Multisymplectic approach to integrable defects in the sine-Gordon model

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Abstract

Ideas from the theory of multisymplectic systems, introduced recently in integrable systems by the author and Kundu to discuss Liouville integrability in classical field theories with a defect, are applied to the sine-Gordon model. The key ingredient is the introduction of a second Poisson bracket in the theory that allows for a Hamiltonian description of the model that is completely equivalent to the standard one, in the absence of a defect. In the presence of a defect described by frozen Bäcklund transformations, our approach based on the new bracket unifies the various tools used so far to attack the problem. It also gets rid of the known issues related to the evaluation of the Poisson brackets of the defect matrix which involve fields at coinciding space point (the location of the defect). The original Lagrangian approach also finds a nice reinterpretation in terms of the canonical transformation representing the defect conditions.

1 Introduction

Integrable systems are a privileged area of Physics and Mathematics where one can test ideas on "toy models" that are sufficiently simple to be amenable to exact analytic treatment but sufficiently complex to capture interesting physical phenomena. When studying a model, the question of defects or impurities is an important (and often difficult) one, for at least two reasons: they represent the departure from an ideal system towards a more realistic situation and they can have dramatic effects on the predicted ideal behaviour. Therefore, there is a strong motivation to study defects/impurities in the context of integrable systems.

Initially, the focus was mainly on quantum (field) theories [1]-[5] and remained concentrated until quite recently [6]-[9] on various quantum systems. The point of view taken in those works was to *maintain* integrability in the presence of defects. The general framework that includes the previous studies was proposed in [10].

The question of integrable defects in classical field theories was considered almost ten years after the publication of the first paper on the topic in integrable QFT. In a series of papers [11]-[12] related to several key models like the sine-Gordon model, the nonlinear Schrödinger (NLS) equation, etc, a Lagrangian approach was proposed where a contribution from the defect is required to compensate for the loss of conservation of the momentum due to the presence of a defect. It was argued that

this is enough to ensure the integrability of a defect model. A crucial observation to support this was that the conditions on the fields that one obtains in this way correspond to Bäcklund transformations frozen at the location of the defect. This approach triggered a strong activity in the analysis of the defect in integrable classical field theories. The observation on frozen Bäcklund transformations was fully exploited in [13] in conjunction with the Lax pair formulation of the general AKNS approach [14] to obtain a generating function of the entire set of modified conserved quantities. This also allowed to answer some questions left open in the Lagrangian formulation like the formulation of the defect conditions directly in terms of the fields of the theory for models like KdV. It also settled the question of integrability in the sense of the presence of an infinite number of conserved quantities. But soon, the question of Liouville integrability became a main issue. The sine-Gordon model was the first model to receive attention [15], followed by a very nice series of papers tackling the question systematically for several models [16, 17, 18]. The procedure in these investigations is based on the a priori assumption that the defect matrix satisfies appropriate Poisson bracket relations formulated in the context of the classical r-matrix approach. A careful regularization is needed in this procedure which yields the so-called "sewing conditions" between the fields in the bulk and those contained in the defect matrix. The consistency of the approach must then be checked a posteriori.

At that stage, there were essentially two approaches to the same question that were not quite reconciled: the Lagrangian/Bäcklund approach and the "sewing conditions" approach. This situation was the motivation for the introduction in [19] of the multisymplectic formalism to study Liouville integrability of classical field theories with a defect. This was done in detail for the case of the NLS equation. The purpose of the present paper is to apply these ideas to another famous prototype of integrable field theory: the sine-Gordon model. It is shown that they apply just as successfully, providing the same unifying picture as for the NLS case and getting rid of the known open problems.

The paper is organised as follows. In Section 2, the two Poisson brackets at the basis the multisymplectic formalism are introduced for the sine-Gordon model and it is shown that two completely equivalent Hamiltonian descriptions exist for this model. Conservation laws and the classical r matrix approach are discussed in the light of the two Poisson brackets, each one corresponding to an independent variable (called space and time variable). In Section 3, we review the sine-Gordon model with a defect from the Lagrangian and Lax pair points of view. We then go on to show how the new formalism allows to prove Liouville integrability of the model with defect directly, with no resort to sewing conditions. The key point is the fact that the defect conditions are reinterpreted as a canonical transformation with respect to the new Poisson structure. It turns out that the generating functional for this canonical transformation is directly built on the defect Lagrangian density originally derived to study problems with defects. The new classical r matrix approach with defect is also derived there and the new monodromy matrix containing the generating function for the conserved quantities in involution is exhibited. Conclusions and outlooks are gathered in Section 4

2 Multisymplectic structure of the sine-Gordon model

2.1 Poisson brackets and Hamiltonian equations

Let us present the multisymplectic structure of the sine-Gordon model by introducing two Poisson brackets on the phase space of the model. The main observation behind the multisymplectic approach to field theory is that the canonical quantization procedure puts emphasis only on the time parameter and, as a consequence, only considers a partial Legendre transformation when defining canonical conjugate coordinates. The idea of the multisymplectic approach is to restore the balance between the independent variables and to consider one Legendre transformation per

independent variable. Although this research area has developed in a rather non systematic way (see e.g. [20] for an attempt to give an account of the various approaches) and into a heavy mathematical formalism, the commonly accepted origin is the so-called De Donder-Weyl formalism [21].

In the traditional approach, given fields ϕ_a depending on the independent variables (x,t)[1] one defines the conjugate momenta π^a as

$$\pi^a = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_a)}\,, \tag{2.1}$$

where $\mathcal{L}$ is the Lagrangian density of the theory. Then, one imposes *equal-time* canonical relations by defining the *space* Poisson brackets as

$$\{\phi_a(x,t_0),\pi^b(y,t_0)\}_S = \delta_a^b\delta(x-y) \quad , \quad \{\phi_a(x,t_0),\phi_b(y,t_0)\}_S = 0 \quad , \quad \{\pi^a(x,t_0),\pi^b(y,t_0)\}_S = 0\,, \tag{2.2}$$

at some initial time t_0. The subcript S on the Poisson bracket indicates that it is equal-time *i.e.* it does not depend on time but only on space. However, the Legendre transformation (2.1) is incomplete and one can also define another set of conjugate momenta by

$$\Pi^a = \frac{\partial \mathcal{L}}{\partial(\partial_x \phi_a)}\,. \tag{2.3}$$

The second bracket is then defined in complete analogy by

$$\{\phi_a(x_0,t),\Pi^b(x_0,\tau)\}_T = \delta_a^b\delta(t-\tau) \quad , \quad \{\phi_a(x_0,t),\phi_b(x_0,\tau)\}_T = 0 \quad , \quad \{\Pi^a(x_0,t),\Pi^b(x_0,\tau)\}_T = 0\,, \tag{2.4}$$

at some fixed "initial" location x_0. These relations can be seen as *equal-space canonical brackets*. The subscript T indicates that this Poisson bracket does not involve space. These two brackets form the basis of the formulation of covariant Poisson brackets for field theories.

We now develop this formalism for the sine-Gordon model. We show that the two Hamiltonian pictures corresponding to the two brackets $\{\ ,\ \}_S$ and $\{\ ,\ \}_T$ yield completely equivalent descriptions of the sine-Gordon model. It is very important to note that the multisymplectic formalism presented here is very different from the well-known bi-Hamiltonian theory of integrable systems [22]. The latter is based on the existence of two compatible equal-time brackets $\{\ ,\ \}_{S1}$ and $\{\ ,\ \}_{S2}$, each of which allows for the description of the *time* evolution of the model. Our equal-space bracket $\{\ ,\ \}_T$ on the other hand yields the *space* evolution of the model.

The sine-Gordon model (in laboratory coordinates) is a relativistic field theory for the scalar field $\phi(x,t)$ with equation of motion given by

$$\phi_{tt} - \phi_{xx} + \frac{m^2}{\beta}\sin\beta\phi = 0\,, \tag{2.5}$$

where m is the mass parameter and β the coupling constant. A Lagrangian density for this equation is

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - \frac{m^2}{\beta^2}(1-\cos\beta\phi)\,. \tag{2.6}$$

From this, applying the Legendre transformations discussed above, we get the conjugate momenta as

$$\pi = \frac{\partial \mathcal{L}}{\partial \phi_t} = \phi_t \quad , \quad \Pi = \frac{\partial \mathcal{L}}{\partial \phi_x} = -\phi_x\,. \tag{2.7}$$

[1] It is enough for our purposes to consider $1+1$ dimensional field theory.

