Exactly solvable potentials of Calogero type for q-deformed Coxeter groups

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Abstract: We establish that by parameterizing the configuration space of a one-dimensional quantum system by polynomial invariants of q-deformed Coxeter groups it is possible to construct exactly solvable models of Calogero type. We adopt the previously introduced notion of solvability which consists of relating the Hamiltonian to finite dimensional representation spaces of a Lie algebra. We present explicitly the G_2^q-case for which we construct the potentials by means of suitable gauge transformations.

1. Introduction

One of the ultimate goals in the study of quantum mechanical systems is to find explicit and possibly exact solutions for the eigensystem of Hamiltonian systems. The Calogero [1, 2, 3] and Sutherland [4, 5, 6, 7] models are some of the well known examples for theories which are integrable and can be solved exactly, classically as well as quantum mechanically. The integrability of the models was established more systematically by relating them to Lie algebraic structures, in the so-called Hamiltonian reduction method [8, 9, 10] or by formulating Lax pairs and zero curvature conditions [11, 12, 13, 14, 15, 16]. Relatively recent [17, 18, 19, 20] the procedure to establish their exact solvability (which is conceptionally different from integrability) was put on a more systematic ground by relating first the coordinates of the configuration space of the Hamiltonians to invariant polynomials. It was shown that the differential operators in these polynomials form a representation for certain algebras, albeit not uniquely. Having an algebraic version of the model, solvability can be established thereafter by noting that the eigenfunctions form a flag which coincides with the finite dimensional representation space of a gl(N)-Lie algebra. This approach has turned out to be successful in many cases and could even be extended to theories which are supersymmetric [21].

In a sequence of publications [22, 23, 24, 25] this procedure was reversed. Instead of starting with a concrete potential for a theory, Haschke and Rühl proposed to start with a Hamiltonian already formulated in terms of invariant polynomials and construct the potential from it. Hence, in this approach the solvability is already build in from the very start and the question is addressed if possibly one obtains new types of potentials which are related to solvable models, which are potentially also integrable. For several examples of models formulated in terms of invariants of the Weyl Group [22, 23, 24] and even for Coxeter groups which are not Weyl groups [25, 26] it was shown that this is indeed possible.

The main purpose of this paper is to demonstrate that this procedure can also be carried out successfully for models which are related to q-deformed Coxeter groups. We demonstrate this for G_2^q. At the same time we show that also for these groups the associated Hamiltonians can be formulated in terms of the gl(N)-Lie algebra generators, hence guaranteeing their solvability.

We shall focus here mainly on the construction of new potentials of Calogero type, adopting to a large extent the point of view of the aforementioned papers. The obvious question of solving the associated Schrödinger problem similar as it has been done for the A_N case in [1, 2, 3] shall not be our concern here. While this is an interesting problem for future work, it appears that it is still open even for almost all of the non-deformed Lie algebras other than the A_N-series.

Our manuscript is organized as follows: In the next section we recall the notion of solvability based on the fact that certain types of Hamiltonians can be formulated in terms of the generators of the Borel subalgebra of the gl(N)-Lie algebra. We show how from this formulation one may systematically construct potentials. In section 3 we assemble the main mathematical properties about polynomial invariants of the Coxeter group, which play the crucial role of coordinates in this context. In section 4 we extend these ideas to the q-deformed Coxeter groups. In section 5 we discuss how certain choices of the pre-potential lead to Cologero type potentials. In sections 6 and 7 we discuss the Calogero model for G_2 and its q-deformed version, respectively, deriving some explicit Calogero type potentials. We state our conclusions in section 8.

2. Construction of exactly solvable potentials

We start by recalling the notion of exact solvability as proposed originally by A. Turbiner [17] about ten years ago. For this we require polynomial spaces of the form

$$V_n = \operatorname{span}\left\{ I_2^{k_2} I_3^{k_3} \ldots I_N^{k_N} \,\middle|\, \sum\nolimits_{i=2}^{N} k_i = n \right\} \,. \tag{2.1}$$

The I_i constitute some generic set of variables not further specified at this point. Evidently, these spaces are embedded into each other $V_0 \subset V_1 \subset V_2 \subset \ldots$, hence forming an infinite flag. A Hamiltonian operator $\mathcal{H}$ acting on such spaces and respecting

$$\mathcal{H} : \; V_n \mapsto V_n \tag{2.2}$$

possesses an infinite family of polynomial eigenfunctions. Therefore, it is natural to refer to such type of Hamiltonians as *exactly solvable.*

It is now a matter of identifying the spaces V_n, which of course allows for numerous solutions. It was noticed that many known models can be fitted into this scheme when one identifies V_n with a finite dimensional representation space of a gl(N)-Lie algebra. A simple representation of this algebra in terms of first order differential operators is found when expressing the usual gl(N)-generators E_{ij} as

$$E_{ij} \equiv J^0_{ij} = I_i \partial_j, \quad E_{i0} \equiv J^-_i = \partial_j, \quad E_{0i} \equiv J^+_i = \kappa I_i - \sum_{k=2}^{N} I_i I_k \partial_k \quad \text{for } \kappa \in \mathbb{R}^+, \tag{2.3}$$

where κ is an arbitrary constant and $\partial_i = \partial/\partial I_i$. It is easy to check that these differential operators satisfy indeed the usual gl(N)-commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} \ . \tag{2.4}$$

According to the representation (2.3) all Hamiltonians which are expressible in terms of the Borel subalgebra of gl(N), i.e. involving only the generators J^0 and J^-, will respect (2.2) and are therefore exactly solvable. Remarkably, it has turned out that many known solvable models of Calogero and Sutherland type can be brought into the general form

$$\mathcal{H} = \sum c^{kl}_{ij} J^0_{ij} J^0_{kl} + \sum \tilde{c}^{k}_{ij} J^0_{ij} J^-_k + \sum \hat{c}_i J^-_i + \sum \check{c}_{ij} J^0_{ij} \ , \tag{2.5}$$

with c^{kl}_{ij}, $\tilde{c}^k_{ij}$, $\hat{c}_i$, $\check{c}_{ij} \in \mathbb{R}$ being some coupling constants.

Unfortunately not all models, in particular the ones we shall discuss below, can be fitted into the gl(N)-framework. Nonetheless, following the same ideology as outlined above, one can appeal to some other algebras which can be realized with different types of differential operators than those provided in (2.3). For our purposes the semi-direct sum $\mathrm{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^{\ell+1}$ will be rather useful [17]. It may be realised by the $\ell + 5$ generators

$$J^1 = \partial_1, \qquad J^2 = I_1 \partial_1 - \frac{\kappa}{3}, \qquad J^3 = I_2 \partial_2 - \frac{\kappa}{3\ell}, \tag{2.6}$$

$$J^4 = I_1^2 \partial_1 + \ell I_1 I_2 \partial_2 - \kappa I_1, \qquad J^{5+i} = I_1^i \partial_2 \qquad \text{for } 1 \leq i \leq \ell, \ \kappa \in \mathbb{R}^+ \ . \tag{2.7}$$

The further condition $\kappa \in \mathbb{Z}^+$ guarantees that the representation is finite dimensional. Expressing now Hamiltonians in terms of the Borel subalgebra of $\mathrm{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^{\ell+1}$, i.e. the J^i for $1 \leq i \leq \ell + 5$ with $i \neq 4$, the flag space of the form

$$\bar{V}_n = \mathrm{span} \left\{ I_1^{k_1} I_2^{k_2} \, | 0 \leq k_1 + \ell k_2 \leq n \right\} \tag{2.8}$$

will be left invariant in the sense (2.2).

The above observations inspired the starting point of the approach in [22, 23, 24, 25] which is the eigenvalue equation for the function $\varphi(\vec{I})$

$$\mathcal{D}\varphi = E\varphi \tag{2.9}$$

with $\mathcal{D}$ being a symmetric Schrödinger operator of the form

$$\mathcal{D} = -\sum_{k,l} \partial_k g_{kl}^{-1} \partial_l + \sum_k r_k \partial_k \ . \tag{2.10}$$

Here g_{kl}^{-1} denotes the inverse of the curvature free symmetric Riemannian tensor $g_{kl} = g_{lk}$, which, in view of (2.5), is at most quadratic in the coordinates I_i. The functions r_k are assumed, again in view of (2.5), to be linear in the coordinates I_i. In many cases these coordinates are taken to be invariant polynomials (for more details see below), albeit sometimes re-parameterizations are needed to guarantee the quadratic and linear dependence of g_{kl} and r_k, respectively.

