$ | $[1, 0^8]$ |
| Dimensions $\dim_k(HH^1(B_i))$ | $[13, 2]$ | $[3, 1, 0]$ | $[1^2, 0^5]$ | $[2, 0^8]$ |
| Sylow $p$-subgroup $P$ of $G$ | $C_4^2 \rtimes C_2^2$ | $3_+^{1+2}$ | $C_5$ | $C_{11}$ |
| $E = N_G(P)/PC_G(P)$ | $\{1\}$ | $C_2 \times C_2$ | $C_4$ | $C_5$ |

*Proof of Theorem 4.5.2.* Let $p = 2$. A defect group of $B_1$ is isomorphic to the Klein 4 group (see [1]), so that $B_1$ is a tame block. The decomposition matrix of $B_1$ is available on `GAP` allowing confirmation that $B_1$ has 3 irreducible Brauer characters, and so 3 simple modules up to isomorphism. Using Erdmann's classification of tame blocks [25, pp.296] one verifies that the decomposition and Cartan matrices of $B_1$ match those of $D(3\mathcal{K})$. We note that by [56, Proposition 5.3] this implies that $HH^1(B_1)$ is a solvable Lie algebra. The isomorphism and dimension of $HH^1(B_1)$ is then given by Holm [33, 2.1(2)], and by Table 4.3 we recover $\dim_k(HH^1(B_0)) = 13$.

To show that $HH^1(B_1)$ is the abelian 2-dimensional Lie algebra, we will use Proposition 2.3.30 to find a Lie algebra basis for $HH^1(kA_4) \cong HH^1(B_1)$. Let $P = C_2 \times C_2$ have generators $x$ and $y$, and $E = C_3$ have a generator $r$ so that $A_4 \cong P \rtimes E$ with $r \cdot x = xy$ and $r \cdot y = x$. One verifies that the action of $E$ on $P$ is Frobenius, so that we have an additive isomorphism $HH^1(k(P \rtimes E) \cong \mathrm{Hom}(P, k)$ coming from the centraliser decomposition. We fix $x$ as the unique conjugacy class representative with centraliser $C_{A_4}(x) = P$ occuring in this decomposition and let $f_1 \in \mathrm{Hom}(P, k)$ be given by the maps that send $x, y$ to 1 and 0 respectively. By Proposition 2.3.30 we obtain a derivation of $kA_4$, $d_1$, which sends $x \mapsto 1 + xy$, $y \mapsto 1 + x$ and $r \mapsto 0$. Similarly if $f_2 \in \mathrm{Hom}(P, k)$ is the map sending both $x$ and $y$ to 1, then Proposition 2.3.30 gives a derivation $d_2$ which maps $x \mapsto 1+y$ and $y \mapsto 1+xy$ and $r \mapsto 0$. One checks that $[d_1, d_2] = 0$, and since $\mathrm{Hom}(P, k)$ has a $k$-basis given by $\{f_1, f_2\}$ we obtain an abelian Lie algebra basis for $HH^1(k(P \rtimes E))$ given by $\{d_1, d_2\}$, completing the proof of (i).

For the blocks in (iii) and (iv), and the non-principal block $B_1$ in (ii), Theorems 4.3.1(iii) and 4.3.2 provide the isomorphisms, and Corollary 4.3.5 provides the dimensions. The dimension of the degree one Hochschild cohomology of the principal block in (ii) is then verified using Table 4.3. To see that this principal block has degree one Hochschild cohomology a solvable Lie algebra, we apply results of Koshitani and Waki [45, Theorem 1] to Theorem 4.2.1: the former gives us the dimensions $\dim_k(\mathrm{Ext}^1_{kG}(S, T))$ for all simple $kG$-modules $S$ and $T$, and this meets the solvability criteria of the latter.

□

**Theorem 4.5.4.** *Let $G = M_{22}$. Then $HH^1(kG)$ is a solvable Lie algebra, and the following isomorphisms of Lie algebras with given dimensions hold.*

*(i) Let $p = 2$. Then $kG$ is indecomposable and $HH^1(kG)$ is a 9-dimensional solvable Lie algebra.*

*(ii) Let $p = 3$. Then there is an isomorphism of 2-dimensional non-abelian Lie algebras*

$$HH^1(B_0) \cong HH^1(k(C_3^2 \rtimes Q_8)),$$

*and there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(B_1) \cong HH^1(kS_3) \cong k.$$

*(iii) Let $p = 5$. Then there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(C_5)^{C_4} \cong k.$$

*(iv) Let $p = 7$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_7)^{C_3}.$$

*(v) Let $p = 11$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_{11})^{C_5}.$$

Table 4.7: Data on the $p$-blocks of $kM_{22}$.

| $G = M_{22}$ | $p=2$ | $p=3$ | $p=5$ | $p=7$ | $p=11$ |
|---|---|---|---|---|---|
| No. of blocks, $B_i$ of $kG$ | 1 | 5 | 8 | 8 | 6 |
| Defects of the $B_i$ | [7] | $[2,1,0^3]$ | $[1,0^7]$ | $[1,0^7]$ | $[1,0^5]$ |
| Dimensions $\dim_k(HH^1(B_i))$ | [9] | $[2,1,0^3]$ | $[1,0^7]$ | $[2,0^7]$ | $[2,0^5]$ |
| Sylow $p$-subgroup $P$ of $G$ | $C_2^4 \rtimes D_8$ | $C_3^2$ | $C_5$ | $C_7$ | $C_{11}$ |
| $E = N_G(P)/PC_G(P)$ | $\{1\}$ | $Q_8$ | $C_4$ | $C_3$ | $C_5$ |

*Proof of Theorem 4.5.4.* By the data in Table 4.7 one sees that for all primes $p \neq 3$ dividing $|G|$ the block structure of $kG$ provides the isomorphism $HH^1(kG) \cong HH^1(B_0)$. The dimensions $\dim_k(HH^1(B_0))$ are therefore simply the same as the dimensions of $HH^1(kG)$, which are calculated using the centraliser decomposition.

In the case $p = 2$, to see that $HH^1(kG)$ is a solvable Lie algebra, we use the quiver for $B_0$ constructed by Hoffman [31] over $\mathbb{F}_2$ along with the fact that the simple $kG$-modules in $B_0$ have as a splitting field $\mathbb{F}_2$ (see [36]) and the result follows by Theorem 4.2.1. We note that in [49] Lempken and Staszewski construct the socle series for the projective indecomposable modules in the principal block of the 3-fold cover of $M_{22}$, from which the result may also be deduced.

In the case $p = 3$ we need only use Theorems 4.3.1 and 4.3.2, and Corollary 4.3.5 to determine the isomorphisms $HH^1(B_1) \cong HH^1(kS_3) \cong HH^1(kC_3)^{C_2}$ and the Lie algebra dimension, which via Table 4.3 allows us to calculate $\dim_k(HH^1(B_0)) = \dim_k(HH^1(kG)) - \dim_k(HH^1(B_1)) = 2$. A Sylow 3-subgroup $P$ is isomorphic to $C_3 \times C_3$, with inertial quotient $E = Q_8$. As before, a result of Rouquier then [65] gives us a stable equivalence of Morita type between $B_0$ and its Brauer correspondent $kN_G(P) = k(P \rtimes E)$, (and again there is derived equivalence between $B_0$ and $kN_G(P)$ constructed explicitly by Okuyama in his unpublished result [62]).

In order to verify that $HH^1(B_0)$ is the unique (up to isomorphism) non-abelian 2-dimensional Lie algebra, we use Proposition 2.3.30 to find a basis for $HH^1(k(P \rtimes E))$ as a Lie algebra.

To show that $HH^1(B_0)$ is the unique (up to isomorphism) non-abelian 2-dimensional Lie algebra, we use Proposition 2.3.30 to find a Lie algebra basis for $HH^1(k(P \rtimes E))$. One verifies that the action of $E$ on $P$ is Frobenius, so that by the centraliser decomposition $HH^1(k(P \rtimes E))$ is isomorphic as a vector space to $\mathrm{Hom}(P,k)$. Let $P$ have as generators $x$ and $y$, and $E$ be generated by $r$ and $s$ such that $sr = r^3s$, and write the action of $E$ on $P$ as $r \cdot x = y^2$, $r \cdot y = x$, $s \cdot x = x^2y$ and $s \cdot y = xy$. A $k$-basis for $\mathrm{Hom}(P,k)$ may be given by the two maps $f_1$ and $f_2$, where $f_1$ sends $x, y \mapsto 2$ and $f_2$ sends $x \mapsto 1$, and $y \mapsto 2$. Then using Proposition 2.3.30 we can find the following two derivations $d_1, d_2 \in \mathrm{Der}(k(P \rtimes E))$ induced by $f_1$ and $f_2$ respectively. The derivation $d_1$ sends

$$x \mapsto 1 + y + 2x^2 + 2xy + xy^2 + 2x^2y^2 \quad \text{and} \quad y \mapsto 1 + x^2 + 2y^2 + xy + 2x^2y + 2xy^2$$

whilst $d_2$ sends

$$x \mapsto 1 + 2x^2 + y^2 + xy + 2xy^2 + 2x^2y \quad \text{and} \quad y \mapsto 1 + x + 2y^2 + 2xy + x^2y + 2x^2y^2.$$

One checks that $[d_1, d_2] = d_2 - d_1$. This completes the $p = 3$ case.

For $p \in \{5, 7, 11\}$, Theorems 4.3.1(iii) and 4.3.2 provide the second isomorphism and Corollary 4.3.5 provides the dimensions. $\square$

**Theorem 4.5.5.** *Let $G = M_{23}$. Then the following isomorphisms of Lie algebras with given dimensions hold.*

*(i) Let $p = 2$. Then there is an isomorphism of solvable 9-dimensional Lie algebras*

$$HH^1(kG) \cong HH^1(B_0).$$

*(ii) Let $p = 3$. Then there is an isomorphism of 2-dimensional Lie algebras*

$$HH^1(B_0) \cong HH^1(k(C_3^2 \rtimes SD_{16})),$$

*and an isomorphism of 3-dimensional simple Lie algebras*

$$HH^1(B_1) \cong HH^1(kC_3).$$

*(iii) Let $p = 5$. Then there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(B_0) \cong HH^1(kC_5)^{C_4},$$

*and there is an isomorphism of 2-dimensional non-abelian Lie algebras*

$$HH^1(B_1) \cong HH^1(kC_5)^{C_2}.$$

*(iv) Let $p = 7$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(B_0) \cong HH^1(B_1) \cong HH^1(kC_7)^{C_3}.$$

*(v) Let $p = 11$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_{11})^{C_5}.$$

*(vi) Let $p = 23$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_{23})^{C_{11}}.$$

Table 4.8: Data on the $p$-blocks of $kM_{23}$

| $G = M_{23}$ | $p = 2$ | $p = 3$ | $p = 5$ | $p = 7$ | $p = 11$ | $p = 23$ |
|---|---|---|---|---|---|---|
| No. of blocks, $B_i$ of $kG$ | 3 | 7 | 10 | 9 | 11 | 14 |
| Defects of the $B_i$ | $[7, 0^2]$ | $[2, 1, 0^5]$ | $[1^2, 0^8]$ | $[1^2, 0^7]$ | $[1, 0^{10}]$ | $[1, 0^4]$ |
| Dimensions $\dim_k(HH^1(B_i))$ | $[9, 0^2]$ | $[2, 3, 0^5]$ | $[1, 2, 0^8]$ | $[2, 2, 0^7]$ | $[2, 0^{10}]$ | $[2, 0^4]$ |
| Sylow $p$-subgroup $P$ of $G$ | $C_2^4 \rtimes D_8$ | $C_3^2$ | $C_5$ | $C_7$ | $C_{11}$ | $C_{23}$ |
| $E = N_G(P)/PC_G(P)$ | $\{1\}$ | $SD_{16}$ | $C_4$ | $C_3$ | $C_5$ | $C_{11}$ |

**Remark 4.5.6.** Although the principal 2-block of $kM_{22}$ and $kM_{23}$ have isomorphic defect groups and first Hochschild cohomology groups of equal dimension, they are not derived equivalent: $kM_{22}$ and $kM_{23}$ have centres of dimension 12 and 17 respectively, whence the principal blocks have centres of dimension 12 and 15 respectively.

*Proof of Theorem 4.5.5.* By the data in Table 4.8 one sees that for $p \in \{2, 11, 23\}$, the block structure of $kG$ provides the isomorphism $HH^1(kG) \cong HH^1(B_0)$ and so the dimensions $\dim_k(HH^1(B_0))$ are therefore the same as the dimensions of $HH^1(kG)$, calculated using the centraliser decomposition. For the primes $p = 11$ and $p = 23$ this can be verified using the isomorphisms of Theorems 4.3.1(iii) and 4.3.2, followed by Corollary 4.3.5. These results also complete the case for $p = 5$ or 7.

To see that the principal 2-block has degree one Hochschild cohomology a solvable Lie algebra we use [57, Theorem 4.7] which constructs the projective indecomposable $\mathbb{F}_2G$-modules and shows us that $\mathrm{Ext}^1_{\mathbb{F}_2G}(S, S) = \{0\}$ and $\dim_{\mathbb{F}_2}(\mathrm{Ext}^1_{\mathbb{F}_2G}(S, T)) \leq 1$ for all simple $\mathbb{F}_2G$-modules $S$ and $T$. The field $\mathbb{F}_2$ is a splitting field for the simple $kG$-modules in $B_0$ (see [36]) and so by Theorem 4.2.1 we get that $HH^1(B_0)$ is solvable.

Now let $p = 3$. A Sylow 3-subgroup $P$ is isomorphic to $C_3 \times C_3$, with inertial quotient $E = SD_{16}$ as in Table 4.8. By a result of Rouquier [65] there is a stable equivalence of Morita type between $B_0$ and its Brauer correspondent $kN_G(P) = k(P \rtimes E)$, (and as before there is derived equivalence between $B_0$ and $kN_G(P)$ constructed explicitly by Okuyama in an unpublished result). Whence $HH^1(B_0) \cong HH^1(k(P \rtimes E))$, the dimensions of which may be calculated using the centraliser decomposition. The non-principal block $B_1$ with non-trivial defect has one Brauer character, so that the inertial quotient of $B_1$ is trivial, and by Theorem 4.3.1 the given isomorphism holds. □

**Remark 4.5.7.** Let $p = 3$. The isomorphism of 3-dimensional simple Lie algebras $HH^1(B_1) \cong HH^1(kC_3)$ agrees with results of Waki [72, Theorem 2.4] and Linckelmann and Rubio y Degrassi [56, Theorem 1.3]: the former shows that $B_1$ contains a unique isomorphism class of simple $kG$-modules with representative $S$, of dimension 231, whose Loewy structure gives $\dim_k(\mathrm{Ext}^1_{kG}(S, S)) = 1$. The latter then tells us that if $HH^1(B_1)$ is simple, then it must be nilpotent.

**Theorem 4.5.8.** *Let $G = M_{24}$. Then the following isomorphisms of Lie algebras with given dimensions hold.*

*(i) Let $p = 2$. Then $kG$ is indecomposable and $HH^1(kG)$ is a 22-dimensional solvable Lie algebra.*

*(ii) Let $p = 3$. Then $\dim_k(HH^1(B_0)) = 6$ and there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(B_i) \cong HH^1(kS_3) \cong k$$

*for $i = 1, \ldots, 4$.*

*(iii) Let $p = 5$. Then there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(B_0) \cong HH^1(B_1) \cong HH^1(kC_5)^{C_4} \cong k,$$

*and there is an isomorphism of 2-dimensional non-abelian Lie algebras*

$$HH^1(B_2) \cong HH^1(kC_5)^{C_2}.$$

*(iv) Let $p = 7$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(B_0) \cong HH^1(B_1) \cong HH^1(B_2) \cong HH^1(kC_7)^{C_3}.$$

*(v) Let $p = 11$. Then there are isomorphisms of 1-dimensional Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong k.$$

*(vi) Let $p = 23$. Then there are isomorphisms of 2-dimensional non-abelian Lie algebras*

$$HH^1(kG) \cong HH^1(B_0) \cong HH^1(kC_{23})^{C_{11}}.$$

**Remark 4.5.9.** Let $p = 3$. Similar to Remark 4.5.3, the result of Usami [71, Proposition 4] gives a perfect isometry between the principal blocks of $kM_{24}$ and $He$, the Held group. On the other hand no derived equivalence seems to exist in the literature and so we cannot determine any further the Lie algebra structure of $HH^1(B_0(kM_{24}))$.

Table 4.9: Data on the $p$-blocks of $kM_{24}$

| $G = M_{24}$ | $p = 2$ | $p = 3$ | $p = 5$ | $p = 7$ | $p = 11$ | $p = 23$ |
|---|---|---|---|---|---|---|
| No. of blocks, $B_i$, of $kG$ | 1 | 8 | 15 | 14 | 16 | 14 |
| Defects of the $B_i$ | [10] | $[3, 1^4, 0]$ | $[1^3, 0^{12}]$ | $[1^3, 0^{11}]$ | $[1, 0^{15}]$ | $[1, 0^{13}]$ |
| Dimensions $\dim_k(HH^1(B_i))$ | [22] | $[6, 1^4, 0]$ | $[1^2, 2, 0^{12}]$ | $[2^3, 0^{11}]$ | $[1, 0^{15}]$ | $[2, 0^{13}]$ |
| Sylow $p$-subgroup $P$ of $G$ | Order 1024 | $3_+^{1+2}$ | $C_5$ | $C_7$ | $C_{11}$ | $C_{23}$ |
| $E = N_G(P)/PC_G(P)$ | $\{1\}$ | $D_8$ | $C_4$ | $C_3$ | $C_{10}$ | $C_{11}$ |

*Proof of Theorem 4.5.8.* For $p = 2$ the case is clear, and we use the centraliser decomposition to compute the dimension of $HH^1(B_0)$. We note that from GAP a Sylow 2-subgroup of $G$ has a structural description isomorphic to $(((D_8^2 \rtimes C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$.

To see that the principal 2-block has degree one Hochschild cohomology a solvable Lie algebra we use [38, Section 5] which provides the socle series' for the projective indecomposable $\mathbb{F}_2G$-modules, from which it may be read that $\mathrm{Ext}^1_{\mathbb{F}_2G}(S, S) = \{0\}$ and $\dim_{\mathbb{F}_2}(\mathrm{Ext}^1_{\mathbb{F}_2G}(S, T)) \leq 1$ for all simple $\mathbb{F}_2G$-modules $S$ and $T$. By James [36], $\mathbb{F}_2$ is a splitting field for $G$ and so by Theorem 4.2.1, $HH^1(B_0)$ is solvable.

Now let $p = 3$. A Sylow 3-subgroup $P$ is isomorphic to $3_+^{1+2}$, the extraspecial group of order 27 with exponent 3. The blocks $B_i$, $i = 1, \ldots, 4$ have 2 Brauer characters so that the inertial quotients are isomorphic to $C_2$ in this case, and the isomorphisms and given dimensions hold by Theorems 4.3.1(iii) and 4.3.2, and Corollary 4.3.5. We calculate $\dim_k(HH^1(B_0)) = \dim_k(HH^1(kG)) - \sum_{i=1}^4 \dim_k(HH^1(B_i))$ using the centraliser and block decomposition of $kG$. This proves (ii).

For all remaining primes, we use the block structure and centraliser decomposition to determine a counting argument, the number of Brauer characters to determine the cyclic inertial quotient $E$, Theorems 4.3.1(iii) and 4.3.2, and Corollary 4.3.5 to get the Lie algebra isomorphisms and dimensions of the first Hochschild cohomology groups of the blocks.

