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Strings from position-dependent noncommutativity

Andreas Fring[•], **Laure Gouba**[◦] **and Frederik G. Scholtz**[◦,*]

[•] *Centre for Mathematical Science, City University London, Northampton Square, London EC1V 0HB, UK*

[◦] *National Institute for Theoretical Physics (NITheP), Stellenbosch 7600, South Africa*

[*] *Institute for Theoretical Physics, Stellenbosch University, Stellenbosch 7600, South Africa*

E-mail: a.fring@city.ac.uk, gouba@sun.ac.za, fgs@sun.ac.za

Abstract: We introduce a new set of noncommutative space-time commutation relations in two space dimensions. The space-space commutation relations are deformations of the standard flat noncommutative space-time relations taken here to have position dependent structure constants. Some of the new variables are non-Hermitian in the most natural choice. We construct their Hermitian counterparts by means of a Dyson map, which also serves to introduce a new metric operator. We propose $\mathcal{PT}$ like symmetries, i.e. antilinear involutory maps, respected by these deformations. We compute minimal lengths and momenta arising in this space from generalized versions of Heisenberg's uncertainty relations and find that any object in this two dimensional space is string like, i.e. having a fundamental length in one direction beyond which a resolution is impossible. Subsequently we formulate and partly solve some simple models in these new variables, the free particle, its $\mathcal{PT}$-symmetric deformations and the harmonic oscillator.

1. Introduction

Noncommutative space-time structures is an old subject dating back over fifty years to Snyder [1]. He introduced noncommutativity in the hope of regularizing the ultra-violet divergencies that plagued quantum field theory at that time, but the discovery of renormalization pushed these ideas on the background. More recently these ideas were revived by the observation of noncommutativity in certain string theories [2] and the compelling arguments for noncommuative space-time structures coming from gravitational stability [3]. This has given rise to intense investigations into noncommutative quantum mechanics, see [4] for an overview, and noncommutative quantum field theories, see [5, 6] for reviews.

This noncommutativity is of the simplest possible type, namely, it is assumed that the Hermitian local coordinates satisfy commutations relations of the type $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, with $\theta^{\mu\nu}$ a constant antisymmetric tensor. However, there are many other possibilities

that cannot be ruled out by present experimental observation. Indeed, even in the very first paper by Snyder [1], this tensor was taken to depend on the coordinates and the momenta. Many different types of possibilities have been explored since then in so-called κ-Poincaré noncommutativity [7], Lie-algebraic approaches [8] , other fuzzy spaces (see [9] for a comprehensive overview) and also more recently in a more generic position dependent approach [10, 11] that takes $\theta^{\mu\nu}$ to be a function of the position coordinates, i.e. assuming $\theta^{\mu\nu}(x)$. In the latter case the consistency of the Jacobi identity involving one momentum and two coordinate variables also requires a change in the mutual commutators between positions and momenta (see e.g. [10, 11]). Relations of such type are common in a more algebraic approach in which Heisenberg's canonical commutation relations or the relations between creation and annihilation operators are directly deformed, e.g. [12, 13, 14]. As deformations of this form will almost inevitably lead to non-Hermitian local coordinates, it was pointed out recently [15] (see also [16]), that therefore these type of structures are related directly to another subject of current interest, namely non-Hermitian Hamiltonian systems with real eigenvalues, see e.g. [17, 18] for recent reviews on the subject.

The main aim of this manuscript is to explore this interrelation further and study the consequences of a simple position dependent deformation of the noncommutative local coordinate commutations relations. Our manuscript is organized as follows: In section 2 we deform the standard relations of flat noncommutative space-time and introduce our new version of dynamical space-time. We argue that the most natural choice for the new variables leads to non-Hermiticity in one position and one momentum variable. By constructing a Dyson map we provide their corresponding set of Hermitian counterparts. We also propose $\mathcal{PT}$-like symmetries respected by the new set of variables. In section 3 we derive a minimal length and a minimal momentum resulting from these deformations. In section 4 we study some simple models formulated in terms of our new set of variables. Our conclusions are drawn in section 5.