Clearly, the operator (2.10) is not of the usual form of a Hamiltonian, that is Laplacian plus potential. In order to extract a potential from this Hamiltonian one has to carry out a gauge transformation $\varphi = e^{\chi}\psi$ to bring the equation (2.9) into the more standard form

$$(-\Delta + V)\psi = E\psi \tag{2.11}$$

involving the Laplace-Beltrami operator in general Riemannian space

$$\Delta = \frac{1}{\sqrt{G}} \sum_{k,l} \partial_k \sqrt{G} g_{kl}^{-1} \partial_l \qquad \text{with} \quad G^{-1} = \det g^{-1} \tag{2.12}$$

and a potential V. Extracting then from the equality $e^{-\chi}\mathcal{D}e^{\chi} = -\Delta + V$ the terms of first and zeroth order in ∂_l, one finds

$$r_k = \sum_l g_{kl}^{-1} \partial_l (2\chi - \ln \sqrt{G}) \tag{2.13}$$

$$V = \sum_k r_k \partial_k \chi - \sum_{k,l} \left[\partial_k (g_{kl}^{-1} \partial_l \chi) + g_{kl}^{-1} \partial_k \chi \partial_l \chi \right], \tag{2.14}$$

respectively. Multiplying now (2.13) with g_{lk}^{-1} and differentiating thereafter with ∂_m one realizes that the right hand side is symmetric under the exchange $m \leftrightarrow l$. Therefore, one deduces immediately for the left hand side the same symmetry

$$\partial_m \sum_l (g_{kl} r_l) = \partial_k \sum_l (g_{ml} r_l) \ . \tag{2.15}$$

This equation constraints the values of r_k and can be solved by

$$r_k = \sum_l g_{kl}^{-1} \partial_l \rho \ . \tag{2.16}$$

The function ρ introduced at this point is referred to as pre-potential. It should be stressed that there is no compelling argument in this approach, which fixes this pre-potential and it remains subject to a convenient ansatz. Substituting (2.16) back into (2.13) and (2.14) one then finds

$$\chi = \frac{1}{2}(\rho + \ln \sqrt{G}) \tag{2.17}$$

$$V = \frac{1}{4} \sum_{k,l} g_{kl}^{-1} \partial_k \rho \partial_l \rho - \frac{1}{4} \sum_{k,l} g_{kl}^{-1} \partial_k (\ln \sqrt{G}) \partial_l (\ln \sqrt{G}) - \sum_{k,l} \partial_k (g_{kl}^{-1} \partial_l \chi) \ . \tag{2.18}$$

It will turn out below that the term in V which involves χ is zero or constant. Provided that $g_{kl}^{-1}\partial_l(\ln\sqrt{G})$ is linear in $\vec{I}$, this would follow directly as a consequence of the assumption already made on r_k, namely that it is linear in the variables $\vec{I}$. In that case we can deduce that this term in the potential would be constant and omitting it just amounts to a constant shift of the ground state energy.

Before we can specify in more detail the ansatz for the pre-potential ρ proposed in [24], we have to gather various facts about invariant polynomials. This will make the suggested ansatz look very natural, albeit not entirely compelling.

3. Polynomial invariants of the Coxeter group

We specify here in more detail the nature of the variables $\vec{I}$ and assemble some of their mathematical properties. First we recall the well known fact, that to each simple root α_i in a root system Δ one can associate a reflection on the hyperplane through the origin orthogonal to α_i

$$\sigma_i(\vec{x}) = \vec{x} - 2\frac{\vec{x}\cdot\alpha_i}{\alpha_i^2}\alpha_i \qquad \text{for } 1 \leq i \leq \ell,\ \vec{x} \in \mathbb{R}^\ell. \tag{3.1}$$

These reflections constitute the Coxeter group $\mathcal{W}$ of rank ℓ or more specifically when $2\alpha\cdot\beta/\beta^2 \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$ a Weyl group. One may then express each vector $\vec{x} \in \mathbb{R}^\ell$ as $\vec{x} = \sum_{i=1}^{\ell} x_i\alpha_i$ and associate to it a polynomial $P(x_1,\ldots,x_\ell)$. The action of the Coxeter group on these polynomials is defined as

$$\sigma_i P(x_1,\ldots,x_\ell) = P(\sigma_i^{-1}(x_1),\ldots,\sigma_i^{-1}(x_\ell)) \ . \tag{3.2}$$

From the defining relations (3.1) and (3.2) it follows directly that by taking the simple roots as a basis for $\mathbb{R}^\ell$ the action of the simple Weyl reflections acquires a particularly simple form

$$\sigma_i P(x_1,...,x_\ell) = P(x_1,...,x_{i-1}, x_i - \sum_j x_j K_{ji}, x_{i+1},...,x_\ell) \ . \tag{3.3}$$

Here K denotes the Cartan matrix $K_{ij} = 2\alpha_i\cdot\alpha_j/\alpha_j^2$. The special set of polynomials which does not change under the action of $\mathcal{W}$, i.e. for which

$$\sigma_i I_s(x_1,\ldots,x_\ell) = I_s(x_1,\ldots,x_\ell) \qquad \text{for all } \sigma_i \in \mathcal{W} \tag{3.4}$$

are the polynomial invariants of the Coxeter group. It turns out that a basic set of linear independent polynomials $\{I_{1+s_1},\ldots,I_{1+s_\ell}\}$ can be graded by the ℓ exponents s_i of the Coxeter group, with $1 \leq i \leq \ell$. The subscripts $1+s_i$ indicate here the degrees of the polynomials. It is this set of basic invariants which one takes as the coordinates of the previously described Hamiltonian system.

Let us now establish and recall some of their main properties, which we shall exploit below:

3.1 Eigenbasis of the Coxeter element

It is clear that the choice of the basis for the coordinates will alter the form of the potential and a priori there is no coordinate system which is more special than another one. However, certain choices make the final expressions very simple and one can take here the search for simplicity as a guiding principle. A particularly suitable choice is the eigenbasis of the Coxeter element. We will see that in this basis the expressions for the polynomial invariants simplify considerably.

Adopting the notations of [27] (see also references therein), we first define the Coxeter element σ in terms of the two special elements of the Weyl group

$$\sigma_{\pm} := \prod_{\alpha_i \in \Phi_{\pm}} \sigma_i \ , \tag{3.5}$$

as $\sigma := \sigma_{-}\sigma_{+}$. Here we have partitioned the set of simple roots into two disjoint sets, say $\alpha_k \in \Phi_+$ and $\beta_k \in \Phi_-$, by associating the values $c_i = \pm 1$ to the vertices i of the Dynkin diagram of the Lie algebra, in such a way that no two vertices related to the same set are linked together. The eigensystem of the Coxeter element can then be brought into the form

$$\sigma v_j = e^{\frac{2\pi i}{h} s_j} v_j \qquad \text{and} \qquad v_j = e^{-i\frac{\pi}{h} s_j} \sum_k \xi_{jk}\alpha_k + \sum_k \xi_{jk}\beta_k \ , \tag{3.6}$$

where we denote by ξ the matrix of left eigenvectors of the Cartan matrix, i.e.

$$\sum_{j=1}^{\ell} \xi_{ij} K_{jk} = 4\sin^2 \frac{\pi s_i}{2h}\ \xi_{ik} \ , \tag{3.7}$$

and the s_i are the aforementioned exponents. Then we implicitly define a variable substitution $\{x_i\} \to \{w_i\}$ by the basis transformation

$$\vec{x} = \sum_i x_i \alpha_i = \sum_{i,j} \zeta_{ij} w_j \alpha_i = \sum_i w_i v_i \ . \tag{3.8}$$

with

$$\zeta_{kj} := \begin{cases} e^{-i\frac{\pi}{h}s_j}\xi_{jk} \ , & \text{for} \ \ \alpha_k \in \Phi_+ \\ \quad \xi_{jk} \ , & \text{for} \ \ \beta_k \in \Phi_- \end{cases} \ . \tag{3.9}$$

Defining then also polynomials in these new variables, we obtain as a consequence of (3.6) the action of the Coxeter element on these polynomials

$$\sigma P(w_1, ..., w_\ell) = P(w_1 e^{\frac{2\pi i}{h}}, w_2 e^{\frac{2\pi i}{h} s_2}, ..., w_\ell e^{-\frac{2\pi i}{h}}) \ . \tag{3.10}$$

Recall here that $s_i + s_{\ell - i} = h$ for $1 \leq i \leq \ell$. Since the Coxeter element is build from simple Weyl reflections it follows from (3.4) that the invariants of the Coxeter group are also invariant under the action of the Coxeter element

$$\sigma I_s(x_1, \ldots, x_\ell) = I_s(x_1, \ldots, x_\ell) \ = I_s\left(\sum_i \zeta_{1i} w_i, \ldots, \sum_i \zeta_{\ell i} w_i\right) \ . \tag{3.11}$$

To be able to compute the action of the Weyl reflections we have to express the polynomials in terms of the x-variables. Nonetheless, by means of (3.8) we can also translate the action of the Weyl reflections in the x-variables to an action in terms of the w-variables which allows for a more concise and possibly generic formulation of the invariants (see below).

