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ABSTRACT
We extend a result due to Kawai on block varieties for blocks with abelian defect groups to blocks with arbitrary defect groups. Kawai's result is a tool to calculate the cohomology variety of a module in a block B of a finite group algebra kG restricted to subgroups of a defect group P, provided that P is abelian. Kawai's result coincides with a Theorem of Avrunin and Scott specialized to modules in the principal block and their restrictions to p-subgroups. J. Rickard raised the question whether Kawai's result can be extended to modules in blocks with arbitrary defect groups. We show that this is indeed the case for modules whose corresponding module over some almost source algebra is fusion stable. We show that this fusion stability hypothesis is automatically satisfied for principal blocks and blocks with abelian defect groups.

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1. Introduction

Throughout this paper, k is an algebraically closed field of prime characteristic p. Given a finite group G, we set $H^*(G) = H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ and denote by $\mathcal{V}_G$ the maximal ideal spectrum of $H^*(G)$. For H a subgroup of G, denote by $\mathrm{res}^G_H : H^*(G) \to H^*(H)$ the restriction map and by $(\mathrm{res}^G_H)^* : \mathcal{V}_H \to \mathcal{V}_G$ the induced map on varieties. For M a finitely generated kG-module, denote by $I_G(M)$ the kernel of the algebra homomorphism $H^*(G) \to \mathrm{Ext}^*_{kG}(M,M)$ induced by the functor $M \otimes_k -$ on the category $\mathrm{mod}(kG)$ of finitely generated kG-modules. Denote by $\mathcal{V}_G(M)$ the closed homogeneous subvariety of $\mathcal{V}_G$ of all maximal ideals of $H^*(G)$ which contain $I_G(M)$. The systematic study of varieties that arise in this way was initiated by Carlson [9, 10], and subsequently taken up by a long list of authors; see Benson [5, Chapter 5] for an overview of what is needed in this paper, as well as further references. The map $(\mathrm{res}^G_H)^*$ sends $\mathcal{V}_H(\mathrm{Res}^G_H(M))$ to $\mathcal{V}_G(M)$, and hence $\mathcal{V}_H(\mathrm{Res}^G_H(M))$ is contained in the inverse image of $\mathcal{V}_G(M)$ under the map $(\mathrm{res}^G_H)^*$. By a result of Avrunin and Scott [2, Theorem 3.1], this inclusion is an equality; that is, we have

$$\mathcal{V}_H(\mathrm{Res}^G_H(M)) = ((\mathrm{res}^G_H)^*)^{-1}(\mathcal{V}_G(M)).$$

Kawai proved in [14, Proposition 5.2] a version of this result for block varieties of modules in blocks with abelian defect groups, and Rickard raised the question whether such a result holds for blocks in general. The purpose of this paper is to extend Kawai's result to a statement on blocks with arbitrary defect groups which at least partially answers Rickard's question, and identifies the main issue—fusion stability—that remains for a complete answer.

Given a block B of kG with a defect group P, an almost source idempotent $i \in B^P$ and associated fusion system $\mathcal{F}$ on P, we denote by $H^*(B)$ the block cohomology; this is the subalgebra of all $\mathcal{F}$-stable elements in $H^*(P)$ (cf. [16, Definition 5.1] or Definition 3.3). We denote by $\mathcal{V}_B$ the maximal ideal spectrum of $H^*(B)$. For Q a subgroup of P, we denote by $r_Q : H^*(B) \to H^*(Q)$ the composition of the inclusion

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$H^*(B) \to H^*(P)$ and the restriction map $\mathrm{res}^P_Q : H^*(P) \to H^*(Q)$. We denote by $r^*_Q : \mathcal{V}_Q \to \mathcal{V}_B$ the map on varieties induced by r_Q. For M a finitely generated B-module, set

$$\mathcal{V}_B(M) = r^*_P(\mathcal{V}_P(iM)).$$

By [14, Corollary 1.2] or [18, Theorem 2.1], this definition is equivalent to the original definition of block varieties in [17, Definition 4.1], and by Lemma 4.1, this definition depends not on i but only on the underlying choice of a maximal B-Brauer pair. Restriction from G to P induces a map $H^*(G) \to H^*(B)$, which in turn induces a finite surjective morphism of varieties $\mathcal{V}_B(M) \to \mathcal{V}_G(M)$. If B is the principal block, this is an isomorphism (see [17, Section 4]). The main motivation for considering $\mathcal{V}_B(M)$ rather than $\mathcal{V}_G(M)$ is that $\mathcal{V}_B(M)$ is invariant under splendid derived and stable equivalences of Morita type between blocks (cf. [17, Theorem 5.5]), while $\mathcal{V}_G(M)$ is in general not even invariant under splendid Morita equivalences (see [18, Section 5]). We have an obvious inclusion

$$\mathcal{V}_P(iM) \subseteq (r^*_P)^{-1}(\mathcal{V}_B(M)).$$

Kawai proved in [14, Proposition 5.2] that if P is abelian, then this inclusion is an equality. We are going to show in Theorem 1.1 that this inclusion becomes an equality for arbitrary P if the kP-module iM is *$\mathcal{F}$-stable* (cf. Definition 2.1), or more generally, if we replace iM by a suitable $\mathcal{F}$-stable kP-module having iM as a direct summand. We will see in Corollary 1.6 that this implies the aforementioned Theorem by Avrunin and Scott provided that M is in the principal block of kG and H is a p-subgroup of G. We refer to Proposition 2.2 for the notion of an $\mathcal{F}$-characteristic P-P-biset. The proofs of the following statements are given in Section 5.

Theorem 1.1. *Let G be a finite group, B a block of kG, P a defect group of B, i an almost source idempotent in B^P, and let $\mathcal{F}$ be the fusion system on P determined by i. Let X be an $\mathcal{F}$-characteristic P-P-biset, and let M be a finitely generated B-module. For every subgroup Q of P we have*

$$\mathcal{V}_Q(kX \otimes_{kP} iM) = (r^*_Q)^{-1}(\mathcal{V}_B(M)).$$

Since X has an orbit isomorphic to P as P-P-biset, it follows that iM is isomorphic to a direct summand of $kX \otimes_{kP} iM$ as a kP-module. Thus we have an inclusion of varieties

$$\mathcal{V}_P(iM) \subseteq \mathcal{V}_P(kX \otimes_{kP} iM).$$

We do not have an example where this inclusion is proper. If iM is $\mathcal{F}$-stable as a kP-module (cf. Definition 2.1), then, by Lemma 4.3, this inclusion is an equality. We also have no example where iM fails to be $\mathcal{F}$-stable.

Corollary 1.2. *Let G be a finite group, B a block of kG, P a defect group of B, and i an almost source idempotent in B^P. Let $\mathcal{F}$ be the fusion system on P determined by i, and let M be a finitely generated B-module. Suppose that the kP-module iM is $\mathcal{F}$-stable. For every subgroup Q of P we have*

$$\mathcal{V}_Q(iM) = (r^*_Q)^{-1}(\mathcal{V}_B(M)).$$

It remains an open question whether there is always at least some almost source idempotent i with the property that iM is fusion-stable for every finitely generated B-module M. We will see in Proposition 4.7 that this is the case if iBi has a P-P-stable k-basis consisting of invertible elements in iBi.

Corollary 1.3. *Let G be a finite group, B a block of kG, P a defect group of B, and i an almost source idempotent in B^P. Let $\mathcal{F}$ be the fusion system on P determined by i. Suppose that iBi has a P-P-stable k-basis contained in $(iBi)^\times$. Let M be a finitely generated B-module M. Then the kP-module iM is $\mathcal{F}$-stable, and for any subgroup Q of P we have*

$$\mathcal{V}_Q(iM) = (r^*_Q)^{-1}(\mathcal{V}_B(M)).$$

Barker and Gelvin conjectured in [3], that every block B with a defect group P should indeed have an almost source algebra with a P-P-stable basis consisting of invertible elements. If true, this would imply that every block B with a defect group P has an almost source idempotent i with the property that iM is an $\mathcal{F}$-stable kP-module, for every finitely generated B-module M, where $\mathcal{F}$ is the fusion system on P determined by i. As mentioned before, this would in turn imply that $\mathcal{V}_P(iM) = \mathcal{V}_P(kX \otimes_{kP} iM)$, by Lemma 4.3. If $\mathcal{F} = N_{\mathcal{F}}(P)$ and i is an actual source idempotent, then it is easy to show that iM is $\mathcal{F}$-stable for any finitely generated B-module M. We deduce the following result.