2. Position dependent noncommutative space-time

We restrict ourselves here to two dimensional space. The most commonly investigated noncommutative space-time is flat obeying in this case the relations

$$\begin{array}{lll} [x_0, y_0] = i\theta, & [x_0, p_{x_0}] = i\hbar, & [y_0, p_{y_0}] = i\hbar, \\ [p_{x_0}, p_{y_0}] = 0, & [x_0, p_{y_0}] = 0, & [y_0, p_{x_0}] = 0, \end{array} \tag{2.1}$$

with $\theta \in \mathbb{R}$. It is well known that a relation to the conventional commutative space-time variables can be achieved by the so-called Bopp-shift in space

$$x_0 = x_s - \frac{\theta}{\hbar} p_{y_s} \qquad \text{and} \qquad y_0 = y_s, \tag{2.2}$$

where the standard coordinates x_s,y_s now commute $[x_s, y_s] = 0$ and all the remaining commutators remain unchanged when replacing the subscript 0 by s in (2.1). Most commonly the Bopp shift is taken to be more symmetrical as $x_0 = x_s - \theta/(2\hbar)p_{y_s}$ and $y_0 = y_s + \theta/(2\hbar)p_{x_s}$, but for several reasons the representation (2.2) will be more convenient for our purposes. We will now explore a simple possibility to deform the relations

(2.1) by introducing a set of new variables X, Y, P_x, P_y of yet unknown properties and convert the constant θ into a function $\theta \to \theta(X,Y)$, by choosing as one possibility $\theta(X,Y) = \theta(1+\tau Y^2)$. As mentioned above consistency of the Jacobi identities requires to alter the remaining commutators. We propose here a simple consistent position dependent and in one case also momentum dependent deformation of (2.1) satisfying all possible permutations of the Jacobi identities[1]

$$\begin{array}{lll} [X,Y] = i\theta(1+\tau Y^2), & [X,P_x] = i\hbar(1+\tau Y^2), & [Y,P_y] = i\hbar(1+\tau Y^2), \\ [P_x,P_y] = 0, & [X,P_y] = 2i\tau Y(\theta P_y + \hbar X), & [Y,P_x] = 0. \end{array} \tag{2.3}$$

Obviously by construction we recover the standard flat noncommutative space-time (2.1) in the limit $\tau \to 0$. For reasons which will become apparent below, we also have to take $\tau \geq 0$. We may now represent the algebra (2.3) in terms of the standard flat Hermitian noncommutative space-time momentum and position operators $x_0, y_0, p_{x_0}, p_{y_0}$ as

$$X = (1+\tau y_0^2)x_0, \quad Y = y_0, \quad P_x = p_{x_0}, \quad \text{and} \quad P_y = (1+\tau y_0^2)p_{y_0}. \tag{2.4}$$

From this representation follows immediately that some of the operators involved are no longer Hermitian. We observe

$$X^\dagger = X + 2i\tau\theta Y, \quad Y^\dagger = Y, \quad P_y^\dagger = P_y - 2i\tau\hbar Y, \quad \text{and} \quad P_x^\dagger = P_x, \tag{2.5}$$

i.e. the X-coordinate and the momentum in Y-direction P_y are not Hermitian. An immediate consequence is that models formulated in terms of the new set of variables will in general also not be Hermitian. However, envoking the often synonymously used concepts of quasi-Hermiticity [19, 20, 21] and pseudo-Hermiticity [22, 23, 24] in the context of studying non-Hermitian systems, one may try to find a similarity transformation, i.e. a Dyson map [25], and convert the non-Hermitian system into a Hermitian one. Whereas usually this is carried out for some concrete Hamiltonian, as this is a common starting point, we can, as suggested in [15], perform this here directly for the set of non-Hermitian observables $\mathcal{O} \neq \mathcal{O}^\dagger$, that is we seek an operator η such that

$$\eta\mathcal{O}\eta^{-1} = o = o^\dagger. \tag{2.6}$$

For the case at hand we find that the Dyson map can be taken to be $\eta = (1+\tau Y^2)^{-1/2}$, such that the new Hermitian variables x, y, p_x, p_y in terms of the standard two dimensional flat noncommutative space-time variables (2.1) become