The associated Hamiltonian densities then read

$$\mathcal{H}_S = \frac{1}{2}\pi^2 + \frac{1}{2}\phi_x^2 + \frac{m^2}{\beta^2}(1-\cos\beta\phi)\,, \tag{2.8}$$

$$\mathcal{H}_T = -\frac{1}{2}\Pi^2 - \frac{1}{2}\phi_t^2 + \frac{m^2}{\beta^2}(1-\cos\beta\phi)\,. \tag{2.9}$$

Given the canonical Poisson brackets

$$\{\phi(x,t_0),\pi(y,t_0)\}_S = \delta(x-y) \quad , \quad \{\phi(x,t_0),\phi(y,t_0)\}_S = 0 \quad , \quad \{\pi(x,t_0),\pi(y,t_0)\}_S = 0\,, \tag{2.10}$$

and

$$\{\phi(x_0,t),\Pi(x_0,\tau)\}_T = \delta(t-\tau) \quad , \quad \{\phi(x_0,t),\phi(x_0,\tau)\}_T = 0 \quad , \quad \{\Pi(x_0,t),\Pi(x_0,\tau)\}_T = 0\,, \tag{2.11}$$

one easily checks that (2.5) is recovered from the following Hamiltonian equations of motion

$$\begin{cases} \phi_t = \{\phi, H_S\}_S\,, \\ \pi_t = \{\pi, H_S\}_S\,, \end{cases} \tag{2.12}$$

or

$$\begin{cases} \phi_x = \{\phi, H_T\}_T\,, \\ \Pi_x = \{\Pi, H_T\}_T\,, \end{cases} \tag{2.13}$$

where

$$H_S = \int \mathcal{H}_S\, dx \quad , \quad H_T = \int \mathcal{H}_T\, dt\,. \tag{2.14}$$

Important remark: Given the natural symmetry of space and time coordindates for the sine-Gordon model, the reader may have the impression that the above discussion (and some of the results to come concerning the Lax pair formulation, integrals of motion and classical r matrix approach) is just a trivial exercise in rewriting the theory under the swap $x \leftrightarrow t$ and the change $\phi \to -\phi$. Our point is that, in the traditional approach, even after such a transformation, one would still use a single Poisson structure describing time evolution of the model. In the present approach, after this transformation, the key point is that the two Poisson brackets still coexist and bring complementary information on the time and space evolution of the system. It is the (co)existence of these Poisson brackets, irrespective of the chosen coordinates, that is the crucial ingredient of the multisymplectic approach. Moreover, in the presence of a defect, no matter what coordinate is declared as the space coordinate, once this is fixed, the traditional approach based on a single Poisson bracket still has the limitations explained in the introduction. In contrast, the present approach is applicable and provides the missing tool, *i.e.* the new equal-space Poisson bracket, to tackle the question of Liouville integrability, as we proceed to explain in the rest of this paper.

2.2 Conservation laws and space and time integrals of motion

Parallel to the traditional Hamiltonian formalism for sine-Gordon, it is well-known that this model is integrable in the sense that it can be formulated via a Lax pair and the associated zero-curvature representation. Following for instance [23], we consider the auxiliary problem

$$\begin{cases} \Psi_x = U\,\Psi\,, \\ \Psi_t = V\,\Psi \end{cases} \tag{2.15}$$

where U and V are two 2×2 matrices depending on x, t and the so-called spectral parameter λ, given by

$$U(x,t,\lambda) = -i\frac{\beta}{4}\pi\,\sigma_3 - ik_0(\lambda)\sin\left(\frac{\beta\phi}{2}\right)\sigma_1 - ik_1(\lambda)\cos\left(\frac{\beta\phi}{2}\right)\sigma_2\,, \tag{2.16}$$

$$V(x,t,\lambda) = i\frac{\beta}{4}\Pi\,\sigma_3 - ik_1(\lambda)\sin\left(\frac{\beta\phi}{2}\right)\sigma_1 - ik_0(\lambda)\cos\left(\frac{\beta\phi}{2}\right)\sigma_2\,. \tag{2.17}$$

The three Pauli matrices are given as usual by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.18}$$

The two function k_0 and k_1 read

$$k_0(\lambda) = \frac{m}{4}(\lambda + \frac{1}{\lambda}) \quad , \quad k_1(\lambda) = \frac{m}{4}(\lambda - \frac{1}{\lambda})\,. \tag{2.19}$$

The compatibility condition $\Psi_{xt} = \Psi_{tx}$ of the auxiliary problem is equivalent to the zero curvature condition

$$U_t - V_x + [U,V] = 0\,, \tag{2.20}$$

which in turn is equivalent to

$$\begin{cases} \pi = \phi_t\,, \\ \Pi = -\phi_x\,, \\ \pi_t + \Pi_x + \frac{m^2}{\beta}\sin\beta\phi = 0\,, \end{cases} \tag{2.21}$$

and hence yields the sine-Gordon equation (2.5).

From the point of view of PDEs, an important characteristic of integrable equations is the existence of an infinite numbers of conservation laws and hence, of conserved quantities. Among many other things, the existence of a Lax pair formulation allows one to find these conservation laws easily and systematically. In Part Two, Chapter II.4 of [23], such a systematic procedure is presented to extract the *conserved quantities* (in time). Here, although we are strongly inspired by this derivation, we want to present a derivation of the *conservation laws* in a way that treats x and t on an equal footing, hence allowing us to extract conserved quantities in time and in space systematically. Another motivation is that conservation laws are more fundamental in the sense that conserved quantities are easily deduced from them by integration over an appropriate domain. Before we continue, we need to specify the chosen functional space and boundary conditions for the fields. When working in the traditional approach, with fixed t and fields depending on x, we assume that

$$\lim_{x\to\pm\infty}\phi(x) = \frac{2\pi Q_\pm}{\beta} = 0 \mod \frac{2\pi}{\beta} \quad , \quad \lim_{|x|\to\infty}\pi(x) = 0\,, \tag{2.22}$$

where the boundary values are approached sufficiently fast *i.e.* in the Schwarz sense. Similarly, when working in the new approach, with fixed x and fields depending on t, we assume that

$$\lim_{t\to\pm\infty}\phi(t) = \frac{2\pi \mathcal{Q}_\pm}{\beta} = 0 \mod \frac{2\pi}{\beta} \quad , \quad \lim_{|t|\to\infty}\Pi(t) = 0\,, \tag{2.23}$$

where the boundary values are approached sufficiently fast *i.e.* in the Schwarz sense. Given our purposes, it is important to note that these two assumptions can coexist. Indeed, solving the initial

value problem in the traditional approach, for initial data satisfying (2.22), naturally provides a large class of solutions, e.g. the N soliton solutions, which are then defined for all $t \in \mathbb{R}$ and satisfy (2.23) at any fixed value of x.

Let us define

$$\Omega = e^{i\frac{\beta\phi}{4}\sigma_3}\,, \tag{2.24}$$

where we will treat Ω either as a function of x for fixed time or vice versa, depending on the chosen picture. Note that

$$\lim_{x\to\pm\infty} U = (-1)^{Q_\pm} U_\infty \quad , \quad \lim_{t\to\pm\infty} U = (-1)^{\mathcal{Q}_\pm} U_\infty\,, \tag{2.25}$$

$$\lim_{x\to\pm\infty} V = (-1)^{Q_\pm} V_\infty \quad , \quad \lim_{t\to\pm\infty} V = (-1)^{\mathcal{Q}_\pm} V_\infty\,, \tag{2.26}$$

where,

$$U_\infty = -ik_1(\lambda)\sigma_2 \quad , \quad V_\infty = -ik_0(\lambda)\sigma_2\,. \tag{2.27}$$

So let us define

$$N = \frac{1}{\sqrt{2}}(\mathbb{I} + i\sigma_1)\,, \tag{2.28}$$

and

$$E_\pm(x,\lambda) = e^{i\frac{\pi}{2}Q_\pm\sigma_3}\, N\, e^{-ik_1(\lambda)x\sigma_3} \quad , \quad \mathcal{E}_\pm(t,\lambda) = e^{i\frac{\pi}{2}\mathcal{Q}_\pm\sigma_3}\, N\, e^{-ik_0(\lambda)t\sigma_3}\,, \tag{2.29}$$

which satisfy

$$\partial_x E_\pm(x,\lambda) = (-1)^{Q_\pm} U_\infty\, E_\pm(x,\lambda)\,, \tag{2.30}$$

$$\partial_t \mathcal{E}_\pm(t,\lambda) = (-1)^{\mathcal{Q}_\pm} V_\infty\, \mathcal{E}_\pm(t,\lambda)\,. \tag{2.31}$$

Let $T(x,y,\lambda)$ and $\mathcal{T}(t,\tau,\lambda)$ denote respectively the space transition matrix (at fixed time) and the time transition matrix (at fixed position). They are solutions of $\Psi_x = U\Psi$ and $\Psi_t = V\Psi$ respectively and normalised by

$$T(x,x,\lambda) = \mathbb{I} \quad , \quad \mathcal{T}(t,t,\lambda) = \mathbb{I}\,. \tag{2.32}$$

The corresponding monodromy matrices are

$$T(\lambda) = \lim_{\substack{x\to\infty \\ y\to-\infty}} E_+^{-1}(x,\lambda)T(x,y,\lambda)E_-(y,\lambda)\,, \tag{2.33}$$

$$\mathcal{T}(\lambda) = \lim_{\substack{t\to\infty \\ \tau\to-\infty}} \mathcal{E}_+^{-1}(t,\lambda)\mathcal{T}(t,\tau,\lambda)\mathcal{E}_-(\tau,\lambda)\,. \tag{2.34}$$

For the purpose of discussing conservations laws and conserved quantities, it is convenient to gauge away the charges at $\pm\infty$ and to consider the following gauge transformed matrices

$$T(x,y,\lambda) = \Omega(x)\widehat{T}(x,y,\lambda)\Omega(y)^{-1} \quad , \quad \mathcal{T}(t,\tau,\lambda) = \Omega(t)\widehat{\mathcal{T}}(t,\tau,\lambda)\Omega(\tau)^{-1}\,. \tag{2.35}$$