Corollary 1.4. *Let G be a finite group, B a block of kG, P a defect group of B, and i an almost source idempotent in B^P. Let $\mathcal{F}$ be the fusion system on P determined by i. Suppose that $\mathcal{F} = N_{\mathcal{F}}(P)$. Let M be a finitely generated B-module M. Then, for any subgroup Q of P, we have*

$$\mathcal{V}_Q(iM) = (r_Q^*)^{-1}(\mathcal{V}_B(M)).$$

It is well-known that if P is abelian, then $\mathcal{F} = N_{\mathcal{F}}(P)$. Thus we obtain Kawai's result mentioned above:

Corollary 1.5 (Kawai [14, Proposition 5.2]). *Let G be a finite group, B a block of kG, P a defect group of B, and i an almost source idempotent in B^P. Suppose that P is abelian. Then for any finitely generated B-module M and any subgroup Q of P we have*

$$\mathcal{V}_Q(iM) = (r_Q^*)^{-1}(\mathcal{V}_B(M)).$$

A block B of kG is *of principal type* if $\mathrm{Br}_Q(1_B)$ is a block of $kC_G(Q)$, for every subgroup Q of P. If B is a block of principal type, then 1_B is an almost source idempotent. Brauer's Third Main Theorem (see e.g. [20, Theorem 6.3.14]) implies that the principal block of kG is of principal type, and hence the principal block idempotent is an almost source idempotent.

Corollary 1.6. *Let G be a finite group, B a block of kG, and P a defect group of B. Suppose that B is of principal type. Then for any finitely generated B-module M and any subgroup Q of P we have*

$$\mathcal{V}_Q(M) = (r_Q^*)^{-1}(\mathcal{V}_B(M)).$$

Corollary 1.6 applies of course also to the principal block B_0 of kG. In that case the block variety $\mathcal{V}_{B_0}(M)$ coincides with the cohomology variety $\mathcal{V}_G(M)$, and hence Corollary 1.6 for prinicipal blocks follows directly from the result [2, Theorem 3.1] of Avrunin and Scott.

It is shown in [6, Theorem 1.1] that if M is indecomposable, then there is a choice of a vertex-source pair (Q, U) of M such that $\mathcal{V}_B(M) = r_Q^*(\mathcal{V}_Q(U))$. For such a choice of (Q, U) we have $\mathcal{V}_Q(U) \subseteq (r_Q^*)^{-1}(\mathcal{V}_B(M))$. This inclusion need not be an equality in general, but it becomes an equality if we replace U by the kQ-module $kX \otimes_{kQ} U$.

Theorem 1.7. *With the notation of Theorem 1.1, suppose that i is a source idempotent and that the B-module M is indecomposable. Let (Q, U) be a vertex-source pair of M such that $Q \leq P$, such that U is isomorphic to a direct summand of iM as a kQ-module, and such that M is isomorphic to a direct summand of $Bi \otimes_{kQ} U$. Regard kX as a kQ-kQ-bimodule. Then we have*

$$\mathcal{V}_Q(kX \otimes_{kQ} U) = (r_Q^*)^{-1}(\mathcal{V}_B(M)).$$

By [15, Proposition 6.3], any indecomposable B-module M has a vertex-source pair (Q, U) satisfying the hypotheses of Theorem 1.7. There are examples where the inclusion $\mathcal{V}_Q(U) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} U)$ is proper, and so tensoring U by kX over kQ in Theorem 1.7 is essential. See Example 6.3.

The strategy to prove Theorem 1.1 is as follows. We first observe that it suffices to prove Theorem 1.1 for $Q = P$. We then apply the Quillen stratification for block module varieties from [18] and adapt the steps in the proof of Kawai's result [14, Proposition 5.2] to the situation at hand.

2. Background on characteristic bisets

Definition 2.1 (cf. [21, Definition 3.3.(1)]). Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A kP-module U is called *$\mathcal{F}$-stable* if for every subgroup Q of P and every morphism $\varphi : Q \to P$ we have an isomorphism of kQ-modules ${}_{\varphi}U \cong \mathrm{Res}^P_Q(U)$. Here ${}_{\varphi}U$ is the kQ-module which is equal to U as a k-vector space, with $u \in Q$ acting as $\varphi(u)$.

For Q a subgroup of a finite group P and $\varphi : Q \to P$ an injective group homomorphism, we denote by $P \times_{(Q,\varphi)} P$ the transitive P-P-biset which is the quotient of $P \times P$ by the equivalence relation $(uv, w) \sim (u, \varphi(v)w)$, where $u, w \in P$ and $v \in Q$. The stabilizer of the image of $(1, 1)$ in the set $P \times_{(Q,\varphi)} P$, regarded as a $P \times P$-set, is the twisted diagonal subgroup $\Delta_{\varphi}(Q) = \{(u, \varphi(u)) \mid u \in Q\}$. In particular, P acts freely on the left and on the right of the set $P \times_{(Q,\varphi)} P$, and the cardinality of this set is $|P| \cdot |P : Q|$. The following result is due to Broto, Levi, and Oliver.

Proposition 2.2. [7, Proposition 2.5] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. There is a finite P-P-biset X with the following properties:*

(i) *Every transitive P-P-subbiset of X is of the form $P \times_{(Q,\varphi)} P$ for some subgroup Q of P and some $\varphi \in$* $\mathrm{Hom}_{\mathcal{F}}(Q, P)$.
(ii) *$|X|/|P|$ is prime to p.*
(iii) *For any subgroup Q of P and any $\varphi : Q \to P$ we have an isomorphism of Q-P-bisets ${}_{\varphi}X \cong {}_{Q}X$ and an isomorphism of P-Q-bisets $X_{\varphi} \cong X_Q$.*

Here ${}_{\varphi}X$ is the Q-P-biset which as a right P-set is equal to X, with $u \in Q$ acting on the left as $\varphi(u)$ on X. The P-Q-biset X_{φ} is defined analogously. The properties (i) and (iii) of X in Proposition 2.2 do not change if we replace X by a disjoint union of finitely many copies of X, and therefore there exists a biset X satisfying the properties (i), (iii) and (ii) replaced by the stronger requirement $|X|/|P| \equiv 1 \pmod p$. Since a P-P-biset of the form $P \times_{(Q,\varphi)} P$ has cardinality $|P| \cdot |P : Q|$, it follows that

$$|X|/|P| \equiv n(X) \pmod p,$$

where $n(X)$ is the number of P-P-orbits in X of length $|P|$. A P-P-biset X satisfying Proposition 2.2 is called an *$\mathcal{F}$-characteristic biset*. (Some authors use this term for bisets satisfying some additional properties; see e. g. [3, Definition 2.1].) Given two P-P-bisets X, X', we denote by $X \times_P X'$ the quotient of the set $X \times X'$ by the equivalence relation $(xu, x') \sim (x, ux')$, where $x \in X$, $x' \in X'$, and $u \in P$. The left and right action of P on $X \times_P X'$ is induced by the left and right action of P on X and X' respectively. We have an obvious kP-kP-bimodule isomorphism $kX \otimes_{kP} kX' \cong k(X \times_P X')$. We record some elementary observations for future reference.