$$\begin{array}{ll} x = \eta X \eta^{-1} = (1+\tau y_0^2)^{\frac{1}{2}} x_0 (1+\tau y_0^2)^{\frac{1}{2}}, & p_x = \eta P_x \eta^{-1} = p_{x_0}, \\ y = \eta Y \eta^{-1} = y_0, & p_y = \eta P_y \eta^{-1} = (1+\tau y_0^2)^{\frac{1}{2}} p_{y_0} (1+\tau y_0^2)^{\frac{1}{2}}. \end{array} \tag{2.7}$$

Our requirement $\tau \geq 0$ ensures here that the Dyson map will be non-singular. By construction, the algebra satisfied by these variables is isomorphic to (2.3)

$$\begin{array}{lll} [x,y] = i\theta(1+\tau y^2), & [x,p_x] = i\hbar(1+\tau y^2), & [y,p_y] = i\hbar(1+\tau y^2), \\ [p_x,p_y] = 0, & [x,p_y] = 2i\tau y(\theta p_y + \hbar x), & [y,p_x] = 0. \end{array} \tag{2.8}$$

[1] That is $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ for $A, B, C \in \{X, Y, P_x, P_y\}$.

Obviously we may also express these variables in terms of the standard commuting space-time variables $x_s, y_s, p_{x_s}, p_{y_s}$ by utilizing (2.2). in principle we could have written down directly the Hermitian representation (2.7) for our deformed algebra and skipped the introduction of the non-Hermitian variables altogether. However, whereas the representation (2.4) is easy to guess it is not obvious how one would construct (2.7) by avoiding the step via the (2.4).

As is well established, given η we can immediately define a new metric operator $\rho = \eta^\dagger \eta$ and an inner product $\langle \quad | \quad \rangle_\rho$, which in terms of the standard inner product $\langle \quad | \quad \rangle$ is defined as

$$\langle \Phi | \Psi \rangle_\rho := \langle \Phi | \rho \Psi \rangle, \tag{2.9}$$

for arbitrary states $\langle \Phi |$ and $| \Psi \rangle$. The operators $\mathcal{O}$ are then Hermitian with respect to this new metric

$$\langle \Phi | \mathcal{O} \Psi \rangle_\rho = \langle \mathcal{O} \Phi | \Psi \rangle_\rho. \tag{2.10}$$

Since in our case the Dyson map η is Hermitian, the metric operator is therefore simply computed to be $\rho = \eta^2 = (1 + \tau Y^2)^{-1}$.

Alternatively one may also exploit Wigner's observation [26] about antilinear operators to investigate the reality of eigenvalue spectra. He found that operators invariant under such transformations possess real eigenvalues when in addition their eigenfunctions also respect this symmetry. $\mathcal{PT}$-symmetry, i.e. a simultaneous parity transformation $\mathcal{P}$ and time reversal $\mathcal{T}$ is a particular examples of such an operator [27]. Let us therefore see how $\mathcal{PT}$-symmetry manifests itself for the above set of variables. We observe for instance that

$$\begin{array}{lllll} \mathcal{PT}: & x_s \mapsto x_s, & y_s \mapsto -y_s, & p_{x_s} \mapsto -p_{x_s}, & p_{y_s} \mapsto p_{y_s}, & i \mapsto -i, \\ & x_0 \mapsto x_0, & y_0 \mapsto -y_0, & p_{x_0} \mapsto -p_{x_0}, & p_{y_0} \mapsto p_{y_0}, & i \mapsto -i, \\ & X \mapsto X, & Y \mapsto -Y, & P_x \mapsto -P_x, & P_y \mapsto P_y, & i \mapsto -i, \\ & x \mapsto x, & y \mapsto -y, & p_x \mapsto -p_x, & p_y \mapsto p_y, & i \mapsto -i, \end{array} \tag{2.11}$$

