THE APPROXIMATE DETERMINANTAL ASSIGNMENT PROBLEM

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Abstract

The Determinantal Assignment Problem (DAP) is one of the central problems of Algebraic Control Theory and refers to solving a system of non-linear algebraic equations to place the critical frequencies of the system to specified locations. This problem is decomposed into a linear and a multi-linear subproblem and the solvability of the problem is reduced to an intersection of a linear variety with the Grassmann variety. The linear subproblem can be solved with standard methods of linear algebra, whereas the intersection problem is a problem within the area of algebraic geometry. One of the methods to deal with this problem is to solve the linear problem and then find which element of this linear space is closer - in terms of a metric - to the Grassmann variety. If the distance is zero then a solution for the intersection problem is found, otherwise we get an approximate solution for the problem, which is referred to as the approximate DAP.

In this thesis we examine the second case by introducing a number of new tools for the calculation of the minimum distance of a given parametrized multi-vector that describes the linear variety implied by the linear subproblem, from the Grassmann variety as well as the decomposable vector that realizes this least distance, using constrained optimization techniques and other alternative methods, such as the SVD properties of the so called Grassmann matrix, polar decompositions and other tools. Furthermore, we give a number of new conditions for the appropriate nature of the approximate polynomials which are implied by the approximate solutions based on stability radius results.

The approximate DAP problem is completely solved in the 2-dimensional case by examining uniqueness and non-uniqueness (degeneracy) issues of the decompositions, expansions to constrained minimization over more general varieties than the original ones (Generalized Grassmann varieties), derivation of new inequalities that provide closed-form non-algorithmic results and new stability radii criteria that test if the polynomial implied by the approximate solution lies within the stability domain of the initial polynomial. All results are compared with the ones that already exist in the respective literature, as well as with the results obtained by Algebraic Geometry Toolboxes, e.g., Macaulay 2. For numerical implementations, we examine under which conditions certain manifold constrained algorithms, such as Newton's method for optimization on manifolds, could be adopted to DAP and we present a new algorithm which is ideal for DAP approximations. For higher dimensions, the approximate solution is obtained via a new algorithm that decomposes the parametric tensor which is derived by the system of linear equations we mentioned before.

Nonemclature

$\mathcal{F}$:	denotes a general field or ring, e.g., $\mathbb{R}, \mathbb{C}$.
a:	denotes a scalar in $\mathcal{F}$.
$\mathcal{F}^n$:	denotes an n dimensional vector space over $\mathcal{F}$.
$\mathcal{V}$:	denotes an ordinary vector space or a vector space/set equipped with a special structure, e.g., varieties, manifolds, etc.
$\bigwedge^k(\mathcal{V})$:	denotes the k-th exterior power of a vector space $\mathcal{V}$.
$\underline{x}$:	denotes a vector or a k-vector (multivector) on $\mathcal{V}$ or $\mathcal{V}_1 \times \cdots \times \mathcal{V}_k$, respectively.
$\underline{a} \wedge \underline{b}$:	denotes the wedge product (exterior product) between two k-vectors in $\bigwedge^k(\mathcal{V})$.
$\mathbb{R}(s)$:	denotes the field of rational functions.
$\mathcal{F}[s]$:	denotes the ring of polynomials over $\mathcal{F}$.
$\mathfrak{I}$:	denotes an ideal of $\mathcal{F}[s]$.
$\mathcal{Q}_{m,n}$:	denotes the set of lexicographically ordered, strictly increasing sequences of m integers from 1,2,...,n.
$\mathcal{F}^{m\times n}$:	denotes the set of $m \times n$ matrices with elements from $\mathcal{F}$, e.g., $\mathbb{R}^{m\times n}$ are all the real $m \times n$ matrices.
A:	denotes a matrix in $\mathcal{F}^{m\times n}$
$\mathcal{N}_r(A), \mathcal{N}_\ell(A)$:	denote the right and the left null space of a matrix A.
$C_k(A)$:	denotes the kth compound matrix of $A \in \mathcal{F}^{m\times n}$, $k \leq \{m, n\}$.
$A \otimes B$:	denotes the Kronecker product between two matrices A and B.
$A * B$:	denotes the Hadamard product between two matrices A and B.

$\underline{e}_1 \otimes \cdots \otimes \underline{e}_n$:	denotes the tensor product of the vectors $\underline{e}_i$ that belong to the vector spaces $\mathcal{V}_i$, respectively, for $i = 1, ..., n$.
$\mathbf{T}$:	denotes a tensor/multi-dimensional array in $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$, for the vector spaces $\mathcal{V}_i$, $i = 1, ..., n$.
$\mathbb{P}^n(\mathcal{F})$:	denotes the n-dimensional projective space for the field $\mathcal{F}$.
$Gr(m, V)$:	denotes the set of all m-dimensional subspaces of an n-dimensional vector space $\mathcal{V}$, called the Grassmannian.
$G_m(\mathcal{V})$ or $G_{m,n}$:	denotes the Grassmann variety of the projective space, implied by the embedding of $Gr(m, V)$ into the projective space $\mathbb{P}\left(\bigwedge^m(V)\right)$.
MIMO:	Multiple inputs-Multiple outputs.
DAP:	Determinantal Assignment Problem.
QPR:	Quadratic Plücker Relations.
CPD:	CANDECOMP/PARAFAC Decomposition (Canonical Decomposition/Parallel Factors).
ALS:	Alternating Least Squares.

Chapter 1

Introduction

Control methodologies are central in the development of engineering design solutions to modern challenging applications. Control Technology is supported by a systems framework, where the study of system properties and the development of solutions to well defined problems are crucial. Control theory provides the backbone of control synthesis methods and control system design. It's main aspects are:

i) Study of systems properties.

ii) Characterization of solvability conditions of exact control problems.

iii) Synthesis methods for control problems.

iv) Design methods.

These aspects are central to the development of design strategies and methodologies since they provide the tools of analysis, formulation of objectives and development of control design approaches and methodologies. Control theory is model dependent and the richest part of it is that dealing with linear, time-invariant, finite dimensional (lumped parameter) systems. Such simple models seem to be appropriate for the Early Design stages where there is neither the scope nor the possibility for detailed modeling.

Control approaches may be classified to those referred to as synthesis and those referred to as design methodologies. Synthesis methodologies are based on well defined models and tackle a well-formulated problems associated with the solution of mathematical problem. These problems aim at producing solvability conditions (defining necessary and/or sufficient conditions) for the existence of solutions expressed frequently as relationships between structural invariants (functions characterizing families of systems under different transformations) as well as desirable system properties. Synthesis methods lead to algorithms for computation of solutions. Design methodologies on the other hand use sets of specifications and rely

on the shaping of performance indicators usually following iterative approaches and they aim to satisfy system properties in an way to satisfy overall design objectives. Bridging the gap between synthesis and design methods involves the development of synthesis methods to a set up where model uncertainty is handled appropriately and approximate solutions to exact synthesis problems is derived, when exact solutions do not exist. This is a very valuable task since it will lead to the development of more powerful design methodologies relying on a combined shaping of performance indicators, handling model uncertainty and optimization based methods for deriving approximate solutions to exact problems.

Amongst the important synthesis methods are those referred to pole assignment, zero assignment, stabilization, etc., under a variety of static or dynamic, centralized or decentralized compensation schemes. Integral part to such methods is the characterization of solvability conditions in term of properties amongst the system invariants and the development of algorithms that provide solution to exact design problems. A major challenge in the development of effective design methodologies is the development of approximate solutions to exact synthesis problems. The need for such developments arises due to issues of model uncertainty, as well as investigating approximate solutions when the exact problems do not have a feasible solution from the engineering viewpoint (when solutions to the exact problems are complex or when there is uncertainty on the realistic values for the design objectives). This thesis aims to extend the potential of frequency assignment synthesis methods by enabling the development of approximate solutions to exact algebraic problems by formulating them as optimization problems that may be tackled by powerful numerical methods.

In general, we distinguish two main approaches in Control Theory. The design methodologies (based on performance criteria and structural characteristics) are mostly of iterative nature and the synthesis methodologies (based on the use of structural characteristics, invariants), are linked to well defined mathematical problems. Of course, there exist variants of the two aiming to combine the best features of the two approaches. The Determinantal Assignment Problem (DAP) belongs to the family of synthesis methods and has emerged as the abstract problem formulation of pole, zero assignment of linear systems [Gia. & Kar. 3], [Kar. & Gia. 5], [Kar. & Gia. 6], [Lev. & Kar. 3]. This approach unifies the study of frequency assignment problems (pole, zero) of multivariable systems under constant, dynamic centralized, or decentralized control structure, has been developed. The Determinantal Assignment Problem (DAP) is equivalent to finding solutions to an inherently non-linear problem and its determinantal character demonstrates the significance of exterior algebra and classical algebraic geometry for control problems. The current thesis aims to develop those aspects of DAP framework that can transform the methodology from a synthesis approach to a design approach that can handle model uncertainty, capable to develop approx-

imate solutions and further empower it with potential for studying stabilization problems. The importance of algebraic geometry for control theory problems has been demonstrated by the work in [Bro. & Byr. 2], [Mart. & Her. 1], etc. The approach adopted in [Gia. & Kar. 3], [Kar. & Gia. 5], [Kar. & Gia. 6], differs from that in [Bro. & Byr. 2], [Byr. 1] in the sense that the problem is studied in a projective, rather than an affine space setting and contrary to that of [Bro. & Byr. 2], [Byr. 1] it can provide a computational approach. The DAP approach relies on exterior algebra, [Mar. 1] and on the explicit description of the Grassmann variety [Hod. & Ped. 1], in terms of the QPR, has the advantage of being computational, and allows the formulation of distance problems, which are required to turn the synthesis method to a design methodology.

The multilinear nature of DAP suggests that the natural framework for its study is that of exterior algebra [Mar. 1]. DAP [Kar. & Gia. 5] may be reduced to a linear problem of zero assignment of polynomial combinants and a standard problem of multi-linear algebra, the decomposability of multivectors [Mar. 1]. The solution of the linear subproblem, whenever it exists, defines a linear space in a projective space of the form $\mathbb{P}^{\binom{n}{m}-1}$ for $n \geq m$, whereas decomposability is characterized by the set of Quadratic Plücker Relations (QPR), which define the Grassmann variety of $\mathbb{P}^{\binom{n}{m}-1}$ [Hod. & Ped. 1]. Thus, the solvability of DAP is reduced to a problem of finding real intersections between the linear variety and the Grassmann variety of the projective space. This Exterior Algebra-Algebraic Geometry method, has provided new invariants (Plücker Matrices and the Grassmann vectors) for the characterization of rational vector spaces, solvability of control problems, ability to discuss both generic and non-generic cases and it is flexible as far as handling dynamic schemes, as well as structurally constrained compensation schemes. The additional advantage of the new framework is that it provides a unifying computational framework for finding the solutions, when such solutions exist. The multilinear nature of DAP has been handled by a "blow up" type methodology, using the notion of degenerate solution and known as "Global Linearisation" [Lev. 1], [Lev. & Kar. 3]. Under certain conditions, this methodology allows the computation of solutions of the DAP problem.

The Determinantal Assignment Problem (DAP) has been crucial in unifying families of frequency assignment as well as stabilization problems which underpin the development of a large number of algebraic synthesis problems but has also led to the introduction of many new challenges and problems of mathematical nature. Amongst these problems we distinguish:

(i) the development of methods for defining real intersections between varieties, a problem linked to realizability of solution in an engineering sense;

(ii) the ability of determining existence of solutions not only in a generic setting,

but also in the context of concrete problems (engineering problems are defined on concrete models);

(iii) computation of solutions to intersection problems (existence of intersections is only part of the problem);

(iv) handling issues of model uncertainty which requires the study of approximate solutions of DAP. The development of criteria for real intersections has been handled by developing cohomology algebra tools [Lev. & Kar. 2].

The issues linked to computation of solutions has led to the development of the Global Linearisation framework [Lev. 1], [Lev. & Kar. 3], which together with the set of Grassmann Invariants [Kar. & Gia. 5] provide the means for addressing problems defined on given models, as well as computing solutions. This framework is by no means completely developed and challenging problems exist such as overcoming difficulties of sensitivity of the Global Linearisation framework and extending it to the case where the models are characterized by uncertainty. The sensitivity issues may be handled by using Homotopy based methodologies [Chow., etc. 1] and by embedding the overall problem into the framework of constrained optimization.

The model uncertainty issues opens up a new area where distance problems such as computing the distance of:

(i) a point from the Grassmann variety;

(ii) a linear variety from the Grassman variety;

(iii) parameterized families of linear varieties from the Grassmann variety;

(iv) relating the latter distance problems with properties of the stability domain [Bar. 1].

The study of these problems relate to classical problems such as spectral analysis of tensors, homotopy methods, constrained optimization, theory of algebraic invariants etc. This thesis addresses the development of DAP along the lines mentioned above, and deals with a number of related mathematical problems which are crucial for the development of control problem solutions. The thesis has an interdisciplinary nature since it is in boundaries between Control and Mathematics. Control Theory defines the problems and the background concepts, Mathematics provides the solutions to well formulated problems and control Engineering deals with the implementation of the solutions of the mathematical problems and the development of algorithms and design methodology.

Development needs beyond the State of the Art: This thesis is based on polynomial matrix theory, exterior algebra and properties of the Grassmann variety of

projective space and aims to introduce an analytic dimension by developing distance problems and optimization tools. Thus, the novelty of this thesis is that it proposes the development of approximate solutions to purely algebraic problems and thus expand the potential of the existing algebraic framework by developing its analytic dimension. The development of the "approximate" dimension of DAP involves the study of a number of problems that can transform the existence results and general computational schemes to tools for control design. There are many challenging issues in the development of the DAP framework and amongst them are its ability to provide solutions even for non-generic cases, handle problems of model uncertainty, as well as providing approximate solutions to the cases where generically there is no solution of the exact problem. The development of the approximate DAP requires a framework for approximation (provided by distance problems) and the formulation of an appropriate constrained optimization problem.

Objectives: The objectives of this thesis are:

(i) To develop tools for the computation of the distance between the Grassmann variety and point and a linear variety and the Grassmann variety of a Projective space, and find approximate solutions of Exterior equations.

(ii) To develop an integrated framework for approximate solutions of DAP and its extension to the case of stabilization problems.

(iii) To develop suitable algorithms that could provide the desirable approximate solutions in higher Grassmann-variety dimensions where tensors are used instead of matrices.

Our research has three aspects which are interlinked and are essential for the development and computation of approximate solutions of DAP. The first is the development of the approximate solutions of exterior equations and the study of all related mathematical problems. The second deals with the development of stability methodologies for general constrained Optimization Problems and finally its application to DAP. The first two are purely mathematical tasks and the last involves their integration to produce solutions to problems of control theory and control design which requires a combination of the two early parts together with control theoretic results to produce a methodology for robust approximate solutions to algebraic synthesis control problems. The main Activity area in which we will work is the area of *Approximate Solutions of Exterior Equations and Distance Problems*: The solution of exterior equations is an integral part of the DAP methodology, since it defines the multi-linear part of the problem. The problem of decomposability of a multivector $\underline{z} \in \bigwedge^m(\mathcal{U})$, where $\mathcal{U}$ is an m-dimensional vector space of $\mathbb{R}^n$, is equivalent to the solvability of the exterior equation

$$\underline{z} = \underline{v}_1 \wedge \cdots \wedge \underline{v}_m,\ \underline{v}_i \in \mathcal{U} \tag{1.1}$$

Note that versions of such equations may be considered, where $\underline{v}_i$ and $\underline{z}$ vectors are polynomial vectors (the latter corresponds to dynamic versions of DAP [Lev. & Kar. 4]). The solvability of such equations is referred to as decomposability of the multi-vector z and are given by the set of quadratics which are known as the Quadratic Plücker Relations (QPRs) [Hod. & Ped. 1] of the space, or of the equivalent projective space $\mathbb{P}^{\dagger}$ and they characterize the Grassmann variety $G(m,n)$ or $G_m(\mathbb{R}^n)$ of $\mathbb{P}^{\binom{n}{m}-1}$, [Hod. & Ped. 1]. Whenever a solution to (1.1) exists, this is a vector space $\mathcal{V}_{\underline{z}} = \text{sp}\{\underline{v}_i\},\ i = 1, ..., m$ and for the control problems defines the corresponding compensator. The overall solution of DAP is reduced to finding common solutions of (1.1) and of a linear equation $P\underline{z} = \underline{a}$, where $\underline{a}$ is a given vector characterizing the assigned polynomial and P an invariant matrix of the given problem (existence of real intersections of the two varieties). Model uncertainty, or non-existence of a solution of (1.1) requires the definition and solution of appropriate problems, which are essential parts in the development of the DAP framework. The expected results will provide the mathematical concepts and tools to develop the new approximate framework for DAP and the research is of pure mathematical nature.

The development of solutions to the research challenges defined before requires addressing specific problems and undertaking research by adopting appropriate methodology which is described below. The overall research is organized in work areas as described below:

(1) Approximate Decomposability Problem (ADP): Assume that for a given vector $\underline{z} \in \mathbb{P}^{\binom{n}{m}-1}$, equation (1.1) does not have solution. Define a vector $\hat{\underline{z}} \in \mathbb{P}^{\binom{n}{m}-1}$ with the least distance from $\underline{z}$, which is decomposable, or equivalently define the distance of $\underline{z}$ from the corresponding Grassmann variety.

(2) Variety Distance Problem (VDP): Given a vector $\hat{\underline{z}} \in \mathbb{P}^{\binom{n}{m}-1}$ and a linear variety $\mathcal{K} := \mathcal{K}(\underline{a}),\ \underline{a} \in \mathbb{R}^n$ of $\mathbb{P}^{\binom{n}{m}-1}$ defined by the solution of $P\underline{z} = \underline{a}$, define the distance of $\mathcal{K}$ from the corresponding Grassmann variety.

(3) Approximate Intersection Problem (AIP): Given a vector $\hat{\underline{z}} \in \mathbb{P}^{\binom{n}{m}-1}$ and a linear variety $\mathcal{K} := \mathcal{K}(\underline{a}),\ \underline{a} \in \mathbb{R}^n$ of $\mathbb{P}^{\binom{n}{m}-1}$ defined by the solution of $P\underline{z} = \underline{a}$, define a vector $\hat{\underline{a}}^t \in \mathbb{R}^n$ such that the linear variety intersects with the Grassmann variety and the following conditions hold true: (i) $\hat{\underline{a}}^t$ has minimum distance has minimum distance from $\underline{a}$; (ii) $\hat{\underline{a}}^t$ has minimum distance from $\underline{a}$ and corresponds to a stable polynomial.

A new framework for searching for approximate solutions has been recently proposed based on the notion of "approximate decomposability" of multi-vectors

[Kar. & Lev. 9]. This approach is based on the characterization of decomposability by the properties of a new family of matrices known as Grassmann Matrices [Kar. & Gia. 6] which has been introduced as an alternative criterion to the standard description of the Grassmann variety provided by the QPRs. This new approach handles simultaneously the question of decomposability and the reconstruction of $\mathcal{V}_{\underline{z}}$. For every $\underline{z} \in \bigwedge^m(\mathcal{U})$ with coordinates $a_\omega,\ \omega \in \mathcal{Q}_{m,n}$, the Grassmann matrix $\Phi_n^m(\underline{z})$ of $\underline{z}$ is defined. In fact it has been shown, that $\text{rank}\Phi_n^m(\underline{z}) \geq n-m$ for all $\underline{z} \neq \underline{0}$ and that $\underline{z}$ is decomposable, if and only if, the equality sign holds. If $\text{rank}\Phi_n^m(\underline{z}) = n-m$ then it was shown that the solution space $\mathcal{V}_{\underline{z}}$ is defined by $\mathcal{V}_{\underline{z}} = \mathcal{N}_r\left(\Phi_n^m(\underline{z})\right)$. The rank based test for decomposability is easier to handle than the QPRs and provides a simple method for the computation of $\mathcal{V}_{\underline{z}}$. The new test provides an alternative formulation for investigation of existence, as well as computation of real solutions of DAP (R-DAP). Solvability of R-DAP is thus reduced to finding a vector such that the rank condition is satisfied.

The study of the above distance problem may be formulated as distance of a Grassmann matrix from the variety of matrices having certain rank and can be studied using approaches such as structural singular values. The characterization of the element with the least distance are the tasks here. This problem is also referred as approximate decomposability and it is a very difficult problem of multi-linear algebra that is not completely solved [Bad. & Kol. 1], [Dela., etc. 1]. In its general form, it is related to several important problems of multi-linear algebra, such as:

(a) Low rank tensor approximation;

(b) Multi-linear singular value decomposition;

(c) Determination of the tensor rank.

The main theme of these problems is to decompose a tensor $\mathbf{T}$ as a sum of rank one tensors, i.e., a sum of decomposable tensors. For the purposes of our work we will consider skew symmetric tensors, i.e, multi-linear tensors $\mathbf{T}$ that arise from determinantal problems and we will try to approximate them by decomposable multi-vectors, i.e., to find vectors $\underline{a}_1, ..., \underline{a}_r$ such that the norm $\|\mathbf{T} - \underline{a}_1 \wedge \cdots \wedge \underline{a}_r\|$ is minimized, where r is the rank of the tensor. This problem can be viewed into two ways, either as a low rank approximation of skew symmetric tensors, or as a distance problem from the Grassmann variety of a projective space that can be formulated in terms of structural singular values, or as nonconvex constrained optimization problem. In this thesis, we will work via the first approach for higher-order Grassmann varieties whereas the second methodology will be discussed for the $G_2(\mathbb{R}^n)$ cases.

In this thesis we will focus on providing a new means for computing approximate solutions for determinantal-type of problems, *without* the use of any generic (i.e., for almost all dynamical systems) or exact solvability conditions and algorithms which are based on them. Note that at first, the genericity problem was thought to be negligible in the sense that it lies in a union of algebraic subsets of lower dimension, [Her. & Mar. 1]. But several authors, e.g., [Ki. 1], [Caro., etc. 1], [Yan. & Ti. 1], [Ki. 2] have showed that under generic pole-assignable conditions, several essential control engineering attributes, e.g., sensitivity, stability, etc., may be lost. The algorithms that were presented for the solution of determinantal-type problems, mostly for the output feedback pole placement problem, [Rav., etc. 2], [Wan. & Ros. 2], [Ki., etc. 2] as well as the approach in [Sot. 1], [Sot. 2] have the same drawback; they are based on Kimuras generic pole assignability condition $m + p > n$ for an m-input, p-output, n-state MIMO system. The algorithms presented in this thesis may provide approximate solutions in any case, independently of generic or exact solvability conditions.

Our work is based on the observation that

(a multivector/matrix/tensor belongs to the Grassmann variety) $\Leftrightarrow$
(the multivector/matrix/tensor satisfies the QPR) $\Leftrightarrow$
(the multivector/matrix/tensor is decomposable) $\Leftrightarrow$
(the multivector/matrix/tensor has rank-1)

The first two equivalences have been well-examined in [Hod. & Ped. 1] and they are considered classic within the context of Algebraic Geometry. A number of some more recent results regarding the different forms these may take is met in [Gee. 1]. The last equivalence, with respect to the rank approach of a multivector, is mostly met in tensor theory; in the simplest case of matrices, a matrix A is said to have rank one if there exist vectors $\underline{a}$, $\underline{b}$ such that

$$A = \underline{a} \times \underline{b}^t \tag{1.2}$$

Consequently, the well-known rank of A (the minimum number of column vectors needed to span the range of the matrix) is the length of the smallest decomposition of A into a sum of such rank-1 outer products, i.e.,

$$A = \underline{a}_1 \times \underline{b}_1^t + \cdots + \underline{a}_n \times \underline{b}_n^t \tag{1.3}$$

In higher dimensions, matrix A is represented by a tensor and the above sum is referred to as *higher-order tensor decomposition*, i.e.,

$$\mathbf{A} = \sum_{i=1}^{r} \mathbf{A}_i \tag{1.4}$$

where $\mathbf{A}_i$ are rank one/decomposable tensors. If the matrix A in the sum (1.3) is skew-symmetric we will show that one of the terms of this sum is the so-called

best rank-one/decomposable approximation and this term is the best approximate solution $\hat{\underline{z}}$ of the determinantal assignment problem written as $P\underline{z} = \underline{a}$, when $\underline{z}$ is decomposable that we mentioned before. For higher dimensions, the sum (1.4) can not guarantee which term is as closest to tensor $\mathbf{A}$, [Kol. & Bad. 3], but at least we may achieve *one decomposable approximation* if $\mathbf{A}$ is a skew-symmetric tensor. In both cases the approximate solution yields an approximate polynomial $\hat{a}(s)$ such that $P\hat{\underline{z}} = \hat{\underline{a}}$. We will study the stability properties of $\hat{a}(s)$ with respect to the approximation $\hat{\underline{z}}$ and stability radius results and we will derive a new criterion for the stability of the approximate solution of DAP. Moreover, these results will be connected with the Grassmann matrix and alternative simplified formulae will be derived, which are completely new for best approximation problems of this form.

For all the 2-dimensional Grassmann varieties, the Approximate Determinantal Assignment problem is completely solved as a manifold constrained optimization problem, where the derivation of the decomposable vector that best approximates the original controller, is based on eigenvalue decompositions (EVD), singular value decompositions or its generalizations, e.g., compact SVD, [Edel., etc. 1] or combinations of them. We present how these methods are expanded in order to yield parameterized approximate solutions in the projective space which is the natural space to examine a problem such as DAP. Uniqueness and non-uniqueness (degeneracy) issues of the decompositions that may arise are also examined in detail. Furthermore, we investigate how the approximate DAP is expanded into constrained minimization over more general varieties than the original ones (Generalized Grassmann varieties) and we derive a new Cauchy-Schwartz type inequality that provides a closed-form non-algorithmic solutions to a wide family of optimization problems related to best decomposable/ rank-1 problems. We provide a new criterion, to test the acceptability of the new approximate solution, i.e., whether the approximation lies in the stability domain of the initial polynomial or not, by using stability radii theory. All results are compared with the ones that already exist in the respective literature (least squares approximations, convex optimization techniques, etc.), as well as with the results obtained by Algebraic Geometry Toolboxes, e.g., Macaulay 2. For numerical implementations, we examine under which conditions certain manifold constrained algorithms, such as Newton's method for optimization on manifolds, could be adopted to DAP and we present a new algorithm which is ideal for DAP approximations.

In higher dimensions, we present for the first time a rank-one tensor approximation algorithm based on the CANDECOMP/PARAFAC decomposition which allows parameters at the entries of the tensor/multivectors that is to be approximated. Our results consider a specific higher order Grassmann variety, the first non-trivial 3- dimensional Grassmann variety. Even though our case does not

involve non-uniqueness issues, several degenerate cases of higher order decompositions may occur for the numeric CANDECOMP/PARAFAC decomposition, due to the fact that the rank of a tensor is not uniquely defined, as in the case of a matrix. This is also the reason that the method does not always guarantee that the approximation implied is actually the "best", but it may be in any case *one* decomposable approximation of the nominal tensor. These problems are much more complicated than the two dimensional case and only special cases of tensors have been examined so far, [Kol. & Bad. 3], [Raj. & Com. 1] regarding their non-uniqueness properties. The parametric decomposition of 3rd- order tensors is among the most important results of this thesis, since all tensor decompositions in the literature work for numerical data exclusively.

More analytically, in Chapter 2, we provide some of the control related mathematical tools and notions. We present several descriptions of linear systems followed by the respective algebraic control theory background. We also discuss some aspects of general feedback configuration and we present the definition of the determinantal assignment problem (DAP) as the unifying problem of all frequency assignment-type problems.

In Chapter 3 we present all the mathematical tools that we are going to use through out this thesis. The purpose of this chapter is to clarify in simple terms, whenever this is possible, all the key mathematical tools that lie behind DAP and the several forms it may take. We begin with the basic concepts of multilinear algebra in order to construct the special space in which determinants are defined and examined, the so-called k-th exterior power implied by an ordinary vector space. This special space is obtained via the properties of the Exterior (or Grassmann) Algebra which is a sub-algebra of the Tensor Algebra. We elaborate on the construction of these sets, since the basic frame of this thesis is based on the concept of tensors and we explain how a tensor/ multi-dimensional array is constructed step-by-step by the vectors of the corresponding vector space. Finally, we present some basic notions and results regarding real affine and projective varieties, in order to define the Grassmann variety, a set whose several properties and equivalent expressions are going to be used excessively in the following chapters.

In Chapter 4, we try to solve DAP with the use of some existing well-known methodologies, before applying our new approximate methodologies, in order to show why a new type of methodology is required for determinantal frequence problems. At first we present the Gröbner basis method which is the most direct technique for solving determinantal type problems, since DAP may be seen as a system of multivariate polynomial equations. We easily see that this methodology can not provide helpful results in higher dimensional cases. The rest of the techniques are separated into algebraic and geometric methodologies. From

the former category, we provide a purely numeric procedure, a full rank algorithm which was built for the output feedback problem where with some simple modifications it is adopted to DAP. However, owing to the nonlinear nature of the problem, the algorithm cannot be guaranteed in all cases to converge to a solution, [Pa. 1]. From the geometric techniques we refer to the Schubert calculus methodology, the global linearization method, [Lev. 1], [Lev. & Kar. 3] and finally to the projective methodologies, [Fal. 1], [Gia. 1], [Kar. & Gia. 5] from which we will introduce the approximate DAP.

In Chapter 5, we interpret the approximate DAP, as a distance problem form the corresponding Grassmann variety for the 2-dimensional and its Hodge-dual case. We start from the simplest Grassmann variety which is described by one QPR only, in order to observe that the least distance from this variety is related to the singular values of a special matrix, the Grassmann matrix, [Kar. & Gia. 6]. The optimization problem is solved via the method of the Lagrange multipliers. We see, that this method can not be easily expanded to the higher dimensions due to the randomly increasing number of the QPR. We then derive a very important formula for 2-vector/2-skew symmetric tensor decomposition, the so called *prime decomposition* where a 2-vector is written as a sum of decomposable vectors, one of which is its "best" approximation, i.e., the one in the Grassmann variety that achieves the least distance. The problem is studied at first via the Euclidean norm and then in the corresponding projective space via the gap metric.

In Chapter 6, we examine the case of degenerate, i.e., repeated eigenvalues in the prime decomposition of the previous chapter. This is a very common procedure in tensor decompositions in general, since equal or special-structured eigenvalues/singular values yield non-uniqueness issues for a decomposition and therefore the approximate solution. After connecting the problem of uniqueness with the uniqueness of matrix least squares distance functions problems, [Hel. & Shay. 1], which helps us derive solid uniqueness criteria for the prime decomposition, we investigate the non-uniqueness case (which has not been thoroughly examined in the respective literature so far, except for special applications in isotropic matrix theory, [Sal. & Cr. 1]) via a completely new approach, the use of *Extremal Varieties*. We prove that when we have degenerate eigenvalues, the approximation implied by the prime decomposition is the worst and the respective gap is calculated. The new varieties are defined in terms of path-wise connectivity, polynomial sums of squares and congugacy-duality properties. These results are applied for the derivation of the best decomposable approximation via some new formulae, which offer for the first time a prototype non-algorithmic approach to rank-approximation problems, where even unsolved or phenomenally unresolved issues, such as the computation of the Lagrange multipliers for manifold constrained problems may be derived in closed-form formulae.

In Chapter 7 we present how the problem of deriving the best decomposable/rank-1 approximation of a multivector for the 2-dimensional case, is actually a special case of the least distance problem between the multivector and the *Generalized Grassmann varieties*, a set that expands the notion of decomposability of the standard Grassmann variety, i.e., for a 2-vector $\underline{a}$, $\underline{a} \wedge \underline{a} = \underline{0}$ corresponds to a decomposable/rank-1 vector, $\underline{a} \wedge \underline{a} \wedge \underline{a} = \underline{0}$ to a sum of 2 decomposable vectors, $\underline{a} \wedge \underline{a} \wedge \underline{a} \wedge \underline{a} = \underline{0}$ to a sum of 3 decomposable vectors, etc. This kind of generalizations are very common within the area of linear-multilinear algebra due to the remarkable applications they may offer; a first generalization along with some useful computational analysis applications is observed in [Gol. & Van. 2], where a best-low rank matrix approximation was achieved for a matrix whose specified columns remained fixed. A generalization for low matrix-rank approximation was also used in [Lu-S., etc. 1] for the design of two dimensional digital filters, whereas several low-rank approximation generalizations of the Eckart-Young theorem, [Eck. & You. 1] were discussed in [Fri. & Tor. 1] and [Kol. 2], among others. Another generalization related to the Grassmann variety is presented in [Rav., etc. 1] via the use of the so called generalized Plücker embedding which was introduced for applications on dynamic output feedback problems. A different but well-known approach views the generalization of the Grassmann variety as the standard Grassmann variety which is preserved under any endomorphism from a vector space to itself, [Kolh. 1]. Nevertheless, this approach is only useful within the context of Lie algebra, which studies these varieties in relation with other algebraic objects, such as the Schur-S polynomials, rings, etc, and not for calculating best approximate solutions on hyper-sets, as in our case. Our approach lies in the concept of expanding the standard exterior algebra/tensor theories, [Hod. & Ped. 1], [Mar. 1] which is also met in the construction of the so-called *generalized Grassmann algebras*, where the properties of the classic exterior (Grassmann) algebra are equipped with multi-linear structures instead of bilinear ones, [Ohn. & Kam. 1], [Kwa. 1]. The most important result however of this chapter is the derivation of a new Cauchy-Schwartz type inequality which is suitable for solving these generalized approximation problems, based on the eigenvalues of the 2-vector. This inequality may cover all classic 2-dimensional decompositions, including degenerate issues (equal or similar structured eigenvalues) and it may be consider prototype, since this is the first time a spectral-type inequality is directly applied to manifold constrained optimization/ best rank-r approximations when $r \geq 1$.

Chapter 8 is the core of this thesis: we solve the approximate DAP and compute the stability properties of the approximate solution, by applying the prime decomposition, the gap metric and several other techniques and formulae that were examined in the previous chapters. We expand the results for parameterized 2-vectors since this was main purpose from the beginning, i.e., best approximate solutions subject to the linear subproblem of DAP. For the computational

construction of the approximate controller we select the non-trivial Grassmann variety $G_{2,5}$, where we show that the problem of the gap minimization from this variety is equivalent to the minimization of a 4-th order polynomial constrained to the unit sphere. This is a very significant result not only for deriving the approximate controller in practical applications of determinantal-type assignment problems without any solvability restrictions, but it may be also seen as a new technique for optimization in the projective space, [Mah. 1]. Note, also that this approach may be also suitable for higher order 2-dimensional Grassmann varieties $G_{2,n}$, where the approximation derived may be considered as a sub-optimal decomposable approximation, a case often met in tensor decompositions and best approximate solutions, [Kol. & Bad. 3]. Furthermore, we examine and we implement for the first time a number of manifold optimization techniques on the approximate DAP; we present how the classic Newton's method is formulated for optimization over Grassmann manifolds, [Edel., etc. 1], [Abs., etc. 2] and we compare the gap we aim to minimize with the Rayleigh quotient, trace-style objective functions used in [Abs., etc. 1], [Abs., etc. 2], [Bas., etc. 1], [Edel., etc. 1]. We show that these algorithms may produce a solution for the approximate DAP only under specific assumptions with regard to the gap and the dimensions of the Grassmann variety. Hence, we built a new algorithm which may work for DAP approximations without special restrictions. Furthermore, we compare our results with the ones obtained by the Numeric Algebraic Geometry Toolbox, Macaulay 2. Finally, new stability criteria are derived with respect to the approximate solution without the calculation of the roots of the approximate polynomial, using stability radius formulae, [Hin. & Pri. 2].

In Chapter 9, we present how to solve the approximate DAP in 3-dimensional Grassmann varieties. In this case the prime decomposition which was studied in the 2-dimensional case, is transformed into the CANDECOMP/PARAFAC (CP) decomposition and the construction of the approximate solution follows the numeric laws and properties of these higher-order tensor SVD-like techniques. We transform the algorithms presented in [Kol. & Bad. 3] that worked only for numerical data, i.e., constant tensors, to algorithms which allow parameters at the entries of the tensor. We then apply a parametric alternating least squares/ CP decomposition (parametric ALS/CPD algorithm) where we imply the parametric decomposable approximation. In order to test the acceptability of the solution (since in the parametric case, contrary to the constant case, the comparison of the implied approximation with the original tensor is not straight-forward), we use the QPR set. With the help of this set we obtain at least one decomposable approximation of the initial controller, since the method does not guarantee in general the optimal solution, [Kol. & Bad. 3]. This is the first result that concerns the computation of the approximate controller in determinantal assignment problems in higher dimensions.

We have tried this thesis to be as complete and independent as possible by presenting all the related background material from the respective areas (algebraic control theory, tensor theory, manifold optimization, rank-1 tensor approximations), so that the nature and the significance of the new results is clear and understandable, even for the reader who is coming in touch with frequency assignment/ determinantal assignment problems, as well as best approximation problems for the first time. The emphasis is given to the examination of manifold constrained optimization techniques and distance calculations on varieties, since this is implied by the nature of the approximate determinantal assignment problem. We wish, that the approach we propose for DAP and its results regarding approximation/optimization on manifolds, may be useful to control theory, algebraic geometry and other scientific fields where conventional methods may be unsatisfactory.

Chapter 2

Preliminary Results from Systems Control Theory

2.1 Introduction

The aim of this chapter is to set the scene for the control theory part of the problem that we will examine in this thesis. We provide a review of background, theoretical control results, basic definitions, fundamental concepts and properties to make this presentation as independent and complete as possible. Nevertheless, a more detailed exposition of the background topics is given in the listed references, [Kar. 4], [Kar. & Mil. 12], [Kar.& Vaf. 13], [Kar. 3], [Hin. & Pri. 1].

In particular, in the first section we present the state space model representation of linear systems that we will use in this thesis. We elaborate on its mathematical features and remark on the general family of models which the state space model belongs.

In Section 2.3, we recall the notion of the transfer function and we explain the notions of poles and zeros which appear in multivariate systems. In the same section we also connect the notions of poles and zeros with the stability, controllability and observability of a dynamic system. In Section 2.4, we demonstrate some of the aspects of the general feedback configuration, in the cases of dynamic feedback for two subsystems and static feedback.

In the last section, we present the Determinantal Assignment Problem (DAP), [Kar. & Gia. 5], [Kar. & Gia. 6], as the unifying form of several frequency assignment problems. We examine their similarities and we present some necessary and sufficient conditions for their solvability.

2.2 State Space Models

In this section we examine the fundamentals of a state space system, following [Kar. & Mil. 12] and [Hin. & Pri. 1]. This concept has evolved as a unification of a variety of notions which have been used in, for example, the classical theory of differentiable dynamical systems, circuit theory and control.

In order to define a dynamical system, i.e., a system that evolves in time, we need to introduce:

i) A *time domain* $\mathcal{T} \subset \mathbb{R}$, so that the variables which describe the behavior of the system are functions of time. The time domain $\mathcal{T}$ may be continuous, i.e., an interval of the form $[0, +\infty)$ or discrete, e.g., $\mathcal{T} = \mathbb{N}$.

ii) The *External variables* of the system, which describe the interactions of the system with the exterior world. These are usually divided into a family $\underline{u} = (u_1, u_2, ...)$ of *inputs* and a family $\underline{y} = (y_1, y_2, ...)$ of *outputs*. By "inputs" we indicate those variables which model the influence of the exterior world on the physical system and can be of different types - either *controlled inputs* or *uncontrolled inputs* (for instance, *disturbances*). By "outputs" we mean those variables with which the system acts on the exterior world. Sometimes the outputs are divided into two (not necessarily mutually disjoint) sets of variables. Those which are actually measured are called *measurements* and those which must be controlled in order to meet specified requirements are called *regulated* (Figure 2.1). The vector spaces of the inputs and outputs

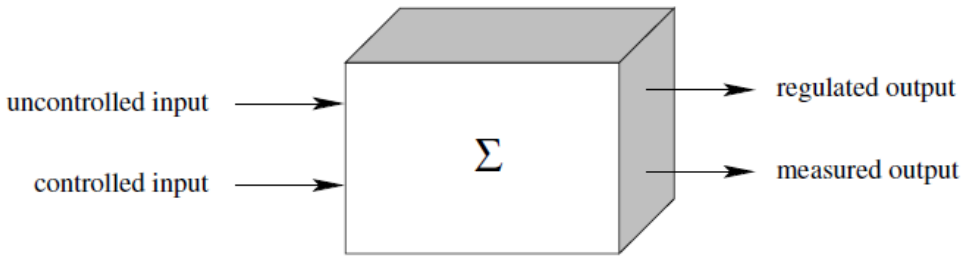


Figure 2.1: External Variables

signals $\mathcal{U}$, $\mathcal{Y}$ are called input-space and output-space, respectively.

iii) The *internal variables* or *states* of the system, which describe processes in the interior of the system. The internal variables of a system can be regarded as a state vector $\underline{x}(t)$ if: **a)** the present state along with the chosen input determine the future states of the system, **b)** at time t, the present state $\underline{x}(t)$ is not influenced by the present and future values $\underline{u}(t_1)$, $t_1 \geq$

t and if **c)** the output value at time t is completely determined by the simultaneous input and state values. The vector space of all states of a system following the previous properties is denoted as $\mathcal{X}$ and is called state-space of the system.

A linear time invariant multivariable system, i.e., if $\{(\underline{x}(t_0),\ \underline{u}(t)),\ t \geq t_0\}$ implies the output $\underline{y}$ and $\{(T_a(\underline{x}(t_0)),\ T_a(\underline{u}(t))),\ t \geq t_0\}$ implies $T_a(\underline{y}_1)$ (where T_a denotes the displacement operator that transfers by $a-$ time units the state, the input and the output vectors to the right-hand side if the input and the output spaces are closed under these displacements) is represented in the time domain by the state variable model

$$S(A,B,C,D): \begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t), & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times k} \\ \underline{y}(t) = C\underline{x} + D\underline{u}(t), & C \in \mathbb{R}^{m\times n},\ D \in \mathbb{R}^{m\times k} \end{cases} \tag{2.1}$$

Matrix A is called the *internal dynamics matrix* of the system and matrices $B,\ C$ are called *input (or actuator)* and *output (or sensor)* matrices respectively, with rank$B = k$, rank$C = m$ and they express the so-called *coupling* of $\underline{u},\ \underline{y}$. Furthermore, if matrix D is equal to the zero matrix, then $S(A,B,C,D)$ is called *strictly proper*.

Time-invariant systems are an equivalence class in the family of linear systems. In general, functions defined on a model which remain the same under certain types of transformations are called system invariants and are usually described by models with the simplest possible structure, i.e., the least number of parameters, which are called canonical forms [Kar. 1]. Canonical forms, corresponding to representation transformations, provide a vehicle for model identification since they contain the minimal number of parameters within a given model structure to be defined. For control analysis and synthesis, aspects of the structure (as it is expressed by the system invariants) characterize the presence or absence of certain system properties; the type and values of invariants provide criteria for solvability of a number of control synthesis problems. In the area of control design, the types and values of invariants frequently impose limitations in what it is possible to achieve. Although the link between system structure and achievable performance, under certain forms of compensation, is not explicitly known, system structure expresses in a way the potential of a system to provide certain solutions to posed control problems. More results with regard to canonical state space representation may be found in [Kar. 1], [Kar.& Vaf. 13].

On the other hand, the choice of a state for a system is not unique. However, there are some choices of state which are preferable to others; in particular, we look for vectors $\underline{x}$ with the least dimension.

Definition 2.2.1. *[Hin. & Pri. 1] The number of the n states, i.e., the coordinates of $\underline{x}$, is defined as the order of the system. The minimal order, i.e., the*

dimension of the smallest state vector, is called the McMillan degree of the system. Furthermore, systems which can be described with a finite number of variables are called finite-dimensional systems or infinite-dimensional otherwise.

In this thesis, we will deal with continuous-invariant time, finite-dimensional, linear systems of the form (4.19). For equivalent representations and for the parametrization problems that these systems appear in specific applications, one may refer to [Kar.& Vaf. 13], [Kar. & Mil. 12].

Remark 2.2.1. *If $s = d/dt$ denotes the derivative operator, then* (4.19) *may be expressed as*

$$P(s)\underline{w}_1(t) = \underline{w}_2 \tag{2.2}$$

where

$$P(s) := \begin{pmatrix} sI_n - A & -B \\ -C & -D \end{pmatrix},\ \underline{w}_1(t) := \begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix},\ \underline{w}_2 := \begin{pmatrix} \underline{0} \\ -\underline{y} \end{pmatrix}$$

The polynomial matrix $P(s)$ is a special type of polynomial matrix, called system matrix pencil, [Kar.& Vaf. 13]. In general, matrix pencils appear as linear operators of the type $sF - G$ where s is an intermediate, frequently representing the Laplace transform variable and they are naturally associated with state-space type problems. They are used for state space calculations and geometric system theory and they provide a unifying framework for the study of the geometric properties of both proper (matrix D is constant) and singular systems (systems of infinite condition number), by reducing problems of singular system theory to equivalent problems of proper system theory, [Kar. 2], [Kal. 1], [Kar. 3]. Pencil matrices have been also used in the well-known Algebraic Eigenvalue Problem [Wilk. 1].

2.3 The Transfer Function Matrix

If one is not interested in the internal dynamics of a system, then the input-output system is basically just a map which associates with any input signal, the corresponding output signal. Specifically, if $\underline{x}(t = 0) = \underline{0}$ are the initial conditions for the states, the Laplace transform of system (4.19) implies,

$$S(A, B, C, D) : \begin{cases} s\underline{x}(s) = A\underline{x}(s) + B\underline{u}(s) \\ \underline{y}(s) = C\underline{x}(s) + D\underline{u}(s) \end{cases} \tag{2.3}$$

where by eliminating the states we obtain

$$\underline{y}(s) = \left(C(sI_n - A)^{-1}B + D\right)\underline{u}(s) \tag{2.4}$$

Definition 2.3.1. *[Kai. 1] The $m \times k$ matrix*

$$G(s) := (C(sI_n - A)^{-1}B + D \tag{2.5}$$

is called the transfer function or transfer matrix of the state space model (4.19) *and describes how the system transforms an input* e^{st} *into the output* $G(s)e^{st}$. *Furthermore,* $G(s)$ *is called proper if* $G(\infty)$ *is a finite constant matrix and strictly proper if* $G(\infty) = 0$*; if at least one entry of* $G(s)$ *is infinity, then* $G(s)$ *is called non proper.*

The set of proper rational functions is denoted by $\mathbb{R}_{\text{prf}}$ and for every $G(s) \in \mathbb{R}_{\text{prf}}^{m\times k}[s]$ there always exists a state space model $S(A, B, C, D)$ such that (2.5) holds. Such state-space models are called *realizations* of $G(s)$.

Definition 2.3.2. *[Kai. 1] The realization with the least possible order is called minimal realization and this order is called the MacMillan degree of* $G(s)$.

Remark 2.3.1. *For an non proper system, it can be shown, [Lew. 1], that it achieves a realization of the form* $E\underline{\dot{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x}$, *when* E *is not invertible.*

It is easily shown that $G(s)$ may take the form $G(s) = N(s)D^{-1}(s)$ which is called *Polynomial Matrix Fractional Description.* Moreover, if the degree of $\det D(s)$ is minimal amongst all other matrix fraction descriptions, then this description is called *irreducible.*

2.3.1 Stability, Controllability and Observability

From the definition of the transfer function and the results on zeros and poles, it is evident that the poles of $G(s)$ must be contained among the eigenvalues of A. In this section, we see that the poles of $G(s)$ are actually contained among the *controllable* and *observable* eigenvalues of A, as only the controllable and observable part of the realization contributes to the transfer function. For this purpose, we briefly summarize at first the basic definitions and results, on controllability, stability and observability for a system $S(A, B, C, D)$.

Definition 2.3.3. *[Hin. & Pri. 1] Let a* $\mathcal{T}$*- time domain dynamical system and* $t_0 \in \mathcal{T}$.

a) A state $\underline{x}_e$ *is called equilibrium state (or equilibrium point) if*

$$\underline{x}(t) = \underline{x}_e,\ \forall t \in \mathcal{T}$$

b) $\underline{x}_e$ *is called asymptotically stable if*

i) $\forall \epsilon > 0,\ \exists \delta_1 > 0 :\ \|\underline{x}(t_0) - \underline{x}_e\| < \delta_1 \Rightarrow \|\underline{x}(t) - \underline{x}_e\| < \epsilon,\ \forall t \geq t_0.$

ii) $\exists \delta_2 > 0 :\ \|\underline{x}(t_0) - \underline{x}_e\| < \delta_2 \Rightarrow \underline{x}(t) \to \underline{x}_e, t \to \infty.$

Condition *(i)* is called *stability in the sense of Lyapunov* for the equilibrium point $\underline{x}_e$, in which case is specifically referred as *attractive point.* An equilibrium point that is not stable in the sense of Lyapunov is called *unstable.*

Theorem 2.3.1. *[Hin. & Pri. 1] A dynamical system is Lyapunov - stable if and only if all eigenvalues of the matrix A of the respective autonomous system, have non positive real part and those whose real part is zero are simple structured. If all eigenvalues of A are negative then $\underline{x}_e$ is asymptotically stable.*

Next, we recall the notion of *controllability*, i.e., whether and how we may choose the input so as to move the system from $\underline{x}(0) = \underline{0}$ to a desired target state $\underline{x}(t_1) = \underline{x}_1$ at a given time t_1.

Definition 2.3.4. *[Kai. 1] Let the system $S(A, B) : \dot{\underline{x}} = A\underline{x} + B\underline{u}$*

i) S(A,B) is called controllable or $(A, B)-$ controllable at a time $t = t_0$ if

$$\forall \underline{x}_0, \underline{x}_1 \in \mathbb{R}^n,\ \exists \underline{u}(t)|_{[t_0,t_1]},\ t_1 \in (t_0, \infty) :\ \underline{x}(t_0) = \underline{x}_0,\ \underline{x}(t_1) = \underline{x}_1$$

Otherwise, it is called uncontrollable.

ii) The eigenvalues $\lambda_i,\ i = 1, 2, ..., n$ of A for which

$$\operatorname{rank}(\lambda_i I_n - A, B) = n \tag{2.6}$$

are called controllable eigenvalues (or modes) of the system.

iii) A system whose all uncontrollable eigenvalues are stable is called stabilizable.

For continuous time systems, the notion of controllability coincides with the notion of *reachability* and the two terms are used equivalently. In case of discrete systems this is not possible, since matrix A may not be invertible.

Observability, on the other hand, is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The observability and controllability of a system are *dual* notions, i.e., controllability provides that an input is available that brings any initial state to any desired final state whereas observability provides that knowing an output trajectory provides enough information to predict the initial state of the system.

Definition 2.3.5. *[Kai. 1] Let $S(A, C)$ be the system (4.19) for $D = 0$.*

i) $S(A, C)$ is called observable or (A,C)- observable at $[t_0, t_F]$ if for an input $\underline{u}(t)$ and an output $\underline{y}(t)$ we get a unique state $\underline{x}(t_0) = \underline{x}_0$. If one of the states of $\underline{x}$ does not satisfy the previous rule, the system is called unobservable.

ii) The eigenvalues $\lambda_i,\ i = 1, 2, ..., n$ of A for which

$$\operatorname{rank}\begin{pmatrix} \lambda_i I_n - A \\ C \end{pmatrix} = n \tag{2.7}$$

are called observable eigenvalues of the system.

iii) A system whose all unobservable eigenvalues are stable is called detectable.

Next we discuss the fundamental notions of poles and zeros of a system which are closely related to the notions described in this section.

2.3.2 Poles and Zeros

The poles and zeros of a system play an important role for its study; Poles could be described as the characteristic of the internal dynamical machinery of the system while zeros are the characteristic of the ways in which this dynamical machinery is coupled to the environment in which the system is embedded, and are associated with specific values of complex frequency at which transmission through the system is blocked. In picturesque terms, poles can be thought of as associated with system resonances coupled to input and output and zeros as associated with anti-resonances at which propagation through the system is blocked.

Loosely speaking, multivariable poles and zeros are resonant and anti-resonant frequencies respectively, that is to say they are frequencies whose transmission explodes with time, or whose transmission is completely blocked. This, of course, is intuitively appealing since it forms a natural extension of the definitions given for the scalar case, where the poles and zeros of a scalar transfer function are defined as the values of the complex frequency s for which the transfer function gain becomes ∞, or 0 correspondingly. The inversion of roles of poles and zeros suggested by their classical complex analysis definition motivates the dynamic (in terms of trajectories) properties of zeros. The physical problem used to define multivariable zeros is the "output zeroing" problem, which is the problem of defining appropriate non-zero input exponential signal vectors and initial conditions which result in identically zero output. Such a problem is the dual of the "zero input" problem defining poles, which is the problem of defining appropriate initial conditions, such that with zero input the output is a nonzero exponential vector signal. Those two physical problems emphasize the duality of the roles of poles and zeros.

Definition 2.3.6. *[Kar. 4] Let the transfer function $G(s)$ of $S(A, B, C, D)$. Then,*

i) $s = s_0$ is a pole of $G(s)$, if the denominator of some entry of $G(s)$ becomes zero at s_0.

ii) $s = s_1$ is a zero of $G(s)$ if $G(s)$ drops rank at $s = s_1$, or equivalently, if there is a rational vector $u(s)$ such that $u(s_1)$ is finite and nonzero and $\lim_{s \to s_1} (G(s)u(s)) = 0$.

Example 2.3.1. *Let*

$$G(s) = \begin{pmatrix} 1 & \frac{1}{s-2} \\ 0 & 1 \end{pmatrix}$$

It is clear that $G(s)$ has a pole at $s = 2$, but it may not be immediately obvious that it also has a zero at $s = 2$. We observe that while s is approaching 2, the second column of $G(s)$ approaches alignment with the first column, so the rank of $G(s)$ approaches 1, i.e., there is a rank drop at $s = 2$. To confirm this, we may choose

$$u(s) = \begin{pmatrix} -1 \\ s-2 \end{pmatrix}$$

Then, $u(2)$ is finite and nonzero and $\lim_{s\to 2}(G(s)u(s)) = 0$.

Furthermore, the multiplicity associated with each pole and zero is not unique as in the single- variable case; each pole is associated with a set of multiplicities, e.g., if

$$G(s) = \text{diag}\left(\frac{s+2}{(s+3)^2}, \frac{s}{(s+2)(s+3)}\right)$$

then we see that $G(s)$ has poles at -3 of multiplicity 2 and 1 respectively. The concepts of pole, and zero have emerged as the key tools of the classical methods of Nyquist-Bode and root locus for the analysis and design of linear, single-input, single-output (SISO) feedback systems. The development of the state space S(A,B,C,D) description, transfer function $G(s)$ description, and complex variable, ($g(s)$, algebraic function) methods for linear multivariable systems has led to a variety of definitions for the zeros and poles in the multivariable case as Definition 2.3.6 and the emergence of many new properties. The variety and diversity in the definitions for the zeros and poles is largely due to the differences between alternative system representations the difference in approaches used, the objectives and types of problems they have to serve. A classic immediate method for the derivation of all poles and zeros without having to test any properties is via the Smith-MacMillan form [Kai. 1];

Theorem 2.3.2. *(Smith-MacMillan) For any $G(s) \in \mathbb{R}^{m\times k}$ there exist non singular, square polynomial matrices $P_1,\ P_2,\ det(P_i) = c_i \in \mathbb{R} \setminus \{0\},\ i = 1, 2$ such that*

$$P_1(s)G(s)P_2(s) = \text{diag}\left(\frac{f_1(s)}{g_1(s)}, \frac{f_2(s)}{g_2(s)}, ..., \frac{f_r(s)}{g_r(s)}\right),\ r = \text{rank}\,(G(s)) \tag{2.8}$$

where $f_i,\ g(i),\ i = 1, 2, ..., r$ are co-prime, monic polynomials such that $f_{i+1} = \lambda_i f_i,\ g_i = \kappa_i g_{i+1},\ i = 1, 2, ..., r-1$ for some $\lambda_i, \kappa_i \in \mathbb{R}$.

Matrices $P_i(s),\ i = 1, 2$ are called *unimodular* and the roots of the numerator polynomials $f_i(s)$ in (2.8) are the *zeros* whereas the roots of the $g_i(s)$ are the *poles* of $G(s)$.

The Smith form, and in some more detail the Kronecker form, [Kar. 4] of the state space system matrix introduce the zero structure of the state space models;

for transfer function models the pole zero structure is introduced by the Smith-McMillan form. Such links reveal the poles as invariants of the alternative system representations under a variety of representation and feedback transformations. The strong invariance of zeros (large set of transformations) makes them critical structural characteristics, which strongly influence the potential of systems to achieve performance improvements under compensation.

From the Polynomial Matrix Fractional Description of $G(s)$ we discussed in section 2.3, we may have:

$$G(s) = N_R(s)D_R^{-1}(s) = D_L(s)N_L^{-1}(s) \tag{2.9}$$

where $N_R(s) \in \mathbb{R}^{m\times k}[s]$, $D_R(s) \in \mathbb{R}^{k\times k}[s]$ and $N_L(s) \in \mathbb{R}^{m\times k}[s]$, $D_L(s) \in \mathbb{R}^{m\times m}[s]$ are the so called *Right Polynomial Matrix Fractional Description* and *Left Polynomial Matrix Fractional Description* of $G(s)$ respectively when matrices D_R, D_L are invertible. This description of the transfer function gives another alternative characterization of the poles and zeros of $G(s)$; the poles of $G(s)$ are the roots of the polynomial $\det(D_L(s)) = c\left(D_R(s)\right), c \in \mathbb{R} \setminus \{0\}$ and the zeros are the roots of the product of the invariant polynomials of $N_L(s)$ or $N_R(s)$.