Lemma 2.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite P-group. Let X, X' be $\mathcal{F}$-characteristic P-P-bisets, and let Y be a P-P-biset satisfying the properties (i) and (ii) of Proposition 2.2. Then the P-P-bisets $X \times_P X'$ and $X \times_P Y \times_P X'$ are $\mathcal{F}$-characteristic bisets. Moreover, the P-P-bisets X and X' are isomorphic to subbisets of $X \times_P X'$.*

Proof. Let Q, R be subgroups of P and $\varphi : Q \to P$ and $\psi : R \to P$ morphisms in $\mathcal{F}$. Using the double coset decomposition $\varphi(Q)\backslash P/R$, an easy verification shows that $(P \times_{(Q,\varphi)} P) \times_P (P \times_{(R,\psi)} P)$ is a unions of P-P-orbits of the form $P \times_{(S,\tau)} P$ for some subgroup S of P and some morphism $\tau : S \to P$. This implies that the bisets $X \times_P X'$ and $X \times_P Y \times_P X'$ satisfy property (i) of Proposition 2.2. One easily checks that $n(X \times_P X') = n(X) \cdot n(X')$ and the analogous statement for $X \times_P Y \times_P X'$, which implies that the bisets $X \times_P X'$ and $X \times_P Y \times_P X'$ satisfy property (ii) of Proposition 2.2, and clearly these two sets inherit property (iii) of Proposition 2.2 from X and X'. The last statement follows from the fact that X and X' have an orbit isomorphic to P as a P-P-biset. □

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let U be a finitely generated kP-module. Let X be an $\mathcal{F}$-characteristic P-P-biset. If U is $\mathcal{F}$-stable, then the last statement in the following lemma shows that every indecomposable direct summand of $kX \otimes_{kP} U$ is a summands of $kP \otimes_{kQ} U$, for some subgroup Q of P.

Lemma 2.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let X be an $\mathcal{F}$-characteristic P-P-biset. Let U be a finitely generated kP-module.*

(i) *The P-P-biset X has an orbit isomorphic to P as a P-P-biset.*
(ii) *The kP-module $kX \otimes_{kP} U$ has a direct summand isomorphic to U.*
(iii) *Let Q, R be subgroups of P, let S be a subgroup of Q, and let $\varphi : S \to R$ be a morphism in $\mathcal{F}$. Set $Y = Q \times_{(S,\psi)} R$. Then $Y \times_R X \cong Q \times_S X$ as Q-P-bisets, and $kY \otimes_{kR} kX \cong kQ \otimes_{kS} kX$ as kQ-kP-bimodules.*
(iv) *The kP-module $kX \otimes_{kP} U$ is $\mathcal{F}$-stable.*
(v) *For any subgroup Q of P and any morphism $\varphi : Q \to P$ in $\mathcal{F}$ the kQ-module ${}_{\varphi}U$ is isomorphic to a direct summand of $\mathrm{Res}^P_Q(kX \otimes_{kP} U)$.*
(vi) *If U is $\mathcal{F}$-stable, then any indecomposable direct summand of the kP-module $kX \otimes_{kP} U$ is isomorphic to a direct summand of $kP \otimes_{kQ} U$ for some subgroup Q of P.*

Proof. Since $|X|/|P|$ is prime to p by Proposition 2.2 (ii), it follows that X has an orbit of length $|P|$. By Proposition 2.2 (i), such an orbit is isomorphic to ${}_{\varphi}P$ for some $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$. It follows from Proposition 2.2 (iii) that X has also an orbit isomorphic to P. This shows (i). It follows from (i) that kX has a direct summand isomorphic to kP as a kP-kP-bimodule, which implies (ii). The statements (iii) and (iv) follow from Proposition 2.2 (iii). Since U is isomorphic to a direct summand of $kX \otimes_{kP} U$ as a kP-module, it follows that ${}_{\varphi}U$ is isomorphic to a direct summand of ${}_{\varphi}kX \otimes_{kP} U \cong \mathrm{Res}^P_Q(kX \otimes_{kP} U)$ as a kQ-module, where the last isomorphism uses the fusion stability property from Proposition 2.2 (iii). This shows (v). By Proposition 2.2 (i), every indecomposable direct summand of $kX \otimes_{kP} U$ is isomorphic to a direct summand of $kP \otimes_{kQ} {}_{\varphi}U$ for some subgroup Q of P and some morphism $\varphi : Q \to P$ in $\mathcal{F}$. Since U is assumed to be $\mathcal{F}$-stable, we have $kP \otimes_{kQ} {}_{\varphi}U \cong kP \otimes_{kQ} U$. Statement (vi) follows. □

Example 2.5. Let G be a finite group and P a Sylow p-subgroup of G. Let $\mathcal{F}$ be the fusion system of G on P; that is, the objects of $\mathcal{F}$ are the subgroups of P and morphisms in $\mathcal{F}$ between two subgroups of P are the injective group homomorphisms induced by conjugation in G. A trivial verification shows that $X = G$, regarded as a P-P-biset, is an $\mathcal{F}$-characteristic biset. In that case we have $kX \otimes_{kP} U = \mathrm{Res}^G_P(\mathrm{Ind}^G_P(U))$, where U is a kP-module. Another easy argument shows that the restriction to kP of any finitely generated kG-module M is $\mathcal{F}$-stable; see the proof of Lemma 4.10.

3. Background on block cohomology varieties

For general background on cohomology varieties see [4, Section 2.25ff], [5, Chapter 5], [11, Chapter 9], and [13, Chapter 8]. For the block analogues of cohomology varieties, see [17]. We need the following well-known facts.

Proposition 3.1. [13, Propositions 8.2.1, 8.2.4], [4, Theorem 2.26.9] *For any subgroup Q of a finite group P, any finitely generated kP-module U and any finitely generated kQ-module V we have*

$$(\mathrm{res}^P_Q)^*(\mathcal{V}_Q(\mathrm{Res}^P_Q(U))) \subseteq \mathcal{V}_P(U),$$

$$(\mathrm{res}^P_Q)^*(\mathcal{V}_Q(V)) = \mathcal{V}_P(\mathrm{Ind}^P_Q(V)),$$

$$\mathcal{V}_P(\mathrm{Ind}^P_Q(\mathrm{Res}^P_Q(U))) \subseteq \mathcal{V}_P(U).$$

We adopt the following abuse of notation: if Q is a subgroup of a finite group P and U a finitely generated kP-module, then we write $\mathcal{V}_Q(U)$ instead of $\mathcal{V}(\mathrm{Res}^P_Q(U))$. The third inclusion in Proposition 3.1 is obviously equivalent to the inclusion

$$\mathcal{V}_P(kP \otimes_{kQ} U) \subseteq \mathcal{V}_P(U).$$

We briefly review block theoretic background, much of which is from [1, 8, 22], referring to [19, 20] for an expository account. We assume familiarity with relative trace maps, the Brauer homomorphism (cf. [19, Theorem 5.4.1]), and (local) pointed groups on G-algebras. One useful technical consequence of Puig's version [19, Theorem 5.12.20] of Green's Indecomposability Theorem [19, Theorem 5.12.3] is the following observation.

Lemma 3.2. *Let G be a finite group, P a p-subgroup of G, and i a primitive idempotent in $(kG)^P$. Let Q be a subgroup of P which is maximal such that $\mathrm{Br}_Q(i) \neq 0$. Then there is a primitive idempotent $j \in i(kG)^Q i$ such that $\mathrm{Br}_Q(j) \neq 0$ and such that*

$$ikG \cong kP \otimes_{kQ} jkG$$

as kP-kG-bimodules.