Then $\widehat{T}$ is a solution of $\Psi_x = \widehat{U}\,\Psi$ where

$$\widehat{U} = -i\frac{\beta}{4}(\phi_x + \pi)\sigma_3 - i\lambda\frac{m}{4}\,\sigma_2 + i\frac{m}{4\lambda}\Omega^{-2}\,\sigma_2\,\Omega^2\,, \tag{2.36}$$

and $\widehat{\mathcal{T}}$ is a solution of $\Psi_t = \widehat{V}\,\Psi$ where

$$\widehat{V} = -i\frac{\beta}{4}(\phi_t - \Pi)\sigma_3 - i\lambda\frac{m}{4}\,\sigma_2 \; - i\frac{m}{4\lambda}\Omega^{-2}\,\sigma_2\,\Omega^2\,. \tag{2.37}$$

Observe that

$$\lim_{x\to\pm\infty}\widehat{U} = U_\infty = \lim_{t\to\pm\infty}\widehat{U} \quad , \quad \lim_{x\to\pm\infty}\widehat{V} = V_\infty = \lim_{t\to\pm\infty}\widehat{V}\,. \tag{2.38}$$

Accordingly, we introduce

$$E_0(x,\lambda) = N\,e^{-ik_1(\lambda)x\sigma_3} \quad , \quad \mathcal{E}_0(t,\lambda) = N\,e^{-ik_0(\lambda)t\sigma_3}\,, \tag{2.39}$$

and the following half-infinite transition matrices

$$\widehat{T}_\pm(x,t,\lambda) = \lim_{y\to\pm\infty}\widehat{T}(x,y,t,\lambda)E_0(y,\lambda) \quad , \quad \widehat{\mathcal{T}}_\pm(x,t,\lambda) = \lim_{\tau\to\pm\infty}\widehat{\mathcal{T}}(t,\tau,x,\lambda)\mathcal{E}_0(\tau,\lambda)\,. \tag{2.40}$$

Finally, note that the corresponding monodromy matrices

$$\widehat{T}(\lambda) \;=\; \lim_{\substack{x\to\infty\\ y\to-\infty}} E_0^{-1}(x,\lambda)\widehat{T}(x,y,\lambda)E_0(y,\lambda)\,, \tag{2.41}$$

$$\widehat{\mathcal{T}}(\lambda) \;=\; \lim_{\substack{t\to\infty\\ \tau\to-\infty}} \mathcal{E}_0^{-1}(t,\lambda)\widehat{\mathcal{T}}(t,\tau,\lambda)\mathcal{E}_0(\tau,\lambda)\,, \tag{2.42}$$

are equal to $T(\lambda)$ and $\mathcal{T}(\lambda)$ respectively, so that they capture the same information about the system. Using the the zero curvature condition, we can study the time evolution of the transition matrix $\widehat{T}(x,y,\lambda)$ and the space evolution of the transition matrix $\widehat{\mathcal{T}}(t,\tau,\lambda)$. Writing the time and space dependence explicitely, we get

$$\partial_t\,\widehat{T}(x,y,t,\lambda) \;=\; \widehat{V}(x,t,\lambda)\widehat{T}(x,y,t,\lambda) - \widehat{T}(x,y,t,\lambda)\widehat{V}(y,t,\lambda) \tag{2.43}$$

$$\partial_x\,\widehat{\mathcal{T}}(t,\tau,x,\lambda) \;=\; \widehat{U}(x,t,\lambda)\widehat{\mathcal{T}}(t,\tau,x,\lambda) - \widehat{\mathcal{T}}(t,\tau,x,\lambda)\widehat{U}(x,\tau,\lambda)\,. \tag{2.44}$$

These equations allows us to deduce the time evolution of $\widehat{T}(\lambda)$ and the space evolution of $\widehat{\mathcal{T}}(\lambda)$,

$$\partial_t\widehat{T}(t,\lambda) \;=\; -ik_0(\lambda)[\sigma_3,\widehat{T}(t,\lambda)]\,, \tag{2.45}$$

$$\partial_x\widehat{\mathcal{T}}(x,\lambda) \;=\; -ik_1(\lambda)[\sigma_3,\widehat{\mathcal{T}}(x,\lambda)]\,. \tag{2.46}$$

This shows that the $(1,1)$ entry of $\widehat{T}(\lambda)$ is time-independent and we denote it $a(\lambda)$. Similarly, the the $(1,1)$ entry of $\widehat{\mathcal{T}}(\lambda)$ is space-independent and we denote it $\mathfrak{a}(\lambda)$.

The key to deriving the conservation laws is the following result that we prove in Appendix A

$$\widehat{T}_-(x,t,\lambda)e^{-ik_0t\sigma_3} = \widehat{\mathcal{T}}_-(x,t,\lambda)e^{-ik_1x\sigma_3}\,. \tag{2.47}$$

We denote the common value by

$$R(x,t,\lambda)e^{-i(k_1x+k_0t)\sigma_3}\,, \tag{2.48}$$

which is then the solution of

$$\begin{cases}\partial_x R = \widehat{U}R + ik_1R\sigma_3\,,\\ \partial_t R = \widehat{V}R + ik_0R\sigma_3\,,\end{cases} \tag{2.49}$$

satisfying

$$\lim_{x\to-\infty} R(x,t,\lambda) = N = \lim_{t\to-\infty} R(x,t,\lambda)\,. \tag{2.50}$$

Generalising the standard approach to the present setting, we can extract the conservation laws as $\lambda \to \infty$ by writing

$$R(x,t,\lambda) = \frac{1}{\sqrt{2}}(\mathbb{I} + \Gamma(x,t,\lambda))e^{Y(x,t,\lambda)}\,, \tag{2.51}$$

where Y is a diagonal matrix with expansion

$$Y(x,t,\lambda) = i\sum_{n=1}^{\infty} \frac{Y_n(x,t)}{\lambda^n}\,, \tag{2.52}$$

and Γ is an off-diagonal matrix with expansion

$$\Gamma(x,t,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma_n(x,t)}{\lambda^n}\,. \tag{2.53}$$

As a consequence, dropping the arguments of the functions for conciseness, we derive the following equations

$$Y_x = \widehat{U}_d + \widehat{U}_o\Gamma + ik_1(\lambda)\sigma_3\,, \tag{2.54}$$
$$Y_t = \widehat{V}_d + \widehat{V}_o\Gamma + ik_0(\lambda)\sigma_3\,, \tag{2.55}$$

as well as the associated space and time Riccati equations for Γ

$$\Gamma_x = \widehat{U}_o + \widehat{U}_d\Gamma - \Gamma\widehat{U}_d - \Gamma\widehat{U}_o\Gamma\,, \tag{2.56}$$
$$\Gamma_t = \widehat{V}_o + \widehat{V}_d\Gamma - \Gamma\widehat{V}_d - \Gamma\widehat{V}_o\Gamma\,, \tag{2.57}$$

where the subscripts d and o denote the diagonal and off-diagonal parts of the matrices respectively. At this stage, the conservation laws are obtained by cross differentiation of (2.54) and (2.55)

$$\left(\widehat{U}_d + \widehat{U}_o\Gamma + ik_1(\lambda)\sigma_3\right)_t = \left(\widehat{V}_d + \widehat{V}_o\Gamma + ik_0(\lambda)\sigma_3\right)_x\,. \tag{2.58}$$

In particular, integrating in x from $-\infty$ (where $Y = 0$) to ∞, (2.58) yields

$$\partial_t\left(\lim_{x\to\infty} Y(x,t,\lambda)\right) = \lim_{\substack{x\to\infty\\ y\to-\infty}} \left[\widehat{V}_d + \widehat{V}_o\Gamma + ik_0(\lambda)\sigma_3\right]_y^x = 0\,. \tag{2.59}$$

We now show that this is of course consistent with (2.45) which shows that the diagonal part of $\widehat{T}(\lambda)$ is time independent. In the process, we also recover the well-known fact that $\ln a(\lambda)$ generates conserved quantities and, as $\lambda \to \infty$,

$$\ln a(\lambda) = i\sum_{n=1}^{\infty} \frac{I_n}{\lambda^n}\,, \tag{2.60}$$

where here,

$$I_n = \lim_{x\to\infty} y_n(x,t)\,, \tag{2.61}$$

where y_n is the $(1,1)$ entry of Y_n in (2.52). Inserting (2.53) into (2.56), we find $\Gamma_0 = i\sigma_1$ and

$$\Gamma_{n+1} = -\frac{2i}{m}\Gamma_{nx}\sigma_3 - \frac{\beta}{m}(\pi+\phi_x)\Gamma_n - \frac{i}{2}\sigma_1(e^{i\beta\phi\sigma_3} - e^{-i\beta\phi\sigma_3})\delta_{n,1} \tag{2.62}$$

$$+\frac{i}{2}\left(\sum_{p=1}^{n}\Gamma_p\,\sigma_1\,\Gamma_{n+1-p} - \sum_{p=0}^{n-1}\Gamma_p\,\sigma_1\,e^{i\beta\phi\sigma_3}\,\Gamma_{n-1-p}\right). \tag{2.63}$$