Remark 2.3.2. *The zeros of $G(s)$ are often called transmission zeros in order to distinguish them from another fundamental category of zeros, the invariant zeros, [MacFar. & Kar. 1], which are the zeros of the polynomial matrix*

$$P(s) := \begin{pmatrix} sI_n - A & -B \\ C & D(s) \end{pmatrix} \tag{2.10}$$

where matrix $P(s)$ is obtained by the necessary and sufficient condition

$$P(s)\begin{pmatrix} \underline{x}_0 \\ \underline{v} \end{pmatrix} = 0 \tag{2.11}$$

which has to hold true in order an input of the form $\underline{u}(t) = \underline{v}H(t)e^{at}$, $a \in \mathbb{C}$ for a system $S\left(A, B, C, D(s)\right)$ to yield

$$\underline{u}(t) = \underline{v}H(t)e^{at},\ \underline{y}(t) \equiv \underline{0},\ t > 0$$

where $\underline{x}_0 \in \mathbb{R}^n$ is the initial state condition vector, $\underline{v} \in \mathbb{R}^k$ and $H(t)$ the Heaviside unit-step function. A third group of zeros was introduced in [Ros. 1] which were named input-output decoupling zeros and they are obtained as the zeros of the invariant polynomial matrices

$$P_1(s) := \left(sI_n - A, B\right),\ P_2(s) := \begin{pmatrix} sI_n - A \\ C \end{pmatrix} \tag{2.12}$$

respectively. The input-decoupling zeros correspond to the zeros of $[D_L(s), N_L(s)]$ *and the output-decoupling zeros to the zeros of* $[D_R(s), N_R(s)]^t$ *for a left-right coprime MFD of* $G(s)$ *since* $N_L(s), N_R(s)$ *have essential parts of Smith form defined by the matrix* $\mathrm{diag}(f_1(s), ..., f_r(s))$ *and* $D_L(s), D_R(s)$ *have essential parts of Smith form defined by* $\mathrm{diag}(g_1(s), ..., g_r(s))$*, [Kar. 2] and for this reason they are also called Smith-zeros. Both of them are associated with the situation where some free modal motion of the system state, of exponential type, is uncoupled from the system's input or output. Note that pencils* $P_1(s)$, $P_2(s)$ *are the same matrices which were used in* (2.6) *and* (2.7) *to test the controllability and the observability of a system, respectively. The sum of the transmission zeros and decoupling zeros is called system zeros of* (2.10)*.*

Finally, we have the following result that connects the notions of controllability and observability of the previous section with the poles and zeros.

Theorem 2.3.3. *[Kar. 4] The following hold true:*

i) If a system is controllable and observable, then the sets of invariant zeros and transmission zeros are the same.

ii) Every uncontrollable eigenvalue is a system zero.

iii) Every unobservable eigenvalue is a system zero.

iv) The spectrum $\sigma\{A - BD^{-1}C\}$ *is precisely equal to the set of the systems transmission zeros.*

v) The poles of the transfer function $G(s)$ *are precisely equal - in location and multiplicity - to the controllable and observable eigenvalues of* A *and the multiplicity indices associated with a pole of* $G(s)$ *are precisely the sizes of the Jordan blocks associated with the corresponding eigenvalue of* A*.*

Every square system (same number of input and outputs) has zeros (finite and/or infinite); however, non-square systems generally do not have zeros and this is an important difference with the poles that exist independent from input, output dimensionalities. Although non-square systems generically have no zeros, they have "almost zeros"; this extended notion expresses "almost pole-zero cancelations" and it is shown in [Kar. 4] that in a number of cases behaves like the exact notion.

Poles and zeros are fundamental system concepts with dynamic, algebraic, geometric, feedback and computational aspects. A detailed account of the above material may be found in [Kar. 4] and [MacFar. & Kar. 1].

2.4 Composite Systems and Feedback

On the general state-space description (4.19), a number of transformations may be applied, which are of the representation type (different coordinate systems) and expressed by coordinate transformations, and of the feedback compensation type; the latter is made up from state, output feedback and output injection. These transformations are denoted below, [Kar.& Vaf. 13]:

(i) $R : k \times k$ input coordinate transformation, $\det R \neq 0$.

(ii) $T : m \times m$ output coordinate transformation, $\det T \neq 0$.

(iii) $Q, Q^{-1} : n \times n$ pair of state coordinate transformations, $\det Q \neq 0$.

(iv) $T : m \times k$ constant output feedback matrix.

(v) $L : n \times k$ state feedback matrix.

(vi) $T : n \times n$ output injection matrix.

The above set of transformations (R, T, Q^{-1}, L, K) when applied on the original system $S(A, B, C, D)$ described by the matrix $P(s)$ in (2.10), i.e.,

$$P(s) = \begin{pmatrix} sI_n - A & -B \\ C & D \end{pmatrix}$$

then a new system $S'(A', B', C', D')$ is produced, described by

$$P'(s) := \begin{pmatrix} Q^{-1} & K \\ 0 & T \end{pmatrix} \begin{pmatrix} sI_n - A & -B \\ C & D \end{pmatrix} \begin{pmatrix} Q & 0 \\ L & R \end{pmatrix} \tag{2.13}$$

As shown in [Kar.& Vaf. 13], $P(s)$, $P'(s)$ are related by a certain form of equivalence, which is defined on matrix pencils. Such transformations are referred to as the *full set of state-space transformation*, or as the *Kronecker set of transformations*. In [Kar.& Vaf. 13] there are various representations the Kronecker set of transformations, such as the so-called $S(A)$ description where the transformations on the $T(s)$ pencils are similar, i.e., $T'(s) = Q^{-1}(sI_n - A)Q$. Before we end this section, we briefly present the feedback compensation on composite systems, in order to generalize some of the results presented in the previous sections.

A *composite system* is a system which consists of a number of several subsystems, describing an original system and a number of controllers. A trivial way of building a composite system from a collection of systems is the so called *direct sum system* where each subsystem can be studied independently of the other, i.e., if $\Sigma_i = (A_i, B_i, C_i, D_i)$ are systems with state space $\mathcal{X}_i$, input space $\mathcal{U}_i$ and

output space $\mathcal{Y}_i$, $i \in \overline{N} := \{1, 2, ..., n\}$, $n \in \mathbb{N}$, the direct sum is the system (A, B, C, D) with state space $\mathcal{X}$, input space $\mathcal{U}$ and output space $\mathcal{Y}$ given by

$$\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i,\ \mathcal{U} = \prod_{i=1}^{n} \mathcal{U}_i,\ \mathcal{Y} = \prod_{i=1}^{n} \mathcal{Y}_i \tag{2.14}$$

and

$$A = \bigoplus_{i=1}^{n} A_i,\ B = \bigoplus_{i=1}^{n} B_i,\ C = \bigoplus_{i=1}^{n} C_i,\ D = \bigoplus_{i=1}^{n} D_i \tag{2.15}$$

Hence, the direct sum is just a collection of uncoupled systems. This is not the case if the subsystems Σ_i are interconnected within the composite system or if the original system is interconnected with itself (feedback). In this section we examine the second case for two subsystems, the original system (plant) and the system of the controller as in the following figure. The input, state and output spaces of the composite system are denoted by $\overline{\mathcal{U}}$, $\overline{\mathcal{X}}$, $\overline{\mathcal{Y}}$ and by $K = (K_{ij})_{i,j\in\overline{N}}$, $K_i^c : \overline{\mathcal{U}} \to \mathcal{U}_i$, $K_i^o : \mathcal{Y}_i \to \overline{\mathcal{Y}}$ we denote the so called *matrix of interconnections* of the subsystems and the *input and output coupling matrices* respectively.

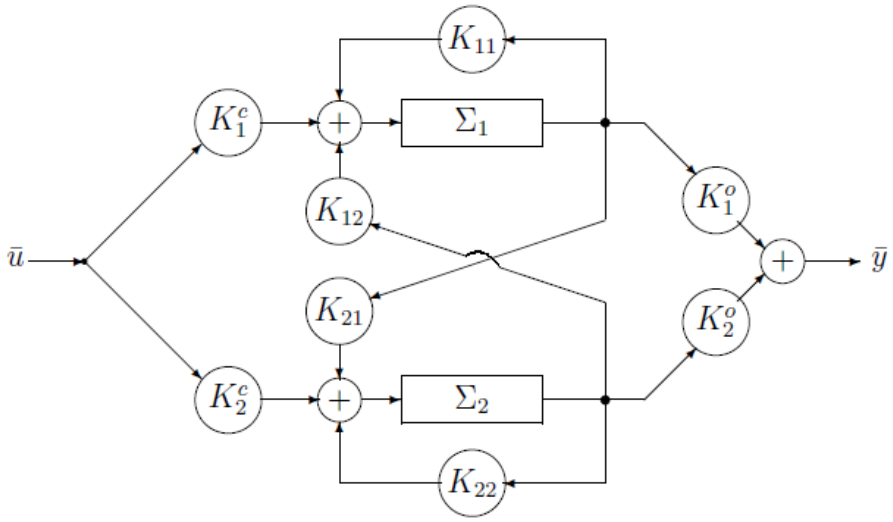


Figure 2.2: General composite system of two subsystems

If we connect the output of Σ_1 to the input of Σ_2 and the output of Σ_2 to the input of Σ_1, then the above configuration 2.2 is called *Dynamic Output Feedback*. Thus, by setting $u_i \equiv \underline{u}_i \in \mathcal{U}$, $y_i \equiv \underline{y}_i \in \mathcal{Y}$ and $\overline{u} \in \overline{\mathcal{U}}$, the couplings $u_2 = y_1$ and $u_1 = y_2 + \overline{u}$ lead to the feedback equations

$$u_1 = C_2x_2 + D_2(C_1x_1 + D_1u_1) + \overline{u},\ u_2 = C_1x_1 + D_1(C_2x_2 + D_2u_2 + \overline{u}) \tag{2.16}$$

These equations can be solved for $\underline{u}_1$ and $\underline{u}_2$ if and only if the matrices $I_{\mathcal{U}_1} - D_2D_1$, or equivalently $I_{\mathcal{U}_2} - D_1D_2$ are invertible. This is the so-called *well-posedness*

condition for the feedback configuration and if it is satisfied, the feedback system is said to be *well defined*. It has input space $\overline{\mathcal{U}} = \mathcal{U}_1$, output space $\overline{\mathcal{Y}} = \mathcal{Y}_1$, state space $\overline{\mathcal{X}} = \mathcal{X}_1 \times \mathcal{X}_2$ and its system equations are given by the data

$$\overline{A} = \begin{pmatrix} A_1 + B_1(I_n - D_2D_1)^{-1}D_2C_1 & B_1(I_n - D_2D_1)^{-1}C_2 \\ B_2(I_n - D_1D_2)^{-1}C_1 & A_2 + B_2(I_n - D_1D_2)^{-1}D_1C_2 \end{pmatrix},$$

$$\overline{B} = \begin{pmatrix} B_1(I_n - D_2D_1)^{-1} \\ B_2(I_n - D_1D_2)^{-1}D_1 \end{pmatrix},$$

$$\overline{C} = \begin{pmatrix} C_1 + D_1(I_n - D_2D_1)^{-1} - D_2C_1 & D_1(I_n - D_2D_1)^{-1}C_2 \end{pmatrix}$$

$$\overline{D} = D_1(I_n - D_2D_1)^{-1}$$

Next theorem provides a necessary and sufficient for a composite system to be proper.

Theorem 2.4.1. *[Vid. 1] Let $G_1(s)$, $G_2(s)$ be the transfer matrices of a plant and its controller.*

i) The transfer function matrix of the composite system is

$$\overline{G}(s) := G_1(s)(I_n - G_2(s)G_1(s))^{-1} \tag{2.17}$$

ii) $\overline{G}$ is proper if and only if

$$t(\infty) \in \mathbb{R} \tag{2.18}$$

where $t(s) = \det(I_n - G_1(s)G_2(s))$.

The next two results [Chen. 1] now generalize the notions of stability, controllability and observability for the general feedback configuration.

Definition 2.4.1. *A composite well-posed system is called internally stable if its respective autonomous system $\dot{\overline{x}} = \overline{A}\overline{x}$ is asymptotically stable.*

Proposition 2.4.1. *Let a composite well-posed system $\overline{\Sigma}$.*

i) $\overline{\Sigma}$ is controllable, if and only if $\{\Sigma_1,\ \Sigma_2\}$ are both controllable.

ii) $\overline{\Sigma}$ is observable, if and only if $\{\Sigma_1,\ \Sigma_2\}$ are both observable.

iii) $\overline{\Sigma}$ is stabilizable, detectable if and only if both $\{\Sigma_1,\ \Sigma_2\}$ are stabilizable, detectable.

The above result is fundamental, since it provides the means for studying internal stability in terms of BIBO stability and it provides the basis for the study of stabilization and pole assignment of the feedback configuration. More results regarding the structural properties of the feedback configuration may be found in [Kar. & Mil. 12].

2.5 Frequency Assignment Problems

The problem of pole placement for systems of the form (4.19) in order to obtain the required stability and other properties, is a core problem in Systems Theory. In this section, we see that these two problems along with other frequency assignment problems such as the design of an asymptotic observer and the problem of zero assignment by squaring down, lie in a wider category of problems, the so called *Determinantal Assignment Problem* [Kar. & Gia. 5], [Kar. & Gia. 6].

Definition 2.5.1. *(Pole Assignment by state feedback) [Won. 1]. Let a state space model of the form* (4.19)*. The Pole Assignment by state feedback problem is defined as the derivation of a matrix* $K \in \mathbb{R}^{k\times n}$, $\text{rank}K = k$, *such that the equation:*

$$\underline{\dot{x}} = (A - BK)\underline{x} + B\underline{u},\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times k} \tag{2.19}$$

has the following characteristic polynomial:

$$\det(sI_n - A + BK) = \det(M(s)\tilde{K}) = a(s) \tag{2.20}$$

where $a(s)$ *is an arbitrary polynomial of* n *degree and* $M(s) = (sI_n - A, B)$, $\tilde{K}^t = (I_n^t, K^t)$.

This problem has been completely solved in [Won. 1], where the following theorem was proved.

Theorem 2.5.1. *A system is (completely) pole assignable by state feedback, if and only if the system is controllable.*

In [Kai. 1] the problem of the *asymptotic observer* has been established as the dual problem of the state feedback problem:

Definition 2.5.2. *(Design of an asymptotic observer) [Kai. 1]. The problem of the asymptotic observer design is defined as the derivation of a matrix* $T \in \mathbb{R}^{n\times m}$, *such that*

$$\det(sI - A + TC) = \det\left(\tilde{T}C(s)\right) a(s) \tag{2.21}$$

where $a(s)$ *is an arbitrary polynomial of* n *degree and* $C^t(s) = ((sI_n - A)^t, C^t)$, $\tilde{T} = (I_n, T)$.

Due to the duality of the state feedback problem and the asymptotic observer's design, the latter is solvable if and only if the system is observable.

On the other hand, the problem of pole placement via output feedback is still an open problem on research, [Kim. 3].

Definition 2.5.3. *(Pole Assignment by constant output Feedback) [Kim. 1]. The problem of pole assignment by constant output feedback is the derivation of a matrix* $F \in \mathbb{R}^{k\times m}$ *such that*

$$\begin{aligned}\det(sI_n - A - BFC) &= \det(sI_n - A)\det(I_n - (sI_n - A)^{-1}BFC) = \\ &= \det(sI_n - A)\det(I_n - G(s)F) = \det\left(F_R T_R(s)\right) = \\ &= a(s) \end{aligned} \tag{2.22}$$

where $a(s)$ *is an arbitrary polynomial of* n *degree and*

$$F_R = (I_k, F) \in \mathbb{R}^{k\times(m+k)},\ T_R(s) = \begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix}$$

are obtained by the Right Polynomial Matrix Fractional Description of the respective transfer function.

The problem has been mostly examined generically, i.e., for almost all polynomials $a(s)$, [Wil. & Hes. 1]. Note that if there exist real matrices F that satisfy the above form (2.22) of the closed-loop characteristic polynomial we refer to the problem as *completely pole assignable by static output feedback*, whereas if there exist open dense sets of real coefficients $(a_1, ..., a_n)$ defined by the polynomial $a(s)$ *for all* matrices F that satisfy (2.22), then we refer to the *generically pole-assignable by static output feedback* problem.

Most authors have examined the problem in terms of solvability and assignabil-

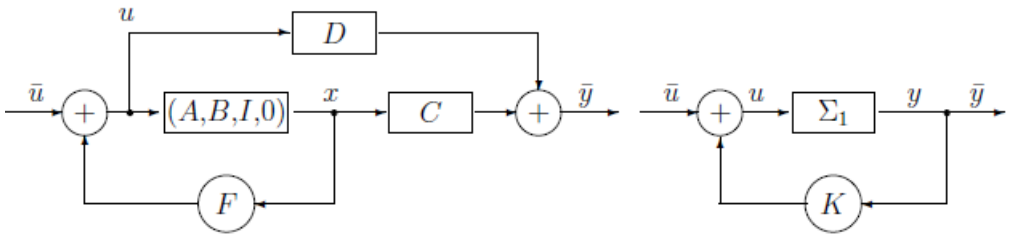


Figure 2.3: Static state and output feedback

ity conditions; From [Dav. & Wan. 1], [Kim. 1] and [Ser. 1] where the sufficient condition $m + k \geq n + 1$ for a generic pole assignment system with m outputs, k inputs and n states was examined, to [Her. & Mar. 1] and [Herm., etc. 1] and the necessary and sufficient condition $mk \geq n$ (which is based on the results presented in [Kim. 1]) and the latter results in [Bro. & Byr. 2], [Wil. & Hes. 1], [Ro. 1], all results concerned the derivation of generic output feedback assignability conditions. Note that the DAP framework which we will introduce next is the first unifying approach that provides a treatment of all these frequency assignment problems.

The methodology proposed in [Lev. & Kar. 2]-[Lev. & Kar. 5] was the first to provide a systematic procedure not only for the derivation of solvability conditions for generic assignability that covered a much wider category of systems, but it could also provide the construction of the desired solution. With the use of a blow-up methodology, known as *Global Linearization* the authors managed to present a unique computational framework that is suitable for computing solutions for a wide variety of determinantal-type assignment problems, contrary to the affine space approach, [Byr. 1], [Mart. & Her. 1]. We will refer to this methodology again in the next chapters with more details, when we will elaborate on the construction of solutions of several determinantal-type pole placement problems, since in this thesis we are also interested in the construction of solutions rather than the derivation of special conditions. The majority of these results and a further discussion concerning the output feedback problem in general, it may be found in the survey paper [Syr., etc. 1].

The *"squaring down" problem* [Kou. & Mac. 1],[Gia. & Kar. 2] is studied in the case of a multi-variable system whose number of measured output variables is greater than the number of control inputs ($m > k$) when we want to combine all outputs together into a new set of outputs, whose number is equal to the number of inputs. It is evident that the solution of the general squaring down problem has significant consequences on the zero structure of the corresponding loop transmission transfer function matrix and therefore that it vitally affects the design procedure. In general, the problem of zero assignment by squaring down is to find a matrix $K \in \mathbb{R}^{k\times m}$ such that the Smith-MacMillan form of $\tilde{G}(s) := KG(s)$ has a given zero structure, i.e., apart from the zeros (if any) inherited from the given rational transfer function $G(s)$, has an additional number of desired zeros and can be considered as a generalization of Rosenbrock's assignment problem [Ros. 1].

Definition 2.5.4. *(Zero Assignment by Squaring Down) [Kou. & Mac. 1]. The zero assignment by squaring down problem is defined as the derivation of a matrix* $K \in \mathbb{R}^{k\times m}$ *such that*

$$\det\left(K \cdot N(s)\right) = a(s) \tag{2.23}$$

where $N(s)$ *is an appropriate matrix and* $a(s)$ *is an arbitrary polynomial of* n *degree.*

The derivation of necessary and sufficient conditions for zero assignment under squaring down was considered in [Kar. & Gia. 7] using projective geometry techniques. In [Kar. & Gia. 7] in particular it has been shown that $k(m-k) \geq \delta+1$, is a necessary and sufficient condition for zero assignment, where δ is the Forney's dynamical degree, [For. 1].

Remark 2.5.1. *In [Apl. 1] it was demonstrated - under specific conditions -how the output feedback problem can be transformed into a squaring down problem and vice versa.*

The problems listed above have been defined for regular state-space systems. Similar problems for extended state-space or descriptor type systems [Ver. 1] may also be formulated in a similar manner. The common formulation of these problems clearly suggests that they are special cases of a more general problem which is known as *Determinantal Assignment Problem.*

Definition 2.5.5. *(The Determinantal Assignment Problem)[Kar. & Gia. 5], [Kar. & Gia. 6]. Let* $M(s) \in \mathbb{R}^{p\times q}[s]$, $q \leq p$ *with* $rank\{M(s)\} = q$ *and also let* $\mathcal{H} = \{H \in \mathbb{R}^{p\times q}, rankH = q\}$. *The problem of finding an* $H \in \mathcal{H}$ *such that*

$$f_M(s, H) = \det(H \cdot M(s)) = a(s) \tag{2.24}$$

where $a(s)$ *is an arbitrary polynomial of an appropriate degree* d, *is called the Determinantal Assignment Problem(DAP).*

Remark 2.5.2. *Equation* (2.24) *is sometimes mentioned as the constant determinantal assignment problem whereas the problem of finding* $H(s)$ *such that* $\det(H(s) \cdot M(s)) = a(s)$ *is referred as the general determinantal assignment problem. However, since all dynamics can be shifted from* $H(s)$ *to* $M(s)$, *the general problem can be described by* (2.24).

Remark 2.5.3. *The degree* d *of* $a(s)$ *depends on the degree of* $M(s)$, *i.e., the polynomial with the maximal degree among the entries of* $M(s)$ *and on the structure of* H. *Through-out this thesis, we have* $d = \deg M(s)$.

As shown in [Kar. & Gia. 6], DAP may be reduced to a linear system of equations where the unknowns are constrained to several quadratic relations. Hence, DAP is solvable when the number of free parameters is greater or equal to the number of constraints. This suggests that DAP has the complexity of a variety intersection problem only in the boundary case when the number of free parameters is equal to the number of constraints. In the case where the degrees of freedom are greater than the constraints, there is a need for alternative methods which explore efficiently the additional degrees of freedom. DAP, being similar to the output feedback pole placement, has allowed the use of the same techniques, provided one takes into consideration the structure of the matrices H and $M(s)$, that may differ from those of the output feedback problem. From that aspect and working via *degenerate feedback gain*, i.e., $\det(H \cdot M(s)) = 0$ for all s, some new solvability conditions have been derived.

Theorem 2.5.2. *[Lev. 1] If* $mk > n$, *then DAP has a solution, for almost all polynomials* $a(s)$.

Remark 2.5.4. *The same condition holds for the static output feedback case, as it was proved in [Lev. & Kar. 2]. More results using this methodology and a generalization for the dynamic output feedback case, may be found in [Lev. & Kar. 3]-[Lev. & Kar. 6].*

In the following chapters, we will approach problem (2.24) in a different way, which is the derivation of a matrix $\hat{H}$ that best approximates matrix H, when the computation of the initial matrix H fails, or when its computation becomes very difficult. As we will see, this problem will be defined as the *Approximate Determinantal Assignment Problem*.

2.6 Conclusions

In this thesis, the emphasis is given on the study of the most general case of frequency assignment problems, which is the Determinantal Assignment Problem. The similarities between DAP and the rest frequency assignment problems, as presented in the previous section, provide a number of well known and established techniques for DAPs further investigation. Our viewpoint in this chapter has been to underpin the nature of dynamic and feedback properties in order to set up the control theory framework of DAP. Notions such as the poles and zeros have thus been central to this approach. Poles express the internal dynamics of the system and they are directly related to stability and other aspects of performance and they are affected by the different types of feedback transformations. On the other hand, zeros are measures of the interaction between internal dynamics and their coupling to inputs and outputs. They are invariant under fundamental feedback transformations and their alternation may be achieved under dynamic compensation, or by design or re-design of the input, output system structure. The theory of invariants and canonical forms of state space systems can be considered from the perspective defined by the theory of matrix pencils, which was briefly presented. Further information about the role and significance of invariants for Control Synthesis and Control Design problems may be found in [Kar. & Mil. 12] and the references therein.

Chapter 3

Exterior Algebra and Algebraic Geometry Tools

3.1 Introduction

In this chapter we present the main mathematical tools that we are going to use in order to transform DAP's equation (2.24) in a way that it could provide concrete solutions, perhaps even in a closed form-formula.

The main idea of this transformation, is that the determinant in (2.24) is viewed as a general multilinear skew symmetric function on the entries of matrix H. This implies that DAP should be approached via Multilinear Algebra methods, where the notions of tensors, tensor products and tensor algebras, play a crucial role for the construction, development and implementation of the respective theory. As we will see, DAP is subject to the laws of a specific sub-algebra of the tensor algebra, called Exterior Algebra, which is the area of study of the skew-symmetric tensors that can be considered as the generalization of antisymmetric matrices.

Furthermore, the roots of the polynomial in (2.24) will provide the link between the problem's skew-symmetry with a relevant algebraic variety. To this purpose, we give some basic background results and properties from the field of Algebraic Geometry with respect to projective algebraic varieties, which will help us define the so called Grassmann variety. The properties of this variety as well as its related features are closely related to DAP, something that will be more clear in the following chapters when DAP will be connected with the solution of a specific optimization problem over the Grassmann variety.

The basic definitions and the background concept of *Exterior Algebra Theory* - a special subalgebra of the tensor algebra whose subspaces are known as *Exterior Powers*- are introduced in Section 3.2, since that would be the study area of

DAP in higher dimensions. The special form that some of the elements of these subspaces have, called *multivectors*, will help us define the so called *decomposable* multivectors, which play a significant role in this thesis. In the same section, we also introduce the notion of the *Hodge -star Duality*, as an important tool for calculations in the area of Exterior algebra.

In the next section we introduce the notion of the *compound matrix*, which it will be used through out this thesis for the derivation of decomposable vectors. Moreover, in Section 3.4 we introduce the Grassmann variety and its related projective space. This variety will be connected with the previous notion of decomposability and will be very helpful in DAPs investigation. Furthermore, we introduce several features of the Grassmann variety, such as the *Quadratic Plücker relations* (QPR).

In Section 3.4.2, we introduce the notion of the *Grassmann matrix* which constitutes an alternative tool for the decomposability of multivectors that we will use later for the solution of the approximate DAP. Finally, in Section 3.5, we present the notion of the Grassmann invariants, such the Plücker matrices, that also constitute an alternative means of testing decomposability.

3.2 Basic Notions of Exterior Algebra

This section provides the most important results from Exterior (or Grassmann) Algebra, i.e., definitions, propositions and theorems that we will need throughout this thesis.

Definition 3.2.1. *[Mar. 1] Let $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_k$ and $\mathcal{W}$ be vector spaces of finite or infinite dimension over a field $\mathcal{F}$ and $0 \leq k < \infty$. If $\mathcal{V}^k := \mathcal{V}_1 \times \mathcal{V}_2 \times ... \times \mathcal{V}_k$ is the set of all the k-tuples of the form $(\underline{x}_1, \underline{x}_2, ..., \underline{x}_k)$, then the map $\underline{f} : \mathcal{V}^k \to \mathcal{W}$ is called k-linear or multilinear of k degree if it is linear in each of its argument, i.e.,*

$$\underline{f}(\underline{x}_1, ..., \underline{x}_i, \lambda\underline{y} + \mu\underline{z}, \underline{x}_{i+1}, ..., \underline{x}_k) = \lambda\underline{f}(\underline{x}_1, ..., \underline{x}_i, \underline{y}, ..., \underline{x}_k) + \mu\underline{f}(\underline{x}_1, ..., \underline{x}_i, \underline{z}, ..., \underline{x}_k)$$

where $\lambda, \mu \in \mathcal{F}$ and $\underline{x}_j, \underline{y}_i \in \mathcal{V}$. If $\mathcal{W} \equiv \mathcal{F}$ then $\underline{f}$ is called k-linear function in $\mathcal{V}$.

The set of multilinear maps as above, forms a vector space denoted by $\mathrm{Hom}_F(\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_k; \mathcal{W})$ and the *dual spaces* of $\mathcal{V}_i$ are defined as $\mathcal{V}_i^* = \mathrm{Hom}(\mathcal{V}_i, F)$ which are called *forms*.

Example 3.2.1. *Let $A = (\underline{a}_1, \underline{a}_2, ..., \underline{a}_n) \in \mathbb{R}^{n\times n}$. Then the determinant $d(A)$ is a function $d : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\underline{d}(\underline{a}_1, ..., \lambda\underline{a}_i + \mu\underline{a}_j, ..., \underline{a}_n) = \lambda\underline{d}(\underline{a}_1, ..., \underline{a}_i, ..., \underline{a}_n) + \mu\underline{d}(\underline{a}_i, ..., \underline{a}_j, ..., \underline{a}_n) \qquad (3.1)$$

Equation (3.1) states that the determinant is an n-linear function in $\mathbb{R}^n$.

The definition of DAP in equation (2.24) suggests that in order to calculate matrix H we need to investigate the analysis of the relevant determinant. The properties of the determinant lead to the conclusion that the determinant in (2.24) should be considered as a skew-symmetric multilinear map.

Definition 3.2.2. *[Mar. 1] Let a k-linear map* $\underline{f} : \mathcal{V}^k \to \mathcal{W}$ *over a field* $\mathcal{F}$.

i) We say that $\underline{f}$ *is alternating if* $\underline{f}(\underline{x}_1, ..., \underline{x}_k) = 0$, *for some* i *with* $\underline{x}_i = \underline{x}_{i+1}$ *and* $1 \leq i \leq k-1$.

ii) We say that $\underline{f}$ *is skew-symmetric if*

$$\underline{f}\left(\underline{x}_{\sigma(1)} + \cdots + \underline{x}_{\sigma(k)}\right) = \text{sign}\sigma \underline{f}(\underline{x}_1, ..., \underline{x}_k) \tag{3.2}$$

for every permutation σ *in the group of permutations* S_k *of k objects.*

Definition 3.2.2 leads to the following Proposition.

Proposition 3.2.1. *[Mar. 1] Let a k-linear map* $\underline{f} : \mathcal{V}^k \to \mathcal{W}$ *over a field* $\mathcal{F}$.

i) Every alternating multilinear map is skew-symmetric.

ii) Every k-linear map $\underline{f}$ *determines a skew symmetric k-linear map* $\underline{g}$ *which is given by*

$$\underline{g} = \sum_{\sigma} \text{sign}\sigma < \underline{\sigma}, \underline{f} > \tag{3.3}$$

where $<,>$ *denotes the inner product of the corresponding vector-form* $\underline{\sigma}$ *of* σ *and* $\underline{f}$.

iii) If $\text{char}(\mathcal{F}) \neq 2$ *then every skew-symmetric multilinear map is alternating.*

Remark 3.2.1. *If* $\dim\mathcal{V} = n$ *and* $k > n$, *then every skew symmetric k-linear map* $\underline{f} : \mathcal{V} \to \mathcal{W}$ *is identically equal to zero.*

By using the previous proposition, we imply the following important result:

Proposition 3.2.2. *[Mar. 1] If* $\underline{f} : \mathcal{V}^k \to \mathcal{W}$ *is an alternating k-linear map such that*

$$\underline{y}_j = \sum_{i=1}^{k} a_{ij}\underline{x}_i,\ 1 \leq j \leq k,\ \underline{y}_j,\ \underline{x}_i \in \mathcal{V} \tag{3.4}$$

then

$$\underline{f}(\underline{y}_1, ..., \underline{y}_k) = \det(a_{ij})\underline{f}(\underline{x}_1, ..., \underline{x}_k) \tag{3.5}$$

It is clear now, that the determinant of an $n \times n$ matrix with entries in $\mathcal{F}$, is an n-linear skew-symmetric function in $\mathcal{F}^n$. Now, many authors ([Gal. 1], [Mar. 1] among many others), in order to obtain the vector space and the respective algebra of determinants, have used the tensor algebra $\mathcal{T}(\mathcal{V})$ that may be defined for all k-linear maps.

Definition 3.2.3. *[Mar. 1] Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$ and let $\otimes$ be the standard tensor product, [Mar. 1]. If*

$$\mathcal{V}_m^{\otimes} := \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{m \text{ times}} \tag{3.6}$$

then

i) The vector space

$$\mathcal{T}(\mathcal{V}) := \bigoplus_{m \geq 0} \mathcal{V}_m^{\otimes},\ \mathcal{V}_0^{\otimes} \equiv \mathcal{F},\ \mathcal{V}_1^{\otimes} \equiv \mathcal{V} \tag{3.7}$$

is called a tensor algebra.

ii) If $\mathfrak{I}$ is the two-sided ideal of $\mathcal{T}(\mathcal{V})$ generated by all tensors of the form $\underline{u} \otimes \underline{u}$, then the k-th Exterior Power of $\mathcal{V}$ is defined as

$$\bigwedge^k(\mathcal{V}) := \mathcal{V}_k^{\otimes}/\mathfrak{I} \cap \mathcal{V}_k^{\otimes} \tag{3.8}$$

iii) The Exterior Algebra (or Grassmann Algebra) of $\mathcal{V}$ is defined as

$$\bigwedge(\mathcal{V}) := \mathcal{T}(\mathcal{V})/\mathfrak{I} = \bigoplus_{k \geq 0}^{k} \bigwedge(\mathcal{V}) \tag{3.9}$$

Proposition 3.2.3. *[Mar. 1]*

i) If $\wedge^k : \mathcal{V}^k \to \bigwedge^k(\mathcal{V})$ is a skew symmetric k-linear map, then the vectors $\wedge^k(\underline{x}_1, ..., \underline{x}_k)$ generate $\bigwedge^k(\mathcal{V})$ for every $\underline{x}_i \in \mathcal{V}$.

ii) If $\underline{g} : \mathcal{V} \to \mathcal{W}$ is any skew symmetric k-linear map, there exists a linear map $\underline{f} : \bigwedge^k(V) \to \mathcal{W}$ such that $\underline{g} = \underline{f} \circ \wedge^k$.

iii) If $\mathrm{Alt}(\mathcal{V}; \mathcal{F})$ denotes the set of all alternating maps over the field $\mathcal{F}$, then

$$\left(\bigwedge^k(\mathcal{V}) \right)^* \cong \mathrm{Alt}(\mathcal{V}; \mathcal{F}) \tag{3.10}$$

The elements of $\bigwedge^k(\mathcal{V})$ are called *alternating k- vectors* or *alternating multivectors*. From the definition of $\mathcal{T}(\mathcal{V})$, the following properties hold:

i) $\bigwedge(\mathcal{V})$ is an associative and anti-commutative graded algebra.

ii) $\bigwedge^{s+r}(\mathcal{V}) = \bigwedge^s(\mathcal{V}) \bigwedge (\bigwedge^r(\mathcal{V}))$ and if $\underline{x} \in \bigwedge^s(\mathcal{V})$, $\underline{y} \in \bigwedge^r(\mathcal{V})$ then

$$\underline{x} \wedge \underline{y} = (-1)^{rs} \underline{y} \wedge \underline{x} \tag{3.11}$$

iv) If $\mathcal{V} = \mathbb{R}^n$, then $\bigwedge^k(\mathbb{R}^n) \simeq \mathbb{R}^{\binom{n}{k}}$.

v) The pair $\left(\bigwedge^k(\mathcal{V}), \wedge^k\right)$ is uniquely defined by $\mathcal{V}$.

We can now construct a basis for the exterior powers $\bigwedge^k(\mathcal{V})$. We need at first to define a total ordering in our bases, therefore we introduce *lexicographical order.*

Proposition 3.2.4. *[Mar. 1] Let $\mathcal{V}$ be an n dimensional vector space over a field $\mathcal{F}$ and let $\{\underline{e}_1, \underline{e}_2, ..., \underline{e}_n\}$ be a basis of $\mathcal{V}$. Then*

i) The set $\{\underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k}\}$ is a basis of $\bigwedge^k(\mathcal{V})$, for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ (lexicographical order).

ii) $\dim\left(\bigwedge^k(\mathcal{V})\right) = \binom{n}{k}$ if $0 \leq k \leq n$ and $\bigwedge^k(\mathcal{V}) = \{0\}$ if $k > n$.

iii) Any multivector $\underline{x} \in \bigwedge^k(\mathcal{V})$ can be uniquely written as

$$\underline{x} = \sum a_{i_1 i_2 \ldots i_k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k} \tag{3.12}$$

where $a_{i_1 i_2 \ldots i_k}$ are the coordinates of $\underline{x}$, with respect to the basis $\{\underline{e}_1, \underline{e}_2, ..., \underline{e}_n\}$ of $\mathcal{V}$ called Plücker coordinates.

Remark 3.2.2. *The dimension of $\bigwedge(\mathcal{V})$ now readily follows; Due to the previous proposition we have that*

$$\dim\left(\bigwedge(\mathcal{V})\right) = \sum_{i=0}^{n} \binom{n}{i} = 2^n \tag{3.13}$$

The following definition is going to be used excessively in the rest of this thesis;

Definition 3.2.4. *[Mar. 1] Let $\mathcal{V}$ be an n dimensional vector space, $k \leq n$ and the k-vector $\underline{x} = \sum a_{i_1 i_2 \ldots i_k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k} \in \bigwedge^k(\mathcal{V})$.*

i) If $\underline{x}$ belongs in the one-dimensional subspaces of $\bigwedge^k(V)$ or equivalently, if it is written in the form

$$\underline{x}_1 \wedge \underline{x}_2 \wedge \cdots \wedge \underline{x}_k \tag{3.14}$$

then it is called decomposable. If $\underline{x}_i$, $i = 1, ..., k$ are linearly independent then $\underline{x}$ is called totally decomposable.

ii) The minimal number of decomposable k-vectors in the expansion

$$\underline{x} = \underline{x}^1 + \underline{x}^2 + \cdots + \underline{x}^n \tag{3.15}$$

where $\underline{x}^i$ are decomposable, is called rank of the k- vector $\underline{x}$.

Remark 3.2.3. *It can be proved that, $\underline{x}_1, ..., \underline{x}_k$ are linearly dependent if and only if $\underline{x}_1 \wedge \underline{x}_2 \wedge \cdots \wedge \underline{x}_k = \underline{0}$.*

We finish the fundamentals of exterior algebra theory with the following definition.

Definition 3.2.5. *[Tol. & Cast. 1] Let $\mathcal{V}$ be an n dimensional vector space over a field $\mathcal{F}$.*

i) The product of two vectors $\underline{x},\ \underline{y} \in \mathcal{V}$ will be denoted as

$$\underline{x} \wedge \underline{y} := \underline{x} \otimes \underline{y} - \underline{y} \otimes \underline{x} \tag{3.16}$$

and will be called exterior (or wedge or Grassmann) product.

ii) Similarly, the product of three vectors $\underline{x}_1,\ \underline{x}_2,\ \underline{x}_3 \in \mathcal{V}$ will be denoted as

$$\begin{aligned} \underline{x}_1 \wedge \underline{x}_2 \wedge \underline{x}_3 : = \underline{x}_1 \otimes \underline{x}_2 \otimes \underline{x}_3 - \underline{x}_2 \otimes \underline{x}_1 \otimes \underline{x}_3 + \underline{x}_2 \otimes \underline{x}_3 \otimes \underline{x}_1 - \\ - \underline{x}_3 \otimes \underline{x}_2 \otimes \underline{x}_1 + \underline{x}_3 \otimes \underline{x}_1 \otimes \underline{x}_2 - \underline{x}_1 \otimes \underline{x}_3 \otimes \underline{x}_2 \end{aligned} \tag{3.17}$$

and will be also called exterior product.

iii) Inductively, if $k \geq 2$ is the number of vectors $\underline{x}_i \in \mathcal{V},\ i = 2, ..., k$ then the wedge product is defined as

$$\underline{x}_1 \wedge \cdots \wedge \underline{x}_k = \delta^{a_1 a_2 \cdots a_k}_{1\ 2\ \cdots\ k} \underline{x}_{a_1} \otimes \cdots \otimes \underline{x}_{a_k} \tag{3.18}$$

where $a_i \in \digamma_n = \{1, 2, ..., n\},\ i \in \digamma_k = \{1, 2, ..., k\},\ a_1 \neq a_2 \neq \cdots \neq a_k$ and $\delta^{(\cdot)}_{(\cdot)}$ are the Kronecker deltas, i.e.,

$$\delta^{a_1 a_2 \cdots a_k}_{1\ 2\ \cdots\ k} = \begin{cases} 1, & \text{if } 1, 2, ..., n \text{ are an even permutation of } a_1 a_2 \cdots a_k \\ -1, & \text{if } 1, 2, ..., n \text{ are an odd permutation of } a_1 a_2 \cdots a_k \\ 0, & \text{otherwise} \end{cases}$$

From this definition, it is clear that

i) $\underline{x} \wedge \underline{x} = \underline{0},\ \forall \underline{x} \in \mathcal{V}$.

ii) $\underline{x} \wedge \underline{y} = -\underline{y} \wedge \underline{x},\ \forall \underline{x},\ \underline{y} \in \mathcal{V}$.

iii) $\underline{x} \wedge \underline{y} \wedge \underline{z} = -\underline{y} \wedge \underline{x} \wedge \underline{z} = \cdots = -\underline{z} \wedge \underline{y} \wedge \underline{x},\ \forall \underline{x},\ \underline{x},\ \underline{z} \in \mathcal{V}$.

3.2.1 The Hodge Star-Operator

As we have mentioned earlier, it is common in tensor algebra and analysis (and in Linear Algebra in general ofcourse) to work on the dual set rather than the set itself - via a corresponding isomorphism - if the former implies information that the given set can not provide. In this sense, the Hodge star- Operator (or Hodge $\star$-operator) has been defined as an isomorphism : $\bigwedge^k(\mathcal{V}) \to \bigwedge^{n-k}(\mathcal{V})$ that generalizes the Laplacian $\Delta : C^k \to C^{k-2},\ \Delta(f) = \nabla^2 f$ in order to be applied to differential forms on a Riemannian manifold.

Definition 3.2.6. *[Mar. 1] Let $\mathcal{V}$ an n dimensional vector space and $0 \leq k \leq n$. If $< \cdot, \cdot >$ denotes the inner product on $\mathcal{V}$, we define*

$$< \underline{u}_1 \wedge \cdots \wedge \underline{u}_k, \underline{v}_1 \wedge \cdots \wedge \underline{v}_k >= \det\left(< \underline{u}_i, \underline{v}_j >\right) \tag{3.19}$$

as an inner product on $\bigwedge^k(\mathcal{V})$.

It is straight-forward to see that if $\{\underline{e}_1, ..., \underline{e}_n\}$ is an orthonormal basis of $\mathcal{V}$, then the basis of $\bigwedge^k(\mathcal{V})$ consisting of the vectors $\underline{e}_\omega,\ \omega = \{i_1, ..., i_k\}$, is an orthonormal basis of $\bigwedge^k(\mathcal{V})$. Since the inner product on $\mathcal{V}$ can induce an inner product on $\mathcal{V}^*$, via isomorphisms, then we can also obtain an inner product on $\bigwedge^k(\mathcal{V}^*)$.

Furthermore, if $\mathcal{V}$ is *oriented* vector space, i.e., its vectors are the linear combinations of the vectors of a selected basis $\{\underline{e}_1, ..., \underline{e}_n\}$ of $\mathcal{V}$, then it is easy to see that $\mathcal{V}^*$ is also oriented by $\{\underline{e}_1^*, ..., \underline{e}_n^*\}$.

Definition 3.2.7. *[Mar. 1] Let $\mathcal{V}$ an oriented n dimensional vector space and $0 \leq k \leq n$. The map*

$$\begin{aligned} &\star : \bigwedge^k(\mathcal{V}) \to \bigwedge^{n-k}(\mathcal{V}), \\ &\star\,(\underline{e}_1, ..., \underline{e}_n) = \underline{e}_{k+1} \wedge \cdots \underline{e}_n \end{aligned} \tag{3.20}$$

is called the Hodge Star-Operator.

Example 3.2.2. *We take two cases for k, n respectively;*

i) If $n \in \mathbb{N},\ k = 0$, then $\star(1) = \underline{e}_1 \wedge \cdots \wedge \underline{e}_n$.

ii) If $n = 4,\ k = 2$, then

$$\underline{x} = \sum_{1 \leq i < j \leq 4} x_{ij} \underline{e}_i \wedge \underline{e}_j,\ \bigwedge^2(\mathbb{R}^4) \ni \underline{x} = (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$$

Therefore

$$\star(\underline{x}) = x_{12} \star (\underline{e}_1 \wedge \underline{e}_2) + x_{13} \star (\underline{e}_1 \wedge \underline{e}_3) + \cdots + x_{34} \star (\underline{e}_3 \wedge \underline{e}_4) =$$

$$= x_{34}(\underline{e}_1 \wedge \underline{e}_2) - x_{24}(\underline{e}_1 \wedge \underline{e}_3) + \cdots + x_{12}(\underline{e}_3 \wedge \underline{e}_4)$$

Thus, the vector representation of $\star(\underline{x})$ *is*

$$\star(\underline{x}) = (x_{34}, -x_{24}, x_{23}, x_{14}, -x_{13}, x_{12})$$

The next result summarizes some important properties of the Hodge Star-Operator.

Proposition 3.2.5. *[Gal. 1]Let* $\mathcal{V}$ *an* n *dimensional vector space and* $0 \leq k \leq n$. *Then*

i) $\star(\star(\underline{x})) = (-1)^{k(n-k)} s\underline{x}$, $\forall\, \underline{x} \in \bigwedge^k(\mathcal{V})$, *where* s *is the signature of the inner product on* $\mathcal{V}$, *i.e., the number of real eigenvalues of the matrix obtained by* $< \cdot, \cdot >$.

ii) $< \underline{x}, \underline{y} >= \star\left(< \underline{x}, \star(\underline{y}) >\right) = \star\left(< \underline{y}, \star(\underline{x}) >\right),\ \forall\, \underline{x}, \underline{y} \in \bigwedge^k(\mathcal{V})$.

iii) $\underline{a}_1 \wedge \cdots \wedge \underline{a}_n = \sqrt{\det\left(< \underline{a}_i, \underline{a}_j >\right)}\underline{e}_1 \wedge \cdots \wedge \underline{e}_n$, *for any orthonormal basis* $\{\underline{e}_1, ..., \underline{e}_n\}$ *and any other basis* $\{\underline{a}_1, ..., \underline{a}_n\}$ *of* $\mathcal{V}$.

3.3 Representation theory of exterior powers of linear maps

In this section, we present the basic results of the general representation of multilinear maps between exterior powers, in order to obtain the form of the matrix representation of a mapping from $\bigwedge^k(\mathcal{V})$ to $\bigwedge^k(\mathcal{W})$. The main result of this section is the derivation of the notion of the compound matrix, which is closely related with the decomposability of a multivector.

Theorem 3.3.1. *[Mar. 1] Let* $\underline{f} : \mathcal{V} \to \mathcal{W}$ *a linear map, where* $\mathcal{V}$, $\mathcal{W}$ *are two finite dimensional vector spaces over a field* $\mathcal{F}$. *Then, there exists a unique homomorphism* $\underline{\tilde{f}} : \bigwedge^k(\mathcal{V}) \to \bigwedge^k(\mathcal{W})$ *such that* $\underline{f}(\underline{v}) = \underline{\tilde{f}}(\underline{v})$, $\forall \underline{v} \in \mathcal{V}$ *and for all* k.

Remark 3.3.1. *Theorem 3.3.1 also suggests a way to construct a multilinear map from* V^k *to* $\bigwedge^k(W)$. *We observe that since* $\underline{f}$ *is a linear map of a vector space* $\mathcal{V}$ *to a vector space* $\mathcal{W}$ *then* $\underline{f}(\underline{x}_1) \wedge \cdots \wedge \underline{f}(\underline{x}_k) \in \bigwedge^k(\mathcal{W})$ *may correspond to the k-tuple* $(\underline{x}_1, ..., \underline{x}_k) \in \mathcal{V}^k$. *This defines a skew-symmetric multilinear map* $\underline{g}$ *such that* $\underline{g} : \mathcal{V}^k \to \bigwedge^k(\mathcal{W})$.

The above results lead to the following important result that describes how the matrix of a map from $\bigwedge^k(\mathcal{V})$ to $\bigwedge^k(\mathcal{W})$ is obtained.

Theorem 3.3.2. *[Mar. 1] Let* $\underline{f} : \mathcal{V} \to \mathcal{W}$ *a linear map and* $\varphi_1 : \bigwedge^k(\mathcal{V}) \to F^{\binom{n}{k}}$, $\varphi_2 : \bigwedge^k(\mathcal{W}) \to F^{\binom{m}{k}}$ *two linear maps such that*

$$\varphi_1(\underline{x}) = \begin{pmatrix} \vdots \\ \lambda_{p_1} \\ \vdots \end{pmatrix}, \ \varphi_2(\underline{y}) = \begin{pmatrix} \vdots \\ \kappa_{p_2} \\ \vdots \end{pmatrix}$$

If $\wedge^k(\underline{f})$ *is a* k*-linear map from* $\bigwedge^k(\mathcal{V})$ *to* $\bigwedge^k(\mathcal{W})$ *and* B *is the matrix representation of the linear map* $\underline{\beta} : \mathcal{F}^m \to \mathcal{F}^n$ *with respect to the elementary bases of* $\mathcal{F}^m$, $\mathcal{F}^n$ *respectively, then the matrix representation* A *of* $\wedge^k(\underline{f})$*, with respect to the elementary bases of* $\bigwedge^k(\mathcal{V})$, $\bigwedge^k(\mathcal{W})$ *respectively, satisfies the equation*

$$A \cdot \begin{pmatrix} \vdots \\ \lambda_{p_1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \kappa_{p_2} \\ \vdots \end{pmatrix} \tag{3.21}$$

and the entries of A *are of the form*

$$\begin{vmatrix} \varrho_{i_1 j_1} & \cdots & \varrho_{i_k j_1} \\ \vdots & \vdots & \vdots \\ \varrho_{i_1 j_k} & \cdots & \varrho_{i_k j_k} \end{vmatrix}$$

where ϱ_{ij} *are the entries of matrix* B*.*

Remark 3.3.2. *It is evident that matrices* A, B *in Theorem 3.3.2 satisfy the equality* $C_p(B) = A$ *and that equivalence relations may be obtained; if we denote* $\wedge^k(\underline{e}')$ *and* $\wedge^k(\underline{\varepsilon}')$ *two different bases for* $\bigwedge^k(\mathcal{V})$, $\bigwedge^k(\mathcal{W})$ *respectively,* B' *the matrix representation of* $\underline{\beta}$ *in terms of the new bases and* Q_1, Q_2 *the matrix representations of maps from the old bases (Theorem 3.3.2) to the new, then*

$$A = C_p(B) = C_p(Q_1)C_p(B')C_p(Q_2) \tag{3.22}$$

Furthermore, if $L(\mathcal{V}, \mathcal{W})$ *is the set of all linear maps from* $\mathcal{V}$ *to* $\mathcal{W}$*, it can be shown, by using the properties of the compound matrices we mentioned before, that the maps* φ_i, $i = 1, 2$ *in Theorem 3.3.2 which associate every map* $\wedge^k(\underline{f})$ *with its matrix representation* A*, constitute an isomorphism* $L\left(\bigwedge^k(\mathcal{V}), \bigwedge^k(\mathcal{W})\right) \to F^{\binom{m}{p}\times\binom{n}{p}}$.

3.3.1 Compound Matrices

The previous results clearly show how a decomposable multivector is related to the minors of a given matrix. For the case of 2-vectors, if $\{\underline{e}_i \otimes \underline{e}_j\}_{(i,j)\in\{1,2,\ldots,n\}}$, $i \neq j$, is a basis of $\mathcal{V} \times \mathcal{V}$, $\dim\mathcal{V} = n$, then

$$\underline{x} \wedge \underline{y} = (x_i\underline{e}_i) \wedge (y_j\underline{e}_j) = (x_i\underline{e}_i) \otimes (y_j\underline{e}_j) - (y_j\underline{e}_j) \otimes (x_i\underline{e}_i)$$

$$= x_i y_j \underline{e}_i \otimes \underline{e}_j - y_j x_i \underline{e}_j \otimes \underline{e}_i = x_i y_j \underline{e}_i \wedge \underline{e}_j$$

$$= x_i y_j \underline{e}_i \wedge \underline{e}_j + x_j y_i \underline{e}_j \wedge \underline{e}_i,\ i < j$$

$$= (x_i y_j - x_j y_i) \underline{e}_i \wedge \underline{e}_j,\ i < j$$

$$= \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \underline{e}_i \wedge \underline{e}_j,\ i < j$$

Thus a decomposable 2-vector may be derived by the 2-minors of a matrix. The results of the previous section may now be simplified by introducing some useful notation and definitions on the sequences of integers and on the submatrices of a given matrix to obtain decomposability for k-vectors in a simplified form as well.

Definition 3.3.1. *(Compound Matrix)[Mar. & Min. 2] Let $\mathcal{Q}_{k,n}$ denote the set of strictly increasing sequences of k integers, $1 \leq k \leq n$, chosen from $1, 2, ..., n$, e.g., $\mathcal{Q}_{2,3} = \{(1,2),(1,3),(2,3)\}$, where the number of sequences of $\mathcal{Q}_{k,n}$ is $\binom{n}{k}$.*

i) If $a, b \in \mathcal{Q}_{k,n}$, we say that a precedes b and we write $a \prec b$, if there exists an integer $t \in [1,k]$ such that $a_1 = b_1, a_2 = b_2, a_{t-1} = b_{t-1}, a_t < b_t$, where $a_i,\ b_i$ denote the elements of a, b respectively. This describes the lexicographic ordering of the elements of $\mathcal{Q}_{k,n}$ which will be designated as $\omega := (i_1, ..., i_n)$ and if $c_1, ..., c_n$ are elements of a field F then $c_\omega := c_{i_1} c_{i_2} \cdots c_{i_1}$.

ii) The k- compound matrix of a matrix $A \in F^{m \times n}$, $1 \leq k \leq \min\{m, n\}$ is a $\binom{m}{k} \times \binom{n}{k}$ matrix whose entries are $det(A(\omega, \omega'))$, where $A(\omega, \omega') \in F^{k \times k}$ denotes the sub-matrix of A which contains the rows $\omega = (i_1, ..., i_k) \in \mathcal{Q}_{k,m}$ and the columns $\omega' = (j_1, ..., j_p) \in \mathcal{Q}_{k,n}$ and it will be designated by $C_k(A)$.

Clearly now, the special case $k = \binom{n}{m}$ implies an $\binom{n}{k}$- dimensional column-vector $C_k(A)$, which is decomposable. Hence, if $A = (\underline{a}_1, \underline{a}_2, ..., \underline{a}_k) \in F^{n \times k},\ 1 \leq k \leq n$ then

$$C_k(A) = \underline{a}_1 \wedge \underline{a}_2 \wedge \cdots \wedge \underline{a}_k \tag{3.23}$$

and the entries of $C_k(A)$ are the Plücker coordinates.

Example 3.3.1. *Let*

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 \end{pmatrix} \in \mathbb{R}^{3 \times 4},\ m = 3,\ n = 4.$$

For $k = 2$, we have that

$$\mathcal{Q}_{2,3} = \{(1,2),(1,3),(2,3)\} := \{\omega_1, \omega_2, \omega_3\}$$

and

$$\mathcal{Q}_{2,4} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\} := \{\omega_1', \omega_2', \omega_3', \omega_4', \omega_5', \omega_6'\}$$

Then

$$C_2(A) = \begin{pmatrix} |A(\omega_1,\omega_1'| & |A(\omega_1,\omega_2'| & |A(\omega_1,\omega_3'| & |A(\omega_1,\omega_4'| & |A(\omega_1,\omega_5'| & |A(\omega_1,\omega_6'| \\ |A(\omega_2,\omega_1'| & |A(\omega_2,\omega_2'| & |A(\omega_2,\omega_3'| & |A(\omega_2,\omega_4'| & |A(\omega_2,\omega_5'| & |A(\omega_2,\omega_6'| \\ |A(\omega_3,\omega_1'| & |A(\omega_3,\omega_2'| & |A(\omega_3,\omega_3'| & |A(\omega_3,\omega_4'| & |A(\omega_3,\omega_5'| & |A(\omega_3,\omega_6'| \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -10 & -15 & -5 & -10 & -5 \\ 1 & -1 & 0 & -5 & -4 & 4 \\ 9 & 6 & 15 & 0 & 9 & 6 \end{pmatrix}$$

For the decomposability case, it is easy to verify in a similar way, that the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

corresponds to the decomposable vector $C_2(B) = (-2,-4,-2)^t$.

The following fundamental theorem will be used in several important parts in the next chapters.

Theorem 3.3.3. *(Binet-Cauchy) If* $A \in F^{m\times n}$, $B \in F^{n\times k}$ *and* $1 \leq k \leq \min\{m,n,k\}$ *then the following equality holds*

$$C_k(A \cdot B) = C_k(A) \cdot C_k(B) \tag{3.24}$$

which expresses in a form of compound matrices the composition law of the exterior powers of linear maps when matrix representations are considered.

Remark 3.3.3. *Other useful properties of the compound matrices which are used next are:*

i) $(C_k(A))^t = C_k(A^t)$, *where* A^t *is the transpose of* A.

ii) $C_k(\lambda A) = \lambda^k C_k(A),\ \lambda \in \mathcal{F}$.

iii) $C_k(I_n) = I_{\binom{n}{k}}$, *where* I_k *is the* $k \times k$ *identity matrix.*

As we will see, the solution that best approximates the real solution of DAP, will be given as a decomposable multivector, say $\underline{x}$. In order to find the matrix H in eqn.(2.24), we have to solve the equation $C_p(H) = \underline{x}$, in terms of H.