Let G be a finite group and B a block of kG; that is, $B = kGb$ for some primitive idempotent b in $Z(kG)$. Thus b is the unit element of B, called the block idempotent of B. Let P be a defect group of B; that is, P is a maximal p-subgroup of G such that kP is isomorphic to a direct summand of B as a kP-kP-bimodule. Equivalently, P is a maximal p-subgroup of G such that $\mathrm{Br}_P(b) \neq 0$. An idempotent $i \in B^P$ is a source idempotent of B if i is a primitive idempotent in the algebra B^P of P-fixed points in B with respect to the conjugation action of P on B, such that $\mathrm{Br}_P(i) \neq 0$, where $\mathrm{Br}_P : (kG)^P \to kC_G(P)$ is the Brauer homomorphism . One of the key properties of a source idempotent i in B^P is that for each subgroup Q of P there is a unique block idempotent e_Q of $kC_G(Q)$ such that $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i) \neq 0$ (cf. [20, Theorem 6.3.3]). More generally, a (not necessarily primitive) idempotent i in B^P is called an almost source idempotent if for each subgroup Q of P there is a unique block idempotent e_Q of $kC_G(Q)$ such that $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i) \neq 0$. By the above, a source idempotent is an almost source idempotent. If i is an almost source idempotent in B^P, then $i = i_0 + i_1$ for some source idempotent i_0 in B^P and some idempotent i_1 in B^P which is orthogonal to i_0. The local point of P containing i_0 is uniquely determined by e_P, hence by i. The extra flexibility of the notion of almost source idempotents is particularly useful if B is the principal block of kG, because - as mentioned earlier - in that case the block idempotent 1_B is an almost source idempotent.

The choice of an almost source idempotent i in B^P determines a fusion system $\mathcal{F} = \mathcal{F}_B(P)$ on P as follows. For Q a subgroup of P, denote by e_Q the unique block idempotent of $kC_G(Q)e_Q$ satisfying $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i) \neq 0$. The objects of $\mathcal{F}$ are the subgroups of P. For two subgroups Q, R of P, a group homomorphism $\varphi : Q \to R$ is a morphism in $\mathcal{F}$ if and only if there exists an element $x \in G$ such that $xQx^{-1} \leq R$, $xe_Qx^{-1} = e_{xQx^{-1}}$, and $\varphi(u) = xux^{-1}$ for all $u \in Q$. See [20, Section 8.5] for more details on fusion systems of blocks and [12] for a general introduction to fusion systems. By the results in [23], the fusion system $\mathcal{F}$ of B defined in this way can be read off the almost source algebra iBi of B; see [20, Theorem 8.7.4]. A subgroup Q of P is *fully $\mathcal{F}$-centralised* if $|C_P(Q)| \geq |C_P(Q')|$ for any subgroup Q' of P which is isomorphic to Q in $\mathcal{F}$. By [20, Proposition 8.5.3], Q is fully $\mathcal{F}$-centralised if and only if $C_P(Q)$ is a defect group of the block $kC_G(Q)e_Q$.

Definition 3.3. [17, Definition 4.1] With the notation above, the block cohomology $H^*(B)$ is the graded subalgebra of $H^*(P)$ consisting of all $\zeta \in H^*(P)$ satisfying for every morphism $\varphi : Q \to R$ in $\mathcal{F}$ the equality $\mathrm{res}^P_Q(\zeta) = \mathrm{res}_\varphi(\mathrm{res}^P_R(\zeta))$. Here $\mathrm{res}_\varphi : H^*(R) \to H^*(Q)$ is the map induced by restriction along the injective group homomorphism $\varphi : Q \to R$.

In other words, $H^*(B)$ is the limit of the contravariant functor on $\mathcal{F}$ sending a subgroup Q of P to $H^*(Q)$ and a morphism $\varphi : Q \to R$ in $\mathcal{F}$ to the induced map $\mathrm{res}_\varphi : H^*(R) \to H^*(Q)$. If B is the principal block of kG, then $H^*(B) \cong H^*(G)$. As mentioned in the introduction, for Q a subgroup of P, we denote by $r_Q : H^*(B) \to H^*(Q)$ the composition of the inclusion $H^*(B) \to H^*(P)$ and the restriction map $\mathrm{res}^P_Q : H^*(P) \to H^*(Q)$.

Lemma 3.4. *With this notation, the following hold for every morphism $\varphi : Q \to R$ in $\mathcal{F}$.*

(i) *We have a commutative diagram of graded algebras*

$$\begin{tikzcd} H^*(R) \arrow[rr, "\mathrm{res}_\varphi"] & & H^*(Q) \\ & H^*(B) \arrow[ul, "r_Q"] \arrow[ur, "r_R"'] & \end{tikzcd}$$

and $H^*(B)$ is universal with this property.

(ii) *The diagram (i) induces a commutative diagram of varieties*

$$\begin{tikzcd} \mathcal{V}_Q \arrow[rr, "\mathrm{res}^*_\varphi"] \arrow[dr, "r^*_Q"'] & & \mathcal{V}_R \arrow[dl, "r^*_R"] \\ & \mathcal{V}_B & \end{tikzcd}$$

(iii) *This diagram in (ii) restricts for any finitely generated kR-module W to a commutative diagram of the form*

$$\begin{tikzcd} \mathcal{V}_Q({}_\varphi W) \arrow[rr, "\mathrm{res}^*_\varphi"] \arrow[dr, "r^*_Q"'] & & \mathcal{V}_R(W) \arrow[dl, "r^*_R"] \\ & \mathcal{V}_B & \end{tikzcd}$$

Proof. Statement (i) is just a reformulation of the definition of $H^*(B)$ as the limit of the functor $Q \mapsto H^*(Q)$ on $\mathcal{F}$. Statement (ii) follows from (i) by passing to maximal ideal spectra, and (iii) is an immediate consequence of (ii). □

For Q a subgroup of P and a finitely generated B-module M set

$$\mathcal{V}_Q^+ = \mathcal{V}_Q \setminus \cup_R \, (\mathrm{res}^Q_R)^*(\mathcal{V}_R)$$

where in the union R runs over the proper subgroups of Q. Set $\mathcal{V}_Q^+(iM) = \mathcal{V}_Q^+ \cap \mathcal{V}_Q(iM)$. The idempotent i need no longer be primitive in B^Q. If J is a primitive decomposition of i in B^Q, then $iM = \oplus_{j\in J} jM$ is a decomposition of iM as a direct sum of kQ-modules. Thus we have

$$\mathcal{V}_Q(iM) = \cup_{j\in J} \, \mathcal{V}_Q(jM)$$

For $j \in J$ set $\mathcal{V}_Q^+(jM) = \mathcal{V}_Q(jM) \cap \mathcal{V}_Q^+$. If $j \in J$ belongs to $\ker(\mathrm{Br}_Q)$, then jM is relatively R-projective for some proper subgroup R of Q, and hence $\mathcal{V}_Q(jM) \subseteq (\mathrm{res}^Q_R)^*(\mathcal{V}_R)$ in that case. Thus

$$\mathcal{V}_Q^+(iM) = \cup_{j\in J^+} \, \mathcal{V}_Q^+(jM)$$

where J^+ is the subset of all $j \in J$ satisfying $\mathrm{Br}_Q(j) \neq 0$ (or equivalently, all $j \in J$ belonging to a local point of Q on iBi). If Q is fully $\mathcal{F}$-centralised, then the conjugation action by $N_G(Q, e_Q)$ on B permutes the local points of Q on iBi, and hence induces an action of the group $\mathrm{Aut}_\mathcal{F}(Q) \cong N_G(Q, e_Q)/C_G(Q)$ on $\mathcal{V}_Q^+(iM)$ (cf. [17, Lemma 4.11]).

We define further the following subvarieties of $\mathcal{V}_B$. We set

$$\mathcal{V}_{B,Q}(M) = r_Q^*(\mathcal{V}_Q(iM)),$$

$$\mathcal{V}_{B,Q}^+(M) = r_Q^*(\mathcal{V}_Q^+(iM)) = \cup_{j\in J^+} r_Q^*(\mathcal{V}_Q(jM)).$$

Denote by $\mathcal{E}$ a set of representatives of the $\mathcal{F}$-isomorphism classes of fully $\mathcal{F}$-centralised elementary abelian subgroups of P. The block variety version of Quillen's cohomology stratification states the following.