3.2 Universal formulae for invariants

It is a natural question to ask, whether there exist formulae which express the invariants in a universal fashion, that is valid for all algebras. Indeed for invariants of degree 2 this is possible and we find

$$I_2(\vec{x}) = \sum_{i=1}^{\ell} t_i x_i^2 + \sum_{i<j} x_i K_{ij} t_j x_j, \quad K_{ij} t_j = K_{ji} t_i, \quad t_i > 1 \, . \tag{3.12}$$

Here the t_i denote the symmetrizers of the Cartan matrix, which could be avoided in the above expression by absorbing them into the roots, which amounts to taking the simple co-roots instead of the simple roots as a basis. For higher degrees we did not succeed to find universal formulae for the invariants.

Changing, however, to the eigenbasis of the Coxeter element it is far more obvious how to write down a universal expression. From (3.10) it is clear that any invariant has to be of the form

$$I_s(\vec{w}) = \sum_{a_1,\ldots,a_\ell=1}^{s} c_s(a_1,\ldots,a_\ell) w_1^{a_1} \ldots w_\ell^{a_\ell} \tag{3.13}$$

where the constants $c_s(a_1,\ldots,a_\ell)$ are constrained as

$$c_s(a_1,\ldots,a_\ell) = \begin{cases} \neq 0 & \text{if } \sum_{i=1}^{\ell} a_i = s, \quad \sum_{i=1}^{\ell} a_i s_i = nh, \quad n \in \mathbb{Z} \\ = 0 & \text{otherwise} \end{cases} \, . \tag{3.14}$$

Consequently, this means for instance that the quadratic invariant has to be of the form

$$I_2 = \sum_i c_2(a_i, a_{\ell-i+1}) w_i w_{\ell-i+1} \, . \tag{3.15}$$

One can proceed similarly for higher degrees, but it is then less obvious how to fix the constants $c_s(a_1,\ldots,a_\ell)$. Hence, for the time being we have to rely on case-by-case studies, but even for explicit algebras the most generic expressions are difficult to find in the literature. See [28] for a complete list.

3.3 Jacobians and Riemannians

It will turn out that a key quantity in this scheme is the Jacobian determinant related to the polynomial invariants

$$J = \det(j) \qquad \text{with } \; j_{kl} = \frac{\partial I_{1+s_k}}{\partial u_l} \, . \tag{3.16}$$

The determinant J is known to possess various important properties, see e.g. [29]:

i) The polynomials $I_{1+s_1},\ldots,I_{1+s_\ell}$ in $u_1,\ldots,u_\ell$ are algebraically independent if and only if $J \neq 0$.

ii) Defining for each root α a linear polynomial

$$p_\alpha(u_1,\ldots,u_\ell) = \sum_{i=1}^{\ell} \tilde{\gamma}_\alpha^{(i)} u_i \qquad \text{with } \tilde{\gamma}_\alpha^{(i)} \in \mathbb{R} , \tag{3.17}$$

such that $p_\alpha(u_1,\ldots,u_\ell) = 0$ defines the hyperplane through the origin orthogonal to α. Then one can factorize J as

$$J = \mu \prod_{\alpha\in\Delta_+} p_\alpha(u_1,\ldots,u_\ell) \qquad \text{with } \mu \in \mathbb{R} , \tag{3.18}$$

where Δ_+ denotes the set of positive roots.

iii) Defining the inverse Riemannian in terms of the basic invariants $\{I_{1+s_1},\ldots,I_{1+s_\ell}\}$

$$g_{kl}^{-1} = \sum_{i=1}^{\ell} \frac{\partial I_k}{\partial u_i}\frac{\partial I_l}{\partial u_i} , \tag{3.19}$$

the Jacobian determinant is related to the determinant of the inverse Riemannian as

$$J^2 = G^{-1} = \det g^{-1} . \tag{3.20}$$

The factorization properties (3.18) and (3.20) for J and G^{-1}, respectively, will make the ansatz for the pre-potential ρ appear very natural.

It is worth pointing out that the choice of the metric (3.19) guarantees that the Laplacian in the variables $\vec{u}$ is flat. This is easily seen by considering the change of the Euclidean metric tensor, i.e. $g(\vec{u})_{mn} = \delta_{mn}$, under a coordinate transformation. For this we just have to multiply

$$g(I)_{ij} = \sum_{m,n} \frac{\partial u^m}{\partial I_i}\frac{\partial u^n}{\partial I_j} g(u)_{mn} = \sum_{m} \frac{\partial u^m}{\partial I_i}\frac{\partial u^m}{\partial I_j}, \tag{3.21}$$

with (3.19). This choice avoids the entire analysis which is needed in this approach to guarantee the flatness of the Laplacian as carried out in [22].

4. Polynomial invariants of the q-deformed Coxeter group

We extend now the previous discussion and seek polynomials which are invariant under q-deformed Weyl reflections. We adopt here the notation of [27], (see also [30]), for more details on q-deformed Weyl reflections and the general context in which they emerged. When acting on a simple root they are defined as

$$\sigma_i^q(\alpha_j) = \alpha_j - (2\delta_{ij} - [I_{ji}]_q)\alpha_i \tag{4.1}$$

with $I = 2 - K$ being the incidence matrix of some Lie algebra and $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ being the standard notation for a q-deformed integer. According to the notions outlined in the previous section the invariants are obviously defined by

$$\sigma_i^q I_s(x_1,\ldots,x_\ell) = I_s(x_1,\ldots,x_\ell) \qquad \text{for } 1 \le i \le \ell . \tag{4.2}$$

Once again for polynomials of degree 2 we can write down a universal formula

$$I_2^q = \sum_{i=1}^{\ell} [t_i]_q x_i^2 + \sum_{i<j} x_i [K_{ij}]_q [t_j]_q x_j \tag{4.3}$$

which is invariant under the q-deformed Weyl reflections. As in the non-deformed case higher invariants are difficult to write down in a universal form. Note also that the q-deformed invariants in the sense of (4.2) are not invariants of the q-deformed Coxeter elements as defined in [30, 27], as the latter does not just consist of transformations σ_i^q. However, we can alter this definition slightly to achieve this, see section 7.

5. Ansatz for the pre-potential

Having fixed a set of basic invariants $\{I_{1+s_1}, \ldots, I_{1+s_\ell}\}$ one assumes that the wavefunctions in (2.9) and (2.11) depend on these coordinates, that is $\varphi \to \varphi(\vec{I})$ and $\psi \to \psi(\vec{I})$. Naturally one can then also view the potential as a function of the invariants, i.e. $V \to V(\vec{I})$ and understand that $\partial_k \equiv \partial/\partial I_k$. Defining the inverse Riemannian as in (3.19) and using (3.20), we can re-write the potential (2.18) in the form

$$V = \frac{1}{4}\sum_{k,l} g_{kl}^{-1} \partial_k \rho \partial_l \rho - \frac{1}{4J^2}\sum_{k,l} g_{kl}^{-1} \partial_k J \partial_l J \; - \frac{1}{2}\sum_{k,l} \partial_k \left[g_{kl}^{-1} \; \left(\partial_l \rho - \frac{1}{J} \partial_l J \right) \right] . \tag{5.1}$$

For the above mentioned reason the last term will usually drop out. It is apparent from this formulation, that re-scaling J by a constant will not alter the potential, a fact which is important with regard to the occurrence of possible coupling constants. To be more specific about the potential one has to choose a suitable pre-potential. In [23, 24, 25] the following ansatz was proposed

$$\rho = \sum_{i=0}^{\ell} \gamma_i \ln P_i(\vec{I}). \tag{5.2}$$

The $P_i(\vec{I})$ for $1 \le i \le \ell$ are defined by the factorization of the determinant of the inverse Riemannian

$$J^2 = G^{-1} = \det g^{-1} = \prod_{i=1}^{\ell} P_i(\vec{I}) \; . \tag{5.3}$$

Evidently, this ansatz (5.3) is inspired by the properties ii) and iii). However, there is an additional significant constraint namely that the $P_i(\vec{I})$ are functions of the invariants $\vec{I}$, which as was argued above is needed to guarantee the solvability. Often one would like to obtain also an additional harmonic confining term proportional to $\sum_i u_i^2$ in V. This is easily achieved by including also a factor of the form $P_0 \sim \exp(\sum_i u_i^2)$ into the ansatz for the pre-potential (5.2). For the reasons outlined in section 2, the entire Hamiltonian, that means also this term, has to be expressed in terms of invariant polynomials. Usually we can take $P_0 = \exp(I_2)$.