Proposition 3.3.1. *[Hod. & Ped. 1] If* $\underline{x} = (..., x_{a_1 \cdots a_m}, ...)^t \in \mathbb{R}^{\binom{n}{m}}$ *is a decomposable vector and* $x_{a_1 \cdots a_m}$ *a nonzero coordinate of* $\underline{x}$*, where* $1 \leq a_1 < \cdots < a_m \leq n$*, then the entries* h_{ij} *of* H *that satisfy the equation* $C_p(H) = \underline{x}$ *have the form*

$$h_{ij} = x_{a_1 \ldots a_{i-1} j a_{i+1} \ldots a_m}, \ i = 1, ..., m, \ j = 1, ..., n \tag{3.25}$$

Example 3.3.2. *Let* $\underline{x} = (x_{123}, x_{124}, x_{125}, x_{134}, x_{135}, x_{145}, x_{234}, x_{235}, x_{245}, x_{345}) \in \bigwedge^3(\mathbb{R}^5)$*. Then*

$$H = \begin{pmatrix} x_{123} & 0 & 0 \\ 0 & x_{123} & 0 \\ 0 & 0 & x_{123} \\ x_{234} & -x_{134} & x_{124} \\ x_{235} & -x_{135} & x_{125} \end{pmatrix}$$

Remark 3.3.4. *The form of* H *in the previous example is not random; it can be proved, [Hod. & Ped. 1], that matrix* H *can be written also as*

$$H = \left(a_{12 \ldots m} I_m, X^t\right)^t \in \mathbb{R}^{m \times n} \tag{3.26}$$

with $x_{ij} = x_{a_1 \ldots a_{i-1} j a_{i+1} \ldots a_m}, \ i = 1, ..., m, \ j = m+1, ..., n.$

3.4 The Grassmann variety

The fact that DAP, as introduced in equation (2.24), requires the poles of a polynomial $a(s)$, suggests that it should be examined in a space where one is interested for the poles of the polynomial and not the polynomial itself. In other words, we need a space where polynomials of the form $p(s) = s - s_1$ and $q(s) = as - as_1$ or the respective vectors of their coefficients, are considered equivalent due to the common root s_1. This implies the embedment of $\bigwedge^k(\mathcal{V})$ in a relevant projective space, where the set of the decomposable k-vectors is viewed as the set of equivalence classes representatives of the so called Grassmann variety. In this section we define the projective space and the Grassmann variety and we present their main properties following.

Definition 3.4.1. *[Hod. & Ped. 1] Let the vector space* $\mathcal{F}^{k+1}$ *of* $(k+1)-$ *tuples of the form* $(x_1, x_2, ..., x_{k+1})$*. Two multivectors* $\underline{x}, \ \underline{y} \in \mathcal{F}^{k+1}$ *will be called equivalent if*

$$(\underline{x}, \underline{y}) \neq (\underline{0}, \underline{0}) \text{ and } \underline{x} = c\underline{y}$$

for some $c \in \mathcal{F} \setminus \{0\}$*.*

This equivalence relation between the vectors can separate them into equivalence classes, where each class consists of all non-zero elements in a one-dimensional subspace of $\mathcal{F}^{k+1}$. This means that the equivalent classes are in one-to-one correspondence with the straight lines through the origin of $\mathcal{F}^{k+1}$.

Definition 3.4.2. *[Hod. & Ped. 1] The set of all equivalence classes of non-zero vectors on $\mathcal{F}^{k+1}$ is called the Projective space of dimension k over $\mathcal{F}$ and is denoted by $\mathbb{P}^k(\mathcal{F})$. Each equivalence class defines a point in the projective space whose coordinates are called homogeneous coordinates. Equivalently, $\mathbb{P}^k(\mathcal{F})$ is the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin $\underline{0}$. If $k = 1,\ 2$ we specifically refer to the projective line and the projective plane, respectively. The complex projective line is also called the Riemann sphere.*

Remark 3.4.1. *In Topology and Differential Geometry, one can see the projective space as the set of all antipodal points $(p, -p)$ of the unit sphere $x^2 + y^2 + z^2 = 1$ or the set of points in the unit sphere such that $p \equiv -p$ [Kob. & Mom. 1]. In other words, lines in the projective space always intersect. We will use this aspect of the projective space in the following chapters.*

Next result, interprets the Plücker coordinates mentioned in Definition 3.3.1, as the homogeneous coordinates of a representative of the related equivalence class in the projective space $\mathbb{P}^k(\mathcal{F})$.

Lemma 3.4.1. *[Hod. & Ped. 1] Let $\underline{f} : \mathcal{V} \to \mathcal{W}$ a linear map, where $\mathcal{V}$, $\mathcal{W}$ are two finite dimensional vector spaces over a field $\mathcal{F}$. Then any two sets of Plücker coordinates which correspond to two different bases of $\mathcal{V}$, with respect to a fixed basis of $\mathcal{W}$, differ by a non-zero scalar factor.*

It is natural to investigate if the opposite direction of this statement also holds true, i.e., if a point - a representative of an equivalence class in the projective space - can be represented via Plücker coordinates. In other words, we want to see if the multi-vector $\underline{a}$, which defines a representative in the projective space via its coordinates a_ω, can be written as a decomposable vector $\underline{a}_1 \wedge \underline{a}_2 \wedge \cdots \wedge \underline{a}_k$ or under which condition this is feasible. To answer that, we need the following results;

Definition 3.4.3. *[Hod. & Ped. 1] Let $\mathcal{F}$ be an algebraically closed field, i.e., $\mathcal{F}$ contains a root for every non-constant polynomial in $\mathcal{F}[x]$. Let also $\mathcal{A}^n$ be an n-affine space over $\mathcal{F}$, i.e., the space obtained by its associated vector space $\mathcal{V}$ over $\mathcal{F}$ without a preferred choice for the origin and $\mathbb{P}^n(\mathcal{F}), \dim\mathbb{P}^n(\mathcal{F}) = n$, be a projective space over F.*

i) If $f_i(x_1, x_2, ..., x_n),\ i = 1, 2, ..., r$ are (not necessarily homogeneous) polynomials over $\mathcal{F}$, then the subset

$$\{(x_1, x_2, ..., x_n) \in \mathcal{A}^n : f_i(x_1, x_2, ..., x_n) = 0,\ i = 1, 2, ..., r\} \tag{3.27}$$

of $\mathcal{A}^n$ is called (affine) algebraic variety.

ii) If $f_i(x_1, x_2, ..., x_{n+1}),\ i = 1, 2, ..., r$ are homogeneous polynomials over $\mathcal{F}$, then the subset

$$\{(x_1, x_2, ..., x_{n+1}) \in \mathbb{P}^n : f_i(x_1, x_2, ..., x_{n+1}) = 0,\ i = 1, 2, ..., r\} \tag{3.28}$$

of $\mathbb{P}^n$ is called projective (algebraic) variety.

iii) The points of the projective space $\mathbb{P}^n(\mathcal{F})$ that satisfy the equations

$$f_i(x_1, x_2, ..., x_{n+1})g_j(x_1, x_2, ..., x_{n+1}) = 0,\ i = 1, 2, ..., r_1,\ j = 1, 2, ..., r_1 \tag{3.29}$$

define the projective variety $\mathcal{V}_1 \cup \mathcal{V}_2$, which is called the union of the projective varieties $\mathcal{V}_1,\ \mathcal{V}_2$, defined respectively by the polynomials f and g.

iv) The subvariety of points common to two projective varieties $\mathcal{V}_1,\ \mathcal{V}_2$ is called the intersection of $\mathcal{V}_1,\ \mathcal{V}_2$ and is denoted by $\mathcal{V}_1 \cap \mathcal{V}_2$ and it is non-empty if $dim\mathcal{V}_1 + dim\mathcal{V}_2 \geq n$.

v) If $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$, where $\mathcal{V}_1,\ \mathcal{V}_2$ are two projective varieties, then $\mathcal{V}$ is called a reducible projective variety and its dimension d is given by

$$d = n - n_e \tag{3.30}$$

where n_e is the minimal number of equations defining the variety $\mathcal{V}$. Otherwise the variety is called irreducible.

Remark 3.4.2. *It can be proved that an algebraic variety does not depend on the coordinate system chosen.*

3.4.1 The Grassmannian and the Plücker Embedding

The map we mentioned at the beginning of this section that helps embed from the affine space to the projective space, is the so called *Plücker embedding.*

Definition 3.4.4. *[Hod. & Ped. 1] Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. The Grassmannian $Gr(m, \mathcal{V})$ is defined as the set of all m-dimensional subspaces of $\mathcal{V}$.*

The Grassmannian admits the structure of an analytic manifold, i.e., the Grassmann manifold. Now, if p is the map $p : Gr(m, \mathcal{V}) \to \mathbb{P}^n(\mathcal{F})$, then the Grassmann variety $G_m(\mathcal{V})$ is the image of the map p in the projective space $\mathbb{P}^n(\mathcal{F})$, i.e., $G_m(\mathcal{V}) = p\,(Gr(m, \mathcal{V}))$.

Definition 3.4.5. *The map $p : Gr(m, \mathcal{V}) \to \mathbb{P}^n(\mathcal{F})$ from the Grassmannian to the projective space $\mathbb{P}^n(\mathcal{F})$ is called the Plücker embedding.*

Proposition 3.4.1. *[Hod. & Ped. 1] Let the Plücker embedding p. Then*

i) p is unique up to scalar multiplication and thus well defined.

ii) p is injective.

Now the definition of the Grassmann variety readily follows.

Definition 3.4.6. *[Hod. & Ped. 1] Let a vector space $\mathcal{V}$ with $\dim\mathcal{V} = n$ and $\mathcal{U}$ an $m-$dimensional subspace of $\mathcal{V}$ over a field $\mathcal{F}$.*

i) The Grassmann variety, denoted as $G_m(\mathcal{V})$ or $G_{n,m}$ or $\Omega(n,m)$, is the irreducible algebraic variety which contains the decomposable vectors of the projective space $\mathbb{P}^{\binom{n}{m}-1}(\mathcal{F})$.

ii) A point in $G_m(\mathcal{V})$ will be called the Grassmann representative of the Grassmann variety.

We know give the definition of the *Quadratic Plücker Relations*(QPRs).

Definition 3.4.7. *(QPRs)[Hod. & Ped. 1] For $1 \leq m \leq n,\ m,\ n \in \mathbb{N}$, let*
- *$\mathcal{G}_{m,n}$ be the set of n^m elements of length m in the form a_i where*

$$1 \leq a_i \leq n,\ i = 1, 2..., m. \tag{3.31}$$

- *$\mathcal{D}_{m,n}$ the set whose elements are obtained by all possible permutations of the elements of $\mathcal{Q}_{m,n}$.*
- *$a(\hat{s}) : k$ denote a sequence in $\mathcal{G}_{m,n}$ such that the a_s term is deleted and replaced by k, i.e.,*

$$a(\hat{s}) : k = (a_1, ..., a_{s-1}, k, a_{s+1}, ..., a_m)$$

- *$\mathcal{S}_n$ the set of all possible permutations of the elements of $\{1, 2, ..., n\}$. If $\varrho : \mathcal{G}_{m,n} \to \mathcal{F}$ is a map such that $\varrho(\gamma\sigma) = sign\sigma \cdot \varrho(\gamma),\ \gamma \in \mathcal{G}_{m,n},\ \sigma \in \mathcal{S}_m$, then for every $a \in \mathcal{D}_{m,n},\ \delta \in \mathcal{D}_{m+1,n}$ the conditions*

$$\sum_{j=1}^{m+1} (-1)^{j-1}\, \varrho\left(a(\hat{i}) : \delta(j)\right) \varrho\left(\delta(\hat{j})\right) = 0 \tag{3.32}$$

are called Quadratic Plucker Relations (QPR) and ϱ is said to satisfy the QPR.

Now, from the definition of a decomposable vector in the previous section, the following important theorem is established.

Theorem 3.4.1. *[Hod. & Ped. 1] If $\mathcal{V}$ is a vector space over a field $\mathcal{F}$, then a multivector in $\bigwedge^m(\mathcal{V})$ is decomposable if and only if, its coordinates satisfy the QPR.*

The previous theorem has given a number of significant results, regarding the algebraic and geometric structure of the Grassmann variety.

Corollary 3.4.1. *[Got. 1],[Die. 1] Let $\mathcal{V}$ be an n dimensional vector space over a field $\mathcal{F}$ and an m dimensional subspace of $\mathcal{V}$, $m < n$. The dimension of the Grassmann variety $G_m(\mathcal{V})$ of the projective space $\mathbb{P}^{\binom{n}{m}-1}(\mathcal{F})$ is given by*

$$\dim\left(G_m(\mathcal{V})\right) \begin{cases} m(n-m), & F \neq \mathbb{C} \\ 2m(n-m), & F = \mathbb{C} \end{cases} \tag{3.33}$$

To end this section, we should mention that the derivation of the QPR that describe the Grassmann variety is a key problem in the field of Algebraic Geometry, due to the complex form of (3.32). In higher dimensions, this formula is difficult to be implemented, even numerically. Moreover, one comes across a second obstacle with respect to their analytical calculation; in the case where $n = 5,\ m = 3$ for example, the QPR are

$$x_1x_6 - x_2x_5 + x_3x_4 = 0, x_1x_9 - x_2x_8 + x_3x_6 = 0, x_1x_{10} - x_4x_8 + x_5x_7 = 0, \quad (3.34)$$

$$x_2x_{10} - x_4x_9 + x_6x_7 = 0, x_3x_{10} - x_5x_9 + x_6x_8 = 0 \quad (3.35)$$

It can be proved that the above set of equations is not minimal, since equations (3.35) may be obtained by the set (3.34). The independent set of QPR which completely describes the Grassmann variety is referred to as the *Reduced Quadratic Plücker Relations* (RQPR) and it can be proved, [Gia. 1], that it is described by the non-trivial relations among the equations

$$C_m(H) = (..., x_{a_1 \cdots a_m}, ...)^t x_{12...m}^{m-1} \quad (3.36)$$

where $\underline{x} = (..., x_{a_1 \cdots a_m}, ...)^t \in \mathbb{R}^{\binom{n}{m}}$ is a decomposable vector with its first coordinate $x_{12...m} \neq 0$ and H the matrix whose entries are given by eqn.(3.25).

3.4.2 The Grassmann Matrix

The Grassmann matrix was introduced in [Kar. & Gia. 6], [Gia. 1] as an alternative way to test the decomposability of a multivector due to the difficulty to calculate in practice equations (3.32). Here we present the main results for the Grassmann matrix in order to prove in the following chapters, that the singular values of the Grassmann matrix provide a direct solution for the approximate DAP.

Definition 3.4.8. *[Kar. & Gia. 6], [Gia. 1] Let $Q_{m,n}$ denote the set of strictly increasing sequences of m integers chosen from $1, ..., n$ and $a(\hat{s})$ denote the sequence $a_1, a_2, ..., a_n$ where the $a(s)$- term is deleted, i.e. $a(\hat{s}) = (a_{1,2}, ..., a_{s-1}, a_{s+1}, ..., a_n)$. Let $\gamma = (i_1, ..., i_{m+1}) \in Q_{m+1,n}$ where $m, n \in \mathbb{N}, m < n$. We define the function φ as follows:*

$$\varphi(i, \gamma) = \begin{cases} 0 & \text{if } i \notin \gamma \\ (-1)^{k-1} a_{\gamma(\hat{k})} & \text{if } i = i_k,\ k = 1, ..., m+1 \end{cases} \quad (3.37)$$

where $a_\omega,\ \omega \in Q_{m+1,n}$, are the coordinates of a vector in $\bigwedge^m(\mathcal{V})$ for an n-dimensional vector space $\mathcal{V}$.

Proposition 3.4.2. *[Kar. & Gia. 6], [Gia. 1] Let $\mathcal{V}$ be an n-dimensional vector space over a field $\mathcal{F}$ and two non-zero vectors $\underline{x} \in \mathcal{V}$, $\underline{a} \in \bigwedge^k(\mathcal{V})$ such that*

$$\underline{x} = \sum_{i=1}^{n} \kappa_i \underline{e}_i, \ \underline{a} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1 i_2 \ldots i_k} \underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k}$$

where $\{\underline{e}_1, ..., \underline{e}_n\}$, $\{\underline{e}_{i_1} \wedge \underline{e}_{i_2} \wedge \cdots \wedge \underline{e}_{i_k}\}$, $1 \le i_1 < i_2 < \cdots < i_k \le n$ are the bases of $\mathcal{V}$ and $\bigwedge^k(\mathcal{V})$, respectively. Then, necessary and sufficient conditions for $\underline{x} \wedge \underline{a} = \underline{0}$ is that

$$\sum_{i=1}^{n} \varphi(i, \gamma) \kappa_i = 0, \ \forall \gamma \in Q_{m+1,n} \tag{3.38}$$

which can be expressed in matrix form as

$$\begin{pmatrix} \phi^1_{\gamma_1} & \phi^2_{\gamma_1} & \ldots & \phi^n_{\gamma_1} \\ \phi^1_{\gamma_2} & \phi^2_{\gamma_2} & \ldots & \phi^n_{\gamma_2} \\ \vdots & \vdots & \ldots & \vdots \\ \phi^1_{\gamma_s} & \phi^2_{\gamma_s} & \ldots & \phi^n_{\gamma_s} \end{pmatrix} \cdot \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.39}$$

where $s = \binom{n}{m+1}$.

Definition 3.4.9. *[Kar. & Gia. 6], [Gia. 1] The matrix*

$$\Phi_{\underline{a}} := \begin{pmatrix} \phi^1_{\gamma_1} & \phi^2_{\gamma_1} & \ldots & \phi^n_{\gamma_1} \\ \phi^1_{\gamma_2} & \phi^2_{\gamma_2} & \ldots & \phi^n_{\gamma_2} \\ \vdots & \vdots & \ldots & \vdots \\ \phi^1_{\gamma_s} & \phi^2_{\gamma_s} & \ldots & \phi^n_{\gamma_s} \end{pmatrix} \tag{3.40}$$

in equation (3.39) *is called the Grassmann matrix of the multi-vector $\underline{a}$.*

Example 3.4.1. *Let $\underline{a} = (a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{34}) \in \wedge^2(\mathbb{R}^4)$. Then the Grassmann matrix of $\underline{a}$ is*

$$\Phi_{\underline{a}} = \begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{pmatrix}$$

Next theorem displays the connection of the decomposability of a multivector and the Grassmann matrix.

Theorem 3.4.2. *[Gia. 1] Let $\mathcal{V}$ an n- dimensional vector space over a field $\mathcal{F}$ and $\Phi_{\underline{a}}$ the Grassmann matrix of a non-zero multivector $\underline{a} \in \bigwedge^m(\mathcal{V})$. Then, the following are equivalent:*

i) $\underline{a}$ is decomposable.

ii) $\dim N_r(\Phi_{\underline{a}}) = m$, *where N_r denotes the left null space of the Grassmann matrix.*

iii) $\text{rank}(\Phi_{\underline{a}}) = n - m$.

iv) $C_{n-m}(\Phi_{\underline{a}}) \neq 0$ *and* $C_{n-m+1}(\Phi_{\underline{a}}) = 0$

Remark 3.4.3. *The above results prove that the Grassmann matrix is a very helpful tool for decomposability problems. Also, $\Phi_{\underline{a}}$ provides the means to reconstruct a subspace $\mathcal{U}$ of $\mathcal{V}$ which corresponds to the decomposable multivector $\underline{a} \in \wedge^m(\mathcal{V})$, by setting $U = N_r(\Phi_{\underline{a}})$. In the next chapters, we will present a new result for the Grassmann matrix, with regard to its Singular Value Decomposition.*

3.5 The Grassmann Invariants of Rational Vector Spaces

The characterization of a rational vector space by a set of invariants has been initiated in [For. 1]. However, the defined set of dynamical indices did not form a complete set of invariants. Although the echelon type minimal basis characterizes completely a rational vector space, the need of an alternative complete characterization has emerged, due to the computational difficulties in finding the echelon type form and the large number of parameters involved in its description. In this section we present the main results for the alternative complete invariants of a rational vector space as introduced in [Kar. & Gia. 5], called *Grassmann Invariants* and we focus particularly on a specific invariant, the *Plücker Matrix*, which is important for DAPs formulation.

Definition 3.5.1. *[Kar. & Gia. 5] Let $\mathcal{M}(s) = \{M(s) : M(s) \in \mathbb{R}^{p\times q}(s)\}$, with* $\text{rank}M(s) = q$, $q \leq p$. *Two matrices $M(s)$, $N(s)$ will be called column equivalent and this shall be denoted by $M(s)\rceil N(s)$ if there exists an invertible rational matrix $Q(s)$ in $\mathbb{R}^{q\times q}(s)$ such that*

$$N(s) = M(s)Q(s) \tag{3.41}$$

Eqn.(3.41) defines an equivalent relationship $\sim$ on $\mathcal{M}(s)$. $\sim(M)$ will denote the equivalence class (orbit) of $M(s) \in \mathcal{M}(s)$ and $\mathcal{M}(s)|\sim$ will denote the set of equivalence classes (quotient orbit) under $\sim$. Clearly, $\mathcal{M}(s)|\sim$ partitions $\mathcal{M}(s)$

and its orbit is a rational vector space $\mathcal{V}_M = \text{colspan}(M(s))$. Since $\dim\mathcal{V}_M = q$ and $\mathcal{V}_M$ is a subspace of $\mathbb{R}^p(s)$, then $\mathcal{M}(s)| \sim\equiv Gr(q, \mathbb{R}^p(s))$.

Definition 3.5.2. *[Kar. & Gia. 5] If $\mathcal{V}_M \in Gr(q, \mathbb{R}^p(s))$, then any non-zero decomposable multivector $\underline{m}(s) = \underline{m}_1(s) \wedge \cdots \wedge \underline{m}_q(s)$, $M(s) = (\underline{m}_1(s), ..., \underline{m}_q(s))$ is called a rational Grassmann representative of $\mathcal{V}_M$. Furthermore, $\underline{m}(s)$ is called reduced if its components $m_\omega(s)$, $\omega = (i_1, .., i_q)$ are coprime.*

Therefore if $\underline{m}(s) = (..., m_\omega(s), ...)^t \in \mathbb{R}^{\sigma+1}$, $\sigma = \binom{p}{q}$, then

$$\begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_\sigma \end{pmatrix} = \begin{pmatrix} p_{00} & p_{10} & \cdots & p_{d0} \\ p_{01} & p_{11} & \cdots & p_{d1} \\ \vdots & \vdots & \cdots & \vdots \\ p_{0\sigma} & p_{1\sigma} & \cdots & p_{d\sigma} \end{pmatrix} \begin{pmatrix} 1 \\ s \\ \vdots \\ s^d \end{pmatrix} \tag{3.42}$$

where $d = \max\{m_i(s),\ i = 1, ..., \sigma\}$. The $(\sigma + 1) \times (d + 1)$ matrix in eqn.(3.42) is called the *basis matrix* of $\underline{m}(s)$ and is denoted by P_d.

Definition 3.5.3. *Let $\underline{n}(s) = \underline{n}_1(s) \wedge \cdots \wedge \underline{n}_d(s)$ a reduced Grassmann representative of $\mathcal{V}_M$ with $\underline{n}(s) = P_d\underline{e}_d(s) := (\underline{p}_0, ..., \underline{p}_d)(1, s, ..., s^d)^t$. Then the polynomial multivector*

$$\underline{g}(\mathcal{V}_M) := \frac{\underline{n}(s)\text{sign}(p_{di})}{\|\underline{p}_d\|} \tag{3.43}$$

where p_{di} is the first nonzero component of $\underline{p}_d$, is defined as the canonical polynomial Grassmann representative of $\mathcal{V}_M$.

The multivector $\underline{g}(\mathcal{V}_M)$ of $\mathcal{V}_M$ is a decomposable vector of $\mathbb{R}^\sigma$. If $\deg\big(\underline{g}(\mathcal{V}_M)\big) = \delta$, where δ is Forney's dynamical order of $\mathcal{V}_M$, [For. 1], i.e., the sum of the invariant dynamical indices $\deg m_i(s)$ of $\mathcal{V}_M$, then

$$\underline{g}(\mathcal{V}_M) = P_\delta\underline{e}_\delta(s),\ \underline{e}_\delta(s) = (1, s, ..., s^\delta)^t \tag{3.44}$$

The $\binom{p}{q} \times (\delta + 1)$ matrix P_δ is referred to as *the Plücker matrix* of $\mathcal{V}_M$.

Remark 3.5.1. *The Plücker matrix is in other words the matrix whose i-th row is formed by the coefficients of the polynomials in the i-th coordinate of $\underline{m}(s)$. Moreover, P_δ provides alternative decomposability criteria for k-vectors; if a polynomial multivector $\underline{p}(s) = P_\delta\underline{e}_\delta(s) := (\underline{p}_0, ..., \underline{p}_\delta)(1, s, ..., s^\delta)^t$ is decomposable, then it can be proved, [Kar. & Gia. 5], that $\underline{p}_0$, $\underline{p}_\delta$ are also decomposable.*

The following theorem is the main result of this section.

Theorem 3.5.1. *[Kar. & Gia. 5] The Plücker matrix P_δ is a complete invariant for $\mathcal{V}_M$.*

Remark 3.5.2. *The Plücker matrix obtains different forms, according to the assignment problem which is under investigation. If for example the pole-assignment by state feedback is studied and we set* $\underline{b}^t(s) = C_n(sI_n - A, B)$*, then*

$$\underline{b}^t(s) = cz_B(s)P(A,B)\underline{e}_{n_1}(s),\ c \in \mathbb{R}^* \tag{3.45}$$

where $z_B(s)$ *corresponds to the uncontrollable eigenvalues of the system or*

$$\underline{b}(s) = cP(A,B)\underline{e}_n(s) \tag{3.46}$$

if the system is controllable. The matrix $P(A,B)$ *is called the controllability Plücker matrix of the system. Other forms of the Plücker matrix may be found in [Gia. 1] for the rest of the frequency assignment problems we discussed in section 2.5.*

3.6 Conclusions

This chapter has given an overall account of the principal definitions and methods used in tensor spaces and algebraic varieties in a space of n dimensions. The theory of algebraic varieties was mainly presented via a theory of varieties in a projective space, whereas a special effort was made to "build up" tensors step-by-step from the components of vectors in an n-dimesional space, in order to illustrate these advanced concepts as simple and complete as possible. Furthermore, we concetrated our study, in a certain variety of the projective space, the Grassmann variety and we highlighted its connection with special matrices such as the Grassmann matrix and the Plücker matrices. These mathematical tools will be implemented in the next chapters, not only for the intepretation of DAP in exterior algebra-algebraic geometry terms, but they will also help us solve the "approximate" DAP when the original problem can not be solved. In lower dimensions, i.e., $2-$vectors, the respective second order tensors are viewed as matrices and DAP and the approximate DAP will be studied via antisymmetric matrices. For k-vectors when $k \geq 3$, we will work via k- order skew-symmetric tensors.

Chapter 4

Methods for Pole Assignment Problems and DAP

4.1 Introduction

Before introducing the approximate DAP methodology, we will briefly present in this chapter some of the most well known techniques that have been used in the past to deal with pole assignment problems as well as the Determinantal Assignment Problem (DAP). From the definition of such problems it is evident that in order to calculate the entries of the desired matrix, one has to expand the determinant of a polynomial matrix and solve afterwards a system of polynomial equations. Thus the area of these problems involves in a natural way the theory of *Gröbner basis*, [Kar. etc. 14], which is amongst the most usually used methods for the solution of systems of polynomial equations.

Moreover, in [Mit., etc. 1], [Mit., etc. 2] a computational framework and several numerical aspects have been presented especially for the DAP case with the use of MATLAB. The work involved the development of an algorithm for the computation of the Grassmann product (exterior product), the specification of a numerical method for the evaluation of the Plücker matrices and an algorithm for the derivation of the numerical solutions of DAP.

DAP includes the case of pole assignment via output feedback which has been also examined in various ways. In general, the problem has been studied mostly under two different perspectives, via *algebraic* and *geometric techniques*. The first category includes all methods within the bounds of linear systems theory and most of them are algorithmic, thus very convenient for design purposes. The characteristic of these techniques is that they do not use all possible degrees of freedom of the matrix that is to be calculated (H in the DAP case) but rely on special forms which simplify the problem. In general, they are based on the so

called *rank-1 design* and the *full-rank design* for H. The former looks for two vectors $\underline{k}_1,\ \underline{k}_2$ such that matrix $K = \underline{k}_1 \times \underline{k}_2$, where $H = (I_n, K)$ and algorithms following this procedure can be found in [Bra. & Pear. 1] and [Chen. 1]. Although these algorithms have considerable simplicity, the resulting closed loop systems have poor disturbance rejection properties compared with their full-rank counterparts. Furthermore, although the rank-1 method has been successfully used in the state feedback problem, its use in the output feedback further reduces the degrees of freedom which are needed for the solvability of the problem, [Kim. 3].

The full-rank designs are more complex and their main objective is to construct a full rank matrix K. The main disadvantage is that most of these algorithms are for *partial* pole placement, i.e., pole assignment under restrictions between the inputs and the outputs - following the generic approaches we mentioned in Section 2.5 -as well as the number of poles that can be assigned. Several other methods for generic and non-generic results on the full-rank algorithms may be found in [Mun. 1].

On the other hand, *geometric techniques* deal not only with the construction of numerical solutions like the algebraic methods do, but also with the understanding of the nature of these problems, focusing on the pole placement via output feedback problem. The term "geometric" is used due to the study of these problems via several relations in the form of maps or intersections of auxiliary geometrical objects, such as linear spaces, algebraic varieties and manifolds. The main geometric approaches can be listed as follows.

i) *Infinitesimal Techniques*, where the output feedback problem is mostly examined via the polynomial map $\phi : F^{\mu} \to F^{d}$, where $F = \mathbb{R}$ or $\mathbb{C}$, μ are the degrees of freedom of H and ϕ maps H to the coefficient vector of the polynomial $a(s)$ in eqn.(2.24). This map was first defined and examined in [Her. & Mar. 1] and [Wil. & Hes. 1] for the output feedback problem where the solvability of the problem was reduced to test if ϕ is onto. If $\mu = km$ and $d = n$, where $k,\ m$ is the number of inputs and outputs respectively and n the degree of $a(s)$, then ϕ maps every $K \in \mathbb{R}^{k\times m}$ such that $H = (I_n, K)$ to $(\lambda_n, ..., \lambda_1),\ a(s) = s^n + \lambda_n s^{n-1} + \cdots + \lambda_1$. A number of properties of the complex and real pole placement map which relate to the dimensions of their images and known system invariants have been also derived in [Lev. & Kar. 7]. The authors have shown that the two dimensions are equal and that their computation is equivalent to determining the rank of the corresponding differential. They have also used a new expression for the differential of ϕ allowing in this way the derivation of new conditions for pole assignability, based on the relationships between the rank of the Plücker matrix and the rank of the differential of the pole placement map.

ii) *Schubert Calculus Techniques*, where the pole placement problem via output feedback is considered as a problem of intersection of hyper-surfaces on a Grassmannian. More analytically, if $\tilde{\mathcal{H}} = \{\text{rowspan}(H) : H \in \mathcal{H}\}$ (actually, the poles of $\det(H \cdot M(s))$ depend only upon the rowspan(H)), then $\tilde{\mathcal{H}}$ is a subset of the Grassmann manifold $G_q(\mathbb{C}^{p+q})$ which contains the q-dimensional subspaces of $\mathbb{C}^{p+q}$. If $s_i,\ i = 1, ..., d$ are the complex conjugate roots of $a(s)$, then we require to find H such that $\det(H \cdot M(s_i)) = 0$. If $\mathcal{S}_i$ contain the q dimensional subspaces V of $\mathbb{C}^{p+q}$ with $\dim(V \cap \text{LKer}(M(s_i))) \geq 1$, then the pole placement problem via output feedback can be viewed as the intersection

$$\mathcal{H}_{\mathbb{C}}(a(s)) = \bigcap_{i=1}^{d} \mathcal{S}_i \cap \tilde{\mathcal{H}}$$

If $d = \dim\mathcal{H}$ then $\mathcal{H}_{\mathbb{C}}(a(s))$ contains finite number of points which may be calculated using Schubert enumerative calculus, [Kl. & Lak. 1]. If this number is odd then a real solution exists, [Bro. & Byr. 2]. The more general case where d is arbitrary was examined in [Gia. & Kar. 3] whereas in [Sot. 1], enumerative geometry techniques were applied for the real Grassmann variety as well.

iii) *Topological Intersection Techniques*, where the *generic* solvability of several pole placement problems, mostly via output feedback, is examined, especially when we are interested in real solutions [Ful. 1].

iv) *Combinatorial Geometric Techniques*, where the solvability of the pole assignment problems is equivalent to finding a q dimensional linear subspace of $\mathbb{R}^{p+q}$ such that it intersects all the p-dimensional subspaces $\text{LKer}(M(s_i))$. A straight forward solution for this problem, which contains a combination of the geometry of linear subspaces and combinatorics involving dimension counting of certain subspaces, may be found in [Ro. 2]. This method was first proposed in [Kim. 1] for the output feedback pole placement problem.

v) *Projective Techniques*, where classical algebraic geometry results are applied for DAP in a projective space - rather than the affine space as in the above cases - to determine the existence of solutions. The approach - introduced in [Kar. & Gia. 5]- relies on exterior algebra to construct the embedding map, called *Plücker embedding* where DAP is reduced to a problem of determining common solutions of a set of linear and quadratic equations which are defined in an appropriate projective space. One major advantage of this framework is that introduces new sets of invariants which may be used to characterize solvability conditions, as well as the derivation of approximate solutions of the initial intersection problem.

Note that for a k-input, m-output and n-state generic, strictly proper system, this procedure has not only implied the sufficient assignability condition $mk > n$, [Lev. 1], [Lev. & Kar. 3], we mentioned in Chapter 2, but also the condition $n_1(m + k) + mk > n + n_1$ for the generic pole placement problem via dynamic controllers of degree n_1, [Lev. & Kar. 3]- [Lev. & Kar. 6]. Furthermore, in [Lev. & Kar. 2] the authors have used Schubert Calculus and cohomology techniques for the Grassmannians (height of the first Whitney class of a Grassmannian, [Kl. & Lak. 1]) to derive a sufficient condition for generic pole assignment by real constant output feedback. Their results had the advantage that the new condition was testable for more cases, since it covered all poles and not only real poles, as in [Byr. 1], [Byr. 3] and the so-called LScat test.

In this chapter we briefly discuss the previous methodologies for the solution of several determinantal assignment problems and how the "approximate" DAP is derived, via the use of projective methodology tools. In particular, in Section 4.2 we refer to the Gröbner Basis Method as an immediate tool for straight-forward calculations where we also we give an example that clarifies the advantages and the disadvantages of this method.

From the *algebraic-techniques* group that deals with numerical solutions, we select the algorithm presented in [Mit., etc. 1] for DAP and the full-rank algorithm in [Pa. 1] for pole assignment via output feedback since this procedure has no extra solvability-assignability conditions except from the existence of the controller. We present this approach in Sections 4.3.1 and 4.3.2. The geometric methods are presented in Section 4.4. Specifically, in Section 4.4.1 we explain how a specific DAP is formed into a Schubert calculus problem.

In Section 4.4.2 we present the algorithm appeared in [Lev. 1], [Lev. & Kar. 3] which belongs in the so called infinitesimal techniques and considers special sequences of feedback compensators which converse to a so called *degenerate compensator*. Finally in 4.4.3, we investigate how DAP is transformed via projective methodologies as this was considered in [Kar. & Gia. 5], where we also introduce the Approximate Determinantal Assignment Problem.

4.2 Direct Calculations on DAP: The Gröbner Basis Method

The Gröbner basis is a very helpful method for obtaining direct solutions on determinantal problems, when the implied system of polynomial equations is relatively easy. In this section, we briefly present the basic notions of the Gröbner basis in order to solve a DAP example via this method and examine the

difficulties in higher dimensions.

Definition 4.2.1. *[Ad. & Lou.1] Let a polynomial ring $F[x_1, ..., x_n]$, $n \in \mathbb{N}$, over a field $\mathcal{F}$ and $\mathfrak{I}$ an ideal of the ring.*

i) A product of the form $x_1^{a_1} \cdots x_n^{a_n}$, $a_i \in \mathbb{N}$, $1 \leq i \leq n$ is called power product. The term-degree of a polynomial $p(x_1, ..., x_n) \in F[x_1, ..., x_n]$ is the sum of the exponents of the term's power product. The leading term of $p(x_1, ..., x_n)$, denoted as $\mathrm{LT}(p)$ is the term with the greatest term-degree and the leading coefficient is the coefficient of the leading term. The power product of the leading term is called leading power product and is denoted by $\mathrm{LP}(p)$.

ii) The degree of a polynomial $p(x_1, ..., x_n) \in F[x_1, ..., x_n]$ is the greatest term-degree of $p(x_1, ..., x_n)$.

The Gröbner basis method is considered the generalization of the Gaussian elimination for linear systems (or the Simplex method for linear programming) where instead of solving the initial polynomial system, one solves a much easier one. Thus, since the Gauss elimination is actually the Euclidean division algorithm, we will need to introduce a division algorithm for multi-variable polynomials as well. In the linear case, the ordering of the terms of polynomials was trivial, i.e., $x^n > x^{n-1} > \cdots > 1$. For multivariate polynomials two orderings are mostly used: the *degree lexicographic order or deglex* and the *degree reverse lexicographic or degrevlex*. Here, we will use the latter, since it is faster in algorithmic computations. Details with regard to the former ordering can be found in [Ad. & Lou.1] and [Bas., etc. 1].

Definition 4.2.2. *(Degree Reverse Lexicographic Ordering)[Ad. & Lou.1]. Let the lex ordering $x_1 > x_2 > x_3$ and $x^a := x_1^{a_1} x_2^{a_2} x_3^{a_3}$. We write $x^a < x^b$ if one of the following conditions hold.*

i) $a_1 + a_2 + a_3 < b_1 + b_2 + b_3$.

ii) $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ and the first coordinates a_i, b_i from the right which are different, satisfy the inequality $a_i > b_i$.

Example 4.2.1. *If $f(x_1, x_2, x_3) = -4x_1^4 x_3 + x_1^3 x_2^4 + 6x_1^2 x_2^2 x_3 - x_1 x_2^3 x_3^3 + 11$ then, the degrevlex term order is*

$$x_1^3 x_2^4 > x_1 x_2^3 x_3^3 > x_1^4 x_3 > x_1^2 x_2^2 x_3 > 11 \tag{4.1}$$

Theorem 4.2.1. *(Division Algorithm in $F[x_1, ..., x_n]$)[Ad. & Lou.1]. Let $P_{1,...,s} := (p_1, ..., p_s)$ be an ordered s-tuple of polynomials in $F[x_1, ..., x_n]$. Then every $p(x_1, ..., x_n)$ can be written as*

$$p(x_1, ..., x_n) = q_1(x_1, ..., x_n)p_1(x_1, ..., x_n) + \cdots + q_s(x_1, ..., x_n)p_s(x_1, ..., x_n) + r(x_1, ..., x_n) \tag{4.2}$$

where $q_i(x_1,...,x_n),\ r(x_1,...,x_n) \in F[x_1,...,x_n]$ *and either* $r(x_1,...,x_n)$ *is the zero polynomial or a linear combination with coefficients in* $\mathcal{F}$*, of monomials, none of which is divisible by any of* $\mathrm{LT}(p_1),...,\mathrm{LT}(p_s)$*. We say that* $r(x_1,...,x_n)$ *is a remainder of* $p(x_1,...,x_n)$ *on division by* $P_{1,...,s}$*.*

Definition 4.2.3. *[Ad. & Lou.1] A finite subset* $\{g_1,...,g_s\}$ *of an ideal* $\mathfrak{I} \subset F[x_1,...,x_n]$ *is called a Gröbner basis or standard basis if*

$$\mathrm{sp}\{\mathrm{LT}(g_1),...,\mathrm{LT}(g_s)\} = \mathrm{sp}\{\mathrm{LT}(\mathfrak{I})\} \tag{4.3}$$

We are now ready to implement the above results to the solution of a Determinantal Assignment Problem.

Example 4.2.2. *We want to calculate a matrix*

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{pmatrix}$$

such that $\det(H \cdot M(s)) = a(s)$ *holds. For*

$$M(s) = \begin{pmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we obtain

$$\det(H \cdot M(s)) = -h_{13}h_{22} + h_{12}h_{23} + (-h_{13}h_{21} + h_{11}h_{23})s$$

If $a(s) = 1 + 2s$*, then the ideal*

$$\mathfrak{I} = \mathrm{sp}\{-h_{13}h_{22} - 1, h_{12}h_{23} - h_{13}h_{21} + h_{11}h_{23} - 2\} \subset \mathbb{R}[h_{11},...,h_{23}]$$

is defined. The Gröbner basis of $\mathfrak{I}$ *(with respect to the degrevlex term order) is the set*

$$\{-1 - h_{13}h_{22} + h_{12}h_{23}, -2 - h_{13}h_{21} + h_{11}h_{23}, h_{11} - 2h_{12} - h_{12}h_{13}h_{21} + h_{11}h_{13}h_{22}\}$$

The solution of this system implies

$$h_{11} = 0,\ h_{21} = -\frac{2}{h_{13}},\ h_{23} = -\frac{1 + h_{13}h_{22}}{h_{12}}$$

Thus, DAP in this case has more than one solution for $h_{12},\ h_{13} \neq 0$*.*

From this example it is clear that, when DAP is solvable, the solution of the implied polynomial system is not straight-forward, since the new system obtained by the Gröbner basis may be even more difficult to solve, than the initial system, especially in higher matrix dimensions. Generally, the method of the Gröbner basis may be used efficiently in lower dimensions pole placement problems, for computational reasons, or when a specific structure for $M(s)$ is known, like in the previous example or when $M(s)$ is sparse. Note that, the Gröbner basis theory has been also used in many other control problems, as in [Lin. 1] and [Kar. etc. 14].

4.3 Algebraic Techniques

Most algebraic techniques involve algorithms that have been designed for the construction of the numerical solution of determinantal assignment-type problems either concern low-dimensions (low-dimensional Grassmann varieties) since the complexity of these problems increases rapidly due to the increasing number of the quadratic Plücker relations, [Hod. & Ped. 1] in higher dimensions or they are constructed for special pole placement problems, such as the output feedback. Here we examine an algorithm designed for the numerical solutuions of DAP, [Mit., etc. 1] and an unconstrained iterative full-rank algorithm for the numerical solutions of a pole placement via output feedback problem.

4.3.1 Numerical Solutions of DAP

As will see more analytically in the Projective techniques section, DAP can be reduced in a *linear subproblem* of the form $\underline{h} \cdot P = \underline{a}$, where $P \in \mathbb{R}^{\binom{p}{q} \times (\delta+1)}$ is the Plücker matrix of the vector space col-span $\{M(s)\}$ with δ being the order of $M(s)$ and a *multilinear subproblem*, where if the linear subproblem is solvable and $\mathcal{K} \subseteq \mathbb{P}^{\binom{p}{q}-1}(\mathbb{R})$ is the respective family of the solution vectors $\underline{h}$, then one has to find whether there exists $\underline{h} \in \mathcal{K}$ such that $\underline{h}$ is decomposable. If such vector exists, then a matrix $H \in \mathbb{R}^{q \times p}$ such that $C_q(H) = c\underline{h},\ c \in \mathbb{R}^*$ has to be determined, [Gia. 1], [Kar. & Gia. 5]. Then in order to calculate a numerical solution for DAP, the authors in [Mit., etc. 1] defined the optimization problem

$$\min \|\underline{h} \cdot P - \underline{a}\|^2 \text{ s.t. } f_i(\underline{h}) := \underline{h}^t Q_i \underline{h} = 0 \tag{4.4}$$

where $\underline{h} \in \mathbb{R}^{\binom{p}{q}}$ is a vector whose coordinates satisfy the (reduced) QPR set and $Q_i \in \mathbb{R}^{\binom{p}{q} \times \binom{p}{q}}$ are appropriate matrices. The numerical solution $H \in \mathbb{R}^{q \times p}$ will then be derived by the solution of $C_q(H) = c\underline{h},\ c \in \mathbb{R}^*$. As explained in [Mit., etc. 1], if $\underline{a}, p, q, \delta, P$ are such that DAP is solvable, then the above constrained optimization problem is solvable and so an exact solution may be found. In fact, if $q(p-q) \geq \delta + 1$, rank$P = \delta + 1$ then (4.4) has a constrained global minimum equal to zero. If $q(p-q) \geq \delta + 1$ then DAP is not solvable, but one can derive a solution of (4.4) in the form $\min \|\underline{h} \cdot P - \underline{a}\|^2 = \varepsilon > 0$ where $\underline{h} \cdot P = \underline{a} + \underline{\varepsilon}$, $\underline{h}^t Q_i \underline{h} = 0$, for $\underline{\varepsilon} \in \mathbb{R}^{\delta+1}$. Other conditions about the solvability and the nature of solutions of (4.4) are given in [Mit., etc. 1], [Mit., etc. 2].

Example 4.3.1. *Let*

$$G(s) = \begin{pmatrix} \dfrac{s+1}{s^3+s^2-1} & \dfrac{s(s+1)}{s^3+s^2-1} \\ \dfrac{s(s+1)}{s^3+s^2-1} & \dfrac{-2}{s^3+s^2-1} \end{pmatrix}$$

Since $m = 2$, $k = 2$, $n = 3$ then $mk = 4 > 3 = n$ and generic pole assignability may be achieved. A right coprime MFD of $G(s)$ will give

$$G(s) = \begin{pmatrix} 0 & -(s+1) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -s & 1 \\ 1 & -s(s+1) \end{pmatrix}^{-1}$$

Hence

$$C_2 \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = C_2 \begin{bmatrix} -s & 1 \\ 1 & -s(s+1) \\ 0 & -(s+1) \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s^3+s^2-1 \\ s^2+s \\ 1 \\ -(s+1) \\ -(s^2+s) \\ -(s+1) \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}$$

and

$$P = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$$

is the corresponding Plücker matrix. Let now $\{-1, -1+i, -1+i\}$ be a set of frequencies which have to be assigned as the closed loop poles of the system. Hence, $a(s) = s^3 + 3s^2 + 4s + 2$ and $\underline{a} = (2, 4, 3, 1)^t$. For these dimensions the corresponding QPR is $h_1h_6 - h_2h_5 + h_3h_4 = 0$ and the optimization problem (4.4) yields a solution $\hat{\underline{h}} = (1, 0.999, -1000, -0.999, 998.001)$. Hence $\underline{\varepsilon} = \hat{\underline{h}} \cdot P - \underline{a} = (-0.002, -0.002, -0.001, 0)^t$ and $(\underline{a}^t + \underline{\varepsilon}^t)(1, s, s^2, s^3)^t = s^3 + 2.999s^2 + 3.998s + 1.998$ which is very close to the desired polynomial $a(s)$.

Note that the optimization problem (4.4) was solved easily in the previous example due to the one QPR that describes the decomposability of the problem. In the next chapters we will present a different approach that could work not only for higher dimensions, but also it will help us derive a solution in closed-form formula for any $\underline{h} = (h_1, h_2, ...)$, $h_i \in \mathbb{R}$, without any generic solvability conditions.

4.3.2 An Unconstrained Iterative Full-Rank Algorithm

In this section we present a different numerical approach, presented in [Pa. 1], about an iterative algorithm used for the pole assignment via output feedback problem which is based on least-squares methodologies.

Let the strictly proper dynamical system

$$S(A,B,C):\begin{cases}\dot{\underline{x}}(t)=A\underline{x}(t)+B\underline{u}(t), & A\in\mathbb{R}^{n\times n},\ B\in\mathbb{R}^{n\times k}\\ \underline{y}(t)=C\underline{x}(t), & C\in\mathbb{R}^{m\times n}\end{cases} \tag{4.5}$$

where rank$C = k \leq n$. From the Polynomial Matrix Fraction Description of the (open-loop) transfer function $G(s)$ we have that $G(s) = N(s)/\Delta(s)$ where $N(s) := C \cdot \text{adj}(sI_n - A)B$ and $\Delta(s) := \det(sI_n - A)$. From the definition of the pole placement via output feedback in section 2.4 we have that the left-hand side of DAP is written as

$$\det(H \cdot M(s)) = \det\left((I_n, K) \cdot (\Delta(s), N(s))^t\right) = \det(sI_n - A + BKC)$$

The placement of the controller $H = (I, K)$ to the system, implies the (closed-loop) transfer function $\overline{G}(s) = \overline{N}(s)/\overline{\Delta}(s)$, for $\overline{N}(s) := C \cdot \text{adj}(sI_n - A + BKC)B$ and $\Delta(s) := \det(sI_n - A + BKC)$. Then

$$BKC = \sum_{i=1}^{k} \underline{b}_i \underline{\kappa}_i C \tag{4.6}$$

where $\underline{b}_i,\ \underline{\kappa}_i$ are the i column of B and the i row of K written in column and row forms respectively. Thus, the characteristic polynomial of $\overline{G}(s)$ is

$$\overline{F}(s) = \det\left(sI_n - A + \sum_{i=1}^{k} \underline{b}_i \underline{\kappa}_i C\right) \tag{4.7}$$

Using the matrix identity [Sand. 1]

$$\underline{v}_1 \text{adj}(M) \underline{v}_1 = \det(M + \underline{v}_2 \underline{v}_1) - \det(M) \tag{4.8}$$

where $M \in \mathbb{R}^{n\times n}$ and $\underline{v}_1,\ \underline{v}_2 \in \mathbb{R}^n$, (4.7) is written as

$$\overline{F}(s) = \underline{\kappa}_1 C \text{adj}\left(sI_n - A + \sum_{i=2}^{k} \underline{b}_i \underline{\kappa}_i C\right) \underline{b}_1 + \det(sI_n - A + \sum_{i=2}^{k} \underline{b}_i \underline{\kappa}_i C) \tag{4.9}$$

Repeating once more we have

$$\overline{F}(s) = \underline{\kappa}_1 C \text{adj}\left(sI_n - A + \sum_{i=2}^{k} \underline{b}_i \underline{\kappa}_i C\right) \underline{b}_1 + \underline{\kappa}_2 C \text{adj}\left(sI_n - A + \sum_{i=3}^{k} \underline{b}_i \underline{\kappa}_i C\right) \underline{b}_2 +$$

$$+ \det(sI_n - A + \sum_{i=3}^{k} \underline{b}_i \underline{\kappa}_i C)$$

By repeated application of the result of (4.8) to $\overline{F}(s)$ we obtain

$$\overline{F}(s) = \sum_{i=1}^{k} \underline{\kappa}_i L_i(s) \underline{b}_i + \det(sI_n - A) \tag{4.10}$$

where $L_i(s) = C\text{adj}\left(sI_n - A + \sum_{i=j+1}^{k} \underline{b}_i \underline{\kappa}_i C\right)$, $j = 1, ..., k-1$ and $L_m(s) = \det(sI_n - A)$. Hence,

$$\overline{F}(s) = F(s) + \sum_{i=1}^{k} \underline{\kappa}_i L_i(s) \underline{b}_i \tag{4.11}$$

It should be noted that in (4.11), only the second term on the righthand side contains the vector $\underline{\kappa}_1$. By interchanging rows of K and the corresponding columns of B, we can obtain expressions similar to (4.11) so that a particular row of K appears in only one term. We can therefore use such equations to obtain expressions in closed form for the effect of a change $\Delta\underline{\kappa}_i$ in the $\underline{\kappa}_i$ row of K, on the closed-loop characteristic polynomial $\overline{F}(s)$. For example, for a change $\Delta\underline{\kappa}_1$ in $\underline{\kappa}_1$, (4.11) becomes

$$\overline{F}(s) + \Delta\overline{F}^1(s) = F(s) + (\underline{\kappa}_1 + \Delta\underline{\kappa}_1) L_1(s) \underline{b}_1 + \sum_{i=2}^{k} \underline{\kappa}_i L_i(s) \underline{b}_i \tag{4.12}$$

From (4.11) and (4.12) we have that $\Delta\overline{F}^1(s) = \Delta\underline{\kappa}_1 L_1(s) \underline{b}_1$ and for a random i-row change we have

$$\Delta\overline{F}^i(s) = \Delta\underline{\kappa}_i) L_1^i(s) \underline{b}_i \tag{4.13}$$

where

$$L_1^i(s) = C\text{adj}\left(sI_n - A + \sum_{j=1}^{i-1} \underline{b}_j \underline{\kappa}_j C + \sum_{j=i+1}^{k} \underline{b}_j \underline{\kappa}_j C\right) \tag{4.14}$$

is a $k \times n$ polynomial matrix which is a function of all the rows of K, except the $\underline{k}_i$ row. By equating coefficients of like powers of s on both sides, equation (4.13) is written as

$$J^i \Delta\underline{\kappa}_i^t = \delta^i \tag{4.15}$$

where $J^i \in \mathbb{R}^{n \times m}$ is the matrix of coefficients obtained from the m-column polynomial vector $L_1^i(s)\underline{b}_i$ and $\delta^i \in \mathbb{R}^n$ is the vector of coefficients of $\Delta\overline{F}^i(s)$, excluding the coefficient of s^n. The least-squares solution of (4.15) in terms of $\underline{\kappa}_i^t$ is

$$\hat{\underline{\kappa}}_i^t = \left(J^i\right)^+ \delta^i \tag{4.16}$$

where $(J^i)^+$ is the Moore-Penrose matrix of J^i. Hence,

$$\Delta\overline{F}^{i+1}(s) = \Delta\overline{F}^{i}(s) - \Delta\underline{\hat{\kappa}}_i(s)\underline{b}_i \Rightarrow \delta^{i+1} = \delta^i - J^i\Delta\underline{\hat{\kappa}}_i^t$$

Thus the algorithm follows the next steps, [Pa. 1]:

i) Let $K := 0$, $\Delta\overline{F}(s) = \overline{F}(s) - F(s)$.

ii) At the i-th step calculate $L_1^i(s)$ by equation (4.14).

iii) Calculate $\Delta\underline{\hat{\kappa}}_{i+1}$ and update the $(i+1)$-row of K. Keep updating until all rows of K have been updated. Let K_F the final approximation of K.

iv) Set $K_0 = K_F$ and repeat until $||\delta||$ is sufficiently small.

v) Return K_F.

Example 4.3.2. *[Pa. 1] Let the transfer function matrix*

$$G(s) = \begin{pmatrix} \dfrac{4-s+s^2}{8+5s-2s^2+s^3} & \dfrac{-2}{8-3s+s^2} \\ \dfrac{2(-2+s)}{8+5s-2s^2+s^3} & \dfrac{-2}{8-3s+s^2} \end{pmatrix}$$

A matrix fraction description of $G(s)$ in the form $G(s) = N(s)D^{-1}(s)$ is given by

$$G(s) = \begin{pmatrix} 4-s+s^2 & -2 \\ 2(-2+s) & -2 \end{pmatrix} \begin{pmatrix} 8+5s-2s^2+s^3 & 0 \\ 0 & 8-3s+s^2 \end{pmatrix}^{-1}$$

Hence, we have

$$M(s) = \begin{pmatrix} 8+5s-2s^2+s^3 & 0 \\ 0 & 8-3s+s^2 \\ 4-s+s^2 & -2 \\ 2(-2+s) & -2 \end{pmatrix},\ H = (I_2, K) = \begin{pmatrix} 1 & 0 & k_{11} & k_{12} \\ 0 & 1 & k_{21} & k_{22} \end{pmatrix}$$

Then

$$F(s) = s^3 - 2s^2 + 5s + 8,\ \overline{F}(s) = s^3 + 30s^2 + 313s + 1014$$

Starting with $K_0 = 0$, the algorithm converges, [Pa. 1], at four iterations,

$$K_1 = \begin{pmatrix} 37.8222 & 170 \\ -119.588 & -53.5227 \end{pmatrix},\ K_2 = \begin{pmatrix} 34.583 & 13.078 \\ -108.469 & -49.7449 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} 32.0103 & 11-8729 \\ -108.408 & -49.7247 \end{pmatrix},\ K_4 = \begin{pmatrix} 32 & 11.8681 \\ -108.407 & -49.7246 \end{pmatrix}$$

Obviously, the last matrix K_4 is the desired controller.

4.4 Geometric Techniques

Geometric methodologies mostly use Schubert calculus/Enumerative Geometry, [Bro. & Byr. 2], [Kl. & Lak. 1], and projective geometry techniques, [Fal. 1], [Kar. & Gia. 5]. The latter, have the significant advantage over the former that they may interpret DAP not only as an intersection problem of hyper-surfaces on a Grassmannian (*Complex DAP*) or as an enumeration problem of the number of the intersections between a general hyper-plane and the curve describing the transfer function of the system [Sot. 2], but also provide the framework for computing real solutions for H or, if this is not possible, its best approximation, due to the QPR set that describes the corresponding Grassmann variety. These issues are discussed in the following sections.

4.4.1 Schubert Calculus

Schubert Calculus/Enumerative Geometry deals with finding the number of points, lines, planes etc., satisfying certain geometric conditions. In the case of determinantal-type assignment problems, the method is very useful since as we have explained in the introduction, these problems may be viewed as intersection problems in the affine (Grassmannians) or in the projective space (Grassmann varieties). Schubert Calculus is an appropriate tool for the study of the problem, since it has been proved that *the intersection of the Grassmann variety $G_{n,m}$ and a certain linear space in $\mathbb{P}(\mathbb{R}^n)$ forms a strictly increasing sequence of linear spaces, whose number may be determined*, [Kl. & Lak. 1].