Theorem 3.5 (cf. [17, Theorem 4.2]). *With the notation above, the following hold.*

(i) *The variety $\mathcal{V}_B(M)$ is a disjoint union*

$$\mathcal{V}_B(M) = \cup_{E\in\mathcal{E}} \mathcal{V}_{B,E}^+(M).$$

(ii) *For each $E \in \mathcal{E}$, the group $\mathrm{Aut}_{\mathcal{F}}(E)$ acts on the variety $\mathcal{V}_E^+(iM)$ and the map r_E^* induces an inseparable isogeny $\mathcal{V}_E^+(iM)/\mathrm{Aut}_{\mathcal{F}}(E) \to \mathcal{V}_{B,E}^+(M)$.*

The decomposition in Theorem 3.5 (i) does not depend on the choice of $\mathcal{E}$; this follows for instance from [17, Lemma 4.7].

4. Almost source idempotents and fusion stable bisets

Let G be a finite group, B a block of kG, P a defect group of B and i an almost source idempotent in B^P, and $\mathcal{F}$ the fusion system of B on P determined by i. Let if i_0 be a source idempotent of B which is contained in iB^Pi (or equivalently, which satisfies $i_0 i = i_0 = ii_0$).

As mentioned above, by [14, Corollary 1.2] or [18, Theorem 2.1], the block variety $\mathcal{V}_B(M)$ of a finitely generated B-module M is equal to $r_P^*(\mathcal{V}_P(i_0M))$, The next Lemma shows that we may use i to calculate $\mathcal{V}_B(M)$. Note that i_0 determines the same fusion system $\mathcal{F}$ on P because $\mathcal{F}$ depends only on the blocks e_Q of $kC_G(Q)$ satisfying $\mathrm{Br}_Q(i)e_Q = \mathrm{Br}_Q(i) \neq 0$, for Q any subgroup of P.

Lemma 4.1. *We have $\mathcal{V}_P(i_0M) \subseteq \mathcal{V}_P(iM)$ and $r_P^*(\mathcal{V}_P(i_0M)) = r_P^*(\mathcal{V}_P(iM)) = \mathcal{V}_B(M)$.*

Proof. Clearly i_0M is a direct summand of iM as a kP-module, whence the first inclusion. Applying r_P^* yields an inclusion of varieties

$$r_P^*(\mathcal{V}_P(i_0M)) \subseteq r_P^*(\mathcal{V}_P(iM)).$$

The left side is the block variety $\mathcal{V}_B(M)$ of M, as noted above. The right side is the union of the varieties $r_P^*(\mathcal{V}_P(i'M))$, where i' runs over a primitive decomposition of i in B^P. Thus, given a primitive idempotent i' in iB^Pi we need to show that $r_P^*(\mathcal{V}_P(i'M))$ is contained in $r_P^*(\mathcal{V}_P(i_0M))$. It follows from Lemma 3.2 that $i'M \cong kP \otimes_{kR} jM$ for some subgroup R of P and some primitive idempotent j in $i'B^Pi'$ satisfying $\mathrm{Br}_R(j) \neq 0$. Thus we have

$$r_P^*(\mathcal{V}_P(i'M)) = r_R^*(\mathcal{V}_R(jM)).$$

If γ' is the point of P on B containing i' and ϵ is the local point of R on B containing j, then R_ϵ is a defect pointed group of $P_{\gamma'}$. Denote by γ the local point of P on B containing i_0. Then R_ϵ is G-conjugate to a local pointed group contained in P_γ. That is, there is $x \in G$ such that

$$R'_{\epsilon'} = {}^xR_\epsilon \leq P_\gamma.$$

Let $j' \in \epsilon'$. Since $R'_{\epsilon'} \leq P_\gamma$ we may choose j' in $i_0B^{R'}i_0$. The map $\varphi : R \to R'$ induced by conjugation with x is a morphism in the fusion system $\mathcal{F}$, because $\mathrm{Br}_R(j)$ and $\mathrm{Br}_{R'}(j')$ are nonzero and belong by construction to the block algebras $kC_G(R)e_R$ and $kC_G(R')e_{R'}$, respectively, so we have ${}^xe_R = e_{R'}$. We clearly have an isomorphism of kR-modules $jM \cong \mathrm{res}_\varphi(j'M)$. The commutative diagram in Lemma 3.4 (iii) implies that

$$r_R^*(\mathcal{V}_R(jM)) = r_{R'}^*(\mathcal{V}_{R'}(j'M).$$

Now $j'M$ is a direct summand of i_0M as a kP-module, and hence we have

$$r_{R'}^*(\mathcal{V}_{R'}(j'M) \subseteq r_{R'}^*(\mathcal{V}_{R'}(i_0M)) = r_P^*((\mathrm{res}^P_{R'})^*(\mathcal{V}_{R'}(i_0M))).$$

By Proposition 3.1 this is contained in $r_P^*(\mathcal{V}_P(i_0M))$, whence the result. □

Lemma 4.2. *Let Q be a subgroup of P and U a finitely generated kQ-module. Let X and X' be $\mathcal{F}$-characteristic P-P-bisets. The following hold.*

(i) *We have $\mathcal{V}_Q(U) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} U)$.*
(ii) *We have $\mathcal{V}_Q(kX \otimes_{kQ} U) = \mathcal{V}_Q(kX' \otimes_{kQ} U)$.*
(iii) *We have $r_Q^*(\mathcal{V}_Q(U)) = r_Q^*(\mathcal{V}_Q(kX \otimes_{kQ} U))$.*

Proof. It follows from Lemma 2.4 (i) that X has a Q-Q-orbit isomorphic to Q, and hence that U is isomorphic to a direct summand of $kX \otimes_{kQ} U$ as a kQ-module. This implies (i). Every Q-P-orbit of X' is of the form $Q \otimes_{(S,\varphi)} P$ for some subgroup S of Q and some morphism $\varphi : S \to P$ in $\mathcal{F}$. Thus, by Lemma 2.4 (iii), every indecomposable direct summand of $kX' \otimes_{kP} kX \otimes_{kQ} U$ as a kQ-module is isomorphic to a direct summand of $kQ \otimes_{kS} kX \otimes_{kQ} U$ for some subgroup S of Q. By Proposition 3.1 we have $\mathcal{V}_Q(kQ \otimes_{kS} kX \otimes_{kQ} U) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} U)$. This shows that $\mathcal{V}_Q(kX' \otimes_{kP} kX \otimes_{kQ} U) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} U)$. By Lemma 2.3, X' is isomorphic to a P-P-subbiset of $X' \times_P X$. Thus $kX' \otimes_{kQ} U$ is isomorphic to a direct summand of $kX' \otimes_{kP} kX \otimes_{kQ} U$ as a kQ-module, and we therefore have $\mathcal{V}_Q(kX' \otimes_{kQ} U) \subseteq \mathcal{V}_Q(kX' \otimes_{kP} kX \otimes_{kQ} U)$. Together we get that $\mathcal{V}_Q(kX' \otimes_{kQ} U) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} U)$. Exchanging the roles of X and X' shows that this inclusion is an equality, whence (ii). By Proposition 2.2 (i), as a kQ-module, $kX \otimes_{kQ} U$ is isomorphic to a direct sum of kQ-modules of the form $kQ \otimes_{kR} {}_\psi U$, with R a subgroup of Q and $\psi : R \to Q$ a morphism in $\mathcal{F}$. By Proposition 3.1 we have

$$\mathcal{V}_Q(kQ \otimes_{kR} {}_\psi U) = (\mathrm{res}^Q_R)^*({}_\psi U).$$

Since $r_R^* = r_Q^* \circ (\mathrm{res}^Q_R)^*$, it follows that

$$r_Q^*(\mathcal{V}_Q(kQ \otimes_{kR} {}_\psi U)) = r_R^*(\mathcal{V}_R({}_\psi U)) = r_{\psi(R)}^*(\mathcal{V}_{\psi(R)}(U))$$

where the last equality uses Lemma 3.4 (iii). Using Proposition 3.1 again we get that

$$r_{\psi(R)}^*\mathcal{V}_{\psi(R)}(U)) = r_R^*((\mathrm{res}^Q_R)^*(\mathcal{V}_R(U))) \subseteq r_Q^*(\mathcal{V}_Q(U)).$$

This proves (iii). □

Lemma 4.3. *Let X be an $\mathcal{F}$-characteristic P-P-biset, and let U be a finitely generated kP-module. If U is $\mathcal{F}$-stable, then $\mathcal{V}_P(U) = \mathcal{V}_P(kX \otimes_{kP} U)$.*

Proof. By Lemma 4.2 we have $\mathcal{V}_P(U) \subseteq \mathcal{V}_P(kX \otimes_{kP} U)$. Assume that U is $\mathcal{F}$-stable. Let U' be an indecomposable direct summand of $kX \otimes_{kP} U$. By Lemma 2.4 (vi), U' is isomorphic to a direct summand of $kP \otimes_{kQ} U$ for some subgroup Q of P. Thus, by Proposition 3.1, we have $\mathcal{V}_P(U') \subseteq \mathcal{V}_P(kP \otimes_{kQ} U) \subseteq \mathcal{V}_P(U)$. This implies $\mathcal{V}_P(kX \otimes_{kQ} U) \subseteq \mathcal{V}_P(U)$. The result follows. □

As a kP-kP-bimodule, iBi is a direct summand of kG. Thus iBi has a P-P-stable k-basis Y.

