Pole Assignment for Symmetric Quadratic Dynamical Systems: An Algorithmic Method

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Abstract

In this article an algorithmic method is proposed for the solution of the pole assignment problem which is associated with a symmetric quadratic dynamical system, in case it is completely controllable. The above problem is proved to be equivalent to two subproblems, one linear and the other multilinear. Solutions of the linear problem must be decomposable vectors, i.e. they must lie in an appropriate Grassmann variety. The proposed method computes a reduced set of quadratic Plucker relations with only three terms each, which describe completely the specific Grassmann variety. Using these relations one can solve the multilinear problem and consequently calculate the feedback matrices which give a solution to the pole assignment problem. Finally, an illustrative example of the proposed algorithmic procedure is given.The advantage of this approach, is that the complete set of feedback solutions is obtained, over which further optimisation can be carried out, if desired.

Keywords:
Control Theory, Pole assignment, Quadratic matrix pencils, Grassmann variety, Plucker relations, numerical algorithm

1. Problem definition and Methodology

The equation of motion for a matrix second-order system, e.g. a structural system with viscus damping and without externally applying forces is expressed as follows:

$$Mq''(t) + Dq'(t) + Kq(t) = 0 \tag{1}$$

Here, M, D, K are the $n \times n$ mass, damping and stiffness matrices respectively. Also $q(t)$ is the displacement vector, $q'(t)$ the velocity vector and $q''(t)$ the acceleration vector. In most applications M, D and K are symmetric matrices. Furthermore, M is positive definite and D, K are positive semi-definite or positive definite.

Separation of variables $q(t) = e^{\lambda t} c$ where c is a constant vector gives us the quadratic eigenvalue problem:

$$P(\lambda_j) c_j = 0, \quad j = 1, 2, \ldots, 2n$$

in which $P(\lambda)$ denotes the quadratic matrix pencil: $P(\lambda) = \lambda^2 M + \lambda D + K$. The eigenvalues and eigenvectors of the pencil govern the free response of the system. The model can be used to identify poorly damped oscillations and resonance phenomena, which we may be able to avoid by the appropriate selection of the matrix parameters.

When a control force is applied the model described by the above equations becomes:

$$Mq''(t) + Dq'(t) + Kq(t) = F(t), \quad F(t) = Bu(t) \tag{2}$$

where $u(t)$ is the $m \times 1$ control input vector and B is an $n \times m$ input matrix.

One of the major concerns for the control engineer is to ensure stability of the control system. Thus, the behaviour of the system is usually modified by applying state feedback control to relocate the troublesome eigenvalues in the complex plane. If the system is (open-loop) unstable, we aim, as a minimum requirement, to stabilise it. If the system is stable, it may be desirable to maintain some degree of relative stability and robustness margins, i.e. ensure that stability is maintained under realistic uncertainty conditions. The notion of stabilization is connected with the problem of relocation of troublesome eigenvalues of the system in equation (2), otherwise called pole placement, and relocation can always be achieved if the above system is completely state controllable. In that case, for any choice of poles, suitable real $m \times n$ matrices F_1 and F_2 can be found such that under the state feedback:

$$u(t) = -F_1 q(t) - F_2 q'(t) \tag{3}$$

the corresponding closed-loop system described by the equation

$$Mq''(t) + (D + BF_2)q'(t) + (K + BF_1)q(t) = 0 \tag{4}$$

has the chosen set of poles. Equivalently, if

$$f(s) = a_0 + a_a s + \ldots + a_{2n-1}s^{2n-1} + s^{2n} = e_{2n}^t(s)\tilde{a} \tag{5}$$

is the required closed-loop polynomial, $e_{2n}^t(s) = [1\ s\ s^2\ \ldots\ s^{2n}]$ and $\tilde{a} = [a_0\ a_1\ a_2\ \ldots\ a_{2n-1}\ 1]$, we want to compute the matrices F_1 and F_2 such that the characteristic polynomial

$$\tilde{\varphi}(s) = \det(s^2M + s(D + BF_2) + (K + BF_1)) \tag{6}$$

of the quadratic pencil

$$P_c := s^2M + s(D + B\ F_2) + (K + B\ F_1) \tag{7}$$

associated with the closed-loop system (4) is equal to:

$$\tilde{\varphi}(s) = (\det M)f(s) \tag{8}$$

Define:

$$Q(s) := [s^2M + sD + K \mid sB \mid B] \tag{9}$$

and

$$V = \begin{bmatrix} I_n \\ \ldots \\ F_2 \\ \ldots \\ F_1 \end{bmatrix} \tag{10}$$