leaves the commutation relations (2.1), (2.3) and (2.8) invariant. We have reflected here only in the y-direction and left the x-direction unaltered. If we wish to have a reflection also in the x-direction, we are forced to change $\theta \mapsto -\theta$. With regard to the standard $\mathcal{PT}$-transformation this would imply that θ has to be taken to be purely imaginary, i.e. $\theta \in i\mathbb{R}$. This is a quite unappealing variant, as this will imply that we have lost the Hermiticity of the original flat space variables x_0 and y_0. This option was investigated in [28]. However, as we argued here that is not necessary in order to ensure real eigenvalues, which is the whole purpose of utilizing this symmetry, as this just requires *any* type of antilinear and involutory operator. In case we would also like to have a reflection in the x-direction, we can alternatively simply define a new map

$$\begin{array}{lllll} \mathcal{P}_\theta\mathcal{T}: & x_s \mapsto -x_s, & y_s \mapsto -y_s, & p_{x_s} \mapsto p_{x_s}, & p_{y_s} \mapsto p_{y_s}, & \theta \mapsto -\theta,\ i \mapsto -i, \\ & x_0 \mapsto -x_0, & y_0 \mapsto -y_0, & p_{x_0} \mapsto p_{x_0}, & p_{y_0} \mapsto p_{y_0}, & \theta \mapsto -\theta,\ i \mapsto -i, \\ & X \mapsto -X, & Y \mapsto -Y, & P_x \mapsto P_x, & P_y \mapsto P_y, & \theta \mapsto -\theta,\ i \mapsto -i, \\ & x \mapsto -x, & y \mapsto -y, & p_x \mapsto p_x, & p_y \mapsto p_y, & \theta \mapsto -\theta.\ i \mapsto -i. \end{array} \tag{2.12}$$

Clearly the newly defined map $\mathcal{P}_\theta\mathcal{T}$ is as required antilinear and involutory, that is $\mathcal{P}_\theta\mathcal{T}^2 = \mathbb{I}$. We stress that in this map the minus sign in $\theta \mapsto -\theta$ is not generated by the antilinear nature of $\mathcal{T}$, but is simply imposed on the real θ.

3. Minimal uncertainties

As is well known in standard flat space-time noncommutativity (2.1), Heisenberg's uncertainty principle applied to a simultaneous measurement of x_0 and y_0, will lead to the fact that they can not be known any longer both at the same time with absolute precision, but we have to satisfy $\Delta x_0 \Delta y_0 \geq \theta/2$. However, we can still measure x_0 precisely, that is we can take the limit $\Delta x_0 \to 0$, when we give up any knowledge about the y_0-direction and allow $\Delta y_0 \to \infty$. The same holds for $x_0 \leftrightarrow y_0$. The consequences are more severe once the right hand sides of the commutation relations in (2.1) cease to be constants, but become functions of the coordinates and/or the momenta. In that case we might encounter for a particular observable say A, that the limit $\Delta A \to 0$ can not be carried out without violating the uncertainly relations, such that ΔA can never be made smaller than a certain value $\Delta A_{\min}$ irrespective of what happens to the other variable involved in the measurement. In that case A can never be known below a precision of its minimal uncertainty $\Delta A_{\min}$. For the system of variables satisfying the commutation relations in (2.3) the uncertainty relations become

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A,B]\rangle_\rho\right| \qquad \text{for} \quad A,B \in \{X,Y,P_x,P_y\}, \tag{3.1}$$

where we have to employ the inner product as defined in (2.9). Starting with a simultaneous X,Y-measurement and following the standard arguments, see e.g. [29, 15], for minimizing the expression (3.1) we have to solve

$$\partial_{\Delta Y} f(\Delta X, \Delta Y) = 0 \qquad \text{and} \qquad f(\Delta X, \Delta Y) = 0, \tag{3.2}$$

for ΔX with $f(\Delta X, \Delta Y)$ defined as

$$f(\Delta X, \Delta Y) = \Delta X \Delta Y - \frac{1}{2}\left|\langle [X,Y]\rangle_\rho\right| = \Delta X \Delta Y - \frac{\theta}{2}\left(1 + \tau \left\langle Y^2\right\rangle_\rho\right), \tag{3.3}$$

$$= \Delta X \Delta Y - \frac{\theta}{2}\left(1 + \tau \langle Y\rangle_\rho^2 + \tau \Delta Y^2\right). \tag{3.4}$$