Lemma 4.4. *Let Y be a P-P-stable basis of iBi. Then Y has a P-P-orbit isomorphic to P, and Y satisfies the property (i) from Proposition 2.2. If in addition i is a source idempotent, then Y satisfies the properties (i) and (ii) from Proposition 2.2.*

Proof. This follows, for instance, from [20, Propositions 8.7.10] together with the fact, due to Puig, that if i is a source idempotent, then $\frac{\dim_k(iBi)}{|P|}$ is prime to p (see e. g. [20, Theorem 6.15.1]). $\square$

It is not known whether i can always be chosen in such a way that Y is an $\mathcal{F}$-characteristic biset. See Proposition 4.7 below for a sufficient criterion for Y to satisfy property (iii) of Proposition 2.2.

Lemma 4.5. *Let X be an $\mathcal{F}$-characteristic P-P-biset, and let Q be a subgroup of P. As a kQ-kP-bimodule, $iBi \otimes_{kP} kX$ is isomorphic to a direct sum of bimodules of the form $kQ \otimes_{kR} kX$, with R running over the subgroups of Q. Moreover, $iBi \otimes_{kP} kX$ has a direct summand isomorphic to kX as a kQ-kP-bimodule.*

Proof. By Lemma 4.4 or by [20, Theorem 8.7.1], as a kQ-kP-bimodule, iBi is isomorphic to a direct sum of bimodules of the form $kQ \otimes_{kR} {}_{\psi}kP$, for some subgroup R of Q and some morphism $\psi : R \to P$ in $\mathcal{F}$. Thus $iBi \otimes_{kP} kX$ is isomorphic to a direct sum of kQ-kP-bimodules of the form $kQ \otimes_{kR} {}_{\psi}kX \cong kQ \otimes_{kR} kX$, where we use the $\mathcal{F}$-stability of X. Since $\mathrm{Br}_P(i) \neq 0$, it follows that iBi has a direct summand isomorphic to kP as a kP-kP-bimodule, hence also as a kQ-kP-bimodule, and therefore $iBi \otimes_{kP} kX$ has a direct summand isomorphic to kX as a kQ-kP-bimodule. The result follows. $\square$

Lemma 4.6. *Let X be an $\mathcal{F}$-characteristic P-P-biset. Let Q be a subgroup of P and W a finitely generated kQ-module. We have*

$$\mathcal{V}_Q(iBi \otimes_{kQ} W) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} W).$$

Proof. Note that kX has a direct summand isomorphic to kP as a kP-kP-bimodule. Thus iBi is isomorphic to a direct summand of $iBi \otimes_{kP} kX$ as a kP-kP-bimodule, hence also as a kQ-kQ-bimodule, and therefore

$$\mathcal{V}_Q(iBi \otimes_{kQ} W) \subseteq \mathcal{V}_Q(iBi \otimes_{kP} kX \otimes_{kQ} W).$$

By Lemma 4.5, as a kQ-module, $iBi \otimes_{kP} kX \otimes_{kQ} W$ is isomorphic to a direct sum of modules of the form $kQ \otimes_{kR} kX \otimes_{kQ} W$ with at least one summand where $R = Q$. Thus the variety $\mathcal{V}_Q(iBi \otimes_{kP} kX \otimes_{kQ} W)$ is contained in the union of varieties of the form $\mathcal{V}_Q(kQ \otimes_{kR} kX \otimes_{kQ} W)$. By Proposition 3.1, these are all contained in $\mathcal{V}_Q(kX \otimes_{kQ} W)$, proving the result. $\square$

Proposition 4.7. *Let G be a finite group, B a block of kG, P a defect group of B and i an almost source idempotent in B^P. Suppose that iBi has a P-P-stable k-basis X which is contained in $(iBi)^\times$. The following hold.*

(i) *If i is a source idempotent, then X is an $\mathcal{F}$-characteristic P-P-biset.*
(ii) *For every subgroup Q of P and any morphism $\varphi : Q \to P$ in $\mathcal{F}$ we have an isomorphism of kQ-B-bimodules ${}_{\varphi}iB \cong iB$.*
(iii) *For every finitely generated B-module M the kP-module iM is $\mathcal{F}$-stable.*

Proof. Statement (i) is proved for instance in [20, Proposition 8.7.11]. Let Q be a subgroup of P and $\varphi : Q \to P$ a morphism in $\mathcal{F}$. By Alperin's Fusion Theorem [20, Theorem 8.2.8], in order to prove (ii) we may assume that Q is $\mathcal{F}$-centric and that φ is an automorphism of Q composed with the inclusion map $Q \leq P$. By [20, Proposition 8.7.10] there exists an element $x \in X$ such that $ux = x\varphi(u)$ for all $u \in Q$. One checks that left multiplication by x on iB is a homomorphism of kQ-B-bimodules ${}_{\varphi}iB \to iB$. Since x is invertible in iBi, this map is an isomorphism, proving (ii). We have $iM \cong iB \otimes_B M$, so (ii) implies (iii). $\square$

It is not known whether every block B with defect group P has at least some almost source idempotent $i \in B^P$ such that the almost source algebra iBi has a P-P-stable basis consisting of invertible elements. See [3] for equivalent reformulations of this problem, as well as a number of cases in which this is true. The following technical observation is a special case of Puig's characterization of fusion in source algebras in [23].

Lemma 4.8. *Let G be a finite group, B a block of kG, P a defect group of B and i a source idempotent in B^P. Denote by $\mathcal{F}$ the fusion system on P determined by i. Let $\varphi \in \mathrm{Aut}(P)$. Then $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ if and only if ${}_{\varphi}iB \cong iB$ as kP-B-bimodules.*

Proof. This is the special case of [20, Theorem 8.7.4.(ii)] applied to the case where $P = Q = R$ and i is an actual source idempotent. □

Proposition 4.9. *Let G be a finite group, B a block of kG, P a defect group of B and i a source idempotent in B^P. Denote by $\mathcal{F}$ the fusion system on P determined by i and suppose that $\mathcal{F} = N_{\mathcal{F}}(P)$. For every finitely generated B-module M the kP-module iM is $\mathcal{F}$-stable.*

Proof. Since $\mathcal{F} = N_{\mathcal{F}}(P)$, it suffices to check the fusion stability condition on iM for automorphisms of P in $\mathcal{F}$. This follows from the obvious kP-isomorphism $iB \otimes_B M \cong iM$ and Lemma 4.8. □

Lemma 4.10. *Let G be a finite group, B a block of kG, P a defect group of B and i an almost source idempotent in B^P. Denote by $\mathcal{F}$ the fusion system on P determined by i. For every finitely generated B-module M the kP-module $\mathrm{Res}^G_P(M)$ is $\mathcal{F}$-stable.*