Then, (2.54) yields

$$Y_n(x,t) = -\frac{m}{4}\int_{-\infty}^{x}\sigma_2\left(\Gamma_{n+1}(\xi,t) - e^{i\beta\phi(\xi,t)\sigma_3}\Gamma_{n-1}(\xi,t) + i\sigma_1\,\delta_{n,1}\right)d\xi\,. \tag{2.64}$$

Hence, noting that for $n \geq 1$, $\Gamma_n(x,t) \to 0$ as $x \to -\infty$, we find, as $\lambda \to \infty$,

$$\widehat{T}(\lambda) = e^{P(\lambda)}\,, \tag{2.65}$$

where

$$P(\lambda) = \lim_{x\to\infty} Y(x,t,\lambda)\,, \tag{2.66}$$

as required. In particular, as $\lambda \to \infty$, $\ln a(\lambda)$ coincides with the $(1,1)$ entry of $\lim_{x\to\infty} Y(x,t,\lambda)$, which is indeed time independent, and (2.60) holds. To complete the analysis, one has to study the behaviour of the previous quantities as $\lambda \to 0$. This is largely simplified by noting that if $\Gamma(x,t,\lambda)$ is solution of (2.56) then $e^{i\beta\phi\sigma_3}\Gamma(x,t,-\frac{1}{\lambda})$ is solution of (2.56) with ϕ changed to $-\phi$ everywhere. Therefore, we deduce that, as $\lambda \to 0$,

$$R(x,t,\lambda) = \frac{1}{\sqrt{2}}(\mathbb{I} + \Gamma(x,t,\lambda))e^{Y(x,t,\lambda)}\,, \tag{2.67}$$

with

$$\Gamma(x,t,\lambda) = \sum_{n=0}^{\infty}(-1)^n\Gamma_n^{'}(x,t)\lambda^n\,, \tag{2.68}$$

and $\Gamma_n^{'}$ is obtained from Γ_n by changing ϕ to $-\phi$. Finally, we obtain

$$Y(x,t,\lambda) = i\sum_{n=0}^{\infty}Y_n^{'}(x,t)\lambda^n\,, \tag{2.69}$$

with

$$Y_0^{'}(x,t) = -\frac{\beta}{2}\int_{-\infty}^{x}\partial_\xi\phi(\xi,t)\,d\xi\,, \tag{2.70}$$

and, for $n \geq 1$,

$$Y_n^{'}(x,t) = -\frac{m}{4}\int_{-\infty}^{x}\sigma_2\left((-1)^{n-1}e^{-\beta\phi(\xi,t)\sigma_3}\Gamma_{n-1}^{'}(\xi,t) - (-1)^{n+1}\Gamma_{n+1}^{'}(\xi,t) - i\sigma_1\,\delta_{n,1}\right)d\xi\,, \tag{2.71}$$

Therefore, as $\lambda \to 0$, one can deduce that

$$\ln a(\lambda) = i\sum_{n=0}^{\infty}I_{-n}\lambda^n \tag{2.72}$$

where

$$I_{-n} = \lim_{x\to\infty} y_n^{'}(x,t)\,, \tag{2.73}$$

and $y_n^{'}$ is the $(1,1)$ entry of $Y_n^{'}$. In particular, an explicit calculation yields

$$I_{-1} - I_1 = \frac{\beta^2}{2m}\int_{\infty}^{\infty}\left[\frac{1}{2}(\pi^2+\phi_x^2) + \frac{m^2}{\beta^2}(1-\cos\beta\phi)\right]dx = \frac{\beta^2}{2m}H_S\,, \tag{2.74}$$

where H_S is the Hamiltonian in (2.14).

Similarly, integrating (2.58) in t from $-\infty$ (where $Y=0$) to ∞, we can perform an analogous analysis in the new picture and extract local conserved quantities in space. Indeed,

$$\partial_x\left(\lim_{t\to\infty} Y(x,t,\lambda)\right) = \lim_{\substack{t\to\infty\\ \tau\to-\infty}}\left[\widehat{U}_d + \widehat{U}_o\Gamma + ik_1(\lambda)\sigma_3\right]_{\tau}^{t} = 0\,. \tag{2.75}$$

In this second picture, one should use (2.57) and (2.55) instead of (2.56) and (2.54) to study the asymptotic behaviour as $\lambda\to\infty$ and $\lambda\to 0$ and extract the conserved quantities. Doing so, as $\lambda\to\infty$ we find

$$\Gamma(x,t,\lambda) = \sum_{n=0}^{\infty}\frac{\gamma_n(x,t)}{\lambda^n}\,, \tag{2.76}$$

with $\gamma_0 = i\sigma_1$ and

$$\begin{aligned}\gamma_{n+1} \;=\; & -\frac{2i}{m}\gamma_{nt}\sigma_3 - \frac{\beta}{m}(\phi_t-\Pi)\gamma_n + \frac{i}{2}\sigma_1(e^{i\beta\phi\sigma_3}-e^{-i\beta\phi\sigma_3})\delta_{n,1}\\ & +\frac{i}{2}\left(\sum_{p=1}^{n}\gamma_p\,\sigma_1\,\gamma_{n+1-p} + \sum_{p=0}^{n-1}\gamma_p\,\sigma_1\,e^{i\beta\phi\sigma_3}\,\gamma_{n-1-p}\right)\,,\end{aligned} \tag{2.77}$$

as well as

$$Y(x,t,\lambda) = i\sum_{n=1}^{\infty}\frac{\mathcal{Y}_n(x,t)}{\lambda^n}\,, \tag{2.78}$$

where

$$\mathcal{Y}_n(x,t) = -\frac{m}{4}\int_{-\infty}^{t}\sigma_2\left(\gamma_{n+1}(x,\tau) + e^{i\beta\phi(x,\tau)\sigma_3}\gamma_{n-1}(x,\tau) - i\sigma_1\,\delta_{n,1}\right)d\tau. \tag{2.79}$$

Therefore, in view of (2.42) and the fact that $\gamma_n(x,t)\to 0$ as $t\to-\infty$ for $n\geq 1$, we obtain that as $\lambda\to\infty$,

$$\ln\mathfrak{a}(\lambda) = i\sum_{n=1}^{\infty}\frac{J_n}{\lambda^n}\,, \tag{2.80}$$

where

$$J_n = \lim_{t\to\infty}\Upsilon_n(x,t)\,, \tag{2.81}$$

and Υ_n is the $(1,1)$ entry of $\mathcal{Y}_n$. So here again, in the new picture, we obtain that $\ln \mathfrak{a}(\lambda)$ generates conserved quantities (in space) J_n. To obtain the behaviour as $\lambda \to 0$, we make a similar observation as before to relate it to the behaviour as $\lambda \to \infty$. In the present case, we find that as $\lambda \to 0$

$$\Gamma(x,t,\lambda) = e^{-i\beta\phi(x,t)\sigma_3}\Gamma^{'}(x,t,\frac{1}{\lambda}), \tag{2.82}$$

where the dot means that $\Gamma^{'}$ solves (2.57) with ϕ changed to $-\phi$. Hence, we deduce

$$\Gamma(x,t,\lambda) = e^{-i\beta\phi(x,t)\sigma_3}\sum_{n=0}^{\infty}\gamma_n^{'}(x,t)\lambda^n, \tag{2.83}$$

where $\gamma_n^{'}$ is γ_n with $\phi \to -\phi$. Accordingly, as $\lambda \to 0$,

$$Y(x,t,\lambda) = i\sum_{n=0}^{\infty}\mathcal{Y}_n^{'}(x,t)\lambda^n, \tag{2.84}$$

with

$$\mathcal{Y}_0^{'}(x,t) = -\frac{\beta}{2}\int_{-\infty}^{t}\phi_\tau(x,\tau)\,d\tau, \tag{2.85}$$

$$\mathcal{Y}_n^{'}(x,t) = -\frac{m}{4}\int_{-\infty}^{t}\sigma_2\left[e^{-i\beta\phi(x,\tau)\sigma_3}\gamma_{n-1}^{'}(x,\tau)+\gamma_{n+1}^{'}(x,\tau)-i\sigma_1\,\delta_{n,1}\right]d\tau\,, n\geq 1\,. \tag{2.86}$$

In particular, as $\lambda \to 0$,

$$\ln \mathfrak{a}(\lambda) = i\sum_{n=0}^{\infty}J_{-n}\lambda^n \tag{2.87}$$

where

$$J_{-n} = \lim_{t\to\infty}\Upsilon_n^{'}(x,t) \tag{2.88}$$

and $\Upsilon_n^{'}$ is the $(1,1)$ entry of $\mathcal{Y}_n^{'}$. As an important check, an explicit calculation yields the new Hamiltonian H_T from (2.9),(2.14), as the following combination of integrals

$$J_1 + J_{-1} = -\frac{\beta^2}{2m}\int_{\infty}^{\infty}\left[\frac{1}{2}(\Pi^2+\phi_t^2)-\frac{m^2}{\beta^2}(1-\cos\beta\phi)\right]dx = \frac{\beta^2}{2m}H_T\,. \tag{2.89}$$