□

*Proof of Theorem 4.1.1.* The result follows from Theorems 4.5.1, 4.5.2, 4.5.4, 4.5.5 and 4.5.8. □

**Remark 4.5.10.** If $G$ is one of the sporadic simple Mathieu groups then it is now immediate from Theorem 4.1.1 that $HH^1(kG) \neq \{0\}$. This agrees, of course, with the general case that

$HH^1(kH) \neq \{0\}$ for all finite groups $H$, which is a direct consequence of the main result by Fleischmann, Janiszczak and Lempken [28]. Explicitly, they show that for each such $G$ and each prime divisor $p$ of $|G|$, $G$ has a $p$-element $x$ such that $x \notin C_G(x)'$, and the non-vanishing then follows from the centraliser decomposition, as $\mathrm{Hom}(C_G(x), k) \neq \{0\}$ for such an element $x$.

On the other hand, both Theorem 4.1.1 and [28] also imply that for all 2-cocycles $\alpha \in Z^2(G; k^\times)$ the Hochschild cohomology group $HH^1(k_\alpha G) \neq \{0\}$. In the first case, this follows from the fact that the twisted group algebra $k_\alpha H$ (see Chapter 5 or [52, Section 1.2]) for a finite group $H$ always arises as a direct sum of blocks of some central group extension of $H$. Alternatively this follows from the results in [28] and the twisted group algebra analogue to the centraliser decomposition,

$$HH^1(k_\alpha H) \cong \bigoplus_g H^1(C_G(g); k_\alpha \hat{g}),$$

([77, Lemma 3.5], which we will return to in Chapter 5) where, as before, $g$ runs over a complete set of conjugacy class representatives of $H$. Here, $k_\alpha \hat{g}$ is a 1-dimensional $kC_H(g)$-module spanned by the image $\hat{g} \in k_\alpha H$ of $g \in H$, with action given by $h \cdot \hat{g} = \alpha(h, g)\alpha(g, h)^{-1}\hat{g}$ for all $h \in C_H(g)$. If $g \in H$ is a $p$-element then $\alpha(h, g) = \alpha(g, h)$ for all $h \in C_H(g)$ (see [40, 2.6.1(iv)]) so that the action of $C_H(g)$ on $k_\alpha \hat{g}$ in the twisted centraliser decomposition of $HH^1(k_\alpha H)$ is trivial. The result for $G$ one of the sporadic simple Mathieu groups now follows from [28], in the same manner as the case when $\alpha = 1$.

We note that more work is necessary to determine the block structure of the $k_\alpha G$ and the dimensions of the first Hochschild cohomology groups of $k_\alpha G$ and its blocks.

**Remark 4.5.11.** It is often useful to know when the Lie algebra $HH^1(A)$ is defined over the prime field $\mathbb{F}_p \subset k$ for some $k$-algebra $A$. By Farrell and Kessar [27, 8.1(ii)] the Morita-Frobenius number for the blocks of the sporadic simple Mathieu groups is 1. Thus by [41] any basic algebra $A$ of these blocks is defined over the prime field $\mathbb{F}_p$; $A$ has a $k$-basis whose structure constants are contained in $\mathbb{F}_p$, or equivalently for each block $B$ and basic algebra $A$ of $B$, there is an $\mathbb{F}_p$-algebra $A'$ such that $A \cong k \otimes_{\mathbb{F}_p} A'$. Consequently, $HH^1(B) \cong HH^1(A) \cong k \otimes_{\mathbb{F}_p} HH^1(A')$ is an isomorphism of Lie algebras; that is the Lie algebra $HH^1(B)$ is defined over $\mathbb{F}_p$.

Moreover, for $G = M_{11}$ or $M_{22}$ one can read from Table 4.4 that the blocks $B$ of $kG$ have a Frobenius number of 1, so that $B \cong k \otimes_k B'$ for some $\mathbb{F}_p$-algebra $B'$: $B$ itself is also defined over $\mathbb{F}_p$ and we see that in this case there is a Lie algebra isomorphism $HH^1(B) \cong k \otimes_{\mathbb{F}_p} HH^1(B')$.

# Chapter 5

# The first Hochschild cohomology of twisted group algebras

This chapter is based on the author's paper [61]. In what follows we will begin by defining and collecting some standard results on twisted group algebras, before proceeding to develop them within the context of the first Hochschild cohomology. We will use these results to complete our case study on the Lie algebra structure of the first Hochschild cohomology of the blocks of $k(C_3^2 \rtimes Q_8)$, whose non-principal block is itself a twisted group algebra, concluding Examples 2.1.14, 2.1.19, 2.3.22, and 2.4.

Then, we will discuss further results on the twisted group algebras of finite simple groups with the aim of proving our main result of this chapter, Corollary 5.1.2, which tells us that such algebras have non-trivial first Hochschild cohomology. As was the case for the main results of Chapter 4, the results of this chapter - amongst other things - add to our understanding of how the structure of blocks is inextricably tied to the structure of its Hochschild cohomology algebra.

We then conclude this chapter with a table of calculations of the dimensions of the first Hochschild cohomology of some twisted finite simple group algebras.

Throughout this chapter, $k$ will be an algebraically closed field of prime characteristic $p$.

## 5.1 Introduction

Given a group $G$ of order divisible by $p$, the non-vanishing of the first Hochschild cohomology of the group algebra $kG$ is a consequence of a result of Fleischmann, Janiszczak and Lempken [28]. There, it is shown that one can always find an element $x \in G$ of order divisible by $p$, whose $p$-part is not contained in $C_G(x)'$, the commutator subgroup of the centraliser of $x$ in $G$; developing the terminology used in [28] we will call such an element $x$ a *weak Non-Schur element*. Consequently, one obtains the non-triviality of $\mathrm{Hom}(C_G(x), k)$ for $x$ a weak Non-Schur element, and combining this with the centraliser decomposition for the Hochschild cohomology of group algebras ([5, Theorem 2.11.2]), one sees that $HH^1(kG)$ is always nonzero. See Proposition 5.5.2 for a full and formal proof of this result.

On the other hand, this is still an open problem for the family of *twisted group algebras*, that is, the algebras obtained by "twisting" the group algebra multiplication by some 2-cocycle. Informally, let $\alpha \in Z^2(G; k^\times)$ and denote by $k_\alpha G$ the $k$-algebra with a $k$-basis given by $X := \{\hat{x} \mid x \in G\}$ and multiplication given by $\hat{x}\hat{y} = \alpha(x, y)\widehat{xy}$ for all $\hat{x}, \hat{y} \in X$. Here, the product $\hat{x}\hat{y}$ is in $k_\alpha G$ whilst $\widehat{xy}$ is the image in $X$ of the product $xy$ in $G$.

The task of finding the first Hochschild cohomology of twisted group algebras can be seen to intersect with the guiding principle of this thesis to investigate the first Hochschild cohomology of blocks, when one observes (as we will) that a twisted group algebra $k_\alpha G$ always arises as a direct sum (or direct product depending on one's point of view) of blocks of some finite central group extension $\hat{G}$ of $G$.

In this chapter we will determine the following result, a slight generalisation to the result of Fleischmann, Janiszczak and Lempken [28], which may be of independent group-theoretic interest.

**Theorem 5.1.1.** *Let $G$ be a finite simple group of order divisible by $p$. Then $G$ contains a weak Non-Schur element $x$ which satisfies $\alpha(g, x) = \alpha(x, g)$ for all $\alpha \in Z^2(G; k^\times)$ and all $g \in C_G(x)$.*

In line with our motivation for this thesis, our motivation for this result lies in one immediate consequence of this theorem, as we obtain the non-vanishing first Hochschild cohomology of twisted finite simple group algebras.

**Corollary 5.1.2.** *Let $G$ be a finite simple group of order divisible by $p$. Then for all 2-cocycles $\alpha \in Z^2(G; k^\times)$, the first Hochschild cohomology group of the twisted group algebra $HH^1(k_\alpha G) \neq \{0\}$.*

Just as with the result of Fleischmann, Janiszczak and Lempken, the proof of Theorem 5.1.1 uses the classification of finite simple groups. What is more, it is interesting to note that no proof of the non-vanishing of the first Hochschild cohomology of group algebras is known at present, that does not use the classification. We are also unaware of a reduction to finite simple groups for the non-vanishing Hochschild cohomology of twisted group algebras.

There has been some recent independent interest in the first Hochschild cohomology of twisted group algebras from Todea [70]. As with our work, Todea takes [28] as a starting point and some similar methods are used, though the main results differ: there, it is shown that for $p$-solvable groups, their twisted group algebras have nonzero first Hochschild cohomology. Consequently, we are able to give a partial answer to a question of Todea ([70, Question 1.3]) in the affirmative, with a reduction to finite simple groups remaining the only barrier to a full affirmative answer.

As mentioned, in addition to generalising the case where $\alpha$ represents the trivial class in $H^2(G; k^\times)$, the non-vanishing of the first Hochschild cohomology of twisted group algebras is a problem of interest in its own right as may be seen when one observes that $k_\alpha G$ always arises as the direct sum of blocks of some central group extension of $G$ (see [34, §1]). Thus showing that $HH^1(k_\alpha G) \neq \{0\}$ in general can also be seen as a stepping stone towards proving the non-vanishing of $HH^1(B)$ for a block $B$ of a finite group algebra with a non-trivial defect group.

As we have seen many times at this point, the first Hochschild cohomology of a $k$-algebra $A$ admits a Lie algebra structure, which extends to a graded Lie algebra structure on $HH^*(A)$, though in this note we are only interested in degree one. As detailed in Chapter 4, our motivation to study the more general problem of the non-vanishing of $HH^1(B)$ comes in part from a wider investigation into the links between the Lie algebra structure of $HH^1(B)$ and the $k$-algebra structure of $B$, examining how each influences the other. In particular, the Lie algebra structure is also expected to provide information useful to the Auslander-Reiten conjecture ([3, Conjecture (5)], see [8, 15, 17, 24, 55, 56, 60, 66] for more examples of this).

The proof of Theorem 5.1.1 amounts to an examination of the weak Non-Schur elements given in the Fleischmann, Janiszczak and Lempken result [28], and determining that in each case the desired property holds. In Section 5.2 we give details of and standard results on twisted group algebras, before developing them in 5.3 within the context of Hochschild cohomology. In Section 5.4 we conclude our case study on the Hochschild cohomology of the blocks of $k(C_3^2 \rtimes Q_8)$, whose non-principal block is itself a twisted group algebra. In Section 5.6 we separate the main result into propositions, showcasing different methods to prove that it holds for specific families of finite simple groups. In Section 5.5 we introduce some technical results on finite simple group that we will need to prove Corollaet 5.1.2 and finally, in Section 5.7 we use `GAP` to calculate some dimensions of Hochschild cohomology groups of twisted finite simple group algebras, showcasing the ease of finding such results.

## 5.2 Twisted group algebras: first results

As we have seen, twisted group algebras are those whose multiplication is twisted by some 2-cocycle. As such, we begin with a review of low degree group cohomology. Let $G$ be a group and $Z$ be an abelian group, written multiplicatively, on which $G$ acts and recall the definitions in Chapter 2 of $H^i(G;Z) = Z^i(G;Z)/B^i(G;Z)$ for $i = 0, 1, 2$. In particular, recall that $\alpha \in Z^2(G;Z)$ if $\alpha : G \times G \to Z$ satisfies the 2-*cocycle identity*, that is, if

$$\alpha(a,b)\alpha(ab,c) = \alpha(a,bc)\,({}^{a}\alpha(b,c)) \tag{5.1}$$

for all $a, b, c \in G$. In addition, we saw that $f \in B^2(G;Z)$ if $f \in Z^2(G;Z)$ is such that there exists some $\gamma : G \to Z$ with the property that

$$f(a,b) = \gamma(a)\,({}^{a}\gamma(b))\,\gamma(ab)^{-1} \tag{5.2}$$

for all $a, b \in G$.

If $\alpha, \beta \in Z^2(G;Z)$ are 2-cocycles, then their pointwise product $\alpha\beta$ defined by

$$(g,h) \mapsto \alpha(g,h)\beta(g,h)$$

for all $g, h \in G$ is again a 2-cocycle. The set $Z^2(G;Z)$ forms an abelian group under this product, with inverse $\alpha^{-1}$ defined by $(g,h) \mapsto \alpha(g,h)^{-1}$, and identity element the trivial cocycle, $(g,h) \mapsto 1_Z$. The set $B^2(G;Z)$ of all 2-coboundaries is then a subgroup of $Z^2(G;Z)$, and the quotient group $H^2(G;Z) = Z^2(G;Z)/B^2(G;Z)$ is the second cohomology group of $G$ with coefficients in $Z$, and we denote by $[\alpha]$ the class in $H^2(G;Z)$ of $\alpha \in Z^2(G;Z)$. We say that 2-cocycles $\alpha$ and $\beta$ are *cohomologous* if $\alpha \in [\beta]$.

**Proposition 5.2.1** ([52, Propositions 1.2.5, 1.2.14])**.** *Let $\alpha \in Z^2(G;Z)$. Then the following hold.*

*(i) For all $g \in G$, we have $\alpha(1,g) = \alpha(1,1)$ and $\alpha(g,1) = {}^{g}\alpha(1,1)$.*

*(ii) There is some $\beta \in Z^2(G;Z)$ such that $\beta(1,1) = 1$ and $\beta$ is cohomologous to $\alpha$.*

*Proof.* To prove (i), we first replace $a = 1, b = 1$ and $c = g$ in the 2-cocycle identity (5.1), that is,

$$\alpha(1,1)\alpha(1,g) = \alpha(1,g)\alpha(1,g),$$

and then again, replacing $a = g, b = 1$ and $c = 1$,

$$\alpha(g,1)\alpha(g,1) = \alpha(g,1)\left({}^{g}\alpha(1,1)\right),$$

from which the results follow.

To prove (ii), first define $\gamma : G \to Z$, $\gamma(g) = \alpha(1,g) = \alpha(1,1)$ by part (i). Then define $\beta : G \times G \to Z$, $\beta(a,b) = \alpha(a,b)\gamma(a)^{-1}\left({}^{a}\gamma(b)\right)^{-1}\gamma(ab) = \alpha(a,b)\left({}^{a}\alpha(1,1)\right)^{-1}$ for all $a, b \in G$. Then, writing $z = \alpha(1,1)$, we have

$$\beta(a,b)\beta(ab,c) = \alpha(a,b)\alpha(ab,c)({}^{a}z)^{-1}({}^{ab}z)^{-1}$$

$$\beta(a,bc)\left({}^{a}\beta(b,c)\right) = \alpha(a,bc)\left({}^{a}\alpha(b,c)\right)({}^{a}z)^{-1}{}^{a}\left({}^{b}z\right)^{-1}$$

Note that associativity of the group action of $G$ on $Z$, ${}^{ab}z = {}^{a}({}^{b}z)$, implies that $({}^{ab}z)^{-1} = {}^{a}\left({}^{b}z\right)^{-1}$. Thus $\beta \in Z^2(G;Z)$, and one sees that $\beta \in [\alpha]$ by definition, whilst $\beta(1,1) = \alpha(1,1)\alpha(1,1)^{-1} = 1_Z$, completing the proof. □

We say that any $\alpha \in Z^2(G;Z)$ such that $\alpha(1,1) = 1_Z$ is a *normalised* 2-cocycle.

Now let $\hat{G}$ be an *extension* of $G$ by $Z$, that is, there is a short exact sequence of groups $1 \to Z \to \hat{G} \to G \to 1$ so that $\hat{G}$ is a group satisfying $\hat{G}/Z \cong G$. We will also refer to the sequence itself as an extension. Without loss of generality, we may assume that the first non-trivial map $Z \to \hat{G}$ in the extension is inclusion and that the second non-trivial map $\hat{G} \to G$ is the canonical projection. A *central extension* of $G$ by $Z$ is an extension $\hat{G}$ such that $\iota(Z) \subseteq Z(\hat{G})$.

Two extensions

$$1 \to Z \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G \to 1,$$

$$1 \to Z \xrightarrow{\iota'} \bar{G} \xrightarrow{\pi'} G \to 1,$$

are called *equivalent* if there exists a homomorphism $f : \hat{G} \to \bar{G}$ such that $f \circ \iota = \iota'$ and $\pi' \circ f = \pi$. One verifies via a rote diagram chasing argument that such an $f$ is an isomorphism, thus we may talk of the equivalence classes of extensions of a given group.

**Theorem 5.2.2** ([40], Corollary 1.1.3)**.** *Let $1 \to Z \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G \to 1$ be a central extension of $G$ by $Z$, and for all $g \in G$ choose a preimage $\hat{g} \in \hat{G}$, so that $\pi(\hat{g}) = g$. Define $\alpha_{\hat{G}} : G \times G \to Z$, $\alpha_{\hat{G}}(a,b) = \hat{a}\hat{b}\widehat{ab}^{-1}$ for all $a, b \in G$. Then we have the following.*

*(i) The map $\alpha_{\hat{G}}$ is in $Z^2(G;Z)$ with respect to the trivial action of $G$ on $Z$.*

*(ii) The class $[\alpha_{\hat{G}}]$ is uniquely determined by the equivalence class of the extension $\hat{G}$.*

*(iii) The extension splits ($\hat{G} = Z \rtimes G$) if and only if $[\alpha_{\hat{G}}] = 1$.*

*(iv) The map $\hat{G} \to [\alpha_{\hat{G}}]$ induces a bijection*

$$\{\text{equivalence classes of central extensions of } G \text{ by } Z\} \longleftrightarrow H^2(G;Z),$$

*where $H^2(G;Z)$ is defined with respect to the trivial action of $G$ on $Z$.*

Thus, any extension of $G$ determines a 2-cocycle, and any 2-cocycle determines an extension of $G$. The former is shown in the statement of the theorem: for any $x \in G$, choose $\hat{x} \in \hat{G}$ such that $\pi(\hat{x}) = x$. Then $\pi(\widehat{xy}) = xy = \pi(\hat{x}\hat{y})$, so that $\widehat{xy}$ and $\hat{x}\hat{y}$ differ by a unique element in $Z$, which we define to be $\alpha(x,y)$:

$$\hat{x}\hat{y} = \alpha(x,y)\widehat{xy}.$$

On the other hand, to see that the latter holds, first take some $\alpha \in Z^2(G;Z)$. As a set, take $\hat{G} = Z \times G$, and define a product on $\hat{G}$ by $(\lambda, x)(\mu, y) = (\alpha(x,y)\lambda\mu, xy)$ for all $\lambda, \mu \in Z$ and all $x, y \in G$. Then $G$ is a group with identity $(\alpha(1,1)^{-1}, 1)$, and we can define $\pi : \hat{G} \to G$, $\pi(\lambda, x) = x$, which is clearly surjective with kernel isomorphic to $Z$, giving us the desired extension. Moreover, if we choose $\hat{x} = (1, x)$ for all $x \in G$, then we obtain $\hat{x}\hat{y} = (1,x)(1,y) = (\alpha(x,y), xy) = (\alpha(x,y), 1)(1, xy) = \alpha(x,y)\widehat{xy}$ showing that this extension does indeed determine $\alpha$.

**Definition 5.2.3.** Let $G$ act trivially on $k^\times = k\backslash\{0\}$ and let $\alpha \in Z^2(G;k^\times)$. We define the *twisted group algebra* $k_\alpha G$ as the $k$-vector space with basis indexed by $G$, $\{\hat{g} \mid g \in G\}$ and multiplication given by $\hat{x}\hat{y} = \alpha(x,y)\widehat{xy}$, extended linearly. Here, the product $\hat{x}\hat{y}$ is in $k_\alpha G$ whilst $\widehat{xy}$ is the image in the given basis of the product $xy$ in $G$.

Associativity of the multiplication in $k_\alpha G$ is a direct consequence of the 2-cocycle identity (5.1) and the trivial action of $G$ on $k^\times$.