Proof. Let Q be a subgroup of P and $\varphi : Q \to P$ a morphism in $\mathcal{F}$. Then there exists an element $x \in G$ such that $\varphi(u) = xux^{-1}$ for all $u \in Q$. Then the map sending $m \in M$ to xm is an isomorphism of kQ-modules $\mathrm{Res}^G_Q(M) \cong {}_{\varphi}M$. □

Remark 4.11. Thanks to Lemma 4.1, the variety $\mathcal{V}_B(M) = r_P^*(\mathcal{V}_P(iM))$ does not depend on the choice of an almost source idempotent i. It is, however, not clear, whether the $\mathcal{F}$-stability of iM depends on the choice of i. The proofs of Lemma 4.8 and Proposition 4.9 given above do require i to be an actual source idempotent. Therefore, if i is a source idempotent, then the module iM in the Corollaries 1.4, 1.5 is $\mathcal{F}$-stable, but without that hypothesis we can only deduce the equalities of varieties stated there but not the $\mathcal{F}$-stability. The connection between the $\mathcal{F}$-stability of iM and the existence of P-P-stable bases of invertible elements in iBi as outlined in Proposition 4.7 suggests that the existence of such a basis is likely to also depend on the choice of i. As mentioned earlier, it is not known whether there is an almost source idempotent i such that a P-P-stable basis of iBi is an $\mathcal{F}$-characteristic biset.

5. Proofs

Proof of Theorem 1.1. Set $U = kX \otimes_{kP} iM$. Note that the kP-module U is $\mathcal{F}$-stable. By Lemma 4.3 we have

$$\mathcal{V}_B(M) = r_P^*(\mathcal{V}_P(iM)) = r_P^*(\mathcal{V}_P(U))$$

and hence we have

$$\mathcal{V}_P(U) \subseteq (r_P^*)^{-1}(\mathcal{V}_B(M)).$$

We observe first that it suffices to show Theorem 1.1 for $Q = P$. Indeed, suppose that

$$\mathcal{V}_P(U) = (r_P^*)^{-1}(\mathcal{V}_B(M)).$$

Let Q be a subgroup of P. By [2, Theorem 3.1] we have

$$\mathcal{V}_Q(U) = ((\mathrm{res}^P_Q)^*)^{-1}(\mathcal{V}_P(U).$$

Since $r_Q = \mathrm{res}^P_Q \circ r_P$, it follows from these two equalities that

$$(r_Q^*)^{-1}(\mathcal{V}_B(M)) = ((\mathrm{res}^P)Q)^*)^{-1}((r_P^*)^{-1}(\mathcal{V}_B(M))) = ((\mathrm{res}^P_Q)^*)^{-1}(\mathcal{V}_P(U)) = \mathcal{V}_Q(U).$$

This shows that it suffices to prove Theorem 1.1 for $Q = P$. We need to show that the inclusion $\mathcal{V}_P(U) \subseteq (r_P^*)^{-1}(\mathcal{V}_B(M))$ is an equality. Let $z \in (r_P^*)^{-1}(\mathcal{V}_B(M))$. We need to show that $z \in \mathcal{V}_P(U)$. By choice of z, we have $z \in \mathcal{V}_P$ and $r_P^*(z) \in \mathcal{V}_B(M)$. Quillen's stratification applied to the kP-module U yields

$$\mathcal{V}_P(U) = \cup_E\ (\mathrm{res}^P_E)^*(\mathcal{V}_E^+(U),$$

where E runs over a set of representatives of the conjugacy classes of elementary abelian subgroups of P. This is a disjoint union.

Quillen's stratification applied to $\mathcal{V}_P$ implies that $z \in \mathcal{V}_{P,E}^+ = (\mathrm{res}^P_E)^*(\mathcal{V}_E^+)$ for some elementary abelian subgroup E of P; that is, we have

$$z = (\mathrm{res}^P_E)^*(s)$$

for some $s \in \mathcal{V}_E^+$. Note that E is unique up to conjugation in P and s is unique up to the action of $N_P(Q)$.

We need to show that E and s can be chosen in such a way that $s \in \mathcal{V}_E^+(U)$. The block variety version of Quillen's stratification, reviewed in Theorem 3.5 and preceding paragraphs, implies that

$$r_P^*(z) = r_F^*(t)$$

for some fully $\mathcal{F}$-centralised elementary abelian subgroup F of P and some $t \in \mathcal{V}_F^+(iM)$. Applying r_*^P to the first equation yields

$$r_P^*(z) = r_E^*(s).$$

This implies that $r_E^*(s) = r_F^*(t)$ in the block variety $\mathcal{V}_B$. The analogue of Quillen's stratification for the block variety $\mathcal{V}_B$ implies that there is an isomorphism $\varphi : E \cong F$ in $\mathcal{F}$ such that $w = \mathrm{res}_\varphi^*(s)$ and t are in the same $\mathrm{Aut}_{\mathcal{F}}(F)$-orbit in $\mathcal{V}_F^+$. That is, after composing φ with a suitable automorphism of F, we may assume that $t = \mathrm{res}_\varphi^*(s)$. Now t belongs to $\mathcal{V}_F^+(iM) \subseteq \mathcal{V}_F^+(U)$. The $\mathcal{F}$-stability of U implies that $s \in \mathcal{V}_E^+(U)$. This completes the proof of Theorem 1.1. □

Just as for Theorem 1.1 it follows from [2, Theorem 3.1] that it suffices to prove any of the five Corollaries to Theorem 1.1 for $Q = P$. Note further that thanks to Lemma 4.1 we may assume that in all of these Corollaries the almost source idempotent is a source idempotent

Proof of Corollary 1.2. This follows from Theorem 1.1 combined with Lemma 4.3. □

Proof of Corollary 1.3. This follows from Corollary 1.2 and Proposition 4.7. □

Proof of Corollary 1.4. This follows from Corollary 1.2 and Proposition 4.9 (here we make use of the fact that i can be assumed to be a source idempotent, by Lemma 4.1). □

Proof of Corollary 1.5. Since P is abelian, it is well-known that $\mathcal{F} = N_{\mathcal{F}}(P)$ (see e. g. [20, Proposition 8.3.8]). Thus Corollary 1.5 follows from Corollary 1.4. □

Remark 5.1. It is shown in [3, Proposition 1.7] that in the situation of Corollaries 1.4, 1.5 the source algebras have P-P-stable bases consisting of invertible elements. Thus these two corollaries follow from this combined with Corollary 1.3.

Proof of Corollary 1.6. By Lemma 4.10, the restriction to P of any finitely generated B-module is $\mathcal{F}$-stable. Since B is assumed to be of principal type, it follows that 1_B is an almost source idempotent of B. Thus Corollary 1.6 follows from Corollary 1.2. □

Corollary 1.6 can also be proved by combining [3, Corollary 2.5] with Corollary 1.3.

Proof of Theorem 1.7. By [6, Theorem 1.1], we have $\mathcal{V}_B(M) = r_Q^*(\mathcal{V}_Q(U))$, and hence we have $\mathcal{V}_Q(U) \subseteq (r_Q^*)^{-1}(\mathcal{V}_B(M)$. By Lemma 4.2 we have $r_Q^*(\mathcal{V}_Q(U)) = r_Q^*(\mathcal{V}(kX \otimes_{kQ} U))$, and therefore

$$\mathcal{V}_Q(kX \otimes_{kQ} U) \subseteq (r_Q^*)^{-1}(\mathcal{V}_B(M)).$$

We need to show that this inclusion is an equality. By Theorem 1.1 we have

$$(r_Q^*)^{-1}(\mathcal{V}_B(M)) = \mathcal{V}_Q(kX \otimes_{kP} iM).$$

By the choice of the vertex-source pair (Q, U) of M, the iBi-module iM is isomorphic to a direct summand of $iBi \otimes_{kQ} U$. Thus we have

$$\mathcal{V}_Q(kX \otimes_{kP} iM) \subseteq \mathcal{V}_Q(kX \otimes_{kP} iBi \otimes_{kQ} U).$$