Let us summarize the results for clarity. Writing,

$$I(\lambda) = \int_{-\infty}^{\infty}(\widehat{U}_d + \widehat{U}_o\Gamma + ik_1(\lambda)\sigma_3)dx\,, \tag{2.90}$$

then

$$\partial_t\, I(\lambda) = 0\,. \tag{2.91}$$

Using the space Riccati equation (2.56) for Γ, one extracts conserved quantities in time by inserting an expansion of Γ as $\lambda \to \infty$ and $\lambda \to 0$. This gives a set I_n, $n \in \mathbb{Z}$ and usual charges are linear combinations of these quantities, e.g. the Hamiltonian H_S is proportional to $I_{-1} - I_1$ and the

topological charge $Q_+ - Q_-$ is proportional to I_0. The $(1,1)$ entry $a(\lambda)$ of the monodromy matrix $\widehat{T}(\lambda) = T(\lambda)$ generates the I_n's according to

$$\ln a(\lambda) = i \sum_{n=1}^{\infty} \frac{I_n}{\lambda^n} \quad , \quad \lambda \to \infty \,, \tag{2.92}$$

$$\ln a(\lambda) = i \sum_{n=0}^{\infty} I_{-n} \lambda^n \quad , \quad \lambda \to 0 \,. \tag{2.93}$$

In the new picture, writing,

$$J(\lambda) = \int_{-\infty}^{\infty} (\widehat{V}_d + \widehat{V}_o \Gamma + i k_0(\lambda) \sigma_3) dx \,, \tag{2.94}$$

then

$$\partial_x \, J(\lambda) = 0 \,. \tag{2.95}$$

Using the time Riccati equation (2.57) for Γ, one extracts conserved quantities in space by inserting an expansion of Γ as $\lambda \to \infty$ and $\lambda \to 0$. This gives a set J_n, $n \in \mathbb{Z}$ and relevant charges in the new picture are linear combinations of these quantities, e.g. the Hamiltonian H_T is proportional to $J_{-1} + J_1$ and the topological charge $\mathcal{Q}_+ - \mathcal{Q}_-$ is proportional to J_0. The $(1,1)$ entry $\mathfrak{a}(\lambda)$ of the monodromy matrix $\widehat{\mathcal{T}}(\lambda) = \mathcal{T}(\lambda)$ generates the J_n's according to

$$\ln \mathfrak{a}(\lambda) = i \sum_{n=1}^{\infty} \frac{J_n}{\lambda^n} \quad , \quad \lambda \to \infty \,, \tag{2.96}$$

$$\ln \mathfrak{a}(\lambda) = i \sum_{n=0}^{\infty} J_{-n} \lambda^n \quad , \quad \lambda \to 0 \,. \tag{2.97}$$

2.3 Classical r-matrix approach for the two Poisson brackets

The combination of the Lax pair approach to conserved quantities together with the Hamiltonian approach to integrable field theories culminates in the so-called classical r-matrix approach [24]. In our context, it gives a convenient way of combining the multisymplectic Hamiltonian approach to the model with the existence of an infinite hierarchy of conserved quantities. The latter appear as functions that are in involution with respect to the Poisson bracket used to describe the model. The novelty here is the classical r matrix approach in terms of the new bracket $\{\ ,\ \}_T$ and the associated role of the conserved quantities in space encoded in $\mathfrak{a}(\lambda)$, the $(1,1)$ entry of the time monodromy matrix $\mathcal{T}(\lambda)$.

2.3.1 The standard approach with $\{\ ,\ \}_S$

Recall that U in (2.16) is given by

$$U(x,\lambda) = -i\frac{\beta}{4}\pi(x)\sigma_3 - i k_0(\lambda) \sin\left(\frac{\beta\phi(x)}{2}\right)\sigma_1 - i k_1(\lambda)\cos\left(\frac{\beta\phi(x)}{2}\right)\sigma_2 \,. \tag{2.98}$$

where we have dropped the time variable since it assumed to be fixed at some initial value in this approach. Then, following a standard calculation (see e.g. [23]), one finds

$$\{U_1(x,\lambda), U_2(y,\mu)\}_S = \delta(x-y)\left[r(\lambda,\mu), U_1(x,\lambda) + U_2(y,\mu)\right] \,, \tag{2.99}$$

where we have used the notation $U_1 = U \otimes \mathbb{1}$, $U_2 = \mathbb{1} \otimes U$ and $r(\lambda, \mu)$ is the classical r-matrix given by

$$r(\lambda, \mu) = f(\lambda, \mu)(\mathbb{1}_2 \otimes \mathbb{1}_2 - \sigma_3 \otimes \sigma_3) + g(\lambda, \mu)(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)\,, \tag{2.100}$$

with

$$f(\lambda, \mu) = -\gamma \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2} \quad , \quad g(\lambda, \mu) = 2\gamma \frac{\lambda \mu}{\lambda^2 - \mu^2} \quad , \quad \gamma = \frac{\beta^2}{16}\,. \tag{2.101}$$

Setting $\lambda = e^{i\alpha}$, $\mu = e^{i\beta}$, it is conveniently rewritten in trigonometric form where it only depends on the difference $\alpha - \beta$ and

$$r(\alpha) = \frac{i\gamma}{\sin \alpha} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos \alpha & -1 & 0 \\ 0 & -1 & \cos \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{2.102}$$

For the transition matrix $T(x, y, \lambda)$, $y < x$, one then finds

$$\{T_1(x, y, \lambda), T_2(x, y, \mu)\}_S = [r(\lambda, \mu), T(x, y, \lambda) \otimes T(x, y, \lambda)]\,, \tag{2.103}$$

Using definition (2.41) for the monodromy matrix and a careful limiting procedure (see Section II.6 of [23]), one obtains the infinite volume Poisson brackets for the monodromy matrix $T(\lambda)$ as

$$\{T_1(\lambda), T_2(\mu)\}_S = r_+(\lambda, \mu)\, T_1(\lambda)\, T_2(\mu) - T_1(\lambda)\, T_2(\mu)\, r_-(\lambda, \mu)\,, \tag{2.104}$$

where

$$r_\pm(\lambda, \mu) = -\frac{\gamma}{2} \begin{pmatrix} \frac{\lambda-\mu}{\lambda+\mu} & 0 & 0 & 0 \\ 0 & p.v.\frac{\lambda+\mu}{\lambda-\mu} & \mp i\pi(\lambda+\mu)\delta(\lambda-\mu) & 0 \\ 0 & \pm i\pi(\lambda+\mu)\delta(\lambda-\mu) & p.v.\frac{\lambda+\mu}{\lambda-\mu} & 0 \\ 0 & 0 & 0 & \frac{\lambda-\mu}{\lambda+\mu} \end{pmatrix}\,. \tag{2.105}$$

One can then conclude that $a(\lambda)$ Poisson commutes with $a(\mu)$

$$\{a(\lambda), a(\mu)\}_S = 0\,. \tag{2.106}$$

This is the key relation showing that the integrals of motion I_n, n$\in \mathbb{Z}$ are in involution with respect to $\{\ ,\ \}_S$. This is taken as the definition of Liouville integrability of the sine-Gordon model.

2.3.2 Classical r-matrix approach for the new bracket $\{\ ,\ \}_T$

In view of the results obtained in Section 2.1, one can easily derive a treatment of the classical r-matrix approach to sine-Gordon with respect to new Poisson bracket $\{\ ,\ \}_T$ that is completely analogous to the standard one reviewed previously. Starting with

$$V(t, \lambda) = i\frac{\beta}{4}\Pi(t)\sigma_3 - ik_1(\lambda)\sin\left(\frac{\beta\phi(t)}{2}\right)\sigma_1 - ik_0(\lambda)\cos\left(\frac{\beta\phi(t)}{2}\right)\sigma_2 \tag{2.107}$$

and using (2.11), one obtains the following result by direct calculation.

Proposition 2.1 *Let the Poisson bracket* $\{\ ,\ \}_T$ *be given by* (2.11) *and* V *be given by* (2.107). *Then,*

$$\{V_1(t,\lambda), V_2(\tau,\mu)\}_T = -\delta(t-\tau)\left[r(\lambda,\mu), V_1(t,\lambda)+V_2(\tau,\mu)\right] , \tag{2.108}$$

with the same classical r*-matrix as in* (2.100).

As a direct consequence, we obtain for the transition matrix the following

Corollary 2.2 *For* $\tau < t$,

$$\{\mathcal{T}_1(t,\tau,\lambda), \mathcal{T}_2(t,\tau,\mu)\}_T = -\left[r(\lambda,\mu), \mathcal{T}(t,\tau,\lambda)\otimes\mathcal{T}(t,\tau,\mu)\right] , \tag{2.109}$$

and,

$$\{\mathcal{T}_1(\lambda), \mathcal{T}_2(\mu)\}_T = -r_+(\lambda,\mu)\,\mathcal{T}_1(\lambda)\,\mathcal{T}_2(\mu) + \mathcal{T}_1(\lambda)\,\mathcal{T}_2(\mu)\,r_-(\lambda,\mu)\,, \tag{2.110}$$

where $r_\pm$ *is given by* (2.105). *In particular,*

$$\{\mathfrak{a}(\lambda), \mathfrak{a}(\mu)\}_T = 0\,. \tag{2.111}$$

We now get that the integrals J_n, $n\in\mathbb{Z}$ generated by $\ln\mathfrak{a}(\lambda)$ are in involution with respect to $\{\ ,\ \}_T$. In this picture, this fact can be taken as a definition of Liouville integrability of the model which is alternative (and equivalent) to the standard one given in the previous section. From the point of the view of the classical r matrix, the two points of view (space or time) are once again equivalent.