This leads to a minimal length for X in a simultaneous X,Y-measurement

$$\Delta X_{\min} = \theta\sqrt{\tau}\sqrt{1 + \tau \langle Y\rangle_\rho^2}. \tag{3.5}$$

There is no nonvanishing minimal length in Y as we may take the limit $\Delta Y \to 0$ without violating the inequality. This means in the two dimensional space spanned by X and Y objects are naturally of string type, being streched out in X-direction where a resolution of its substructure beyond the absolute minimal value $\Delta X_{\min}$, that is $\theta\sqrt{\tau}$, is completely impossible. This even holds when sacrifycing all knowledge about the Y-direction. On the other hand in the Y-direction a complete resolution can be achieved when all information about the X-direction is given up.

Arguing in the same way we do not encounter any minimal length or minimal momentum in a simultaneous X, P_x-measurement. However, in a simultaneous Y, P_y-measurement we find a minimal momentum

$$\Delta\left(P_y\right)_{\min} = \hbar\sqrt{\tau}\sqrt{1+\tau\left\langle Y\right\rangle_\rho^2}, \tag{3.6}$$

whereas once again there is no minimal length in Y.

The argumentation for a simultaneous X, P_y-measurement is less straightforward as we encounter terms of the type $\left\langle YP_y\right\rangle_\rho$ and $\left\langle YX\right\rangle_\rho$, which can not be treated in the same manner. However, since the behaviour of X and P_y is linear on both sides of the inequality in both cases, we do not expect a minimal length or a minimal momentum to arise in this circumstance.

4. Models in position dependent noncommutative space-time

As mentioned, any Hamiltonian depending on the new variables X and P_y will obviously no longer be Hermitian. We will now study some examples, starting by formulating them in terms of these variables, then computing some equivalent formulations and subsequently solving some of the models in their most convenient form.

4.1 The free particle

The simplest Hamiltonian one can envisage in these variable is the free particle. A priori it is not even clear if the free particle Hamiltonian in these variables still describes a free particle in the standard sense as the non-Hermitian nature might alter this property even in this simple case. In two dimensions the free particle Hamiltonian reads

$$\mathcal{H}_{\mathrm{f}}(X,Y,P_x,P_y) = \frac{1}{2m}(P_x^2+P_y^2). \tag{4.1}$$

It now depends on our preferences whether we wish to treat the model in these variables, but with a changed metric as decribed in section 2 or if we transform the Hamiltonian into standard flat non-commutative space. Using the relations (2.4) we can convert (4.1) into

$$\mathcal{H}_{\mathrm{f}}(x_0,y_0,p_{x_0},p_{y_0}) = \frac{1}{2m}\left[p_{x_0}^2+(1+\tau y_0^2)^2p_{y_0}^2-2i\hbar\tau y_0(1+\tau y_0^2)p_{y_0}\right]. \tag{4.2}$$

As is apparent this Hamiltonian is non-Hermitian and we still need to change the metric as decribed above when we wish to compute expectation values or other physical quantities. Yet another possibility is to map this Hamiltonian to a Hermitian one, which may then be treated in the conventional way. In analogy to (2.6) we can achieve this by means of a similarity transformation. Since all our variables are converted into Hermitian ones by the same Dyson map, this will also hold for any function in these variables, as for instance the Hamiltonian. Thus another possibility to consider $\mathcal{H}_{\mathrm{f}}$ is in terms of the Hermitian variables introduced in (2.7)

$$h_{\mathrm{f}}(x,y,p_x,p_y) = \eta\mathcal{H}_{\mathrm{f}}\eta^{-1} = \frac{1}{2m}(p_x^2+p_y^2). \tag{4.3}$$