Substituting the ansatz (5.3) into (2.18), the potential acquires the form

$$V = \frac{1}{4}\sum_{i,j,k,l}\left(\gamma_i\gamma_j - \frac{1}{4}\right) g_{kl}^{-1}\partial_k(\ln P_i)\partial_l(\ln P_j) - \frac{1}{2}\sum_{i,k,l}\left(\gamma_i - \frac{1}{2}\right)\partial_k\left[g_{kl}^{-1}\partial_l(\ln P_i)\right] \ . \quad (5.4)$$

As was pointed out in [24], it will turn out that the terms with $i \neq j$ in the first term are constants (even zero) most of the time, which therefore can be dropped safely by just shifting the ground state. The other motivation for the ansatz (5.3) is that one would like the two terms in (5.1) or (2.18) to combine naturally.

Comparing now (5.3), (3.18) and (3.20) we can proceed and exploit the fact that the P_i factorize further into linear polynomials

$$P_i = \prod_{\alpha\in\Delta_+^{(i)}} [p_\alpha(\vec{u})]^2 \ . \quad (5.5)$$

The relation (5.5) is here the defining relation for the set of positive roots $\Delta_+^{(i)}$. Exploiting the fact that the $p_\alpha(\vec{u})$ are linear in $\vec{u}$, see (3.17), and changing from the invariant polynomials as coordinates to the $\vec{u}$-variables, we obtain

$$V = \sum_{i,j}\left[\left(\gamma_i\gamma_j - \frac{1}{4}\right)\sum_{\alpha\in\Delta_+^{(i)},\beta\in\Delta_+^{(j)}} \frac{\sum_k \tilde{\gamma}_\alpha^{(k)}\tilde{\gamma}_\beta^{(k)}}{p_\alpha(\vec{u})p_\beta(\vec{u})}\right] \ . \quad (5.6)$$

Recall that the $\tilde{\gamma}_\alpha^{(k)}$ are defined in (3.17). The expression (5.6) constitutes a general formula for potentials when starting with *any* Coxeter group. This structure will survive in different coordinate systems, as clearly the $p_\alpha(\vec{u})$ remain linear after linear coordinate transformations. Often one changes the coordinate system by using an explicit representations for the roots in an orthogonal basis, which one may find in various places of the literature, e.g. [29]. Then (5.6) enables one to write down directly the potentials associated to any Coxeter group. In practice it turns out that only the diagonal terms in the sum survive, i.e. $\alpha = \beta$, such that the potentials will always be of Calogero type. We omitted in (5.6) the term resulting from the last term in (5.4) and also a possible constant.

6. From G_2-polynomial invariants to the Calogero models

In [19] it was shown that the Calogero models for three particles exhibit an underlying G_2-structure, which can be exploited to establish their solvability. In [24] this procedure was reversed and it was shown that the approach outlined in section 2 and 3 indeed yields potentials of the Calogero type when one starts with a G_2-structure. Here we recall briefly the procedure, mainly to set the scene for the q-deformed treatment below, but also to establish a few facts not pointed out so far. In particular, in [19] as well as in [24] not the most general G_2-invariants were used. As we will show below the most generic invariants involve some arbitrary constants. The obvious question to ask is whether one obtains a new type of potential when using the procedure outlined above in terms of these generic coordinates, possibly involving additional coupling constants.

Let us start with the action of the G_2-Weyl reflections on the simple roots

$$\sigma_1(\alpha_1) = -\alpha_1 \qquad\qquad \sigma_1(\alpha_2) = 3\alpha_1 + \alpha_2 \tag{6.1}$$

$$\sigma_2(\alpha_1) = \alpha_1 + \alpha_2 \qquad\qquad \sigma_2(\alpha_2) = -\alpha_2 \ . \tag{6.2}$$

Then for a general vector $\vec{x} = x_1\alpha_1 + x_2\alpha_2$ in $\mathbb{R}^2$ we have

$$\sigma_1(\vec{x}) = (3x_2 - x_1)\alpha_1 + x_2\alpha_2 \tag{6.3}$$

$$\sigma_2(\vec{x}) = x_1\alpha_1 + (x_1 - x_2)\alpha_2, \tag{6.4}$$

such that

$$\sigma_1 P(x_1, x_2) = P(3x_2 - x_1, x_2) \tag{6.5}$$

$$\sigma_2 P(x_1, x_2) = P(x_1, x_1 - x_2) \ . \tag{6.6}$$

The equations (6.5) and (6.6) follow also directly from (3.3).

Using (6.5) and (6.6) we can now generate solutions to the equation (3.4), i.e. construct the invariant polynomials. The procedure is straightforward. We simply write down the most generic expression for a potential candidate for a polynomial invariant $I_s(\vec{x})$ of degree s similar to the form as in (3.13) with arbitrary coefficients c_s, but now in terms of the x-variables. Acting then with all simple Weyl reflections on this polynomial and demanding invariance (3.4) leads to a system of equations which determine the c_s. Depending on the degree and the algebra this might not yield enough equations to fix all constants and one ends up with expressions still involving free parameters. In this manner we find as generic invariants

$$I_2 = \kappa_2 \left(\frac{1}{3}x_1^2 + x_2^2 - x_1 x_2 \right) \tag{6.7}$$

$$I_6 = \kappa_6 \left(-\frac{2}{27}x_1^6 + \frac{2}{3}x_1^5 x_2 - \frac{5}{3}x_1^4 x_2^2 + 5x_1^2 x_2^4 - 6x_1 x_2^5 + 2x_2^6 \right) + \tilde{\kappa}_6 I_2^3 \ . \tag{6.8}$$

Here κ_2, κ_6 and $\tilde{\kappa}_6$ remain arbitrary constants. We see that besides an overall constant, which is naturally always present, I_6 also involves an additional free parameter $\tilde{\kappa}_6$.

A first restriction on possible values the constants might take comes from the fact that we want I_2 and I_6 to be algebraically independent. To establish this we compute first the Jacobian determinant for these invariants according to the definition (3.16)

$$J = \frac{2}{3}\kappa_2\kappa_6 x_1 x_2 (x_1 - x_2)(x_1 - 2x_2)(x_1 - 3x_2)(2x_1 - 3x_2) \ . \tag{6.9}$$

As we expect from property i) stated in section 3 and the explicit expression in (6.8) we have to keep $\kappa_6 \neq 0$ in order to guarantee the algebraic independence of I_2 and I_6. Obviously, for $\kappa_6 = 0$ we have $I_6 = \tilde{\kappa}_6 I_2^3$. We also note that $\tilde{\kappa}_6$ can remain completely arbitrary in this context, but as we see below, we can not simply set it to zero for our purposes.

Alternatively, we can compute the Jacobian determinant by an entirely different formula, namely (3.18), and thus confirm the computation which led to (6.9). To be able to use (3.18), we recall that the positive roots of G_2 are, (see e.g. [31, 29]))

$$\Delta_+^{G_2} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\} \ , \tag{6.10}$$

such that the hyperplanes result to

$$\begin{array}{lll} p_{\alpha_1} = 2x_1 - 3x_2, & p_{\alpha_1+\alpha_2} = x_1 - 3x_2, & p_{2\alpha_1+\alpha_2} = x_1, \\ p_{3\alpha_1+\alpha_2} = x_1 - x_2, & p_{\alpha_2} = x_1 - 2x_2, & p_{3\alpha_1+2\alpha_2} = x_1. \end{array} \tag{6.11}$$

Assembling the p_α into the product (3.18) the result confirms our findings (6.9) with $\mu = (2/3)\kappa_2\kappa_6$. Notice, that the constants κ_2 and κ_6 organize in a separate factor in J, such that different choices, apart from $\kappa_2 = 0$ or $\kappa_6 = 0$, will not alter the polynomial structure of J. From (5.1) we deduce that in the second term of the potential the overall factor in J just cancels, such that the constants κ_i completely drop out from this term.