Definition 4.4.1. *[Kl. & Lak. 1] Let $\mathcal{A}_0 \subsetneqq \mathcal{A}_1 \subsetneqq \cdots \subsetneqq \mathcal{A}_m$ be a strictly increasing sequence of $m+1$ linear spaces in $\mathbb{P}(\mathbb{R}^n)$. An m- dimensional linear space $\mathcal{L} \in \mathbb{P}(\mathbb{R}^n)$ is said to satisfy the Schubert condition if*

$$\dim(\mathcal{A}_i \cap \mathcal{L}) \geq i, \ \forall i = 0, ...m \tag{4.17}$$

The sets of all such spaces $\mathcal{L}$ are denoted as $\Omega(\mathcal{A}_0, ..., \mathcal{A}_m) \equiv \Omega(a_0, ..., a_m)$, where $\dim\mathcal{A}_i = a_i$ and in Schubert Calculus terminology are called *Schubert Varieties* and their study is important since they form minimal bases for the Grassmann varieties $G_{n,m}$. Next theorem will help us interpret DAP as an enumerative geometry problem.

Theorem 4.4.1. *[Kl. & Lak. 1] The Schubert Variety $\Omega(\mathcal{A}_0, ..., \mathcal{A}_m)$ is the intersection of the Grassmann variety $G_{n,m}$ and a certain linear space in $\mathbb{P}(\mathbb{R}^n)$. Furthermore,*

$$\Omega(a_0, ..., a_m) = \begin{vmatrix} \sigma(a_0) & \cdots & \sigma(a_0 - m) \\ \vdots & \cdots & \vdots \\ \sigma(a_m) & \cdots & \sigma(a_m - m) \end{vmatrix} \tag{4.18}$$

where $\sigma(h) = \Omega(h, n-m+1, ..., n), \ h = 0, ..., n-m$.

Eqn.(4.18) is known as *the determinantal formula of Schubert Calculus* and it can be shown that several determinantal assignment problems can be written in the form (4.18). Next, we present how this is achieved for a specific determinantal-type problem, the dynamic output pole placement problem, [Hub. & Ver. 1]. Let the system

$$S(A,B,C,D):\left\{\begin{array}{ll}\dot{\underline{x}}(t)=A\underline{x}(t)+B\underline{u}(t), & A\in\mathbb{R}^{n\times n},\ B\in\mathbb{R}^{n\times k}\\ \underline{y}(t)=C\underline{x}+D\underline{u}(t), & C\in\mathbb{R}^{m\times n},\ D\in\mathbb{R}^{m\times k}\end{array}\right. \tag{4.19}$$

with $D=0$ and the dynamic compensator described by

$$\left\{\begin{array}{ll}\dot{\underline{z}}=F\underline{z}+G\underline{y}, & F\in\mathbb{R}^{q\times q},\ G\in\mathbb{R}^{q\times k}\\ \underline{u}=H\underline{z}+K\underline{y}, & H\in\mathbb{R}^{m\times q},\ K\in\mathbb{R}^{m\times k}\end{array}\right. \tag{4.20}$$

for $\underline{z}\in\mathbb{R}^q$. After eliminating $\underline{u}$, $\underline{y}$ between the two systems, we have that

$$\left(\begin{array}{c}\dot{\underline{x}}\\ \dot{\underline{z}}\end{array}\right)=\left(\begin{array}{cc}A+BKC & BH\\ GC & F\end{array}\right)\left(\begin{array}{c}\underline{x}\\ \underline{z}\end{array}\right) \tag{4.21}$$

The behavior of this closed-loop system is determined by the $n+q$ eigenvalues of the matrix in (4.21). For a plant given by the matrix triplet (A,B,C) and $n+q$ eigenvalues, the dynamic pole placement problem asks for the matrix quadruples (F,G,H,K) which determine the dynamic compensators that yield closed-loop systems with a specific set of eigenvalues. The dynamic pole placement problem can be formulated as a geometric problem by rewriting its characteristic polynomial as follows.

$$\begin{aligned}\varphi(s)&=\det\left(s\left(\begin{array}{cc}I_n & 0\\ 0 & I_q\end{array}\right)-\left(\begin{array}{cc}A+BKC & BH\\ GC & F\end{array}\right)\right)=\\ &=\det\left(\begin{array}{cccc}sI_n-A-BKC & -BH & BK & -B\\ -GC & sI_q-F & G & 0\\ 0 & 0 & I_k & 0\\ 0 & 0 & 0 & I_m\end{array}\right)=\\ &=\det\left(\begin{array}{cccc}sI_n-A & 0 & 0 & -B\\ 0 & sI_q-F & G & 0\\ C & 0 & I_k & 0\\ 0 & -H & K & I_m\end{array}\right)=\\ &=\det\left(\begin{array}{cc}I_k & C(sI_n-A)^{-1}B\\ H(sI_q-F)^{-1}G+K & I_m\end{array}\right)\times\\ &\times\det(sI_n-A)\times\det(sI_q-F)\end{aligned} \tag{4.22}$$

The first determinant in (4.22) represents the intersection of m-dimensional vector spaces (m-planes) defined by the given triplet (A,B,C) with p- dimensional spaces determined by the unknown quadruple (F,G,H,K). As these p-planes

depend on the variable s, they have maximal minors of degree q and we call them degree q-maps. Thus the dynamic pole placement problem is equivalent to the computation of all degree q-maps into the Grassmannian of p-planes that meet n given m-planes at prescribed s-values. It is easy to see that for each specification of an eigenvalue λ_i the condition that the characteristic polynomial (4.22) vanish at $s = \lambda_i$ enforces one polynomial condition on the set of degree q-maps.

The off-diagonal entries in the first matrix in eqn.(4.22) are the transfer functions of the original system and the compensator. If we consider the coprime factorizations

$$C(sI_n - A)^{-1}B = N(s)D^{-1}(s),\ H(sI_q - F)^{-1}G + K = P(s)Q^{-1}(s)$$

and since $n,\ q$ are the respective McMillan degrees, we have

$$\det D(s) = \det(sI_n - A),\ \det Q(s) = \det(sI_q - F)$$

Hence,

$$\varphi(s) = \det \begin{pmatrix} Q(s) & N(s) \\ P(s) & D(s) \end{pmatrix} \tag{4.23}$$

The first column of this 2 by 2 block matrix represents the Hermann-Martin curve, [Mart. & Her. 1] $\gamma : \mathbb{P}^1 \to G_k(\mathbb{R}^{m+k})$ of the compensator and the second column the Hermann-Martin curve $\Gamma : \mathbb{P}^1 \to G_m(\mathbb{R}^{m+k})$ of the original system. Then the pole placement problem is interpreted as the derivation of rational curves Γ such that

$$\gamma(s_i) \cap \Gamma(s_i) \neq \emptyset,\ i = 1, ..., n + q \tag{4.24}$$

When $q = 0$ we are solving the *static pole placement problem* and we are looking for maps of degree 0 (i.e., constant maps) which meet a specific set of given m-planes. In this case the characteristic polynomial (4.22) has degree n and we can find solution planes whenever n is less than the dimension mk of the space of k-planes in $(m + k)$-dimensional space. A static compensator is then represented by the matrix

$$\begin{pmatrix} I_k \\ K \end{pmatrix}$$

whose column space is just a point in the Grassmann variety $G_k(\mathbb{R}^{m+k})$. The number of intersections of problem (4.24) has been studied in [Sot. 2], [Byr. 2], [Wan. & Ros. 2], [Bro. & Byr. 2], implying a number of conditions, such as

$$\frac{1!2! \cdots (k-1)!(mk)!}{m!(m+1)! \cdots (m+k-1)!} \tag{4.25}$$

for the number of static compensators for a generic linear system of McMillan degree mk.

Moreover the authors in [Lev. & Kar. 2] have also used Schubert calculus techniques and the cohomology of Grassmannians, i.e., intersection theory for Grassmannians to provide a non-factorial approach to the problem of pole assignment by output feedback, by calculating the height the first Whitney class of an appropriate Grassmannian, [Kl. & Lak. 1]. The method uses vector bundles (vectors parameterized by another manifold or variety which in our case is the Grassmann variety) where the Whitney classes (also known as Stiefel-Whitney classes) represent the set of the invariants of the vector bundle that describe whether the vector bundle is an everywhere independent set of sections or not and the respective height represents the least integer a such that the manifold or the variety is covered by $a+1$ open sets, each of which is contractible to a point in the manifold/variety. The advantage of their method with respect to the factorial form is that the height approach is testable for more k-input, m-output, n-state proper systems. More on this method may be found in [Lev. & Kar. 2]and [Lev. 1].

Even though this geometric method helps to the understanding DAPs nature and provides very helpful results with regard to the number of real and complex solutions (for $\mathbb{P}^1(\mathbb{R})$ or $\mathbb{P}^1(\mathbb{C})$ respectively), it does not imply the form or the construction of solutions. Next methods, show how we may approach a solution using similar geometric techniques.

4.4.2 Global Asymptotic Linearization around Singular Solutions.

Here we present the method proposed in [Lev. 1] and [Lev. & Kar. 3] for constructing pole placing real (constant and dynamic) output feedback compensators for proper plants. This is the most complete methodology for determinantal-type assignment problems regarding solvability conditions as well as construction of the solutions.

As we have already seen, equation (2.24) can be written as

$$a(s) = \det\left\{(I_n, K)\begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix}\right\} \tag{4.26}$$

where $H = (I_n, K)$ and $N_R(s)D_R^{-1}(s)$ is a coprime matrix fractional description of the corresponding transfer function of an open-loop proper system.

Definition 4.4.2. *[Bro. & Byr. 2], [Byr. & Stev. 3] Let $K \in \mathbb{R}^{k\times m}$ be a compensator as in the above analysis for a k-input, m-output, n-state system.*

i) The map $\rho(K) = (s_1, ..., s_n)$ is called the root-locus map.

ii) If ρ can be defined for all compensators K, then the system is called non-degenerate. Otherwise, if there exists at least one K for which ρ is not defined then the dynamic system is called degenerate.

iii) The points where the feedback configuration of the (composite) system has a singularity, i.e., it is not well-posed, are called degenerate points or degenerate feedback gain.

iv) The degenerate feedback gain has the form $\text{rowspan}(A,B) \in G_{k,m+k}$ *if*

$$\det\left\{(A,B)\begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix}\right\} = 0,\ \forall s \tag{4.27}$$

Next proposition, provides the means for calculating all degenerate points of a given system.

Proposition 4.4.1. *[Lev. 1] The following are equivalent.*

i) $\underline{v} = \text{rowspan}(A,B) \in \mathbb{R}^k$ *corresponds to a degenerate gain.*

ii) $\exists\ \underline{m}(s) \in \mathbb{R}^{(m+k)\times 1} : (A,B)\underline{m}(s) = 0,\ \forall s.$

iii) $\exists\ \underline{m}(s) \in \mathbb{R}^{(m+k)\times 1} : \text{rank}C_{\underline{m}(s)} \leq m$, *where* $C_{\underline{m}(s)}$ *is the matrix of coefficients of* $\underline{m}(s)$.

If we consider the composite gain sequences $S_\epsilon = (A,B) + \epsilon(A',B')$ with $\det(A + \epsilon A') \neq 0$, then $S_\epsilon \to 0$, as $\epsilon \to 0$. Thus, the corresponding DAP is formed as

$$\det\left\{(A,B) + \epsilon(A',B')\begin{pmatrix} D_R(s) \\ N_R(s) \end{pmatrix}\right\} = a_\epsilon(s) \tag{4.28}$$

Since the roots of a polynomial do not change if we multiply (or divide) by a number, it is more appropriate to consider the coefficient vector of the polynomial modulo dilations (multiplications by scalar). In this way, to examine the convergence of $a_\epsilon(s)$ when $\epsilon \to 0$, we will regard the coefficient vector $\underline{a}_\epsilon \in \mathbb{R}^{n+1}$ as a sequence $\text{span}\{\underline{a}_\epsilon \in \mathbb{P}(\mathbb{R}^n)\}$.

Theorem 4.4.2. *[Lev. 1] Let S_ϵ a sequence of finite gains converging to $\underline{v} =$* $\text{rowspan}(A,B)$. *Then the corresponding sequence of closed loop polynomial vectors* $\text{span}\{\underline{a}_\epsilon\}$ *converges to* $\text{span}\{\underline{a}\} \in \mathbb{P}(\mathbb{R}^n)$ *as* $\epsilon \to 0$. *Furthermore the function that maps the direction* (A',B') *to* $\text{span}\{\underline{a}\}$ *is linear.*

Theorem 4.4.3. *[Lev. 1] Let* $(b_{ij})_{1\leq i\leq k,\ 1\leq j\leq k+m} := (A',B')$. *Then*

$$a(s) = \sum b_{ij}a_{ij}(s),\ \underline{a} = \text{vec}(b_{ij})L_{\underline{v}} \tag{4.29}$$

where $a_{ij}(s)$ is the determinant of a $k \times k$ polynomial matrix having the same rows as $AD_R(s) + BN_R(s)$, except the i-th row which is replaced by the j-th row

of $M(s) = (D_R(s)^t, N_R(s)^t)^t$ *and* $L_{\underline{v}} \in \mathbb{R}^{k(m+k)\times(n+1)}$ *is the matrix representation of the linear function that maps the direction* b_{ij} *to the coefficient vector* $\underline{a}$ *of the initial polynomial* $a(s)$*.*

For a given degenerate point, the arbitrary prime polynomial assignability by sequences of feedback compensators converging to this point, depends readily on $L_{\underline{v}}$. In fact,

Corollary 4.4.1. *[Lev. 1] An arbitrary prime polynomial can be assigned via a sequence of feedback compensators converging to a degenerate point, if and only if* $\text{rank}L_{\underline{v}} = n+1$*. In that case, the appropriate direction can be found by solving* $\underline{a} = \text{vec}(b_{ij})L_{\underline{v}}$ *in terms of* $\text{vec}(b_{ij})$*.*

This analysis suggests the following procedure for the construction of pole placing compensators.

i) Construct a degenerate point $\underline{v} = \text{rowspan}(A, B)$.

ii) Calculate matrix $L_{\underline{v}}$.

iii) If $\text{rank}L_{\underline{v}} = n+1$, solve the linear equation $\underline{a} = \text{vec}(b_{ij})L_V$ with direction $(b_{ij})_{1\le i\le k,\ 1\le j\le k+m} := (A', B')$. Else, return to step (i).

iv) The one-parameter family $K_\epsilon = (A + \epsilon A')^{-1}(B + \epsilon B')$ of $k \times m$ matrices is the family of real constant feedback compensators placing the poles of the system at the given set, as $\epsilon \to 0$.

v) Select a small enough ϵ to approach the given closed loop pole polynomial as close as possible.

Example 4.4.1. *[Lev. 1] Let*

$$M(s) = \left(\begin{array}{cc} s^3 & 0 \\ 1 & s^2 \\ \hline 1+s^2 & s+1 \\ s+3 & s \\ s+1 & 1 \end{array}\right) = \left(\begin{array}{c} D_R(s) \\ N_R(s) \end{array}\right)$$

Since we want $\det(AD_R(s)+BN_R(s)) = 0$*, a degenerate point* $\underline{v} = \text{rowspan}(A, B)$ *is defined by*

$$(A, B) = \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right)$$

The polynomials $a_{ij}(s)$ *are given by*

$a_{11}(s) = 0,\ a_{12}(s) = -s^4 + 2s^3 + 3s^2,\ a_{13}(s) = -s^3 + s^2 + 5s + 3,\ a_{14}(s) = -s^3 + 2s^2 + 3s,$
$a_{15}(s) = -s^2 + 2s + 3,\ a_{21}(s) = 0,\ a_{22}(s) = s^5,\ a_{23}(s) = s^4 + s^3,\ a_{24}(s) = s^4,\ a_{25}(s) = s^3$

Therefore,

$$L_{\underline{v}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & 5 & 3 \\ 0 & 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where we see that rank$L_{\underline{v}} = 6$, i.e., every prime polynomial can be assigned via a certain sequence of feedback compensators converging to $\underline{v}$. Suppose we want to assign the (stable) polynomial $s^5 + 5s^4 + 10s^3 + (117/9)s^2 + (51/9)s + 1$. To achieve this, we have to solve the system

$$\mathrm{vec}(b_{ij})L_{\underline{v}} = (1, 5, 10, \frac{117}{9}, \frac{51}{9})$$

with respect to $vec(b_{ij}) = (b_{11}, ..., b_{15}, b_{21}, ..., b_{25})$. One of the solutions is

$$vec(b_{ij}) = (0, \frac{10}{3}, 0, \frac{5}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{25}{3}, \frac{15}{3})$$

which implies the direction

$$(A', B') = b_{ij} = \frac{1}{3}\begin{pmatrix} 0 & 10 & 0 & 5 & 1 \\ 0 & 3 & 0 & 25 & 15 \end{pmatrix}$$

Hence,

$$(A + B) + \epsilon(A', B') = \begin{pmatrix} 1 & \frac{10\epsilon}{3} & 0 & \frac{5\epsilon}{3} & \frac{\epsilon}{3} \\ 0 & \epsilon & 1 & \frac{25\epsilon - 3}{3} & 5\epsilon - 1 \end{pmatrix}$$

Then, the sequence of 2×3 feedback compensators is given by

$$K_\epsilon = (A + \epsilon A')^{-1}(B + \epsilon B') = \begin{pmatrix} -\frac{10}{3} & \frac{5(6 - 47\epsilon)}{9} & \frac{10 - 49\epsilon}{3} \\ \frac{1}{\epsilon} & \frac{25\epsilon - 3}{3\epsilon} & \frac{5\epsilon - 1}{\epsilon} \end{pmatrix}$$

Then

$$\lim_{\epsilon \to 0} \det\{(I_2, K)M(s)\} = a(s)$$

since

$$\det\{(I_2, K)M(s)\} = s^5 + 5s^4 + \left(10 - 191\frac{\epsilon}{9}\right)s^3 + \frac{117 - 902\epsilon}{9}s^2 + \frac{51 + 235\epsilon}{9}s + 1 + 33\epsilon$$

4.4.3 Projective Techniques: The Approximate DAP

Even though the previous geometric techniques have the important advantage that they transform DAP into an intersection problem between a linear variety and the Grassmann variety, their results concern generic assignability. In this section, we present a projective technique that implies approximate solutions which are as "close" to the real compensator as possible, regardless generic-non generic or assignability conditions. The concept of this method lies in the description of the Grassmann variety via its corresponding QPR set.

If matrices H and $M(s)$ in DAP equation (2.24) are written as

$$H = \begin{pmatrix} \underline{h}_1^t \\ \underline{h}_2^t \\ \vdots \\ \underline{h}_q^t \end{pmatrix}, \; M(s) = (\underline{m}_1(s), \underline{m}_2(s), ..., \underline{m}_q(s))$$

then, by the Binet-Cauchy Theorem, eqn.(2.24) is written as

$$C_q(H) \cdot C_q(M(s)) =< \underline{h}, \underline{m}(s) >= \sum_{\omega \in Q_{p,q}} h_\omega m_\omega(s) = a(s) \tag{4.30}$$

where $\underline{h} := \underline{h}_1^t \wedge \cdots \wedge \underline{h}_q^t \in \mathbb{R}^{1\times\binom{p}{q}}$, $\underline{m}(s) := \underline{m}_1(s) \wedge \cdots \wedge \underline{m}_q(s) \in \mathbb{R}^{\binom{p}{q}\times 1}$ and $h_\omega, m_\omega(s), \omega = (i_1, ..., i_q)$ are the coordinates of $\underline{h}, \underline{m}(s)$ respectively, for the set $Q_{p,n}$ of the strictly increasing integers p chosen from 1,...,n. Eqn. (4.30) suggests that the coefficients of $\det(H \cdot M(s))$ can be considered not only as multilinear skew-symmetric functions on the entries of H, but also as linear functions on the components h_ω. Hence, DAP is reduced to:

i) *Linear subproblem.* Suppose that $\underline{h}$ is free. Find the conditions under which vectors $\underline{h} \in \mathbb{R}^{1\times\binom{p}{q}}$ exist such that

$$< \underline{h}, \underline{m}(s) >= a(s) \Leftrightarrow \underline{h} \cdot P = \underline{a} \tag{4.31}$$

where $P \in \mathbb{R}^{\binom{p}{q}\times(\delta+1)}$ is the Plücker matrix of the vector space col-span $\{M(s)\}$, i.e., the matrix whose i-th row is formed by the coefficients of the polynomials in the i-th coordinate of $\underline{m}(s)$ and δ is the order of $M(s)$.

ii) *Multilinear subproblem.* Assume that the above linear subproblem is solvable and $\mathcal{K} \subseteq \mathbb{P}^{\binom{p}{q}-1}(\mathbb{R})$ is the family of the solution vectors $\underline{h}$ of (4.31). Then find whether there exists $\underline{h} \in \mathcal{K}$ such that $\underline{h}$ is decomposable, i.e., solve the equation

$$\underline{h} = \underline{h}_1 \wedge \cdots \wedge \underline{h}_q \tag{4.32}$$

or equivalently, find which $\underline{h}$ satisfies the set of QPR. If such vector exists, determine the matrix $H \in \mathbb{R}^{q\times p}$ such that $C_q(H) = c\underline{h}, \; c \in \mathbb{R}^*$.

Therefore, since the decomposability of the multi-linear problem is equivalent to finding which $\underline{h}$ satisfies the set of QPR, [Hod. & Ped. 1], [Mar. 1], if $\tilde{H}$ is the set of solutions of (2.24), then

$$\tilde{H} = \mathcal{K} \cap G_q(\mathbb{R}^n) \tag{4.33}$$

If intersection (4.33) is empty, the second best approach to the solution of DAP, is the calculation of the closest point of the Grassmann variety to $\mathcal{K}$, i.e.,

$$\min_{\underline{h} \in G_q(\mathbb{R}^n)} \text{dist}(\underline{h}, \mathcal{K}) \tag{4.34}$$

where $\mathcal{K}$ is described by the solutions of the linear system (4.31) and $G_q(\mathbb{R}^n)$ by the set of QPR. Eqn.(5.1) represents a relaxation of the exact intersection problem, and it is referred to as the *approximate DAP*, which makes sense as long as the stability of the resulting polynomial that is produced can be guaranteed. This extension makes the investigation relevant to problems where there are no real intersections and thus approximate solutions are sought, which is the main purpose of this thesis.

Remark 4.4.1. *Note that in [Lev. & Kar. 2] DAP was connected for the first time with the Schubert calculus methodology as well, by defining the feedback matrices that correspond to DAP as representations of a special sequence of subvarieties of the Grassmann variety, the so-called Schubert varieties.*

4.5 Conclusions

The main purpose of this chapter was to review a number of well-known techniques to solve the Determinantal Assignment Problem (DAP), to discuss its difficulties and therefore introduce the Approximate DAP. We reviewed all relevant approaches and results, from the algebraic and geometric point of view and we applied them on several forms of DAP, such as the pole assignment by dynamic output feedback, among others. We have also provided all necessary unifying terminology and background definitions, before any applications and as we have seen, the immediate methods, such as the Gröbner basis, are useful only for lower dimensions, the algebraic methods usually use an algorithmic approach to the problem and geometric techniques are based on topological or algebrogeometric intersection theory.

The latter category has produced a number of results which are rather qualitative and oriented towards a search for generic solvability conditions. The approximate DAP, has been defined as an alternative method to overcome these genericity issues; instead of solving DAP when a certain property holds for all systems of k-inputs, m-outputs and n-states except, possibly, for some belonging

to a "negligible" set, we try to calculate the controller that best approximates the initial compensator. Note that, in [Lev. 1] the genericity of a system has been also defined (in an algebrogeometric aspect) as a subvariety of strictly lower dimension, of the set of systems which is given the structure of a variety. On the other hand, the Schubert Calculus/Enumerative Geometry techniques may help us recognize the number of intersections between such varieties but lack on the construction of solutions, since they do not get advantage of the QPR set that defines the Grassmann variety which may in turn write the solution being sought as $\underline{h}_1 \wedge \underline{h}_1 \cdots \wedge \underline{h}_r$ (Schubert Calculus and Geometric techniques deal with the existence of generic solutions and they do not provide computations). This can be surpassed by the so called Global Asymptotic Linearization method of the pole placement map of a proper system or by the construction of approximate solutions of the initial DAP that the general theory of Projective Geometry provides.

The Projective techniques as we saw, are based on the Plücker embedding - a natural embedding for determinantal problems. In [Lev. 1], the Plücker embedding was used to find necessary and sufficient conditions for the DAP, to parameterize the set (variety) of linear time, invariant systems of k-inputs, m-outputs and n-states as an algebraic manifold and to produce new system invariants for the solution of generic and exact problems. In this thesis, we will use the Projective techniques to construct approximate solutions of DAP, when all the above procedures fail or when their verification is difficult, due to the complexity of the problem.

Chapter 5

Minimum Distance of a 2-vector from the Grassmann varieties $G_2(\mathbb{R}^n)$

5.1 Introduction

As we have already mentioned, the solution of the optimization problem

$$\min_{\underline{h}\in G_q(\mathbb{R}^n)} \text{dist}(\underline{h}, \mathcal{K}) \tag{5.1}$$

where $\mathcal{K}$ is described by the solutions of the linear system (4.31) in the linear subproblem of DAP and $G_q(\mathbb{R}^n)$ by the set of QPR, referred to as the *approximate DAP* is the central objective of this thesis, as well as the stability properties of the "approximate" polynomial $\hat{a}(s)$ that corresponds to an approximation $\hat{\underline{h}}$ which are very important for the perturbed solutions to be acceptable. Our approach views the problem as a minimization problem between a solution $\underline{z}(\underline{x})$ of the linear subproblem (4.31) and the Grassmann variety $G_2(\mathbb{R}^n)$, i.e.,

$$\min_{\underline{x}} g(\underline{z}(\underline{x}), G_2(\mathbb{R}^n)) \tag{5.2}$$

where g is an appropriate gap function implied by the gap metric gap$(\cdot)$, [Wey. 1], between the parameterized multivector $\underline{z}(\underline{x})$ in the projective space and the Grassmann variety with $\underline{x}$ being the vector whose components are the free parameters of (4.31) that describe $\mathcal{K}$. In order to solve problem (8.2) we need at first to solve

$$\min_{\underline{z}} g(\underline{z}, G_2(\mathbb{R}^n)) \tag{5.3}$$

when $\underline{z}$ is fixed, i.e., least distance of a *point* from the Grassmann variety.

In this chapter, we examine problem (5.3) and the methodology used for the computation of the minimizer will be based on the *best decomposable approximation of multivectors*, [Kar. & Lev. 9]. Note that the results from the solution of (5.3) in this chapter will be applied to the solution of problem $\min_{\underline{h}\in G_q(\mathbb{R}^n)} \text{dist}(\underline{h}, \mathcal{K})$ in Chapter 8, where we will expand the results obtained for the fixed 2-vector $\underline{z}$ to $\underline{z}(\underline{x})$, the parameterized vector describing the linear variety $\mathcal{K}$.

Our approach here uses for the first time the spectral analysis of the matrix representation of the given multivector $\underline{z}$ which implies the derivation of the QPR set in an analytical form providing all results in closed-form formulae without generic or exact solvability conditions.

More analytically in Section 5.2, the least distance of a point $\underline{z}$ from the Grassmann variety $G_2(\mathbb{R}^4)$ is studied, i.e.,

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \text{when} \quad \underline{z} \in \bigwedge^2(\mathbb{R}^4) \text{ is a fixed vector and } \underline{x} \text{ is decomposable} \tag{5.4}$$

This starting case will enable us to understand the nature of the problem and its difficulties as the dimensions of the Grassmann varieties increase and it will help us determine the tools which are going to be used for the general case $G_2(\mathbb{R}^n)$. In this chapter, we also present the first important result of this thesis; the minimizer of (5.4), which is calculated via the Lagrange multipliers method, coincides with the least singular value of the related Grassmann matrix $\Phi_{\underline{z}}$.

In Section 5.3 we study problem (5.4) for the $\bigwedge^2(\mathbb{R}^n)$ and $\bigwedge^{n-2}(\mathbb{R}^n)$ cases and it is divided in three parts; In 5.3.1, the prime decomposition of a fixed 2-vector is introduced, based on the spectral analysis of the skew-symmetric matrix-form of the multivector, along with some properties which may imply a number of important results, such as the derivation of the QPR set in the form $\underline{z} \wedge \underline{z} = \underline{0}$. Moreover, the prime decomposition is applied for the solution of the distance problem:

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \text{when} \quad \underline{z} \in \bigwedge^2(\mathbb{R}^n) \text{ is a fixed vector and } \underline{x} \text{ is decomposable} \tag{5.5}$$

In section 5.3.2 the Hodge dual problem of (5.5) is studied, i.e.,

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \text{when} \quad \underline{z} \in \bigwedge^{n-2}(\mathbb{R}^n) \text{ is a fixed vector and } \underline{x} \text{ is decomposable} \tag{5.6}$$

with the use of the properties of the Hodge-$\star$ operator. In 5.3.3 we generalize the result of Section 5.2 and we prove the very significant result that the least singular value of the Grassmann matrix is equal to the minimizer of (5.5), not

only for the $\bigwedge^2(\mathbb{R}^4)$ case, but also for $\bigwedge^2(\mathbb{R}^n)$. In the same section, we prove that the the skew-symmetric matrix $T_{\underline{z}}$ is a special case of the Grassmann matrix, [Kar. & Gia. 6], [Kar. & Lev. 9].

In Section 5.4 problem (5.5) is studied in the projective space; Since, DAP requires a procedure where the poles of the systems are viewed independently of their respective polynomials, i.e., $p(s) = s - s_1$ and $q(s) = as - as_1$ are equivalent, due to the same pole s_1, any distances should be considered in a relevant projective space $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$, rather than the affine space $\mathcal{A}^n$ or $\mathbb{R}^n$. In 5.4.1 we define the natural metric between two "points"(representatives of the respective equivalent classes) of the projective space, i.e., the gap metric, and the gap function g between a "point" and a subspace of $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$ which we will use to calculate the distance between a point and the Grassmann variety $G_2(\mathbb{R}^n)$. Furthermore, we provide a unifying representation of the different forms of metrics used in $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$ and $G_2(\mathbb{R}^n)$ in terms of the gap metric.

Finally, in Section 5.4.2, we solve the key optimization problem of this thesis $\min_{\underline{z}} g(\underline{z}, G_2(\mathbb{R}^n))$ when $\underline{z}$ is fixed and we provide some simplifications of the solutions in the case of specific Grassmann varieties.

5.2 The $\bigwedge^2(\mathbb{R}^4)$ case

In this section the problem of the least distance of a multivector in $\bigwedge^2(\mathbb{R}^4)$ from the set of decomposable vectors in $\bigwedge^2(\mathbb{R}^4)$, which is denoted by $D_{\wedge^2(\mathbb{R}^4)}$ is studied, i.e.,

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \text{when} \quad \underline{z} \in \bigwedge^2(\mathbb{R}^4) \text{ is a fixed vector and } \underline{x} \text{ is decomposable} \tag{5.7}$$

Equivalently, the constraint concerning the decomposability of $\underline{x}$ is written as $QPR(\underline{x}) = 0$. This case provides the first straight-forward results for constrained optimization problems over the manifolds and will help us not only for the solution of the problem in the respective projective space and $G_2(\mathbb{R}^n)$ but also to overcome the difficulties that occur in higher dimensions.

Lemma 5.2.1. *The Quadratic Plücker Relations (QPR) that describe $D_{\wedge^2(\mathbb{R}^4)}$ as well as the Grassmann variety $G_2(\mathbb{R}^4)$ of the projective space $\mathbb{P}^5(\mathbb{R})$ are given by the single QPR*

$$c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} = 0 \tag{5.8}$$

Proof. Let the vector space $\mathcal{U}$ with $\dim\mathcal{U} = 4$ and a 2-dimensional subspace $\mathcal{V}$ of $\mathcal{U}$. Then a basis $\{\underline{v}_1, \underline{v}_2\}$ of $\mathcal{V}$ can be extended to a basis $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ of $\mathcal{U}$.

Hence, any vector $\underline{v}$ in $\bigwedge^2(\mathcal{U})$ is written as

$$\underline{v} = c\underline{v}_1 \wedge \underline{v}_2 = \sum_{1\leq i<j\leq 4} c_{ij}\underline{v}_i \wedge \underline{v}_j \tag{5.9}$$

where c_{ij} is a set of Plücker coordinates of $\mathcal{V}$. It can be proved using several Exterior Algebra techniques, e.g., [Lev., etc. 9],[Gee. 1] that any 2-vector $\underline{v}$ is decomposable if and only if $\underline{v} \wedge \underline{v} = \underline{0}$. Later in this chapter, the condition $\underline{v}\wedge\underline{v} = 0$ will be proved in a rather straight-forward way, via a specific multivector decomposition, called the prime decomposition. Thus

$$\begin{aligned}
&\underline{v} \wedge \underline{v} = 0 \Leftrightarrow \left(\sum_{1\leq i<j\leq 4} c_{ij}\underline{v}_i \wedge \underline{v}_j\right) \wedge \left(\sum_{1\leq i<j\leq 4} c_{ij}\underline{v}_i \wedge \underline{v}_j\right) = 0 \Leftrightarrow \\
&c_{12}c_{34}\underline{v}_1 \wedge \underline{v}_2 \wedge \underline{v}_3 \wedge \underline{v}_4 + \cdots + c_{34}c_{12}\underline{v}_3 \wedge \underline{v}_4 \wedge \underline{v}_1 \wedge \underline{v}_2 = 0 \Leftrightarrow \\
&2(c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23})\underline{v}_1 \wedge \underline{v}_2 \wedge \underline{v}_3 \wedge \underline{v}_4 = 0
\end{aligned}$$

Since $\underline{v}_1 \wedge \underline{v}_2 \wedge \underline{v}_3 \wedge \underline{v}_4 \neq \underline{0}$ (because $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ are linearly independent) we obtain eqn.(5.8), which is the Quadratic Plücker Relation that defines a hypersurface in the 5-dimensional projective space $\mathbb{P}^5(\mathbb{R})$, i.e., the Grassmann variety $G_2(\mathbb{R}^4)$. □

Due to the single QPR (5.8), the solution of (5.7) can be directly derived by classical Lagrangian theory and the singular values of the Grassmann matrix.

Let $\underline{z} = (a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{34}) \in \bigwedge^2(\mathbb{R}^4)$. The Grassmann matrix of $\underline{z}$ is given by

$$\Phi(\underline{z}) = \begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{pmatrix}$$

The singular values of $\Phi(\underline{z})$ can be calculated in terms of $\underline{z}$ as follows.

Lemma 5.2.2. *[Kar. & Lev. 9] The four singular values $\sigma_4 \geq \sigma_3 \geq \sigma_2 \geq \sigma_1$ of $\Phi(\underline{z})$ satisfy:*

$$\sigma_1^2 = \sigma_2^2 = \frac{\|\underline{z}\|^2 - \sqrt{\|\underline{z}\|^4 - 4QPR(\underline{z})^2}}{2} \tag{5.10}$$

$$\sigma_3^2 = \sigma_4^2 = \frac{\|\underline{z}\|^2 + \sqrt{\|\underline{z}\|^4 - 4QPR(\underline{z})^2}}{2} \tag{5.11}$$

Proof. The squared singular values of $\Phi(\underline{z})$ can be calculated as the eigenvalues of the matrix $\Phi(\underline{z}) \cdot \Phi(\underline{z})^t$. Therefore,

$$\det\left(\lambda I_4 - \Phi(\underline{z}) \cdot \Phi(\underline{z})^t\right) = (\lambda^2 - \sum_{1\leq i<j\leq 4} (a_{ij} \cdot \lambda) + a_{12}^2a_{34}^2 + a_{13}^2a_{24}^2 + a_{14}^2a_{23}^2 +$$

$$+ 2a_{12}a_{34}a_{14}a_{23} - 2a_{12}a_{34}a_{13}a_{24} - 2a_{13}a_{24}a_{14}a_{24})^2 =$$
$$= \left(\lambda^2 - \|\underline{z}\|^2\lambda + QPR(\underline{z})^2\right)^2$$

Thus, the four roots of the characteristic polynomial are two pairs of double roots. The first pair corresponds to the smallest root of $\lambda^2 - \|\underline{z}\|^2\lambda + QPR(\underline{z})^2 = 0$ and the second one, to the lowest root of the same equation. Thus (5.10), (5.11) hold true. □

Lemma 5.2.3. *The two lowest singular values of $\Phi(\underline{z})$ are zero, if, and only if $\underline{z}$ is decomposable, in which case $\sigma_3 = \sigma_4 = \|\underline{z}\|$.*

Proof. This follows directly from (5.10). □

Theorem 5.2.1. *The solution of the optimization problem* (5.7) *is given by*

$$\underline{x}_0 = \frac{1}{1 - \frac{\sigma_2^4}{QPR^2(\underline{z})}} \cdot \left(I_6 - \frac{\sigma_2^2}{QPR(\underline{z})} \cdot J\right) \underline{z} \tag{5.12}$$

where σ_2 is the second singular value of the Grassmann matrix $\Phi(\underline{z})$ and J is given by

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proof. The Lagrangian for the minimization problem (5.7) is given by

$$\sum_{i=1}^{6}(x_i - z_i)^2 + \lambda\left(x_1x_6 - x_2x_5 + x_3x_4\right) \tag{5.13}$$

The first order conditions are given by

$$2(x_1 - z_1) + \lambda x_6 = 0,\ 2(x_2 - z_2) - \lambda x_5 = 0,\ 2(x_3 - z_3) + \lambda x_4 = 0,$$
$$2(x_4 - z_4) + \lambda x_3 = 0,\ 2(x_5 - z_5) - \lambda x_2 = 0,\ 2(x_6 - z_6) + \lambda x_1 = 0$$

and $QPR(\underline{x}) = 0$. The first six equations can be written collectively

$$\left(I_6 + \frac{\lambda}{2}J\right)\underline{x} = \underline{z} \tag{5.14}$$

But since $J^2 = I_6$, we have that

$$\left(I_6 + \frac{\lambda}{2}J\right)^{-1} = \frac{1}{1 - \frac{\lambda^2}{4}} \cdot \left(I_6 - \frac{\lambda}{2}J\right) \tag{5.15}$$

Therefore, the solution of the first order conditions is given by (5.14):

$$\underline{x} = \frac{1}{1 - \frac{\lambda^2}{4}} \cdot \left(I_6 - \frac{\lambda}{2} J \right) \underline{z} \tag{5.16}$$

If we substitute (5.16) to $QPR(\underline{x}) = 0$ we get

$$QPR(\underline{z})\lambda^2 - 2\|\underline{z}\|^2\lambda + 4QPR(\underline{z}) = 0 \tag{5.17}$$

which provides two solutions in terms of λ:

$$\lambda_{1,2} = \frac{\|\underline{z}\|^2 \mp \sqrt{\|\underline{z}\|^4 - 4QPR(\underline{z})^2}}{QPR(\underline{z})}$$

or equivalently

$$\lambda_1 = \frac{2\sigma_2^2}{QPR(\underline{z})},$$

$$\lambda_2 = \frac{2\sigma_4^2}{QPR(\underline{z})}$$

This gives rise to two possible solutions:

$$\underline{x}_1 = \frac{1}{1 - \frac{\sigma_2^4}{QPR^2(\underline{z})}} \cdot \left(I_6 - \frac{\sigma_2^2}{QPR(\underline{z})} \cdot J \right) \underline{z} \tag{5.18}$$

or

$$\underline{x}_2 = \frac{1}{1 - \frac{\sigma_4^4}{QPR^2(\underline{z})}} \cdot \left(I_6 - \frac{\sigma_4^2}{QPR(\underline{z})} \cdot J \right) \underline{z} \tag{5.19}$$

The minimization problem (5.7) has a global minimum (in the projective space $\mathbb{P}^5(\mathbb{R})$) which defines the direction of the smooth projective variety defined by the equation $QPR(\text{sp}\{\underline{z}\}) = 0$ with the smallest angle from $\text{sp}\{\underline{z}\}$. Lifting it to $\mathbb{R}^6$, this direction a line of smooth points of the variety defined by $QPR(\underline{z}) = 0$ in $\mathbb{R}^6$ (except $\underline{0}$ which is not obviously a solution) and therefore the first order conditions have to be satisfied. Thus, the global minimizer is given by (5.18) or (5.19). It is sufficient therefore, to find which vector out of $\underline{x}_1$ and $\underline{x}_2$ has the smallest distance from $\underline{z}$. To this end we set

$$a := \frac{\lambda}{2}$$

where λ is any of the roots of equation (5.17). Hence, the number a satisfies the following:

$$QPR(\underline{z})a^2 - 2\|\underline{z}\|^2 a + 4QPR(\underline{z}) = 0 \tag{5.20}$$

If a_1, a_2 are the roots of (5.20) we have that $a_1 \cdot a_2 = 1$ (a_1, a_2 correspond to the lowest and the largest singular values, respectively). Therefore, the distances between $\underline{x}_1$ and $\underline{x}_2$ from $\underline{z}$ are

$$\|\underline{x}_1 - \underline{z}\|^2 = \left\| \frac{1}{1-a_1^2} \cdot (I_6 - a_1 \cdot J)\, \underline{z} - \underline{z} \right\|^2,$$

$$\|\underline{x}_2 - \underline{z}\|^2 = \left\| \frac{1}{1-a_2^2} \cdot (I_6 - a_2 \cdot J)\, \underline{z} - \underline{z} \right\|^2$$

But generally

$$\left\| \frac{1}{1-a^2} \cdot (I_6 - a \cdot J)\, \underline{z} - \underline{z} \right\|^2 = \left\| \left(\frac{1}{1-a^2} - 1 \right) \underline{z} - \frac{a}{1-a^2} J\underline{z} \right\|^2 =$$
$$= \frac{a^2}{(1-a^2)^2} \|a\underline{z} - J\underline{z}\|^2$$

Since $a_1 \cdot a_2 = 1$ we have that

$$\frac{a_1^2}{(1-a_1^2)^2} = \frac{a_2^2}{(1-a_2^2)^2}$$

Therefore, it is sufficient to check which of the two distances $\|a_1\underline{z} - J\underline{z}\|^2$ and $\|a_2\underline{z} - J\underline{z}\|^2$ has the smallest value. Now, we have

$$\|a\underline{z} - J\underline{z}\|^2 = \underline{z}^t (aI_6 - J)(aI_6 - J)\underline{z} =$$
$$= (a^2+1)\|\underline{z}\|^2 - 4aQPR(\underline{z})$$

Since a satisfies (5.20) we have that

$$a^2 + 1 = \frac{\|\underline{z}\|^2}{QPR(\underline{z})} a$$

Therefore,

$$\|a\underline{z} - J\underline{z}\|^2 = \left(\frac{\|\underline{z}\|^4}{QPR(\underline{z})} - 4QPR(\underline{z}) \right) a =$$
$$= \frac{\|\underline{z}\|^4 - 4QPR^2(\underline{z})}{QPR(\underline{z})} a$$

Hence, $\|\underline{z}\|^4 - 4QPR^2(\underline{z}) \geq 0$ and since $a_1 = \lambda_1/2,\ a_2 = \lambda_2/2$ we have

$$\|a_1\underline{z} - J\underline{z}\|^2 = \frac{\|\underline{z}\|^4 - 4QPR^2(\underline{z})}{QPR^2(\underline{z})} \sigma_2^2 \leq$$
$$\leq \frac{\|\underline{z}\|^4 - 4QPR^2(\underline{z})}{QPR^2(\underline{z})} \sigma_4^2 =$$

$$= \|a_2\underline{z} - J\underline{z}\|^2$$

Thus, $\underline{x}_1$ corresponds to the global minimum of the optimization problem (5.7). □

Next theorem is the main result for the solution of problem (5.7). It depicts an alternative solution to (5.7), since, as it states, the distance of the given vector $\underline{z}$ from $D_{\wedge^2(\mathbb{R}^4)}$ is exactly equal to the second (the smallest) singular value of the Grassmann matrix $\Phi(\underline{z})$.

Theorem 5.2.2. *The distance between $\underline{z} \in \bigwedge^2(\mathbb{R}^4)$ and $D_{\wedge^2(\mathbb{R}^4)}$ defined by the equation QPR($\underline{x}$)=0, is exactly the second singular value of the Grassmann matrix $\Phi(\underline{z})$.*

Proof. We only need to calculate the norm $\|\underline{x}_1 - \underline{z}\|$. Using the previous theorem, we have that

$$\|\underline{x}_1 - \underline{z}\|^2 = \frac{a_1^2}{\left(1 - a_1^2\right)^2} \frac{\|\underline{z}\|^4 - 4QPR^2(\underline{z})}{QPR(\underline{z})} a_1 \tag{5.21}$$

But

$$\frac{a_1^2}{\left(1 - a_1^2\right)^2} = \frac{1}{\left(a_1 - \frac{1}{a_1}\right)^2}$$

and $a_1 = \left(\|\underline{z}\|^2 - \sqrt{\|\underline{z}\|^4 - 4QPR(\underline{z})^2}\right)/2QPR(\underline{z})$. Hence

$$\frac{1}{\left(a_1 - \frac{1}{a_1}\right)^2} = \frac{\|\underline{z}\|^4 - 4QPR^2(\underline{z})}{QPR^2(\underline{z})} \tag{5.22}$$

If we substitute equation (5.22) to (5.21) we have that

$$\|\underline{x}_1 - \underline{z}\|^2 = a_1 QPR(\underline{z}) = \frac{\lambda_1}{2} QPR(\underline{z}) =$$
$$= \frac{\sigma_2^2}{QPR(\underline{z})} QPR(\underline{z}) = \sigma_2^2$$

which proves the result. □

Example 5.2.1. *Let $\mathbb{R}^6 \simeq \wedge^2(\mathbb{R}^4) \ni \underline{z} = (10, 2, 15, 3, 1, -20)$. Then $\underline{z}$ is not decomposable since $QPR(\underline{z}) = -157 \neq 0$ or $\sigma_2 = 5.91722 \neq 0$, if we calculate the singular values of the corresponding Grassmann matrix $\Phi(\underline{z})$. The decomposable multi-vector $\underline{x}_1$ that has the least distance from $\underline{z}$ is defined as:*

$$\underline{x}_1 = \frac{1}{1 - \frac{\sigma_2^4}{QPR^2(\underline{z})}} \cdot \left(I_6 - \frac{\sigma_2^2}{QPR(\underline{z})} \cdot J\right) \underline{z} =$$
$$= (5.82963, 1.86999, 16.4892, 6.67734, 0.582963, -18.6999)$$

Then

$$\|\underline{z} - \underline{x}_1\| = 5.91722 = \sigma_2$$

5.3 Minimization in $\bigwedge^2(\mathbb{R}^n)$ and $\bigwedge^{n-2}(\mathbb{R}^n)$

The solution of problem (5.7) has been achieved due to the description of $G_2(\mathbb{R}^4)$ by the single QPR (5.8) which has allowed the direct calculation of the Lagrange multipliers. When one more dimension is added to the problem, i.e., $\bigwedge^2(\mathbb{R}^5)$, the respective minimization problem becomes instantly more difficult, since the decomposability criterion $\underline{v} \wedge \underline{v} = 0 \Leftrightarrow \underline{v}$ is decomposable (or the QPR formula (3.32)) implies 5 QPR. Indeed, if a multivector $\underline{z} = (z_1, z_2, z_3, ..., z_{10}) \in \bigwedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$ is written as

$$\begin{aligned}\underline{z} = z_1 e_1 \wedge e_2 + z_2 e_1 \wedge e_3 + z_3 e_1 \wedge e_4 + z_4 e_1 \wedge e_5 + z_5 e_2 \wedge e_3 + \\ + z_6 e_2 \wedge e_4 + z_7 e_2 \wedge e_5 + z_8 e_3 \wedge e_4 + z_9 e_3 \wedge e_5 + z_{10} e_4 \wedge e_5\end{aligned}$$

then $\underline{z}$ is decomposable if and only if:

$$\begin{aligned}\underline{z} \wedge \underline{z} = 0 \Leftrightarrow 2(z_1 z_8 - z_2 z_6 + z_3 z_5) e_1 \wedge e_2 \wedge e_3 \wedge e_4 + 2(z_1 z_9 - z_2 z_7 + z_4 z_5) e_1 \wedge e_2 \wedge e_3 \wedge \\ \wedge e_5 + 2(z_1 z_{10} - z_3 z_7 + z_4 z_6) e_1 \wedge e_2 \wedge e_4 \wedge e_5 + 2(z_2 z_{10} - z_3 z_9 + z_4 z_8) e_1 \wedge e_3 \wedge e_4 \wedge \\ \wedge e_5 + 2(z_5 z_{10} - z_6 z_9 + z_7 z_8) e_2 \wedge e_3 \wedge e_4 \wedge e_5 = 0\end{aligned}$$

The second order homogeneous equations

$$z_1 z_8 - z_2 z_6 + z_3 z_5 = 0, z_1 z_9 - z_2 z_7 + z_4 z_5 = 0, z_1 z_{10} - z_3 z_7 + z_4 z_6 = 0,$$

$$z_2 z_{10} - z_3 z_9 + z_4 z_8 = 0, z_5 z_{10} - z_6 z_9 + z_7 z_8 = 0$$

are the *Quadratic Plücker Relations*(QPR) [Hod. & Ped. 1] that define the Grassmann variety of the projective space $P^9(\mathbb{R})$. Clearly, the Lagrange multipliers method can not be directly implemented as in case $G_2(\mathbb{R}^4)$. In this section, we present how the minimization problem

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \text{when} \quad \underline{z} \in \bigwedge^2(\mathbb{R}^n) \text{ is a fixed 2-vector and } \underline{x} \text{ is decomposable} \tag{5.23}$$

is solved, via *best decomposable approximation of multivectors* methods, [Kar. & Lev. 9], where $\underline{z}$ is decomposed into a sum of decomposable 2-vectors.

5.3.1 The Prime Decomposition of 2-vectors and Least Distance in $\bigwedge^2(\mathbb{R}^n)$

In this section, we connect a 2-vector $\underline{z}$ with its corresponding skew-symmetric matrix, defined as $T_{\underline{z}}$. The spectral analysis of $T_{\underline{z}}$ will imply the best decomposable 2-vector of $\underline{z}$ which we will refer as the *prime decomposition*.

Definition 5.3.1. *Let* $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$, $\underline{y}_1,\ \underline{y}_2 \in \mathbb{R}^n$ *and* $f(\underline{y}_1, \underline{y}_2) =< \underline{z}, \underline{y}_1 \wedge \underline{y}_2 >$ *be a bilinear form, from* $\mathbb{R}^n \times \mathbb{R}^n$ *to* $\mathbb{R}$ *with respect to* $\underline{y}_1,\ \underline{y}_2$. *We define as* $T_{\underline{z}} \in \mathbb{R}^{n\times n}$ *the matrix representation of* f, *i.e.,*

$$< \underline{z}, \underline{y}_1 \wedge \underline{y}_2 >= \underline{y}_1^t \cdot T_{\underline{z}} \cdot \underline{y}_2 \tag{5.24}$$

Next lemma connects the components of the 2-vector $\underline{z}$ with the entries of $T_{\underline{z}}$ where its skew-symmetry is implied.

Lemma 5.3.1. *For every multivector* $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$, $T_{\underline{z}}$ *is given by*

$$T_{\underline{z}} = \begin{pmatrix} 0 & z_{12} & z_{13} & \cdots & z_{1,n} \\ -z_{12} & 0 & z_{23} & \cdots & z_{2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -z_{1,n-1} & \cdots & -z_{n-2,n-1} & 0 & z_{n-1,n} \\ -z_{1,n} & -z_{2,n} & \cdots & -z_{n-1,n} & 0 \end{pmatrix} \tag{5.25}$$

Proof. If $\{\underline{e}_i \wedge \underline{e}_j\}_{1\le i<j\le n}$ an orthonormal basis of $\bigwedge^2(\mathbb{R}^n)$ then we have that $\underline{z} = \sum_{1\le i<j\le n} z_{ij}\underline{e}_i \wedge \underline{e}_j$, where z_{ij} are the components of $\underline{z}$ known as *Plücker coordinates*, [Hod. & Ped. 1]. Therefore,

$$f(\underline{e}_i, \underline{e}_j) =< \underline{z}, \underline{e}_i \wedge \underline{e}_j >= z_{ij},\ f(\underline{e}_j, \underline{e}_i) =< \underline{z}, \underline{e}_j \wedge \underline{e}_i >= -z_{ij}$$

and $f(\underline{e}_i, \underline{e}_i) = 0$. Hence, $T_{\underline{z}}$ readily follows. □

Lemma 5.3.2. *The eigenvalues of* $T_{\underline{z}}$ *are purely imaginary.*

Proof. Let $T_{\underline{z}}\underline{v} = \lambda\underline{v}$, where $\underline{v}$ is an eigenvector of $T_{\underline{z}}$ and A^* denote the conjugate transpose of A, i.e., $A^* = \bar{A}^t$. Then, since $T_{\underline{z}}$ is real and skew-symmetric, i.e., $T_{\underline{z}}^t = -T_{\underline{z}}$, we have that

$$\begin{aligned}
&T_{\underline{z}}\underline{v}^* = \lambda^*\underline{v}^* \Rightarrow \\
&(T_{\underline{z}}\underline{v}^*)^t\underline{v} = (\lambda^*\underline{v}^*)^t\underline{v} \Rightarrow \\
&(\underline{v}^*)^t\, T_{\underline{z}}^t\underline{v} = \lambda^*\,(\underline{v}^*)^t\,\underline{v} \Rightarrow \\
&(\underline{v}^*)^t\,(-T_{\underline{z}})\underline{v} = \lambda^*\,(\underline{v}^*)^t\,\underline{v} \Rightarrow \\
&(\underline{v}^*)^t\,(-\lambda)\underline{v} = \lambda^*\,(\underline{v}^*)^t\,\underline{v} \Rightarrow \\
&-\lambda\|\underline{v}\|^2 = \lambda^*\|\underline{v}\|^2 \Rightarrow \\
&-\lambda = \lambda^*
\end{aligned}$$

□

Remark 5.3.1. *Lemma 5.3.2 and standard spectral analysis methods for skew-symmetric matrices, [Bel. 1], [Gan. 1], imply that* $T_{\underline{z}}$ *has* $k := [n/2]$ *imaginary eigenvalues, i.e.,* $\pm i\sigma_1, ..., \pm i\sigma_k$ *with* $\sigma_k \ge \sigma_{k-1} \ge \cdots \ge \sigma_1 \ge 0$, *corresponding to the complex eigenvectors* $\underline{e}_{2k} \pm i\underline{e}_{2k-1}, ..., \underline{e}_2 \pm i\underline{e}_1$ *when* $n = 2k$ *and* $0, \pm i\sigma_1, ..., \pm i\sigma_k$, $\underline{e}_{2k+1}, \underline{e}_{2k} \pm i\underline{e}_{2k-1}, ..., \underline{e}_2 \pm i\underline{e}_1$ *when* n *is odd, where* $\{\underline{e}_j\}_{j=1}^{2k}$, $\{0, \underline{e}_j\}_{j=1}^{2k+1}$ *are orthonormal basis for* $\mathbb{R}^n$ *when* $n = 2k,\ n = 2k+1$, *respectively.*

For the rest of this thesis, σ_i - whenever they are used - they will denote the imaginary parts of the eigenvalues of $T_{\underline{z}}$ and $\underline{e}_i$ the corresponding vectors obtained by the eigenvectors of $T_{\underline{z}}$. Using the above results and Remark 5.3.1, the following proposition is obtained.

Proposition 5.3.1. *Any* $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$ *is decomposed as:*

$$\underline{z} = \sigma_k \underline{x}_k + \sigma_{k-1}\underline{x}_{k-1} + ... + \sigma_1 \underline{x}_1 \tag{5.26}$$

where $\underline{x}_k := \underline{e}_{2k} \wedge \underline{e}_{2k-1}, ..., \underline{x}_1 := \underline{e}_2 \wedge \underline{e}_1$.

Proof. Following [Gan. 1], $T_{\underline{z}}$ is written as

$$\left[\underline{e}_{2k}, \underline{e}_{2k-1}, \dots, \underline{e}_2, \underline{e}_1\right] \cdot \begin{bmatrix} 0 & \sigma_k & 0 & \cdots & 0 & 0 \\ -\sigma_k & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sigma_{k-1} & \cdots & 0 \\ 0 & 0 & -\sigma_{k-1} & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_1 \\ 0 & 0 & 0 & \cdots & -\sigma_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_{2k}^t \\ \underline{e}_{2k-1}^t \\ \vdots \\ \underline{e}_2^t \\ \underline{e}_1^t \end{bmatrix}$$

if $n = 2k$, or

$$\left[\underline{e}_{2k+1}, \underline{e}_{2k}, \dots, \underline{e}_2, \underline{e}_1\right] \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sigma_k & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\sigma_k & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{k-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\sigma_{k-1} & 0 \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \sigma_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\sigma_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_{2k+1}^t \\ \underline{e}_{2k}^t \\ \vdots \\ \underline{e}_2^t \\ \underline{e}_1^t \end{bmatrix}$$

if $n = 2k + 1$. When $n = 2k$, we obtain:

$$\begin{aligned} T_{\underline{z}} &= \sigma_k(\underline{e}_{2k}\underline{e}_{2k-1}^t - \underline{e}_{2k-1}\underline{e}_{2k}^t) + \cdots + \sigma_1(\underline{e}_2\underline{e}_1^t - \underline{e}_1\underline{e}_2^t) = \\ &= \sigma_k T_{\underline{e}_{2k}\wedge\underline{e}_{2k-1}} + \cdots + \sigma_1 T_{\underline{e}_2\wedge\underline{e}_1} \end{aligned}$$

Hence, $\underline{z} = \sigma_k\underline{x}_k + \sigma_{k-1}\underline{x}_{k-1} + ... + \sigma_1\underline{x}_1$. The proof is similar for the extra zero eigenvalue, when $n = 2k + 1$. □

The decomposition defined by (5.26) will be referred as *the prime decomposition* of $\underline{z}$. Clearly, with some simple alternations it may be considered as the standard SVD for the $\bigwedge^2(\mathbb{R}^n)$ case. Other interesting forms of multivector-tensor decompositions may be found in, [Del. 1], [Yok. 1], [Kol. & Bad. 3] and the references therein. We may now give a proof of the decomposability condition $\underline{z} \wedge \underline{z} = \underline{0}$ based on the prime decomposition.