By construction this Hamiltonian is Hermitian. Yet another option is to relate this version to the standard Hermitian flat noncommutative variable in (2.1). We find

$$\begin{aligned} h_{\rm f}(x_0,y_0,p_{x_0},p_{y_0}) &= \frac{1}{2m}\left[p_{x_0}^2+(1+\tau y_0^2)^{1/2}p_{y_0}(1+\tau y_0^2)p_{y_0}(1+\tau y_0^2)^{1/2}\right], \\ &= \frac{1}{2m}\left[p_{x_0}^2+(1+\tau y_0^2)^2p_{y_0}^2\right]+\frac{\hbar^2\tau^2}{m}y_0^2-\frac{\tau\hbar}{2m}. \end{aligned} \tag{4.4}$$

Apparently we have converted the free particle into a harmonic oscillator like potential in one direction, due to the emergence of the y_0^2-term. However, the mixed term in y_0 and p_{y_0} will in fact compensate for this interaction, having the effect that this Hamiltonian still allows for a continuous spectrum. According to (2.3) the non-Hermitian momenta P_x, P_y, or likewise p_x, p_y, still commute and therefore we can find simultaneous eigenfunctions for both operators. Consequently the eigenfunction factorizes $\psi(X,Y)=\varphi(X)\varphi(Y)$ and since the eigenvalues for P_x^2 are continuous in the infinite plane, also the spectrum for P_y^2 can not be discrete in this setting. We should stress that (4.1), (4.2), (4.3) and (4.4) are just different points of view to describe the same type of physics, but care needs to be taken in the selection of meaningful observables.

Let us now solve this model in its variant (4.4). Appealing to the nonsymmetric Bopp-shift in the form (2.2) and using the fact that in position space we can represent the momenta as differential operators $p_{y_s}=-i\hbar\partial_{y_s}$, $p_{x_s}=-i\hbar\partial_{x_s}$ we may re-write the eigenvalue equation

$$h_{\rm f}\varphi(x_0)\varphi(y_0)=E\varphi(x_0)\varphi(y_0) \tag{4.5}$$

as a decoupled second order differential equation in the two variables x_0 and y_0

$$-\frac{\hbar^2}{2m}\left[\partial_{x_0}^2+(1+\tau y_0^2)^2\partial_{y_0}^2-2\tau^2y_0^2+\frac{\tau}{\hbar}\right]\varphi(x_0)\varphi(y_0)=E\varphi(x_0)\varphi(y_0). \tag{4.6}$$

Here lies another reason for adopting the non-symmetric form (2.2). This form still guarantees the decoupling of the two sets of variables, whereas the more symmetric version will lead to a mixing of the x_0 and y_0 variables. Equation (4.6) is solved by

$$\varphi(x_0) = c_1\sin kx_0+c_2\cos kx_0 \tag{4.7}$$

$$\varphi(y_0) = \sqrt{1+\tau y_0^2}\left[\tilde{c}_1P_1^{\mu}\left(iy_0\sqrt{\tau}\right)+\tilde{c}_2Q_1^{\mu}\left(iy_0\sqrt{\tau}\right)\right] \tag{4.8}$$

with continuous eigenenergy

$$E(k)=\frac{k^2\hbar^2}{2m}+\frac{\tau\hbar}{2m}, \tag{4.9}$$

parameterized by $k\in\mathbb{R}$ and $\mu=\sqrt{3+k^2/\tau}$. The functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are associated Legendre polynomials of the first and second kind, respectively, and $c_1,c_2,\tilde{c}_1,\tilde{c}_2$ are integration constants. Notice that the limit $\tau\to 0$ to the undeformed case is nontrivial in this case as we had to introduce a variable transformation involving $1/\sqrt{\tau}$ in order to convert (4.6) into the conventional form of the differential equation solvable by associated Legendre polynomials.