Then we can rewrite equation (7) as:

$$P_c(s) = Q(s)V \tag{11}$$

Using compound matrices and applying the Binet-Cauchy theorem we can write,

$$\tilde{\varphi}(s) = \det(P_c(s)) \equiv C_n(P_c(s)) = C_n(Q(s)V) = C_n(Q(s))C_n(V) \tag{12}$$

We now define:

$$P := P(M, D, K, B) \in \mathbb{R}^{(2n+1)\times\ell} \tag{13}$$

Since $C_n(Q(s)) := e_{2n}^t(s)P$, the columns of P are the coefficient vectors of the elements of $C_n(Q(s))$, which are polynomials of maximum degree $2n$ and

$$g := C_n(V) = C_n\left(\begin{bmatrix} I_n \\ \dots \\ F_2 \\ \dots \\ F_1 \end{bmatrix}\right) \in \mathbb{R}^{\ell} \tag{14}$$

where $\ell := \binom{n+2m}{n}$.

Combining equations (5), (8) and (12) we get

$$e_{2n}^t(s)Pg = e_{2n}^t(s)(\det M)\tilde{a} := e_{2n}^t(s)a$$

which in turn implies that:

$$Pg = a \tag{15}$$

2. Techniques from algebraic geometry and algorithmic solution

Before we state the theorems on which the construction of the algorithm is based, we need the following notation and definitions [21], [24], [25]. Let the binomial coefficient k be

$$k := \binom{n}{m}, n, m \in \mathbb{N},\ n \geq m$$

Denote the set of the first n natural numbers as

$$T_n := \{\kappa \in \mathbb{N}^* : 1 \leq \kappa \leq n\}$$

with $\mathbb{N}^*$ the set of naturals without zero.
Let the set of elements (sequences), where the elements $(a_j, 1 \leq j \leq m)$ are not necessarily different

$$T_m^n := \{(a_1, \dots, a_m) :\ a_j \in \mathcal{T}_n\ , 1 \leq j \leq m\}$$

Also, let the set of elements (sequences), where the elements of every sequence are in strictly increasing order, be denoted as

$$\tilde{T}_m^n := \{(a_1, \dots, a_m)\ \in \mathcal{T}_m^n\ : a_\kappa \neq a_\lambda,\ \text{if}\ \kappa \neq \lambda\}$$

Denote by D_m^n all m-tuples $< a_1, a_2, \ldots, a_m >$ such that $(a_1, \ldots, a_m) \ \in \tilde{T}_m^n$ and $a_\kappa < a_\lambda$ if $\kappa < \lambda$. We can order the elements of the set D_m^n as follows:

$$< a_1, a_2, \ldots, a_m > \ \prec \ < j_1, j_2, \ldots, j_m >$$

if and only if there exists $\kappa \in \mathcal{T}_n$ such that $a_\kappa < j_\kappa$ and $a_\lambda = j_\lambda$ for every $\lambda < \kappa$. The above relation is a total ordering called lexicographical ordering. Using the lexicographical ordering, we can now characterize the k elements of D_m^n by

$$a_0 = < 1, 2, \ldots, m-1, m >$$

$$a_1 = < 1, 2, \ldots, m-1, m+1 >$$

$$\vdots$$

$$a_{k-1} = < n-m+1, \ldots, n-1, n >$$

Note that $a_0 \prec a_1 \prec \ldots \prec a_{k-1}$. Next, we define the map

$$\delta : \tilde{T}_m^n \to N \ \times \ D_m^n$$

such that:

$$(a_1, a_2, \ldots, a_m) \xrightarrow{\delta} (\lambda, < a'_1, a'_2, \ldots, a'_m >)$$

where λ is the number of permutations needed to order the elements of $(a_1, a_2, \ldots, a_m)$ in normal ordering $< a'_1, a'_2, \ldots, a'_m >$.

If $[x_0, x_1, \ldots, x_{k-1}]^T \in \mathcal{P}^{k-1}(\mathbb{R})$, we define the following maps:

$$\phi \ : \ D_m^n \ \to [x_0, x_1, \ldots, x_{k-1}]^T$$

by $a_\kappa \xrightarrow{\phi} x_\kappa, \ 0 \le \kappa \le k-1$, where it is clear from the above definition of ϕ that $\phi^{-1}(x_\kappa) = a_\kappa$ and

$$\tilde{\phi} : N \times D_m^n \to \mathbb{R}$$

by

$$(\lambda, a_\kappa) \xrightarrow{\tilde{\phi}} (-1)^\lambda \phi(a_\kappa) = (-1) x_\kappa$$