**Proposition 5.2.4.** *Let $\alpha \in Z^2(G;k^\times)$ and consider the twisted group algebra $k_\alpha G$. Then we have the following.*

*(i) The unit element in $k_\alpha G$ is the element $\alpha(1,1)\widehat{1_G}$.*

*(ii) If $x \in G$, then the multiplicative inverse of $\hat{x} \in k_\alpha G$ is equal to*

$$\hat{x}^{-1} = \alpha(1,1)^{-1}\alpha(x,x^{-1})^{-1}\widehat{x^{-1}}.$$

*Proof.* Let $x \in G$. Then $\alpha(1,1)^{-1}\hat{1}\hat{x} = \alpha(1,1)^{-1}\alpha(1,x)\hat{x} = \hat{x}$ by Proposition 5.2.1, proving (i). Now observe that $\alpha(1,1)^{-1}\alpha(x,x^{-1})^{-1}\widehat{x^{-1}}\hat{x} = \alpha(1,1)^{-1}\alpha(x,x^{-1})^{-1}\alpha(x,x^{-1})\hat{1} = \alpha(1,1)^{-1}\hat{1}$, so that (ii) follows from (i). □

**Proposition 5.2.5** ([52], Proposition 1.2.6)**.** *Let $\alpha, \beta \in Z^2(G;k^\times)$. Then there is a $k$-algebra isomorphism $k_\alpha G \cong k_\beta G$ if and only if $\alpha$ and $\beta$ are cohomologous in $H^2(G;k^\times)$.*

Let us recapitulate so far. It is well known that there is a bijective correspondence between the cohomology classes of 2-cocycles $[\alpha] \in H^2(G;k^\times)$, the equivalence classes of central extensions $1 \to k^\times \to H \to G \to 1$ and the twisted group algebras $k_\alpha G$. Moreover, for such an $\alpha$ one can always construct a central extension of the form $1 \to Z \to \hat{G} \to G \to 1$, for some finite subgroup $\hat{G}$ of $H$ and some (necessarily cyclic of $p'$-order) subgroup $Z$ of $k^\times$, (see [52, Proposition 1.2.18] for a formal statement of this form). With this notation, multiplication in $\hat{G}$ is equivalent to multiplication between elements of the basis of $k_\alpha G$ indexed by $G$, and so we make no distinction between the two. In particular, for each $x \in G$, we write $\hat{x}$ to mean both the image of $x$ in $k_\alpha G$ or a preimage of $x$ in $\hat{G}$. Note that this means that we may always restrict to the case where $\alpha$ takes values in $Z$.

One might now ask, how does this all relate to the blocks of $k\hat{G}$? The answer lies in an idempotent which we have seen before, in Corollary 2.1.12.

**Proposition 5.2.6** ([52], Propositions 1.2.17, 1.2.18)**.** *Let $\hat{G}$ be a central extension of $G$ by a cyclic $p'$ subgroup $Z$ of $k^\times$, $1 \to Z \to \hat{G} \xrightarrow{\pi} G \to 1$. Let $\chi : Z \to k^\times$ be a group homomorphism and for any $x \in G$ choose a preimage $\hat{x} \in \hat{G}$. Denote by $\beta \in Z^2(G; Z)$ the 2-cocycle determined by $\hat{x}\hat{y} = \beta(x, y)\widehat{xy}$ for all $x, y \in G$, and let $\alpha \in Z^2(G; k^\times)$ be the 2-cocycle defined by $\alpha = \chi \circ \beta$. Then the element*

$$e = \frac{1}{|Z|} \sum_{z \in Z} \chi(z)^{-1} z$$

*is a central idempotent in $k\hat{G}$, and we have an isomorphism of $k$-algebras*

$$k\hat{G}e \cong k_\alpha G,$$

*given by the map sending $z\hat{x}e \mapsto \chi(z)x$.*

Note that the idempotent $e$ in the proposition is not necessarily primitive. Thus we may take a primitive decomposition $e = b_1 + \cdots + b_m$ and observe that $\bigoplus_{i=1}^{m} k\hat{G}b_i \cong k_\alpha G$, that is, $k_\alpha G$ is a direct sum of blocks of $k\hat{G}$, as previously mentioned.
We can extend these isomorphisms further.

**Theorem 5.2.7** ([34], Theorem 1.1)**.** *Let $G$, $\hat{G}$, $Z$ and $\alpha$ be as in the statement of Proposition 5.2.6. Then, with $|Z| = m$, we have an isomorphism of $k$-algebras (respectively of left/right ideals of $k\hat{G}$)*

$$k\hat{G} \cong \prod_{i=0}^{m-1} k_{\alpha^i} G \cong \bigoplus_{i=0}^{m-1} k_{\alpha^i} G. \tag{5.3}$$

Let us now introduce some Hochschild cohomology. Taking Hochschild cohomology of (5.3), gives an isomorphism of $k$-vector spaces,

$$HH^1(k\hat{G}) \cong \bigoplus_{i=0}^{m-1} HH^1(k_{\alpha^i} G), \tag{5.4}$$

since, as we have seen already, the Ext functor is additive in both arguments [75, Proposition 3.3.4] and vanishes when applied to modules belonging to different direct factors (see [19, Section 3.4]). This will see regular applications in the proof of Proposition 5.7.1: in particular it will be used to calculate the dimensions of the degree one Hochschild cohomology of twisted simple group algebras for small $Z$, and one notes that on comparing $k$-vector space dimensions in (5.4), we have

$$\dim_k(HH^1(k\hat{G})) = \sum_{i=0}^{m-1} \dim_k(HH^1(k_{\alpha^i} G)) = \dim_k(HH^1(kG)) + \sum_{i=1}^{m-1} \dim_k(HH^1(k_{\alpha^i} G)),$$

which leads to

$$\dim_k(HH^1(k\hat{G})) - \dim_k(HH^1(kG)) = \sum_{i=1}^{m-1} \dim_k(HH^1(k_{\alpha^i}G)). \tag{5.5}$$

What is more, it is easily seen that $k_{\alpha^{-1}}G \cong (k_\alpha G)^{\mathrm{op}}$ as $k$-algebras and $HH^n(A) \cong HH^n(A^{\mathrm{op}})$ as Lie algebras for an arbitrary $k$-algebra $A$ and for all $n$, which can be used to simplify (5.4) and (5.5) in explicit cases, as this results in an equality of dimensions $\dim_k(HH^1(k_\alpha G)) = \dim_k(HH^1(k_{\alpha^{-1}}G))$. Moreover, the centraliser decomposition allows for the easy computation of the dimensions on the left hand side of (5.5).

## 5.3 Twisted group algebras: $HH^1$

What can be said of the dimensions of the $HH^1$ of twisted group algebras? In that direction, we can try to generalise from un-twisted group algebras. In fact, as it happens, many results on group algebras do indeed generalise nicely to the twisted group algebra setting, and in some cases both settings are simply specific examples of the more general *Hopf algebra* setting. As for finding dimensions, we have a generalisation of Theorem 2.3.16, the centraliser decomposition.

**Theorem 5.3.1** ([77, Lemma 3.5]). *There is a canonical isomorphism of graded $k$-modules*

$$HH^*(k_\alpha G) \cong \bigoplus_x H^*(C_G(x); k_\alpha \hat{x}), \tag{5.6}$$

*where $x$ runs over a complete set of conjugacy class representatives of $G$, the subgroup $C_G(x) \subseteq G$ is the centraliser of $x$ in $G$ and $k_\alpha \hat{x}$ is the 1-dimensional $kC_G(x)$-module spanned by the element $\hat{x} \in k_\alpha G$, the image of the element $x \in G$ in the basis of $k_\alpha G$ given by $X = \{\hat{a} \mid a \in G\}$, with action given by $g \cdot \hat{x} = \alpha(g,x)\alpha(x,g)^{-1}\hat{x}$ for all $g \in C_G(x)$.*

Note that setting $[\alpha] = [1] \in H^2(G; k^\times)$ above recovers the standard "untwisted" centraliser decomposition. We remark that the action of $C_G(x)$ on $k_\alpha \hat{x}$ is the "twisted conjugation" action:

$$\hat{g}\hat{x} = \alpha(g,x)\widehat{gx} = \alpha(g,x)\widehat{xg} = \alpha(g,x)\alpha(x,g)^{-1}\hat{x}\hat{g},$$

whence $g \cdot \hat{x} = \hat{g}\hat{x}\hat{g}^{-1}$ for all $g \in C_G(x)$.

We remark that the proof of Theorem 5.3.1 is simply an adaption of the proof of Theorem 2.3.16 to the twisted group algebra case.

*Proof of Theorem 5.3.1.* Fix $n \geq 0$ and $A = k_\alpha G$ with a basis given by $\{\hat{g} \mid g \in G\}$. We will write $A^e$ for $A \otimes_k A^{\mathrm{op}} \cong k_\alpha G \otimes_k k_{\alpha^{-1}}G$, and let $\Delta$ be the group algebra over the diagonal subgroup $G \times G$, so that $\Delta \cong kG$ as $k$-algebras and $\Delta$ is isomorphic to a subalgebra of $A^e$ (see [52, Corollary 1.6.11]). By [10, Lemma 3.3] we have an isomorphism of $A^e$-modules,

$$\mathrm{Ind}_\Delta^{A^e}(k) = A^e \otimes_\Delta k \cong A,$$

given by the map sending $a \otimes b \otimes 1_k \in A^e \otimes_\Delta k$ to $ab \in A$.

Now, by definition, $HH^n(A) = \mathrm{Ext}^n_{A^e}(A; A)$ which as above is isomorphic to $\mathrm{Ext}^n_{A^e}(\mathrm{Ind}^{A^e}_{\Delta}(k); A)$. This in turn is isomorphic to $\mathrm{Ext}^n_{\Delta}(k; \mathrm{Res}^{A^e}_{\Delta}(A))$ by the Eckmann-Shapiro Lemma (see Benson [4, Corollary 2.8.4] for a proof of this), whence $HH^n(A) \cong H^n(G; A)$ where $G$ acts on $A$ by conjugation. We decompose $A$ as a direct sum of $\Delta$-modules, $A \cong \bigoplus_x k_\alpha \mathcal{C}_x$ where $x$ runs over a complete set of conjugacy class representatives of $G$ and $k_\alpha \mathcal{C}_x$ is the subspace of $A$ spanned by the image of the conjugacy class containing $x$, in $A$: note $G = \bigcup_x \mathcal{C}_x$ is a disjoint union whence the decomposition is a direct sum.

For $x$ a fixed conjugacy class representative, $\mathcal{C}_x$ is naturally in a set-wise bijection with $G/C_G(x)$, the set of left co-sets of $C_G(x)$ in $G$. Whence we have that each $k_\alpha \mathcal{C}_x$ is isomorphic as an $A$-module to $k_\alpha(G/C_G(x))$, with action given by left multiplication: writing $\bar{g}$ for $\hat{g}C_G(x) \in k_\alpha(G/C_G(x))$ we have that $\hat{h} \cdot \bar{g} = \alpha(h, g)\overline{hg}$ for all $\hat{h} \in A$. On the other hand, there is an isomorphism of $A$-modules

$$k_\alpha(G/C_G(x)) \cong \Delta \otimes_{\Delta(C_G(x))} k_\alpha \hat{x} = \mathrm{Ind}^{\Delta}_{\Delta(C_G(x))}(k_\alpha \hat{x}),$$

where $k_\alpha \hat{x}$ is given as in the statement of the theorem, given by the map sending $\bar{g} \mapsto \hat{g} \otimes \hat{g} \otimes \hat{x} \in \Delta \otimes_{\Delta(C_G(x))} k_\alpha \hat{x}$. A further use of the Eckmann-Shapiro Lemma then shows that

$$\mathrm{Ext}^n_{\Delta}(k; \mathrm{Ind}^{\Delta}_{\Delta(C_G(x))}(k_\alpha \hat{x})) \cong \mathrm{Ext}^n_{\Delta(C_G(x))}(\mathrm{Res}^{\Delta}_{\Delta(C_G(x))}(k); k_\alpha \hat{x}) = H^n(C_G(x); k_\alpha \hat{x}),$$

with action of $C_G(x)$ on $k_\alpha \hat{x}$ given by "twisted conjugation":

$$g \cdot \hat{x} = \hat{g}\hat{x}\hat{g}^{-1} = \alpha(g, x)\alpha(gx, g^{-1})\alpha(g, g^{-1})^{-1}\alpha(1, 1)^{-1}\hat{x},$$

for all $g \in C_G(x)$. By the two-cocycle identity, utilising the fact that $g$ centralises $x$, one verifies that the coefficient of $\hat{x}$ in the action is indeed that given in the statement of the theorem: $\alpha(g, x)\alpha(x, g)^{-1}$. Since cohomology commutes with direct sums the proof is now complete: for

$$HH^n(A) \cong H^n(kG; A) \cong H^n(kG; \bigoplus_x k_\alpha \mathcal{C}_x) \cong \bigoplus_x H^n(kG; k_\alpha \mathcal{C}_x) \cong \bigoplus_x H^n(C_G(x); k_\alpha \hat{x}).$$

□

As the twisted centraliser decomposition is only a slight generalisation of the un-twisted version, a natural question is how does one relate to the other? In particular, one might wonder when the cohomology groups on the right hand side of (5.6) are simply groups of homomorphisms? Of course, this is the case when $\alpha$ represents the trivial class in $H^2(G; k^\times)$, but as it turns out we can go even further than this. We start in this direction with a seemingly innocuous definition.

**Definition 5.3.2.** Following well known terminology [34, 40], we say that an element $x \in G$ is *$\alpha$-regular* if $\alpha(g, x) = \alpha(x, g)$ for all $g \in C_G(x)$.

Note that this occurs precisely when the action of $C_G(x)$ on $k_\alpha \hat{x}$ in Theorem 5.3.1 is trivial, or equivalently when $H^1(C_G(x); k_\alpha \hat{x}) \cong \mathrm{Hom}(C_G(x), k)$. It is useful, therefore, to find situations when this occurs, giving weight to the need for its own definition. Our next results are all in this direction, relating the structure of $C_G(x)$ to the situation when $x$ is $\alpha$-regular, as well as explicitly finding situations where $\alpha$-regularity occurs.

For the remainder of this section we fix the following: $x$ is an element of $G$, and $\alpha \in Z^2(G; k^\times)$ is such that the cohomology class $[\alpha]$ corresponds to a central extension $1 \to Z \to \hat{G} \to G \to 1$.

We also denote by $C_{\hat{G}}(x) \leq \hat{G}$ the preimage of $C_G(x)$ under the surjection $\hat{G} \to G$. We will write $\lambda_\alpha : C_G(x) \to k^\times$ for the homomorphism inducing the action of $C_G(x)$ on $k_\alpha \hat{x}$, defined by $g \mapsto \alpha(g,x)\alpha(x,g)^{-1}$ for all $g \in C_G(x)$. Note that two preimages $\hat{g}, \hat{h} \in \hat{G}$ of an element $g \in G$ will differ by an element of $Z$, that $C_{\hat{G}}(\hat{x})$ is normal in $C_{\hat{G}}(x)$, and that we have a well-defined action of $C_{\hat{G}}(x)$ on $k_\alpha \hat{x}$ given by $\hat{g} \cdot \hat{x} = \lambda_\alpha(g)\hat{x}$ for all $\hat{g} \in C_{\hat{G}}(x)$, which is trivial on restriction to $C_{\hat{G}}(\hat{x})$. Let $N = \ker(\lambda_\alpha) = \{g \in C_G(x) \mid \alpha(g,x) = \alpha(x,g)\}$, and notice that $C_{\hat{G}}(\hat{x})$ consists of all elements $\hat{g} \in \hat{G}$ such that the image $g$ is in $C_G(x)$ and has the property $\alpha(g,x) = \alpha(x,g)$.

The following proposition gives us a good handle on how the extension parametrised by $\alpha$ affects the structure of $C_G(x)$ and the spaces $H^1(C_G(x); k_\alpha \hat{x})$ in the twisted centraliser decomposition, and we remark that this as well as the remaining results of this section are the author's own original work.

**Proposition 5.3.3.** *With the notation above, we have the following.*

*(i) There are isomorphisms of groups $C_{\hat{G}}(\hat{x})/Z \cong N$, and $C_{\hat{G}}(x)/Z \cong C_G(x)$.*

*(ii) There is an equality of subgroups $C_{\hat{G}}(\hat{x}) = C_{\hat{G}}(x)$ if and only if $x$ is $\alpha$-regular.*

*(iii) The group $H := C_{\hat{G}}(x)/C_{\hat{G}}(\hat{x})$ is isomorphic to a cyclic $p'$-subgroup of $k^\times$.*

*(iv) There are isomorphisms of $k$-vector spaces*

$$H^1(C_G(x); k_\alpha \hat{x}) \cong H^1(C_{\hat{G}}(x); k_\alpha \hat{x}) \cong H^1(C_{\hat{G}}(\hat{x}); k_\alpha \hat{x})^H \cong H^1(N; k_\alpha \hat{x})^H,$$

*where the final two terms are the fixed points under the action of $H$ induced by $C_{\hat{G}}(x)$ on $k_\alpha \hat{x}$.*

**Remark 5.3.4.** The action of $H$ on $H^1(C_{\hat{G}}(\hat{x}); k_\alpha \hat{x})$ is induced by the action of $C_{\hat{G}}(x)$ on $k_\alpha \hat{x}$: ${}^{\hat{g}}f : \hat{a} \mapsto \lambda_\alpha(g) f(\hat{g}^{-1}\hat{a}\hat{g})$ for all $\hat{g} \in C_{\hat{G}}(x)$, $\hat{a} \in C_{\hat{G}}(\hat{x})$ and $f \in H^1(C_{\hat{G}}(\hat{x}); k_\alpha \hat{x})$, with the action on $H^1(N; k_\alpha \hat{x})$ defined similarly. Note that

$$\mathrm{Hom}(C_{\hat{G}}(\hat{x}), k) \cong H^1(C_{\hat{G}}(\hat{x}); k_\alpha \hat{x}) \cong H^1(N; k_\alpha \hat{x}) \cong \mathrm{Hom}(N, k),$$

though we keep the notation as in the statement of the proposition because in each case $H$ acts possibly non-trivially.

For the remainder of this section we fix $H$ as defined in the statement of the proposition.

*Proof of Proposition 5.3.3.* On restriction to $C_{\hat{G}}(\hat{x})$ and $C_{\hat{G}}(x)$, the surjection $\hat{G} \to G$ with kernel $Z$ gives, respectively, the desired isomorphisms in (i), from which (ii) is immediate (alternatively note that the action of $C_{\hat{G}}(\hat{x})$ on $k_\alpha \hat{x}$ is trivial). The third isomorphism theorem proves (iii), noting that $H \cong C_G(x)/N$ injects into $k^\times$.