4.2 $\mathcal{PT}$ and $\mathcal{P}_\theta\mathcal{T}$-extensions of the free particle

As explained above we still have a good chance to have well defined models with real eigenvalues when our Hamiltonian remains invariant with respect to an antilinear involutory symmetry. Let us therefore in the spirit of deforming Hermitian models add some additional terms to the free particle Hamiltonian

$$\mathcal{H}_{\rm fPT}(X,Y,P_x,P_y)=\frac{1}{2m}(P_x^2+P_y^2)+\lambda(iY)^nP_y^m,\qquad \text{with } n,m\in\mathbb{N}_0,\lambda\in\mathbb{R}. \tag{4.10}$$

Clearly, this Hamiltonian remains invariant with respect to the $\mathcal{PT}$ as well as the $\mathcal{P}_\theta\mathcal{T}$-symmetry, i.e. $[\mathcal{PT},\mathcal{H}_{\rm fPT}]=[\mathcal{P}_\theta\mathcal{T},\mathcal{H}_{\rm fPT}]=0$. Since P_x also commutes with the added term, we may still apply the argument of the previous subsection and construct simultaneous eigenstates for P_x^2 and the remaining term. For instance, for $n=m=1$ the corresponding differential equation in position space becomes

$$-\frac{\hbar^2}{2m}\left[\partial_{x_0}^2+(1+\tau y_0^2)^2\partial_{y_0}^2-2\tau^2y_0^2+\lambda y_0(1+\tau y_0^2)\partial_{y_0}\right]\varphi(x_0)\varphi(y_0)=E\varphi(x_0)\varphi(y_0), \tag{4.11}$$

where we take E directly in the form (4.9). The solution for $\varphi(x_0)$ will remain the same, whereas $\varphi(y_0)$ results now to

$$\varphi(y_0)=\left(1+\tau y_0^2\right)^\kappa\left[\tilde{c}_1P_\nu^\mu\left(iy_0\sqrt{\tau}\right)+\tilde{c}_2Q_\nu^\mu\left(iy_0\sqrt{\tau}\right)\right] \tag{4.12}$$

with

$$\nu=\frac{1}{2}+\frac{\lambda}{4\tau},\quad \mu=\frac{\sqrt{\lambda^2+2\lambda\tau+9\tau^2}}{2\tau}-\frac{1}{2}\quad\text{and}\quad \kappa=\frac{\sqrt{\lambda^2+4\lambda\tau+4\tau(k^2+3\tau)}}{2\tau}. \tag{4.13}$$

We can take here two different points of view: On one hand we may choose λ to be a generic constant, which would imply that the model (4.10) still remains non-Hermitian in the limit $\tau\to 0$. On the other hand we can identify $\lambda=\tau$, such that the limit $\tau\to 0$ will reduce $\mathcal{H}_{\rm fPT}$ to a Hermitian Hamiltonian. In that case the constants in (4.13) simplify to $\nu=3/4$, $\mu=\sqrt{3}-1/2$ and $\kappa=1/2\sqrt{17+4k^2/\tau}$. Once again the limit $\tau\to 0$ is nontrivial.

4.3 The harmonic oscillator

The next natural complication of our previous examples would be the two dimensional harmonic oscillator

$$\mathcal{H}_{\rm ho}(X,Y,P_x,P_y)=\frac{1}{2m}(P_x^2+P_y^2)+\frac{m\omega^2}{2}(X^2+Y^2). \tag{4.14}$$

Obviously also this model can be re-written in terms of the flat commuting variables obeying (2.1)

$$\begin{aligned}\mathcal{H}_{\rm ho}(x_0,y_0,p_{x_0},p_{y_0})=&\frac{1}{2m}\left[p_{x_0}^2+(1+\tau y_0^2)^2p_{y_0}^2-2i\hbar\tau y_0(1+\tau y_0^2)p_{y_0}\right]\\&+\frac{m\omega^2}{2}\left[(1+\tau y_0^2)^2x_0^2+2i\theta\tau y_0(1+\tau y_0^2)x_0+y_0^2\right].\end{aligned} \tag{4.15}$$

Since $\mathcal{H}_{\rm ho}$ is evidently non-Hermitian, we have to employ a similarity transformation and convert it in the same spirit as in the previous section into a Hermitian Hamiltonian

$$h_{\rm ho}(x,y,p_x,p_y)=\frac{1}{2m}(p_x^2+p_y^2)+\frac{m\omega^2}{2}(x^2+y^2). \tag{4.16}$$