Proposition 5.3.2. *Any 2-vector $\underline{z}$ is decomposable if and only if $\underline{z} \wedge \underline{z} = \underline{0} \in \wedge^4(\mathbb{R}^n)$.*

Proof. ($\Rightarrow$) If $\underline{z}$ is decomposable then $\underline{z} = \underline{a} \wedge \underline{b}$, $\underline{a},\ \underline{b} \in \mathbb{R}^n$. Hence, $\underline{z} \wedge \underline{z} = \underline{a} \wedge \underline{b} \wedge \underline{a} \wedge \underline{b} = \underline{0}$.

($\Leftarrow$) If $\underline{z} \wedge \underline{z} = 0$, then from the prime decomposition we have that $\sigma_i \sigma_j = 0$, for all $(i, j), j > i$ pairs. This means that $k-1$ the number σ_i have to be zero. Due to $0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_k$, we have that $\sigma_1, ..., \sigma_{k-1} = 0$. Therefore, $\underline{z} = \sigma_k \cdot \underline{x}_k$.

□

The solution of the minimization problem defined by (5.2.1) is considered next.

Theorem 5.3.1. *Let $\underline{x} \in \bigwedge^2(\mathbb{R}^n)$ be decomposable and $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$. Then the best decomposable approximation of $\underline{z}$ is $\hat{\underline{z}} = \sigma_k \underline{x}_k$ and*

$$\|\underline{z} - \hat{\underline{z}}\| = \sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2} \tag{5.27}$$

Proof. Let $D_{\wedge^2(\mathbb{R}^n)}$ denote the subsets of decomposable vectors in $\wedge^2(\mathbb{R}^n)$ and $\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}$. Then,

$$\min_{\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}} \|\underline{x} - \underline{z}\|^2 = \min_{\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}} \left\{ \|\underline{z}\|^2 + \|\underline{x}\|^2 - 2 < \underline{z}, \underline{x} > \right\} \tag{5.28}$$

Thus (5.28) is minimized at $\underline{x}_1 = < \underline{z}, \underline{x} > \underline{x} / \|\underline{x}\|^2$ for some decomposable vector $\underline{x}$. Hence,

$$\min_{\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}} \|\underline{x} - \underline{z}\|^2 = \min_{\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}, \|\underline{x}\| = 1} \left\{ \|\underline{z}\|^2 - < \underline{z}, \underline{x} >^2 \right\} = \|\underline{z}\|^2 - \max_{\underline{x} \in D_{\wedge^2(\mathbb{R}^n)}, \|\underline{x}\| = 1} < \underline{z}, \underline{x} >^2$$

We are therefore aiming to maximize $< \underline{z}, \underline{x} >$ when $\underline{x} = \underline{y}_1 \wedge \underline{y}_2$ where $\underline{y}_1,\ \underline{y}_2$ are orthonormal. Thus,

$$\begin{aligned} \max_{\underline{y}_1, \underline{y}_2} | < \underline{z}, \underline{y}_1 \wedge \underline{y}_2 > | &= \max_{\underline{y}_1, \underline{y}_2} < \underline{y}_1^t T_{\underline{z}}, \underline{y}_2 > = \max_{\|\underline{y}_1\| = 1} \left\langle \underline{y}_1^t T_{\underline{z}}, \frac{\underline{y}_1^t T_{\underline{z}}}{\|\underline{y}_1^t T_{\underline{z}}\|} \right\rangle = \\ &= \max_{\|\underline{y}_1\| = 1} \|\underline{y}_1^t T_{\underline{z}}\| = \sigma_k \end{aligned}$$

Thus, (5.28) implies

$$\begin{aligned} \|\underline{x} - \underline{z}\|^2 &\geq \sigma_k^2 + \sigma_{k-1}^2 + \cdots + \sigma_1^2 + \|\underline{x}\|^2 - 2\sigma_k \cdot \|\underline{x}\| \geq \\ &\geq \sigma_{k-1}^2 + \sigma_{k-2}^2 + \cdots + \sigma_1^2 + (\sigma_k - \|\underline{x})\|)^2 \geq \\ &\geq \sigma_{k-1}^2 + \sigma_{k-2}^2 + \cdots + \sigma_1^2 \end{aligned}$$

Hence, $\hat{\underline{z}} = \sigma_k \underline{x}_k$ realizes the least distance from all decomposable multivectors, which is

$$\|\underline{z} - \hat{\underline{z}}\| = \sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2} \tag{5.29}$$

□

Corollary 5.3.1. *(Properties of the Prime Decomposition) Let* $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$ *and let us denote*

$$\underbrace{\underline{z} \wedge \underline{z} \wedge \cdots \wedge \underline{z}}_{k-factors} \equiv \underline{z}^{\wedge k} \tag{5.30}$$

1) If $\sigma_1, \sigma_2, ..., \sigma_k$ *are the imaginary parts of the eigenvalues of* $T_{\underline{z}}$*, then*

$$\underline{z} = \sum_{i=1}^{k} \sigma_i \underline{x}_i,\ \underline{z} \wedge \underline{z} = 2! \sum_{j>i} \sigma_i \sigma_j \underline{x}_i \wedge \underline{x}_j, ..., \tag{5.31}$$

$$\underline{z}^{\wedge \mu} = \mu! \sum_{1 \le i_1 < ... < i_\mu \le k} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_\mu} \underline{x}_{i_1} \wedge \underline{x}_{i_2} \wedge ... \wedge \underline{x}_{i_\mu},\ 2 \le \mu \le k, ..., \tag{5.32}$$

$$\underline{z}^{\wedge k} = k! \sigma_1 \sigma_2 \cdots \sigma_k \underline{x}_1 \wedge \underline{x}_2 \wedge ... \wedge \underline{x}_k \tag{5.33}$$

2) The characteristic polynomial of $T_{\underline{z}}$ *is given by*

$$\varphi(\lambda) = \begin{cases} \lambda^n + \|\underline{z}\|^2 \lambda^{n-2} + \dfrac{\|\underline{z} \wedge \underline{z}\|^2}{(2!)^2} \lambda^{n-4} + \cdots \dfrac{\|\underline{z}^{\wedge k}\|^2}{(k!)^2}, & n = 2k \\ \lambda^n + \|\underline{z}\|^2 \lambda^{n-2} + \dfrac{\|\underline{z} \wedge \underline{z}\|^2}{(2!)^2} \lambda^{n-4} + \cdots \dfrac{\|\underline{z}^{\wedge k}\|^2}{(k!)^2} \lambda, & n = 2k+1 \end{cases} \tag{5.34}$$

Proof. 1) For k=2 and with the use of the prime decomposition we have that

$$\begin{aligned} \underline{z} \wedge \underline{z} &= \left(\sum_{i=1}^{k} \sigma_i \underline{x}_i \right) \wedge \left(\sum_{i=1}^{k} \sigma_i \underline{x}_i \right) = \\ &= \sigma_k \sigma_{k-1} \underline{x}_k \wedge \underline{x}_{k-1} + ... + \sigma_k \sigma_1 \underline{x}_k \wedge \underline{x}_1 + \\ &+ \sigma_{k-1} \sigma_k \underline{x}_{k-1} \wedge \underline{x}_k + ... + \sigma_{k-1} \sigma_1 \underline{x}_{k-1} \wedge \underline{x}_1 + ... \\ &+ \sigma_2 \sigma_k \underline{x}_2 \wedge \underline{x}_k + ... + \sigma_2 \sigma_1 \underline{x}_2 \wedge \underline{x}_1 + \\ &+ \sigma_1 \sigma_k \underline{x}_1 \wedge \underline{x}_k + ... + \sigma_1 \sigma_2 \underline{x}_1 \wedge \underline{x}_2 = \\ &= 2! \sum_{j>i} \sigma_i \sigma_j \underline{x}_i \wedge \underline{x}_j \end{aligned}$$

Suppose that (5.32) holds true. Then

$$\underline{z}^{\wedge(\mu+1)} = \underline{z}^{\wedge \mu} \wedge \underline{z} = \left(\mu! \sum_{1 \le i_1 < ... < i_\mu \le k} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_\mu} \underline{x}_{i_1} \wedge \underline{x}_{i_2} \wedge ... \wedge \underline{x}_{i_\mu} \right) \wedge \sum_{i=1}^{k} \sigma_i \underline{x}_i =$$

$$= (\mu+1)\mu! \sum_{1\leq i_1<\ldots<i_{\mu+1}\leq k} \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{\mu+1}}\underline{x}_{i_1}\wedge\underline{x}_{i_2}\wedge\ldots\wedge\underline{x}_{i_{\mu+1}}$$

which proves the result.

2) If $n=2k$, from the spectral decomposition of $T_{\underline{z}}$ we have that

$$\begin{aligned}\varphi(\lambda) &= (\lambda^2+\sigma_1^2)(\lambda^2+\sigma_2^2)\cdots(\lambda^2+\sigma_k^2) = \\ &= \lambda^n + (\sigma_1^2+\cdots+\sigma_k^2)\lambda^{n-1}+\cdots+(\sigma_1^2\cdots\sigma_k^2)\end{aligned}$$

Similarly, if n is odd:

$$\begin{aligned}\varphi(\lambda) &= (\lambda^2+\sigma_1^2)(\lambda^2+\sigma_2^2)\cdots(\lambda^2+\sigma_k^2)\lambda = \\ &= \lambda^n + (\sigma_1^2+\cdots+\sigma_k^2)\lambda^{n-1}+\cdots+(\sigma_1^2\cdots\sigma_k^2)\lambda\end{aligned}$$

The result now follows, due to the equations (5.31)-(5.33).

□

Corollary 5.3.1 implies a very important result; the optimization problem (5.23) is directly solved via the components of the given 2-vector $\underline{z}$ only implying the minimizer of (5.23) without applying any algorithmic procedures that usually follow this kind of manifold constrained optimization problems, [Abs., etc. 1], [Abs., etc. 2], [Bas., etc. 1].

Example 5.3.1. *Let* $\underline{z} = (2,-8,1,5,0,11,-3,7,-1,6)^t \in \mathbb{R}^{10}$. *The skew-symmetric matrix* $T_{\underline{z}}$ *for this vector is*

$$T_{\underline{z}} = \begin{pmatrix} 0 & 2 & -8 & 1 & 5 \\ -2 & 0 & 0 & 11 & -3 \\ 8 & 0 & 0 & 7 & -1 \\ -1 & -11 & -7 & 0 & 6 \\ -5 & 3 & 1 & -6 & 0 \end{pmatrix}$$

The spectral decomposition canonical of $T_{\underline{z}}$ *implies that*

$$\begin{aligned}&\underline{r} = (-0.0785093, 0.094211, 0.549565, 0.204124, 0.800795)^t, \\ &\underline{b}_1 = (0.395816, -0.0925693, -0.0410099, 0.900056, -0.151586)^t, \\ &\underline{b}_2 = (-0.0182749, -0.613107, -0.616617, 0, 0.493507)^t, \\ &\underline{a}_1 = (0.236471, 0.760887, -0.527125, 0, 0.29542)^t, \\ &\underline{a}_2 = (-0.883693, 0.166455, -0.195497, 0.38501, -0.0701948)^t\end{aligned}$$

and $\sigma_2 = 8.16558, \sigma_4 = 15.5988$. *Therefore,*

$$\begin{aligned}\sigma_2\cdot\underline{a}_1\wedge\underline{a}_2 = \; &(5.81187, -4.18115, 0.743424, 1.99617, -0.49817, \\ &2.3921, -0.83766, -1.65719, 0.77373, -0.928749)^t\end{aligned}$$

and

$$\sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = \ (-3.81187, -3.81885, 0.256576, 3.00383, 0.49817, \\ 8.6079, -2.16234, 8.65719, -1.77373, 6.92875)^t$$

Hence,

$$\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)^t = \underline{z}$$

as we were expecting from the prime decomposition of $\underline{z}$*. Also,*

$$\underline{x} = \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$$

is the best decomposable approximation of $\underline{z}$ *and*

$$\begin{aligned} \|\underline{x} - \underline{z}\| =& \|(5.81187, -4.18115, 0.743424, 1.99617, -0.49817, 2.3921, \\ & - 0.83766, -1.65719, 0.77373, -0.928749)\| = 8.16558 = \\ & = \sigma_2 \end{aligned}$$

which is the least distance of $\underline{z}$ *from the set of all decomposable vectors.*

5.3.2 Approximation in $\bigwedge^{n-2}(\mathbb{R}^n)$

The results of the previous section may now be extended to the dual case. In fact, the previous results imply calculations over $G_{n-2}(\mathbb{R}^n)$ and rely on the Hodge $\star$-operator, [Jost. 1], [Mar. 1]. This operator was firstly defined in order to generalize the notion of the Laplacian on Riemannian manifolds.

Definition 5.3.2. *The Hodge* $\star$*-operator, for any positively oriented orthonormal basis of an* n*-dimensional vector space* V *is*

$$* : \bigwedge^m(V) \to \bigwedge^{n-m}(V) \tag{5.35}$$

such that

$$(\underline{a} \wedge \underline{b})^* = < \underline{a}, \underline{b} > \underline{w} \tag{5.36}$$

where $\underline{a}, \underline{b} \in \bigwedge^m(V)$, $\underline{w} \in \bigwedge^n(V)$ *and* $m \leq n$.

We first consider some background results, [Jost. 1], [Mar. 1].

Lemma 5.3.3. *The Hodge* $\star$*-operator is linear, one to one, onto and an isometry.*

Lemma 5.3.4. *Let* $D_{\wedge^2(\mathbb{R}^n)}$, $D_{\wedge^{n-2}(\mathbb{R}^n)}$ *denote the subsets of decomposable vectors in* $\bigwedge^2(\mathbb{R}^n)$, $\bigwedge^{n-2}(\mathbb{R}^n)$ *respectively. Then* $\star : D_{\wedge^2(\mathbb{R}^n)} \to D_{\wedge^{n-2}(\mathbb{R}^n)}$ *is also one to one and onto.*

Next we use conjugacy in general vector spaces to study the minimization problem for $G_{n-2}(\mathbb{R}^n)$.

Proposition 5.3.3. *Let $\mathcal{U},\ \mathcal{V}$ be two finite dimensional vector spaces and $T : \mathcal{U} \to \mathcal{V}$ a linear, "1-1", onto isometry. If $\mathcal{U}_1,\ \mathcal{V}_1$ are two isometric subsets of $\mathcal{U},\ \mathcal{V}$ respectively through T and*

$$f(\underline{u}) := \arg\min_{\underline{u}_1 \in \mathcal{U}_1} \|\underline{u} - \underline{u}_1\|,\ g(\underline{u}) := \arg\min_{\underline{v}_1 \in \mathcal{V}_1} \|\underline{u} - \underline{v}_1\| \tag{5.37}$$

then

$$f(\underline{u}) = T^{-1}(g(T(\underline{u}))) \tag{5.38}$$

Proof. Following the standard properties of operators and duality in Banach spaces, [Roc. 1], we have that

$$\begin{aligned} f(\underline{u}) &= \arg\min_{\underline{u}_1 \in \mathcal{U}_1} \|T(\underline{u}) - T(\underline{u}_1)\| = T^{-1}\left(\arg\min_{T(\underline{u}_1) \in \mathcal{V}_1} \|T(\underline{u}) - T(\underline{u}_1)\|\right) = \\ &= T^{-1}(\arg\min_{\underline{v}_1 \in \mathcal{V}_1} \|T(\underline{u}) - \underline{v}_1\|) = T^{-1}(g(T(\underline{u}))) \end{aligned}$$

□

The above result may be described by the commutative diagram:

$$\begin{array}{ccc} & T & \\ \mathcal{U} & \longrightarrow & \mathcal{V} \\ \downarrow f & & g \downarrow \\ \mathcal{U}_1 & \longrightarrow & \mathcal{V}_1 \\ & T & \end{array}$$

In our case, $\mathcal{U},\ \mathcal{V}$ are represented by $\bigwedge^2(\mathbb{R}^n),\ \bigwedge^{n-2}(\mathbb{R}^n)$ and $\mathcal{U}_1,\ \mathcal{V}_1$ by $D_{\wedge^2(\mathbb{R}^n)}$ and $D_{\wedge^{n-2}(\mathbb{R}^n)}$, respectively. Then the diagram above, shows how the minima for $D_{\wedge^{n-2}(\mathbb{R}^n)}$ and $D_{\wedge^2(\mathbb{R}^n)}$ in Proposition 5.3.3 may be derived from each other. Hence, if $T \equiv \star$ the following result is established.

Corollary 5.3.2. *For every $\underline{z} \in \bigwedge^{n-2}(\mathbb{R}^n)$ the following equality holds:*

$$\min_{\underline{a}_1,..,\underline{a}_{n-2} \in \mathbb{R}^n} \|\underline{z} - \underline{a}_1 \wedge ... \wedge \underline{a}_{n-2}\| = \min_{\underline{b}_1,\underline{b}_2 \in \mathbb{R}^n} \|\underline{z}^* - \underline{b}_1 \wedge \underline{b}_2\| \tag{5.39}$$

The above may be illustrated by the following example:

Example 5.3.2. *Let $\underline{z} = (6, 1, 7, -3, -11, 0, -5, 1, 8, 2)^t \in \bigwedge^3(\mathbb{R}^5) \simeq \mathbb{R}^{10}$. Then $\underline{z}^* = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)^t \in \bigwedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$. Hence,*

$$T_{\underline{z}^*} = \begin{pmatrix} 0 & 2 & -8 & 1 & 5 \\ -2 & 0 & 0 & 11 & -3 \\ 8 & 0 & 0 & 7 & -1 \\ -1 & -11 & -7 & 0 & 6 \\ -5 & 3 & 1 & -6 & 0 \end{pmatrix}$$

From the canonical form of $T_{\underline{z}^*}$, we have

$\underline{r} = (-0.0785093, 0.094211, 0.549565, 0.204124, 0.800795)^t$,
$\underline{b}_1 = (0.395816, -0.0925693, -0.0410099, 0.900056, -0.151586)^t$,
$\underline{b}_2 = (-0.0182749, -0.613107, -0.616617, 0, 0.493507)^t$,
$\underline{a}_1 = (0.236471, 0.760887, -0.527125, 0, 0.29542)^t$,
$\underline{a}_2 = (-0.883693, 0.166455, -0.195497, 0.38501, -0.0701948)^t$
and $\sigma_2 = 8.16558, \sigma_4 = 15.5988$. *Therefore,*

$$\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 = (5.81187, -4.18115, 0.743424, 1.99617, -0.49817, 2.3921, -0.83766, -1.65719, 0.77373, -0.928749)^t$$

and

$$\sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = (-3.81187, -3.81885, 0.256576, 3.00383, 0.49817, 8.6079, -2.16234, 8.65719, -1.77373, 6.92875)^t$$

Hence,

$$\sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2 = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)^t = \underline{z}^*$$

and $\underline{x} = \sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2$ *is the best decomposable approximation of* $\underline{z}^*$. *Thus*

$$(\sigma_4 \cdot \underline{b}_1 \wedge \underline{b}_2)^* = \sigma_4 \cdot \underline{a}_1 \wedge \underline{a}_2 \wedge \underline{r} = (6.92874, 1.77373, 8.65719, -2.16234, -8.60789, 0.498171, -3.00383, 0.256576, 3.81885, -3.81187)^t$$

is the best decomposable approximation of $\underline{z}$.

Next we present an alternative way for the calculation of the best decomposable approximation of a 2-vector via the Grassmann matrix, [Kar. & Gia. 6], [Kar. & Lev. 9].

5.3.3 Best approximation and the Grassmann Matrix

As we have proved, optimization problem (5.23) is closely connected to the calculation of $\sigma_k, \sigma_{k-1}, ..., \sigma_1$ which are the imaginary parts of the eigenvalues of the skew symmetric matrix $T_{\underline{z}}$. In order to achieve their calculation, we implemented the spectral analysis of $T_{\underline{z}}$ which in this case leads to two Jordan-blocks matrices. In this section we present a new way that simplifies significantly the calculations of $\sigma_k, \sigma_{k-1}, ..., \sigma_1$, similar to the $\bigwedge^2(\mathbb{R}^4)$ case. Specifically, we prove that the least singular value of the Grassmann matrix coincides with the least distance from the set of all decomposable vectors. Furthermore, we prove that the Grassmann matrices constitute a wider category than the skew-symmetric matrices, since $T_{\underline{z}}$ is a special form of a Grassmann matrix. First, we give an equivalent definition of the Grassmann matrix.

Definition 5.3.3. *For every* $\underline{z} \in \mathbb{R}^n, \underline{y} \in \mathbb{R}^m$ *where* $n > m$, $n, m \in \mathbb{N}$, *the matrix* $\Phi_{\underline{z}} \in \mathbb{R}^{n \times m}$ *such that*

$$\Phi_{\underline{z}} \cdot \underline{y} = \underline{z} \wedge \underline{y} \tag{5.40}$$

is called the Grassmann matrix of $\underline{z}$.

Now, we will calculate the singular values of $\Phi_{\underline{z}}$.

From Definition 5.3.3 we have that

$$\|\Phi_{\underline{z}} \cdot \underline{y}\| = \|\underline{z} \wedge \underline{y}\| \tag{5.41}$$

Now,

$$\begin{aligned}\|(\underline{z} \wedge \underline{y}\|^2 &= \|(\sigma_k \underline{b}_1 \wedge \underline{b}_2 + ... + \sigma_1 \underline{w}_1 \wedge \underline{w}_2) \wedge (y_1 \underline{b}_1 + ... + y_n \underline{w}_2)\|^2 = \\ &= \sigma_k^2(y_3^2 + y_4^2 + ... + y_n^2) + \sigma_{k-1}^2(y_1^2 + y_5^2 + ... + y_n^2) + ... + \sigma_1^2(y_1^2 + y_3^2 + ... + y_{n-2}^2) = \\ &= y_1^2(\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2) + y_2^2(\sigma_k^2 + \sigma_{k-2}^2 + ... + \sigma_1^2) + ... + y_n^2(\sigma_k^2 + \sigma_{k-1}^2 + ... + \sigma_2^2)\end{aligned}$$

Therefore, if we set

$$H_i := \sigma_k^2 + \sigma_{k-1}^2 + ... + \sigma_{i-1}^2 + \sigma_{i+1}^2 + ... + \sigma_1^2 \tag{5.42}$$

where this sum denotes that the i-entry is missing, then the squared singular values of the Grassmann matrix are

$$\|\underline{z}\|^2,\ H_1,\ H_1,\ H_2,\ H_2, ..., H_k,\ H_k,$$

if $n = 2k + 1$ or

$$H_1,\ H_1,\ H_2,\ H_2, ..., H_k, H_k,$$

if $n = 2k$.

Corollary 5.3.3. *The minimum singular value of the Grassmann matrix is* $\sqrt{H_k}$ *which is equal to the least minimum distance from the set of all decomposable vectors. If* $H_k = 0$ *then the given multivector is decomposable and the distance is zero.*

Proof. This is evident from the above calculations of the squared singular values H_i. □

Example 5.3.3. *Let* $\underline{z} = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)$ *as in Example 5.3.1.*

Then the Grassmann matrix for $\underline{z}$ *is*

$$\begin{pmatrix} 0 & 8 & 2 & 0 & 0 \\ 11 & -1 & 0 & 2 & 0 \\ -3 & -5 & 0 & 0 & 2 \\ 7 & 0 & -1 & -8 & 0 \\ -1 & 0 & -5 & 0 & -8 \\ 6 & 0 & 0 & -5 & 1 \\ 0 & 7 & -11 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 6 & 0 & 3 & 11 \\ 0 & 0 & 6 & 1 & 7 \end{pmatrix}$$

The Singular Value Decomposition for this matrix gives the following singular values

$$\lambda_1 = 17.6068,\ \lambda_2 = \lambda_3 = 8.16558,\ \lambda_4 = \lambda_5 = 15.5988$$

In deed, we observe that the smallest singular value is 8.16558 which coincides with σ_2 *as we saw in Example 5.3.1.*

Remark 5.3.2. *The characteristic polynomial* $\varphi(\lambda)$ *of* $\Phi^t\Phi$ *may also be obtained if specific decomposable sets are given. If n=5 then*

$$\Phi = \begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 & 0 \\ a_{24} & -a_{14} & 0 & a_{12} & 0 \\ a_{25} & -a_{15} & 0 & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} & 0 \\ a_{35} & 0 & -a_{15} & 0 & a_{13} \\ a_{45} & 0 & 0 & -a_{15} & a_{14} \\ 0 & a_{34} & -a_{24} & a_{23} & 0 \\ 0 & a_{35} & -a_{25} & 0 & a_{23} \\ 0 & a_{45} & 0 & -a_{25} & a_{24} \\ 0 & 0 & a_{45} & -a_{35} & a_{34} \end{pmatrix}$$

for $\underline{z} = (z_1, z_2, .., z_9, z_{10}) \equiv (a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}, a_{35}, a_{45})$. *Hence,*

$$\varphi(\lambda) = det(\lambda \cdot I_5 - \Phi^t\Phi) = (\lambda - \|z\|^2)\left[(\lambda - \sigma_2^2)(\lambda - \sigma_4^2)\right]^2 = (\lambda - \|z\|^2)\left[\lambda^2 - (\sigma_2^2 + \sigma_4^2)\lambda + \sigma_2^2 \cdot \sigma_4^2\right]^2$$

Therefore, we have that

$$\varphi(\lambda) = \left(\lambda^2 - \|z\|^2 \cdot \lambda + \sum (QPR)^2\right)^2 \cdot (\lambda - \|z\|^2)$$

Next, we show how the skew - symmetric matrix $T_{\underline{z}}$ is related to the Grassmann matrix $\Phi_{\underline{z}}$. We will see that actually $T_{\underline{z}}$ is a Grassmann matrix. This is a very important result, since the spectral analysis of $T_{\underline{z}}$ gives two forms of Jordan blocks matrices instead of one simpler matrix, such as $\Phi_{\underline{z}}$. This can also be helpful in higher dimensions, since $T_{\underline{z}}$ acts as a tensor - making any calculations more complicated - while $\Phi_{\underline{z}}$ is still a matrix.

Theorem 5.3.2. *The skew-symmetric matrix $T_{\underline{z}}$ has the form of a Grassmann matrix.*

Proof. From Definition 5.3.3 we have that

$$< \underline{z}, \underline{y}_1 \wedge \underline{y}_2 >= \underline{y}_1^t \cdot T_{\underline{z}} \cdot \underline{y}_2 =< \underline{y}_1^t \cdot T_{\underline{z}}, \underline{y}_2 > \tag{5.43}$$

But also,

$$< \underline{z}, \underline{y}_1 \wedge \underline{y}_2 >= \left(\underline{z}^* \wedge \underline{y}_1 \wedge \underline{y}_2\right)^* =< \left(\underline{z}^* \wedge \underline{y}_1\right)^*, \underline{y}_2 > \tag{5.44}$$

Therefore,

$$\left(\underline{z}^* \wedge \underline{y}_1\right)^* = \left(\underline{y}_1^t \cdot T_{\underline{z}}\right)^t = -T_{\underline{z}}\underline{y}_1 \tag{5.45}$$

since $T_{\underline{z}}^t = -T_{\underline{z}}$. Now, if $\Phi_{\underline{z}}^1$ is a Grassmann matrix such that $\Phi_{\underline{z}}^1 \cdot \underline{y}_1 = \left(\underline{z}^* \wedge \underline{y}_1\right)^*$ then we have that

$$-T_{\underline{z}}\underline{y}_1 = \Phi_{\underline{z}}^1 \underline{y}_1 \text{ or } T_{\underline{z}} = -\Phi_{\underline{z}}^1 \tag{5.46}$$

which proves the result. □

5.4 Optimization in the Projective Space

In this section, the minimization problem (5.23) is studied over the projective space $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$. The decomposability of 2-vectors in the subset $D_{\wedge^2(\mathbb{R}^n)}$ in $\bigwedge^2(\mathbb{R}^n)$ will be now expressed by the Grassmann variety $G_2(\mathbb{R}^n)$ (the Plücker embedding we saw in Chapter 3, allows the embedment in the projective space) and the distances $\|\cdot\|$ will be computed via the natural metric of $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$, the gap metric gap$(\cdot)$, [Wey. 1]. Then problem (5.23) will be formulated as $\min_{\underline{x}} g(\underline{z}(\underline{x}), G_2(\mathbb{R}^n))$ and $\min_{\underline{x}} g(\underline{z}, G_2(\mathbb{R}^n))$ if $\underline{z}$ is fixed, where g is the gap function between a multivector and the Grassmann variety, implied by gap$(\cdot)$. This section deals with the second optimization problem and the fixed case of the 2-vector (the general parameterized case $\underline{z}(\underline{x})$ will be examined in Chapter 8) and we also present the basic forms that the gap metric may take, i.e., angle metric, chordal distance, etc.

5.4.1 The Gap Metric

The gap metric has its origin in projective geometry, [Wey. 1] as the natural metric for counting distances in a general projective space $\mathbb{P}^n(F)$ and in functional analysis, [Kre. & Kras. 1] where it was used in the perturbation theory of linear operators. For control theory applications it is usually used as an appropriate tool for the study of uncertainty in feedback systems, [Sak. 1]. In our case, we will utilize the definitions and the unitary properties of the gap metric as introduced in [Wey. 1] for a real projective space, which are similar with the ones

regarding the Fubini study norm for complex projective spaces measurements, [Kob. & Mom. 1].

Definition 5.4.1. *[Wey. 1] Let $\underline{x}, \underline{z}$ be two "points" (representatives of the corresponding equivalent classes* $\text{span}\{\underline{x}\}$, $\text{span}\{\underline{z}\}$*) in* $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$ *and* $\mathcal{U}$ *a plane in* $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$*, with* $\binom{n}{2}-1$ *coordinates each. Then*

i) The distance between $\underline{x}, \underline{z}$ is given by

$$\text{gap}(\underline{x}, \underline{z}) := \left(\frac{\sum_{i<j}^{\binom{n}{2}-1} |x_i z_j - x_j z_i|^2}{\sum_{i=1}^{\binom{n}{2}-1} |x_i|^2 \cdot \sum_{i=1}^{\binom{n}{2}-1} |z_i|^2} \right)^{1/2} \tag{5.47}$$

ii) The distance between $\underline{x}$,$\mathcal{U}$ is given by

$$g(\underline{x},\mathcal{U}) := \left(\frac{\sum_{i=1}^{\binom{n}{2}-1} |x_i u_i|^2}{\sum_{i=1}^{\binom{n}{2}-1} |x_i|^2 \cdot \sum_{i=1}^{\binom{n}{2}-1} |u_i|^2} \right)^{1/2} \tag{5.48}$$

Equivalently therefore we have the following definition.

Definition 5.4.2. *The gap metric* gap *between any lines* $\text{span}\{\underline{x}\}, \text{span}\{\underline{z}\}$ *in the projective space is given by*

$$\text{gap}(\underline{x}, \underline{z}) = |\sin(\underline{x}\hat{,}\underline{z})| = \min_{\lambda} \left\| \frac{\underline{z}}{\|\underline{z}\|} - \frac{\underline{x}}{\|\underline{x}\|} \cdot \lambda \right\| \tag{5.49}$$

Remark 5.4.1. *The fact that* gap *is a metric function is evident, since*

a) $\text{gap}(\underline{x}_1, \underline{x}_2) \geq 0$*, where the equality holds for* $\underline{x}_1 = \underline{x}_2$

b) $\text{gap}(\underline{x}_1, \underline{x}_2) = \text{gap}(\underline{x}_2, \underline{x}_1)$

c) $(\underline{x}_1\hat{,}\underline{x}_2) \leq (\underline{x}_1\hat{,}\underline{x}_3) + (\underline{x}_3\hat{,}\underline{x}_2)$

The main tool for distance calculations in this thesis, as we have already mentioned, would be the gap metric and the distance function g between a point and the Grassmann variety. In the rest of this section, we examine the special forms that the gap metric may take. First, we examine the case where the gap is restrained in $G_m(\mathbb{R}^n)$. In this case, it is usually referred as the *angle metric* which is defined via the Grassmann representatives, [Gia. 1], [Kar. & Gia. 5], [Kar. & Lev. 10], we saw in Chapter 3.

Definition 5.4.3. *[Gia. 1], [Kar. & Gia. 5] Let $\mathcal{V}$ be an n-dimensional vector space over $\mathbb{R}$ and $\underline{x} = (x_0, ..., x_p) \in \bigwedge^m(\mathcal{V})$, $p = \binom{n}{m} - 1$ a Grassmann representative.*

i) The decomposable multivector $\underline{\tilde{x}}$ *given by*

$$\underline{\tilde{x}} = \begin{cases} \dfrac{|\underline{x}_p|\underline{x}}{\|\underline{x}\|}, & \underline{x}_p \neq \underline{0} \\ \\ \dfrac{|\underline{x}_i|\underline{x}}{\|\underline{x}\|}, & \underline{x}_p = \underline{0} \end{cases} \tag{5.50}$$

where $\underline{x}_i$ *is the first non-zero coordinate of* $\underline{x}$*, is called the normal Grassmann representative of* $G_m(V)$*.*

ii) The angle between two m*-dimensional subspaces* $\mathcal{V}_1,\ \mathcal{V}_2$ *of* V *is defined as*

$$\left(\mathcal{V}_1\hat{,}\mathcal{V}_2\right) = \arccos\left|(\underline{\tilde{x}}_1\hat{,}\underline{\tilde{x}}_2)\right| \tag{5.51}$$

and it is a metric for the Grassmann manifold.

Remark 5.4.2. *It can be shown, [Gia. 1], that the normal Grassmann representative* $\underline{\tilde{x}}$ *is unique with* $\|\underline{\tilde{x}}\| = 1$*, where* $\|\cdot\|$ *is the Euclidean norm. Thus, all normal Grassmann representatives belong to the unit sphere in* $\bigwedge^m(\mathcal{V})$ *and more precisely, to the upper hemisphere and half of the equator of the unit sphere.*

Next metric is the standard metric for computations on the Grassmann variety.

Definition 5.4.4. *[Lu. 1] Let* A_1, A_2 *be basis matrices of* $\mathcal{V}_1,\ \mathcal{V}_2 \in G_{n,m}$*. Then, the function* ℓ *define as*

$$\ell(\mathcal{V}_1, \mathcal{V}_2) = \arccos\left(\det\left(X_1 X_2^t X_2 X_1^t\right)\right)^{1/2} \tag{5.52}$$

is a metric on $G_{n,m}$*, where* $X_1 = (A_1^t A_1)^{-1/2} A_1^t,\ X_2 = (A_2^t A_2)^{-1/2} A_2^t$*.*

In [Gia. 1] is was proved that the angle metric and ℓ are equal. We prove the same for the gap metric gap, as well.

Proposition 5.4.1. *The gap metric* gap *is also a metric on* $G_{n,m}$ *with*

$$\text{gap}(\underline{x}_1, \underline{x}_2) = \det\left(X_1 X_2^t X_2 X_1^t\right) \tag{5.53}$$

where $X_i, i = 1, 2$ *as mentioned in Definition 5.4.4.*

Proof. Due to the Cauchy-Binet theorem, we have that

$$\det\left(X_1 X_2^t X_2 X_1^t\right) = C_m(X_1) C_m(X_2^t) C_m(X_2) C_m(X_1^t) \tag{5.54}$$

By the definitions of $X_1,\ X_2$ we have that

$$C_m(X_1) = \frac{\underline{x}_1^t}{\|\underline{x}_1\|},\ C_m(X_2) = \frac{\underline{x}_2^t}{\|\underline{x}_2\|}$$

where $\underline{x}_1 = C_m(A_1),\ \underline{x}_2 = C_m(A_2)$. Therefore,

$$\text{gap}(\underline{x}_1, \underline{x}_2) = \sin(\mathcal{V}_1, \mathcal{V}_2) \tag{5.55}$$

and

$$\ell(\mathcal{V}_1, \mathcal{V}_2) = \arccos\left|(\tilde{\underline{x}}_1, \hat{\tilde{\underline{x}}}_2)\right|,\ \tilde{\underline{x}}_1 := \frac{\underline{x}_1^t}{\|\underline{x}_1\|},\ \tilde{\underline{x}}_2 := \frac{\underline{x}_2^t}{\|\underline{x}_2\|} \tag{5.56}$$

and the result follows due to (5.55). □

Remark 5.4.3. *It has been proved, [Sak. 1], that if the previous distance is considered as the gap between the graphs $G(\cdot)$ of the closed operators $K_1,\ K_2$ of two systems, when these are viewed as closed subspaces of the Hilbert space $H \times H$, then*

$$\text{gap}\,(G(K_1), G(K_2)) = \sup_{u \in D(K_1)} \inf_{v \in D(K_1)} \frac{\|\underline{u} - \underline{v}\|^2 \|K_1\underline{u} - K_2\underline{v}\|^2}{\left(\|\underline{u}\|^2 + \|K_1\underline{u}\|^2\right)^{1/2}} \tag{5.57}$$

where $D(\cdot)$ is the domain of the respective operator. Therefore, (5.49) *and* (5.57) *are equivalent expressions of the gap metric.*

A second approach of the gap occurs when the angles between $m-$planes in an n-dimensional vector space $\mathcal{V}$ are calculated instead of lines or subspaces. Then the notions of the gap metric for the projective space and the ℓ metric for $G_{n,m}$ are generalized into the so-called *chordal distance* which is met in the study of geodesics in the Grassmann manifold, [Wong. 1].

Definition 5.4.5. *[Wong. 1] Let $\mathcal{V}$ an n-dimensional vector space over a field $\mathcal{F}$.*

i) An m-plane is m-dimensional subset of $\mathcal{V}$ which does not need to pass through the origin. If $m = n - 1$, them the $m-$plane is called hyper plane.

ii) The angles θ_i between two m-planes $\mathcal{U}_1,\ \mathcal{U}_2$ are the non-zero stationary values of the angle between the non-zero vectors $\underline{u}_i$ in $\mathcal{U}_1$ and their orthogonal projection $\underline{v}_i$ onto $\mathcal{U}_2$, respectively. The number of angles θ_i between $\mathcal{U}_1$ and $\mathcal{U}_2$ is m, of which at most $r = \min(n, m)$ can be nonzero.

iii) The square root of the sum of the squares of the $m-$angles between two consecutive m-planes $U_1,\ U_2$, denoted as $Ch(\mathcal{U}_1, \mathcal{U}_2)$, is called the Chordal distance and it is a metric on $G_m(\mathcal{V})$.

In the following results we prove how the gap is connected to the metric Ch.

Lemma 5.4.1. *If $\theta_i,\ i = 1, ..., m$ is the angle between the vectors $\underline{u}_i \in \mathcal{U}_1,\ \underline{u}_2 \in \mathcal{U}_2$ and $\text{span}\{\underline{u}_i\}$, $\text{span}\{\underline{v}_i\}$, their respective lines, then*

$$Ch(\mathcal{U}_1, \mathcal{U}_2) = \sqrt{\text{gap}^2(\underline{u}_1, \underline{v}_1) + \text{gap}^2(\underline{u}_2, \underline{v}_2) + \cdots + \text{gap}^2(\underline{u}_m, \underline{v}_m)} \tag{5.58}$$

Proof. This follows directly from Definitions 5.4.2 and 5.4.5. □

Remark 5.4.4. *It can be proved, [Ko. & Le. 1], that when $m \leq n/2$ then,*

$$Ch(U_1, U_2) = \frac{1}{2} \left\| AA^t - BB^t \right\|_F \tag{5.59}$$

where $A,\ B$ matrices in $F^{n\times m}$ that generate $U_1,\ U_2$. The form (5.59) for the chordal distance is widely used in complex MIMO systems, [Ko. & Le. 1].

Remark 5.4.5. *The chordal distance may be transformed into a suitable metric for $\mathbb{P}^n(\mathbb{C})$; if the stereographic projection of a complex number $u = a + bi$ to the sphere $x^2 + y^2 + (z - 1/2)^2 = 1/4$ implies the point $A(x, y, z)$, then, as shown in [Kal. 1],*

$$x = \frac{a|u|}{2\,(1 + |u|^2)},\ y = \frac{b|u|}{2\,(1 + |u|^2)},\ z = \frac{|z|^2}{1 + |z|^2} \tag{5.60}$$

Hence, if $s \equiv (x, y, z), s' \equiv (x', y', z')$ then

$$Ch(s, s') = |s - s'|\sqrt{(1 + |s|^2)(1 + |s'|^2)},\ Ch\,(s, \infty) = \frac{1}{\sqrt{1 + |s|^2}} \tag{5.61}$$

Equation (5.60) is the main tool for the study of systems eigenvalue sensitivity under perturbations, [Stew. 1].

Remark 5.4.6. *Other forms of the gap metric which are usually met in distance problems between two subspaces $\mathcal{V}_1,\ \mathcal{V}_2$ in a projective space are*

i) $\mathrm{gap}(Y_1, Y_2) = ||\underline{\theta}||_2$ *(arc length)*

ii) $\mathrm{gap}_{\mathrm{p2}}(Y_1, Y_2) = \|\sin(\underline{\theta})\|_2$ *(projection F -norm),*

iii) $\mathrm{gap}_{\mathrm{c2}}(Y_1, Y_2) = 2\|\sin(\frac{1}{2\underline{\theta}})\|_\infty$ *(chordal norm),*

iv) $\mathrm{gap}_{\mathrm{cF}}(Y_1, Y_2) = 2\|\sin(\frac{1}{2\underline{\theta}})\|_2$ *(chordal Frobenius-norm),*

v) $\mathrm{gap}_{\mathrm{p2}}(Y_1, Y_2) = \|\sin(\underline{\theta})\|_\infty$ *(projection 2-norm) and*

vi) $\mathrm{gap}_{\mathrm{FS}}(Y_1, Y_2) = \arccos\left(\prod_i \cos\theta_i\right)$ *(Fubini-Study norm)*

where $\theta_i,\ i = 1, ..., k$ are the principal angles between $\mathcal{V}_1,\ \mathcal{V}_2$, spanned by the columns of the $p \times k$ orthonormal matrices $Y_1,\ Y_2$ for some $0 < k \leq p$ and $\underline{\theta} = (\theta_1, ..., \theta_2)$. Except the last case, all metrics consider the $||\cdot||_\infty,\ ||\cdot||_2$ norms, whereas the Fubini study norm is used for complex projective spaces [Kob. & Mom. 1]. More details on these metrics as well as other equivalent forms may be found in [Edel., etc. 1].

5.4.2 Least Distance over $G_2(\mathbb{R}^n)$

Let g denote the gap between a point and the Grassmann variety in the projective space $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$ and $\text{gap}(\cdot,\cdot)$ the gap metric between two points in $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$. Then problem (5.23) is interpreted as the derivation of the best decomposable approximation representative $\hat{\underline{x}}$ of a given 2-vector representative $\underline{z}$ for the respective equivalence classes. In the following, for notational simplicity, we identify $\underline{z} \equiv \text{span}\{\underline{z}\}$ and as we have already mentioned, we follow [Wey. 1] for the definitions of the gap metric between points, i.e., $\text{gap}(\underline{z},\underline{x}) = |\sin(\underline{z},\hat{\underline{x}})|$ and the gap function g between a point and a subspace of the projective space.

Theorem 5.4.1. *The gap function g between $\underline{z}$ and $G_2(\mathbb{R}^n)$ is equal to*

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \tag{5.62}$$

Proof. From the definition of the gap between a point and a subspace in $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$, [Wey. 1], we have that

$$\begin{aligned} g\left(\underline{z}, G_2(\mathbb{R}^n)\right) :&= \min_{\underline{x}\in G_2(\mathbb{R}^n)} \text{gap}(\underline{z},\underline{x}) = \min_{\underline{x}\in G_2(\mathbb{R}^n)} \sqrt{1 - \frac{<\underline{z},\underline{x}>^2}{\|\underline{x}\|^2\|\underline{z}\|^2}} = \\ &= \left(1 - \frac{1}{\|\underline{z}\|^2} \max_{\underline{x}\in G_2(\mathbb{R}^n)} \frac{<\underline{z},\underline{x}>^2}{\|\underline{x}\|^2}\right)^{\frac{1}{2}} = \sqrt{1 - \frac{\sigma_k^2}{\|\underline{z}\|^2}} = \\ &= \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \end{aligned}$$

due to Theorem 5.3.1. □

Clearly, the minimization of g and consequently the multilinear subproblem of DAP are equivalent to the maximization of the largest eigenvalue of $T_{\underline{z}}$, since $g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{1 - \sigma_{\max}^2/\|\underline{z}\|^2}$. Note, that eigenvalue optimization problems of this form are very important in the field of matrix theory, [Gol. & Van. 2], [Horn. & Joh. 1], [Xia. 1] and in most cases they are usually addressed algorithmically, [Le. & Over. 1], [Sha. & Fa. 1]. Specifically it has been shown that the optimization of an eigenvalue of a symmetric matrix is strongly connected to the minimization of the sum of the eigenvalues of the matrix, [Le. & Over. 1] whereas in [Sha. & Fa. 1], first and second order conditions were given, regarding the existence of solutions of such eigenvalue optimization problems. These results have led to *Overton's algorithm*, [Le. & Over. 1] which is an alternation of the standard Newton's algorithm, constructed specifically for eigenvalue optimization problems. Some results concerning eigenvalue optimization for a general matrix A may be found also in [Le. & Over. 1], but for a special structure of A.

A number of control theory problems addressed via the Overton's algorithm may be found in [Boy., etc. 1]. Clearly now, Theorem 6.3.1 may be regarded as a new tool for closed-form solutions for the optimization of the largest eigenvalue of a skew-symmetric matrix.

Theorem 5.4.2. *The maximum possible gap of a multi-vector* $\underline{z} \in \wedge^2(\mathbb{R}^n)$ *from* $G_2(\mathbb{R}^n)$ *is equal to*

$$\max_{\underline{z}} g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{1 - \frac{1}{k}} \tag{5.63}$$

Proof. Using the result of Theorem 6.3.1 we have that

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\sqrt{\sigma_k^2 + \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}} = \frac{1}{\sqrt{1 + \frac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}} \leq$$

$$\leq \frac{1}{\sqrt{1 + \frac{\sigma_k^2}{\sigma_k^2 + \sigma_k^2 + \sigma_k^2 + ... + \sigma_k^2}}} = \frac{1}{\sqrt{1 + \frac{1}{k-1}}} = \sqrt{1 - \frac{1}{k}}$$

□

Remark 5.4.7. *The use of eigenvalues/singular values for minimum-maximum distance problems concerning subspaces, lines, sets, etc., is a standard technique examined by several authors, e.g., [Gol. & Van. 2], [Xia. 1]. In general, the authors rewrite the angle between two subspaces in the form*

$$\sin\theta = \text{dist}\left(\text{range}(Q_a), \text{range}(Q_0)\right) = \sigma_{\max} Q_b^t Q_0$$

where $Q = [Q_a, Q_b]$, $\hat{Q} = [Q_0, Q_1]$ *are two orthogonal matrices whose columns span the two subspaces respectively, with* $Q_a,\ Q_0 \in \mathbb{R}^{n \times r}$ *and* $Q_a^t Q_a = Q_0^t Q_0 = I_r$ *and they examine its upper and lower bounds. Our approach has answered this upper-lower bound problem for the Grassmann variety case in the projective space.*

Formula (6.6) can be further simplified if the set of QPR that describe the respective Grassmann varieties is given. If for instance $n = 5$ (similarly for $n = 4$ and the single QPR $\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0$) then the minimum gap is given by the next theorem.

Theorem 5.4.3. *The minimum gap between a 2-vector* $\underline{z} \in \wedge^2(\mathbb{R}^5)$ *and* $G_2(\mathbb{R}^5)$ *is given by*

$$g_\wedge(\underline{z}, G_2(\mathbb{R}^5)) = \frac{\sqrt{\sum QPR^2(\underline{z})}}{\sum_{i=1}^{10} z_i^2} \tag{5.64}$$

Proof. The minimum gap is $g(\underline{z}, \hat{\underline{z}}) = \sigma_1/\|\underline{z}\| \equiv g(\underline{z})$, with $\hat{\underline{z}} = \sigma_2 \underline{e}_1 \wedge \underline{e}_2,\ \sigma_2 \geq \sigma_1$. From the prime decomposition of $\underline{z}$ we have that $\|\underline{z} \wedge \underline{z}\| = \|2\sigma_1\sigma_2 \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \wedge \underline{e}_4\|$. Hence, if

$$g_\wedge := \frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2} \tag{5.65}$$

we have that

$$g_\wedge = \frac{2\sigma_1\sigma_2}{\|\underline{z}\|^2} = \frac{2\sqrt{\|\underline{z}\|^2 - \sigma_1^2}\sigma_1}{\|\underline{z}\|^2} = 2\sqrt{1 - \left(\frac{\sigma_1}{\|\underline{z}\|}\right)^2}\frac{\sigma_1}{\|\underline{z}\|} =$$
$$= 2\sqrt{1 - g^2(\underline{z})}g(\underline{z}) = 2\sqrt{g^2(\underline{z})(1 - g^2(\underline{z}))}$$

Since the function $f(x) = \sqrt{x^2(1-x^2)}$ is increasing at $[0, \sqrt{2}/2]$ we have that the minimization of g is equivalent to the minimization of $g_\wedge$. In other words, the norm of the QPR over the norm of $\underline{z}$ can be used for minimization in $\wedge^2(\mathbb{R}^5)$, i.e.,

$$g_\wedge(\underline{z}, G_2(\mathbb{R}^5)) = \frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2} = \frac{\sqrt{\sum QPR^2(\underline{z})}}{\sum_{i=1}^{10} z_i^2} \tag{5.66}$$

□

Theorem 5.4.3 is one of the most important results in this thesis; if a specific Grassmann variety is given, all distance computations are expressed in terms of $\underline{z}$, which is a remarkable simplification for Grassman optimization problems.

5.5 Conclusions

In this chapter, a first approach has been made to the solution of the approximate determinantal assignment problem by considering the optimization problem $\min_{\underline{x}} \|\underline{x} - \underline{z}\|$ when $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$ or $\underline{z} \in \bigwedge^{n-2}(\mathbb{R}^n)$ is a fixed vector and $\underline{x}$ is decomposable. This problem will be generalized in the next chapters for a parameterized 2-vector $\underline{z}(\underline{x})$ which it will describe the whole linear variety $\mathcal{K}$ (implied by the linear subproblem of DAP) and thus the approximate DAP will be completely solved in the 2-dimensional case, provided the stability of the approximate polynomial yield by the approximate solution.

The initial case $n = 4$ was solved via the Lagrange multipliers method and has provided the important result that the solution of the above minimization problem can be derived by the least singular value of the corresponding Grassmann matrix. On the other hand, the general case $n > 4$ has been proved more difficult since the QPR that describe the set of decomposable vectors $D_{\wedge^2(\mathbb{R}^n)}$ in $\bigwedge^2(\mathbb{R}^n)$ do not allow the implementation of the Lagrange multipliers method. To overcome this obstacle, we have used the 2-tensor, skew-symmetric matrix form

$T_{\underline{z}}$ of a 2-vector $\underline{z}$ and by applying its spectral decomposition, the so called prime decomposition of $\underline{z}$ was obtained. The latter has provided remarkable help to the above problem, by implying its solution in closed form formulae for the $\bigwedge^2(\mathbb{R}^n)$ as well as $\bigwedge^{n-2}(\mathbb{R}^n)$.

Furthermore, the result of the least distance via the lowest singular value of the Grassmann matrix has been also confirmed for the all cases $n \geq 4$, and the antisymmetric matrix $T_{\underline{z}}$ has been proved to be a special case of the Grassmann matrix. These results have helped us to solve the same distance problem in the projective space $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$, which is the natural space for DAP and it was shown that $g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = (\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2})/\|\underline{z}\|$ where g is the gap function between a "point" $\underline{z} \in \mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$ and the Grassmann variety $G_2(\mathbb{R}^n)$ of $\mathbb{P}^{\binom{n}{2}-1}(\mathbb{R})$ (the embedment of $D_{\wedge^2(\mathbb{R}^n)}$ to the projective space $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$ via the Plücker embedding, [Mar. 1]). The gap function g and the related gap metric as well as the several forms that they may take were discussed in section 5.4.1. This analysis has helped us connect our formulae with several other optimization problems, such as the optimization of an eigenvalue of a matrix (which is addressed algorithmically in most cases) and subspace distance problems.

The central result of this chapter was given in Theorem 6.44, where we proved that if specific Grassmann varieties are given, e.g., n=5, then the gap g may be written as

$$\frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2} = \frac{\sqrt{\sum QPR^2(\underline{z})}}{\|\underline{z}\|^2} \tag{5.67}$$

which is very important, since the distance is uniquely expressed in terms of the components of the fixed point $\underline{z}$.

Chapter 6

Degenerate Cases: Maximum Distance from the Grassmann varieties $G_2(\mathbb{R}^n)$ and the algebrogeometric structure of $G_2(\mathbb{R}^5)$

6.1 Introduction

In the previous chapter we saw that in order to solve the approximate DAP as a distance optimization problem between a linear variety and the Grassmann variety we need at first to solve the optimization problem of a fixed 2-vector from the related Grassmann variety. The solution has been achieved via the so-called prime decomposition of a nominal 2-vector $\underline{z}$. The case where the eigenvalues of $T_{\underline{z}}$, i.e., the matrix form of $\underline{z}$, are distinct may be examined via eigenvalue-spectral analysis methodologies, as in [Lev., etc. 9] that lie within the wider area which investigates *approximate decomposability problems*, [Kar. & Lev. 9]. Actually the authors in [Lev., etc. 9] have proposed a 2-vector approach for such problems that has led to a number of closed-form solutions, contrary to the algorithmic-numerical approach in [Eck. & You. 1], [Gol., etc. 1] or in [Dela., etc. 1], [Kol. & Bad. 3], [Sav. & Li.1] for higher dimensions. On the other hand, if the eigenvalues of a nominal matrix, such as $T_{\underline{z}}$ in our case, have multiplicities, the implied approximations may be non-unique, a case which is still under investigation, [Kru. 1] and [Ten. & Sid. 1].

In this chapter we connect for the first time the non-uniqueness/degenerate eigenvalues of a 2-vector decomposition in $\bigwedge^2(\mathbb{R}^n)$, i.e., $\sigma_k = \cdots = \sigma_1$, to a new variety whose distance from the Grassmann variety $G_2(\mathbb{R}^n)$ is maximal. We refer to this

variety as $\mathcal{V}_1$ and we calculate its maximum gap from $G_2(\mathbb{R}^n)$, which actually implies the worst possible decomposable approximation. New significant results are also implied with regards to the topology of the Grassmann variety $G_2(\mathbb{R}^n)$ of the decomposable 2-vectors in the projective space.

Furthermore, we study the case of the non-trivial Grassmann variety $G_2(\mathbb{R}^5)$ separately (the first Grassmann variety that is not described by a single Quadratic Plücker Relation, [Hod. & Ped. 1]) where new algebrogeometric properties are obtained, enhancing in this way the algebraic-geometric results for low-order Grassmann varieties which is an important problem on its own, [Smi. 1], [Rid. 1], [San., etc. 1], [Eis., etc. 1], [Har. & Law. 1], [Mor. 1].

This chapter is organized as follows: In Section 6.2 some fundamental results are given regarding the uniqueness of matrix decompositions, best approximation/least distance problems and distinct eigenvalues. We explain how these are connected to the prime decomposition and the approximate DAP. In Section 6.3 we attempt to solve for the first time the least distance problem from the Grassmann variety in the case of degeneracy, i.e., repeated eigenvalues, via the use of algebraic geometry techniques and algebraic varieties, instead of numerical approaches that usually follow the standard SVD. We define the new variety as $\mathcal{V}_1$, and we derive the equation that describes it. It is proved that this variety, consists of those 2-vectors whose imaginary parts σ_i are equal and whose gap from $G_2(\mathbb{R}^n)$ is maximum. Thus our approach may give closed-form formulae even for the case of multiple eigenvalues.

In Section 6.4 we elaborate on the of study $\mathcal{V}_1$ and the degeneracy problem that may appear in the prime decomposition via algebrogeometric techniques this time; we select the $n = 5$ case, which is the least studied among the Grassmann varieties $G_2(\mathbb{R}^n)$ and we connect $G_2(\mathbb{R}^5)$ with $\mathcal{V}_1$. Analytically, in 6.4.1, the path-wise connectivity of $\mathcal{V}_1$ is proved and in 6.4.2 it is showed that $G_2(\mathbb{R}^5) \cup \mathcal{V}_1$ may be written as a Polynomial Sum of Squares. In 6.4.3 the complementarity of $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$ is examined in the related projective space and in 6.4.4 two new conjugates of a 2-vector are given.

The above duality and conjugacy results between $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$ are applied in Section 6.5 for the solution of two problems; in 6.5.1 we derive the Lagrange multipliers for the best decomposable approximation problem and in 6.5.2 for the derivation of the polar decomposition of a 2-vector. Note that while these kind of problems are viewed algorithmically by most researchers, our approach implies closed form solutions only in terms of the components of the 2-vector. Finally, in Section 6.6 we present how the previous analysis helps us obtain sensitivity criteria, with respect to the gap metric.

6.2 Basic Uniqueness Results

Let $r, n \in \mathbb{N}$ be two positive integers with $2r \leq n$. Let $\mathcal{V}_{r,n,T_{\underline{z}}}$ denote the variety of skew-symmetric matrices $T_{\underline{z}} \in \mathbb{R}^{n\times n}$ for a 2-vector $\underline{z}$, as this was introduced in the previous section with rank $2r$. If A is a fixed $n \times n$ skew-symmetric matrix, then we may consider the distance function $d : \mathcal{V}_{r,n,T_{\underline{z}}} \to \mathbb{R}_+$ such that

$$d(T_{\underline{z}}) = \|T_{\underline{z}} - A\|^2 \tag{6.1}$$

where $\|X\|^2 = \mathrm{tr}XX^t$ is the squared Frobenius norm. Next theorem provides an important uniqueness result.