So far, we have only two coordinates. To incorporate a three body interaction we need one more coordinate. Let us therefore choose an orthogonal basis in $\mathbb{R}^3$ for the simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$, with $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$ (see e.g. [29]). Then we can introduce a new set of variables via the relation

$$\vec{x} = x_1\alpha_1 + x_2\alpha_2 = (x_1 - 2x_2)\varepsilon_1 + (x_2 - x_1)\varepsilon_2 + x_2\varepsilon_3 = y_1\varepsilon_1 + y_2\varepsilon_2 + y_3\varepsilon_3 \ , \tag{6.12}$$

with the built-in constraint $y_1 + y_2 + y_3 = 0$. In these variables the action of the Weyl group on the same polynomials becomes

$$\sigma_1 P(y_1, y_2, y_3) = P(y_2, y_1, y_3) \tag{6.13}$$

$$\sigma_2 P(y_1, y_2, y_3) = P(-y_1, -y_3, -y_2) \ . \tag{6.14}$$

In principle, we could proceed as above, i.e. writing down generic expression with arbitrary coefficients and use directly (6.13) and (6.14) to find invariant polynomials in the y-variables. However, since we have increased the number of coordinates also the amount of unknown coefficients grows and we will end up with polynomials involving many more free constants than just the three we are left with when using the x-variables. Instead, as we know the invariants already, we can also use in (6.7) and (6.8) directly the substitutions $x_1 \to -y_1 - 2y_2$ and $x_2 \to -y_1 - y_2$, such that we obtain

$$I_2 = \kappa_2(y_2^2 + y_1^2 + y_1 y_2)/3 \tag{6.15}$$

$$\begin{aligned} I_6 = {} & \frac{1}{27}(\tilde{\kappa}_6 - 2\kappa_6)(y_1^6 + y_2^6) + \frac{1}{9}(\tilde{\kappa}_6 - 2\kappa_6)(y_1^5 y_2 + y_1 y_2^5) \\ & + \frac{1}{9}(5\kappa_6 + 2\tilde{\kappa}_6)(y_1^4 y_2^2 + y_1^2 y_2^4) + \frac{1}{27}(40\kappa_6 + 7\tilde{\kappa}_6) y_1^3 y_2^3. \end{aligned} \tag{6.16}$$

Only for the special choice $\kappa_2 = -3$, $\tilde{\kappa}_6 = 2\kappa_6$ and $\kappa_6 = 1$ the expressions (6.15) and (6.16) reduce to the coordinates λ_1, λ_2 used in [19]. Notice, that in the y-variables the invariants become symmetric polynomials [32].

In order to reproduce the Calogero potentials we need to make yet another coordinate transformation and introduce Jacobi relative coordinates

$$y_i = z_i - \frac{1}{3}\sum_{j=1}^{3} z_j \ , \tag{6.17}$$

which separate off the center of mass motion. The constraint in the y-coordinates is now replaced by $z_1 + z_2 + z_3 = 3Z$, where Z is constant. In these coordinates we compute the inverse Riemannian (3.19) to

$$g_{ij}^{-1} = \sum_{k=1}^{3} \frac{\partial I_i}{\partial z_k} \frac{\partial I_j}{\partial z_k} = \kappa_2 \begin{pmatrix} 2/3 I_2 & 2I_6 \\ 2I_6 & 6I_2^2/\kappa_2^3 \; \left[2\tilde{\kappa}_6 I_6 + I_2^3(4\kappa_6^2 - \tilde{\kappa}_6^2)/\kappa_2^3\right] \end{pmatrix}_{ij} . \tag{6.18}$$

Apparently g_{ij}^{-1} is not quadratic in the variables I_i, which as we discussed is a necessary requirement to be able to bring the Hamiltonian into the form (2.5) and hence ensuring the solvability of the model. However, for $\tilde{\kappa}_6 = 2\kappa_6$ one can choose a different set of variables $I_2 = \tau_2$, $I_6 = \tau_3^2$, see [19], such that $\partial_i g_{ij}^{-1} \partial_j$ is of the desired form in the τ variables. Alternatively, one can also use the representation of a different algebra to establish solvability [19]. Let us from now on take $\tilde{\kappa}_6 = 2\kappa_6$ and $\kappa_2 = 1$ but leaving κ_6 arbitrary. Then we compute from (6.18)

$$G^{-1} = \det g^{-1} = 4I_6 \left(4\kappa_6 I_2^3 - I_6\right) . \tag{6.19}$$

For the pre-potential we make now an ansatz according to (5.3), where we also include the previously mentioned P_0-term

$$P_0 = e^{I_2}, \qquad P_1 = 4\kappa_6 I_2^3 - I_6, \qquad P_2 = I_6 . \tag{6.20}$$

From this we compute with formula (5.4) the potential to

$$V = \frac{\gamma_0^2}{6} I_2 + 3\lambda_1 \kappa_6 \frac{I_2^2}{4\kappa_6 I_2^3 - I_6} + 3\lambda_2 \kappa_6 \frac{I_2^2}{I_6} \tag{6.21}$$

$$= \frac{1}{2}\omega^2 \sum_{k=1}^{3} z_k^2 + \lambda_1 \sum_{1 \le i < j \le 3} \frac{1}{(z_i - z_j)^2} + 3\lambda_2 \sum_{\substack{1 \le i < j \le 3 \\ i,j \ne k}} \frac{1}{(z_i + z_j - 2z_k)^2} , \tag{6.22}$$

where the coupling constants are $\omega = \gamma_0/(3\sqrt{2})$ and $\lambda_i = 2\gamma_i^2 - 1/2$ for $i = 1, 2$. The potential in the form (6.22) corresponds to so-called rational G_2-model [33], which reduces to the Calogero model [1, 2, 3] when the three-particle interaction is switched off, i.e. for $\lambda_2 \to 0$. Notice that the coupling constants γ_0, γ_1 and γ_2 which enter the scheme through the ansatz for the pre-potential just reparameterize the coupling constants of the G_2-model. Note also that the constant κ_6 has dropped out completely, such that any choice, apart from $\kappa_6 = 0$, will yield the same potential (6.22). Hence, the ambiguity in the choice of the invariant polynomials as coordinates has no bearing on the physics.

The invariants acquire a particularly simple form when we use the eigenbasis of the Coxeter element. The transformations outlined above yield for the G_2-case

$$x_1 = \sqrt{3}(e^{-i\pi/6} w_1 - e^{-i\pi 5/6} w_2), \quad x_2 = w_1 + w_2 , \tag{6.23}$$

such that the invariants simplify considerably

$$I_2 = \kappa_2 w_1 w_2 \quad \text{and} \quad I_6 = \kappa_6 (w_1^6 + w_2^6) + \tilde{\kappa}_6 w_1^3 w_2^3 . \tag{6.24}$$

We then find for the Jacobian

$$J = -6\kappa_2\kappa_6(w_1^6 - w_2^6) \tag{6.25}$$
$$= -6\kappa_2\kappa_6(w_1 + w_2)(w_1 - w_2)(w_1^2 - w_1w_2 + w_2^2)(w_1^2 + w_1w_2 + w_2^2) , \tag{6.26}$$

which could of course be used to construct the potential in these variables. As this type of factorization of J involves quadratic polynomials we will end up with potentials not quite of Calogero type. To achieve this we would have to factorize the last two terms further involving complex coefficients, but in that case the individual two particle interactions terms would be complex. Solvability is only guaranteed when we can express the factors in terms of the invariant polynomials.

7. Exactly solvable potentials from q-deformed G$_2$-polynomial invariants

Let us now extend the previous analysis to the q-deformed case. To commence we need to evaluate the q-deformed Weyl reflections σ_i^q as defined in (4.1) for which we require the q-deformed Cartan matrix. In our conventions it reads for the G_2-case

$$K_q = \begin{pmatrix} 2 & -1 \\ -[3]_q & 2 \end{pmatrix} . \tag{7.1}$$

With K_q at hand we can now seek invariant polynomials according to the definitions (4.1) and (4.2). We proceed in the same manner as for the non-deformed case and start with generic expressions $I_s^q(\vec{x})$ for polynomials of degree s as in (3.13) and fix the constants as outlined above. For generic deformation parameters q we did not find invariants. However, if we parameterize the q's as

$$q^2 = \frac{1}{2}\left(1 + 2\cos\frac{2\pi}{h}\right) + \sqrt{\left(1 + 2\cos\frac{2\pi}{h}\right)^2 - 4} \tag{7.2}$$

with h being some integer, the q-deformed Cartan matrix becomes

$$(K_q)_{ij} = \frac{2\alpha_i^q \cdot \alpha_j^q}{\alpha_j^q \cdot \alpha_j^q} = \begin{pmatrix} 2 & -1 \\ -4\cos^2\frac{\pi}{h} & 2 \end{pmatrix}_{ij} . \tag{7.3}$$

Implicitly, we used here (7.3) to define some q-deformed roots α_i^q. Clearly for $h = 3$ we recover the Cartan matrix of A_2, for $h = 4$ we obtain the one of C_2 and $h = 6$ corresponds to G_2. For the values (7.2) of q we find there exist always the invariants I_2^q and I_h^q. From the above mentioned arguments this suggests that the exponents of this algebra G_2^q are 1 and $h-1$. This assertion is supported by the observation that the formula for eigenvalues of the Cartan matrix (3.7) still holds for the q-deformed case (7.3) when taking $s_1 = 1$ and $s_2 = h - 1$.