To ease notation, for the remainder of the proof we will write $k\hat{x} = k_\alpha \hat{x}$. To prove (iv), we use the fundamental exact sequence, Eq. (2.17), induced by the short exact sequence $1 \to C_{\hat{G}}(\hat{x}) \to C_{\hat{G}}(x) \to H \to 1$ to see that

$$0 \to H^1(H; (k\hat{x})^{C_{\hat{G}}(\hat{x})}) \to H^1(C_{\hat{G}}(x); k\hat{x}) \to H^1(C_{\hat{G}}(\hat{x}); k\hat{x})^H \to H^2(H; (k\hat{x})^{C_{\hat{G}}(\hat{x})}) \to \cdots . \quad (5.7)$$

As $H$ is cyclic of order coprime to the exponent of $k\hat{x}$ (viewed as an additive group) and $C_{\hat{G}}(\hat{x})$ acts trivially on $k\hat{x}$, one sees that the second and fifth terms in (5.7) vanish, showing that $H^1(C_{\hat{G}}(x); k\hat{x}) \cong H^1(C_{\hat{G}}(\hat{x}); k\hat{x})^H$. The fundamental exact sequence induced by $1 \to N \to C_G(x) \to H \to 1$, that is, on replacing the groups $C_{\hat{G}}(\hat{x})$ with $N$ and $C_{\hat{G}}(x)$ with $C_G(x)$ in (5.7), shows that $H^1(C_G(x); k\hat{x}) \cong H^1(N; k\hat{x})^H$. We make one further use of the fundamental exact sequence, this time induced by the short exact sequence $1 \to Z \to C_{\hat{G}}(x) \to C_G(x) \to 1$:

$$0 \to H^1(C_G(x); (k\hat{x})^Z) \to H^1(C_{\hat{G}}(x); k\hat{x}) \to H^1(Z; k\hat{x})^{C_G(x)} \to \cdots ,$$

which, since $Z$ is of $p'$-order and acts trivially on $k\hat{x}$, shows that $H^1(C_G(x); k\hat{x}) \cong H^1(C_{\hat{G}}(x); k\hat{x})$, completing the proof. $\square$

Arguably the most important result concerning what an $\alpha$-regular element might look like is the following.

**Lemma 5.3.5.** *Suppose that $x$ has order coprime to $|Z|$. Then $x$ is $\alpha$-regular, and the action of $C_G(x)$ on $k_\alpha\hat{x}$ in Theorem 5.3.1 is trivial. In particular, any $p$-element is $\alpha$-regular.*

*Proof.* The group $Z$ is isomorphic to a cyclic subgroup of $k^\times$ of order $m$, say, and suppose that the order of $x$ is $r$. Note that $m$ is coprime to $p$. Consider the subgroup of $\hat{G}$, $S := \langle \hat{x}, Z\rangle$. Then $S$ is an extension of $Z$ by a group of coprime index, $1 \to Z \to S \to \langle x\rangle \to 1$. By the Schur-Zassenhaus Lemma this extension splits, $S \cong \langle x\rangle \rtimes Z$ and in fact equals the trivial extension, since $S$ is abelian. Whence we may choose a lift $\hat{x} \in \hat{G}$ of the same order as $x$.

Now let $g \in C_G(x)$, so that $\hat{g}\hat{x}\hat{g}^{-1} = z\hat{x}$ for some $z \in Z$. With $m = |Z|$ one sees that $\hat{g}\hat{x}^m\hat{g}^{-1} = \hat{x}^m$, and $\hat{g} \in C_{\hat{G}}(\hat{x}^m)$. On the other hand, $m$ is coprime to the order of $x$, so that $\langle \hat{x}^m\rangle = \langle \hat{x}\rangle$. Since $\hat{g} \in C_{\hat{G}}(\hat{x}^m)$, $\hat{g}$ commutes with all $y \in \langle \hat{x}^m\rangle = \langle \hat{x}\rangle$, whence $C_{\hat{G}}(\hat{x}^m) = C_{\hat{G}}(\hat{x})$. We have that $\hat{g}\hat{x} = \hat{x}\hat{g}$ and the result follows. $\square$

Thus we have that if $x$ is a $p$-element, then $H^1(C_G(x); k_\alpha\hat{x}) \cong \mathrm{Hom}(C_G(x), k)$ in the twisted centraliser decomposition, and the structure of $\mathrm{Hom}(C_G(x), k)$ is certainly much easier to understand. As we have mentioned, Fleischmann, Janiszczak and Lempken [28] have shown that there are certain families of groups that have $p$-elements such that $\mathrm{Hom}(C_G(x), k) \neq \{0\}$: the proposition above therefore suggests that we are very much on our way to showing our non-vanishing result of Corollary 5.1.2.

**Lemma 5.3.6.** *Let $x$ be such that $C_G(x)$ is at least one of the following: a cyclic group, a $p$-group or a group whose Sylow subgroups are all cyclic. Then $x$ is $\alpha$-regular, and in particular the action of $C_G(x)$ on $k_\alpha\hat{x}$ in Proposition 2.3.16 is trivial.*

*Proof.* In general, if $K$ is a group that is either cyclic, a $p$-group or a group whose Sylow subgroups are all cyclic then it is a classical result of group cohomology that $H^2(K; k^\times) = \{1\}$ [39, Corollary 9.4.4]. Consequently if $C_G(x)$ is one such group, then in each case the restriction of $\alpha$ to $C_G(x)$ is a coboundary: $\alpha(g, h) = 1$ for all $g, h \in C_G(x)$. Recalling that the action of $C_G(x)$ on $k_\alpha\hat{x}$ is given by $g \cdot \hat{x} = \alpha(g, x)\alpha(x, g)^{-1}\hat{x}$, the result follows. $\square$

**Proposition 5.3.7.** *Suppose $x$ is such that its centraliser $C_G(x)$ is abelian of order divisible by $p$. Then $H^1(C_G(x); k_\alpha\hat{x}) \neq \{0\}$ if and only if $x$ is $\alpha$-regular.*

*Proof.* Note that if $C_G(x)$ is abelian of order divisible by $p$, then $\mathrm{Hom}(C_G(x), k) \neq \{0\}$. In one direction the proof is now clear: if the action of $C_G(x)$ is trivial, then $H^1(C_G(x); k_\alpha\hat{x}) \cong \mathrm{Hom}(C_G(x), k)$.

In the other direction, suppose that $H^1(C_G(x); k_\alpha\hat{x}) \neq \{0\}$, then via the isomorphisms of Proposition 5.3.3(iv) we have $H^1(N; k_\alpha\hat{x})^H \neq \{0\}$. Choose a non-zero 1-cocycle $f : N \to k_\alpha\hat{x}$. As $C_G(x)$ is abelian, we have that $f$ is fixed by the action of $H$ if and only if $\lambda_\alpha(g)f(a) = f(a)$ for all $g \in C_G(x)$ and $a \in N$. As $f$ is non-zero one sees that this can occur if and only if the action of $C_G(x)$ is trivial. This completes the proof. $\square$

**Remark 5.3.8.** From this proof it is easy to see that more generally, for $C_G(x)$ not necessarily abelian it is still the case that for all $f \in Z^1(C_G(x); k_\alpha\hat{x})$, $f \equiv 0$ on $Z(C_G(x))$.

## 5.4 Example. (The Lie algebra structure of $HH^1\left(k_\alpha(C_3^2 \rtimes C_2^2)\right)$)

In this section we will complete our case study, concluding the work that we have seen in Examples 2.1.14, 2.1.19, 2.3.22 and 2.4. So far, in those examples, we have seen that when $\mathrm{char}(k) = 3$, the group $G = (C_3 \times C_3) \rtimes Q_8$ has two blocks, $B_0$ and $B_1$, each with a defect group $C_3 \times C_3$, and that the principal block $B_0$ is isomorphic to the group algebra over the group $H = (C_3 \times C_3) \rtimes (C_2 \times C_2)$. In addition, we have have found the dimensions $\dim_k(HH^1(kG)) = 14$, $\dim_k(HH^1(B_0)) = 6$ and $\dim_k(HH^1(B_1)) = 8$, and we have determined a Lie algebra basis for $HH^1(B_0)$ as well as some of its structural Lie algebra properties. We will finish by determining how those properties differ for $HH^1(B_1)$. As a summary of our results, we state the following theorem: statements (i), (ii) and (iii) have already been shown, except for the solvability of $HH^1(B_1)$ in (ii). In this section we will show statement (iv) from which this solvability is an immediate result.

**Theorem 5.4.1.** *Let $\mathrm{char}(k) = 3$, $E = C_2 \times C_2$, $L = Q_8$ and $Z = Z(L) \cong C_2$. Let $P = C_3 \times C_3$ and consider a faithful action of $E$ on $P$, where the first copy of $C_2$ in $E$ inverts a generator of the first copy of $C_3$ in $P$ and commutes with the second copy of $C_3$, and where the second copy of $C_2$ in $E$ inverts a generator of the first copy of $C_3$ and commutes with the second copy of $C_3$. Let $G = P \rtimes L$ in the central extension $1 \to Z \to G \to P \rtimes E \to 1$. Then we have the following.*

*(i) The group algebra $kG$ has 2 blocks, $kG = B_0 \times B_1$, and each block has a defect group equal to $P$.*

*(ii) The Lie algebras $HH^1(kG)$, $HH^1(B_0)$ and $HH^1(B_1)$ have dimensions 14, 6 and 8 respectively and are solvable.*

*(iii) The Lie algebra $HH^1(B_0)$ has a 4-dimensional derived Lie subalgebra, and has derived length and 3-toral rank both equal to 2.*

*(iv) The Lie algebra $HH^1(B_1)$ has a 6-dimensional derived Lie subalgebra, and has derived length equal to three and 3-toral rank both equal to two. In addition, there is a maximal 3-toral Lie subalgebra of $HH^1(B_1)$ which is a complement to the derived Lie subalgebra of $HH^1(B_1)$.*

To obtain a Lie algebra basis for $HH^1(B_1)$ we construct derivations from the twisted centraliser decomposition, employing a method similar to that of Example 2.4. We do so by using the following result; this result is another example of a situation where we can generalise from the un-twisted group algebra case. In particular, we are generalising Proposition 2.3.30, which will allow us to find class representatives of derivations in the Hochschild cohomology of a twisted group algebra.

**Proposition 5.4.2.** *Let $\{g_1 = 1, g_2, \ldots, g_\ell\}$ be a complete set of conjugacy class representatives of $G$, let $G_i = C_G(g_i)$ for $i = 1, \ldots, \ell$. Let $f \in Z^1(G_i; k_\alpha \hat{g}_i)$ for some $i$. Then there is a derivation $d = d_f \in \mathrm{Der}(k_\alpha G)$, defined, for all $g \in G$ by*

$$d(\hat{g}) = \mathrm{cor}_{G_i}^G(f)(g)\hat{g},$$

*and extended linearly to $k_\alpha G$.*

*Proof.* The details of this proof are just a slight generalisation of the proof of Proposition 2.3.30 so we do not go into as much detail as there. As in Theorem 5.3.1 we have that $HH^1(k_\alpha G) = \bigoplus_{i=1}^{\ell} H^1(G_i, k_\alpha \hat{g}_i)$. Let $f$ be such that $[f] \in H^1(G_i, k_\alpha \hat{g}_i)$, then by the inclusion of $k_\alpha \hat{g}_i \subseteq k_\alpha G$ we may view $[f]$ as an element of $H^1(G_i; k_\alpha G)$. In particular $[\mathrm{cor}_{G_i}^G(f)] \in H^1(G; k_\alpha G) \cong HH^1(k_\alpha G)$. The map sending $[f]$ to $[\mathrm{cor}_{G_i}^G(f)]$ is easily seen to be a $k$-module homomorphism. This has its inverse given by the map sending $[d] \in H^1(G; k_\alpha G)$ to $[\hat{\pi}_x(\mathrm{res}_{C_G(x)}^G(d))] \in H^1(C_G(x); k_\alpha \hat{x})$ for some $x \in G$, where $\hat{\pi}_x$ is the map $Z^1(C_G(x); k_\alpha G) \to Z^1(C_G(x); k_\alpha \hat{x})$ induced by

$$\pi_x : k_\alpha G \to k_\alpha \hat{x}, \quad \pi_x(\sum_{g \in G} \lambda_g \hat{g}) = \lambda_x \hat{x}.$$

One checks that for arbitrary $\gamma \in Z^1(G; k_\alpha G)$ there is a derivation $d_\gamma \in \mathrm{Der}(k_\alpha G)$, given by $d_\gamma(\hat{g}) = \gamma(g)\hat{g}$, and that the map sending $\gamma$ to $d_\gamma$ has inverse sending a derivation $D \in \mathrm{Der}(k_\alpha G)$ to the 1-cocycle in $Z^1(G; k_\alpha G)$ defined by $g \mapsto D(\hat{g})\hat{g}^{-1}$ for all $g \in G$. A simple verification shows that under these maps, $B^1(G; k_\alpha G)$ is sent to $\mathrm{IDer}(k_\alpha G)$ and vice versa. □

As we will see, when calculating the Lie bracket on a twisted group algebra $HH^1(k_\alpha G)$ we often find that inner derivations arise as a result. Of course, this means that the particular commutator in question evaluates to zero in $HH^1(k_\alpha G)$, however it would be remiss of us not to consider and check the inner derivations as well. Given an un-twisted group algebra $kG$, calculating inner derivations is simple, and if for some $x \in G$, $f_x$ is the inner derivation $f_x(g) = gx - xg$ for all $g \in G$, then it is of course clear that $f_x(g) = 0$ for all $g \in C_G(x)$. The twisted counterpart to this result is not quite as general, and perhaps unsurprisingly involves $\alpha$-regular elements.

**Lemma 5.4.3.** *Let $\alpha \in Z^2(G; k^\times)$, let $x \in G$ and consider that inner derivation $f_x : k_\alpha G \to k_\alpha G$, $f_x(\hat{g}) = \hat{g}\hat{x} - \hat{x}\hat{g}$. If $x$ is $\alpha$-regular then $f_x(\hat{g}) = 0$ for all $g \in C_G(x)$.*

*Proof.* We evaluate that

$$f_x(\hat{g}) = \hat{g}\hat{x} - \hat{x}\hat{g} = \alpha(g,x)\widehat{gx} - \hat{x}\hat{g} = \alpha(g,x)\widehat{xg} - \hat{x}\hat{g} = \alpha(g,x)\alpha(x,g)^{-1}\hat{x}\hat{g} - \hat{x}\hat{g} = 0,$$

where the final equality holds as $x$ is $\alpha$-regular. □

Let us return to the setting of Theorem 5.4.1 with all notation carried over from there: we fix this notation for the remainder of this section. Our aim is to investigate the Lie algebra structure of $HH^1(B_1)$. As noted in the previous examples, $B_1$ arises as a twisted group algebra over the group $H = P \rtimes E$, $k_\alpha H$ for some $\alpha \in Z^2(H; k^\times)$: let us begin by discussing this. First observe that this follows directly from Theorem 5.2.7. Applied to our situation it is immediate that $kG \cong kH \times k_\alpha H$, and as we know that $B_0 \cong kH$ that forces $B_1 \cong k_\alpha H$.

We can be more explicit than this: we will make use of Proposition 5.2.6 to find a representative $\alpha$ of the class in $H^2(H; k^\times)$ that defines the multiplication in the twisted group algebra $k_\alpha H$, though to do so we need to fix some more notation. First recall that $E = C_2 \times C_2$ has generators $r$ and $s$, $P = C_3 \times C_3$ is generated by $a$ and $b$, and there is a faithful conjugation action of $E$ on $P$ that we write as $r \cdot a = a^2$, $s \cdot b = b^2$, $[a, s] = [b, r] = 1$.

Fix a presentation of $L$,

$$L = \langle u, v \mid u^4 = 1, u^2 = v^2, vuv^{-1} = u^{-1} \rangle \cong Q_8.$$

With this setup, there is a canonical surjection $\pi : L \to E$ sending $u, v$ to $r, s$ respectively, with kernel $Z = \langle u^2 \rangle$. As in Proposition 5.2.6 we make a choice of preimages of $E$ in $L$: we let $\widehat{1_E} = 1_L$, $\hat{r} = u$, $\hat{s} = v$ and $\widehat{rs} = uv$. We use this to construct the 2-cocycle $\beta \in Z^2(E; Z)$ defined by $\hat{x}\hat{y} = \beta(x, y)\widehat{xy}$ for all $x, y \in E$. Note that our choice $\widehat{1_E} = 1_L$ explicitly means that $\beta$ is normalised.

This gives the following table, where the first column represents the first argument of $\beta$ and the first row represents the second argument of $\beta$.

Table 5.1: The 2-cocycle $\beta \in Z^2(E; Z)$

| $\beta(-,-)$ | $1$ | $r$ | $s$ | $rs$ |
|---|---|---|---|---|
| $1$ | $1$ | $1$ | $1$ | $1$ |
| $r$ | $1$ | $u^2$ | $1$ | $u^2$ |
| $s$ | $1$ | $u^2$ | $u^2$ | $1$ |
| $rs$ | $1$ | $1$ | $u^2$ | $u^2$ |

Continuing with the construction provided by Proposition 5.2.6, the two irreducible characters of $Z$, namely the trivial character $\chi_0$ and the sign character $\chi_1$ which sends $u^2 \mapsto -1$ in $k^\times$, may be composed with $\beta$ to give a 2-cocycle $\chi_i \circ \beta \in Z^2(E; k^\times)$ $i = 0, 1$. Evidently $\chi_0 \circ \beta$ represents the trivial class, and we let $\alpha = \chi_1 \circ \beta$. The blocks of $kL$ are in bijective correspondence with the characters of $Z$, and so we have two block idempotents of $kL$ given by

$$b_0 = \frac{1}{|Z|}(\chi_0(1)1 + \chi_0(u^2)u^2) = \frac{1}{2}(1 + u^2),$$
$$b_1 = \frac{1}{|Z|}(\chi_1(1)1 + \chi_1(u^2)u^2) = \frac{1}{2}(1 - u^2),$$

and we have that $kLb_0 \cong kE$ and $kLb_1 \cong k_\alpha E$. This second algebra isomorphism is made explicit when one considers that as a set,

$$L \cdot b_1 = \{b_1, ub_1, vb_1, uvb_1, -b_1, -ub_1, -vb_1, -uvb_1\},$$

so that $\{b_1, ub_1, vb_1, uvb_1\}$ is a $k$-basis for $kLb_1$. Now compare Table 5.2 below (which follows immediately from Table 5.1 above and that $\alpha = \chi_1 \circ \beta$) to Table 5.3 below that: it is clear that $kLb_1 \cong k_\alpha E$.

Table 5.2: Multiplication table for $k_\alpha E$

| $\cdot$ | $1$ | $r$ | $s$ | $rs$ |
|---|---|---|---|---|
| $1$ | $1$ | $r$ | $s$ | $rs$ |
| $r$ | $r$ | $-1$ | $rs$ | $-s$ |
| $s$ | $s$ | $-rs$ | $-1$ | $r$ |
| $rs$ | $rs$ | $s$ | $-r$ | $-1$ |

Table 5.3: Multiplication table for $kLb_1$

| $\cdot$ | $b_1$ | $ub_1$ | $vb_1$ | $uvb_1$ |
|---|---|---|---|---|
| $b_1$ | $b_1$ | $ub_1$ | $vb_1$ | $uvb_1$ |
| $ub_1$ | $ub_1$ | $-b_1$ | $uvb_1$ | $-vb_1$ |
| $vb_1$ | $vb_1$ | $-uvb_1$ | $-b_1$ | $ub_1$ |
| $uvb_1$ | $uvb_1$ | $vb_1$ | $-ub_1$ | $-b_1$ |

The situation may now be inflated to $G = P \rtimes L$ and $H = P \rtimes E$ noting that $Z$ acts trivially on $P$, and we see that as before, writing $B_i = kGb_i$ for $i = 0, 1$, we have

$$kG = B_0 \times B_1 \cong kH \times k_\alpha H.$$

Before we go any further, we remark that the multiplication table for $k_\alpha H$ follows immediately from Table 5.2: the only products in $k_\alpha H$ that give rise to a non-trivial twist by $\alpha$ are induced from the multiplication in $k_\alpha E$. To be precise, we have the multiplication table for $k_\alpha H$ as given in Table 5.4: there, $x$ and $y$ vary over all elements of $P$, so that each entry is in fact a $9 \times 9$ block of entries, and for ease of notation we denote the conjugation action of $E$ on $P$ by $g \cdot h$ for all $g \in E$ and $h \in P$. On comparing Table 5.4 with Table 5.2 one sees that the multiplication twists by $-1$ follow directly from $k_\alpha E$ as expected.