Theorem 6.2.1. *[Hel. & Shay. 1] Let a skew-symmetric matrix $A \in \mathbb{R}^{n\times n}$ with distinct eigenvalues and m such that $n = 2m$ or $n = 2m + 1$.*

i) Distance d has exactly $\binom{m}{r}$ critical points.

ii) Every critical point of d is non-degenerate, i.e., the Hessian matrix at these points is invertible.

iii) The index of a critical point, i.e., the number of negative eigenvalues of the Hessian matrix at the critical point, associated with a permutation $\underline{\mu}$ ($\mu_1 < \cdots < \mu_r$ and $\mu_{r+1} < \cdots < \mu_k$) is given by

$$2\mathrm{Card}\{(i, j) : \mu_i < \mu_j,\ 1 \leq i \leq r,\ r + 1 \leq j \leq k\} \tag{6.2}$$

where Card denotes the cardinality (number of elements) of the above set $\{i, j\}$.

The above theorem implies a number of important results for the prime decomposition;

i) When the eigenvalues of a skew-symmetric matrix are distinct, there is a unique global minimum of d. Hence, the prime decomposition as well as the best decomposable approximation of a 2-vector are also unique.

ii) From the prime decomposition we easily see that the critical points of d are the rank $2n$ matrices of the form

$$X = \begin{cases} \mathrm{diag}(c_1L, ..., c_mL), & n = 2m \\ \mathrm{diag}(c_1L, ..., c_mL, 0), & n = 2m + 1 \end{cases} \tag{6.3}$$

where

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{6.4}$$

iii) If Eig(iX), Eig(iA) are the sets of the eigenvalues of X and A that have a positive imaginary part respectively, then the subvariety $\tilde{V}$ of $\mathcal{V}_{r,n,T_{\underline{z}}}$ that contains those critical points X for which Eig(iX) $\subseteq$ Eig(iA), is a fiber bundle with base manifold $G_r(\mathbb{R}^m)$ and the Euclidean space $\mathbb{R}^{r^2}$ as fiber, i.e., $\tilde{V}$ is written as a topological structure $\left(\mathcal{V}_{r,n,T_{\underline{z}}}, G_r(\mathbb{R}^m), \varphi, \mathbb{R}^{r^2}\right)$ with $\varphi : \mathcal{V}_{r,n,T_{\underline{z}}} \to G_r(\mathbb{R}^m)$, being a continuous surjection, [Hel. & Shay. 1]. In other words, all the critical points of d are contained in $\tilde{V}$. Conclusion (iv) in Theorem 6.2.1, states that every critical point has an even index. Then by using the formula :

Remark 6.2.1. *The problem of the distance of a point from a variety (or a manifold) has been also connected with the maximum number of cuts of these varieties before they separate into two pieces, i.e., the Betti number, since as shown in [Hel. & Shay. 1], the number of critical points which have index* $2m$ *is equal to* $(2m)^{(th)}$ *Betti number of the Grassmann manifold* $G_r(\mathbb{R}^m)$*. Then with the use of formula*

$$\frac{(c^n-1)\cdots(c^{n-m+1}-1)}{(c^m-1)\cdots(c-1)} = \sum_{j=0}^{m(n-m)} \lambda_{n,m}(j)c^j \tag{6.5}$$

it was proved in [Kolh. 1] that the set of all odd Betti numbers of the Grassmann variety is zero, but only for a finite field $\mathcal{F}$ *with* c *elements. Other connections between distances and cuts of surfaces (dimensional holes) have been examined in [Chee. 1], [Chee. 2], whereas some more recent results place the Betti numbers within the concept of the so-called Jacobian criteria for complete intersections, that may provide strong intersection information, [Avr. & Her. 1].*

Now, when σ_i in the prime decomposition are equal it is easy to verify that the prime decomposition is not unique and equivalently the above distance problem does not have a unique solution. The problem as far the distance function d is concerned and the case of degenerate singular values has been studied in [Stew. 2] among others via the notion of the so-called *numerical rank of a matrix*. Most approaches for the problem of degeneracy follow the numerical laws and variations of the standard SVD. In the next section, we try for the first time to connect the problem of degeneracy and best decomposable approximations with algebraic geometry techniques, via the use of specific algebraic varieties. This approach will enable us to imply closed form solutions, even for this case.

6.3 The Extremal variety $\mathcal{V}_1$ of $G_2(\mathbb{R}^n)$

In this section we show that the gap g in eqn.(6.6) is maximized when the imaginary eigenvalues of a 2-vector are equal. Thus we may obtain distance criteria even in the case of degeneracy.

In particular, we define this set as $\mathcal{V}_1$, we derive the equation that describes it and we calculate its distance from a random 2-vector.

Theorem 6.3.1. *The gap g between $\underline{z}$ and $G_2(\mathbb{R}^n)$ is equal to*

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \tag{6.6}$$

which is achieved at $\hat{\underline{z}} = \sigma_k \underline{x}_k$.

Proof. We have that

$$\begin{aligned} g\left(\underline{z}, G_2(\mathbb{R}^n)\right) :&= \min_{\underline{x} \in G_2(\mathbb{R}^n)} \text{gap}(\underline{z}, \underline{x}) = \min_{\underline{x} \in G_2(\mathbb{R}^n)} \sqrt{1 - \frac{<\underline{z}, \underline{x}>^2}{\|\underline{x}\|^2 \|\underline{z}\|^2}} = \\ &= \left(1 - \frac{1}{\|\underline{z}\|^2} \max_{\underline{x} \in G_2(\mathbb{R}^n)} \frac{<\underline{z}, \underline{x}>^2}{\|\underline{x}\|^2}\right)^{\frac{1}{2}} = \sqrt{1 - \frac{\sigma_k^2}{\|\underline{z}\|^2}} = \\ &= \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} \end{aligned}$$

since

$$\begin{aligned} \max_{\underline{y}_1, \underline{y}_2} | <\underline{z}, \underline{y}_1 \wedge \underline{y}_2> | &= \max_{\underline{y}_1, \underline{y}_2} <\underline{y}_1^t T_{\underline{z}}, \underline{y}_2> = \max_{\|\underline{y}_1\|=1} \left\langle \underline{y}_1^t T_{\underline{z}}, \frac{\underline{y}_1^t T_{\underline{z}}}{\|\underline{y}_1^t T_{\underline{z}}\|} \right\rangle = \\ &= \max_{\|\underline{y}_1\|=1} \|\underline{y}_1^t T_{\underline{z}}\| = \sigma_k \end{aligned}$$

□

Theorem 6.3.2. *The maximum gap of a 2-vector $\underline{z}$ from the Grassmann variety $G_2(\mathbb{R}^n)$ is equal to*

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{1 - \frac{1}{k}}$$

Proof. In Theorem 6.3.1 we saw that $g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}/\|\underline{z}\|$. Hence,

$$\begin{aligned} g\left(\underline{z}, G_2(\mathbb{R}^n)\right) &= \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\sqrt{\sigma_k^2 + \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}} = \frac{1}{\sqrt{1 + \frac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}} \leq \\ &\leq \frac{1}{\sqrt{1 + \frac{\sigma_k^2}{\sigma_k^2 + \sigma_k^2 + \sigma_k^2 + ... + \sigma_k^2}}} = \frac{1}{\sqrt{1 + \frac{1}{k-1}}} = \sqrt{1 - \frac{1}{k}} \end{aligned}$$

□

Theorem 6.3.3. *A multi-vector $\underline{z} \in \wedge^2(\mathbb{R}^n)$ has the maximum distance from the Grassmann variety $G_2(\mathbb{R}^n)$, if and only if*

$$\underline{z} = \sigma(\underline{x}_k + \underline{x}_{k-1} + ... + \underline{x}_1) \tag{6.7}$$

Proof. ($\Rightarrow$) From the previous theorem, a multi-vector $\underline{z}$ that has the maximum distance from $G_2(\mathbb{R}^n)$ satisfies the equation

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{1}{\sqrt{1 + \frac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}} = \sqrt{1 - \frac{1}{k}}$$

Therefore,

$$\frac{\sigma_k^2}{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2} = \frac{1}{k-1}$$

Equivalently

$$\underbrace{\sigma_k^2 + \sigma_k^2 + ... + \sigma_k^2}_{k-1 \text{ times}} = \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2$$

This means that the two sums of $k-1$ numbers are equal to each other while $0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_k$. Hence,

$$\sigma_k = \sigma_{k-1} = \sigma_{k-2} = ... = \sigma_1 = \sigma$$

and the result follows due to the prime decomposition of $\underline{z}$.
($\Leftarrow$) Let $\sigma_k = \sigma_{k-1} = \sigma_{k-2} = ... = \sigma_1 = \sigma$. Hence,

$$g\left(\underline{z}, G_2(\mathbb{R}^n)\right) = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\|\underline{z}\|} = \frac{\sqrt{\sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}}{\sqrt{\sigma_k^2 + \sigma_{k-1}^2 + \sigma_{k-2}^2 + ... + \sigma_1^2}} =$$

$$= \sqrt{\frac{(k-1)\sigma^2}{k\sigma^2}} = \sqrt{1 - \frac{1}{k}}$$

□

Definition 6.3.1. *The extremal variety*

$$\mathcal{V}_1 \equiv Extr\left(G_2(\mathbb{R}^n)\right) \tag{6.8}$$

is the variety containing all points that achieve the maximum distance from the Grassmann variety $G_2(\mathbb{R}^n)$.

Just like $G_2(\mathbb{R}^n)$ is described by the Quadratic Plucker relations, (QPR), we aim to find the equation describing $\mathcal{V}_1$.

Theorem 6.3.4. *Let $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$. The variety $\mathcal{V}_1 = Extr\left(G_2(\mathbb{R}^n)\right)$ is described by the equation*

$$\|\underline{z} \wedge \underline{z}\| = \sqrt{2\left(1 - \frac{1}{k}\right)} \cdot \|\underline{z}\|^2 \tag{6.9}$$

Proof. ($\Rightarrow$) If $\underline{z} \in \mathcal{V}_1$ then, due to Theorem 6.3.3, $\underline{z}$ is written as

$$\underline{z} = \sigma\underline{x}_1 + \sigma\underline{x}_2 + ... + \sigma\underline{x}_k \tag{6.10}$$

Therefore, $\|\underline{z}\|^2 = k\sigma^2$ and $\|\underline{z} \wedge \underline{z}\| = 2\sigma^2\sqrt{k(k-1)/2}$ because

$$\begin{aligned}\|\underline{z} \wedge \underline{z}\|^2 &= 4\sigma^4\|\underline{x}_1 \wedge \underline{x}_2 + \cdots + \underline{x}_{k-1} \wedge \underline{x}_k\|^2 = \\ &= 4\sigma^4 < \underline{x}_1 \wedge \underline{x}_2 + \cdots + \underline{x}_{k-1} \wedge \underline{x}_k, \underline{x}_1 \wedge \underline{x}_2 + \cdots + \underline{x}_{k-1} \wedge \underline{x}_k >= \\ &= 4\sigma^4\left((k-1) + (k-2) + \cdots + 1\right) = 4\sigma^4\frac{k(k-1)}{2}\end{aligned}$$

From these two equations we obtain

$$\|\underline{z} \wedge \underline{z}\| = \frac{\sqrt{2k(k-1)}}{k} \cdot \|\underline{z}\|^2 = \sqrt{2\left(1 - \frac{1}{k}\right)} \cdot \|\underline{z}\|^2$$

($\Leftarrow$) First we prove that

$$\underline{z}^{\wedge\mu} = \mu! \sum_{1 \le i_1 < ... < i_\mu \le k} \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_\mu}\underline{x}_{i_1} \wedge \underline{x}_{i_2} \wedge ... \wedge \underline{x}_{i_\mu},\ 2 \le \mu \le k \tag{6.11}$$

Indeed, for $\mu = 2$ and with the use of the prime decomposition we obtain:

$$\begin{aligned}\underline{z} \wedge \underline{z} &= \left(\sum_{i=1}^{k} \sigma_i\underline{x}_i\right) \wedge \left(\sum_{i=1}^{k} \sigma_i\underline{x}_i\right) = \\ &= \sigma_k\sigma_{k-1}\underline{x}_k \wedge \underline{x}_{k-1} + ... + \sigma_k\sigma_1\underline{x}_k \wedge \underline{x}_1 + \\ &+ \sigma_{k-1}\sigma_k\underline{x}_{k-1} \wedge \underline{x}_k + ... + \sigma_{k-1}\sigma_1\underline{x}_{k-1} \wedge \underline{x}_1 + ... \\ &+ \sigma_2\sigma_k\underline{x}_2 \wedge \underline{x}_k + ... + \sigma_2\sigma_1\underline{x}_2 \wedge \underline{x}_1 + \\ &+ \sigma_1\sigma_k\underline{x}_1 \wedge \underline{x}_k + ... + \sigma_1\sigma_2\underline{x}_1 \wedge \underline{x}_2 = \\ &= 2! \sum_{j>i} \sigma_i\sigma_j\underline{x}_i \wedge \underline{x}_j\end{aligned}$$

Suppose that (6.11) holds true. Then

$$\underline{z}^{\wedge(\mu+1)} = \underline{z}^{\wedge\mu} \wedge \underline{z} = \left(\mu! \sum_{1 \le i_1 < ... < i_\mu \le k} \sigma_{i_1} \cdots \sigma_{i_\mu}\underline{x}_{i_1} \wedge ... \wedge \underline{x}_{i_\mu}\right) \wedge \sum_{i=1}^{k} \sigma_i\underline{x}_i =$$

$$= (\mu+1)\mu! \sum_{1\leq i_1<\ldots<i_{\mu+1}\leq k} \sigma_{i_1}\cdots\sigma_{i_{\mu+1}}\underline{x}_{i_1}\wedge\ldots\wedge\underline{x}_{i_{\mu+1}}$$

which proves the result. Now, if $\|\underline{z}\wedge\underline{z}\| = \sqrt{2\left(1-\frac{1}{k}\right)}\cdot\|\underline{z}\|^2$ then with the use of formula (6.11), we have that

$$(k-1)\sum_{i=1}^{k}\sigma_i^4 - 2\sum_{\substack{i=1\\k>j>i}}^{k}\sigma_i^2\sigma_j^2 = 0 \tag{6.12}$$

But

$$\sum_{\substack{i=1\\k>j>i}}^{k}(\sigma_i^2-\sigma_j^2)^2 = \underbrace{(\sigma_1^4+\sigma_2^4+\ldots+\sigma_k^4)+\ldots+(\sigma_1^4+\sigma_2^4+\ldots+\sigma_k^4)}_{k-1 \text{ times}} - 2\sum_{\substack{i=1\\k>j>i}}^{k}\sigma_i^2\sigma_j^2 \tag{6.13}$$

Therefore, from equations (6.12), (6.13) we have that

$$\sigma_i = \sigma_j,\ \forall i = 1,2,\ldots,k,\ k>j>i.$$

□

Corollary 6.3.1. *Let $\underline{z} = \sigma(\underline{x}_k + \underline{x}_{k-1} + \ldots + \underline{x}_1)$ as in equation (6.7). Then $\mathcal{V}_1$ can be also described by the equation*

$$\left(\frac{\|\underline{z}^{\wedge\mu}\|}{\mu!}\right)^2 = \left(\frac{\|\underline{z}^2\|}{k}\right)^{\mu}\cdot\binom{k}{\mu} \tag{6.14}$$

Proof. Again, with the use of (6.11), for $\mu\leq k$, we have that

$$\|\underline{z}^{\wedge\mu}\|^2 = (\mu!)^2\sigma^{2\mu}\cdot\binom{k}{\mu} \tag{6.15}$$

From (6.7) we have that $\|\underline{z}\|^2 = k\sigma^2$ and the result readily follows. □

We now may solve the distance problem between a 2-vector and $\mathcal{V}_1$.

Theorem 6.3.5. *Let the prime decomposition of a 2-vector $\underline{z}$ be of the form $\underline{z} := \sum_{i=1}^{k}\sigma_i\underline{x}_i\wedge\underline{y}_i$. The distance between $\underline{z}$ and $\mathcal{V}_1$ is equal to*

$$d(\underline{z},\mathcal{V}_1) = \sum_{i=1}^{k}(\sigma_i-\bar{\sigma})^2 \tag{6.16}$$

and is realized for $\underline{v}_0 = \bar{\sigma}\sum_{i=1}^{k}\underline{x}_i\wedge\underline{y}_i$, where $\bar{\sigma} = \sum_{i=1}^{k}\sigma_i/n$.

Proof. Let $\underline{v} \in \mathcal{V}_1$. Then $\underline{v} = \sigma \sum_{i=1}^{k} \underline{a}_i \wedge \underline{b}_i$. Therefore,

$$\begin{aligned}\|\underline{z} - \underline{v}\|^2 &= \left\| \sum_{i=1}^{k} \sigma_i \underline{x}_i \wedge \underline{y}_i - \sigma \sum_{i=1}^{k} \underline{a}_i \wedge \underline{b}_i \right\|^2 = \\ &= \sum_{i=1}^{k} \sigma_i^2 + n\sigma^2 - 2 < \sum_{i=1}^{k} \sigma_i \underline{x}_i \wedge \underline{y}_i, \sigma \sum_{i=1}^{k} \underline{a}_i \wedge \underline{b}_i > \geq \\ &\geq \sum_{i=1}^{k} \sigma_i^2 + n\sigma^2 - 2\sigma \sum_{i=1}^{k} \sigma_i := f(\sigma)\end{aligned}$$

Since $f(\sigma)$ is a quadratic it is minimized when $f'(\sigma) = 0$, i.e., $\sigma = \sum_{i=1}^{k} \sigma_i / n := \bar{\sigma}$. In this case we have that

$$f(\sigma) = \sum_{i=1}^{k} (\sigma_i - \bar{\sigma})^2 \tag{6.17}$$

and its minimizer is $\underline{v}_0 = \bar{\sigma} \sum_{i=1}^{k} \underline{x}_i \wedge \underline{y}_i$. □

We complete the study of $\mathcal{V}_1$ for the Grassmann variety $G_2(\mathbb{R}^n)$ by examining the case where at least two σ_i in the prime decomposition are equal. If we name such a variety $\mathcal{V}_2$ we show that $\mathcal{V}_1$ is actually a sub-variety of $\mathcal{V}_2$.

Theorem 6.3.6. *Let $\mathcal{V}_2$ be the variety of all 2-vectors of the form $\underline{z} = \sigma_k \underline{x}_k + \sigma_{k-1} \underline{x}_{k-1} + ... + \sigma_1 \underline{x}_1$, where at least two σ_i are equal.*

i) The equation defying $\mathcal{V}_2$ is

$$det(S_{f,f'}) = 0 \tag{6.18}$$

where $S_{f,f'}$ is the Sylvester matrix for the polynomial

$$f_{\underline{z}}(x) = x^k - \|\underline{z}\|^2 x^{k-1} + \left(\frac{\|\underline{z} \wedge \underline{z}\|}{2!} \right)^2 x^{k-2} + ... + (-1)^k \left(\frac{\|\underline{z}^{\wedge k}\|}{k!} \right)^2 \tag{6.19}$$

ii) $\mathcal{V}_1$ is a sub-variety of $\mathcal{V}_2$.

Proof. i) Since $\underline{z}$ has at least two σ_i equal, then $f(x)$ and $f'(x)$ must have a common root. But the Sylvester matrix for two polynomials $P_1(x) = a_m x^m + \cdots + a_0$, $P_2(x) = b_n x^n + \cdots + b_0$ is defined as the $(m+n) \times (m+n)$ matrix formed by filling the matrix, beginning with the upper left corner with the coefficients of P_1, then shifting down one row and one column to the right and filling in the coefficients starting there until they hit the right side; this process is also repeated for the coefficients of P_2. Therefore, the result of the proposition follows.

ii) Let $\underline{z} \in \mathcal{V}_1$. Due to equation (6.14), polynomial $f_{\underline{z}}(x)$ can be written as

$$f(x) = x^k - \|\underline{z}\|^2 x^{k-1} + \cdots (-1)^\mu \left(\frac{\|\underline{z}^2\|}{k}\right)^\mu \cdot \binom{k}{\mu} + \cdots (-1)^k \left(\frac{\|\underline{z}^2\|}{k}\right)^k =$$
$$= \left(x - \frac{\|\underline{z}^2\|}{k}\right)^k$$

This means that $f(x)$ has multiple roots, thus $f(x)$ and $f'(x)$ have common roots, which proves the result.

□

Example 6.3.1. *Let* $\underline{z} \in \wedge^2(\mathbb{R}^5)$ *and* $a := -\|\underline{z}\|^2$, $b := \frac{\|\underline{z}\wedge\underline{z}\|^2}{4}$. *Then*

$$det\,(S_{f,f'}) = 0 \Leftrightarrow \begin{vmatrix} 1 & a & b \\ 2 & a & 0 \\ 0 & 2 & a \end{vmatrix} = -a^2 + 4b = -\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|^2 = 0$$

Hence, $\underline{z}$ *is an element of* $\mathcal{V}_1$.

Example 6.3.2. *If* $\underline{z} \in \wedge^2(\mathbb{R}^6)$ *and similarly* $a := -\|\underline{z}\|^2$, $b := \frac{\|\underline{z}\wedge\underline{z}\|^2}{4}$, $c := -\frac{\|\underline{z}\wedge\underline{z}\wedge\underline{z}\|^2}{36}$ *then*

$$det\,(S_{f,f'}) = 0 \Leftrightarrow \begin{vmatrix} 1 & a & b & c & 0 \\ 0 & 1 & a & b & c \\ 3 & 2a & b & 0 & 0 \\ 0 & 3 & 2a & b & 0 \\ 0 & 0 & 3 & 2a & b \end{vmatrix} = -a^2b^2 + 4b^3 + 4a^3c - 18abc + 27c^2 = 0$$

6.4 Properties of $\mathcal{V}_1$: the $G_2(\mathbb{R}^5)$ case

In this section, we resent the $\mathcal{V}_1$ approach which, contrary to the previous algorithmic approach, will give us a new perspective of the Grassmann variety in the projective space and the approximate decomposability problem in cases of degeneracy (σ_i are all equal), since, as we will show $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$ are complementary varieties. We work with $G_2(\mathbb{R}^5)$, since it is among the ones in $G_2(\mathbb{R}^n)$ which has been the least examined, as shown in [San., etc. 1]:

Grassmann Variety	Plücker Coordinates	QPR	Level of knowledge
$G_2(\mathbb{R}^4)$	6	1	Excellent: $G_2(\mathbb{R}^4) = S^2 \times S^2$(the Cartesian product of two 2-spheres)
$G_2(\mathbb{R}^6)$	15	15	Very good: Spectrahedron
$G_2(\mathbb{R}^8)$	28	70	Very good: Spectrahedron
$G_2(\mathbb{R}^{10})$	45	210	Very good: Spectrahedron
$G_2(\mathbb{R}^{12})$	66	495	Very good: Spectrahedron
$G_2(\mathbb{R}^{13})$	78	715	Very good: Spectrahedron
$G_3(\mathbb{R}^6)$	20	35	Good: Not a Spectrahedron
$G_3(\mathbb{R}^7)$	35	140	Good : Not a Spectrahedron
$G_3(\mathbb{R}^8)$	56	420	Good : Not a Spectrahedron
$G_3(\mathbb{R}^9)$	84	1050	Good : Not a Spectrahedron
$G_4(\mathbb{R}^8)$	70	721	Poor: Very Difficult

(The spectrahedron refers to the Grassmann orbitrope of the Grassmannian and it is defined as

$$\mathcal{S}_p = \{\underline{x} \in \mathbb{R}^n | A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0 \text{ for some symmetric matrices } A_i\} \tag{6.20}$$

where $\succ$ denotes a positive-definite matrix).

In the rest of this section we elaborate on the properties of $\mathcal{V}_1$ for the $G_2(\mathbb{R}^5)$ case and the several forms of connection between them and with their related projective space. Variety $\mathcal{V}_1$ is described through-out the section by $\|\underline{z}\|^2 = \|\underline{z} \wedge \underline{z}\|$, whereas $\underline{z}^*$ denotes the Hodge-star operator on $\underline{z}$.

6.4.1 The path-wise connectivity of $\mathcal{V}_1$

For the $G_2(\mathbb{R}^5)$ case the definition of $\mathcal{V}_1$ takes the following form.

Definition 6.4.1. *If* $\mathcal{V} := \left\{\underline{z} \in \bigwedge^2(\mathbb{R}^5) : \underline{z} = \sigma \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2),\ \sigma \in \mathbb{R}\right\}$ *where* $\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2$ *arise from the spectral decomposition of* $\underline{z}$*, then*

$$\mathcal{V}_1 = \left\{sp\{\underline{v} \in \mathbb{P}^9(\mathbb{R})\} : \underline{v} \in \mathcal{V}\right\} \tag{6.21}$$

Definition 6.4.2. *Let* $\underline{z} \in \bigwedge^2(\mathbb{R}^5)$*. If* $\underline{z} \wedge \underline{z} \neq \underline{0}$ *we define the vector* $\underline{r}_{\underline{z}}$ *as*

$$\underline{r}_{\underline{z}} := \frac{(\underline{z} \wedge \underline{z})^*}{\|\underline{z} \wedge \underline{z}\|} \tag{6.22}$$

Theorem 6.4.1. *Let* $\underline{z} \in \bigwedge^2(\mathbb{R}^5)$. *If* $\underline{z} \wedge \underline{z} \neq 0$ *then*

$$\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2 = \frac{1}{4} \cdot \left\|\underline{z} \wedge \underline{r}_{\underline{z}} + \underline{z}^*\right\|^2 \cdot \left\|\underline{z} \wedge \underline{r}_{\underline{z}} - \underline{z}^*|\right\|^2 \tag{6.23}$$

Proof. We have that

$$\begin{aligned}\left\|\underline{z} \wedge \underline{r}_{\underline{z}} + \underline{z}^*\right\|^2 &= \left\|\underline{z} \wedge \underline{r}_{\underline{z}}\right\|^2 + \|\underline{z}^*\|^2 + 2 < \underline{z} \wedge \underline{r}_{\underline{z}}, \underline{z}^* >= \\ &= 2\|\underline{z}\|^2 + 2 < \underline{z} \wedge \underline{r}_{\underline{z}}, \underline{z}^* >\end{aligned}$$

Similarly, $\left\|\underline{z} \wedge \underline{r}_{\underline{z}} - \underline{z}^*\right\|^2 = 2\|\underline{z}\|^2 - 2 < \underline{z} \wedge \underline{r}_{\underline{z}}, \underline{z}^* >$. Therefore,

$$\begin{aligned}\left\|\underline{z} \wedge \underline{r}_{\underline{z}} + \underline{z}^*\right\|^2 \cdot \left\|\underline{z} \wedge \underline{r}_{\underline{z}} - \underline{z}^*|\right\|^2 &= 4\left(\|\underline{z}\|^4 - < \underline{z} \wedge \underline{r}_{\underline{z}}, \underline{z}^* >^2\right) = \\ &= 4\left(\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z} \wedge \underline{r}_{\underline{z}}\|^2\right) = \\ &= 4\left(\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2\right)\end{aligned}$$

due to the Definition 6.4.2. □

Remark 6.4.1. *From Theorem 6.4.1 we have that, if* $\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2 = 0$ *then either* $\underline{z} \wedge \underline{r}_{\underline{z}} = \underline{z}^*$ *or* $\underline{z} \wedge \underline{r}_{\underline{z}} = -\underline{z}^*$. *However, the second set of equations cannot be solved, since by applying* $\underline{z}\wedge$ *at both sides we get:*

$$\underline{z} \wedge \underline{z} \wedge \frac{(\underline{z} \wedge \underline{z})^*}{\|\underline{z} \wedge \underline{z}\|} = -\underline{z} \wedge \underline{z}^* \Rightarrow \|\underline{z} \wedge \underline{z}\| = -\|\underline{z}\|^2$$

which is satisfied if and only if, $\underline{z} = 0$. *Therefore,*

$$sp\{\underline{z}\} \in \mathcal{V}_1 \Leftrightarrow \begin{cases} \underline{z} \wedge \underline{z} \neq 0 \\ \underline{z} \wedge \underline{r}_{\underline{z}} = \underline{z}^* \end{cases}$$

Proposition 6.4.1. *Let* $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$, $\sigma_1 \neq 0$, $\sigma_2 \geq \sigma_1$. *The set* $\{\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2, \underline{r}_{\underline{z}}\}$, *is a positively oriented basis of* $\mathbb{R}^5$, *where* $\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2$ *as in Definition 6.4.1 and* $\underline{r}_{\underline{z}}$ *as in Definition 6.4.2.*

Proof. We have that $\underline{z} \wedge \underline{z} = 2\sigma_1\sigma_2 \underline{a}_1 \wedge \underline{a}_2 \wedge \underline{b}_1 \wedge \underline{b}_2$. Therefore, $\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{b}_1 \wedge \underline{b}_2 = \sigma_1\sigma_2 \underline{z} \wedge \underline{z}/2$. Then,

$$\begin{aligned}\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{b}_1 \wedge \underline{b}_2 \wedge \underline{r}_{\underline{z}} &= \frac{1}{2\sigma_1\sigma_2} \underline{z} \wedge \underline{z} \wedge \frac{(\underline{z} \wedge \underline{z})^*}{\|\underline{z} \wedge \underline{z}\|} = \\ &= \frac{1}{2\sigma_1\sigma_2} \frac{\|\underline{z} \wedge \underline{z}\|^2}{\|\underline{z} \wedge \underline{z}\|} \wedge \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \wedge \underline{e}_4 \wedge \underline{e}_5 = \\ &= \frac{\|\underline{z} \wedge \underline{z}\|}{2\sigma_1\sigma_2} \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \wedge \underline{e}_4 \wedge \underline{e}_5 = \\ &= +1 \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \wedge \underline{e}_4 \wedge \underline{e}_5\end{aligned}$$

□

Remark 6.4.2. *Due to Proposition 6.4.1 , from now on, we can assume that the matrix* $U_{\underline{z}} := \left(\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2, \underline{r}_{\underline{z}}\right)$ *, has determinant equal to* $+1$*, i.e.,* $U_{\underline{z}} \in SO_5(\mathbb{R})$.

Theorem 6.4.2. *The set* $\mathcal{V}_1$ *is path-wise connected.*

Proof. Let $sp\{\underline{z}_1\}, sp\{\underline{z}_2\}$ two elements of $\mathcal{V}_1$. Then,

$$\underline{z}_1 = \frac{1}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2)\,,\ \underline{z}_2 = \frac{1}{\sqrt{2}} \cdot (\underline{a}_3 \wedge \underline{a}_4 + \underline{b}_3 \wedge \underline{b}_4)$$

are two representatives, where $\underline{a}_3, \underline{a}_4, \underline{b}_3, \underline{b}_4$ have the same properties with $\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2$, i.e. they are orthonormal. We consider their corresponding spectral matrices:

$$U_{\underline{z}_1} := \left(\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2, \underline{r}_{\underline{z}_1}\right),\ U_{\underline{z}_2} := \left(\underline{a}_3, \underline{a}_4, \underline{b}_3, \underline{b}_4, \underline{r}_{\underline{z}_2}\right) \tag{6.24}$$

both in $SO_5(\mathbb{R})$. Now, let $U := U_{\underline{z}_1}^{-1} \cdot U_{\underline{z}_2} \in SO_5(\mathbb{R})$. We consider a skew-symmetric matrix A, such that $e^A = U$. Then the path $U(t) = U_{\underline{z}_1} \cdot e^{At}$, connects $U_{\underline{z}_1}, U_{\underline{z}_2}$ in $SO_5(\mathbb{R})$. Indeed,

$$\det\left(U(t)\right) = \det\left(U_{\underline{z}_1}\right) \cdot e^{(trA)t} = 1 \cdot 1 = 1,\ U(0) = U_{\underline{z}_1}, U(1) = U_{\underline{z}_1} \cdot U_{\underline{z}_1}^{-1} \cdot U_{\underline{z}_2} = U_{\underline{z}_2} \tag{6.25}$$

Therefore, if $U(t) := (\underline{a}_1(t), \underline{a}_2(t), \underline{b}_1(t), \underline{b}_2(t), \underline{r}(t))$, then

$$\underline{z}(t) = 1/2 \cdot (\underline{a}_1(t) \wedge \underline{a}_2(t) + \underline{b}_1(t) \wedge \underline{b}_2(t)) \tag{6.26}$$

connects $\underline{z}_1, \underline{z}_2$ in $\mathcal{V}_1$. □

Example 6.4.1. *We will construct a path between two elements of* $\mathcal{V}_1$*,*

$$\underline{z}_1 = \frac{1}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2)\,,\ \underline{z}_2 = \frac{1}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 - \underline{b}_1 \wedge \underline{b}_2)$$

For these elements we have $U_{\underline{z}_1} := (\underline{a}_1, \underline{a}_2, \underline{b}_1, \underline{b}_2, \underline{r})\,,\ U_{\underline{z}_2} := (\underline{a}_3, \underline{a}_4, \underline{b}_3, \underline{b}_4, -\underline{r})$. *Therefore*

$$U = U_{\underline{z}_1}^{-1} \cdot U_{\underline{z}_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and since $e^A = U$*, then*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -\frac{\pi}{\sqrt{2}} \\ 0 & 0 & -\frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & 0 \end{pmatrix},\ e^{At} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1+\cos\pi t}{2} & \frac{1-\cos\pi t}{2} & \frac{\sin\pi t}{\sqrt{2}} \\ 0 & 0 & \frac{1-\cos\pi t}{2} & \frac{1+\cos\pi t}{2} & -\frac{\sin\pi t}{\sqrt{2}} \\ 0 & 0 & -\frac{\sin\pi t}{\sqrt{2}} & \frac{\sin\pi t}{\sqrt{2}} & \cos\pi t \end{pmatrix}$$

Hence, $U(t) = U_{\underline{z}_1} \cdot e^{At} = (\underline{a}_1(t), \underline{a}_2(t), \underline{b}_1(t), \underline{b}_2(t), \underline{r}(t))$, *where*

$$\begin{aligned}
&\underline{a}_1(t) \equiv \underline{a}_1, \underline{a}_2(t) \equiv \underline{a}_2, \\
&\underline{b}_1(t) \equiv \frac{1+\cos\pi t}{2} \cdot \underline{b}_1 + \frac{1-\cos\pi t}{2} \cdot \underline{b}_2 - \frac{\sin\pi t}{\sqrt{2}} \cdot \underline{r}, \\
&\underline{b}_2(t) \equiv \frac{1-\cos\pi t}{2} \cdot \underline{b}_1 + \frac{1+\cos\pi t}{2} \cdot \underline{b}_2 + \frac{\sin\pi t}{\sqrt{2}} \cdot \underline{r}, \\
&\underline{r}(t) \equiv \frac{\sin\pi t}{\sqrt{2}} \cdot \underline{b}_1 - \frac{\sin\pi t}{\sqrt{2}} \cdot \underline{b}_2 + \cos\pi t
\end{aligned} \tag{6.27}$$

Hence, $\underline{z}(t)$ *is given by*

$$\underline{z}(t) = \frac{1}{\sqrt{2}} \cdot \left(\underline{a}_1(t) \wedge \underline{a}_2(t) + \underline{b}_1(t) \wedge \underline{b}_2(t)\right),$$

where $\underline{z}(t) \in \mathcal{V}_1$, $\underline{z}(0) = \underline{z}_1$, $\underline{z}(1) = \underline{z}_2$, $\forall t \in [0,1]$

Remark 6.4.3. *In the previous example the two elements could be also connected by the path* $\underline{z}(t) = \left(\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge (\cos\pi t) \cdot \underline{b}_2 + (\sin\pi t) \cdot \underline{r}\right)/\sqrt{2}$, $t \in [0,1]$.

6.4.2 Polynomial Sum of Squares

In this section we proceed the investigation for the algebrogeometric structure of $\mathcal{V}_1$ and specifically we examine to what extent the polynomial $\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2$ implied by the equation $\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\| = 0$ that describes $\mathcal{V}_1$ may be written as a *Sum of Squares(SOS)*. If this is possible, then we can have a set of polynomial equations describing the variety, just like the QPR describe the Grassmann variety, [Hod. & Ped. 1], [Mar. 1]. It can be proved, using the Matlab SOSTOOLS toolbox, that $\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2$ can not be written as a SOS. Instead, we will prove that this is feasible for the variety $G_2(\mathbb{R}^5) \cup \mathcal{V}_1$.

Definition 6.4.3. *Let the* 10×2 *matrix* $A = (\underline{z} \wedge (\underline{z} \wedge \underline{z})^*, \underline{z}^*)$ *for* $\underline{z} \in \wedge^2(\mathbb{R}^5)$. *The fourth degree homogeneous polynomials* $f_i(\underline{z}), i = 1, ..., 45$ *are defined as the* 2×2 *minors of A, i.e.,* $C_2(A) = (f_1(\underline{z}), f_2(\underline{z}, ..., f_{45}(\underline{z})))^t$

Theorem 6.4.3. *Let* $\underline{z} \in \bigwedge^2(\mathbb{R}^5)$. *Then* $\|\underline{z} \wedge \underline{z}\|^2 \cdot (\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2) = \sum_{i=1}^{45} f_i^2(\underline{z})$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{45} f_i^2(\underline{z}) &= C_2(A^t) \cdot C_2(A) = det\left(A^t \cdot A\right) = \\
&= \begin{vmatrix} \|\underline{z} \wedge (\underline{z} \wedge \underline{z})^*\|^2 & < \underline{z} \wedge (\underline{z} \wedge \underline{z})^*, \underline{z}^* > \\ < \underline{z} \wedge (\underline{z} \wedge \underline{z})^*, \underline{z}^* > & \|\underline{z}\|^2 \end{vmatrix} = \\
&= \|\underline{z}\|^2 \cdot \|\underline{z} \wedge (\underline{z} \wedge \underline{z})^*\|^2 - \|\underline{z} \wedge \underline{z} \wedge (\underline{z} \wedge \underline{z})^*\|^2 = \\
&= \|\underline{z}\|^2 \cdot \|\underline{z}\|^2 \cdot \|\underline{z} \wedge \underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|^4 = \|\underline{z} \wedge \underline{z}\|^2 \cdot \left(\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2\right)
\end{aligned}$$

□

Corollary 6.4.1. *The polynomial* $\|\underline{z} \wedge \underline{z}\|^2(\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2)$ *whose zero locus defines the variety* $G_2(\mathbb{R}^5) \cup \mathcal{V}_1$ *is a polynomial SOS.*

Proof. It is not difficult to show that $\|\underline{z} \wedge \underline{z}\| = 0$ defines not only $G_2(\mathbb{R}^5)$, but also $G_2(\mathbb{R}^n)$. If $\underline{z} \in G_2(\mathbb{R}^n)$, then $\underline{z} = \underline{a} \wedge \underline{b}$, $\underline{a},\ \underline{b} \in \mathbb{R}^n$. Hence, $\underline{z} \wedge \underline{z} = \underline{a} \wedge \underline{b} \wedge \underline{a} \wedge \underline{b} = \underline{0}$. Conversely, if $\underline{z} \wedge \underline{z} = \underline{0}$, then from formulae (6.11) we have that $\sigma_i \sigma_j = 0$, for all $(i, j), j > i$ pairs. This means that $k - 1$ the number σ_i have to be zero. Due to $0 \leq \sigma_1 \leq \sigma_2 \leq ... \leq \sigma_k$, we have that $\sigma_1, ..., \sigma_{k-1} = 0$. Therefore, $\underline{z} = \sigma_k \cdot \underline{x}_k$. The result now follows since $\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2 = 0$ defines $\mathcal{V}_1$. □

Now, the next corollary is evident.

Corollary 6.4.2. *The equations* $f_i(\underline{z}) = 0$ *define the variety* $G_2(\mathbb{R}^5) \cup \mathcal{V}_1$.

6.4.3 The complementarity of $G_2(\mathbb{R}^5)$ and $\mathcal{V}_1$

In this section we prove that the gaps between a fixed 2-vector $\underline{z}$ and the varieties $G_2(\mathbb{R}^5), \mathcal{V}_1$ are complementary. First we will need the $\mathcal{V}_1$-representation of a 2-vector. This definition may be implied by Theorem 6.3.5, i.e., the 2-vector $\underline{v}_1 \in \mathcal{V}_1$ that best approximates a fixed vector $\underline{z}$ whose $G_2(\mathbb{R}^5)$-prime decomposition is $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$, is given by $\underline{v}_1 = (\sigma_2 + \sigma_1) \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2)/2$.

Definition 6.4.4. *The* $\mathcal{V}_1$*-decomposition of* $\underline{z}$ *is defined as*

$$\underline{z} = \frac{\sigma_2 + \sigma_1}{2} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2) + \frac{\sigma_2 - \sigma_1}{2} \cdot (\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2)$$

From Theorem (6.3.5) and for $n = 5$ we see that the gap $g(\underline{z}, V_1)$ between a 2-vector $\underline{z}$ and $\mathcal{V}_1$ is given by

$$g(\underline{z}, V_1) = \frac{\sigma_2 - \sigma_1}{\sqrt{2}\sqrt{\sigma_2^2 + \sigma_1^2}} = \frac{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{\sqrt{2}\|\underline{z}\|} = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\|\underline{z} \wedge \underline{z}\|}{\|\underline{z}\|^2}} \tag{6.28}$$

This observation leads to the following important results.

Theorem 6.4.4. *If* θ_1, θ_2 *are the gap angles of* $\underline{z}$ *from the varieties* $G_2(\mathbb{R}^5)$ *and* $\mathcal{V}_1$ *respectively, then*

$$\theta_1 + \theta_2 = \frac{\pi}{4}$$

Proof. The $G_2(\mathbb{R}^5)$- decomposition of any $\underline{z} \in \mathbb{R}^{10}$ is

$$\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 = \|\underline{z}\| \cdot \left(\frac{\sigma_2}{\|\underline{z}\|} \underline{b}_1 \wedge \underline{b}_2 + \frac{\sigma_1}{\|\underline{z}\|} \underline{a}_1 \wedge \underline{a}_2 \right) =$$

$$= \|\underline{z}\| \left(\cos\theta_1 \underline{b}_1 \wedge \underline{b}_2 + \sin\theta_1 \underline{a}_1 \wedge \underline{a}_2 \right)$$

since

$$\sin^2(\underline{z}\hat{,}\underline{x}) = \frac{\|\underline{z}\|^2 \cdot \|\underline{x}\|^2 - <\underline{z},\underline{x}>^2}{\|\underline{z}\|^2 \cdot \|\underline{x}\|^2} = \frac{\|\underline{z}\|^2 \cdot \sigma_2^2 - \sigma_2^4}{\|\underline{z}\|^2 \cdot \sigma_2^2} = \frac{\sigma_1^2}{\|\underline{z}\|^2}$$

for $\underline{x} = \sigma_2 \underline{b}_1 \wedge \underline{b}_2$ and $\underline{z} = \sigma_1 \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \underline{b}_1 \wedge \underline{b}_2$. Also, from Definition 6.4.4 we have that

$$\underline{z} = \|\underline{z}\| \cdot \left(\frac{\sigma_2 + \sigma_1}{\sqrt{2} \cdot \|\underline{z}\| \sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2) + \frac{\sigma_2 - \sigma_1}{\sqrt{2} \cdot \|\underline{z}\| \sqrt{2}} \cdot (\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2) \right) =$$

$$= \|\underline{z}\| \cdot \left(\frac{\cos\theta_2}{\sqrt{2}} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2) + \frac{\sin\theta_2}{\sqrt{2}} \cdot (\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2) \right),$$

where $0 \leq \theta_1, \theta_2 \leq \frac{\pi}{4}$. Therefore, $\sin\theta_1 = g\left(\underline{z}, G_2(\mathbb{R}^5)\right), \sin\theta_2 = g\left(\underline{z}, V_1\right)$. Thus, by eqn.(6.28) we have

$$\sin\theta_2 = \frac{\sigma_2 - \sigma_1}{\sqrt{2}\|\underline{z}\|} = \frac{1}{\sqrt{2}} (\cos\theta_1 - \sin\theta_1) = \sin\left(\frac{\pi}{4} - \theta_1\right)$$

Hence, $\theta_1 + \theta_2 = \pi/4$, since $0 \leq \theta_1, \theta_2 \leq \frac{\pi}{4}$. □

Corollary 6.4.3. $G_2(\mathbb{R}^5) = \mathrm{Extr}(\mathcal{V}_1) \Leftrightarrow \mathcal{V}_1 = \mathrm{Extr}(G_2(\mathbb{R}^5)$.

We complete the results regarding $\mathcal{V}_1$ and its general description in the next theorem.

Theorem 6.4.5. *Let $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$ be the prime decomposition of a 2-vector $\underline{z} \in \bigwedge^2(\mathbb{R}^5)$ such that $\|\underline{z}\|^2 = \|\underline{z} \wedge \underline{z}\|$. Then, $\underline{z}$ may be decomposed as $\underline{z} = \sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2$, where the decomposition is not unique. All orthonormal representations for a fixed $\underline{z}$ are parameterized by the two dimensional projective space $P^2(\mathbb{R})$.*

Proof. Let $\underline{z} = \sigma_1 \cdot \underline{a}'_1 \wedge \underline{a}'_2 + \sigma_2 \cdot \underline{b}'_1 \wedge \underline{b}'_2$. Because $\sigma_1^2 + \sigma_2^2 = \|\underline{z}\|^2 = \|\underline{z} \wedge \underline{z}\| = 2\sigma_1\sigma_2$ we get that

$$\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 = 0 \Rightarrow \sigma_1 = \sigma_2 \equiv \sigma$$

Since $\underline{z} \wedge \underline{z}$ is common for the two representations and non zero we have that $\mathrm{colspan}[\underline{b}_1, \underline{b}_2, \underline{a}_1, \underline{a}_2] = \mathrm{colspan}[\underline{b}'_1, \underline{b}'_2, \underline{a}'_1, \underline{a}'_2]$. We now consider a matrix U such that $[B, A] \cdot U = [B', A']$, $U = [U_1, U_2]$, where $B = [\underline{b}_1, \underline{b}_2]$, $A = [\underline{a}_1, \underline{a}_2]$, $B' = [\underline{b}'_1, \underline{b}'_2]$, $A' = [\underline{a}'_1, \underline{a}'_2]$. Then we get $\underline{b}'_1 \wedge \underline{b}'_2 = C_2[B, A] \cdot C_2[U_1]$ and $\underline{a}'_1 \wedge \underline{a}'_2 = C_2[B, A] \cdot C_2[U_2]$, where $C_2[U_1], C_2[U_2] \in \bigwedge(\mathbb{R}^4)$. Hence, if $\underline{x} := C_2[U_1]$ then $\underline{x}^* := C_2[U_2]$. Therefore,

$$\sigma \cdot \underline{a}'_1 \wedge \underline{a}'_2 + \sigma \cdot \underline{b}'_1 \wedge \underline{b}'_2 = C_2[B, A]\,(\sigma\underline{x} + \sigma\underline{x}^*) \tag{6.29}$$

and

$$\sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2 = C_2[B, A]\,(\sigma\underline{e}_1 + \sigma\underline{e}_6) \tag{6.30}$$

Thus if, $C_2[B,A]\,(\sigma\underline{x}+\sigma\underline{x}^*) = C_2[B,A]\,(\sigma\underline{e}_1+\sigma\underline{e}_6)$, by taking the left inverse matrix of $C_2[B,A]$ we have that $\sigma\underline{x}+\sigma\underline{x}^* = \sigma\underline{e}_1+\sigma\underline{e}_6$. Hence, by applying the Hodge star operator we obtain

$$\sigma\cdot(\underline{x}+\underline{x}^*) = \sigma\cdot(\underline{e}_1+\underline{e}_6) \tag{6.31}$$

If $\underline{x} = (x_1,x_2,x_3,x_4,x_5,x_6)$, then $\underline{x}^* = (x_6,-x_5,x_4,x_3,-x_2,x_1)$ and from eqn.(6.31),we have that

$$x_1+x_6 = 1,\ x_2-x_5 = 0,\ x_3+x_4 = 0 \tag{6.32}$$

Also, since $\underline{x}$ is decomposable in $\bigwedge^2(\mathbb{R}^4)$, it satisfies the unique Quadratic Plucker Relation

$$x_1x_6 - x_2x_5 + x_3x_4 = 0 \tag{6.33}$$

Therefore, eqn.(6.33) due to equations (6.32) and the fact that $\|\underline{x}\| = 1$, is equivalent to

$$\left(x_1-\frac{1}{2}\right)^2 + x_2^2 + x_3^2 = \frac{1}{4} \tag{6.34}$$

Hence, the representation of $\underline{z}$ is not unique. Also, the pair $(\underline{x}^*,\underline{x})$ corresponds to $\sigma\cdot\underline{a}_1\wedge\underline{a}_2+\sigma\cdot\underline{b}_1\wedge\underline{b}_2$, whereas $(\underline{x},\underline{x}^*)$ to $\sigma\cdot\underline{b}_1\wedge\underline{b}_2+\sigma\cdot\underline{a}_1\wedge\underline{a}_2$, which are the same representatives. Hence, we have identified $\underline{x},\underline{x}^*$ i.e. the antipodal points of the sphere above, which gives rise to the projective space $P^2(\mathbb{R})$. □

6.4.4 Conjugacy on the varieties $\mathcal{V}$ and $\mathcal{V}_1$

The decomposition $\underline{z} = \sigma_1\cdot\underline{a}_1\wedge\underline{a}_2+\sigma_2\cdot\underline{b}_1\wedge\underline{b}_2$, $\sigma_2>\sigma_1>0$ of a 2-vector $\underline{z}$ as a sum of two perpendicular decomposable vectors allows us to define a reflection $\bigwedge^2(\mathbb{R}^5)\to\bigwedge^2(\mathbb{R}^5)$ through the variety $\underline{z}\wedge\underline{z} = \underline{0}$. Due to Theorem 6.4.4 and Corollary 6.4.3, the vector $\underline{x} = \sigma_1\cdot\underline{a}_1\wedge\underline{a}_2$ realizes the smallest distance from the variety defined by $\underline{z}\wedge\underline{z} = \underline{0}$ in $\bigwedge^2(\mathbb{R}^5)$, which means that $\underline{x}$ is perpendicular to the tangent space of the variety defined by $\underline{z}\wedge\underline{z} = \underline{0}$ at the point $\sigma_2\cdot\underline{b}_1\wedge\underline{b}_2$. Now, if $\hat{\underline{z}}$ denotes the *reflection* of $\underline{z}$ which is orthogonal to $\underline{x}$, through the plane, then

$$\begin{aligned}\hat{\underline{z}} &= \underline{z} - 2\cdot\frac{<\underline{z},\underline{x}>}{\|\underline{x}\|^2}\cdot\underline{x} = \underline{z} - 2\cdot\frac{\sigma_1^2}{\sigma_1^2}\cdot\sigma_1\underline{a}_1\wedge\underline{a}_2 = \\ &= \sigma_2\cdot\underline{b}_1\wedge\underline{b}_2 - \sigma_1\cdot\underline{a}_1\wedge\underline{a}_2 = \sigma_2\cdot\underline{b}_1\wedge\underline{b}_2+\sigma_2\cdot\underline{a}_2\wedge\underline{a}_1\end{aligned}$$

Definition 6.4.5. *Let $\underline{z}\in\bigwedge^2(\mathbb{R}^5)$. The vector $\hat{\underline{z}} = \sigma_2\cdot\underline{b}_1\wedge\underline{b}_2-\sigma_1\cdot\underline{a}_1\wedge\underline{a}_2$ is called the* (directed) conjugate *of $\underline{z}$.*

Proposition 6.4.2. *Let $\mathcal{V}$ as in Definition 6.4.2. Then, conjugacy ($\hat{}$) is a well defined isometry $\bigwedge^2(\mathbb{R}^{10})\backslash\mathcal{V}\to\bigwedge^2(\mathbb{R}^{10})\backslash\mathcal{V}$, whose fixed points are given by the set $\{\underline{z}\in\bigwedge^2(\mathbb{R}^{10}) : \underline{z}\wedge\underline{z} = \underline{0}\}$.*

Proof. Since any 2-vector $\underline{z}$ can be written as $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$, then $\hat{\underline{z}}$ is defined uniquely as $\hat{\underline{z}} = \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 + \sigma_1 \cdot \underline{a}_2 \wedge \underline{a}_1$ and $\|\hat{\underline{z}}\|^2 = \|\underline{z}\|^2 = \sigma_1^2 + \sigma_2^2$. If $\underline{z} = \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$ then $\hat{\underline{z}} = \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$. Conversely, if $\underline{z} \notin \mathcal{V}$ and $\underline{z} = \hat{\underline{z}}$ then $\underline{z} = \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$ which proves the proposition. □

Remark 6.4.4. *From Definition 6.4.5 and Theorem 6.13 and since the prime decomposition of $\underline{z}$ may be written as $\underline{z} = (\underline{z} + \hat{\underline{z}})/2 + (\underline{z} - \hat{\underline{z}})/2$, the closest vector to $\underline{z} \in \bigwedge^2(\mathbb{R}^5)$ from $G_2(\mathbb{R}^5)$ may also be written as $(\underline{z} + \hat{\underline{z}})/2$,*

Theorem 6.4.6. *Conjugacy $(\hat{})$ cannot be continuously extended to $\mathcal{V}$.*

Proof. If $\underline{z} = \sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2$ and $\underline{z}_n := (\sigma + 1/n) \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2$, then

$$\hat{\underline{z}}_n \to \sigma \cdot \underline{a}_1 \wedge \underline{a}_2 - \sigma \cdot \underline{b}_1 \wedge \underline{b}_2 \tag{6.35}$$

and if $\underline{w}_n := \sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + (\sigma + 1/n) \cdot \underline{b}_1 \wedge \underline{b}_2$, then

$$\hat{\underline{w}}_n \to -\sigma \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma \cdot \underline{b}_1 \wedge \underline{b}_2 \tag{6.36}$$

Therefore, $\underline{z}_n, \underline{w}_n \to \underline{z}$ while $\lim_{n\to\infty} \hat{\underline{z}}_n \neq \lim_{n\to\infty} \hat{\underline{w}}_n$. □

Remark 6.4.5. *It is easy to prove the following formulas, relating $\underline{z}$ and $\hat{\underline{z}}$.*

1) $< \underline{z}, \hat{\underline{z}} >= \sigma_2^2 - \sigma_1^2$.

2) $\underline{z} \wedge \hat{\underline{z}} = \underline{0}$.

3) $\hat{\underline{z}} \wedge \hat{\underline{z}} = -\underline{z} \wedge \underline{z}$.

4) $\sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 = (\underline{z} + \hat{\underline{z}})/2$, $\sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 = (\underline{z} - \hat{\underline{z}})/2$.

5) $\hat{\hat{\underline{z}}} = z$, *i.e.,* $(\hat{})$ *is an involution.*

We now define the second form of conjugacy with respect to $\mathcal{V}_1$.

Definition 6.4.6. *The $\mathcal{V}_1$- conjugate of a 2-vector $\underline{z} \in \bigwedge^(\mathbb{R}^5)$ is defined as*

$$\breve{\underline{z}} = \sigma_1 \underline{b}_1 \wedge \underline{b}_2 + \sigma_2 \underline{a}_1 \wedge \underline{a}_2 \tag{6.37}$$

Remark 6.4.6. *Similarly to the directed conjugate $\hat{\underline{z}}$, due to the above definition, the closest vector to $\underline{z} \in \wedge^2(\mathbb{R}^5)$ from the variety $\mathcal{V}_1$ is also written in the form $(\underline{z} + \breve{\underline{z}})/2$.*

We observe that the $\mathcal{V}_1$- conjugacy $(\breve{})$ leaves $\mathcal{V}_1$ fixed, because $(\sigma_2 \underline{b}_1 \wedge \underline{b}_2 + \sigma_1 \underline{a}_1 \wedge \underline{a}_2)\breve{} = \sigma_1 \underline{b}_1 \wedge \underline{b}_2 + \sigma_2 \underline{a}_1 \wedge \underline{a}_2$, but it is not well defined at $G_2(\mathbb{R}^5)$, since if

$$\underline{z}_n = \left(1 + \frac{1}{n}\right) \underline{b}_1 \wedge \underline{b}_2 + \frac{1}{n} \underline{a}_1 \wedge \underline{a}_2 \to \underline{b}_1 \wedge \underline{b}_2, \; \underline{w}_n = \left(1 + \frac{1}{n}\right) \underline{b}_1 \wedge \underline{b}_2 + \frac{1}{n} \underline{d}_1 \wedge \underline{d}_2 \to \underline{b}_1 \wedge \underline{b}_2$$

then

$$\underline{\breve{z}}_n = \frac{1}{n}\underline{b}_1 \wedge \underline{b}_2 + \left(1 + \frac{1}{n}\right)\underline{a}_1 \wedge \underline{a}_2 \to \underline{a}_1 \wedge \underline{a}_2$$

and

$$\underline{\breve{w}}_n = \frac{1}{n}\underline{b}_1 \wedge \underline{b}_2 + \left(1 + \frac{1}{n}\right)\underline{d}_1 \wedge \underline{d}_2 \to \underline{d}_1 \wedge \underline{d}_2 \neq \underline{a}_1 \wedge \underline{a}_2$$

We can extend the $\mathcal{V}_1$- conjugation to the Grassmann variety if we define

$$\mathcal{X} := \overline{\mathbb{P}^9(\mathbb{R}) \setminus G_2(\mathbb{R}^5)} = \left\{(\underline{z}, \underline{r}) \in \mathbb{P}^9(\mathbb{R}) \times G_2(\mathbb{R}^5) : \underline{z}^* \wedge \underline{r} = 0\right\} \tag{6.38}$$

The following theorem completes the study of $\mathcal{V}_1$ for the $n = 5$ case.