As we argued in the previous section, it is difficult to find generic expressions for the invariants in the x-variables. However, as we will see in the eigenbasis of the Coxeter element this task simplifies drastically. According to (3.8) and (3.9) we have the transformations

$$\vec{x} = \begin{pmatrix} 1 + e^{-\frac{2\pi i}{h}} & 1 + e^{\frac{2\pi i}{h}} \\ 1 & 1 \end{pmatrix}\vec{w} \qquad \Leftrightarrow \qquad \vec{w} = \frac{i}{2\sin\frac{2\pi}{h}}\begin{pmatrix} 1 & -1 - e^{\frac{2\pi i}{h}} \\ -1 & 1 + e^{-\frac{2\pi i}{h}} \end{pmatrix}\vec{x} , \tag{7.4}$$

such that the q-deformed Weyl reflections in the w-variables simplify to

$$\sigma_1^q(w_1) = w_2, \quad \sigma_1^q(w_2) = w_1, \quad \sigma_2^q(w_1) = e^{\frac{2\pi i}{h}} w_2, \quad \sigma_2^q(w_2) = e^{-\frac{2\pi i}{h}} w_1 \ . \tag{7.5}$$

The σ_1^q-transformations dictate that the invariants have to be symmetric in w_1, w_2 and the σ_2^q-transformations constrain their overall degree. With (7.5) we can easily find the most generic expressions for the invariants

$$I_2^q = \kappa_2 w_1 w_2 \qquad \text{and} \qquad I_h^q = \kappa_h (w_1^h + w_2^h) \ + \tilde{\kappa}_h (w_1 w_2)^{h/2}. \tag{7.6}$$

where $\tilde{\kappa}_h = 0$ for h being an odd integer. We can now transform back to the x-variables and confirm for instance the generic formula (4.3) for the invariant of degree 2, which still takes on a fairly simple form

$$I_2^q = \frac{\kappa_2}{4\sin^2(2\pi/h)} \left(x_1^2 + 4\cos^2\frac{\pi}{h} x_2^2 - 4\cos^2\frac{\pi}{h} x_1 x_2 \right) \ . \tag{7.7}$$

On the other hand, the expressions for the I_h^q are already quite cumbersome, albeit it is clear how to construct them from (7.6) and (7.4).

As we saw in the previous section it was crucial to change the coordinate system yet further to recover the Calogero potentials in the usual form. We proceed here similarly. Let us choose first an orthogonal basis for the two simple q-deformed roots in $\mathbb{R}^3$

$$\alpha_1^q = (\sqrt{3}\cos\frac{2\pi}{h} + \sin\frac{2\pi}{h}, -2\sin\frac{2\pi}{h}, \sin\frac{2\pi}{h} - \sqrt{3}\cos\frac{2\pi}{h})/\sqrt{3} \tag{7.8}$$

$$\alpha_2^q = (-\sqrt{3} - \sqrt{3}\cos\frac{2\pi}{h} - \sin\frac{2\pi}{h}, 2\sin\frac{2\pi}{h}, \sqrt{3} + \sqrt{3}\cos\frac{2\pi}{h} - \sin\frac{2\pi}{h})/\sqrt{3} \ . \tag{7.9}$$

The inner products of these roots are $\alpha_1^q \cdot \alpha_1^q = 2$, $\alpha_2^q \cdot \alpha_2^q = 8\cos^2\frac{\pi}{h}$ and $\alpha_1^q \cdot \alpha_2^q = -4\cos^2\frac{\pi}{h}$ such that we recover the q-deformed Cartan matrix according to (7.3). Of course the choices (7.8) and (7.9) are not uniquely determined. As an additional selection criterion we demand that I_2^q will be of an analogous form (6.15) as in the non-deformed case for all choices of h, such that it will be ensured that we can express g_{22}^{-1} in terms of I_2^q. At the same time this will ensure that $P_0 = \exp(I_2^q)$ yields the harmonic confining potential, similarly as for the standard G_2-case. The above choice for the simple roots induces a definition for new variables

$$\vec{y} = x_1 \alpha_1^q + x_2 \alpha_2^q \ , \tag{7.10}$$

which satisfy the constraint $y_1 + y_2 + y_3 = 0$. In turn this means that we can replace in (7.7)

$$\vec{x} = -\frac{1}{2} \begin{pmatrix} 2 & (1+\sqrt{3}\cot\frac{\pi}{h}) \\ 2 & (1+\sqrt{3}\cot\frac{2\pi}{h}) \end{pmatrix} \vec{y} \tag{7.11}$$

such that I_2^q is indeed always of the form (6.15)

$$I_2^q = \frac{\kappa_2}{4\sin^2(2\pi/h)} \left(y_1^2 + y_2^2 + y_1 y_2 \right) = \frac{3\kappa_2}{8\sin^2(2\pi/h)} \left(\sum_{i=1}^{3} z_i^2 - 3Z^2 \right) \ . \tag{7.12}$$

Again it is obvious how to obtain the expressions for I_h^q, but they turn out to be more cumbersome. From (7.6) it is apparent that the case of even and odd Coxeter number exhibit different behaviour. We treat them now separately and supply for each case an explicit example.

7.1 Even Coxeter numbers, h=8

As the obvious difference between the odd and even case we found that in the even case the additional constant $\tilde{\kappa}_h$ is entering the procedure. We shall see that there are also other more profound differences. We present the case $h = 8$ in detail. For this the relation (7.2) simply yields

$$q^2 = \frac{1}{2}(1+\sqrt{2}) + \sqrt{2\sqrt{2}-1}, \qquad \text{and} \qquad [3]_q = 2+\sqrt{2}. \tag{7.13}$$

We can now take the invariants as given by the relations (7.6) and carry out the substitutions $\vec{w} \to \vec{x} \to \vec{y} \to \vec{z}$ specified above. In terms of the Jacobian relative coordinates z_i the inverse Riemannian (3.19) results to

$$g_{ij}^{-1} = \sum_{k=1}^{3} \frac{\partial I_i}{\partial z_k}\frac{\partial I_j}{\partial z_k} = \kappa_2 \begin{pmatrix} I_2 & 4I_8 \\ 4I_8 & 16 I_2^3/\kappa_2^4 \; \left[2\tilde{\kappa}_8 I_8 + I_2^4(\tilde{\kappa}_8^2 - 4\kappa_8^2)/\kappa_2^4\right] \end{pmatrix}_{ij} . \tag{7.14}$$

At this point we are facing a similar problem as in the non-deformed case, that is $h = 6$, namely that g_{ij}^{-1} is not of degree two in the variables I_i. Whereas for $h = 6$ one may find a suitable variable transformation, we did not succeed in this case. Nonetheless, the solvability of the model may now be guaranteed now by relating the model to $\mathrm{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^4$, rather than gl(N), see (2.6) and (2.7).

Keeping from now on $\tilde{\kappa}_8 = 2\kappa_8$ and also $\kappa_2 = 1$, we can bring the Hamiltonian into the desired form for the Schrödinger operator (2.5), (2.10)

$$\begin{aligned} \mathcal{D} = & -J^2 J^1 - 8J^3 J^1 - 64\kappa_8 J^3 J^8 + \left[4(\gamma_1+\gamma_2) - 5 - \frac{11}{9}\kappa_8\right] J^1 \\ & -16\kappa_8 \left[(1-\gamma_2) + \frac{4\kappa_8}{9}\right] J^8 + \gamma_0 J^2 + 4\gamma_0 J^3 \; . \end{aligned} \tag{7.15}$$

To turn this operator into the standard form (2.11) we follow the procedure outlined in section 2. First, we compute

$$G^{-1} = \det g^{-1} = -16 I_8 (I_8 - 4\kappa_8 I_2^4) \; . \tag{7.16}$$

Before we proceed further to compute the potential, let us see if we still have a relation between J and G^{-1} of the type (3.20) for the q-deformed algebra. In particular, we wish to see whether a relation of the type (3.18) still holds. For this purpose we need first of all a notion of positive q-deformed roots. We assume that these roots are generated in a similar way as the ordinary roots, i.e. by repeated action of the Coxeter element. Defining a q-deformed version of this[1]

$$\sigma_q = \sigma_2^q \sigma_1^q \tag{7.17}$$

[1]This q-deformed Coxeter element differs slightly from the one defined in [30, 27], as here there is no τ-transformation involved. The Coxeter element in [30, 27] is only of order h up to some factors of q.

we compute the entire set of q-deformed roots Δ_q lying in the orbits of the simple q-deformed roots α_i^q, $\sigma_q^1(\alpha_i^q)$, $\sigma_q^2(\alpha_i^q)$, ...