Finally, if we denote $\beta := (a_1, a_2, \ldots, a_m) \in \mathcal{T}_m^n$, we define the map

$$g : T_m^n \to \mathbb{R}$$

by

$$g : T_m^n \to \mathbb{R} : g(\beta) = \begin{cases} (\tilde{\phi} \circ \delta)(\beta), & \text{if } \beta \in \tilde{T}_m^n \\ 0, & \text{if } \beta \in T_m^n \backslash \tilde{T}_m^n \end{cases}$$

We can now construct the Plucker relations: For every group of $m-1$ indices

$$< t_1, t_2, \ldots, t_{m-1} > \in \ \tilde{T}_{m-1}^n$$

and from every group of $m+1$ indices

$$< p_1, p_2, \ldots, p_{m+1} > \in \ \tilde{T}_{m+1}^n$$

we define for $\kappa = 1, \ldots, m+1$,

$$\beta_\kappa := (t_1, t_2, \ldots, t_{m-1}, p_\kappa)$$

and

$$\gamma_\kappa := (p_1, p_2, \ldots, p_{\kappa-1}, p_{\kappa+1}, \ldots, p_{m+1}),$$

for $\kappa = 1, \ldots, m+1$ and the corresponding Plucker relation is as follows

$$\sum_{\kappa=1}^{m+1} (-1)^\kappa g(\beta) g(\gamma_\kappa) = 0$$

Next we state the following theorems which extract from the whole set of quadratic Plucker relations a reduced set of relations which have simple form and describe completely the Grassmann variety of the corresponding projective space.

Theorem 1 [25]: Assume that $x = [x_0, x_1, \ldots, x_{k-1}]^t \ \in \Omega(m,n)$ and $\phi^{-1}(x_\kappa) =< a_1, a_2, \ldots, a_m > \in D_m^n$ with $a_2 < n - m + 1$. Then, there exists a three-term Plucker relation $\sigma(x_\kappa, x_{\kappa 1}, x_{\kappa 2}, x_{\kappa 3}, x_{\kappa 4}, x_{\kappa 5}) = 0$ where $\kappa < \kappa_i, \ i = 1, 2, \ldots, 5$.

Theorem 2 [25]: Assume that $x = [x_0, x_1, \ldots, x_{k-1}]^t \ \in \Omega(m,n)$. The full number of the coordinates x_κ having the property $\phi^{-1}(x_\kappa) =< a_1, a_2, \ldots, a_m >$ with $a_2 < n - m + 1$ is

$$r := \binom{n}{m} - (n-m)m - 1$$

Corollary 1 [25]: For every $x = [x_0, x_1, \ldots, x_{k-1}]^t \in \Omega(m,n)$, there exists a set S of three-term quadratic Plucker relations, of the form

$$\sigma_i(x_{\kappa i}, x_{\lambda i}, x_{\mu i}, x_{\nu i}, x_{\xi i}, x_{\rho i}) = 0$$

where $1 \leq i \leq r$ with

$$\kappa_i \leq \min(\lambda_i, \mu_i, \nu_i, \xi_i, \rho_i), \text{ for } 1 \leq i \leq r,$$

and

$$\kappa_j < \kappa_{j+1}, \ 1 \leq i \leq r-1.$$

Theorem 3 [25]: The three-term quadratic Plucker relations given by the set S of Corollary 1 describe completely the Grassmann variety $\Omega(m,n)$ of the projective space $P^{k-1}(\mathbb{R})$.

Next, we propose the following algorithm which computes the Reduced Set of Quadratic Plucker Relations (RSQPR).

Algorithm RSQPR

- Step 1: Read the dimensions n, m.
- Step 2: Compute $k = \binom{n}{m}$.
- Step 3: Repeat for $k = 0, 1, \ldots, k-1$. a. Find the κ-th order multi-index $< a_1, a_2, \ldots, a_m >$
  b. If $\lambda_2 < n - m + 1$ then
  b1. Find indices $j_1, j_2 \in T_n - \{a_1, a_2, \ldots, a_m\}$ such that $j_1 > a_2$ and $j_2 > a_2$.
  b2. Define $a_{m+1} := j_2$, $\beta_\rho := (a_3, \ldots, a_m, j_1, a_\rho)$ and $\gamma_\rho := (a_1, \ldots, a_{\rho-1}, a_{\rho+1}, \ldots, a_{m+1})$.
  b3. Type Plucker relation

$$\sum (-1)^\rho \cdot g(\beta_\rho) \cdot g(\gamma_\rho) = 0$$

- Step 4: End.