Theorem 6.4.7. *The conjugacy* $(\breve{})$ *is well defined at* $\mathcal{X} = \overline{\mathbb{P}^9(\mathbb{R}) \setminus G_2(\mathbb{R}^5)}$.

Proof. $\mathcal{X}$ contains the points $(\underline{z}, (\underline{z} \wedge \underline{z})^*)$ and the points of the "blow-up" at $G_2(\mathbb{R}^5)$, i.e., the points $(\underline{b}_1 \wedge \underline{b}_2, \underline{r})$ such that $< \underline{r}, \underline{b}_i >= 0$, $i = 1, 2$. Thus, $(\breve{})$ is well defined for $\underline{z} \in \mathbb{P}^9(\mathbb{R}) \setminus G_2(\mathbb{R}^5)$ as $(\breve{\underline{z}}, (\breve{\underline{z}} \wedge \breve{\underline{z}})^*)$ and at the "blow -up", as $(\underline{b}_1 \wedge \underline{b}_2, \underline{r})\breve{} = ((\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{r})^*, \underline{r})$. Hence, the $\mathcal{V}_1$- conjugation $(\breve{})$ is well defined at $\mathcal{X}$ and

$$(\underline{z}_n, (\underline{z}_n \wedge \underline{z}_n)^*) \to (\underline{b}_1 \wedge \underline{b}_2, (\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{a}_1 \wedge \underline{a}_2)^*)$$

$$(\underline{w}_n, (\underline{w}_n \wedge \underline{w}_n)^*) \to (\underline{b}_1 \wedge \underline{b}_2, (\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{d}_1 \wedge \underline{d}_2)^*)$$

i.e., they converge to same point unless we are on the fiber through $\underline{b}_1 \wedge \underline{b}_2$. □

Remark 6.4.7. *We may easily prove similar formulae to Remark 6.5.1 for the* $\mathcal{V}_1$- *conjugation* $(\breve{})$ *as well.*

6.5 Applications and Alternative Decompositions

We now apply the previous congugacy-duality results to solve the best decomposable approximation problem via the method of the Lagrange multipliers and to derive a special form of the polar decomposition.

6.5.1 Calculation of the Lagrange multipliers

In this section we propose how to calculate the Lagrange multipliers of the basic minimization problem of this thesis:

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \text{when} \quad \underline{x} \quad \text{is decomposable} \tag{6.39}$$

Note, that this the first time the Lagrange multiplies are given in closed-form formulae for this kind of manifold constrained optimization problems, since most approaches, follow the SVD methodology, as we have mentioned before, due to the complexity of the constraints. Here we address (6.39) using the previous conjugacy-dual analysis tools results for the $\bigwedge^2(\mathbb{R}^5)$ case.

Lemma 6.5.1. *Let $\underline{z}, \underline{x} \in \bigwedge^2(\mathbb{R}^5)$ and $\underline{x}$ be decomposable. Then, minimization problem* (6.39) *is equivalent to*

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad when \quad \frac{1}{2} \cdot \underline{x} \wedge \underline{x} = 0 \tag{6.40}$$

Proof. This is straight forward by Corollary 6.4.1. □

Theorem 6.5.1. *The Lagrange multiplies vector for a 2-vector $\underline{z}$ is given by*

$$\underline{r}_1^* = -\frac{\sigma_1}{\sigma_2} \cdot \underline{r}_{\underline{z}} + \mu_1 \underline{b}_1 + \mu_2 \underline{b}_2 \tag{6.41}$$

where μ_1, μ_2 are free real numbers, σ_i, $\underline{b}_i$, $i = 1, 2$ are obtained by the prime decomposition of $\underline{z}$ and $\underline{r}_{\underline{z}}$ as in Definition 6.4.2.

Proof. If $\underline{r}_1 \in \mathbb{R}^5$ the Lagrange multiplier considered in $\bigwedge^4(\mathbb{R}^5)$, then the Lagrangian is: $L(\underline{x}, \underline{r}_1) = (\|\underline{x} - \underline{z}\|^2)/2 - < \underline{r}_1, (\underline{x} \wedge \underline{x})/2 >$. Hence,

$$\begin{aligned} dL(\underline{x}, \underline{r}_1) = 0 \Rightarrow &< \underline{x} - \underline{z}, d\underline{x} > - < \underline{r}_1, \underline{x} \wedge d\underline{x} >= 0 \Rightarrow \\ \Rightarrow &< \underline{x} - \underline{z}, d\underline{x} > - < (\underline{r}_1^* \wedge \underline{x})^*, d\underline{x} >= 0 \end{aligned}$$

Therefore,

$$< \underline{x} - \underline{z} - (\underline{r}_1^* \wedge \underline{x})^*, d\underline{x} >= 0 \Rightarrow \underline{z} = \underline{x} - (\underline{r}_1^* \wedge \underline{x})^*$$

On the other hand, the solution of problem (6.39) is $\underline{x} = \sigma_2 \underline{b}_1 \wedge \underline{b}_2$ for $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$. Therefore, by (6.22), we have that

$$\underline{z} = \sigma_1 \cdot (\underline{r}_{\underline{z}} \wedge \underline{b}_1 \wedge \underline{b}_2)^* + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 = \left(\frac{\sigma_1}{\sigma_2} \underline{r}_{\underline{z}} \wedge \underline{x} \right)^* + \underline{x} \tag{6.42}$$

Thus,

$$-(\underline{r}_1^* \wedge \underline{x})^* = \left(\frac{\sigma_1}{\sigma_2} \underline{r}_{\underline{z}} \wedge \underline{x} \right)^* \Leftrightarrow \underline{r}_1^* = -\frac{\sigma_1}{\sigma_2} \cdot \underline{r}_{\underline{z}} + \mu_1 \underline{b}_1 + \mu_2 \underline{b}_2,$$

where μ_1, μ_2 are free real numbers. Hence the set of the Lagrange multipliers is a 2-dimension affine subspace of $\mathbb{R}^5$. □

Remark 6.5.1. *From eqn.*(6.42) *we obtain*

$$\underline{z} - \underline{x} = \frac{\left(\underline{r}_{\underline{z}} \wedge \underline{x} \right)^* \sigma_1}{\sigma_2} \tag{6.43}$$

If we conceder a 10×10 matrix $R_{\underline{z}}$ such that $\left(\underline{r}_{\underline{z}} \wedge \underline{x} \right)^* = R_{\underline{z}} \cdot \underline{x}$, then eqn.(6.43) *is written as*

$$\left(\frac{\sigma_2}{\sigma_4} R_{\underline{z}} + I_{10} \right) \cdot \underline{x} = \underline{z} \tag{6.44}$$

Thus, we have a new way to calculate the best approximation $\underline{x}$ from $\underline{z}$, i.e.,

a) If $\sigma_2 > \sigma_1$ then $\underline{x}$ must be unique. Hence,

$$\underline{x} = \left(\frac{\sigma_1}{\sigma_2} R_{\underline{z}} + I_{10}\right)^{-1} \underline{z} = \left(\frac{\sigma_1^2 R_{\underline{z}}^2 - \sigma_1\sigma_2 R_{\underline{z}}}{\sigma_2^2 - \sigma_1^2} + I_{10}\right) \underline{z}$$

b) If $\sigma_2 = \sigma_1$ then $\underline{x}$ is not unique, therefore we have that the right kernel of $(\sigma_1/\sigma_2 R_{\underline{z}} + I_{10})$ is $\{\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2, \underline{b}_1 \wedge \underline{a}_1 + \underline{b}_2 \wedge \underline{a}_2, \underline{b}_2 \wedge \underline{a}_1 - \underline{b}_1 \wedge \underline{a}_2\}$. Thus,

$$\underline{x} = \underline{x}_0 + \kappa_1(\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2) + \kappa_2(\underline{b}_1 \wedge \underline{a}_1 + \underline{b}_2 \wedge \underline{a}_2) + \kappa_3(\underline{b}_2 \wedge \underline{a}_1 - \underline{b}_1 \wedge \underline{a}_2),$$

where $\underline{x}_0$ is a particular solution and $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ such that the vector $\underline{x}$ is decomposable.

c) If $\sigma_1 = 0$ then $\underline{x} = \underline{z}$.

Proposition 6.5.1. *The affine subspace $\underline{r}_1^*$ satisfies the following equation:*

$$\left(T_{\underline{z}}^2 + \sigma_2^2 I_5\right) \underline{x} = -\sigma_1\sigma_2 \cdot \underline{r}_{\underline{z}} \tag{6.45}$$

Proof. It is easy to see that $(-\sigma_1/\sigma_2) \cdot \underline{r}_{\underline{z}}$ is a specific solution of eqn.(6.45). Furthermore, from the spectral analysis of $T_{\underline{z}}$ we have $\ker(T^2 + \sigma_2^2 I_5) = \mathrm{sp}\{\underline{b}_1, \underline{b}_2\}$ and the conclusion is evident. □

Example 6.5.1. *We shall calculate the Lagrange multipliers for the minimization problem*

$$\min_{\underline{x}} \|\underline{x} - \underline{z}\| \quad \textit{when} \quad \underline{x} \in \bigwedge^2(\mathbb{R}^5) \quad \textit{is decomposable}$$

The Quadratic Plücker Relations for this problem are given by Corollary 6.4.1:

$$x_1x_8 - x_2x_6 + x_3x_5 = 0,\ x_1x_9 - x_2x_7 + x_4x_5 = 0,\ x_1x_{10} - x_3x_7 + x_4x_6 = 0,$$

$$x_2x_{10} - x_3x_9 + x_4x_8 = 0,\ x_5x_{10} - x_6x_9 + x_7x_8 = 0$$

If $\underline{y} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ the vector of the five Lagrange multipliers $\lambda_1, ..., \lambda_5$ we want to calculate, then the Lagrangian is:

$$\begin{aligned} L(\underline{x}, \underline{y}) = \frac{1}{2}\left[(x_1 - z_1)^2 + (x_2 - z_2)^2 + ... + (x_{10} - z_{10})^2\right] - \lambda_1(x_1x_8 - x_2x_6 + x_3x_5) - \\ - \lambda_2(x_1x_9 - x_2x_7 + x_4x_5) - \lambda_3(x_1x_{10} - x_3x_7 + x_4x_6) - \lambda_4(x_2x_{10} - x_3x_9 + x_4x_8) - \\ - \lambda_5(x_5x_{10} - x_6x_9 + x_7x_8) \end{aligned}$$

Therefore, we have the following system of 15 equations:

$$\frac{\partial L}{\partial x_1} = \lambda_3x_{10} - r_1x_8 - \lambda_2x_9 + (x_1 - z_1) = 0, ..., \frac{\partial L}{\partial x_{10}} = -\lambda_3x_1 - \lambda_4x_2 - \lambda_5x_5 + (x_{10} - z_{10}) =$$

$$= 0, \frac{\partial L}{\partial \lambda_1} = -x_1x_8 + x_2x_6 - x_3x_5 = 0, ..., \frac{\partial L}{\partial \lambda_5} = -x_{10}x_5 - x_7x_8 + x_6x_9 = 0$$

Now, let $\underline{z} = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)$. *By eliminating* $\lambda_1, ..., \lambda_5$ *from this system we obtain two solutions for* $\underline{x}$*:*

$$\underline{x}_1 = (-3.81187, -3.81885, 0.256576, 3.00383, 0.49817,$$
$$8.6079, -2.16234, 8.65719, -1.77373, 6.92875)$$

$$\underline{x}_2 = (5.81187, -4.18115, 0.743424, 1.99617, -0.49817,$$
$$2.3921, -0.83766, -1.65719, 0.77373, -0.928749)$$

The first solution achieves the minimum distance we want. The substitution of $\underline{x}_1$ *to the above system implies:*

$$\lambda_1 = -0.434745 + 0.78802\lambda_4 - 0.567264\lambda_5,$$
$$\lambda_2 = 0.202979 - 0.067309\lambda_4 - 2.258183\lambda_5$$
$$\lambda_3 = -0.243646 - 1.00183\lambda_4 + 0.130689\lambda_5$$

where $\lambda_4, \lambda_5 \in \mathbb{R}$. *Therefore,*

$$\underline{y} = \underline{u}_0 + r_4 \cdot \underline{u}_1 + r_5 \cdot \underline{u}_2$$

where

$$\underline{u}_0 = (0, 0, -0.24364, -0.20297, -0.43474),$$
$$\underline{u}_1 = (0, -1, -1.00183, 0.067309, 0.78802)$$
$$\underline{u}_2 = (1, 0, 0.13068, 2.25818, -0.56726)$$

We must now prove that $\underline{u}_0 + r_4 \cdot \underline{u}_1 + r_5 \cdot \underline{u}_2 = -(\sigma_1/\sigma_2) \cdot \underline{r} + \mu_1 \underline{b}_1 + \mu_2 \underline{b}_2,\ \mu_1, \mu_2 \in \mathbb{R}$ *where,*

$$\underline{r} = (-0.0785093, 0.094211, 0.549565, 0.204124, 0.800795)^t$$
$$\underline{b}_1 = (0.395816, -0.0925693, -0.0410099, 0.900056, -0.151586)^t$$
$$\underline{b}_2 = (-0.0182749, -0.613107, -0.616617, 0, 0.493507)^t$$
$$\underline{a}_1 = (0.236471, 0.760887, -0.527125, 0, 0.29542)^t$$
$$\underline{a}_2 = (-0.883693, 0.166455, -0.195497, 0.38501, -0.0701948)^t$$

and $\sigma_1 = 8.16558, \sigma_2 = 15.5988$ *are obtained by the spectral analysis of*

$$T_{\underline{z}} = \begin{pmatrix} 0 & 2 & -8 & 1 & 5 \\ -2 & 0 & 0 & 11 & -3 \\ 8 & 0 & 0 & 7 & -1 \\ -1 & -11 & -7 & 0 & 6 \\ -5 & 3 & 1 & -6 & 0 \end{pmatrix}$$

But these two affine subspaces are equal, since

$$\underline{r}_0 = -0.523475\underline{r} - 0.106799\underline{b}_1 - 0.0643132_1,\ -0.523475 = -\frac{\sigma_1}{\sigma_2}$$

and $\underline{b}_1 \wedge \underline{b}_2 \parallel \underline{u}_1 \wedge \underline{u}_2 \Leftrightarrow \underline{u}_1 \wedge \underline{u}_2 = -0.244369\underline{b}_1 \wedge \underline{b}_2$*, because*

$$\begin{aligned}\underline{u}_1 \wedge \underline{u}_2 = (&-0.24436, -0.24481, 0.016448, 0.19256, 0.031936,\\ &0.55183, -0.13862, 0.55499, 0.113709, 0.44184),\end{aligned}$$

$$\begin{aligned}\underline{b}_1 \wedge \underline{b}_2 = (&1, 1.00183, -0.0673098, -0.788021, -0.130689,\\ &-2.25818, 0.567265, -2.27111, 0.465318, -1.81768)\end{aligned}$$

6.5.2 The Polar Decomposition

The polar decomposition has been characterized as the generalization of the representation $re^{i\theta}$ of complex numbers to matrices. In matrix form, e.g., [Gan. 1], [Gol. & Van. 2], [Hig. 1] among others, it states that for a matrix $A \in \mathbb{C}^{m\times n}$, $m \geq n$, there exists a matrix $U \in \mathbb{C}^{m\times n}$ and a unique Hermitian positive semi-definite matrix $H \in \mathbb{C}^{n\times n}$ such that $A = UH$, $(\bar{U})^t U = I_n$. Moreover, if rank$A = n$ then H is positive definite and U is uniquely defined. Polar decomposition problems are mainly addressed via numerical analysis methods [Hig. 1]. However, the conjugacy analysis we presented earlier may provide a very helpful polar decomposition in a 2-vector form, which does not require any numerical methods, since the related components may be directly computed via specific closed-form formulae. Indeed, if $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$, is the prime decomposition of a 2-vector in $\wedge^2(\mathbb{R}^5)$ then, due to its gaps between the Grassmann variety and $\mathcal{V}_1$, $\underline{z}$ may be written as $\underline{z} = \|\underline{z}\|\,(\cos\theta \cdot \underline{b}_1 \wedge \underline{b}_2 + \sin\theta \cdot \underline{a}_1 \wedge \underline{a}_2)$. Then for the directed and the $\mathcal{V}_1$- conjugates of $\underline{z}$ we have

$$\hat{\underline{z}} = \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 - \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 = \|\underline{z}\|\,(\cos\theta \cdot \underline{b}_1 \wedge \underline{b}_2 - \sin\theta \cdot \underline{a}_1 \wedge \underline{a}_2) \tag{6.46}$$

$$\breve{\underline{z}} = \sigma_1 \cdot \underline{b}_1 \wedge \underline{b}_2 + \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 = \|\underline{z}\|\,(\sin\theta \cdot \underline{b}_1 \wedge \underline{b}_2 + \cos\theta \cdot \underline{a}_1 \wedge \underline{a}_2) \tag{6.47}$$

where, $0 < \theta < \frac{\pi}{4}$. Now, if we consider the complex number $\|\underline{z}\|e^{i\theta}$ and the complex vector $\underline{b}_1 \wedge \underline{b}_2 + i \cdot \underline{a}_1 \wedge \underline{a}_2$ we have that $\|\underline{z}\|e^{i\theta}(\underline{b}_1 \wedge \underline{b}_2 + i \cdot \underline{a}_1 \wedge \underline{a}_2) = \hat{\underline{z}} + i\breve{\underline{z}}$ and $\|\underline{z}\|e^{-i\theta}(\underline{b}_1 \wedge \underline{b}_2 + i \cdot \underline{a}_1 \wedge \underline{a}_2) = \underline{z} + i\hat{\underline{z}}$. The 2-vector definition of the polar decomposition is now evident.

Definition 6.5.1. *The polar decomposition of a multi-vector* $\underline{z} \in \wedge^2(\mathbb{R}^5)$ *is given by*

$$\underline{z} = Re\left(\|\underline{z}\|e^{-i\theta}(\underline{b}_1 \wedge \underline{b}_2 + i \cdot \underline{a}_1 \wedge \underline{a}_2)\right) = Re\left(\|\underline{z}\|e^{i\theta}(\underline{b}_1 \wedge \underline{b}_2 - i \cdot \underline{a}_1 \wedge \underline{a}_2)\right) \tag{6.48}$$

Therefore, given the coordinates of a 2-vector $\underline{z}$, one has to calculate the components $\underline{b}_1 \wedge \underline{b}_2,\ \underline{a}_1 \wedge \underline{a}_2$ to derive the respective polar form. We need the following result first.

Proposition 6.5.2. *If* $\underline{z} \wedge \underline{z} \neq 0$ *then*

$$\breve{\underline{z}} = \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* = \left(\underline{z} \wedge \frac{(\underline{z} \wedge \underline{z})^*}{\|\underline{z} \wedge \underline{z}\|}\right)^* \tag{6.49}$$

Proof. We have that

$$\begin{aligned}\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* &= \left((\sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2) \wedge \underline{r}_{\underline{z}}\right)^* = \\ &= \sigma_2 \left(\underline{b}_1 \wedge \underline{b}_2 \wedge \underline{r}_{\underline{z}}\right)^* + \sigma_1 \left(\underline{a}_1 \wedge \underline{a}_2 \wedge \underline{r}_{\underline{z}}\right)^* = \\ &= \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_1 \cdot \underline{b}_1 \wedge \underline{b}_2 = \breve{\underline{z}}\end{aligned} \tag{6.50}$$

since $\underline{r}_{\underline{z}} = (\underline{z} \wedge \underline{z})^* / \|\underline{z} \wedge \underline{z}\|$ □

Theorem 6.5.2. *(Closed-Form Formulae of the main Components) The imaginary parts* σ_i *and their respective orthonormal eigenvectors* $\underline{a}_i,\ \underline{b}_i,\ i = 1, 2$ *of a 2-vector* $\underline{z} \in \wedge^2(\mathbb{R}^5)$ *satisfy the following formulae.*

1)
$$\sigma_2 = \frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} + \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2},$$
$$\sigma_1 = \frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} - \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2}$$

2)
$$\underline{b}_1 \wedge \underline{b}_2 = \frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} + \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right),$$
$$\underline{a}_1 \wedge \underline{a}_2 = \frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} - \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right)$$

Proof. Let $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$, $\sigma_2 > \sigma_1 > 0$ the prime decomposition of $\underline{z} \in \wedge^2(\mathbb{R}^5)$. Then

1) As we have seen $\sigma_1^2 + \sigma_2^2 = \|\underline{z}\|^2,\ \|\underline{z} \wedge \underline{z}\| = 2\sigma_1\sigma_2$. Therefore,

$$\frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} + \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2} = \frac{|\sigma_1 + \sigma_2| + |\sigma_1 - \sigma_2|}{2} = \sigma_2$$

since $\sigma_2 > \sigma_1 > 0$. Similarly,

$$\frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} - \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2} = \frac{|\sigma_1 + \sigma_2| - |\sigma_1 - \sigma_2|}{2} = \sigma_1$$

2) In Proposition 6.5.2 we saw that

$$\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* = \breve{\underline{z}} = \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_1 \cdot \underline{b}_1 \wedge \underline{b}_2$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} + \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right) = \\
&= \frac{\sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 + \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_1 \cdot \underline{b}_1 \wedge \underline{b}_2}{2\sigma_2 + 2\sigma_1} + \\
&+ \frac{\sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 - \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2 - \sigma_1 \cdot \underline{b}_1 \wedge \underline{b}_2}{2\sigma_2 - 2\sigma_1} = \\
&= \frac{\underline{b}_1 \wedge \underline{b}_2}{2} + \frac{\underline{b}_1 \wedge \underline{b}_2}{2} = \underline{b}_1 \wedge \underline{b}_2
\end{aligned}$$

Similarly, we obtain that

$$\frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} - \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right) = \underline{a}_1 \wedge \underline{a}_2$$

□

Clearly, the previous result does not only provide the polar decomposition of a 2-vector, but also implies its prime decomposition as well as the respective conjugacy forms, in terms of its coordinates only without any SVD-like numerical applications. We verify this with the following general example.

Example 6.5.2. *Let* $\underline{z} = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6)^t \in \wedge^2(\mathbb{R}^5) \simeq \mathbb{R}^{10}$ *as in Example 6.5.1. Then*

$$\|\underline{z}\|^2 = 310,\ \|\underline{z} \wedge \underline{z}\| = \|2 \cdot (QPR_1, QPR_2, QPR_3, QPR_4, QPR_5)\| = 104\sqrt{6}$$

Therefore,

$$\frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} + \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2} = 15.5988 = \sigma_2$$

and

$$\frac{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|} - \sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}{2} = 8.16558 = \sigma_1$$

Furthermore, if $\underline{z} = (z_1, z_2, ..., z_{10})$ *then*

$$\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* = \frac{2}{\|\underline{z} \wedge \underline{z}\|}\left(\left(\sum_{1 \le i < j \le 5} a_{ij} \cdot \underline{e}_i \wedge \underline{e}_j\right) \wedge \left(\sum_{1 \le i < j \le 5} QPR_i \cdot \underline{e}_i\right)\right)^* =$$

$$= \frac{2}{\|\underline{z} \wedge \underline{z}\|} \cdot (z_8 QPR_1 + z_{10} QPR_3 + z_9 QPR_2, -(z_6 QPR_1 - z_{10} QPR_4 + z_7 QPR_2),$$
$$z_5 QPR_1 - z_9 QPR_4 - z_7 QPR_3, -(-z_5 QPR_2 - z_8 QPR_4 - z_6 QPR_3),$$
$$z_3 QPR_1 + z_{10} QPR_5 + z_4 QPR_2, -(z_2 QPR_1 + z_9 QPR_5 - z_4 QPR_3),$$
$$-z_2 QPR_2 + z_8 QPR_5 - z_3 QPR_3, z_1 QPR_1 + z_7 QPR_5 + z_4 QPR_4,$$
$$-(-z_1 QPR_2 + z_6 QPR_5 + z_3 QPR_4), z_1 QPR_3 + z_5 QPR_5 + z_2 QPR_4)$$

Hence

$$\left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^* = (9.10708, -9.98638, 1.55448, 5.38574, -0.690882,$$
$$, 9.07567, -2.73212, 1.36606, 0.549565, 1.85282)$$

Thus,

$$\frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} + \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right) = (-0.244369, -0.244816, 0.0164484, 0.192568,$$
$$, 0.0319364, 0.55183, -0.138622, 0.55499, -0.113709, 0.444184) = \underline{b}_1 \wedge \underline{b}_2$$

Hence,

$$\sigma_2 \underline{b}_1 \wedge \underline{b}_2 = (-3.81187, -3.81885, 0.256576, 3.00383, 0.49817,$$
$$, 8.6079, -2.16234, 8.65719, -1.77373, 6.92875)$$

which is the best decomposable approximation of $\underline{z}$*. Also*

$$\frac{1}{2}\left(\frac{\underline{z} + \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|}} - \frac{\underline{z} - \left(\underline{z} \wedge \underline{r}_{\underline{z}}\right)^*}{\sqrt{\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|}}\right) = (0.711752, -0.512046, 0.0910436, 0.244461,$$
$$-0.0610085, 0.292949, -0.102584, -0.202948, 0.094755, -0.11374) = \underline{a}_1 \wedge \underline{a}_2$$

Hence, $\sigma_1 \underline{a}_1 \wedge \underline{a}_2 + \sigma_4 \underline{b}_1 \wedge \underline{b}_2 = (2, -8, 1, 5, 0, 11, -3, 7, -1, 6) = \underline{z}$ *and due to Theorem 6.4.4 (*$\sin\theta = \sigma_1/\|\underline{z}\|$, $\cos\theta = \sigma_2/\|\underline{z}\|$*), the polar form of* $\underline{z}$ *is easily verified. Finally,* $\hat{\underline{z}} =$ $(9.62374, 0.362305, -0.486848, 1.00766, 0.99634,$ $6.2158, -1.32468, 10.3144, -2.54746, 7.8575)$ *and* $\breve{\underline{z}} =$ $(9.10708, -9.98638, 1.55448, 5.38574,$ $-0.690882, 9.07567, -2.73212, 1.36606, 0.549565, 1.85282)$.

6.6 Gap Sensitivity

Next we examine the sensitivity properties of the various distances we calculated in the previous sections. The sensitivity will be calculated in terms of perturbation of $\underline{z}$ of differentials of related functions. Through out this section, σ_1, σ_2 will be functions of the multi-vector $\underline{z}$ as these were proved in Proposition 6.5.2.

Theorem 6.6.1. *Let* $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2,\ \sigma_2 > \sigma_1 > 0$. *Then*

$$\nabla\sigma_2 = \underline{b}_1 \wedge \underline{b}_2,\ \nabla\sigma_1 = \underline{a}_1 \wedge \underline{a}_2 \tag{6.51}$$

Proof. From the prime decomposition, we easily obtain

$$\|\underline{z}\|^2 = \sigma_2^2 + \sigma_1^2 \Rightarrow < \underline{z}, d\underline{z} >= \sigma_2 d\sigma_2 + \sigma_1 d\sigma_1$$

and since $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$ we have that

$$\sigma_2 < \underline{b}_1 \wedge \underline{b}_2, d\underline{z} > +\sigma_1 < \underline{a}_1 \wedge \underline{a}_2, d\underline{z} >= \sigma_2 d\sigma_2 + \sigma_1 d\sigma_1 \tag{6.52}$$

Hence,

$$\begin{aligned}
\|\underline{z} \wedge \underline{z}\| = 2\sigma_1\sigma_2 \Rightarrow d\|\underline{z} \wedge \underline{z}\| = 2(\sigma_1 d\sigma_2 + \sigma_2 d\sigma_1) \Rightarrow \\
\Rightarrow \frac{d < \underline{z} \wedge \underline{z}, \underline{z} \wedge \underline{z} >}{2\|\underline{z} \wedge \underline{z}\|} = 2(\sigma_1 d\sigma_2 + \sigma_2 d\sigma_1) \Rightarrow \\
\Rightarrow \frac{< \underline{z} \wedge \underline{z}, \underline{z} \wedge d\underline{z} >}{\|\underline{z} \wedge \underline{z}\|} = \sigma_1 d\sigma_2 + \sigma_2 d\sigma_1 \Rightarrow \\
\Rightarrow \left\langle \frac{((\underline{z} \wedge \underline{z})^* \wedge \underline{z})^*}{\|\underline{z} \wedge \underline{z}\|}, d\underline{z} \right\rangle = \sigma_1 d\sigma_2 + \sigma_2 d\sigma_1
\end{aligned}$$

and again from the prime decomposition we get

$$\sigma_1 < \underline{b}_1 \wedge \underline{b}_2, d\underline{z} > +\sigma_2 < \underline{a}_1 \wedge \underline{a}_2, d\underline{z} >= \sigma_1 d\sigma_2 + \sigma_2 d\sigma_1 \tag{6.53}$$

Because $\sigma_1 \neq \sigma_2$, from equations 6.52, 6.53 we have that

$$< \underline{b}_1 \wedge \underline{b}_2, d\underline{z} >= d\sigma_2,\ < \underline{a}_1 \wedge \underline{a}_2, d\underline{z} >= d\sigma_1 \tag{6.54}$$

Hence,

$$\nabla\sigma_2 = \underline{b}_1 \wedge \underline{b}_2,\ \nabla\sigma_1 = \underline{a}_1 \wedge \underline{a}_2 \tag{6.55}$$

□

Corollary 6.6.1. *For every 2-vector* $\underline{z} \in \bigwedge^2(\mathbb{R}^5),\ \|\underline{z}\|^2 \neq \|\underline{z} \wedge \underline{z}\|$, *we have that*

$$\underline{z} = \sigma_1 \nabla\sigma_2 + \sigma_2 \nabla\sigma_1 \tag{6.56}$$

Now we calculate, in terms of differentials, the sensitivity of the gap $g\left(\underline{z}, G_2(\mathbb{R}^5)\right) = \sigma_1/\|\underline{z}\|$ in terms of perturbations of $\underline{z}$.

Theorem 6.6.2. *Let* $g(\underline{z}) = \sigma_1/\|\underline{z}\|$ *the gap of a 2-vector* $\underline{z} = \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$ *from* $G_2(\mathbb{R}^5)$ *and* $\mathrm{sp}\{\underline{z}\} \in \mathbb{P}^9(\mathbb{R})$ *where* $\|\underline{z}\| = 1$. *If we consider a perturbation of* $\underline{z}$ *such that*

$$d\|\underline{z}\| = 0$$

then

$$dg(\underline{z}) =< \underline{a}_1 \wedge \underline{a}_2, d\underline{z} > \tag{6.57}$$

Proof. We have that

$$dg(\underline{z}) = d\frac{\sigma_1}{\|\underline{z}\|} = \frac{\|\underline{z}\| d\sigma_1 - \sigma_1 d\|\underline{z}\|}{\|\underline{z}\|^2} = d\sigma_1 = <\underline{a}_1 \wedge \underline{a}_2, d\underline{z}>$$

due to Theorem 6.6.1 and the fact that $d\|\underline{z}\| = 0$, since $\|\underline{z}\| = 1$. □

Theorem 6.6.3.

$$d(\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|) = 2 < \underline{z} - (\underline{z} \wedge \underline{r}_{\underline{z}})^*, d\underline{z} >$$

and

$$d(\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|) = 2 < \underline{z} + (\underline{z} \wedge \underline{r}_{\underline{z}})^*, d\underline{z} >$$

Proof.

$$\begin{aligned} d(\|\underline{z}\|^2 - \|\underline{z} \wedge \underline{z}\|) &= 2 < \underline{z}, d\underline{z} > - \frac{d < \underline{z} \wedge \underline{z}, \underline{z} \wedge \underline{z} >}{2\|\underline{z} \wedge \underline{z}\|} = \\ &= 2 < \underline{z}, d\underline{z} > - \frac{4 < \underline{z} \wedge \underline{z}, \underline{z} \wedge d\underline{z} >}{2\|\underline{z} \wedge \underline{z}\|} = \\ &= 2 < \underline{z}, d\underline{z} > - 2\frac{< ((\underline{z} \wedge \underline{z})^* \wedge \underline{z})^*, d\underline{z} >}{\|\underline{z} \wedge \underline{z}\|} = \\ &= 2 < \underline{z} - (\underline{z} \wedge \underline{r}_{\underline{z}})^*, d\underline{z} > \end{aligned}$$

Similarly we obtain

$$d(\|\underline{z}\|^2 + \|\underline{z} \wedge \underline{z}\|) = 2 < \underline{z} + (\underline{z} \wedge \underline{r}_{\underline{z}})^*, d\underline{z} >$$

□

6.7 Conclusions

The main purpose of this chapter was to study the best decoposable approximation problem, i.e., approximate solutions for the multilinear subproblem of DAP, in the case where the uniqueness of the prime decomposition fails. As we have seen, the prime decomposition is unique if the imaginary parts σ_i are distinct and equivalently the best decoposable approximation implied by the prime decomposition is also unique and the critical points of the related distance-minimization problem are $\binom{m}{r}$, for $n = 2m$ or $n = 2m + 1$ and $2r \leq n$, for an $n \times n$ skew-symmetric matrix.

On the other hand, if we have repeated eivenvalues (degenerate eigenvalues) it is easy to verify that the prime decomposition is not unique. Then the approximation problem may be addressed via the extremal variety $\mathcal{V}_1$ approach, i.e., the

variety with the maximum distance from $G_2(\mathbb{R}^n)$ that contains those 2-vectors $\underline{z}$ whose imaginary parts σ_i are all equal. For the special case $n = 5$ we proved a new result regarding the geometry of the Grassmann variety, that $G_2(\mathbb{R}^5)$ and its corresponding extremal variety $\mathcal{V}_1$ are complementary sets.

These results have helped us calculate via congugacy and Hodge-duality properties the Lagrange multipliers of the best decomposable approximation problem in a closed form formula, which is a new result for manifold-constrained optimization problems. This analysis has led to the most significant result of this chapter and one of the most important results of the thesis, that the prime decomposition of a 2-vector $\underline{z}$, and similar decompositions such as the polar decomposition, are written *as functions of the components of* $\underline{z}$. Thus, given any 2-vector in $\bigwedge^2(\mathbb{R}^5)$ one can calculate its best decomposable approximation, its worst decomposable approximation (degeneracy case) or its polar decomposition via a simple calculation of the formulae in Theorem 6.5.2. We summarize the most important new results in the following table.

-	Variety $G_2(\mathbb{R}^5)$	Variety $\mathcal{V}_1$
Decomposition of $\underline{z}$	$\sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2 + \sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$	$\frac{\sigma_2+\sigma_1}{2} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2) + \frac{\sigma_2-\sigma_1}{2} \cdot (\underline{b}_1 \wedge \underline{b}_2 - \underline{a}_1 \wedge \underline{a}_2)$
Equation defying variety	$\|\underline{z} \wedge \underline{z}\|^2 = 0 \Leftrightarrow \sigma_1 = 0$	$\|\underline{z}\|^4 - \|\underline{z} \wedge \underline{z}\|^2 = 0 \Leftrightarrow \sigma_1 = \sigma_2$
Distance from variety	σ_2	$\frac{\sigma_2-\sigma_1}{\sqrt{2}}$
Closest vector to $\underline{z}$ from variety	$\sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2$	$\frac{\sigma_2+\sigma_1}{2} \cdot (\underline{a}_1 \wedge \underline{a}_2 + \underline{b}_1 \wedge \underline{b}_2)$
Gap of $\underline{z}$ from variety	$\frac{\sigma_1}{\|\underline{z}\|}$	$\frac{\sigma_2-\sigma_1}{\|\underline{z}\|\sqrt{2}}$
SOS	Yes	No
Extremal variety	$\mathcal{V}_1$	$G_2(\mathbb{R}^5)$
Definition of the conjugate space	$\overline{\mathbb{P}^9(\mathbb{R}) \setminus \mathcal{V}_1}$	$\overline{\mathbb{P}^9(\mathbb{R}) \setminus G_2(\mathbb{R}^5)}$
Conjugate of $\underline{z}$	$\sigma_2 \cdot \underline{b}_1 \wedge \underline{b}_2 - \sigma_1 \cdot \underline{a}_1 \wedge \underline{a}_2$	$\sigma_1 \cdot \underline{b}_1 \wedge \underline{b}_2 + \sigma_2 \cdot \underline{a}_1 \wedge \underline{a}_2$

The examination of the uniqueness or the non-uniqueness of the prime decomposition presented in this chapter is necessary for the complete investigation of the approximate DAP, since the case $\sigma_1 = \cdots = \sigma_k$ implies that the approximate solution of DAP is not unique and for some Grassmann varieties as $G_2(\mathbb{R}^5)$, the representation of all solutions may be found. Moreover, the approach presented here has an independent mathematical importance on its own, since it provides a new technique for decomposability (and low rank problems in general) with degenerate eigenvalues. In the next chapter we will continue this investigation by generalizing the best decomosability problem between a 2-vector and the Grassmann variety we saw in the previous chapter and the spectral degeneracy case examined in this chapter, by constructing a wider linear spectral set containing the Grassmann variety $G_2(\mathbb{R}^n)$ as well as $\mathcal{V}_1$, that covers all possible aspects in

approximate decomposability problems. This will help us provide not only the most complete presentation for best decomposable approximation problems, but also deal with the approximate solutions of DAP under any singularities that may arise.

Chapter 7

General Approximate Decomposability Problems

7.1 Introduction

We have seen in the previous chapters how the approximate DAP is formed as an optimization problem between the Grassmann variety and a linear variety described by a parameterized multivector of the form $\underline{z}(\underline{x})$, where $\underline{z}$ is a multivector whose coordinates are functions of $\underline{x}$, i.e., the free parameters implied by the linear subproblem of DAP. We have solved this optimization problem by considering $\underline{z}$ as a stable/non-parameterized 2-vector at first and we examined the case where the corresponding matrix of z has equal eigenvalues, something that has yield the worst solution of the minimization problem between the 2-vector and the Grassmann variety. Before expanding $\underline{z}$ into $\underline{z}(\underline{x})$ and calculating the stability properties of the new approximate polynomial implied by the approximate solution - which in turn will give the complete solution of the approximate DAP in the $\bigwedge^2(\mathbb{R}^n)$ case - we will show in this chapter that the problem of approximating a bivector by one on the Grassmann variety as well as the approximation by a 2-vector on $\mathcal{V}_1$, i.e., the worst approximation for DAP, are special cases of two wider approximation problems on sets that satisfy certain decomposability conditions, which we denote as $G_2^{\lambda}(\mathbb{R}^n)$ and $G_{\lambda_1,\ldots,\lambda_k}$. The former is obtained via the generalization of $\underline{z} \wedge \underline{z} = \underline{0}$, that describes the Grassmann variety $G_2(\mathbb{R}^n)$, into the λ-times wedge product $\underline{z} \wedge \cdots \wedge \underline{z} = \underline{0}$, for a 2-vector $\underline{z}$, while the second concerns the case of degenerate eigenvalues, where σ_i are required to have a special analogy, a situation often met in manifold optimization problems. By examining these cases, we provide a complete investigation for the approximate DAP for all cases, even when the 2-vector is subject to conditions which imply several pathologies to the approximate solution.

The calculation of the distance of a point from a general set $\mathcal{W}$ has been examined

in [Tho. 1], [Gold. 1] for surface smoothing and curve-surface selection problems, in the case where $\mathcal{W}$ is a manifold as well as in [Bro. 1] and [Tur., etc. 1] for the calculation of the "closest" example to a query in a database. In general, the formulation of the problem involves the solution of

$$\min \operatorname{dist}(\underline{z}, \mathcal{W}) \tag{7.1}$$

where "dist" is an appropriate distance function. In our case "dist" is the gap function g of the projective space we introduced in Chapter 5. The significance of the approach we introduce here lies in the fact that if $\mathcal{W}$ has the structure of $G_2^{\lambda}(\mathbb{R}^n)$ and $G_{\lambda_1,\ldots,\lambda_k}$ then problem (7.1) is directly solved in closed-form formulae - whereas in most cases this manifold optimization problem is approached algorithmically, within the algorithmic framework that has been constructed for optimization involving sets of multivectors, manifolds, varieties, etc., [Cox. 1] - and thus a number of approximate decomposabilty problems are easily solved, such as approximate multivectors with repeated eigenvalues. Note that, if $\mathcal{W}$ is a Grassmann variety then (7.1) is also usually addressed algorithmically in two ways:

(i) either as a minimization problem of a cost function over the Grassmann manifold with the use of special metrics, geodesics and combinations of them to count distances, [Sav. & Li.1], [Abs., etc. 2], [Edel., etc. 1], or

(ii) as a special case of the *Higher Order SVD* , [Kol. & Bad. 3], [Sav. & Li.1] in the concept of determining a tensor by another one of lower-rank, i.e., for a 2-vector $\underline{a}$, $\underline{a}\wedge\underline{a} = \underline{0}$ corresponds to a decomposable vector (equivalently a rank-2 approximation for the respective matrix, [Kol. & Bad. 3]), $\underline{a}\wedge\underline{a}\wedge\underline{a} = \underline{0}$ to a sum of 2 decomposable vectors (rank-4 approximation), $\underline{a}\wedge\underline{a}\wedge\underline{a}\wedge\underline{a} = \underline{0}$ to a sum of 3 decomposable vectors (rank 6 approximation), etc.

However, the $m = 2$ case is studied by most authors separately, because it reduces the problem to lower-rank approximation for matrices, providing great simplifications as far as the solutions and the understanding of the nature of the problem are concerned, and also provides a number of expansions with useful applications; in [Gol. & Van. 2] a first generalization along with some applications have been discussed, where a best-low rank matrix approximation was achieved for a matrix whose specified columns remained fixed. A generalization for low matrix-rank approximation was used in [Lu-S., etc. 1] for the design of two dimensional digital filters, whereas several low-rank approximation generalizations were discussed in [Fri. & Tor. 1] and [Kol. 2].

In this chapter we present a new approach for this kind of problems by introducing the generalized sets $G_2^{\lambda}(\mathbb{R}^n)$ and $G_{\lambda_1,\ldots,\lambda_k}$ that not only provide a thorough investigation of the approximate DAP in all cases (degenerate eigenvalues,

approximations in cases of special rank) but also provides a number of closed-form solutions for problem (7.1) which covers a class of equivalent low-rank and approximation problems for skew-symmetric matrices, [Jia. 1], [Tre. 1] and approximation on sets with spectral degeneracies, such as the set of critical points in [O'Neil. 1].

The mathematical background of these generalizations lies in the expansion of the standard Exterior (Grassmann) Algebra and its properties, which were discussed in Chapter 3, mostly in three ways: (i) the generalized Grassmann algebras that use the so-called Generalized Grassmann numbers instead of the standard Grassmann numbers, [Ohn. & Kam. 1], (ii) generalized Grassmann-like algebras equipped with multi-linear structures instead of bilinear ones, [Kwa. 1] and (iii) generalizations of the basic notions related to the Grassmannian, such as the Plücker embedding, [Rav., etc. 1].

Following the 2-vector approach introduced in [Lev., etc. 9] and [Lev., etc. 9] and with the use of Von Neumann's trace Inequality, i.e., $\text{trace}(AB) \leq \sigma_1 s_1 + \cdots + \sigma_n s_n$, where σ_i, s_i are the singular values (in increasing order) of any $n \times n$ matrices A and B, [Mir. 1] which we interpret in a 2-vector form, we examine and provide closed-form solutions to the problems

$$\text{dist}(\underline{z}, G_2^{\lambda}(\mathbb{R}^n)),\ \text{dist}(\underline{z}, G_{\lambda_1,\ldots,\lambda_k}) \tag{7.2}$$

where $\underline{z} \in \wedge^2(\mathbb{R}^n)$, $G_2^{\lambda}(\mathbb{R}^n)$ is the Generalized Grassmann variety containing the Grassmann variety $G_2(\mathbb{R}^n)$ and $G_{\lambda_1,\ldots,\lambda_k}$ a general set with eigenvalue multiplicities. Furthermore, the generalization we propose in the second problem may be also used as a new tool for the study of isotropic matrices, where matrices whose eigenvalues are all equal are studied for applications to robotics, [Sal. & Cr. 1].

We should note that these forms of Grassmann generalizations are new and constitute a natural expansion of the standard exterior algebra/tensor theory concept, [Hod. & Ped. 1], [Mar. 1] and the generalization from bi-linearity to k-linearity which is usually met in category (ii) we mentioned before. The approach proposed to these distance problems has not been examined before in such general form and it may cover all optimization problems that concern optimization over $G_2(\mathbb{R}^n)$ as well as the low-rank approximation problems for skew-symmetric matrices in [Jia. 1] and [Tre. 1] and the set of critical points in [O'Neil. 1].

The chapter is organized as follows. In Section 7.2 we present some background results regarding the generalizations of the standard Grassmann algebra, i.e., the Exterior Algebra we showed in Chapter 3, since we need a wider framework to establish our expansions. We briefly present in three subsections, the main theory of all three categories we mentioned above, which are mostly used to set up the mathematical foundations of the expansions that follow.

In Section 7.3 we present a very important result of this thesis, the new Cauchy-Schwartz Type Inequality in $\bigwedge^2(\mathbb{R}^n)$. We implement this inequality to solve two general distance problems; In 7.4.1 we examine the minimum distance problem between a 2-vector and the varieties $G_2^{\lambda}(\mathbb{R}^n)$ where the best decomposable approximate solution for the approximate DAP may now be derived directly as a special case. In Section 7.4.2 we study the same problem between a 2-vector and the set $G_{\lambda_1,...,\lambda_k}$. Finally, several examples are given to point out the significance of these general varieties.

7.2 Basic Grassmann-Generalizations

Our generalizations of the standard Grassmann variety are based on the primary definitions of the Grassmann varieties and decomposability as these were introduced in [Hod. & Ped. 1], [Mar. 1]. Similarly to the standard Grassmann varieties, which were introduced and developed via the Exterior Algebra (Grassmann Algebra) rules and properties in Chapter 3, the concept behind these generalizations follows the principles of the so-called Generalized Grassmann Algebra. Next we briefly present the main properties of that algebra, as well as the structure of the most general form of these kind of algebras, the so-called Generalized Grassmann-like Algebras, from which the former may be obtained. At the end of the section we briefly present a generalization of the Plücker embedding, which is usually met in dynamical systems applications. First we recall the basic notions of a generalized Grassmann algebra as introduced in [Ohn. & Kam. 1].

7.2.1 The Generalized Grassmann Algebra

The (ordinary) Grassmann algebra is the exterior algebra we saw in Chapter 3 of a vector space which is spanned by its generators. The n numbers $x_1, ..., x_n$ in the Grassmann algebra are said to be (ordinary) Grassmann numbers if they anti-commute with each other, i.e., $x_i x_j = -x_j x_i$ but they commute ordinary any other real number a, i.e., $x_i a = a x_i$.

Remark 7.2.1. *It can be shown that the Grassmann numbers may be also represented by a special form of matrices, [Ohn. & Kam. 1], which are interpreted as the linear operators corresponding to the exterior multiplication on the Grassmann algebra itself.*

Definition 7.2.1. *[Ohn. & Kam. 1] The generalized Grassmann numbers are defined as the numbers x_i, $i = 1, ..., n$ that satisfy the conditions*

$$x_i x_j = \eta_{ij} x_j x_i, \tag{7.3}$$

where

$$\eta_{ij} = \begin{cases} -, & i = j \\ +, & i \neq j \end{cases} \tag{7.4}$$

The algebra of the generalized Grassmann numbers is called generalized Grassmann algebra and it is denoted as $\mathcal{G}_{GA}$.

Remark 7.2.2. *A different generalization has been attempted in [Det. 1], where a similar generalized Grassmann algebra was derived as a special case of the Clifford algebras of polynomials.*

Similarly to the standard exterior algebra, which is characterized by determinants and skew-symmetric tensors, the generalized Grassmann algebra is studied via the so-called generalized Grassmann matrices and tensors and the generalized Grassmann determinant.

Definition 7.2.2. *[Ohn. & Kam. 1] Let* M_{ij} *denote the* ij *element of a matrix* M *and let a Grassmann number* x_m *with* $i \leq m \leq j$.

i) If $M_{ij}x_m = \eta_{im}\eta_{jm}x_m M_{ij}$ *and* $M_{ij}M'_{m\ell}x_m = \eta_{im}\eta_{i\ell}\eta_{jm}\eta_{j\ell}x_m M'_{m\ell}M_{ij}$ *then* M *is called a General Grassmann 2-tensor and the quantity* M_{ij} *has the same commutation property as that of* $x_i x_j$ *when commuted with other quantities.*

ii) Any quantities $T^{i_1 i_2 \cdots}_{j_1 j_2 \cdots}$ *which have the same commutation property as that of* $x_{i_1}x_{i_2}\cdots x_{j_1}x_{j_2}\cdots$ *when commuted with other quantities will be referred to as generalized Grassmann* k*-tensors, for* $k > 2$.

iii) The generalized Grassmann determinant of M *is defined as*

$$(M_{1j_1}x_{j_1})(M_{2j_2}x_{j_2})\cdots(M_{nj_n}x_{j_n}) \equiv detM \cdot x_1 x_2 \cdots x_n \tag{7.5}$$

The above definition is a natural expansion of the exterior algebra obtained by the tensor product in Chapter 3. The advantage of this generalization is that matrix M may be replaced by a polynomial matrix $M(s)$ whose elements are all differentiable with respect to a real parameter s. Then it can be proved that

$$\frac{d}{ds}M(s) = \sum_{i=1}^{n} \det\left(1 - \delta_{im}M(s)_{ij} + \delta_{im}\frac{d}{ds}M(s)_{ij}\right),\ i \leq m \leq j \tag{7.6}$$

This approach provides many applications to systems theory, such as the so-called Fermi systems, whose state space may be considered as the generalized Grassmann algebra $\mathcal{G}_{GA}$.

7.2.2 Generalized Grassmann-like Algebras

A second well-known expansion method is also based on the notion of the Grassmann algebra.

Definition 7.2.3. *[Kwa. 1] Let $\mathcal{D}_{n,p}$ be the group generated by its $\omega, \gamma_1, ..., \gamma_n$ elements satisfying*

$$\omega\gamma_i = \gamma_i\omega,\ \omega^p = \gamma_i^p = 1,\ \gamma_i\gamma_j = \omega\gamma_j\gamma_i,\ p \geq 1 \tag{7.7}$$

for $i < j$ and $i, j = 1, ..., n$. Then we define an algebra $\mathcal{A}$ as $\mathcal{A} = \mathcal{Z}_p \oplus \mathcal{Z}_p \oplus \cdots \oplus \mathcal{Z}_p$, where $\mathcal{Z}_p$ is a subgroup of the center of $\mathcal{D}_{n,p}$, i.e., the set of all elements of $\mathcal{D}_{n,p}$ that commute with all other elements.

In the case of $\mathcal{A}$ admitting a $\mathcal{A}_0$ subgroup of index 2, i.e., half of the elements of $\mathcal{A}$ lie in $\mathcal{A}_0$, one comes up with the following definition.

Definition 7.2.4. *[Kwa. 1] Let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$. Then $\epsilon_0 : \mathcal{A} \times \mathcal{A} \to \mathbb{C}^*$ is said to be the Grassmann commutation factor if*

$$\epsilon_0(a, b) = \begin{cases} -1, & a, b \in \mathcal{A}_1 \\ 1, & \text{otherwise} \end{cases} \tag{7.8}$$

Similarly to the exterior algebra, one may define a δ-symmetric algebra $\mathcal{S}_\delta$ of a vector space $\mathcal{V}$, such that $\mathcal{S}_\delta := \mathcal{T}/\mathfrak{I}_\delta$, where $\mathcal{T}$ is the tensor algebra of $\mathcal{V}$ and $\mathfrak{I}_\delta$ is an ideal of $\mathcal{T}$ generated by the elements $\underline{x}_i \otimes \underline{y}_j - \underline{y}_j \otimes \underline{x}_i\delta(a, b)$ for $(\underline{x}_i), \underline{y}_j \in \mathcal{V} \times \mathcal{V}$, where $\delta : \mathcal{A} \times \mathcal{A} \to \mathbb{C}^*,\ \delta(a, a) = 1,\ a \in \mathcal{A}$.

Definition 7.2.5. *[Kwa. 1] Let p be an epimorphism of the $\mathcal{A}$-graded associative algebras, i.e., vector spaces with an increasing sequence of subsets $\mathcal{F}_i$ which are written as the direct sum $\oplus_{\mathbb{N}} (\mathcal{F}_n/\mathcal{F}_{n-1})$. Then $\mathcal{S}_\delta/\ker p$ is called a generalized Grassmann-like algebra.*

Definition 7.2.5 clearly constitutes a generalization of the ordinary Exterior Algebra. If $\mathcal{A} = \mathcal{Z}_2 \oplus \mathcal{Z}_2 \oplus \cdots \oplus \mathcal{Z}_2$, (i.e., $\mathcal{A}$ is the subgroup of the well-known group of Dirac γ matrices for Euclidean spaces, which was the first group within the context of Lie algebra that was studied for generalization, [Morr. 1]), then for $\mathcal{A} = \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 0)\} \cup \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$ and $\delta \equiv 1$, $\mathcal{S}_{\epsilon_0}$ is the usual Grassmann (Exterior) algebra.

7.2.3 The Generalized Plücker Embedding

A different category of generalizations, involves the generalization of the closely related notions to a Grassmannian, as in [Rav., etc. 1], where the standard Plücker embedding, [Mar. 1] was expanded into the so called Generalized Plücker embedding. The authors defined the standard Grassmann variety $G_m(\mathbb{R}^n)$ as $G_m(\mathbb{R}^n) = \mathcal{K}^0_{m,n-m}$ and generalized aftewards to $\mathcal{K}^q_{m,n-m}$, where q was the McMillan degree of all m-input, $(n - m)$-output transfer functions.

Definition 7.2.6. *[Rav., etc. 1] Let $\mathcal{X}$ denote the space whose each element can be considered as an equivalence class of matrices $M = (M_{ij}(s,t))_{1\leq i\leq m, 1\leq j\leq n}$ where each $M_{ij}(s,t)$ is a homogeneous polynomial of degree q_i with $\sum q_i = q$. The image of the map assigning the $m \times m$ ninors of each matrix $M \in \mathcal{X}$ to a projective space of the form $\mathbb{P} = \mathbb{P}\left(\bigwedge^m V \otimes H^0(\mathbb{P}^1)\right)$, where H^0 is the corresponding Sobolev space of the projective line $\mathbb{P}^1$, is called the Generalized Plücker embedding, denoted as $\mathcal{K}^q_{m,n-m}$.*

The authors tried to generalize the classical methods used to compute the degree of the Grassmannian, to compute the degree of $\mathcal{K}^q_{m,n-m}$ when space $\mathcal{X}$ was considered as a compactification of the space of all m-input, $(n-m)$-output transfer functions of McMillan degree q. Then the ordinary Grassmann variety $G_m(\mathbb{R}^n)$ is naturally defined as $G_m(\mathbb{R}^n) := \mathcal{K}^0_{m,n-m}$ and with the use of the cell-decomposition of an intermediate space $\mathcal{A}^q_{m,n-m}$, i.e., disjoint unions of spaces homeomorphic to an n-sphere, [Rav., etc. 1], the expansion to $K^q_{m,n-m}$ was achieved.

Theorem 7.2.1. *[Rav., etc. 1] The degree of $K^q_{m,n-m}$ is equal to the number of maximal totally ordered subsets of $\mathcal{I}_{(n-m+1,n-m+2,\ldots,n;q)}$, where $\mathcal{I}_{(\underline{a};q)}$ is the set of vectors $\underline{a} = (a_1, ..., a_m)$ such that $1 \leq a_1 < \cdots < a_m,\ a_m - a_1 < n$.*

The result of the above theorem implies that the degree of $\mathcal{K}^q_{m,n-m}$ is also equal to the degree of the pole placement map in the critical dimension, i.e., the dimensionality of a space at which the nature of the states of a system change, offering a great deal of applications to systems theory, [Wan. & Ros. 2] and mathematical physics, [Vaf. 1].

Note that the expansions described in these three sections, actually try to give a wider range of the properties ralated to the Grassmann variety and its mathematical framework, rather than the variety itself. Our approach will be based on the expansion of the condition $\underline{z} \wedge \underline{z} = \underline{0}$ that describes the QPR of $G_2(\mathbb{R}^n)$ into $\underline{z} \wedge \underline{z} \wedge \cdots \wedge \underline{z} = \underline{0}$.

7.3 Spectral Inequalities

In this section we show how a specific spectral inequality may be used to solve optimization problems where the constraints satisfy the Grassmann generalizations presented in the previous sections.

7.3.1 Background Matrix and Eigenvalue Inequalities

The use or the derivation of inequalities has always been a key problem for many researchers that deal with constrained optimization techniques. One of the

first spectral inequalities examined, was the Wielandt inequality

$$|\underline{x}^t A \underline{y}| \leq \sqrt{(\underline{x}^t A \underline{x})\left(\underline{y}^t A \underline{y}\right)} \tag{7.9}$$

that was regarded as a version of the Cauchy-Schwarz inequality for matrices asserting that A is a positive definite matrix and $\underline{x},\ \underline{y}$ any real vectors. In [Has. 1], several generalizations of (7.9) were presented, with regard to the eigenvalues of A. The second most well known spectral inequality is Von Neumann's trace inequality

$$\text{trace}(AB) \leq \sigma_1 s_1 + \cdots + \sigma_n s_n \tag{7.10}$$

where $\sigma_i,\ s_i$ are the singular values (in increasing order) of any $n \times n$ matrices A and B, [Mir. 1]. Since then a number of generalizations or special forms have been discovered such as the following in the case orthogonal vectors.