σ_q^0	α_1^q	α_2^q
σ_q^1	$-\alpha_3^q = -(\alpha_1^q + \alpha_2^q)$	$\alpha_6^q = (2+\sqrt{2})\alpha_1^q + (1+\sqrt{2})\alpha_2^q$
σ_q^2	$-\alpha_4^q = -(1+\sqrt{2})\alpha_1^q - \sqrt{2}\alpha_2^q$	$\alpha_7^q = (2+2\sqrt{2})\alpha_1^q + (1+\sqrt{2})\alpha_2^q$
σ_q^3	$-\alpha_5^q = -(1+\sqrt{2})\alpha_1^q - \alpha_2^q$	$\alpha_8^q = (2+\sqrt{2})\alpha_1^q + \alpha_2^q$
σ_q^4	$-\alpha_1^q$	$-\alpha_2^q$

. (7.18)

Note that the order of the Coxeter element (7.17) is indeed $h = 8$, i.e. $\sigma_q^8 = 1$. We adopt now the same notion for positive and negative roots as in the non-deformed case, that is we call $\alpha = \sum n_i \alpha_i^q$ a positive root if all coefficients n_i are positive. With this notion the set of the $2h$ roots can be separated equally into h positive and h negative roots. We have verified this statement up to $h = 20$ of even Coxeter numbers, which strongly suggests that it holds in general. Using now (7.18) we can compute the hyperplanes through the origin to all positive q-deformed roots

$$\begin{array}{ll} p_{\alpha_1^q} = 2x_1 - (2+\sqrt{2})x_2 & p_{\alpha_2^q} = x_1 - 2x_2 \\ p_{\alpha_3^q} = \sqrt{2}x_1 - (2+\sqrt{2})x_2 & p_{\alpha_6^q} = \sqrt{2}x_1 - 2(1+\sqrt{2})x_2 \\ p_{\alpha_4^q} = x_2 & p_{\alpha_7^q} = x_1 \\ p_{\alpha_5^q} = x_1 - x_2 & p_{\alpha_8^q} = (2+\sqrt{2})x_1 - 2(1+\sqrt{2})x_2 \quad . \end{array} \tag{7.19}$$

Comparing now with (7.16) we have once again a relation between J and $G^{-1}$of the type (3.20) where J can be expressed as a product of hyperplanes (3.18)

$$G^{-1} = \frac{\kappa_8^2}{4(3+2\sqrt{2})} \prod_{\alpha^q \in \Delta_q^+} (p_{\alpha^q})^2 \tag{7.20}$$

The two factors in (7.16) admit yet a further interpretation. Organizing the roots into two sets Δ_s^q and Δ_l^q of short and long roots, respectively, we find the identities

$$-\frac{\kappa_8}{4} \prod_{\alpha^q \in \Delta_s} (p_{\alpha^q})^2 = I_8 - 4\kappa_8 I_2^4 \quad \text{and} \quad \frac{\kappa_8}{16(3+2\sqrt{2})} \prod_{\alpha^q \in \Delta_l} (p_{\alpha^q})^2 = I_8 \ . \tag{7.21}$$

According to (5.3) we make now the following ansatz for the pre-potential

$$P_0 = e^{I_2}, \qquad P_1 = I_8 - 4\kappa_8 I_2^4, \qquad P_2 = I_8. \tag{7.22}$$

From formula (5.4) and including also the P_0-term we then compute the potential to

$$V = \frac{\gamma_0^2}{4} I_2 - 16\lambda_1 \kappa_8 \frac{I_2^3}{I_8 - 4\kappa_8 I_2^4} + 16\lambda_2 \kappa_8 \frac{I_2^3}{I_8} \tag{7.23}$$

with $\lambda_i = (\gamma_i^2 - 1/4)$ for $i = 1, 2$. Using the above mentioned identities or directly (5.6), we can also re-write this potential in terms of the z-variables. First of all we compute

$$P_1 = \frac{1}{4^2 3^3} (z_1 - z_3)^2 (z_1 + z_3 - 2z_3)^2 \prod_{\varepsilon = \pm 1} \left[(1+\varepsilon\sqrt{3})z_1 + (1-\varepsilon\sqrt{3})z_3 - 2z_2 \right]^2 \tag{7.24}$$

$$P_2 = \frac{1}{4^3 3^4} \prod_{\varepsilon, \bar{\varepsilon} = \pm 1} \left[(1 - \bar{\varepsilon}\sqrt{3} - \varepsilon\sqrt{6})z_1 + (1 + \bar{\varepsilon}\sqrt{3} + \varepsilon\sqrt{6})z_3 - 2z_2 \right]^2 \ . \tag{7.25}$$

We find

$$
\begin{aligned}
V \;=\; & \frac{1}{2}\omega^2 \sum_{k=1}^{3} z_k^2 + \frac{\lambda_1}{(z_1 - z_3)^2} + \frac{3\lambda_1}{(z_1 + z_3 - 2z_2)^2} \\
& + \sum_{\varepsilon=\pm 1} \frac{6\lambda_1}{\left[(1+\varepsilon\sqrt{3})z_1 + (1-\varepsilon\sqrt{3})z_3 - 2z_2\right]^2} \\
& + \sum_{\varepsilon,\bar{\varepsilon}=\pm 1} \frac{6(2+\varepsilon\bar{\varepsilon}\sqrt{2})\lambda_2}{\left[(1-\bar{\varepsilon}\sqrt{3}-\varepsilon\sqrt{6})z_1 + (1+\bar{\varepsilon}\sqrt{3}+\varepsilon\sqrt{6})z_3 - 2z_2\right]^2}
\end{aligned}
\tag{7.26}
$$

with $\omega = \gamma_0\sqrt{3}/(2\sqrt{2})$. We omitted here a constant which contains the center of mass coordinate. Remarkably, all off-diagonal terms, that is terms in (5.4) with $i \neq j$, cancel each other. This potential has a very similar structure as the usual Calogero potentials (6.22), but it involves now deformed two and three-particle interactions. We find similar structures for higher values of h. Responsible for this structure is the fact that we can still factorize J, and therefore G^{-1} in terms of products of hyperplanes as in (7.20). Remarkably, these potentials are all exactly solvable by construction.

7.2 Odd Coxeter numbers, h=5

The structure for theories with odd values of the Coxeter number is somewhat different. Let us consider $h = 5$ in more detail. In that case the relation for the deformation parameter (7.2) simply yields a root of unity

$$
q = e^{i\pi/10} \qquad \text{and} \qquad [3]_q = \frac{3+\sqrt{5}}{2}. \tag{7.27}
$$

Replacing now in equation (7.6) the variables $\vec{w} \to \vec{x}$, the invariant of degree 5 in the x-variables is still not too lengthy, unlike for greater values of h, and reads in this case

$$
I_5^q = \kappa_5 \left(\frac{1}{2}(3-\sqrt{5})x_1^4 x_2 - 2x_1^3 x_2^2 + (1+\sqrt{5})x_1^2 x_2^3 - \frac{1}{2}(1+\sqrt{5})x_1 x_2^4 \right) . \tag{7.28}
$$

Obviously I_2^q and I_5^q are algebraically independent as one is of even and the other of odd degree, respectively. As for the even case, we can proceed and carry out in (7.28) the substitutions $\vec{x} \to \vec{y} \to \vec{z}$ such that the invariants are expressed in terms of the Jacobian relative coordinates z_i. In these coordinates the inverse Riemannian (3.19) results to

$$
g_{ij}^{-1} = \sum_{k=1}^{3} \frac{\partial I_i}{\partial z_k}\frac{\partial I_j}{\partial z_k} = \kappa_2(5-\sqrt{5}) \begin{pmatrix} \frac{1}{5}I_2 & \frac{1}{2}I_5 \\ \frac{1}{2}I_5 & 5I_2^4\kappa_5^2/\kappa_2^5 \end{pmatrix}_{ij} . \tag{7.29}
$$