Table 5.4: Multiplication table for $k_\alpha H$

| $\cdot$ | $y$ | $yr$ | $ys$ | $yrs$ |
|---|---|---|---|---|
| $x$ | $xy$ | $xyr$ | $xys$ | $xyrs$ |
| $xr$ | $x(r \cdot y)r$ | $-x(r \cdot y)$ | $x(r \cdot y)rs$ | $-x(r \cdot y)s$ |
| $xs$ | $x(s \cdot y)s$ | $-x(s \cdot y)rs$ | $-x(s \cdot y)$ | $x(s \cdot y)r$ |
| $xrs$ | $x(rs \cdot y)rs$ | $x(rs \cdot y)s$ | $-x(rs \cdot y)r$ | $-x(rs \cdot y)$ |

With a good description of $B_1$ as a twisted group algebra, we will now apply Theorem 5.3.1 to verify what we have already seen via a counting argument, that the dimension $\dim_k(HH^1(B_1)) = 8$. To do this, we will need to construct some explicit 1-cocycles and 1-coboundaries in (some of) the first cohomology spaces $H^1(C_H(x); k_\alpha \hat{x})$.

We summarise the details of the basis for $HH^1(k_\alpha H)$ constructed above, in Table 5.5 below. There, the entries on the first row are the basis elements constructed, and the first column a basis of $k_\alpha H$. The entries within the table denote the element of $k_\alpha H$ to which the element in the first column is sent by the derivation in the first row, and a "." denotes 0. Note the similarities with Table 2.1, as well as the subtle differences, and of course, the addition of two more derivations.

Table 5.5: Class representatives of a basis for $HH^1(k_\alpha H)$ as a Lie algebra

| | $d_a$ | $d_b$ | $d_{as}$ | $d_{br}$ | $d_{ab,1}$ | $d_{ab,2}$ | $d_r$ | $d_s$ |
|---|---|---|---|---|---|---|---|---|
| $\hat{a}$ | $\widehat{a^2}+2$ | . | $(\widehat{a^2}+1)\left(\sum\limits_{i=0}^{2}\widehat{b^i}\right)\hat{s}$ | . | $(\widehat{a^2}+2)(\hat{b}+\widehat{b^2})$ | . | . | $\left(\sum\limits_{i=0}^{2}\widehat{b^i}\right)\widehat{as}$ |
| $\hat{b}$ | . | $\widehat{b^2}+2$ | . | $\left(\sum\limits_{i=0}^{2}\widehat{a^i}\right)(\widehat{b^2}+1)\hat{r}$ | . | $(\hat{a}+\widehat{a^2})(\widehat{b^2}+2)$ | $\left(\sum\limits_{i=0}^{2}\widehat{a^i}\right)\widehat{br}$ | . |
| $\hat{r}$ | . | . | . | . | . | . | . | . |
| $\hat{s}$ | . | . | . | . | . | . | . | . |

We will proceed to detail the construction of this basis. Begin by recalling that

$$
\begin{aligned}
HH^1(kH) &\cong \bigoplus_x \operatorname{Hom}(C_H(x), k), \\
&= \operatorname{Hom}(H,k) \oplus \operatorname{Hom}(C_3 \times S_3, k)^{\oplus 2} \oplus \operatorname{Hom}(D_{12}, k)^{\oplus 2} \oplus \operatorname{Hom}(C_3 \times C_3, k) \\
&\qquad \oplus \operatorname{Hom}(C_6, k)^{\oplus 2} \oplus \operatorname{Hom}(C_2 \times C_2, k),
\end{aligned}
$$

where $C_H(a) = \langle a, b, s\rangle \cong C_3 \times S_3$, $C_H(b) = \langle a, b, r\rangle \cong C_3 \times S_3$, $C_H(ab) = \langle a, b\rangle \cong C_3 \times C_3$, $C_H(as) = \langle a, s\rangle \cong C_6$ and $C_H(br) = \langle b, r\rangle \cong C_6$. In the twisted setting, however, we will also see contributions from the centralisers $C_H(r) = \langle br, s\rangle \cong D_{12}$ and $C_H(s) = \langle as, r\rangle \cong D_{12}$, and must still consider $C_H(rs) = \langle r, s\rangle \cong C_2 \times C_2$. Thus, we have a twisted centraliser decomposition given by

$$
\begin{aligned}
HH^1(k_\alpha H) &\cong \bigoplus_x H^1(C_H(x); k_\alpha \hat{x}), \\
&\cong H^1(H; k_\alpha \hat{1}) \oplus H^1(C_H(a); k_\alpha \hat{a}) \oplus H^1(C_H(b); k_\alpha \hat{b}) \\
&\qquad \oplus H^1(C_H(r); k_\alpha \hat{r}) \oplus H^1(C_H(s); k_\alpha \hat{s}) \oplus H^1(C_H(ab); k_\alpha \widehat{ab}) \\
&\qquad\quad \oplus H^1(C_H(as); k_\alpha \widehat{as}) \oplus H^1(C_H(br); k_\alpha \widehat{br}) \oplus H^1(C_H(rs); k_\alpha \widehat{rs}).
\end{aligned}
$$

Now, $\alpha(g,1) = \alpha(1,1) = \alpha(1,g)$ for all $g \in H$, so that $H^1(H; k_\alpha \hat{1}) \cong \operatorname{Hom}(H,k)$ which we have already seen is $\{0\}$. We also have that $\alpha$ is trivial on $P$ so that $H^1(C_H(ab); k_\alpha \widehat{ab}) \cong \operatorname{Hom}(C_3 \times C_3, k)$ (alternatively this follows from Lemma 5.3.5), and we have already seen in Example 2.3.22 that this has dimension 2. It also follows from Lemma 5.3.5 that $a$ and $b$ are $\alpha$-regular, so that

$H^1(C_H(a); k_\alpha \hat{a}) \cong H^1(C_H(b); k_\alpha \hat{b}) \cong \mathrm{Hom}(C_3 \times S_3, k)$, each contributing 1 to the total dimension of $HH^1(k_\alpha H)$. Using Lemma 5.3.6, in particular noting that $C_H(as)$ and $C_H(br)$ are both cyclic, we have that $as$ and $br$ are $\alpha$-regular and $H^1(C_H(as); k_\alpha \widehat{as}) \cong H^1(C_H(br); k_\alpha \widehat{br}) \cong \mathrm{Hom}(C_6, k)$, each contributing 1 to the total dimension of $HH^1(k_\alpha H)$. To summarise so far, we have seen that

$$\begin{aligned} HH^1(k_\alpha H) \cong{} & \mathrm{Hom}(C_3 \times S_3, k)^{\oplus 2} \oplus \mathrm{Hom}(C_3 \times C_3, k) \oplus \mathrm{Hom}(C_6, k)^{\oplus 2} \\ & \oplus H^1(C_H(r); k_\alpha \hat{r}) \oplus H^1(C_H(s); k_\alpha \hat{s}) \oplus H^1(C_H(rs); k_\alpha \widehat{rs}), \end{aligned}$$

has dimension at least 6, and contains a copy of $HH^1(kH) \cong \mathrm{Hom}(C_3 \times S_3, k)^{\oplus 2} \oplus \mathrm{Hom}(C_3 \times C_3, k) \oplus \mathrm{Hom}(C_6, k)^{\oplus 2}$.

We now turn to the final 3 cohomology groups, for which the action of the centraliser is non-trivial: this is seen immediately from Table 5.1 (after identifying $u^2$ with $-1$). The action of $C_H(r) = \langle br, s \rangle$ on $k_\alpha \hat{r}$ is given by $br \cdot \hat{r} = \hat{r}$ and $s \cdot \hat{r} = -\hat{r}$. The latter action comes directly from Table 5.1. The former action is seen by noting first that $r \cdot \hat{r} = \hat{r}$ in Table 5.1 (or, of course, noting that $r \cdot \hat{r} = \alpha(r,r)\alpha(r,r)^{-1}\hat{r} = \hat{r}$), so that $br \cdot \hat{r} = b \cdot \hat{r}$, and then observing that we already know that $\alpha(b, r) = \alpha(r, b)$ since $b$ is $\alpha$-regular: $r \in C_H(b)$ acts trivially on $k_\alpha \hat{b}$.

Now let $f \in Z^1(C_H(r); k_\alpha \hat{r})$, and note that $Z(C_H(r)) = \langle r \rangle$, so that by Remark 5.3.8, we have the useful property $f(r) = 0$. One also notes that $f(1) = f(s^2) = f(s) + s \cdot f(s) = f(s) - f(s) = 0$. We will now proceed to use the fundamental 1-cocycle property that $f(gh) = f(g) + g \cdot f(h)$ for all $g, h \in C_H(r)$, so that in particular $f(gr) = f(g)$ for all $g \in C_H(r)$.

Suppose that $f(s) = \lambda \hat{r}$ and $f(b) = \mu \hat{r}$ for some $\lambda, \mu \in k$. It is now easily verified that $f$ is completely defined on $C_H(r) = \{1, r, s, rs, b, br, bs, brs, b^2, b^2r, b^2s, b^2rs\}$ by

$$\begin{aligned} 1, r &\mapsto 0, \\ s, rs &\mapsto \lambda \hat{r}, \\ b, br &\mapsto \mu \hat{r}, \\ b^2, b^2r &\mapsto 2\mu \hat{r}, \\ bs, brs &\mapsto (\lambda + \mu)\hat{r}, \\ b^2s, b^2rs &\mapsto (\lambda + 2\mu)\hat{r}. \end{aligned}$$

Setting each of $\lambda$ to 1 and $\mu$ to 0, then vice versa, gives two linearly independent 1-cocycles, which we label $f_1$ and $f_2$ respectively: these form a $k$-basis for $Z^1(C_H(r); k_\alpha \hat{r})$ as a $k$-vector space. Whence, $f_1$ sends $1, r, b, br, b^2, b^2r \mapsto 0$ and $s, rs, bs, brs, b^2s, b^2rs \mapsto \hat{r}$, whilst $f_2$ sends $1, r, s, rs \mapsto 0$, sends $b, br, bs, brs \mapsto \hat{r}$ and sends $b^2, b^2r, b^2s, b^2rs \mapsto 2\hat{r}$.

It is straightforward to verify that $f_1(g) = (g - 1) \cdot \hat{r}$ for all $g \in C_H(r)$. On the other hand, suppose there is some $\eta \in k$ such that $f_2(g) = \eta(g - 1) \cdot \hat{r}$ for all $g \in C_H(r)$. Then $f_2(b) = \eta(b - 1) \cdot \hat{r} = \eta(b \cdot \hat{r} - \hat{r}) = 0$, a contradiction as $f_2(b) = \hat{r}$. Whence $f_1 \in B^2(C_H(r); k_\alpha \hat{r})$, $f_2 \notin B^2(C_H(r); k_\alpha \hat{r})$, and we see that $H^1(C_H(r); k_\alpha \hat{r})$ is 1-dimensional with a basis given by the class with one choice of representative as $f_2$.

An argument almost verbatim to the one just given, shows that $\dim_k(H^1(C_H(s); k_\alpha \hat{s})) = 1$. By relabeling $b$ with $a$ and swapping the roles of $r$ and $s$, the argument runs through in the same manner - this is due to the symmetry of the action of $E$ on $P$.

It remains to deal with $H^1(C_H(rs); k_\alpha \widehat{rs})$, though this is relatively simple to sort out. First observe (as in the previous two cases) that by Table 5.1 the action of $C_H(rs)$ is non-trivial on

$k_\alpha\widehat{rs}$. Next, since $C_H(rs) = E$, we have $Z(C_H(rs)) = C_H(rs)$. Finally, by Remark 5.3.8 (or indeed by the latter half of the proof of Proposition 5.3.7) any cocycle $f \in H^1(C_H(rs); k_\alpha\widehat{rs})$ must be identically equal to 0 on $C_H(rs)$ - in other words, $H^1(C_H(rs); k_\alpha\widehat{rs}) = \{0\}$.

Whence we have a $k$-vector space isomorphism

$$\begin{aligned} HH^1(k_\alpha H) \cong \operatorname{Hom}(C_3 \times S_3, k)^{\oplus 2} \oplus \operatorname{Hom}(C_3 \times C_3, k) \oplus \operatorname{Hom}(C_6, k)^{\oplus 2} \\ \oplus H^1(C_H(r); k_\alpha\hat{r}) \oplus H^1(C_H(s); k_\alpha\hat{s}), \end{aligned} \tag{5.8}$$

and the dimension of $HH^1(B_1)$ is, as expected

$$\begin{aligned} \dim_k(HH^1(B_1)) &= \dim_k(HH^1(k_\alpha H)) \\ &= \sum_x \dim_k(H^1(C_H(x), k_\alpha\hat{x})) \\ &= 2 \times 1 + 1 \times 2 + 2 \times 1 + 1 + 1 \\ &= 8. \end{aligned}$$

What is more, there is a copy of $HH^1(kH)$ sitting inside $HH^1(k_\alpha H)$ (as vector spaces),

$$HH^1(k_\alpha H) \cong HH^1(kH) \oplus H^1(C_H(r); k_\alpha\hat{r}) \oplus H^1(C_H(s); k_\alpha\hat{s}), \tag{5.9}$$

which will speed up our next and ultimate goal: to find a Lie algebra basis of $HH^1(k_\alpha H)$. We already have a Lie algebra basis for $HH^1(kH)$, and whilst it does not immediately transfer to a Lie algebra basis of $HH^1(k_\alpha H)$, it certainly helps - as we will see.

To find a Lie algebra basis for $HH^1(k_\alpha H)$ we will make use of the twisted centraliser decomposition of $HH^1(k_\alpha H)$, Equation (5.8) above, and Proposition 5.4.2. This proposition tells us that, starting with a $k$-vector space basis of the cohomology groups on the right hand side of (5.8), these can be induced to a derivation in $HH^1(k_\alpha H)$.

To that end, we begin with $H^1(C_H(r); k_\alpha\hat{r})$. As we have seen, this has a 1-dimensional basis with a class representative that we will relabel now as $f_r$, which sends $1, r, s, rs \mapsto 0$, sends $b, br, bs, brs \mapsto \hat{r}$ and sends $b^2, b^2r, b^2s, b^2rs \mapsto 2\hat{r}$. As in the proposition, via the inclusion $k_\alpha\hat{r} \subset k_\alpha H$, $f_r$ may be viewed as an element of $H^1(C_H(r); k_\alpha H)$. Now we corestrict $f_r$ to an element of $H^1(H; k_\alpha H)$, recalling that the action of $H$ on $k_\alpha H$ is given by *twisted conjugation*, that is, for all $g \in H$ and $\hat{x} \in k_\alpha H$,

$$g \cdot \hat{x} = \hat{g}\hat{x}\hat{g}^{-1} = \alpha(g, x)\alpha(gx, g^{-1})\alpha(g, g^{-1})^{-1}\alpha(1, 1)^{-1}\hat{y},$$

where $y = gxg^{-1}$. As we have already seen, $\alpha(1, 1) = 1$, and since $\alpha(x, y) = \pm 1$ for all $x, y \in H$ this simplifies to

$$g \cdot \hat{x} = \alpha(g, x)\alpha(gx, g^{-1})\alpha(g, g^{-1})\hat{y} \tag{5.10}$$

where $y = gxg^{-1}$ as before.

A set of coset representatives of $H_r := C_H(r)$ in $H$ is given by $\{1, a, a^2\}$. Since $f_r(r) = f_r(s) = 0$ it is simple to verify that $\operatorname{cor}_{H_r}^H(f_r)(r) = \operatorname{cor}_{H_r}^H(f_r)(s) = 0$. We also have that $\operatorname{cor}_{H_r}^H(f_r)(a) = f_r(1) + a \cdot f_r(1) + a^2 \cdot f_r(1) = 0$. On the other hand,

$$\operatorname{cor}_{H_r}^H(f_r)(b) = \hat{r} + a \cdot \hat{r} + a^2 \cdot \hat{r}. \tag{5.11}$$

Thus we need to know how $a$ acts on $\hat{r}$, and in particular, by Equation (5.10), we need to find $\alpha(a,r), \alpha(ar,a^2)$ and $\alpha(a,a^2)$ for the action of $a$ on $\hat{r}$, and $\alpha(a^2,r), \alpha(a^2r,a)$ and $\alpha(a^2,a)$ for the action of $a^2$ on $\hat{r}$ (though one, of course, follows from the other). For this we can quickly use as look-up table the multiplication table for $k_\alpha H$ given in Table 5.4. On the other hand, it can be productive to be careful, precise, specific and explicit: we do so here.

We first extend our choice of preimages of $E$ in $L$ given there, to the preimages of $H$ in $G$. Recall that

$$H = \langle a,b,r,s \mid a^3 = b^3 = r^2 = s^2 = 1, [a,b] = [r,s] = [a,s] = [b,r] = 1, rar = a^2, sbs = b^2\rangle,$$

and we also fix a presentation of $G$ as

$$G = \langle c,d,u,v \mid c^3 = d^3 = u^4 = 1, u^2 = v^2, [c,d] = [c,v] = [d,u] = 1, [u,c] = c, [v,d] = d, [v,u] = u^2\rangle,$$

so that in the central extension $1 \to Z \to G \to H \to 1$ the map $\pi : G \to H$ with kernel $Z = \langle u^2\rangle$ is defined by $\pi(c) = a, \pi(d) = b, \pi(u) = r$ and $\pi(v) = s$. Now, for $i,j = 0,1,2$ and $m,n = 0,1$ let $w = a^ib^jr^ms^n$ be an arbitrary element of $H$. Then we fix the preimage of $w$ in $G$ as $\widehat{w} = c^id^ju^mv^n \in G$. With this we extend the construction of the 2-cocycle $\beta \in Z^2(E;Z)$ seen in Table 5.1 to a 2-cocycle in $Z^2(H;Z)$, which we abusively denote by $\beta$. Recall that $\beta$ is defined by $\hat{x}\hat{y} = \beta(x,y)\widehat{xy}$ for all $x,y \in H$, and that $\alpha(x,y) = 1$ if $\beta(x,y) = 1$, and $\alpha(x,y) = -1$ if $\beta(x,y) = u^2$.

With this setup, it is now simple to check that $\alpha(a,r) = \alpha(ar,a^2) = \alpha(a^2,r) = \alpha(a^2r,a) = 1$, whilst we have already seen that $\alpha(a,a^2) = \alpha(a^2,a) = 1$ since $\alpha$ is trivial on $P$. Thus, Equation (5.11) becomes

$$\begin{aligned}\operatorname{cor}_{H_r}^H(f_r)(b) &= \hat{r} + \widehat{ara^2} + \widehat{a^2ra},\\ &= \hat{r} + \widehat{a^2r} + \widehat{ar}.\end{aligned}$$

By Proposition 5.4.2, we therefore have our first derivation in $HH^1(k_\alpha H)$, given by $d_r(\hat{a}) = d_r(\hat{r}) = d_r(\hat{s}) = 0$, and

$$\begin{aligned}d_r(\hat{b}) &= \operatorname{cor}_{H_r}^H(f_r)(b)\hat{b}\\ &= \hat{r}\hat{b} + \widehat{a^2r}\hat{b} + \widehat{ar}\hat{b},\\ &= \widehat{br} + \widehat{a^2br} + \widehat{abr},\end{aligned}$$

where the third equality is easily seen when one checks that, by the construction above, $\alpha(r,b) = \alpha(ar,b) = \alpha(a^2r,b) = 1$.