3 Sine-Gordon model with a defect: Liouville integrability

3.1 Defect conditions as "frozen" Bäcklund transformations

3.1.1 Review of the Lagrangian approach

Viewing a defect in space as an internal boundary condition on the fields and their time and space derivatives at a given point, the fruitful idea of *frozen* Bäcklund transformations, originally noticed in [11], is a convenient way of introducing integrable defects in classical field theories described by a Lax pair. Initially, starting from a Lagrangian density of the form

$$\mathcal{L} = \theta(-x)\mathcal{L}_{\widetilde{\phi}} + \theta(x)\mathcal{L}_\phi - \delta(x)\left(\frac{1}{2}(\widetilde{\phi}\phi_t - \phi\widetilde{\phi}_t) - \mathcal{B}\right) , \tag{3.1}$$

where, without loss of generality, the location of the defect has been chosen to be $x=0$, it was required that the defect conditions between the sine-Gordon fields ϕ and $\widetilde{\phi}$ on either side of the defect were such that the associated modified momentum was conserved in time. This led to a solution for $\mathcal{B}$ of the form

$$\mathcal{B} = \frac{2m}{\beta^2}\left(\sigma\cos\beta\left(\frac{\widetilde{\phi}+\phi}{2}\right) + \sigma^{-1}\cos\beta\left(\frac{\widetilde{\phi}-\phi}{2}\right)\right) \tag{3.2}$$

and to the defect conditions, at $x=0$,

$$\begin{cases} \widetilde{\phi}_x - \phi_t = \frac{m}{\beta}\left(\sigma\sin\beta\left(\frac{\widetilde{\phi}+\phi}{2}\right) + \sigma^{-1}\sin\beta\left(\frac{\widetilde{\phi}-\phi}{2}\right)\right) , \\ \widetilde{\phi}_t - \phi_x = \frac{m}{\beta}\left(\sigma\sin\beta\left(\frac{\widetilde{\phi}+\phi}{2}\right) - \sigma^{-1}\sin\beta\left(\frac{\widetilde{\phi}-\phi}{2}\right)\right) . \end{cases} \tag{3.3}$$

These conditions also ensured that the modified Hamiltonian and first higher integral were also conserved. They were recognized as frozen Bäcklund transformations of the sine-Gordon model at the location of the defect.

3.1.2 Review of the (traditional) Lax pair approach

Later, this observation was exploited in [13] for a systematic derivation of the generating function of the modified conserved quantities for all models in the AKNS hierarchies [14]. In particular, the sine-Gordon model in light cone coordinates was discussed there. Here, for our purposes, we present the procedure of [13] but for sine-Gordon in laboratory coordinates. The central ingredient is the so-called Bäcklund or defect matrix L which is required to be a solution of

$$L_t = VL - L\widetilde{V} \quad , \quad x = 0 \, , \tag{3.4}$$

where V is the time Lax matrix (2.17) and $\widetilde{V}$ the same time Lax matrix with ϕ, Π replaced by $\widetilde{\phi}, \widetilde{\Pi}$. With an appropriate solution of L, on the one hand one reproduces the defect conditions (3.3) and, on the other hand, L can be used to connect the transition matrices of both theories and obtain the monodromy matrix of the theory on the full line with a defect at $x = 0$. Assuming that the fields ϕ, π, Π describe the model for $x > 0$ and the fields $\widetilde{\phi}, \widetilde{\pi}, \widetilde{\Pi}$ describe the theory for $x < 0$, in the notations of Section 2.2, we see that $\widehat{T}_+^{-1}(0, t, \lambda)$ describes the positive half-line $(0, \infty)$ at time t while $\widehat{\widetilde{T}}_-(0, t, \lambda)$ describes the negative half-line $(-\infty, 0)$ at time t. Therefore, the system on the line with a defect encoded in L is described by the following monodromy matrix

$$\mathcal{M}_S(t, \lambda) = \widehat{T}_+^{-1}(0, t, \lambda) \, \widehat{L}(t, \lambda) \, \widehat{\widetilde{T}}_-(0, t, \lambda) \, , \tag{3.5}$$

where

$$\widehat{L}(t, \lambda) = \Omega(t)^{-1} L(t, \lambda) \widetilde{\Omega}(t) \tag{3.6}$$

is the gauged Bäcklund/defect matrix. A direct computation then yields

$$\partial_t \mathcal{M}_S(t, \lambda) = -ik_0 [\sigma_3, \mathcal{M}_S] \, , \tag{3.7}$$

which show that the diagonal part of $\mathcal{M}_S$ is automatically time independent for any solution L of (3.4). Let us denote by

$$\mathcal{I}(\lambda) = \ln(\mathcal{M}_S)_d \, , \tag{3.8}$$

the logarithm of the diagonal part of $\mathcal{M}_S$. We recover the analog of the results of [13] in the following form

$$\mathcal{I}(\lambda) = I_+(\lambda) + \widetilde{I}_-(\lambda) + I_{defect}(\lambda) \tag{3.9}$$

where

$$I_+(\lambda) = \int_0^\infty \left(\widehat{U}_d - \Gamma \widehat{U}_o + ik_1 \sigma_3 \right) dx \quad , \quad \widetilde{I}_-(\lambda) = \int_{-\infty}^0 \left(\widehat{\widetilde{U}}_d - \widetilde{\Gamma} \widehat{\widetilde{U}}_o + ik_1 \sigma_3 \right) dx \, , \tag{3.10}$$

and

$$I_{defect}(\lambda) = \ln \frac{1}{2} \left(\widehat{L}_d - \Gamma \widehat{L}_o + \widehat{L}_o \widetilde{\Gamma} - \Gamma \widehat{L}_d \widetilde{\Gamma} \right) \, . \tag{3.11}$$

As explained in Section 2.2, in this picture, Γ, resp. $\widetilde{\Gamma}$, is determined from the space Riccati equation (2.56) involving $\widehat{U}$, resp. $\widehat{\widetilde{U}}$.

These are general results valid for any Bäcklund matrix satisfying (3.4). In the present case, the connection with the defect conditions (3.3) goes as follows. We found that the following solution for L is such that (3.4) is equivalent to (3.3),

$$L(t,\lambda) = \Omega\widetilde{\Omega}^{-1} - \frac{i\sigma}{\lambda}\widetilde{\Omega}^{-1}\sigma_2\Omega\,, \tag{3.12}$$

where all the fields ϕ and $\widetilde{\phi}$ are understood as depending on t only for fixed $x = 0$. The quantity $I_{defect}(\lambda)$ is the generating matrix of the so-called defect contributions to the hierarchy of conserved quantities for the model with an integrable defect given by the condition (3.4). In the special case where L is given by (3.12), it gives the defect contributions corresponding to the defect conditions (3.3). Of course, it can be checked that the present results agree with the results obtained in [11, 13].

3.1.3 Lax pair approach in the new picture

In the new picture, for $x > 0$, the system is described by the time monodromy matrix $\widehat{\mathcal{T}}(x,\lambda)$ defined as in (2.42) For $x < 0$, it is described by the time monodromy matrix $\widehat{\widetilde{\mathcal{T}}}(x,\lambda)$, defined similarly, but with ϕ, π, Π replaced by $\widetilde{\phi}, \widetilde{\pi}, \widetilde{\Pi}$. Recall that the space evolution of these monodromy matrices is given by (2.46). The theory on the positive half-line is assumed to be continuous as $x \to 0^+$ and the theory on the negative half-line is assumed to be continuous as $x \to 0^-$. At $x = 0$, the defect conditions (3.3) relating the fields on each side, and encoded in (3.4), translate into the following connection formula for the time transition matrices $\widehat{\mathcal{T}}(t,\tau,0,\lambda)$ and $\widehat{\widetilde{\mathcal{T}}}(t,\tau,0,\lambda)$

$$\widehat{\mathcal{T}}(t,\tau,0,\lambda) = \widehat{L}(t,\lambda)\widehat{\widetilde{\mathcal{T}}}(t,\tau,0,\lambda)\widehat{L}(\tau,\lambda)^{-1}\,. \tag{3.13}$$

In turn, this gives,

$$\widehat{\mathcal{T}}(0,\lambda) = B_+(\lambda)\widehat{\widetilde{\mathcal{T}}}(0,\lambda)B_-^{-1}(\lambda)\,, \tag{3.14}$$

where

$$B_\pm(\lambda) = \frac{\lambda}{\lambda + i\sigma}\left(\mathbb{I} - \frac{i\sigma}{\lambda}(-1)^{\widetilde{\mathcal{Q}}_\pm + \mathcal{Q}_\pm}\sigma_3\right)\,. \tag{3.15}$$

In particular, the generating functions of the conserved quantities are related by

$$\ln \mathfrak{a}(\lambda) = \ln \widetilde{\mathfrak{a}}(\lambda) + \ln C(\lambda)\,, \tag{3.16}$$

where

$$C(\lambda) = \frac{1}{\lambda + i\sigma}(\lambda - i\sigma(-1)^{\widetilde{\mathcal{Q}}_+ + \mathcal{Q}_+})(\lambda + i\sigma(-1)^{\widetilde{\mathcal{Q}}_- + \mathcal{Q}_-})\,. \tag{3.17}$$

Hence, we find that the new Hamiltonians H_T and $\widetilde{H}_T$ are related by

$$H_T = \widetilde{H}_T + \frac{2m}{\beta^2}(\sigma + \frac{1}{\sigma})\left((-1)^{\widetilde{\mathcal{Q}}_+ + \mathcal{Q}_+} - (-1)^{\widetilde{\mathcal{Q}}_- + \mathcal{Q}_-}\right)\,. \tag{3.18}$$

3.2 Defect conditions as canonical transformations

We are now ready to tackle the crux of the matter and to show that the integrable defect conditions for sine-Gordon considered so far in the literature, and given here by (3.3), are in fact canonical transformations between the fields (ϕ, Π) and $(\widetilde{\phi}, \widetilde{\Pi})$ with respect to the new Poisson structure given by $\{\ ,\ \}_T$.