Using the representation (2.7) we may of course also re-express this Hamiltonian in term of the flat commuting variables obeying (2.1)

$$\begin{aligned}
h_{\rm ho}(x_0,y_0,p_{x_0},p_{y_0}) &= \frac{1}{2m}\left[p_{x_0}^2+(1+\tau y_0^2)^{1/2}p_{y_0}(1+\tau y_0^2)p_{y_0}(1+\tau y_0^2)^{1/2}\right]\\
&\quad+\frac{m\omega^2}{2}\left[(1+\tau y_0^2)^{1/2}x_0(1+\tau y_0^2)x_0(1+\tau y_0^2)^{1/2}+y_0^2\right],\\
&= \frac{1}{2m}\left[p_{x_0}^2+(1+\tau y_0^2)^2p_{y_0}^2\right]+\frac{m\omega^2}{2}\left[1-2\theta^2\tau^2-2\frac{\hbar^2\tau^2}{m^2\omega^2}\right]y_0^2\\
&\quad+\frac{m\omega^2}{2}(1+\tau y_0^2)^2x_0^2+2i\tau y_0(1+\tau y_0^2)\left[m\omega^2\theta x_0-\frac{\hbar}{m}p_{y_0}\right]\\
&\quad-\frac{\tau}{2}\left(m\omega^2\theta^2+\frac{\hbar}{m}\right).
\end{aligned} \tag{4.17}$$

Clearly this is a far more complicated model to solve with the same method as in the previous sections, as the system viewed as a differential equation no longer decouples in x_0 and y_0. We leave the construction of solutions for this model by alternative means for future work.

5. Conclusions

We have provided a new version of noncommutative space-time in two dimensions, which is dynamical in the sense that the x,y-commutation relations acquire a position dependent structure constant. An immediate consequence of this deformation of the common flat commutation relations was that some of the natural variables associated to these new commutation relations are non-Hermitian. As we have shown this is not dictated by the commutation relations themselves as there exist an isomorphic algebra in terms of Hermitian operators (2.8). However, these variables do not constitute the natural starting point and only emerge when the Dyson map has been constructed. This in turn will also give rise to a new natural metric, which has to be used to compute physical quantities. We encounter here the well known problem that this metric might not be unique and there could be other possibilities related to different types of physical observables.

Previous attempts [10, 11] to construct dynamical deformations of (2.1) were based on a quantization procedure of a concrete classical system in the presence of constraints, thus providing a nice physical scenario in which such type of deformed spaces might arise. However, the resulting algebra is only valid in tems of Dirac brackets up to the imposed constraints, whereas our algebra (2.1) is selfconsistent, model independent and entirely placed into a quantum mechanical setting.

The interesting physical consequence we found is that any object in this two dimensional space will be string like, as we found that one direction is inevitably bounded by

the quantity $\theta\sqrt{\tau}$, beyond which any further localization is not only impossible but even meaningless. As the constant value indicates, i.e. being explicitly dependent on τ, this is a direct consequence of our very starting point, namely the position dependent deformation.

We have also analysed the various possibilities to implement $\mathcal{PT}$-symmetry and generalized versions of it for our deformed noncommutative space-time. Our proposed maps have even consequences for the flat version of it, as we argue that one does not have to take θ to be purely imaginary. This avoids the unappealing feature that in doing this one looses the Hermiticity of x_0 and y_0. Having the two types of antilinear involutory maps, $\mathcal{PT}$ and $\mathcal{P}_\theta\mathcal{T}$, would allow us also to investigate further extensions of any model. Assuming for some function $f(y, p_x, p_y)$ that $[\mathcal{PT}, f(y, p_x, p_y)] = 0$ or $[\mathcal{P}_\theta\mathcal{T}, f(y, p_x, p_y)] = 0$ we have now the option to add terms of the type $x^n f(y, p_x, p_y)$ or $(ix)^n f(y, p_x, p_y)$, respectively, without violating this symmetry.

Clearly there are many interesting immediate problems arising from our investigations, such as the investigation of further possibilities of consistent deformations, the constuction of the solution for the harmonic oscillator, the study of additional models in terms of our newly proposed variables, deformations of dispersion relations resulting from the models considered here and generalizations to fully fledged field theory setting.

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