Now we turn our attention to $H_s := C_H(s)$. The isomorphism of cohomology groups

$$H^1(H_r; k_\alpha\hat{r}) \cong H^1(H_s; k_\alpha\hat{s})$$

allows us to quickly define a 1-cocycle $f_s : H_s \to k_\alpha\hat{s}$ sending $1,r,s,rs \mapsto 0$, sending $a,as,ar,ars \mapsto \hat{s}$ and sending $a^2,a^2s,a^2r,a^2rs \mapsto 2\hat{s}$, and one may follow the arguments above to arrive at the conclusion that $\operatorname{cor}_{H_s}^H(f_s)(r) = \operatorname{cor}_{H_s}^H(f_s)(s) = \operatorname{cor}_{H_s}^H(f_s)(b) = 0$ and that

$$\begin{aligned}\operatorname{cor}_{H_s}^H(f_s)(a) &= \hat{s} + b\cdot\hat{s} + b^2\cdot\hat{s},\\ &= \hat{s} + \widehat{bsb^2} + \widehat{b^2sb},\\ &= \hat{s} + \widehat{b^2s} + \widehat{bs},\end{aligned}$$

so that again, by Proposition 5.4.2 we have a derivation in $HH^1(k_\alpha H)$ which we label $d_s$, that sends $\hat{b}, \hat{r}, \hat{s} \mapsto 0$ and

$$d_s(\hat{a}) = \widehat{as} + \widehat{ab^2s} + \widehat{abs}.$$

We have dealt with the cohomology groups with a non-trivial action - now we turn our attention to those with trivial action, namely the decomposition of the copy of $HH^1(kH)$ inside $HH^1(k_\alpha H)$. We can take the work done towards the construction of the derivations of $HH^1(kH)$, Table 2.1, as a starting point to find the next derivations, and combine these with Proposition 5.4.2. As we have seen, however, we must take care to check for non-trivial actions arising in the corestriction map.

First consider $H^1(C_H(a); k_\alpha \hat{a}) \cong \mathrm{Hom}(C_H(a), k)$ and as usual let $H_a = C_H(a) = \langle a, b, s \rangle$. There, we had a basis of $\mathrm{Hom}(C_H(a), k)$ given by $\{f_a\}$, which sent $a \mapsto 1$ and $b, s \mapsto 0$. View this as an element of $H^1(C_H(a); k_\alpha \hat{a})$ so that $f_a(b) = f_a(s) = 0$ and $f_a(a) = \hat{a}$. As in Example 2.4 we have

$$\begin{aligned} \mathrm{cor}_{H_a}^H(f_a)(a) &= f_a(a) + r \cdot f_a(a^2), \\ &= \hat{a} + 2r \cdot \hat{a}. \end{aligned}$$

We must take care here: $r \cdot \hat{a} = \alpha(r,a)\alpha(ra,r)\alpha(r,r)\widehat{a^2}$. We already know that $\alpha(r,r) = -1$ and one verifies that $\alpha(r,a) = 1$. On the other hand, $\widehat{ra}\hat{r}$ and $\widehat{rar}$ differ by $u^2$: $\widehat{ra}\hat{r} = ucu = c^2u^2 = \widehat{a^2}u^2 = \widehat{rar}u^2$, so that $\beta(ra,r) = u^2$ and $\alpha(ra,r) = -1$. Thus

$$\mathrm{cor}_{H_a}^H(f_a)(a) = \hat{a} + 2\widehat{a^2},$$

as in the untwisted case where the analogous corestriction map sent $a \mapsto a + 2a^2$. We have our third derivation, and we abuse notation by labeling it the same as in the untwisted case, $d_a : k_\alpha H \to k_\alpha H$, $d_a(\hat{b}) = d_a(\hat{r}) = d_a(\hat{s}) = 0$ and

$$\begin{aligned} d_a(\hat{a}) &= \mathrm{cor}_{H_a}^H(f_a)(a)\hat{a}, \\ &= \hat{a}\hat{a} + 2\widehat{a^2}\hat{a}, \\ &= \widehat{a^2} + 2, \end{aligned}$$

(here, $1 = 1\hat{1}$).

We follow the analogous route to construct a derivation from $H^1(C_H(b); k_\alpha \hat{b}) \cong \mathrm{Hom}(C_H(b), k)$. Here, with $H_b := C_H(b)$ and $f_b : C_H(b) \to k_\alpha \hat{b}$ the cocycle sending $b \mapsto \hat{b}$ and $a, r, s \mapsto 0$, we have $\mathrm{cor}_{H_b}^H(f_b)(b) = \hat{b} + 2s \cdot \hat{b}$. The action of $s$ on $\hat{b}$ is given by $s \cdot \hat{b} = \alpha(s,b)\alpha(sb,s)\alpha(s,s)\widehat{b^2}$, and as in the previous case, $\alpha(s,b) = 1$ whilst $\widehat{sb}\hat{s}$ and $\widehat{sbs}$ differ by $v^2 = u^2$, so that $\alpha(sb,s) = -1$. We have already seen that $\alpha(s,s) = -1$ and one checks that by Proposition 5.4.2 this gives a fourth derivation $d_b \in HH^1(k_\alpha H)$ defined by $d_b(\hat{a}) = d_b(\hat{r}) = d_b(\hat{s}) = 0$ and

$$d_b(\hat{b}) = \widehat{b^2} + 2.$$

Continuing in this manner, checking all twisted conjugations actions on $k_\alpha H$ carefully, we determine that the following are two more linearly independent derivations of $k_\alpha H$

$$d_{as}(\hat{a}) = (\widehat{a^2}+1)\left(\sum_{i=0}^{2}\widehat{b^i}\right)\hat{s},$$
$$d_{br}(\hat{b}) = \left(\sum_{i=0}^{2}\widehat{a^i}\right)(\widehat{b^2}+1)\hat{r},$$

induced from the bases of $\mathrm{Hom}(C_H(as), k)$ and $\mathrm{Hom}(C_H(br), k)$ respectively (note that they differ from their untwisted counterparts in Table 2.1). Similarly, we reach the final two derivations of $k_\alpha H$ in our choice of basis,

$$d_{ab,1}(\hat{a}) = (\widehat{a^2}+2)(\hat{b}+\widehat{b^2}),$$
$$d_{ab,2}(\hat{b}) = (\hat{a}+\widehat{a^2})(\widehat{b^2}+2),$$

induced from 2-dimensional basis of $\mathrm{Hom}(C_H(ab), k)$, this time equivalent to their untwisted counterparts in Table 2.1.

Our final task is to determine the Lie algebra structure of $HH^1(k_\alpha H)$. For this we will need some inner derivations, and therefore Lemma 5.4.3. Recall that $\dim_k(\mathrm{IDer}(k_\alpha H)) = \dim_k(k_\alpha H) - \dim_k(Z(k_\alpha H))$. The centre of a twisted group algebra, perhaps unsurprisingly, has a basis given by the conjugacy class sums of $\alpha$-regular elements (see [40, Theorem 2.6.3] for the specifics of this). In our case we have that $1, a, b, ab, as$ and $br$ are all $\alpha$-regular, so that $\dim_k(\mathrm{IDer}(k_\alpha H)) = 36-6 = 30$.

For $x \in H$, let $f_x \in \mathrm{IDer}(k_\alpha H)$ denote the inner derivations $f_x(\hat{g}) = \hat{g}\hat{x} - \hat{x}\hat{g}$ for all $g \in H$. Using Lemma 5.4.3 to help us, it is a routine and trivial task to calculate the inner derivations: we do not reproduce those calculations here. Instead, we will hand pick the inner derivations that we require: let $F \in \mathrm{IDer}(k_\alpha H)$ be the inner derivation

$$F = 2f_{abrs} + f_{a^2brs} + f_{ab^2rs} + 2f_{a^2b^2rs}.$$

Then, on calculating the Lie bracket on the basis of $HH^1(k_\alpha H)$ as given in Table 5.5, one verifies that

$$F = [d_{as}, d_{br}] = [d_r, d_{as}] = [d_{br}, d_s] = [d_s, d_r].$$

Of course, in $HH^1(k_\alpha H)$, the inner derivation $F$ identifies to 0. Calculating the Lie bracket on the basis for $HH^1(k_\alpha H)$ we have just constructed, we obtain Table 5.6 below, given in the usual form.

From this table, writing $\mathcal{L} = HH^1(k_\alpha H)$, one verifies that a basis for $\mathcal{L}'$ is given by

$$\{d_{as}, d_{br}, d_a + d_{ab,1}, d_b + d_{ab,2}, d_r, d_s\},$$

that a basis for $\mathcal{L}''$ is given by

$$\{d_s + d_{as}, d_r + d_{br}\},$$

and that $\mathcal{L}''' = \{0\}$. The derivations $d_a$ and $d_b$ are 3-toral (as in the untwisted case) and form the basis of a maximal 3-toral Lie subalgebra of $\mathcal{L}$, which we denote by $\mathcal{H}$: observe that $\mathcal{L} = \mathcal{L}' \oplus \mathcal{H}$. What is more $(\mathcal{L}')^{[p]} = \{d^p \mid d \in \mathcal{L}'\} = \{0\}$.

We remark that we have now shown the final statement in Theorem 5.4.1 holds.

Table 5.6: The Lie bracket relations on the class representatives of a basis of $HH^1(k_\alpha H)$

| $[-,-]$ | $d_a$ | $d_b$ | $d_{as}$ | $d_{br}$ | $d_{ab,1}$ | $d_{ab,2}$ | $d_r$ | $d_s$ |
|---|---|---|---|---|---|---|---|---|
| $d_a$ | . | . | $2d_s$ | $d_{br}$ | . | $d_{ab,2}+d_b$ | $d_r$ | $2d_{as}$ |
| $d_b$ | | . | $d_{as}$ | $2d_r$ | $d_{ab,1}+d_a$ | . | $2d_{br}$ | $d_s$ |
| $d_{as}$ | | | . | . | $2d_s$ | $2d_{as}+d_s$ | . | . |
| $d_{br}$ | | | | . | $2d_{br}+d_r$ | $2d_r$ | . | . |
| $d_{ab,1}$ | | | | | . | $d_a+d_{ab,1}+2(d_b+d_{ab,2})$ | $d_{br}$ | $d_{as}$ |
| $d_{ab,2}$ | | | | | | . | $d_{br}$ | $d_{as}$ |
| $d_r$ | | | | | | | . | . |
| $d_s$ | | | | | | | | . |

**Remark 5.4.4.** In the work of Benson, Kessar and Linckelmann [7] they compute the first Hochschild cohomology of the *quantum complete intersection* algebra

$$A = k\langle x, y \mid x^p = y^p = xy - qyx = 0\rangle,$$

for $p = \mathrm{char}(k)$ an odd prime and $q \in k^\times$ an element of order dividing $p-1$. As we have seen, with $p = 3$ and $q = -1$ this algebra is a basic algebra for the block $B_1$ in our example, and so $HH^1(A) \cong HH^1(B_1)$ as $k$-vector spaces. Thus we may verify directly that the dimension we have just calculated, $\dim_k(HH^1(B_1)) = 8$, as well as all the Lie algebraic results on $\mathcal{L}, \mathcal{L}'$ and $\mathcal{H}$, agree with those shown more generally by Benson, Kessar and Linckelmann in [7, Theorem 1.1].

## 5.5 Twisted group algebras and finite simple groups

In this section we continue to develop results on twisted group algebras, focusing now more on the twisted group algebras over the finite simple groups. The aim is to develop the technology to prove Theorem 5.1.1 and Corollary 5.1.2. We remark that the remainder of this chapter is the author's own published work [61].

**Definition 5.5.1.** Following the terminology of [28], we make the following definitions.

(1) We say that $G$ satisfies the *weak Non-Schur property* $W(p)$ if there is some $x \in G$ such that its $p$-part is not contained in $C_G(x)'$.

(2) We say that $G$ satisfies the *strong Non-Schur property* $S(p)$ if there is a $p$-element $x \in G$ such that $x$ is not contained in $C_G(x)'$, and we call such an element $x$ a *strong Non-Schur element*.

Evidently we have $S(p)$ implies $W(p)$. What is more, the main result of [28] proves that $W(p)$ holds for all finite groups $G$. As noted in the introduction, this may be used in conjunction with Proposition 2.3.16, setting $[\alpha] = [1]$ in $H^2(G; k^\times)$, to show that $HH^1(kG) \neq \{0\}$ for all finite groups. We record this fact here.

**Proposition 5.5.2.** *Let $G$ satisfy $W(p)$ for some weak Non-Schur element $x$. Then* $\mathrm{Hom}(C_G(x), k)$ *is nonzero. In particular, $HH^1(kG) \neq \{0\}$ for all finite groups $G$.*

*Proof.* Recall that $O^p(G)$ denotes the smallest normal subgroup of $G$ such that the quotient group $G/O^p(G)$ is a $p$-group. Let $A = C_G(x)/C_G(x)'$, which is of order divisible by $p$, since $G$ satisfies $W(p)$. Then with $R = A/O^p(A)$, the quotient $R/\Phi(R)$ is a non-trivial elementary abelian $p$-group with rank equal to $\dim_k(\mathrm{Hom}(C_G(x), k))$. The second statement then follows from [28] and Proposition 2.3.16. □

As mentioned, Fleischmann, Janiszczak and Lempken show in [28] that every finite group of order divisible by $p$ satisfies $W(p)$. In addition, they show that it is not true in general that $W(p)$ implies $S(p)$. Indeed, were it the case that all finite groups satisfied $S(p)$ then one could easily show that $HH^1(k_\alpha G) \neq \{0\}$ for *all* finite groups $G$ and all $\alpha \in Z^2(G; k^\times)$, via the following result.

**Lemma 5.5.3.** *Let $G$ satisfy $S(p)$. Then for all $\alpha \in Z^2(G; k^\times)$, every strong Non-Schur element $x$ is $\alpha$-regular. In particular, $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* On the one hand, $x$ is a $p$-element satisfying $\mathrm{Hom}(C_G(x), k) \neq \{0\}$ by Proposition 5.5.2. On the other, by Lemma 5.3.5 $x$ is $\alpha$-regular, so $H^1(C_G(x); k_\alpha\hat{x}) \cong \mathrm{Hom}(C_G(x), k)$, and the result follows by Proposition 2.3.16. □

It is easy to show a slight reformulation of the proposition above: any group satisfying $S(p)$ also satisfies $\dim_k(HH^1(k\hat{G})) > \dim_k(HH^1(kG))$, where $\hat{G}$ is the central extension of $G$ defined by $\alpha$, so that the left hand side of (5.5) is nonzero, and therefore some power of $\alpha$, say $\alpha^i$, gives us $HH^1(k_{\alpha^i}G) \neq \{0\}$.

One notes that since strong Non-Schur elements are by default *alpha*-regular, it is somewhat tautological to refer to strong Non-Schur elements as *strong Non-Schur $\alpha$-regular* elements. On the other hand we will continue to do so as it distinguishes clearly this more powerful property from the weak Non-Schur case, which may or may not be $\alpha$-regular.

**Lemma 5.5.4** ([28], Lemma 1.2)**.** *Let $P$ be a Sylow $p$-subgroup of $G$ and suppose that $Z(P)$ is not contained in $P'$. Then $G$ satisfies $S(p)$. In particular, $G$ satisfies $S(p)$ if it has abelian Sylow $p$-subgroups.*

The following result, innocuous as it appears, is crucial to proving the non-vanishing of the first Hochschild cohomology for the twisted group algebras of $E_6(q)$ and ${}^2E_6(q^2)$, for some prime power $q$.

**Theorem 5.5.5** ([13], Corollary 3)**.** *Let $G$ be a group, then the following are equivalent.*

*(i) The group $G$ is soluble with all Sylow subgroups abelian.*

*(ii) For every prime $p$ and every two $p$-elements $x$ and $y$ in $G$, the group commutator $[x, y]$ is a $p'$-element.*

The final results of Section 5.2 are required to show the non-vanishing of the first Hochschild cohomology for the twisted group algebra of $E_7(q)$ for some prime power $q$: here we need to consider the restriction $Z \cong C_2$. We start with a useful lemma.

**Lemma 5.5.6.** *Let $P$ be an abelian $p$-group with $p > 2$, $A \cong C_2$ and consider a semidirect product $P \rtimes A$. Then there exists a decomposition of $P$ as direct product of abelian $p$-groups such that for any generator $y$ of $P$ in this decomposition and for $z$ the non-trivial element of $A$, either $z$ commutes with $y$, or $y$ is sent to its inverse under conjugation with $z$.*

*Proof.* As $|A|$ is coprime to $|P|$ we have a decomposition of abelian $p$-groups $P = C_P(A) \times [P, A]$ (see for example [35, Theorem 4.34]). Now let $y$ and $z$ be as above, and notice that since $zyz \in P$, we have $u := y(zyz) = (zyz)y$. On conjugating $u$ by $z$ we have $zuz = z(zyzy)z = yzyz = u$, so that $u \in C_P(A)$. If $y \in C_P(A)$ then we are done, so we may assume that $y \in [P, A]$. Whence $u \in [P, A]$, and therefore $u \in C_P(A) \cap [P, A] = \{1\}$. One sees that $u = yzyz = 1$, so that $zyz = y^{-1}$, and the proof is complete. $\square$

In the sequel, when determining the Hochschild cohomology in degree one for the twisted group algebra of $E_7(q)$ for some prime power $q$, we first prove that it is nonzero. This is then shown to imply that any weak Non-Schur element is $\alpha$-regular in $E_7(q)$, necessary in order to complete Theorem 5.1.1; note that this is in contrast to the other finite simple groups for which $\alpha$-regularity is shown first. To do this, we required the following two propositions.

**Proposition 5.5.7.** *Let $Z \cong C_2$, $p > 2$ and suppose that $x \in G$ is a weak Non-Schur element such that $C_{\hat{G}}(\hat{x})$ is abelian. Then $H^1(C_G(x); k_\alpha \hat{x}) \neq \{0\}$ and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* We may assume that $x$ is not $\alpha$-regular, so that by Proposition 5.3.3, $H := C_{\hat{G}}(x)/C_{\hat{G}}(\hat{x}) \cong C_G(x)/N \cong Z$ (as $\alpha$ takes values in $Z$). Let $H = \langle z \rangle$. Recall also by Proposition 5.3.3 that we have isomorphisms of $k$-modules $H^1(C_G(x); k_\alpha \hat{x}) \cong H^1(N; k_\alpha \hat{x})^H$ and $H^1(N; k_\alpha \hat{x}) \cong \mathrm{Hom}(N, k_\alpha \hat{x})$. By assumption $C_{\hat{G}}(\hat{x})$ is abelian, whence $N \cong C_{\hat{G}}(\hat{x})/Z$ is abelian; write $N = P \times M$ for $P$ the Sylow $p$-subgroup of $N$ and $M$ of order coprime to $p$. We have a short exact sequence $1 \to P \to \hat{P} \to H \to 1$ for some $\hat{P} \leq C_G(x)$ which splits by the Schur-Zassenhaus lemma (recall that $p$ is odd).

We write $P = \langle a_1, \dots a_n \mid a_i^{r_i} = [a_i, a_j] = 1, i, j = 1, \dots, n \rangle$, where under the action of $H$ on $P$ in the fixed point space $\mathrm{Hom}(P, k_\alpha \hat{x})^H$ we make the choice (as in Lemma 5.5.6) that each $a_i$ either commutes with $z$ or is sent to its inverse under the action of $z$. Then $\mathrm{Hom}(N, k_\alpha \hat{x}) \cong \mathrm{Hom}(P, k_\alpha \hat{x})$ has dimension $n$, and we may choose a $k$-basis for the latter as $\{f_1, \dots, f_n\}$, where $f_i$ is the map sending $a_i$ to $\hat{x}$ and $a_j$ to $0$ for $j \neq i$.