The idea is to adapt the results obtained in [25, 26] to the present multisymplectic approach. In doing so, we obtain a new interpretation of the defect density in the Lagrangian (3.1)

$$\mathcal{L}_{defect} = \frac{1}{2}(\widetilde{\phi}\phi_t - \phi\widetilde{\phi}_t) - \mathcal{B} \tag{3.19}$$

as the density for the generating functional of the canonical transformation correponding to the defect conditions. Let us first review briefly the idea of [25, 26]. Given that one deals with an integrable model, one consider transformations that leave not only the form of Hamilton's equations invariant but also the form of all the integrals of motion. The usual requirement is that the transformation $(q(x), p(x)) \mapsto (Q(x), P(x))$, $H \mapsto K$ be such that[2]

$$\int (P\, dQ)\, dx - K\, dt = \int (p\, dq)\, dx - H\, dt + dF\,, \tag{3.20}$$

for some F called the generating functional of the canonical transformation. Restricting our attention to F of the form

$$F = S[q, p, Q, P] - Et \tag{3.21}$$

for some constant E, one gets in particular,

$$K(Q, P) = H(q, p) + E\,, \tag{3.22}$$

and, assuming that the new variables do not depend explicitely on t and taking q and Q as the functionally independent variables,

$$p = \frac{\delta S}{\delta q}\quad,\quad P = -\frac{\delta S}{\delta Q}\,. \tag{3.23}$$

For an integrable PDE, [25, 26] extend (3.22) to all the integrals of the motion and require

$$K_n(Q, P) = H_n(q, p) + E_n\,, \tag{3.24}$$

or, equivalently, for the densities,

$$\mathcal{K}_n(Q, P) = \mathcal{H}_n(q, p) + \partial_x \mathcal{E}_n(q, p, Q, P, q_x, p_x, Q_x, P_x, \dots)\,, \tag{3.25}$$

in which case

$$E_n = [\mathcal{E}_n]_{-\infty}^{\infty}\,. \tag{3.26}$$

Guided with this principle, they considered several well-known integrable PDEs and were able to find in each case a solution for S such that the transformation formula (3.23) yields precisely the well-known Bäcklund transformations for the model under consideration.

[2] At this point, we use generic notations for the canonical variables and the Hamiltonians to illustrate the main ingredients of the approach in [25, 26].

For our purposes, the role of the two independent variables x and t can easily be interchanged in the previous general discussion. Therefore, we consider tranformations $(q(t), p(t)) \mapsto (Q(t), P(t))$, $H_T \mapsto K_T$ such that

$$\int (P\, dQ)\, dt - K_T\, dx = \int (p\, dq)\, dt - H_T\, dx + dF_T\,, \tag{3.27}$$

for some F_T which we regard as the generating functional of the canonical transformation with respect to the new bracket $\{\ ,\ \}_T$. One also requires that all the integrals of motion (in space now) K_{n_T} and H_{n_T} satisfy

$$K_{n_T}(Q, P) = H_{n_T}(q, p) + E_{n_T}\,. \tag{3.28}$$

Choosing F_T of the form

$$F_T = S_T[q, p, Q, P] - E_T x \tag{3.29}$$

for some constant E_T, we get

$$K_T(Q, P) = H_T(q, p) + E_T\,, \tag{3.30}$$

and, assuming that the new variables do not depend explicitely on x and taking q and Q as the functionally independent variables,

$$p = \frac{\delta S}{\delta q} \quad , \quad P = -\frac{\delta S}{\delta Q}\,. \tag{3.31}$$

Let us apply this approach to the situation of the sine-Gordon model in the presence of a defect described by (3.3) at $x = 0$. For $x < 0$, the model is described by the fields $\widetilde{\phi}, \widetilde{\Pi}$ and the integrals of motion $\widetilde{J}_n$, play the role of K_{n_T}. For $x > 0$, the model is described by the field ϕ, Π and the integrals of motion J_n, play the role of H_{n_T}.

Therefore, in view of (3.16), condition (3.28) is easily seen to hold in our context. One simply expands $C(\lambda)$ as $\lambda \to \infty$ or $\lambda \to 0$ to identify the constants E_{n_T} of (3.28) from the defect contributions to J_n and $\widetilde{J}_n$. To complete the discussion about the canonical nature of the defect conditions, we simply have to find a generating functional S_T such that eqs (3.31) reproduce eqs (3.3). In the present situation, eqs (3.31) take the form

$$\Pi(0, t) = \frac{\delta S_T}{\delta \phi(0, t)} \quad , \quad \widetilde{\Pi}(0, t) = -\frac{\delta S_T}{\delta \widetilde{\phi}(0, t)}\,. \tag{3.32}$$

Choosing

$$S_T[\phi, \Pi, \widetilde{\phi}, \widetilde{\Pi}] = \int_{-\infty}^{\infty} \mathcal{L}_{defect}\, dt = \int_{-\infty}^{\infty} \left(\frac{1}{2}(\widetilde{\phi}\phi_t - \phi\widetilde{\phi}_t) - \mathcal{B} \right) dt \tag{3.33}$$

where $\mathcal{B}$ is given by eq (3.2), (3.32) become

$$\begin{cases} \Pi(0, t) = \frac{\partial \mathcal{L}_{defect}}{\partial \phi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}_{defect}}{\partial \phi_t}\,, \\ \widetilde{\Pi}(0, t) = -\left(\frac{\partial \mathcal{L}_{defect}}{\partial \widetilde{\phi}} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}_{defect}}{\partial \widetilde{\phi}_t} \right)\,. \end{cases} \tag{3.34}$$

It can be verified by direct calculation that these are exactly the defect conditions (3.3). Summarizing, we have proved the following

Proposition 3.1 *The defect conditions* (3.3) *are canonical transformations for the Poisson bracket* $\{\ ,\ \}$. *They can be conveniently written as*

$$\Pi(0,t) = \frac{\delta S_T}{\delta\phi(0,t)} \quad , \quad \widetilde{\Pi}(0,t) = \frac{\delta S_T}{\delta\widetilde{\phi}(0,t)}\,, \tag{3.35}$$

where S_T *given in* (3.33) *and is the time integral of the defect Lagrangian density* $\mathcal{L}_{defect}$. *The form of all the conserved quantities (in space) is preserved in the sense of* (3.28).

Recalling that on the solutions of the equations of motion, $\Pi = \phi_x$ and $\widetilde{\Pi} = \widetilde{\phi}_x$, we note that eqs (3.34) are exactly the defect conditions obtained in the Lagrangian approach by applying the variational principle to $\mathcal{L}$ in (3.1).

3.3 Liouville integrability: classical r-matrix approach with defect

From the results of Sections 3.1.3 and 3.2, the natural objects to study the classical r-matrix approach for the full problem with a defect, while avoiding the usual problem, are the Poisson bracket $\{\ ,\ \}_T$ and the following monodromy matrix $\mathcal{M}_T(x,\lambda)$ defined by

$$\mathcal{M}_T(x,\lambda) = \begin{cases} B_+(\lambda)\widehat{\widetilde{\mathcal{T}}}(x,\lambda)B_-^{-1}(\lambda) \quad , \ x \leq 0\,, \\ \widehat{\mathcal{T}}(0,\lambda) = B_+(\lambda)\widehat{\widetilde{\mathcal{T}}}(0,\lambda)B_-^{-1}(\lambda) \quad , \quad x = 0\,, \\ \widehat{\mathcal{T}}(0,\lambda) \quad , \quad x \geq 0\,, \end{cases} \tag{3.36}$$

where $B_\pm$ is given in (3.15). Note that in the traditional approach, one would consider instead $\mathcal{M}_S$ in (3.5) to describe the system and then, one would try to compute its Poisson brackets with respect to $\{\ ,\ \}_S$. The source of all complications with this approach is the evaluation of Poisson brackets involving $L(t,\lambda)$. Instead, here, in view of Corollary 2.2, we obtain immediately, for all $x \in \mathbb{R}$,