Now iBi is isomorphic to a direct summand of $iBi \otimes_{kP} X$ as a kP-kQ-bimodule, and hence we get an inclusion

$$\mathcal{V}_Q(kX \otimes_{kP} iBi \otimes_{kQ} U) \subseteq \mathcal{V}_Q(kX \otimes_{kP} iBi \otimes_{kP} kX \otimes_{kQ} U).$$

Let Y be a P-P-stable k-basis of iBi, so that $iBi \cong kY$ as kP-kP-bimodule. By Lemma 4.4, Y satisfies the properties (i) and (ii) from Proposition 2.2. It follows from Lemma 2.3, that the set $X \times_P Y \times_P X$ is an $\mathcal{F}$-characteristic P-P-biset. Thus, by Lemma 4.2 we have an equality

$$\mathcal{V}_Q(kX \otimes_{kP} iBi \otimes_{kP} kX \otimes_{kQ} U) = \mathcal{V}_Q(kX \otimes_{kQ} U).$$

Together this shows the inclusion

$$(r_Q^*)^{-1}(\mathcal{V}_B(M) \subseteq \mathcal{V}_Q(kX \otimes_{kQ} U).$$

This completes the proof of Theorem 1.7. □

6. Examples

With the notation of Theorem 1.1, we do not know of an example where the inclusion $\mathcal{V}_P(iM) \subseteq \mathcal{V}_P(kX \otimes_{kP} iM)$ is strict. As mentioned earlier, this inclusion is an equality if the kP-module iM is stable with respect to the fusion system $\mathcal{F}$ on P determined by the almost source idempotent i. The following example describes a finitely generated kP-module U such that the inclusion $\mathcal{V}_P(U) \subseteq \mathcal{V}_P(kX \otimes_{kP} U)$ is strict. In that example the kP-module U is not isomorphic to iM for any M.

Example 6.1. Suppose that p is odd. Let Q, R be cyclic groups of order p, and let u, v be a generator of Q, R, respectively. Set $P = Q \times R$. Let τ be the automorphism of order 2 of P which exchanges u and v (identified to their images in P). Set $V = \mathrm{Ind}_Q^P(k)$ and $W = \mathrm{Ind}_R^P(k)$. Since τ exchanges Q and R, it follows that V and W are exchanged by τ; that is, $W \cong {}_\tau V$ and $V \cong {}_\tau W$. Set $L = P \rtimes \langle \tau \rangle$ and denote by $\mathcal{F}$ the fusion system of L on P. We have

$$\mathrm{Res}_P^L \mathrm{Ind}_P^L(V) \cong \mathrm{Res}_P^L \mathrm{Ind}_P^L(W) \cong V \oplus W.$$

By Proposition 3.1 we have

$$\mathcal{V}_P(V) = (\mathrm{res}_Q^P)^*(\mathcal{V}_Q),$$

$$\mathcal{V}_P(W) = (\mathrm{res}_R^P)^*(\mathcal{V}_R).$$

Since Q, R are different cyclic subgroups of P, the varieties $\mathcal{V}_P(V)$ and $\mathcal{V}_P(W)$ are different lines in $\mathcal{V}_P$. Note that kL has a unique block $B = kL$ and that $H^*(L) = H^*(B)$ is the subalgebra of τ-stable elements in $H^*(P)$, or equivalently, the subalgebra of $\mathcal{F}$-stable elements in $H^*(P)$. The P-P-biset $X = L$ is an $\mathcal{F}$-characteristic biset. Since $L = P \cup P\tau$, it follows that

$$kX \otimes_{kP} V = V \oplus W \cong \mathrm{Res}_P^L \mathrm{Ind}_P^L(V)$$

from which we get a strict inclusion

$$\mathcal{V}_P(V) \subsetneq \mathcal{V}_P(kX \otimes_{kP} V) = \mathcal{V}_P(V) \cup \mathcal{V}_P(W).$$

Denote by $r_P : H^*(L) \to H^*(P)$ the inclusion map, and by $r_P^* : \mathcal{V}_P \to \mathcal{V}_L$ the induced map on varieties. By Proposition 3.1 we have

$$r_P^*(\mathcal{V}_P(V)) = \mathcal{V}_L(\mathrm{Ind}_P^L(V)) = r_P^*(\mathcal{V}_P(W)).$$

By [2, Theorem (3.1)], applied to $\mathrm{Ind}_P^L(V)$, we have

$$(r_P^*)^{-1}(\mathcal{V}_L(\mathrm{Ind}_P^L(V))) = \mathcal{V}_P(V \oplus W) = \mathcal{V}_P(V) \cup \mathcal{V}_P(W).$$

Since the action on $\mathcal{V}_P$ induced by τ exchanges $\mathcal{V}_P(V)$ and $\mathcal{V}_P(W)$, it follows that $r_P^*(\mathcal{V}_P(V)) = r_P^*(\mathcal{V}_P(W))$. Thus $\mathcal{V}_P(V)$ and $\mathcal{V}_P(W)$ are both contained in $(r_P^*)^{-1}(r_P^*(\mathcal{V}_P(V)))$. This shows that we have a strict inclusion $\mathcal{V}_P(V) \subsetneq (r_P^*)^{-1}(r_P^*(\mathcal{V}_P(V)))$.

Remark 6.2. The Example 6.1 contradicts the inclusion $\supseteq$ in the statement of [24, Theorem 2.2]. While the inclusion $\subseteq$ in [24, Theorem 2.2] holds in the generality as stated there, for the reverse inclusion one needs some extra hypotheses. With the notation of [24, Theorem 2.2], the following hypotheses, communicated to the author by C.-C. Todea, are sufficient for the reverse inclusion: $\mathcal{F}_1$and $\mathcal{F}_2$ are saturated fusion systems of finite groups $G_1 \leq G_2$ on $P_1 \leq P_2$ and U is a finitely generated kG_2-module.

Example 6.3. We adapt the previous example to show that tensoring by kX over Q in Theorem 1.7 is necessary if Q is a proper subgroup of P, even possibly when B is a nilpotent block. Let $p = 2$ and Q be a Klein four group. Write $Q = \langle s \rangle \times \langle t \rangle$ with involutions s, t. The group $\mathrm{GL}_2(k)$ acts on kQ in the obvious way (by sending s, t to shifted cyclic subgroups). Let $W = kQ/\langle t \rangle$; this is a 2-dimensional kQ-module with vertex $\langle t \rangle$, hence periodic of period 1. Since there are only finitely many isomorphism classes of kQ-modules with cyclic vertex, it follows that ${}_\tau W$ has vertex Q for almost all $\tau \in \mathrm{GL}_2(k)$. Set $P = Q \rtimes \langle u \rangle$ for some involution u satisfying $usu = t$ (so that P is a dihedral group). Choose $\tau \in \mathrm{GL}_2(k)$ such that $U = {}_\tau W$ has vertex Q and such that $c_u \circ \tau \neq \tau$, where c_u is conjugation by u regarded as an automorphism of kQ. Set $M = \mathrm{Ind}_Q^P(U)$ and $U' = {}_{c_u}U$. Then $\mathrm{Res}_Q^P(M) \cong U \oplus U'$. Both (Q, U) and (Q, U') are vertex-source pairs of M. Since U, U' are periodic, the choice of τ implies that the varieties $\mathcal{V}_Q(U)$ and $\mathcal{V}_Q(U')$ are different lines in $\mathcal{V}_Q$. The fusion system $\mathcal{F}$ is in this situation the trivial fusion system $\mathcal{F}_P(P)$, and the set $X = P$, as a P-P-biset, is a characteristic biset of $\mathcal{F}$. Thus, as a kQ-module, we have $kX \otimes_{kQ} U \cong \mathrm{Res}_Q^P(\mathrm{Ind}_Q^P(U)) \cong U \oplus U'$, and since the varieties $\mathcal{V}_Q(U)$ and $\mathcal{V}_Q(U')$ are different, it follows that $\mathcal{V}_Q(U)$ is properly contained in $\mathcal{V}_Q(kX \otimes_{kQ} U)$.

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