Finally, suppose we have computed the coordinates of a decomposable vector as it is seemed below

$$x = [x_0, x_1, \ldots, x_{k-1}]^t \in \Omega(m,n)$$

using the quadratic Plucker relations given by the above algorithm. Then we would like to reconstruct $H \in \mathbb{R}^{n\times m}$ which has the property

$$C_m(H) = x \tag{16}$$

The following Proposition is helpful.

Proposition 1 [25]: Let $x = [x_0, x_1, \dots, x_{k-1}]^t \in \mathbb{R}^k$ be a decomposable vector and $x_p \neq 0$ for an index p, $1 \leq p \leq k-1$, such that

$$\phi^{-1}(x_p) =< a_1, a_2, \dots, a_m >$$

Then, the elements h_{ij} of matrix $H \in \mathbb{R}^{n\times m}$ which satisfies equation (16) are given by

$$h_{ij} = g((a_1, a_2, \dots, a_{j-1}, i, a_{j+1}, \dots, a_m)) \tag{17}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$.

3. Numerical Example

The benefits of the proposed algorithmic procedure are clarified by the following example. We consider the system of the form (2), with

$$M = 10I_3,\ K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix},\ D = 0_{3\times 3},\ B = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

The open-loop e-values are $\pm 3.6039i, \pm 2.4940i, \pm 0.8901i$. Suppose we want to shift these to $-1, -2, -3, -4, -5, -6$, using the state feedback law (3) where $F_1,\ F_2 \in \mathbb{R}^{1\times 3}$. In this case the problem consists of finding matrices $F_1,\ F_2$ such that the closed-loop system (4) has a charactristic polynomial

$$\tilde{\phi}(s) = (\det M) f(s)$$

where $f(s) = \prod_{i=1}^{6}(s - \lambda_i) = \prod_{i=1}^{6}(s + i)$.

After some computations, we arrive at a linear system of the form (15) which is $Pg = a$, the solution of which is given by:

$$g_0 = 1,\ g_1 = 2932.5,\ g_2 = -4318.571,\ g_3 = 3345$$

$$g_4 = -5737.143,\ g_6 = 1447.5,\ g_7 = -2705.714$$

Variables $g_5,\ g_8,\ g_9$ can be chosen arbitrarily since they do not appear in equations of the linear system. Now we must solve a multinear subproblem, which consists of the following three equations, the so-called reduced set of quadratic Plucker relations:

$$g_4 g_6 - g_3 g_7 + g_0 g_9 = 0$$

$$-g_5 g_6 + g_3 g_8 - g_1 g_9 = 0$$

$$g_5 g_7 - g_4 g_8 + g_2 g_9 = 0$$

These equations are generated by the proposed algorithm. Using a symbolic package i.e. Mathematica and substituting the solutions of the linear system to the above set of quadratic relations the following solutions are obtained:

$$g_5 = -2378500,\ g_8 = -1683375,\ g_9 = -746100$$

Now, from equation (15) in combination with Proposition 1, we obtain the feedback matrices

$$F_1 = [g_7, -g_4, g_2] = [-2705.714\ ,\ 5737.143\ ,\ -4318.571]$$

and

$$F_2 = [g_6, -g_3, g_1] = [1447.5\ ,\ -3345\ ,\ 2932.5]$$

In order to avoid cumbersome calculations, a numerical software package can be used to verify that the closed-loop system (4) has the following poles:

$$\tilde{\lambda}_1 = -1.000000000047105$$

$$\tilde{\lambda}_2 = -1.999999999594935$$

$$\tilde{\lambda}_3 = -3.000000000111104$$

$$\tilde{\lambda}_4 = -3.999999999929748$$

$$\tilde{\lambda}_5 = -5.000000000132230$$

$$\tilde{\lambda}_6 = -6.000000000120886$$

Note that whatever the starting guess for the approximation of the eigenvalues, the error $|\lambda_i - \tilde{\lambda}_i|$, for $i = 1, \ldots, 6$, is always smaller than 10^{-9}.

4. Conclusion

An efficient algorithmic framework has been proposed for the solution of the pole-assignment problem of symmetric quadratic systems. This involves the computation of a reduced set of quadratic Plucker relations describing completely the Grassmann variety of the corresponding projective space. The extraction of this reduced set has been achieved by the use of a simple criterion based on the correspondence between the coordinates of a decomposable vector and lexicographical orderings. The minimum number of the linear independent quadratic Plucker relations which describe completely the Grassmann variety is:

$$V_{eq} = \binom{n}{m} - m(n-m) - 1$$

Each equation of this reduced set is homogeneous and contains only three terms. This fact has beneficial influence on the overall complexity of the problem which is mainly due to its non-linearity. The method can be extended to the solution of other problems of control theory having a similar multi-linear nature [1], [2], [3], [14].

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