$$\begin{aligned} \{\mathcal{M}_{T1}(x,\lambda), \mathcal{M}_{T2}(x,\mu)\}_T = & -r_+(x,\lambda,\mu)\,\mathcal{M}_{T1}(x,\lambda)\,\mathcal{M}_{T2}(x,\mu) \\ & +\mathcal{M}_{T1}(x,\lambda)\,\mathcal{M}_{T2}(x,\mu) r_-(x,\lambda,\mu)\,, \end{aligned} \tag{3.37}$$

where

$$r_\pm(x,\lambda,\mu) = \left(e^{ik_1(\lambda)x\sigma_3} \otimes e^{ik_1(\mu)x\sigma_3}\right) r_\pm(\lambda,\mu) \left(e^{-ik_1(\lambda)x\sigma_3} \otimes e^{-ik_1(\mu)x\sigma_3}\right) \tag{3.38}$$

and $r_\pm$ is given by (2.105). This immediately implies

$$\{\mathfrak{a}(\lambda), \mathfrak{a}(\mu)\}_T = 0 \quad , \quad \{\widetilde{\mathfrak{a}}(\lambda), \widetilde{\mathfrak{a}}(\mu)\}_T = 0\,, \tag{3.39}$$

as desired. In particular, the integrals J_n are in involution with respect to $\{\ ,\ \}_T$ and the same is true of the integrals and $\widetilde{J}_n$. We can conclude that the sine-Gordon model with an integrable defect given by (3.3) is Liouville integrable.

4 Conclusions

We used the idea of multisymplectic formalism to discuss Liouville integrability of the sine-Gordon model with a (Bäcklund) defect, thereby showing that the ideas introduced in [19], and illustrated explicitely there for the nonlinear Schrödinger equation, apply equally well to another famous example of integrable field theory. The advantages of the multisymplectic formalism are threefold:

- it naturally introduces a new Poisson structure for the model, in terms of which the various approaches to integrability in the presence of a defect given by a frozen Bäcklund transformation are unified. The bottom line is that a Bäcklund defect is simply a special type of canonical transformation performed at a given point in space, such that all the integrals of motion (in space) are preserved in form. Of course, to see this, one had first to identify the appropriate Poisson structure to be used. The Lagrangian approach produces the density appearing in the generating functional describing the defect conditions as canonical transformations with respect to the new bracket. The Lax pair approach produces the defect/Bäcklund matrix in terms of which the transformation formulas from the old variables to the new variables can be calculated explicitely, as well as all the corresponding transformation formulas between old and new conserved quantities.

- The new Poisson structure allows one to study Liouville integrability of the model with defect directly, by using a "dual" picture to the traditional one, whereby the system is described by a time monodromy matrix instead of the usual space monodromy matrix. When there is no defect, the two pictures are completely equivalent and the choice of coordinate is a matter of preference, dictated by the traditional notion of time evolution of a system. The introduction of a defect in space naturally distinguishes between space and time coordinates. In this case, the use of the new Poisson brackets is no longer a simple relabelling of coordinates but appears to be the appropriate way of treating the question of Liouville integrability in the presence of a defect.

- The use of the new Poisson brackets shows an interesting duality in the classical r matrix approach. The time part of the Lax pair satisfies the same Poisson algebra as the space part of the Lax pair (up to a minus sign). While this might no be so surprising for the sine-Gordon model studied in this paper (given the natural symmetric role of x and t for this model), the same fact was found also for the nonlinear Schrödinger equation in [19]. It is an interesting question to know if this holds for other well-known integrable systems with a Lax pair. This would suggest a connection between the multisymplectic formalism and integrable systems that goes well beyond the particular focus on defects that we have studied here.

Given that the method of the classical r matrix goes over very naturally in our formalism, an interesting question would be to revisit the question of an integrable defect at the quantum level and its relation to the quantum R matrix. For the sine-Gordon model, some aspects of this problem have already been investigated in [27, 28] and quantum transmission matrices were found. Ultimately, a challenging problem would be to extend our results beyond the purely transmitting Bäcklund defects and find classical systems which correspond to a classical limit of the Reflection-Transmission algebras [4, 10].

Appendix

A Proof of Eq. (2.47)

We prove

$$\widehat{T}_-(x,t,\lambda)e^{-ik_0t\sigma_3} = \widehat{\mathcal{T}}_-(x,t,\lambda)e^{-ik_1x\sigma_3}\,. \tag{A.1}$$

Recall that, from the definition of $T(x,y,\lambda)$, we obtain that $\widehat{T}(x,y,t,\lambda)$ satisfies

$$\partial_x\widehat{T}(x,y,t,\lambda) = \widehat{U}(x,t,\lambda)\widehat{T}(x,y,t,\lambda)\,, \tag{A.2}$$

$$\partial_t\widehat{T}(x,y,t,\lambda) = \widehat{V}(x,t,\lambda)\widehat{T}(x,y,t,\lambda) - \widehat{T}(x,y,t,\lambda)\widehat{V}(y,t,\lambda)\,. \tag{A.3}$$

Let us set

$$\Psi(x,t,\lambda) = e^{i\varphi(x,t,\lambda)\sigma_3} N^{-1} \widehat{T}_-(x,t,\lambda) e^{-ik_0 t\sigma_3} \tag{A.4}$$

where $\widehat{T}_-(x,t,\lambda)$ is defined in (2.40) and

$$\varphi(x,t,\lambda) = k_1(\lambda)x + k_0(\lambda)t\,. \tag{A.5}$$

Then, one derives

$$\partial_x\,\Psi(x,t,\lambda) = e^{i\varphi(x,t,\lambda)\sigma_3}\mathcal{U}(x,t,\lambda)e^{-i\varphi(x,t,\lambda)\sigma_3}\Psi(x,t,\lambda)\,, \tag{A.6}$$

$$\partial_t\,\Psi(x,t,\lambda) = e^{i\varphi(x,t,\lambda)\sigma_3}\mathcal{V}(x,t,\lambda)e^{-i\varphi(x,t,\lambda)\sigma_3}\Psi(x,t,\lambda)\,, \tag{A.7}$$

where

$$\mathcal{U}(x,t,\lambda) = N^{-1}(\widehat{U}(x,t,\lambda) - U_\infty)N\,, \tag{A.8}$$

$$\mathcal{V}(x,t,\lambda) = N^{-1}(\widehat{V}(x,t,\lambda) - V_\infty)N\,. \tag{A.9}$$

Consequently, note that,

$$\lim_{x\to-\infty}\Psi(x,t,\lambda) = \mathbb{I}\,, \tag{A.10}$$

and therefore we can write

$$\Psi(x,t,\lambda) = \mathbb{I} + \int_{-\infty}^{x} e^{i\varphi(\xi,t,\lambda)\sigma_3}\mathcal{U}(\xi,t,\lambda)e^{-i\varphi(\xi,t,\lambda)\sigma_3}\Psi(\xi,t,\lambda)\,d\xi\,. \tag{A.11}$$

Given that

$$\lim_{t\to-\infty}\mathcal{U}(x,t,\lambda) = 0\,, \tag{A.12}$$

we deduce that

$$\lim_{t\to-\infty}\Psi(x,t,\lambda) = \mathbb{I}\,. \tag{A.13}$$

Similarly, from the definition of $\mathcal{T}(t,\tau,\lambda)$ and $\widehat{\mathcal{T}}_-(t,x,\lambda)$ in (2.40), if we set

$$\Phi(x,t,\lambda) = e^{i\varphi(x,t,\lambda)\sigma_3} N^{-1} \widehat{\mathcal{T}}_-(x,t,\lambda) e^{-ik_1 x\sigma_3}\,, \tag{A.14}$$

we find that

$$\partial_x\,\Phi(x,t,\lambda) = e^{i\varphi(x,t,\lambda)\sigma_3}\mathcal{U}(x,t,\lambda)e^{-i\varphi(x,t,\lambda)\sigma_3}\Phi(x,t,\lambda)\,, \tag{A.15}$$

$$\partial_t\,\Phi(x,t,\lambda) = e^{i\varphi(x,t,\lambda)\sigma_3}\mathcal{V}(x,t,\lambda)e^{-i\varphi(x,t,\lambda)\sigma_3}\Phi(x,t,\lambda)\,, \tag{A.16}$$

and

$$\lim_{t\to-\infty}\Phi(x,t,\lambda) = \mathbb{I}\,. \tag{A.17}$$

So we can write

$$\Phi(x,t,\lambda) = \mathbb{I} + \int_{-\infty}^{t} e^{i\varphi(x,\tau,\lambda)\sigma_3}\mathcal{V}(x,\tau,\lambda)e^{-i\varphi(x,\tau,\lambda)\sigma_3}\Phi(x,\tau,\lambda)\,d\tau\,, \tag{A.18}$$

and then deduce

$$\lim_{x\to-\infty}\Phi(x,t,\lambda) = \mathbb{I}\,. \tag{A.19}$$

Hence, both Ψ and Φ satisfy the same differential equations with same boundary conditions: they must be equal, which is precisely (A.1). Of course, a similar statement can be made for $\widehat{T}_+(x,t,\lambda)$ and $\widehat{\mathcal{T}}_+(x,t,\lambda)$.

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