Once again we have the problem that g_{ij}^{-1} is not of degree two in the variables I_i. As for $h = 8$ we can relate once more to the $\mathrm{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^5$algebra, see (2.6) and (2.7). From now on we keep $\kappa_2 = 1$ and bring the Hamiltonian into the desired form (2.5), (2.10)

$$
\mathcal{D} = (\sqrt{5}-5)\left(\frac{1}{5}\,J^2J^1 + J^1J^3 + 5\kappa_5^2 J^5 J^9 - \left(\gamma_1 - \frac{7}{10} - \frac{2\kappa}{3}\right)J^1 - \frac{\gamma_0}{5}J^2 + \frac{\gamma_0}{2}\,J^3\right) \tag{7.30}
$$

We proceed similarly as in the previous subsection and compute from (7.29)

$$G^{-1} = \det g^{-1} = \frac{5}{2}(\sqrt{5}-3)(I_5^2 - 4\kappa_5^2 I_2^5) \ . \tag{7.31}$$

In order to obtain Calogero type potentials it is vital to factorize G^{-1} further into linear polynomials. Let us proceed analogously as for even Coxeter numbers and compute the orbits of the q-deformed Coxeter element (7.17). We find

σ_q^0	α_1^q	α_2^q
σ_q^1	$\alpha_3^q = -(\alpha_1^q + \alpha_2^q)$	$\alpha_7^q = \frac{1}{2}(3+\sqrt{5})\alpha_1^q + \frac{1}{2}(1+\sqrt{5})\alpha_2^q$
σ_q^2	$\alpha_4^q = -\frac{1}{2}(1+\sqrt{5})\alpha_1^q - \frac{1}{2}(\sqrt{5}-1)\alpha_2^q$	$\alpha_8^q = \ \ \frac{1}{2}(1+\sqrt{5})\alpha_1^q$
σ_q^3	$\alpha_5^q = \frac{1}{2}(\sqrt{5}-1)\alpha_2^q$	$\alpha_9^q = -\frac{1}{2}(1+\sqrt{5})(\alpha_1^q + \alpha_2^q)$
σ_q^4	$\alpha_6^q = \frac{1}{2}(1+\sqrt{5})\alpha_1^q + \alpha_2^q$	$\alpha_{10}^q = -\frac{1}{2}(3+\sqrt{5})\alpha_1^q - \alpha_2^q$
σ_q^5	α_1^q	α_2^q

(7.32)

Note that the order of the Coxeter element (7.17) is still h, that is in this case $\sigma_q^5 = 1$. For odd values of the Coxeter number we can still separate the roots into positive and negative roots, but now the negative roots can no longer be obtained by reversing the signs of all positive roots. There are now positive roots without a negative counterpart. Unfortunately, as a consequence of this the factorization property (3.18) does no longer hold in its stated form. Nonetheless, one can still select some hyperplanes obtained from the root system (7.32) and factorize G^{-1}, albeit now the selection principle does no longer favour the positive roots and is less clear. We compute

$$\begin{array}{ll}
p_{\alpha_1^q} = x_1 - \frac{1}{4}(3+\sqrt{5})x_2 & p_{\alpha_2^q} = x_1 - 2x_2 \\
p_{\alpha_3^q} = x_1 - (2+\sqrt{5})x_2 & p_{\alpha_7^q} = x_1 + \frac{1}{2}(1+\sqrt{5})x_2 \\
p_{\alpha_4^q} = \frac{1}{2}(1+\sqrt{5})x_1 - x_2 & p_{\alpha_8^q} = x_1 - \frac{1}{4}(3+\sqrt{5})x_2 \\
p_{\alpha_5^q} = x_1 - 2x_2 & p_{\alpha_9^q} = x_1 + \frac{1}{2}(1-\sqrt{5})x_2 \\
p_{\alpha_6^q} = \frac{1}{2}(\sqrt{5}-1)x_1 + x_2 & p_{\alpha_{10}^q} = x_1 - \frac{1}{2}(\sqrt{5}-1)x_2
\end{array} \tag{7.33}$$

and construct from this

$$J = p_{\alpha_1^q} p_{\alpha_2^q} p_{\alpha_3^q} p_{\alpha_7^q} p_{\alpha_{10}^q} \ . \tag{7.34}$$

Now, unlike as in the even case, the splitting into long and short roots no longer corresponds to factors in terms of hyperplanes.
For the pre-potential we make now the ansatz

$$P_0 = e^{I_2}, \qquad P_1 = I_5^2 - 4\kappa_5^2 I_2^5. \tag{7.35}$$

With formula (5.4) we then compute the potential in terms of invariant polynomials to

$$V = \frac{\gamma_0^2}{4}\left(1 - \frac{1}{\sqrt{5}}\right) I_2 + 5(\sqrt{5}-5)\lambda\kappa_5^2 \frac{I_2^4}{I_5^2 - 4\kappa_5^2 I_2^5}, \tag{7.36}$$

with $\lambda = (\gamma_1^2 - 1/4)$. Once again we can also re-write V in terms of the z-variables. First we factorize

$$P_1 = \frac{5^2}{3^4 4^5}(z_1 - z_3)^2 \prod_{\bar{\varepsilon},\varepsilon=\pm 1}\left[(1+\sqrt{3+\frac{6\varepsilon}{\sqrt{5}}})z_{2+\bar{\varepsilon}} + (1-\sqrt{3+\frac{6\varepsilon}{\sqrt{5}}})z_{2-\bar{\varepsilon}} - 2z_2\right]^2 \tag{7.37}$$

from which we deduce the potential to

$$V = \frac{1}{2}\omega^2 \sum_{k=1}^{3} z_k^2 + \frac{\lambda}{(z_1 - z_3)^2} + \sum_{\bar{\varepsilon},\varepsilon=\pm 1} \frac{6(1+\varepsilon/\sqrt{5})\lambda}{\left[(1+\sqrt{3+\frac{6\varepsilon}{\sqrt{5}}})z_{2+\bar{\varepsilon}} + (1-\sqrt{3+\frac{6\varepsilon}{\sqrt{5}}})z_{2-\bar{\varepsilon}} - 2z_2\right]^2} \tag{7.38}$$

with $\omega^2 = 3(3-\sqrt{5})\gamma_0^2/20$. Once again all off-diagonal terms cancel each other and as in the even case this potential is also of Calogero type. We find similar types of potentials for higher values of the Coxeter number.

8. Conclusions

It has been shown previously that solvability of certain types of Hamiltonians can be established [17, 18, 19, 20] by relating the differential operators inside the Hamiltonians, that is essentially the Laplace operator, to a representation of the gl(N)-Lie algebra. This formulation can be made very systematic by associating the differential structure to polynomial invariants of Coxeter groups. This led the authors of [22, 23, 24, 25] to propose a procedure which allows to construct solvable Hamiltonians by taking the structure of the polynomial invariants as a starting point. Here we showed that this procedure can be extended successfully to polynomial invariants of q-deformed Coxeter groups. We constructed some potentials resulting from these type of invariants. Due to the fact that the Jacobian determinant can still be factorized in terms of linear polynomials the resulting potentials are of Calogero type.

There are several open issues and unanswered questions which deserve further investigations. Clearly it would be interesting to carry out the outlined procedure explicitly for other algebras than G_2^q. The presented example indicates that one can expect similar structures beyond G_2^q. Eventually one should aim at a unified formulation, as opposed to case-by-case studies, analogously to the non-deformed case as indicated in section 5. Crucial will be here the factorization of the Jacobian determinant J. In the presented example this works nicely for even Coxeter numbers, but for odd h the example hints that one possibly has to employ a different q-deformed Coxeter transformation in order to obtain a definite criterion for the selection of the hyperplanes which yields the factorization of J.

To achieve a unified formulation it will be important to have systematic and generic expressions for the polynomial invariants [28]. In the above analysis we have seen that the choice of a suitable basis is absolutely crucial for this task. The favoured one is the eigenbasis of the Coxeter element as we have demonstrated.

Since we have shown that one can extend the approach from Coxeter to q-deformed Coxeter groups, it is also natural to suspect that one might as well employ it for reflection groups the general type introduced in [34].

Naturally it appears also possible to construct potential of Sutherland type by using different types of coordinates [28].

To find the explicit wavefunction for the above Hamiltonians is now also an obvious question to ask. Following the quoted literature there is a straightforward procedure to construct them from the above mentioned results. In this context one might also address technical questions as for instance self-adjointness similar as it has been done in Calogero's original work for hardcore boundary conditons [1, 2, 3]. For slightly more general boundary conditions, see e.g. [35]. Conceptionally one should stress that solvability in the sense provided here is far more constructive with regard to this question than integrability. The latter usually just guarantees the existence of exact solutions, whereas solvability is already tied closely to the explicit solutions.

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