If $H$ acts non-trivially on $P$, let $a_m$ be one such generator that is sent to its inverse under the action of $H$, $z a_m z = a_m^{-1}$. Viewing $H \subseteq k$ as $H = \{\pm 1\}$, one sees that $H$ acts on $k_\alpha \hat{x}$ as $z \cdot \lambda \hat{x} = -\lambda \hat{x}$ for all $\lambda \in k$. Putting this together, the action of $H$ on $\mathrm{Hom}(P, k_\alpha \hat{x})$ sends $f_m$ to ${}^z f_m$, which maps $a_m \mapsto z \cdot f_m(z a_m z) = -f_m(a_m^{-1}) = f_m(a_m)$ and $a_i \mapsto -f_m(z a_i z) = 0 = f_m(a_i)$ for all $i \neq m$. With this construction, we have $f_m \in \mathrm{Hom}(P; k_\alpha \hat{x})^H \cong H^1(C_G(x); k_\alpha \hat{x})$, and we are done.

Otherwise, if $H$ acts trivially on $P$, then we may write $C_G(x) = P \times \hat{M}$ where, writing $S$ for the Sylow 2-subgroup of $N$, $N = P \times M = P \times (M' \times S)$ for some subgroup $M'$ of odd order coprime to $p$, and $\hat{S}$ such that $\hat{S}/S \cong H$, we have $\hat{M} = M' \rtimes \hat{S}$. Whence $H^1(C_G(x); k_\alpha \hat{x}) \cong H^1(P; k_\alpha \hat{x})$ (by the Künneth formula) which we have already seen has dimension $n > 0$. $\square$

**Proposition 5.5.8.** *Let $Z \cong C_2$, $p > 2$ and suppose that $x \in G$ is a weak Non-Schur element such that $C_{\hat{G}}(\hat{x})$ is a direct product of cyclic groups of even order. Then $H^1(C_G(x); k_\alpha \hat{x}) \neq \{0\}$ if and only if $x$ is $\alpha$-regular.*

*Proof.* One direction is clear: if $x$ is $\alpha$-regular then $H^1(C_G(x); k_\alpha \hat{x}) \cong \mathrm{Hom}(C_G(x), k)$, and $C_G(x) \cong C_{\hat{G}}(\hat{x})/Z$ is abelian. As $x$ is a weak Non-Schur element we have $\mathrm{Hom}(C_G(x), k) \neq \{0\}$.

In the other direction we show the contrapositive: suppose that $x$ is not $\alpha$-regular. This gives us that $C_{\hat{G}}(x)/C_{\hat{G}}(\hat{x}) \cong C_G(x)/N \cong Z$ (as $\alpha$ takes values in $Z$) and recall that $C_{\hat{G}}(\hat{x})/Z \cong N$ is abelian. Whence we may write $N = \langle a_1, \dots, a_n \mid a_i^{m_i} = [a_i, a_j] = 1, i, j = 1, \dots, n \rangle$ with $m_i$ even for all $i$, and $C_G(x) = \langle N, s \mid a_i^{m_i} = [a_i, a_j] = s^t = 1, sa_is^{-1}, s^2 \in N, i, j = 1, \ldots, n\rangle$. Recall that $N$ acts trivially on $k_\alpha\hat{x}$, and with $k \supseteq \{\pm 1\} = Z \cong C_G(x)/N = \langle sN\rangle$, we may view the action of $s$ on $k_\alpha\hat{x}$ as sending $\lambda\hat{x} \mapsto -\lambda\hat{x}$ for all $\lambda \in k$.

If $H^1(C_G(x); k_\alpha\hat{x}) \neq \{0\}$ choose a nonzero $f \in Z^1(C_G(x); k_\alpha\hat{x})$. One sees that $f(1) = 0$, for $f(1) = f(1 \cdot 1) = f(1) + 1 \cdot f(1) = 2f(1)$. Now, since $m_i$ is even for all $i = 1, \ldots, n$ and $\mathrm{char}(k)$ is odd, we have $f(a_i) = 0$, for $0 = f(1) = f(a_i^{m_i}) = m_if(a_i)$ where the final equality holds since $N$ acts trivially on $k_\alpha\hat{x}$. It must therefore be the case that $f$ is nonzero on $s$, and without loss of generality we make the choice that $f(s) = \hat{x}$; it is now clear that $Z^1(C_G(x); k_\alpha\hat{x})$ is one-dimensional with a $k$-basis given by $\{f\}$.

On the other hand, we will show that $f \in B^1(C_G(x); k_\alpha\hat{x})$, for there exists some $\lambda \in k$ such that $f(g) = (g-1) \cdot \lambda\hat{x}$ for all $g \in C_G(x)$. Let $\lambda = (-2)^{-1}$ which exists as $\mathrm{char}(k)$ is odd. If $g = a_i$ for some $i$ then $f(g) = 0 = (g-1) \cdot \lambda\hat{x}$ as $N$ acts trivially on $k_\alpha\hat{x}$, whilst if $g = s$ then $f(s) = \hat{x} = -2\lambda\hat{x} = (s-1) \cdot \lambda\hat{x}$.

We arrive at a contradiction and so it must be the case that $H^1(C_G(x); k_\alpha\hat{x}) = \{0\}$; the contrapositive proof is complete, whence the result. □

## 5.6 The proof of Theorem 5.1.1

Throughout this section we fix $k$ to be (algebraically closed) of characteristic $p$ dividing the order of a finite group $G$; all notation is as in Sections 5.1, 5.2, 5.3 and 5.5. We begin with abelian groups.

**Lemma 5.6.1.** *Let $G$ be an abelian group, $x \in G$ a $p$-element. Then $x$ is an $\alpha$-regular, strong Non-Schur element of $G$. In particular, $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* Such an $x$ is a strong Non-Schur element by virtue of the fact that $C_G(x)' = G' = \{1\}$. The result follows from Lemma 5.5.3. □

In fact, we can show something a little stronger.

**Theorem 5.6.2.** *Suppose $O^p(G)$ is a proper subgroup of $G$. Then $HH^1(k_\alpha G) \neq \{0\}$ for all $\alpha \in Z^2(G; k^\times)$.*

*Proof.* Let $R = G/O^p(G)$ be the largest $p$-group quotient of $G$, then $R/\Phi(R)$ is a non-trivial elementary abelian $p$-group. Whence

$$\dim_k(\mathrm{Hom}(G, k)) = \dim_k(\mathrm{Hom}(R/\Phi(R), k)) \neq 0.$$

Any cocycle $\alpha \in Z^2(G; k^\times)$ satisfies

$$\alpha(g, 1) = \alpha(1, 1) = \alpha(1, g)$$

for all $g \in G$ [52, Proposition 1.2.5], so on choosing the conjugacy class representative $x = 1$ in the decomposition of Proposition 2.3.16, one sees that the action of $G = C_G(x)$ on $k_\alpha\hat{x}$ is trivial. The result follows. □

**Proposition 5.6.3.** *Let $G$ be a finite group of order divisible by $p$ and $q$ a prime power. Suppose further that $G$ is one of the following.*

- *An alternating group $A_n$ for $n \geq 5$.*
- *A sporadic simple group.*
- *A Chevalley group of classical type, $Gl_n(q), Sl_n(q), PSl_n(q), U_n(q), SU_n(q), PSU_n(q)$, $SO_{2n+1}(q), P\Omega_{2n+1}(q), Sp_{2n}(q), PSp_{2n}(q), SO^+_{2n}(q), P\Omega^+_{2n}(q), SO^-_{2n}(q)$ or $P\Omega^-_{2n}(q)$.*

*Then for all $\alpha \in Z^2(G; k^\times)$, $G$ contains weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* This is an immediate consequence of [28, Propositions 2.1,2.2 and 3.2], and in fact, aside from the exceptions $(G, p) \in \{(Ru, 3), (J_4, 3), (Th, 5)\}$ these groups all satisfy $S(p)$, and so the result holds in these cases.

For the exceptions we have the following: the Schur multiplier of $J_4$ and $Th$ is trivial whence $H^2(G; k^\times)$ is also, and there is nothing to show. For the case $G = Ru$, one can find an element $x \in G$ of order divisible by $p$ such that $C_G(x) = \langle x \rangle$ [28, Proposition 2.2]. The result follows from Lemma 5.3.6.

□

We now turn to the finite groups of Lie type.

**Lemma 5.6.4.** *Let $G$ be one of ${}^2G_2(3^{2n+1})$ for some $n \geq 1$, the simple group ${}^2G_2(3)'$, ${}^3D_4(q^3)$, ${}^2F_4(2^{2n+1})$, $E_8(q)$ or the simple Tits group ${}^2F_4(2)'$, for some prime power $q$. Then for all $\alpha \in Z^2(G; k^\times)$, $G$ has weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* One sees this by noting that in all cases the Schur multiplier is trivial, then applying Proposition 5.5.2. □

**Proposition 5.6.5.** *Let $G$ be a finite group of Lie type, defined over a field of characteristic $r$ with $q = r^m$ for some $m$. Let $\alpha \in Z^2(G; k^\times)$. Then we have the following.*

*(i) If $p = r$ then $G$ contains strong Non-Schur $\alpha$-regular elements.*

*(ii) Suppose that $p \neq r$. If $W$ is the Weyl group of $G$ and $p$ does not divide the order of $W$, then $G$ contains strong Non-Schur $\alpha$-regular elements.*

*In each case, $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* This is an immediate consequence of [28]: if $p = r$ then by [11, E.II.,III.] and [28, Lemma 3.1(1)] $G$ contains a $p$-element $x$ whose centraliser is abelian of order divisible by $p$, and we are done by Lemma 5.5.3. If $p \neq r$, and $p$ does not divide $|W|$, then by [11, E.II.,III.] and [28, Lemma 3.1(2)] the Sylow $p$-subgroups of $G$ are abelian, and we are done by Lemma 5.5.4. □

**Proposition 5.6.6.** *Let $G$ be a Suzuki group ${}^2B_2(2^{2n+1})$ for some $n \geq 1$. Then for all $\alpha \in Z^2(G; k^\times)$, $G$ contains weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* If $n > 1$ then the Schur multiplier of $G$ is trivial and there is nothing to show. If $n = 1$ then $G = {}^2B_2(8)$ has Schur multiplier isomorphic to the Klein-four group. We may assume by Proposition 5.6.5 that $p \neq 2$ and $p$ divides the order of the Weyl group $W$ of $G$. On the other hand, the Weyl group of a root system of $B_2$ type has order 8, completing the proof. □

**Proposition 5.6.7.** *Let $G$ be a Chevalley group $G_2(q)$ for some prime power $q > 2$, or the simple group $G_2(2)'$. Then for all $\alpha \in Z^2(G; k^\times)$, $G$ contains weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* There is an isomorphism $G_2(2)' \cong PSU_3(3^2)$ and so we are done in this case by Proposition 5.6.3. Now suppose $G = G_2(q)$, $q > 2$. The Weyl group of $G$ has order $2^2 \cdot 3$ [28, Lemma 3.1] so by Proposition 5.6.5 we only need to consider $p = 2, 3$. The Schur multiplier of $G$ is trivial except when $q = 3$ or 4, in which case it is of order 3 or 2 respectively [18].

First let $q = 3$. By Proposition 5.6.5 we have $p = 2$ (otherwise we are done). Let $\alpha \in Z^2(G; k^\times)$. One verifies in `GAP` that $G$ has an element $x$ of order 8, with centraliser $C_G(x) = \langle x \rangle$, and we are done by Lemma 5.3.6: $G$ satisfies $S(p)$.

Now let $q = 4$, so that $p = 3$. In this case, using `GAP` one can find an element $x$ of order 15 with $C_G(x) = \langle x \rangle$, and we are again done by Lemma 5.3.6, though $G$ only satisfies $W(p)$ in this case. □

Note that these calculations agree with the more general calculations done by Fleischmann, Janiszczak and Lempken in [28, Proposition 4.1].

**Proposition 5.6.8.** *Let $G$ be a Chevalley group $F_4(q)$ for some prime power $q$. Then for all $\alpha \in Z^2(G; k^\times)$, $G$ contains weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* The Schur multiplier of $G$ is trivial in all cases except $q = 2$. The Weyl group of $F_4$ type has order $2^7 \cdot 3$, so by Proposition 5.6.5 we only need to check $G = F_4(2)$ and $p = 3$. By [28, Proposition 4.1], there is some weak Non-Schur $x \in G$ such that $C_G(x)$ has order 21 and in fact one verifies (for example in `GAP`) that $C_G(x) \cong C_{21}$. By Lemma 5.3.6 the action of $C_G(x)$ on $k_\alpha \hat{x}$ is trivial. This completes the proof. □

**Proposition 5.6.9.** *Let $G$ be the untwisted exceptional Chevalley group $E_6(q)$ or its twisted counterpart ${}^2E_6(q^2)$ for some prime power $q$. Then for all $\alpha \in Z^2(G; k^\times)$, $G$ contains weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* Let $G$ be $E_6(q)$ (respectively ${}^2E_6(q^2)$) as in the statement, let $\alpha \in Z^2(G; k^\times)$ and denote by $\mathbf{G}$ a simple, simply-connected algebraic group of type $E_6$ defined over a field of characteristic $r > 0$. Let $\hat{G}$ be the group of fixed points of $\mathbf{G}$ under a standard (respectively non-standard) Frobenius map $F$, so that we have a central extension $1 \to Z \to \hat{G} \to G \to 1$ corresponding to the cohomology class of $\alpha$. Both $Z$ and the Schur multiplier $M(G)$ of $G$ are isomorphic to the cyclic group of order $\gcd(3, q-1)$ (respectively of order $\gcd(3, q+1)$, except for the case when $q = 2$ we have $M(G) \cong C_2 \times C_2 \times C_3$) [18]. Whence we may assume $q \equiv 1 (\text{mod } 3)$ (respectively $q \equiv 2 (\text{mod } 3)$) and $Z \cong C_3$, otherwise $\alpha = 1$ and $k_\alpha G \cong kG$ (recall that $\alpha$ takes values in $Z$). We may also assume therefore that $p \neq 3$, as $Z$ is cyclic of $p'$-order. The Weyl group $W$ of type $E_6$ is isomorphic to $PSp_4(3) \rtimes C_2$ of order $2^7 \cdot 3^4 \cdot 5$ [28], and so by Proposition 5.6.5 we may restrict to the cases that $p$ is equal to either 2 or 5.

Fix $x \in G$ to be the weak Non-Schur element found in [28]. The element $\hat{x} \in \hat{G}$ which maps to $x \in G$ is chosen to be a regular, semisimple element: an element of $r$-divisible order with centraliser in $\hat{G}$ a finite torus, whence abelian, of order divisible by $p$. We will show that the action of $C_G(x)$ on $k_\alpha \hat{x}$ is trivial in the twisted centraliser decomposition, which will give us our result.

The order of $C_{\hat{G}}(\hat{x})$ is given in [28] as $(q-1)(q^3+1)(q^2+1)$ (respectively $(q+1)(q^3-1)(q^2+1)$), and one may verify (for example by using [11, G, Table 2]) that $C_{\hat{G}}(\hat{x})$ is indeed a maximal torus in $\hat{G}$ of this order. The maximal tori of $\hat{G}$ have been classified by Derizioutis and Fakiolas [21] and on locating $T_{20}$ in [21, Table II] one sees that $C_{\hat{G}}(\hat{x})$ is in fact cyclic. Whence by Proposition 5.3.3(i) $N$ must be cyclic (recall that $N = \ker(\lambda_\alpha)$ as in Section 5.2). What is more, all Sylow $\ell$-subgroups of $C_{\hat{G}}(x)$ and $C_G(x)$ are cyclic except for possibly when $\ell = 3$ (see for example [34, Proposition 2.2]).

Now, since $\alpha$ takes values in $Z$ we have that $H := C_{\hat{G}}(x)/C_{\hat{G}}(\hat{x}) \cong C_G(x)/N$ is isomorphic to the trivial group or to $C_3$. If $H$ is trivial, then we are done by Proposition 5.3.3. If $H \cong C_3$ then $C_{\hat{G}}(x)$ and $C_G(x)$ are metacyclic (and therefore also solvable). We will show that in this case all Sylow subgroups of $C_G(x)$ are abelian, and then use Theorem 5.5.5 to arrive at a contradiction, proving that $H$ is trivial and therefore $x$ must be $\alpha$-regular.

Let $S$ be a Sylow 3-subgroup of $C_{\hat{G}}(x)$ of order $3^c$, for some integer $c$. Then $S$ contains both $Z$ and the Sylow 3-subgroup of $C_{\hat{G}}(\hat{x})$ which is cyclic of order $3^{c-1}$. Whence $S$ contains a cyclic subgroup of index 3, and so by a result of Brown [16, Lemma IV(4.2)], $S$ is isomorphic to one of $C_{3^c}$, $C_{3^{c-1}} \times C_3$ or $C_{3^{c-1}} \rtimes C_3$ with the action of a generator of $C_3$ raising a generator of $C_{3^{c-1}}$ to the power $1 + 3^{c-2}$. A Sylow 3-subgroup of $C_G(x)$ is obtained by taking the quotient $S/Z$, giving $C_{3^{c-1}}$ or $C_{3^{c-2}} \times C_3$ as the only possibilities, and so any Sylow 3-subgroup of $C_G(x)$ is abelian.

Note that if $S/Z \cong C_{3^{c-1}}$ then we are done, for in this case all Sylow subgroups of $C_G(x)$ are cyclic whence $[\alpha]$ restricts to the trivial class on $C_G(x)$ (see Lemma 5.3.6), so we may assume that $S/Z \cong C_{3^{c-2}} \times C_3$, although the following will actually show that we only need $S/Z$ to be abelian to arrive at a contradiction.

As $C_G(x)$ is metacyclic, it admits a presentation

$$\langle a, b \mid a^{|N|} = 1, b^3 = a^t, bab^{-1} = a^s \rangle,$$

where $s^3 \equiv 1 (\mathrm{mod}\ |N|)$ and $st \equiv t (\mathrm{mod}\ |N|)$. We have that $a$ is a generator of $N$ and $b$ maps to a generator $\bar{b}$ of $C_3$ under the isomorphism $C_G(x)/N \cong C_3$. Without loss of generality we make the choice that $b$ is a 3-element. Note that we may assume 3 divides $|N|$ for otherwise $c = 2$ and the Sylow 3-subgroups of $C_G(x)$ have order 3. So we may choose an element of order 3, $a^i$ for some integer $i$, in $N$. Now consider the commutator $[a^i, b] = a^i b a^{-i} b^{-1} = a^i (a^i)^{-s}$, which by Theorem 5.5.5 has order coprime to 3. One verifies that this is not the case, for $[a^i, b]^3 = a^{3i}(a^{3i})^{-s} = 1$, and we arrive at our contradiction. Whence $H$ is trivial, implying $x$ is $\alpha$-regular by Proposition 5.3.3, and the result follows. □

**Proposition 5.6.10.** *Let $G$ be the exceptional Chevalley group $E_7(q)$. Then for all $\alpha \in Z^2(G; k^\times)$, $G$ contains weak Non-Schur $\alpha$-regular elements, and in particular $HH^1(k_\alpha G) \neq \{0\}$.*

*Proof.* Let $\alpha \in Z^2(G; k^\times)$ and denote by $\mathbf{G}$ a simple, simply-connected algebraic group of type $E_7$ defined over a field of characteristic $r > 0$, and $\hat{G}$ be the group of fixed points of $\mathbf{G}$ under a standard Frobenius map $F$, so that we have a central extension $1 \to Z \to \hat{G} \to G \to 1$ corresponding to the cohomology class of $\alpha$, where $Z$ is a cyclic subgroup of $k^\times$ of $p'$-order. Both $Z$ and the Schur multiplier $M(G)$ of $G$ are isomorphic to the cyclic group of order $\gcd(2, q-1)$ [18], whence we may assume $q$ is odd and $Z \cong M(G) \cong C_2$, otherwise $Z^2(G; k^\times) = \{1\}$ and $k_\alpha G \cong kG$. We may also assume that $p \neq 2$, else $H^2(G; k^\times) \cong M(G)_{p'}$ is trivial and again there is nothing to show.

The Weyl group $W$ of type $E_7$ is isomorphic to $C_2 \times Sp_6(2)$ of order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ [28] and so by Proposition 5.6.5 we restrict to the cases that $p$ is equal to one of $3, 5$ or $7$. It is shown in [28] that there is some weak Non-Schur element $x \in G$. The element $\hat{x} \in \hat{G}$ which maps to $x \in G$ is chosen so that $C_{\hat{G}}(\hat{x})$ is abelian, of even order divisible by $p$.

Now, since $\alpha$ takes values in $Z$ we have that $H := C_{\hat{G}}(x)/C_{\hat{G}}(\hat{x}) \cong C_G(x)/N$ is isomorphic to the trivial group or to $C_2$. If $H$ is trivial, then we are done by Proposition 5.3.3. If $H \cong C_2$ then we are in the situation where Propositions 5.5.7 and 5.5.8 apply; the former shows that $HH^1(k_\alpha G) \neq \{0\}$ and the latter then tells us that in this case this is equivalent to the $\alpha$-regularity of $x$, completing the proof. □

*Proof of Theorem 5.1.1.* The proof now follows from Theorem 5.6.2 and Propositions 5.6.3, 5.6.6, 5.6.7, 5.6.8, 5.6.9, 5.6.10 and Lemma 5.6.4. □

*Proof of Corollary 5.1.2.* By Theorem 5.1.1, $G$ contains weak Non-Schur $\alpha$-regular elements. On the one hand such an element $x$ gives $\mathrm{Hom}(C_G(x), k) \neq \{0\}$ by Proposition 5.5.2, and on the other $\mathrm{Hom}(C_G(x), k) \cong H^1(C_G(x); k_\alpha \hat{x})$ since $x$ is $\alpha$-regular. The result now follows from the twisted centraliser decomposition, Proposition 2.3.16. □

## 5.7 The dimensions of the $HH^1$ of twisted group algebras

Using the results of Section 5.2, in particular Proposition 5.2.6 and the isomorphisms given in (5.3) and (5.4), the comments that follow these isomorphisms, the (untwisted) centraliser decomposition, and the `GAP` code found in Appendix A.1 we are able to calculate the dimensions of the first Hochschild cohomology of some finite simple group algebras and their twisted counterparts. Since the cover $\hat{G}$ is an extension of $G$ by a relatively small central subgroup $Z$ in each case below, finding the dimensions of $HH^1(k_\alpha G)$ reduces to some elementary arithmetic.

We will now describe how these dimensions are calculated, separating cases based on the group structure of $Z$. Let $Z = C_2$, which accounts for the majority of the cases we have been able to calculate. The isomorphism of $k$-algebras given by (5.3) tells us that $k\hat{G} \cong kG \times k_\alpha G$ in this case. Taking Hochschild cohomology, one obtains the $k$-vector space isomorphism of (5.4),

$$HH^1(k\hat{G}) \cong HH^1(kG) \oplus HH^1(k_\alpha G)$$

which may then be used to show that

$$\dim_k(HH^1(k\hat{G})) - \dim_k(HH^1(kG)) = \dim_k(HH^1(k_\alpha G)),$$

as in (5.2). The dimensions on the left hand side of this equality are then calculated individually using the centraliser decomposition and the `GAP` code in Appendix A.1. Here, an algorithmic procedure is applied to determine a description of the centraliser decomposition of $HH^1(k\hat{G})$ in terms of the $k$-vector spaces $\mathrm{Hom}(C_{\hat{G}}(\hat{x}), k)$, where $\hat{x}$ runs over a complete set of conjugacy class representatives of $\hat{G}$, and the dimensions of these vector spaces are calculated - see the example below, when $\hat{G} = C_6.A_7$. The same procedure is applied to $HH^1(kG)$ and the difference in dimensions of $HH^1(k\hat{G})$ and $HH^1(kG)$ is calculated, giving $\dim_k(HH^1(k_\alpha G))$.

When $Z = C_3$, the method follows in a similar fashion. We have $k\hat{G} \cong kG \times k_\alpha G \times k_{\alpha^{-1}} G$, so that

$$HH^1(k\hat{G}) \cong HH^1(kG) \oplus HH^1(k_\alpha G)^{\oplus 2},$$

which may be seen by noting that $k_{\alpha^{-1}}G \cong (k_\alpha G)^{\mathrm{op}}$ whence $HH^1(k_{\alpha^{-1}}G) \cong HH^1(k_\alpha G)$ (see Section 5.2). This gives

$$\dim_k(HH^1(k\hat{G})) - \dim_k(HH^1(kG)) = 2 \cdot \dim_k(HH^1(k_\alpha G)).$$

The dimensions on the left hand side are then calculated again using the `GAP` code in [60, Appendix A], and we are able to determine the dimension of $HH^1(k_\alpha G)$.

When $Z = C_4$, $C_6$ or in one case $C_{12}$ we have to be a little more careful. In some of these cases, one finds that by Corollary 5.1.2, the only option is that the dimensions of the $HH^1(k_{\alpha^i}G)$, $i = 1, \ldots, |Z|-1$, are all equal to 1; take, for example, the case where $G = A_7$, $Z = C_6$ and $p = 5$. Here, the `GAP` code calculates $\dim_k(HH^1(k\hat{G})) = 6$. This is done by first by computing the centralisers $C_{\hat{G}}(\hat{g}_i)$, $i = 1, \ldots, 40$, of a complete set of conjugacy class representatives $\{\hat{g}_i \mid i = 1, \ldots, 40\}$ of $\hat{G}$ as isomorphic to the following: six copies of $\hat{G}$, two copies of $C_3 \times C_6$, three copies of $C_3 \times (C_3 \rtimes C_8)$, six copies of $C_{24}$, twelve copies of $C_{42}$, six copies of $C_{30}$, two copies of $C_3 \times C_3 \times Q_8$ and three copies of $C_{12} \times C_3$. Then, for each conjugacy class representative $\hat{g}_i$, the groups $R_i := C_{\hat{G}}(\hat{g}_i)/O^5(C_{\hat{G}}(\hat{g}_i))$ are computed: trivial in all cases except for the six copies of $C_{\hat{G}}(\hat{g}_i) \cong C_{30}$, in which case $R_i \cong C_5$. Next, the elementary abelian 5-groups $R_i/\Phi(R_i)$ are computed for all $i$ (trivial except for the six copies of $C_5$), and a sum of the total elementary abelian 5-ranks is computed for all $i$, giving the total sum of dimensions of the $\mathrm{Hom}(C_{\hat{G}}(\hat{g}_i), k)$ in the centraliser decomposition to be 6.

The algorithm proceeds as above to find also that $\dim_k(HH^1(kG)) = 1$ which gives a sum of dimensions

$$\sum_{i=1}^{5} \dim_k(HH^1(k_{\alpha^i}G)) = \dim_k(HH^1(k\hat{G})) - \dim_k(HH^1(kG)) = 5.$$

By our main corollary of this chapter, the non-vanishing of $HH^1(k_\alpha G)$ for $G$ a finite simple group, one therefore obtains $\dim_k(HH^1(k_{\alpha^i}G)) = 1$ for $i = 0, \ldots, 5$. On the other hand, in this same example with instead the prime $p = 7$, we have that via the inclusion $C_3 \hookrightarrow C_6$, $\alpha^2$ and $\alpha^{-2}$ correspond exactly to the entry of the table for which $Z = C_3$, and via the inclusion $C_2 \hookrightarrow C_6$, $\alpha^3$ corresponds to the earlier entry for which $Z = C_2$. Consequently, subtracting the dimensions of $\dim_k(HH^1(k_\alpha G))$ for the entries $Z = C_3$ and $Z = C_2$ from the dimension of $HH^1(k\hat{G})$ we are able to conclude that the dimensions of $HH^1(k_{\alpha^i}G)$, $i = 0, \ldots, 5$ are equal to 2.

In fact, the only occurrence in our table for which the dimensions $\dim_k(HH^1(k_{\alpha^i}G))$ are not all equal for all $i$, is when $G$ is the Mathieu group $M_{22}$, $Z = C_4$ and $p = 3$, and so we make a record of this here. In this case one sees, using the inclusion $C_2 \hookrightarrow C_4$ and same reasoning as in the previous paragraph, that $\dim_k(HH^1(k_{\alpha^i}G))$ is equal to $3, 2, 3, 2$ for $i = 0, 1, 2, 3$ respectively.

We remark that our list was chosen by working our way through the groups whose character tables and further details are given in The ATLAS [18], discarding those groups along the way for which `GAP` could neither construct nor perform the necessary calculations on. The group notation used is as in The ATLAS; all other notation follows as previously.

**Proposition 5.7.1.** *Let $G$ be one of the groups given in the first column of Table 5.7, and $\hat{G}$ be a central extension of $G$ by a cyclic $p'$-group $Z$. Let $M(G)$ be the Schur multiplier of $G$, so that there is an isomorphism of group $H^2(G; k^\times) \cong M(G)_{p'}$, the $p'$-part of $M(G)$. Then for all $\alpha \in Z^2(G; k^\times)$ corresponding to a faithful character $Z \to k^\times$, the dimensions of the first Hochschild cohomology of $k_\alpha G$ are given as in the final column of Table 5.7.*

Table 5.7: Dimensions of the first Hochschild cohomology of some finite simple group algebras

| $G$ | $M(G)$ | $Z$ | $p$ | $\dim_k(HH^1(k\hat{G}))$ | $\dim_k(HH^1(kG))$ | $\dim_k(HH^1(k_\alpha G))$ |
|---|---|---|---|---|---|---|
| $A_5$ | $C_2$ | $C_2$ | 3 | 2 | 1 | 1 |
| | | | 5 | 4 | 2 | 2 |
| $L_3(2)$ | $C_2$ | $C_2$ | 3 | 2 | 1 | 1 |
| | | | 7 | 4 | 2 | 2 |
| $A_6$ | $C_6$ | $C_2$ | 3 | 8 | 4 | 4 |
| | | | 5 | 4 | 2 | 2 |
| | | $C_3$ | 2 | 9 | 3 | 3 |
| | | | 5 | 6 | 2 | 2 |
| | | $C_6$ | 5 | 12 | 2 | 2 |
| $L_2(11)$ | $C_2$ | $C_2$ | 3 | 5 | 2 | 3 |
| | | | 5 | 4 | 2 | 2 |
| | | | 7 | 4 | 2 | 2 |
| $L_2(13)$ | $C_2$ | $C_2$ | 3 | 5 | 2 | 3 |
| | | | 7 | 6 | 3 | 3 |
| | | | 13 | 4 | 2 | 2 |
| $L_2(17)$ | $C_2$ | $C_2$ | 3 | 8 | 4 | 4 |
| | | | 17 | 4 | 2 | 2 |
| $A_7$ | $C_6$ | $C_2$ | 3 | 9 | 5 | 4 |
| | | | 5 | 2 | 1 | 1 |
| | | | 7 | 4 | 2 | 2 |
| | | $C_3$ | 2 | 17 | 5 | 6 |
| | | | 5 | 3 | 1 | 1 |
| | | | 7 | 6 | 2 | 2 |
| | | $C_6$ | 5 | 6 | 1 | 1 |
| | | | 7 | 12 | 2 | 2 |
| $L_2(19)$ | $C_2$ | $C_2$ | 3 | 8 | 4 | 4 |
| | | | 5 | 9 | 4 | 5 |
| | | | 19 | 4 | 2 | 2 |
| $L_2(23)$ | $C_2$ | $C_2$ | 3 | 11 | 5 | 6 |
| | | | 11 | 10 | 5 | 5 |
| | | | 23 | 4 | 2 | 2 |
| $L_2(27)$ | $C_2$ | $C_2$ | 3 | 12 | 6 | 6 |
| | | | 7 | 13 | 6 | 7 |
| | | | 13 | 12 | 6 | 6 |
| $L_2(29)$ | $C_2$ | $C_2$ | 3 | 14 | 7 | 7 |
| | | | 5 | 14 | 7 | 7 |
| | | | 7 | 13 | 6 | 7 |
| | | | 29 | 4 | 2 | 2 |
| $L_2(31)$ | $C_2$ | $C_2$ | 3 | 14 | 7 | 7 |
| | | | 5 | 14 | 7 | 7 |
| | | | 31 | 4 | 2 | 2 |

| $G$ | $M(G)$ | $Z$ | $p$ | $\dim_k(HH^1(k\hat{G}))$ | $\dim_k(HH^1(kG))$ | $\dim_k(HH^1(k_\alpha G))$ |
|---|---|---|---|---|---|---|
| $A_8$ | $C_2$ | $C_2$ | 3 | 14 | 7 | 7 |
| | | | 5 | 6 | 3 | 3 |
| | | | 7 | 4 | 2 | 2 |
| $L_3(4)$ | $C_4^2 \times C_3$ | $C_2$ | 3 | 4 | 2 | 2 |
| | | | 5 | 4 | 2 | 2 |
| | | | 7 | 4 | 2 | 2 |
| | | $C_3$ | 2 | 30 | 10 | 10 |
| | | | 5 | 6 | 2 | 2 |
| | | | 7 | 6 | 2 | 2 |
| | | $C_6$ | 5 | 12 | 2 | 2 |
| | | | 7 | 12 | 2 | 2 |
| $U_4(2)$ | $C_2$ | $C_2$ | 3 | 39 | 20 | 19 |
| | | | 5 | 2 | 1 | 1 |
| $Sz(8)$ | $C_2^2$ | $C_2$ | 5 | 2 | 1 | 1 |
| | | | 7 | 6 | 3 | 3 |
| | | | 13 | 6 | 3 | 3 |
| $M_{12}$ | $C_2$ | $C_2$ | 3 | 7 | 4 | 3 |
| | | | 5 | 4 | 2 | 2 |
| | | | 11 | 4 | 2 | 2 |
| $A_9$ | $C_2$ | $C_2$ | 3 | 25 | 12 | 13 |
| | | | 5 | 7 | 4 | 3 |
| | | | 7 | 2 | 1 | 1 |
| $J_2$ | $C_2$ | $C_2$ | 3 | 13 | 7 | 6 |
| | | | 5 | 18 | 10 | 8 |
| | | | 7 | 2 | 1 | 1 |
| $S_6(2)$ | $C_2$ | $C_2$ | 3 | 28 | 16 | 12 |
| | | | 5 | 6 | 3 | 3 |
| | | | 7 | 2 | 1 | 1 |
| $A_{10}$ | $C_2$ | $C_2$ | 3 | 29 | 15 | 14 |
| | | | 5 | 9 | 5 | 4 |
| | | | 7 | 6 | 3 | 3 |
| $L_3(7)$ | $C_3$ | $C_3$ | 2 | 38 | 12 | 13 |
| | | | 7 | 21 | 7 | 7 |
| | | | 19 | 18 | 6 | 6 |
| $G_2(3)$ | $C_3$ | $C_3$ | 2 | 33 | 11 | 11 |
| | | | 7 | 3 | 1 | 1 |
| | | | 13 | 6 | 2 | 2 |
| $S_4(5)$ | $C_2$ | $C_2$ | 3 | 24 | 12 | 12 |
| | | | 5 | 39 | 20 | 19 |
| | | | 13 | 6 | 3 | 3 |
| $U_3(8)$ | $C_3$ | $C_3$ | 2 | 45 | 15 | 15 |
| | | | 7 | 27 | 9 | 9 |
| | | | 19 | 18 | 6 | 6 |

| $G$ | $M(G)$ | $Z$ | $p$ | $\dim_k(HH^1(k\hat{G}))$ | $\dim_k(HH^1(kG))$ | $\dim_k(HH^1(k_\alpha G))$ |
|---|---|---|---|---|---|---|
| $A_{11}$ | $C_2$ | $C_2$ | 3 | 37 | 20 | 17 |
| | | | 5 | 13 | 7 | 6 |
| | | | 7 | 7 | 4 | 3 |
| | | | 11 | 4 | 2 | 2 |
| $HS$ | $C_2$ | $C_2$ | 3 | 10 | 5 | 5 |
| | | | 5 | 15 | 8 | 7 |
| | | | 7 | 2 | 1 | 1 |
| | | | 11 | 4 | 2 | 2 |
| $G_2(4)$ | $C_2$ | $C_2$ | 3 | 24 | 14 | 10 |
| | | | 5 | 22 | 12 | 10 |
| | | | 7 | 6 | 3 | 3 |
| | | | 13 | 4 | 2 | 2 |
| $M_{22}$ | $C_{12}$ | $C_2$ | 3 | 6 | 3 | 3 |
| | | | 5 | 2 | 1 | 1 |
| | | | 7 | 4 | 2 | 2 |
| | | | 11 | 4 | 2 | 2 |
| | | $C_3$ | 2 | 29 | 9 | 10 |
| | | | 5 | 3 | 1 | 1 |
| | | | 7 | 6 | 2 | 2 |
| | | | 11 | 6 | 2 | 2 |
| | | $C_4$ | 3 | 10 | 3 | 2 |
| | | | 5 | 4 | 1 | 1 |
| | | | 7 | 8 | 2 | 2 |
| | | | 11 | 8 | 2 | 2 |
| | | $C_6$ | 5 | 6 | 1 | 1 |
| | | | 7 | 12 | 2 | 2 |
| | | | 11 | 12 | 2 | 2 |
| | | $C_{12}$ | 5 | 12 | 1 | 1 |
| | | | 7 | 24 | 2 | 2 |
| | | | 11 | 24 | 2 | 2 |

# Appendix A

# Appendix

## A.1 Using GAP to calculate the dimension of the first Hochschild cohomology group of a finite group algebra

Throughout this section let $G$ be a finite group and $k$ a field of characteristic $p$ dividing the order of $G$. The following work was first written by the author in [60], and details how one can quickly and easily run a basic GAP code to first find a description of the centraliser decomposition of $G$ and then find the $k$-vector space dimension of $HH^1(kG)$.

Once a group $G$ is defined in GAP, the command `div` below gives the set of prime divisors of $|G|$, $\pi(G)$. The functions following `div` then take as input either $G$, or $G$ and a prime $p \in \pi(G)$, and output the following lists;

1. The centralisers of a complete set of conjugacy class representatives of $G$,
2. For each conjugacy class representative $x \in G/\sim$, the groups $C_G(x)/O^p(C_G(x)) = R$,
3. The elementary abelian $p$-groups $R/\Phi(R)$,
4. The rank of $R/\Phi(R)$,

respectively. The final function then produces a list of dimensions $\dim_k(HH^1(kG))$, with one entry for each $p \in \pi(G)$.

```
div:=PrimeDivisors(Size(G));

ListFpCentralisers:=function(X);
> return List(ConjugacyClasses(X),i->
        Image(IsomorphismFpGroup(Centralizer(X,Representative(i)))));
> end;;

ListMaxPQuot:=function(X,p);
> return List(ListFpCentralisers(X),i -> Image(EpimorphismPGroup(i,p)));
> end;;
```

```
ListElAb:=function(X,p);
> return List(ListMaxPQuot(X,p), i -> i/FrattiniSubgroup(i));
> end;;

ListPRank:=function(X,p);
> return List(ListElAb(X,p), i -> RankPGroup(i));
> end;;

dimHH1:=function(X);
> return List(div,i->Sum(ListPRank(X,i)));
> end;
```

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