Proposition 7.3.1. *[Has. 1] Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix and let $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^{n\times 1}$ be two vectors such that $\|\underline{x}\| = \|\underline{y}\|$ and $\underline{x}^t\underline{y} = 0$. Assume that the eigenvalues of A, in increaing order, are $\lambda_1 \leq \cdots \leq \lambda_n$. Then*

$$|\underline{x}^t A \underline{y}| \leq \frac{\lambda_n - \lambda_1}{2\left(\lambda_n - \lambda_1\right)}\left(\underline{x}^t A \underline{x} + \underline{y}^t A \underline{y}\right) \tag{7.11}$$

Furthermore, if A is symmetric then

$$\min_{\substack{i,j=1\\ i\neq j}} \left\{ \frac{|\lambda_i - \lambda_j|}{2} \right\} \leq |\underline{x}^t A \underline{y}| \leq \frac{\lambda_n - \lambda_1}{2} \tag{7.12}$$

In [And. 1], one may find more different expressions of inequalities (7.11), (7.12), such as the first inequality connecting the eigenvalues λ_i of a general matrix A in $\mathbb{C}^{n\times n}$ with its singular values σ_i, i.e.,

$$2\sum_{i=1}^{\ell}\left\{\sigma_i^2(Re(A)) - |Re(\lambda_i(A))|^2\right\} \geq \sum_{i=1}^{\ell}\left\{\sigma_{n-i+1}^2(A) - |\lambda_i(A)|^2\right\},\ \ell = 1, ..., n \tag{7.13}$$

that may be implemented in a series of control theory problems, most of which concern linear cases, [Boy., etc. 1].

A very important result for further investigation of this kind of inequalities and their applications was presented in [Thom. 1], where the most well known matrix spectral inequalities as above, were connected with algebraic topology techniques, i.e., Schubert calculus and minimax problems on Grassmann manifolds. The results of this approach were expanded in [Rid. 1], where the inverse eigenvalue problem of defining the set of linear inequalities that describe the possible spectrum of a matrix $A_1 + \cdots + A_r$, where A_i are Hermitian was examined. The

problem was successfully connected with the maximum number of the linear subspaces of $\mathbb{C}^n$, i.e., the complex Grassmann manifold.

However, a large number of inverse eigenvalue problems (such as DAP) have not yet been examined via matrix-eigenvalue inequalities, but they are still almost implicitely studied via numerical analysis procedures, [Abs., etc. 2], [Bas., etc. 1], [Edel., etc. 1], [Smi. 1]. The main reason is that the majority of such inequalities, as explained in [Mar. & Min. 2], involves matrices that belong to convex sets, whereas the Grassmannian, as we saw in the previous chapter, is a non-convex set, on which several convexity properties may be recognised, but just for the affine case [Bus. 1], [Good. & Pol. 1], something that worked for the eigenvalue problem in [Rid. 1].

In the next section, we provide a 2-vector form of Von Neumann's trace inequality, $\text{trace}(AB) \leq \sigma_1 s_1 + \cdots + \sigma_n s_n$, [Mir. 1], which can solve a number of approximate decomposabilty problems which are viewed as distance problems form sets described by general decomposability and other conditions that follow the expansion properties we discussed at the previous section.

7.3.2 The bivector form of Von Neumann's trace inequality

It is not difficult to imply a 2-vector form (since the generalized sets in our case involve vector descriptions) of Von Neumann's trace inequality since

$$2\langle \underline{z}_1, \underline{z}_2 \rangle_{\wedge^2(\mathbb{R}^n)} = \text{trace}(T^t_{\underline{z}_1} T_{\underline{z}_2})$$

However, an alternative and independent proof may be given in the 2-vector case, that takes into consideration the skew-symmetry of $\wedge^2(\mathbb{R}^n)$ as follows.

Lemma 7.3.1. *Let $\underline{z}_1, \underline{z}_2 \in \wedge^2(\mathbb{R}^2)$ and $k = [n/2]$. If*

$$\underline{z}_1 = \sum_{i=1}^{k} \sigma_i \underline{x}_{2i-1} \wedge \underline{x}_{2i} \text{ and } \underline{z}_2 = \sum_{i=1}^{k} s_i \underline{y}_{2i-1} \wedge \underline{y}_{2i} \tag{7.14}$$

are two prime decompositions of $\underline{z}_1$, $\underline{z}_2$ respectively, where $\sigma_k \geq \sigma_{k-1} \geq \cdots \geq \sigma_1 \geq 0$, $s_k \geq s_{k-1} \geq \cdots \geq s_1 \geq 0$ are the imaginary parts of the eigenvalues of $T_{\underline{z}_1}$, $T_{\underline{z}_2}$ and $\underline{x}_i$, $\underline{y}_i$ their respective orthonormal eigenvectors, then

$$< \underline{z}_1, \underline{z}_2 > \leq \sum_{i=1}^{k} \sigma_i s_i \tag{7.15}$$

Proof. We assume first that n is even, i.e. $n = 2k$. Without loss of generality we may assume that $\underline{y}_i = \underline{e}_i = (0, 0, 0, ...0, 1, 0, ..., 0)$ in the i-th entry. We also set the matrix

$$Z := [\underline{x}_1, \underline{x}_2, ..., \underline{x}_n] \tag{7.16}$$

which is an orthonormal matrix, i.e., $Z^t \cdot Z = Z \cdot Z^t = I_n$. Then

$$< \underline{z}_1, \underline{z}_2 >= \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_i s_j < \underline{x}_{2i-1} \wedge \underline{x}_{2i}, \underline{e}_{2i-1} \wedge \underline{e}_{2i} >= \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_i s_j X_{ij} \tag{7.17}$$

where

$$X_{ij} = \begin{vmatrix} x_{2j-1,2i-1} & x_{2j-1,2i} \\ x_{2j,2i-1} & x_{2j,2i} \end{vmatrix} \tag{7.18}$$

Then

$$X_{ij} = \begin{vmatrix} x_{2j-1,2i-1} & x_{2j-1,2i} \\ x_{2j,2i-1} & x_{2j,2i} \end{vmatrix} \leq \frac{x^2_{2j-1,2i-1} + x^2_{2j-1,2i} + x^2_{2j,2i-1} + x^2_{2j,2i}}{2} := K_{ij} \tag{7.19}$$

Thus,

$$< \underline{z}_1, \underline{z}_2 >\leq \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_i s_j K_{ij} \tag{7.20}$$

Furthermore, since Z is an orthonormal matrix we have that $\sum_i x^2_{i,j} = \sum_j x^2_{i,j} = 1$ which proves that $\sum_i K_{ij} = \sum_j K_{ij} = 1$. Hence, $K = (K_{ij}) \in \mathcal{D}_s$ where $\mathcal{D}_s$ is the set of doubly stochastic matrices. Therefore, we have to solve the following Linear Programming Maximization Problem

$$\max_{K \in \mathcal{D}_s} F(K) \text{ where } F(K) = \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_i s_j K_{ij} \tag{7.21}$$

The solution of the problem must contain at least one of the vertices of the polytope of the doubly stochastic matrices, [Mars., etc. 1]. The vertices of the polytope are the matrix representations of all permutations which we denote by K_π. We have therefore to calculate $F(K_\pi)$ and find the one that attains the maximal value. Note, however, that

$$F(K_\pi) = \sum_{i=1}^{k} \sigma_i s_{\pi(i)} \tag{7.22}$$

which is less or equal to $\sigma_1 s_1 + \sigma_2 s_2 + \cdots + \sigma_k s_k$, due to the rearrangement inequality, [Mars., etc. 1]:

$$a_{\sigma_{(1)}} b_1 + a_{\sigma_{(2)}} b_2 + \cdots + a_{\sigma_{(n)}} b_n \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where σ is any permutation and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n, 0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$. Therefore

$$< \underline{z}_1, \underline{z}_2 > \leq \max_{K \in \mathcal{D}_s} \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_i s_j K_{ij} \leq \sum_{i=1}^{k} \sigma_i s_i \tag{7.23}$$

If $n = 2k+1$, the proof follows exactly the same steps, with the difference that the matrix Z is now $Z = [\underline{x}_1, \underline{x}_2, ..., \underline{x}_{2k}]$ with $Z^t Z = I_{2k}$. The optimization problem (7.21) in this case becomes

$$\max_{K \in \overline{\mathcal{D}_s}} F(K) \text{ where } F(K) = \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_i s_j K_{ij} \tag{7.24}$$

where $\overline{\mathcal{D}_s}$ is the set of all matrices that satisfy the properties

$$\sum_i K_{ij} \leq 1, \ \sum_j K_{ij} \leq 1 \tag{7.25}$$

We may follow the same procedures to solve problem (7.24) as we did for problem (7.21) where we get $K \in \overline{\mathcal{D}_s}$. □

Eqn. (7.15) directly implies

$$< \underline{z}_1, \underline{z}_2 > \leq \sum_{i=1}^{k} \sigma_i s_i \leq \left(\sum_{i=1}^{k} \sigma_i^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{k} s_i^2 \right)^{1/2} \tag{7.26}$$

Inequality (7.26) is a refinement of the Cauchy-Schwartz inequality. These kind of inequalities share the same interest between many researchers, i.e., [And. 1], [Horn. & Joh. 1], [Hor. & Mat. 2], [Mar. & Min. 2], [Wey. 1] among many others. As we will show next, (7.26) may be directly applied to a wide category of best-approximation problems, which are interpreted as distance problems from special sets and varieties.

Remark 7.3.1. *The main difficulty in the derivation of inequalities concerning skew-symmetric matrices only -contrary to the symmetric matrices case- is based on the multi-linear nature and the properties of* $\bigwedge^2 (\mathbb{R}^n)$. *The most well known results with regard to skew-symmetric matrices inequalities may be found in [Wey. 2], where it was shown that the eigenvalues* λ_i *of a skew-symmetric matrix* A *and the eigenvalues* κ_i *of* $\bar{A}^t A$, *where* $\bar{A}^t$ *is the conjugate Hermitian of* A, *satisfy the inequalities*

$$\lambda_1^2 \leq \kappa_1, \ \lambda_1^2 \lambda_2^2 \leq \kappa_1 \kappa_2, ..., \lambda_1^2 \cdots \lambda_n^2 \leq \kappa_1 \cdots \kappa_n$$

7.4 Applications to General Distance Problems

In this section we apply the previous inequality, for the solution of the problems dist$(\underline{z}, G_2^{\lambda}(\mathbb{R}^n))$ and dist$(\underline{z}, G_{\lambda_1,...,\lambda_k})$, where $G_2^{\lambda}(\mathbb{R}^n)$ may be considered as a Generalized Grassmann variety and , $G_{\lambda_1,...,\lambda_k}$ a general variety of degenerate eigenvalues. Note that this is the first time, that an inequality of this form is implemented to best decomposable/low-rank approximation problems in such general forms.

7.4.1 Minimum Distance from the $G_2^{\lambda}(\mathbb{R}^n)$ varieties

We aim to calculate the distance of a given multivector $\underline{z}$ from the varieties described by $\underline{z} \wedge \underline{z} \wedge \underline{z} = 0$, $\underline{z} \wedge \underline{z} \wedge \underline{z} \wedge \underline{z} = 0$ and in general the λ-times wedge product variety $\underline{z}^{\wedge\lambda} = 0$.

Definition 7.4.1. *The varieties noted as $G_2^{\lambda}(\mathbb{R}^n)$ where $\lambda \in \{2, 3, .., k\}$, are described by:*

$$G_2^{\lambda}(\mathbb{R}^n) = \left\{\underline{x} \in \wedge^2(\mathbb{R}^2) : \underline{x}^{\wedge\lambda} = 0\right\} \tag{7.27}$$

The above definition complies with the rules of expansion to λ-linearity we saw in Definition in Section 7.2.2 (for instance, the Generalized Grassmann 2-tensor in Definition 7.2.2 is written as the λ-times wedge product $\underline{x}^{\wedge\lambda}$, in our case) and the standard properties of decomposability that a Grassmann variety should have, as we saw in Chapter 3. Furthermore, from Definition 7.4.1 we see that $G_2^{\lambda}(\mathbb{R}^n) \in \text{Hilb}(n)$, where Hilb$(n)$ is the set of Hilbert schemes on an n-projective space, i.e., a scheme (topological space along with the commutative rings of all its open sets, i.e., an "enlargement" of the notion of algebraic variety) that is the parameter space for the closed subschemes in the corresponding projective space $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$. Note that Grothendieck was the first to construct the Hilbert scheme as a subscheme of the Grassmann variety by the vanishing of various determinats. More details on the Grassmann varieties when they are viewed as Hilbert schemes may be found in [Nit. 1]. Thus, our definition agrees with the standard background definitions and results that should characterize a Grassmann variety.

Theorem 7.4.1. *(Least Distance from the Generalized Grassmann Varieties) Let $\underline{z} = \sigma_1 \underline{a}_1 \wedge \underline{b}_1 + \sigma_2 \underline{a}_2 \wedge \underline{b}_2 + ... + \sigma_k \underline{a}_k \wedge \underline{b}_k$ be the prime decomposition of a 2-vector $\underline{z}$ and $\underline{a}_i$, $\underline{b}_i$ be the respective orthonormal vectors obtained by the spectral analysis of $T_{\underline{z}}$. Then,*

$$\min \text{dist}\left(\underline{z}, G_2^{\lambda}(\mathbb{R}^n)\right) = \sqrt{\sigma_{k-\lambda+1}^2 + \sigma_{k-\lambda}^2 + ... + \sigma_1^2} \tag{7.28}$$

which it is achieved at $G_2^{\lambda}(\mathbb{R}^n) \ni \underline{x}_0 = \sigma_{k-\lambda+2} \underline{a}_{k-\lambda+2} \wedge \underline{b}_{k-\lambda+2} + ... + \sigma_k \underline{a}_k \wedge \underline{b}_k$.

Proof. We have:

$$\|\underline{z}-\underline{x}\|^2 = \|\underline{z}\|^2 + \|\underline{x}\|^2 - 2 < \underline{z}, \underline{x} >= \sum_{i=1}^{k} \sigma_i^2 + \sum_{i=k-\lambda+1}^{k} s_i^2 - 2 < \underline{z}, \underline{x} > \geq$$

$$\geq \sum_{i=1}^{k} \sigma_i^2 + \sum_{i=k-\lambda+1}^{k} s_i^2 - 2(\sigma_k s_k + ... + \sigma_1 s_1) \text{ (due to Theorem 7.3.1)}$$

$$= \sum_{i=k-\lambda+1}^{k} (\sigma_i - s_i)^2 + \sigma_{k-\lambda}^2 + ... + \sigma_1^2 \geq \sigma_{k-\lambda}^2 + ... + \sigma_1^2$$

Thus $\|\underline{z}-\underline{x}\|^2 \geq \sigma_{k-\lambda}^2 + ... + \sigma_1^2$ for all $\underline{x} \in G_2^{\lambda}(\mathbb{R}^n)$ and obviously the multivector $\underline{x}_0 \in G_2^{\lambda}(\mathbb{R}^n)$ that realizes the least distance from $G_2^{\lambda}(\mathbb{R}^n)$ is $\underline{x}_0 = \sigma_{k-\lambda+2}\underline{a}_{k-\lambda+2} \wedge \underline{b}_{k-\lambda+2} + ... + \sigma_k \underline{a}_k \wedge \underline{b}_k$. □

Remark 7.4.1. *A different approach of generalization of the classical Grassmann variety $G_m(\mathbb{R}^n)$ has been presented in [Kolh. 1], via the use of endomorphisms and a special Grassmann variety, the so-called Lagrangian Grassmannian, a smooth manifold of Lagrangian subspaces (subspaces which are equal to their othogonal complements, $\mathcal{V} = \mathcal{V}^{\perp}$, but not necessarily $\mathcal{V} \cap \mathcal{V}^{\perp} = \emptyset$) of a real sympletic vector space $\mathcal{V}$ (a vector space equipped with a bilinear form φ which is skew-symmetric, i.e., $\varphi(\underline{x}, \underline{y}) = -\varphi(\underline{y}, \underline{x})$, totally isotropic, i.e., $\varphi(\underline{x}, \underline{x}) = 0$, $\forall \underline{x} \in \mathcal{V}$ and nondegenerate, i.e., if $\varphi(\underline{x}, \underline{y}) = 0$, $\forall \underline{y} \in \mathcal{V}$ then $\underline{x} = \underline{0}$.) Then with the help of this variety (the simplest case of which is the subspace spanned by the vectors $\underline{x}_1, ..., \underline{x}_n$ for the $\mathbb{R}^{2n}$ case, $n \in \mathbb{N}$) the Generalized Grassmann variety was defined as the set of all m-dimensional subspaces of $\mathbb{R}^n$ that are preserved under any endomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. However, this approach is only useful within the Lie algebra approach of the Grassmann varieties, which studies $G_m(F^n)$ in relation with other algebraic objects, such as the Schur-S polynomials, rings, etc, and not for deriving best approximate solutions on hyper-sets that may include the Grassmann variety, such as $G_2^{\lambda}(\mathbb{R}^n)$ in our case.*

7.4.2 Least Distance from the Varieties $G_{\lambda_1,...,\lambda_k}$

The case where the eigenvalues or the singular values of a matrix have a similar structure among them is a problem often met in matrix applications, from matrix decompositions, where uniqueness issues arise, to robotics and isotropic matrix theory, [Sal. & Cr. 1], for matrices whose eigenvalues are all equal. In this section we approximate a 2-vector by one whose imaginary parts σ_i of the corresponding eigenvectors satisfy the general degeneracy property $\sigma_i = \rho\lambda_i$, for $0 \leq \lambda_1 \leq \cdots \leq \lambda_k$ and $\rho > 0$.

Definition 7.4.2. *Let $\underline{z} \in \wedge^2(\mathbb{R}^n)$ and let $\underline{z} = \sum_{i=1}^{k} \sigma_i \underline{x}_i$ be its prime decomposition, $\underline{x}_i := \underline{e}_{2i} \wedge \underline{e}_{2i-1}$, where $\underline{e}_i$ are obtained by the spectral analysis of $T_{\underline{z}}$. If*

$\lambda_i,\ i=1,...,k$ *are real numbers such that* $0 \leq \lambda_1 \leq \cdots \leq \lambda_k$ *and* $\rho > 0$, *then we define the set*

$$G_{\lambda_1,...,\lambda_k} := \left\{ \underline{z} \in \wedge^2(\mathbb{R}^n) : \underline{z} = \sum_{i=1}^{k} \sigma_i \underline{x}_i,\ \textit{such that}\ \sigma_i = \rho\lambda_i,\ i=1,...,k \right\} \quad (7.29)$$

From Definition 7.4.2, we easily verify that the case of decomposable vectors $D_{\wedge^2(\mathbb{R}^n)}$ is a special case of $G_{\lambda_1,...,\lambda_k}$, i.e., $D_{\wedge^2(\mathbb{R}^n)} = G_{0,0,...,1}$.

Remark 7.4.2. *Since* $G_{\lambda_1,...,\lambda_k} = G_{\mu_1,...,\mu_k}$ *if and only if* $(\lambda_1,...,\lambda_k) = \rho(\mu_1,...,\mu_k)$ *for some* $\rho > 0$, *then without loss of generality we may assume that* $\|\underline{\lambda}\| = 1$ *where* $\underline{\lambda} = (\lambda_1,...,\lambda_k)$.

Next theorem is very important for solving general minimization problems with respect to $G_{\lambda_1,...,\lambda_k}$.

Theorem 7.4.2. *Let* $\underline{z} \in \wedge^2(\mathbb{R}^n)$ *and let* $\underline{z} = \sum_{i=1}^{k} \sigma_i \underline{x}_i$ *be its prime decomposition. If* $0 \leq \lambda_1 \leq \cdots \leq \lambda_k$ *is an increasing sequence of real numbers with* $\|\underline{\lambda}\| = 1,\ \lambda = (\lambda_1,...,\lambda_k)$ *then*

$$\min \operatorname{dist}(\underline{z}, G_{\lambda_1,...,\lambda_k}) = \left\{ \sum_{i=1}^{k} \sigma_i^2 - \left(\sum_{i=1}^{k} \sigma_i \lambda_i \right)^2 \right\}^{1/2} \quad (7.30)$$

The closest vector from $G_{\lambda_1,...,\lambda_k}$ *realizing this distance is given by*

$$\underline{x}_0 = \left(\sum_{i=1}^{k} \sigma_i \lambda_i \right) \cdot \left(\sum_{i=1}^{k} \lambda_i \underline{x}_i \right) \quad (7.31)$$

where $\underline{x}_i = \underline{e}_{2i} \wedge \underline{e}_{2i-1}$ *and* $\underline{e}_i$ *are obtained by the eigenvectors of* $T_{\underline{z}}$.

Proof. Let $\underline{x} \in G_{\lambda_1,...,\lambda_k}$. Then,

$$\min_{\underline{x} \in G_{\lambda_1,...,\lambda_k}} \|\underline{x} - \underline{z}\|^2 = \min_{\underline{x} \in G_{\lambda_1,...,\lambda_k}} \left\{ \|\underline{z}\|^2 + \|\underline{x}\|^2 - 2 < \underline{z}, \underline{x} > \right\} \quad (7.32)$$

Thus (7.32) is minimized at $\underline{x}_1 =< \underline{z}, \underline{x} > \underline{x}/\|\underline{x}\|^2$ for some $\underline{x} \in G_{\lambda_1,...,\lambda_k}$. Hence,

$$\min_{\underline{x} \in G_{\lambda_1,...,\lambda_k}} \|\underline{x} - \underline{z}\|^2 = \min_{\underline{x} \in G_{\lambda_1,...,\lambda_k}, \|\underline{x}\|=1} \left\{ \|\underline{z}\|^2 - < \underline{z}, \underline{x} >^2 \right\} =$$
$$= \|\underline{z}\|^2 - \max_{\underline{x} \in G_{\lambda_1,...,\lambda_k}, \|\underline{x}\|=1} < \underline{z}, \underline{x} >^2$$

However, from the Cauchy-Schwartz type equation (7.26), we have that

$$< \underline{z}, \underline{x} >^2 \leq \left(\sum_{i=1}^{k} \sigma_i \lambda_i \right)^2 \quad (7.33)$$

where the maximum is achieved when $\underline{x} = \sum_{i=1}^{k} \lambda_i \underline{x}_i$. Hence, in our case

$$\underline{x}_1 = \frac{< \underline{z}, \underline{x} > \underline{x}}{\|\underline{x}\|^2} = \left(\sum_{i=1}^{k} \sigma_i \lambda_i \right) \cdot \left(\sum_{i=1}^{k} \lambda_i \underline{x}_i \right) \tag{7.34}$$

which implies the minimum distance

$$\text{dist}\,(\underline{z}, G_{\lambda_1,\ldots,\lambda_k}) = \left\{ \|\underline{z}\|^2 - < \underline{z}, \underline{x} >^2 \right\}^{1/2} = \left\{ \sum_{i=1}^{k} \sigma_i^2 - \left(\sum_{i=1}^{k} \sigma_i \lambda_i \right)^2 \right\}^{1/2} \tag{7.35}$$

□

We now present 3 characteristic examples.

Example 7.4.1. *If $\lambda_k = 1$ and $\lambda_i = 0$ for $i < k$, then we see that*

$$\text{dist}\,(\underline{z}, G_{0,\ldots,0,1}) = \left\{ \sum_{i=1}^{k} \sigma_i^2 - \sigma_k^2 \right\}^{1/2} = \text{dist}\left(\underline{z}, D_{\wedge^2(\mathbb{R}^n)}\right) \tag{7.36}$$

which is the best decomposable approximation problem, i.e., $D_{\wedge^2(\mathbb{R}^n)} \ni \underline{x}_{\min} = \sigma_k \underline{e}_{2k} \wedge \underline{e}_{2k-1}$.

Example 7.4.2. *Let $\lambda_1 = \cdots = \lambda_k = 1/\sqrt{k}$. Then*

$$\text{dist}\,(\underline{z}, G_{\lambda_1,\ldots,\lambda_k}) = \left\{ \sum_{i=1}^{k} \sigma_i^2 - \frac{\left(\sum_{i=1}^{k} \sigma_i\right)^2}{k} \right\}^{1/2} = \sqrt{\frac{\sum_{i>j} (\sigma_i - \sigma_j)^2}{k}}$$

where the least distance is achieved at $G_{\lambda_1,\ldots,\lambda_k} \ni \underline{x}_{\min} = k^{-1} \left(\sum_{i=1}^{k} \sigma_i\right) \sum_{i=1}^{k} \underline{x}_i$ where $\underline{x}_i = \underline{e}_{2i} \wedge \underline{e}_{2i-1}$.

Example 7.4.3. *Let $\lambda_1 = \cdots = \lambda_{k-2} = 0,\ \lambda_{k-1} = \lambda_k = 1/\sqrt{2}$. Then*

$$\text{dist}\,(\underline{z}, G_{\lambda_1,\ldots,\lambda_k}) = \left\{ \sum_{i=1}^{k-2} \sigma_i^2 - \frac{(\sigma_k + \sigma_{k-1})^2}{2} \right\}^{1/2} = \sqrt{\sum_{i=1}^{k-2} \sigma_i^2 + \frac{(\sigma_k - \sigma_{k-1})^2}{2}}$$

where the minimum is obtained at $G_{\lambda_1,\ldots,\lambda_k} \ni \underline{x}_{\min} = (\sigma_k + \sigma_{k-1})(\underline{e}_{2k} \wedge \underline{e}_{2k-1} + \underline{e}_{2k-2} \wedge \underline{e}_{2k-3})$.

7.5 Conclusions

There is a wide variety of problems arising in system and control theory that can be reduced to a handful of standard convex or quasiconvex optimization problems, i.e., optimization of a real-valued objective function defined on an interval or on a convex space, such that the inverse image of any set of the form $\{x \in \mathbb{R} : x < a,\ a \in \mathbb{R}\}$ is a convex set. These problems, as explained in [Boy., etc. 1], involve matrix inequalities and vary from Lyapunov stability problems to eigenvalue problems, for the minimization of the maximum eigenvalue of a matrix that depends affinely on a variable, subject to an linear matrix inequality constraint. However, only a few inverse eigenvalue problems have been connected with the derivation of matrix or matrix spectral inequalities, such as the pole placement via state feedback, [Boy., etc. 1] and the problem of defining the set of linear inequalities that describe the possible spectrum of a matrix which is written as a sum of Hermitian matrices, [Rid. 1]. The latter problem was examined via the Schubert calculus methods we saw in Chapter 3.

In this chapter we tried to connect the Approximate DAP with some spectral inequalities and specifically with Von Neumann's trace inequality, [Mir. 1] where we showed that its 2-vector form (7.26) is suitable not only for solving best approximation problems on Grassmann varieties, but it may solve any distance problem from the varieties described by $\underline{z} \wedge \cdots \wedge \underline{z} = \underline{0}$ which may be regarded as a Generalization of the standard Grassmann variety sets of 2-vectors with degenerate eigenvalues.

The generalization to these varieties has been established by the primary definitions of decomposability, [Hod. & Ped. 1], [Mar. 1] and the concept of generalizing the standard Exterior (Grassmann) Algebra to the so-called Generalized Grassmann Algebras, e.g., [Kwa. 1], [Ohn. & Kam. 1]. Note also that our generalization follows the definitions and laws of Grothendieck, who viewed the Grassmannian as a Hilbert scheme and constructed it by the vanishing of various determinants, [Nit. 1]. Hence, $G_2^{\lambda}(\mathbb{R}^n)$ may also be considered in Hilb(n) of $\mathbb{P}^{\binom{n}{2}}(\mathbb{R})$, i.e., a closed subscheme of the projective subspace.

This chapter completes the examination of DAP in all pathological cases that may arise during the process of approximation, such as matrices with equal of repeated eigenvalues or matrices of specific rank for the $\bigwedge^2(\mathbb{R}^n)$ case and we may now proceed to the expansion of $\underline{z}$ into $\underline{z}(\underline{x})$ that covers the whole linear variety $\mathcal{K}$ of the linear subproblem of DAP we saw in Chapter 3 and expansion to higher dimensions, i.e., $\bigwedge^3(\mathbb{R}^n)$. It is also worth mentioning that the results presented here may be considered prototype, since this is the first time the solution of such general approximation problems covers optimization techniques for the

standard Grassmann variety $G_2(\mathbb{R}^n)$, the Generalized Grassmann varieties and general manifold constrained optimization problems and low-rank techniques.

Chapter 8

Solutions of the Approximate Determinantal Assignment Problem in $\bigwedge^2(\mathbb{R}^n)$

8.1 Introduction

As we have already seen, the multilinear nature of DAP has suggested an Exterior Algebra framework for its study. Specifically, DAP may be reduced to a linear problem of zero assignment of polynomial combinants, described by the solutions of the linear equations $\underline{h}^t P = \underline{a}^t$ where P is the corresponding Plücker matrix and a standard problem of multi-linear algebra expressed by the decomposability of the multivector $\underline{h}$. The solution of the linear sub-problem, defines a linear space in a projective space whereas decomposability is characterized by the set of *Quadratic Plücker Relations* (QPR), i.e., the set of quadratic equations defining the Grassmann variety of a related projective space. Thus, the solvability of DAP is reduced to a problem of finding real intersections between the linear variety and the Grassmann variety. This novel Exterior Algebra-Algebraic Geometry method, has provided, as we have seen a number of invariants, such as the Plücker Matrices and Grassmann vectors, suitable for the characterization of rational vector spaces and the solvability of control problems, in both generic and non-generic cases, and it is flexible as far as handling dynamic schemes, as well as structurally constrained compensation schemes, [Kar. & Gia. 5]. An additional advantage of this framework is that it provides a unifying computational framework for finding the solutions, when such solutions exist.

The above approach for the study of DAP in a projective, rather than an affine space setting provides a computational approach that relies on exterior algebra and on the explicit description of the Grassmann variety in terms of the QPR, which allows its formulation as a distance problem between varieties in the (real)

projective space. This may transform the problem of exact intersection to a problem of "approximate intersection", i.e., small distance -via a suitable metric- between varieties, thus transform the exact DAP synthesis method to a DAP design methodology, where approximate solutions to the exact problem are sought. This enables the derivation of solutions, even for non-generic cases and handles problems of model uncertainty, as well as approximate solutions to the cases where generically there is no solution of the exact problem.

In this chapter we consider the distance problem

$$\text{argmin}_{\underline{h}\in G_m(\mathbb{R}^n)}\text{dist}(\underline{h},\mathcal{K}) \tag{8.1}$$

where $\mathcal{K}$ is described by the solutions of a system of linear equations and $G_m(\mathbb{R}^n)$ is the Grassmann variety described by the set of QPR, as a relaxation of the exact intersection problem, which is referred to as the *approximate DAP*. Note that in order for the problem to make sense, the polynomial implied by the approximate solution must satisfy certain conditions, such as stability and distance criteria from the original stable polynomial. This extension makes the investigation relevant to problems where there are no real intersections and thus approximate solutions are sought. Note that a solution to the approximate problem produces an approximate polynomial that will be assigned and this requires studying the stability properties of this perturbed polynomial, which are very important for the perturbed solutions to be acceptable.

We elaborate on the solution of problem (8.1) and the stability properties of the "approximate" polynomial $\hat{a}(s)$ that corresponds to an approximation $\hat{\underline{h}}$. Our approach views the problem as a minimization problem between a solution $\underline{h}\equiv\underline{z}(\underline{x})$ of a linear problem and the Grassmann variety $G_2(\mathbb{R}^n)$, i.e.,

$$\min_{\underline{x}} g(\underline{z}(\underline{x}),G_2(\mathbb{R}^n)) \tag{8.2}$$

where g is the gap, between the parameterized multivector $\underline{z}(\underline{x})$ in the projective space and the Grassmann variety with $\underline{x}$ being the vector of free parameters that describe the linear variety $\mathcal{K}$. It is shown that the solution of (8.2) is implied by the solution of $\min_{\underline{z}} g(\underline{z},G_2(\mathbb{R}^n))$, i.e., least distance of a *fixed 2-vector* from the Grassmann variety.

The methodology used for the computation of the above minimizers is based on the *best decomposable approximation of multivectors*, in the vicinity of a parametric multivector in the projective space. Specifically, in Section 8.2 we present the generalization of $\underline{z}$ to $\underline{z}(\underline{x})$, where $\underline{x}$ is the vector whose coordinates are the free parameters of $\underline{z}^tP=\underline{a}^t$. For the $n=5$ case, we show that the minimization problem between the corresponding Grassmann variety and the linear variety is equivalent to the minimization of a fourth degree polynomial constrained to the

unit sphere.

In Section 8.3, stability results are examined with respect to the nominal polynomial $a(s)$; In particular, in 8.3.1 we elaborate on the stability criteria which are usually used for similar stability criteria, such as the Hermite and the Hankel matrices, the Bézoutiants, Kharitonov's theorem and stability results in terms of stability radius, whereas in 8.3.2 we implement the latter methodology for a new stability criterion for the approximate DAP that does not require the calculation of the poles.

Section 8.4 is the core of this chapter where some of the most important results of this thesis are exposed; in 8.4.2 and 8.4.1 it is shown that algorithms such as Newton's method on the Grassmann manifold and the Macaulay 2 Algebraic Geometry toolbox may provide solutions for problem (8.1) only in special cases that act restricting for the approximate DAP. This leads us to build in 8.4.3 a new algorithim more appropriate for the approximate DAP.

Finally, in Section 8.5 we apply these results to a special form of DAP, the zero assignment by squaring down problem in order to view the special modifications.

8.2 Least Distance between the linear variety $\mathcal{K}$ and $G_2(\mathbb{R}^n)$

In this section, $\underline{z}$ is obtained as a function $\underline{z}(\underline{x})$ of the degrees of freedom $\underline{x}$ that describe the linear variety $\mathcal{K}$ of the linear system (4.31). Note that such linear varieties are functions of the coefficients of the polynomial that is to be assigned.

Proposition 8.2.1. *Let $\underline{x} = (x_1, ..., x_r)$, $r \in \mathbb{N}$. The least distance between $\mathcal{K}$ and $G_2(\mathbb{R}^n)$ is given by*

$$\min_{\underline{x}} \sqrt{\sigma_{k-1}^2(\underline{x}) + \sigma_{k-2}^2(\underline{x}) + ... + \sigma_1^2(\underline{x})} \quad \text{subject to } \sum_{i=1}^{k} \sigma_i^2(\underline{x}) = 1 \tag{8.3}$$

where $k := [n/2]$, $\sigma_i(\underline{x})$ is the real part of the $i-th$ eigenvalue of $T_{\underline{z}}$ and $\underline{z}(\underline{x})$ represents the parametric form of the linear variety $\mathcal{K}$.

Proof. Let the determinantal assignment problem $\underline{z}^t P = \underline{a}^t$. Since the poles of the system remain the same under scalar multiplication, we are interested for the general solution of $\underline{z}^t P = \lambda \underline{a}^t$, $\lambda \in \mathbb{R}$. Therefore, if $\underline{z}_0^t$ is a particular solution of DAP then a general solution of the linear problem is given by

$$\underline{z}^t = \lambda \underline{z}_0^t + \underline{\kappa}^t V = [\lambda, \underline{\kappa}^t] \begin{bmatrix} \underline{h}_0^t \\ V \end{bmatrix} \tag{8.4}$$

where $V \in \mathbb{R}^{(r-1)\times\binom{n}{2}}$ is the matrix representation of $\mathcal{N}_\ell(P)$ and $\underline{\kappa}^t \in \mathbb{R}^{r-1}$ for $r = \dim\left(\text{row-span}\left\{\left[\begin{array}{c}\underline{h}_0^t\\ V\end{array}\right]\right\}\right)$. Hence, if $(\underline{v}_1, \underline{v}_2, ..., \underline{v}_r)^t$ is an orthonormal basis of the row-span$\left\{\left[\begin{array}{c}\underline{z}_0^t\\ V\end{array}\right]\right\}$, then the general solution of the linear problem is given by

$$\underline{z} = x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_r\underline{v}_r \equiv \underline{z}(\underline{x}) \tag{8.5}$$

Thus, $\|\underline{z}\| = \|\underline{x}\| = 1$ and the result follows from Theorem 6.3.1. □

Similarly to Theorem 6.3.1, eqn.(8.3) can be further simplified if specific Grassmann varieties are given. The following theorem is one of the main results of this article, which characterizes the way the distance between the linear variety $\mathcal{K}$ and specific Grassmann varieties may be computed.

Theorem 8.2.1. *The least distance problem between the linear variety $\mathcal{K}$ and $G_2(\mathbb{R}^5)$ is equivalent to the minimization of a 4th order homogeneous polynomial $F(\underline{x})$, constrained to the unit sphere.*

Proof. We saw that the case $n = 5$ has implied that the minimization of $g_\wedge = \|\underline{z}\wedge\underline{z}\|/\|\underline{z}\|^2$, which may be used instead of the minimization of the gap g. Hence, problem (8.3) can be transformed into

$$\min_{\underline{z}} \|\underline{z} \wedge \underline{z}\|^2 \quad \text{subject to } \|\underline{z}\| = 1 \tag{8.6}$$

Therefore, due to eqn.(8.5) we obtain

$$\begin{aligned}\|\underline{z} \wedge \underline{z}\|^2 &= \left\|\left(\sum_{i=1}^{r} x_i\underline{v}_i\right) \wedge \left(\sum_{i=1}^{r} x_i\underline{v}_i\right)\right\|^2 = \left\|\sum_{i,j=1}^{r} x_i x_j \underline{v}_i \wedge \underline{v}_j\right\|^2 = \\ &= \left\langle \sum_{i,j=1}^{r} x_i x_j \underline{v}_i \wedge \underline{v}_j, \sum_{\varrho,\mu=1}^{r} x_\varrho x_\mu \underline{v}_\varrho \wedge \underline{v}_\mu \right\rangle = \\ &= \sum_{1\le i,j,\varrho,\mu\le r} x_i x_j x_\varrho x_\mu \langle \underline{v}_i \wedge \underline{v}_j, \underline{v}_\varrho \wedge \underline{v}_\mu\rangle \end{aligned} \tag{8.7}$$

Eqn.(8.7) is a 4th order homogeneous polynomial $F(\underline{x})$ in terms of $x_i,\ x_j,\ x_\varrho,\ x_\mu$. Thus (8.3) is written as

$$\min_{\underline{x}} F(\underline{x}) \quad \text{subject to } \|\underline{x}\| = 1 \tag{8.8}$$

□

If $\underline{z} = \underline{z}(\underline{x})$ implied by the solution of (8.3) or (8.8) is decomposable, then DAP is solved precisely. If $\underline{z}$ is not decomposable, its prime decomposition is implemented in order to obtain the best decomposable approximation $\hat{\underline{z}}$.

8.3 Stability Methodologies

As we have already seen, the asymptotic stability of the continuous time system (4.19), requires that all the eigenvalues lie in open left half-plane. As stated in [Hin. & Pri. 1], these spectral stability criteria were already known in the 19th century, but in the absence of systematic solution procedures for algebraic equations of order $n \geq 5$ and without computers for their approximate solution, these criteria could only be verified for lower dimensional systems. It was therefore a problem of fundamental importance, both for mathematical stability theory and its applications, to express the spectral stability criteria by verifiable conditions on the coefficients of the characteristic polynomial. In [Hin. & Pri. 1] it is also mentioned how many leading mathematicians have contributed to this field and developed methods for determining the number of roots of a polynomial in certain locations of the complex plane (e.g., the real axis, the upper half-plane). In this section we present some of the most important methods and results which have been obtained in this field, such as the notions of the *Routh-Hurwitz stability criterion*, *Hermite matrices*, *Hankel matrices*, *the Bézoutiants*, *Kharitonov's theorem* and the notion of *stability radius*. For the purposes of our work we will use the last methodology. The results discussed here play an important role, not only in stability analysis, but also in other areas of systems theory such as realization theory and model reduction.

8.3.1 Stability of Real and Complex Polynomials

The following definition is considered well-known.

Definition 8.3.1. *A polynomial is said to be a Hurwitz polynomial or Hurwitz stable if all of its roots lie in the open left half complex plane, noted as* $\mathbb{C}_-$.

One of the most well-known mathematical tests used for the determination of a Hurwitz polynomial is the *Routh-Hurwitz stability criterion* which began from Routh's recursive algorithm for the determination of the roots of the characteristic polynomial of a linear system and whether they have negative real parts or not and it was given in an equivalent matrix form by Hurwitz who showed that a polynomial is stable if and only if the sequence of determinants of the principal submatrices of the matrix formed by the coefficients of the polynomial are positive.

Theorem 8.3.1. *(The Routh-Hurwitz stability criterion)[Hin. & Pri. 1] Let an* n*-degree polynomial* $p(s) = a_0 + a_1 s^{n-1} + \cdots + a_n$ *and its corresponding* $n \times n$

square (Hurwitz) matrix

$$H := \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & \cdots & \cdots & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & \cdots & \ddots & \cdots & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & \cdots & \cdots & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & \cdots & \cdots & \cdots & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix} \tag{8.9}$$

Then $p(s)$ is Hurwitz iff all the leading principal minors of H are positive, i.e.,

$$a_1 > 0,\ a_2a_1 - a_0a_3 > 0,\ a_3a_2a_1 - a_0a_3^2 > 0, \cdots$$

Thus, the approximate solution of DAP will involve the solution of a constrained minimization problem where the constraints should satisfy the QPR of the Grassmann variety as well as the above inequalities, which in our case are parametrized by the free parameters $\underline{x}$ defining the linear variety $\mathcal{K}$. Optimization problems of this form are usually addressed algorithmically via semi-definite programming methods, since the Hurwitz inequalities form a semi-algebraic variety. However, the fact that the approximate DAP deals with parametrized multivectors and thus the coefficients of the approximate polynomial will depend on $\underline{x}$, these algorithms can not provide solutions since they are built for numerical and not symbolic data. In the next section we will present a suitable algorithm for these kind of problems whereas stability will be examined with the help of the stability radius, which is significantly easier than the semi-algebraic sets stability approach when algorithmic computations are involved. In the rest of this section we present some other methods which are used for polynomial stability testing and we thoroughly examine the stability radius formulae which we will use.

The following proposition associates every arc in $C^* := \mathbb{C} \setminus \{0\}$, i.e., every segment of a differential curve, with the so-called (continuous) argument function (recall that every $z \in C^*$ has a polar representation $z = |z|e^{i\theta}$ where θ is an argument of z, denoted as $\theta := \arg(z)$).

Proposition 8.3.1. *[Bur. 1] Given an arbitrary interval $\mathcal{I} \subset \mathbb{R}$ and a continuous function (curve) $\gamma : \mathcal{I} \to C^*$, there exists a continuous function $\theta : \mathcal{I} \to \mathbb{R}$ such that*

$$\gamma(t) = |\gamma(t)|e^{i\theta(t)},\ t \in \mathcal{I} \tag{8.10}$$

Since the exponential function is a homomorphism of the additive group $\mathbb{C}$ onto the multiplicative group $\mathbb{C}^*$, the continuous function θ is uniquely determined by (8.10) up to an additive constant $2\pi\lambda,\ \lambda \in \mathbb{Z}$.

Definition 8.3.2. *[Hin. & Pri. 1]Given an arbitrary interval $\mathcal{I} \subset \mathbb{R}$ and a continuous function $\gamma : \mathcal{I} \to \mathbb{C}^*$, any continuous function $\theta : \mathcal{I} \to \mathbb{R}$ satisfying* (8.10)*, is called an argument function which is denoted as $\theta(\cdot) := \arg(\gamma(\cdot))$. If $\mathcal{I} = [a, b]$, then the net change of the argument of $\gamma(t)$ as t moves from a to b is given by*

$$\Delta_a^b \gamma(t) := \theta(b) - \theta(a) \tag{8.11}$$

Proposition 8.3.2. *(Complex Polynomial Stability)[Hin. & Pri. 1] Let a polynomial $a(s) \in \mathbb{C}[s]$ of n degree without zeros on the imaginary axis. Then $a(s)$ is Hurwitz stable if and only if*

$$\lim_{k\to\infty} \Delta_{-k}^{k} \arg\left(a(i\omega)\right) = n\pi \tag{8.12}$$

where $i\omega \in \mathbb{C} \setminus \mathbb{R}$.

In the DAP case we are interested in real polynomials. If $a(s) \in \mathbb{R}[s]$ then $a(-i\omega) = \overline{a(i\omega)}$ and the previous stability criterion may take the following form.

Proposition 8.3.3. *(Real Polynomial Stability) Let $a(s) \in \mathbb{R}[s]$. Then $a(s)$ is Hurwitz stable if and only if*

$$\lim_{k\to\infty} \Delta_{0}^{k} \arg\left(a(i\omega)\right) = n\pi/2 \tag{8.13}$$

Remark 8.3.1. *Some additional stability criteria may be shown which only hold for real polynomials. One of the simplest and most useful is the necessary (but not sufficient) criterion which implies the Hurwitz stability of a real polynomial if all of its coefficients are non-zero and of the same sign. In particular, if $a(s)$ is a monic real Hurwitz polynomial all its coefficients must be positive.*

Despite these important results, the use of the net change Δ is not very useful for practical calculations. In practice the stability of a polynomial is mostly examined via the properties of specific matrices or by expressing the polynomial into equivalent polynomial formulae. These methodologies usually involve: (i) Hermite forms, (ii) Bézoutiants, (iii) Hankel matrices, (iv) Kharitonov's theorem and (v) stability radius formulae. For the DAP case we will work in the next section via the last category, i.e., stability radius. We briefly present the main results of the rest of the categories mentioned above, since as we will see, they are all related. A more thorough presentation may be found in [Hin. & Pri. 1].

Theorem 8.3.2. *(Hermite) Let the n-degree complex polynomial* $a(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$ *where* $\alpha_i \in \mathbb{C}$. *Then* $a(s)$ *is Hurwitz stable if and only if the Hermite matrix*

$$H(a) := h_{ij} = \sum_{m=1}^{\min\{i,n+1-j\}} (-1)^{m-1} \left(\overline{\alpha_{i-m}\alpha_{j+m-1}} - (-1)^{i+j-1}\alpha_{i-m}\overline{\alpha_{j+m-1}} \right) \tag{8.14}$$

for $i, j \in \{1, ..., n\}$, *is positive definite.*

Theorem 8.3.3. *(Bézoutiant) Let a complex polynomial* $a(s)$ *of n-degree be written as* $a(s) = a_r(s) + ia_\ell(s)$, *where* $a_r(s) = s^n + \lambda_{n-1}s^{n-1} + \cdots + \lambda_1 s + \lambda_0$ *and* $a_\ell(s) = s^n + \kappa_{n-1}s^{n-1} + \cdots + \kappa_1 s + \kappa_0$, *where* $\lambda_i,\ \kappa_i \in \mathbb{R}$. *Then* $a(s)$ *is Hurwitz stable if and only if the Bézoutiant (or Bézout matrix)*

$$B_n(a_r, a_\ell) := b_{ij} = \sum_{m=1}^{\min\{i,n+1-j\}} (-1)^{m-1}\kappa_{i-m}\lambda_{j+m-1} - \lambda_{i-m}\kappa_{j+m-1},\ i, j \in \{1, ..., n\} \tag{8.15}$$

is positive definite.

Remark 8.3.2. *It can be proved, [Hin. & Pri. 1], that the Hermite matrix* $H(a)$ *and the Bézoutiant* $B_n(a_r, a_\ell)$ *are congugate matrices in the sence that* $H(a) = 2DB_n(a_r, a_\ell)\bar{D}^t$, *where* $D = \mathrm{diag}(1, i, ..., i^{n-1})$.

Theorem 8.3.4. *(Hankel Matrices) Let* $a(s)$ *be an n-degree complex polynomial written as* $a(s) = a_r(s) + ia_\ell(s)$. *Then* $a(s)$ *is Hurwitz stable if and only if the Hankel matrix*

$$Hk_n(f(a)) := \begin{bmatrix} f(a)_1 & f(a)_2 & \cdots & f(a)_n \\ f(a)_2 & f(a)_3 & \cdots & f(a)_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ f(a)_n & f(a)_{n+1} & \cdots & f(a)_{2n-1} \end{bmatrix} \tag{8.16}$$

where

$$f\left(a(s)\right) := \begin{cases} a_r(s)/a_\ell(s), & \deg a_r \le \deg a_\ell \\ -a_\ell(s)/a_r(s), & \deg a_\ell < \deg a_r \end{cases} \tag{8.17}$$

is positive definite.

In [Hin. & Pri. 1] more results connecting the Hankel matrices to Hermite and Bézout matrices may be found as well as their applications to systems stability and simplified results for the real polynomial case. In particular, a real n degree polynomial $a(s)$ written as a sum of two polynomials $u(s),\ v(s)$ such that $a(s) = u(s^2) + s \cdot v(s^2)$ is Hurwitz stable if and only if the matrix $Hk_n(u/v)$ is positive definite and all the coefficients of $a(s)$ have the same sign.

On the other hand, Kharitonov's theorem deals with the following problem; If for a real monic polynomial all it is known is that its coefficients lie within certain bounds, how can one decide whether it is Hurwitz stable. In order to answer this question Kharitonov found necessary and sufficient conditions for the stability of the whole set of monic polynomials whose coefficients belong to prescribed intervals. Such sets are known as *interval polynomials*.

Theorem 8.3.5. *(Kharitonov) Let $a(s) = s^n + \sum_{i=1}^{n} \alpha_i s^{n-i}$ be an n-degree real monic polynomial with $0 < \beta_i \leq \alpha_i \leq \gamma_i$, where β_i and γ_i are known real scalars which is written in terms of its even and odd parts as $a(s) = u(s^2) + s \cdot v(s^2)$. Let also the 4 polynomials $k_1(s) = u_1(s^2) + s \cdot v_1(s^2)$, $k_2(s) = u_2(s^2) + s \cdot v_2(s^2)$, $k_3(s) = u_1(s^2) + s \cdot v_2(s^2)$ and $k_4(s) = u_2(s^2) + s \cdot v_1(s^2)$, where $u_1(s^2) = \beta_n + \gamma_{n-2}s^2 + \beta_{n-4}s^4 + \cdots$, $u_2(s^2) = \gamma_n + \beta_{n-2}s^2 + \gamma_{n-4}s^4 + \cdots$, $v_1(s) = \beta_{n-1} + \gamma_{n-3}s^2 + \beta_{n-5}s^4 + \cdots$, $v_2(s) = \gamma_{n-1} + \beta_{n-3}s^2 + \gamma_{n-5}s^4 + \cdots$. Then $a(s)$ is Hurwitz stable if and only if k_i, $i = 1, ..., 4$ are Hurwitz stable.*

Kharitonov's theorem is considered one of the corner-stone results of polynomial stability for system and control theory. More details as well as applications of this theorem may be found in [Das. 1] and [Hin. & Pri. 1].

Example 8.3.1. *Let the tranfer function*

$$G(s) = \frac{a_0 + a_1 s}{a_2 s^2 + a_3 s^3 + a_4 s^4}$$

with $a_0 \in [x_0, y_0]$, $a_1 \in [x_1, y_1]$, $a_2 \in [x_2, y_2]$, $a_3 \in [x_3, y_3]$, $a_4 \in [x_4, y_4]$. The characteristic polynomial is then

$$a(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4$$

Thus, $k_{\min}^{\text{even}}(s) = x_0 + y_2 s^2 + x_4 s^4$, $k_{\max}^{\text{even}}(s) = y_0 + x_2 s^2 + y_4 s^4$, $k_{\min}^{\text{odd}}(s) = x_1 s + y_3 s^3$, $k_{\max}^{\text{odd}}(s) = y_1 s^2 + x_3 s^3$. Hence, the Kharitonov polynomials are $k_1(s) = x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4$, $k_2(s) = x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4$, $k_3(s) = y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4$ and $k_4(s) = y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4$. Then $a(s)$ is Hurwitz if $k_i(s)$ are Hurwitz, i.e., if the coefficients are positive (otherwise multiply by -1) and

$$x_1 y_2 y_3 > x_1^2 x_4 + y_3^2 x_0,\ y_1 y_2 x_3 > y_1^2 x_4 + x_3^2 x_0$$

$$x_1 x_2 y_3 > x_1^2 y_4 + y_3^2 y_0,\ y_1 x_2 x_3 > y_1^2 y_4 + x_3^2 y_0$$

which are implied by the Routh-Hurwitz stability criterion we saw at the beginning of this section.

Hence, in our case if we want to solve $\det(H \cdot M(s)) = a(s)$ when $a(s)$ is stable, we just have to restrict ourselves only to the four Kharitonov's polynomials k_i, and not on the entire domain of stable polynomials, created by the semi-algebraic variety of the Routh-Hurwitz criterion. In other words, DAP may be reduced to a

linear subproblem described by the linear variety $\mathcal{K}$, the multilinear subproblem described by the QPRs of the related Grassmann variety and the semi-algebraic variety implied by the four polynomials k_i whose stability is described by the non-linear inequalities that the coefficients of $a(s)$ must satisfy. Even though this approach provides significant simplifications to the approximate solutions under assumed stability, the corresponding optimization problem still remains difficult due to complexity of the constraints that now do not just satisfy the QPRs, but also the non-linear inequalities of Kharitonov's stability sub-domain which as we have mentioned before are parametrized by the free parameters $\underline{x}$ of the linear variety $\mathcal{K}$. The optimization problem from the Grassmann variety when $a(s)$ is stable may be further simplified with the use of the stability radius.

The stability radius method tests which variations of the coefficient vector of $a(s)$ can be tolerated without destroying the property that all its roots are in a prescribed region of the complex plane, i.e., $\mathbb{C}_-$. The method originates from Kharitonov's theorem we saw in Theorem 8.3.5, since it is also based on separating the even and the odd parts of $a(s)$ and it is the appropriate tool for the approximate solution of DAP and the stability of its "approximate" polynomial.

Definition 8.3.3. *(Stability Radius)[Hin. & Pri. 2] Let $a(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$ be an n-degree monic polynomial with its coefficient vector $\underline{\alpha} = (\alpha_0, ..., \alpha_{n-1}) \in \mathbb{R}^n$ $(\mathbb{C}^n)$. Let also a pertubation of the coefficients of the form*

$$\alpha_{j-1}(\underline{d}) = \alpha_{j-1} - \sum_{i=1}^{\ell} d_i c_{ij},\ j \in \{1, ..., n\} \tag{8.18}$$

where $\underline{d} = (d_1, ..., d_\ell) \in \mathbb{R}^\ell$ $(\mathbb{C}^\ell)$ is an unknown disturbance vector and $C = (c_{ij}) \in F^{\ell \times n}$ is a given matrix. The stability radius for $a(s, \underline{\alpha})$ is given by

$$r_{\mathbb{F}}(\underline{\alpha}, C, \mathbb{C} \setminus \mathbb{C}_-) = \inf \left\{ \|\underline{d}\|; \underline{d} \in \mathbb{F}^\ell,\ \exists \lambda \in \mathbb{C} \setminus \mathbb{C}_- : a(\lambda, \underline{\alpha}(\underline{d})) = 0 \right\} \tag{8.19}$$

Remark 8.3.3. *A different formulation of the above stability radius definition is given in [Gr. & Lan. 1] where the polynomial problem is reformulated as a matrix problem, using companion matrix theory and the polynomial pseudozero set introduced in [Mos. 1].*

Next we elaborate on Definition 8.3.3 to implement it to an approximate DAP algorithm.

8.3.2 New Stability Criteria for the Approximate DAP

In this section we assume that the solution $\underline{z}(\underline{x})$ is not a decomposable multivector, i.e., the gap between $\underline{z}(\underline{x})$ and the Grassmann variety in not zero. Then a perturbed polynomial $\hat{a}(s)$ is derived such that $\hat{\underline{z}}^t P = \hat{\underline{a}}^t$, with respect to the

initial problem. We examine the stability properties of the resulting polynomial $\hat{a}(s)$ and its distance from the nominal polynomial $a(s)$, which we intended to assign. We need a more explicit form for the real stability radius. We follow the stability radius methodology proposed in [Hin. & Pri. 2].

We suppose that $\mathcal{F} = \mathbb{R}$, $C \in \mathbb{R}^{\ell \times n}$ is a given matrix and $a(s, \underline{\alpha})$ is a real polynomial having all its roots in $\mathbb{C}_-$, with $\underline{\alpha} \in \mathbb{R}^n$. To characterize the real stability radius we denote by $d(\underline{y}, \mathbb{R}_{\underline{v}})$ the distance of a point $\underline{y} \in \mathbb{R}^\ell$ from the linear space $\mathbb{R}_{\underline{v}}$ spanned by $\underline{v} \in \mathbb{R}$. Then

$$d^2(\underline{y}, \mathbb{R}_{\underline{v}}) \begin{cases} \|\underline{y}\|^2 - \dfrac{< \underline{y}, \underline{v} >}{\|\underline{v}\|^2}, & \underline{v} \neq \underline{0} \\ \|\underline{y}\|^2, & \underline{v} = \underline{0} \end{cases} \tag{8.20}$$

Then the following result may be established.

Proposition 8.3.4. *[Hin. & Pri. 2] Let $(\alpha, C) \in \mathbb{R}^{1 \times n} \times \mathbb{R}^{\ell \times n}$ and suppose that all the roots of $a(s, \underline{\alpha})$ lie in $\mathbb{C}_-$. Then*

$$r_{\mathbb{R}}(\underline{\alpha}, C; \mathbb{C} \setminus \mathbb{C}_-) = \max_{s \in \partial(\mathbb{C} \setminus \mathbb{C}_-)} d\left(G_R(s), \mathbb{R}_{G_L(s)}\right)^{-1} \tag{8.21}$$

where $\partial(\mathbb{C} \setminus \mathbb{C}_-)$ is the border of $\mathbb{C} \setminus \mathbb{C}_-$ and $G_R(s), G_L(s) \in \mathbb{R}^{\ell \times 1}$ such that

$$G_R(s) + iG_L(s) = \frac{1}{a(s, \underline{\alpha})} \left(\sum_{j=1}^{n} c_{1j} s^{j-1}, \sum_{j=1}^{n} c_{2j} s^{j-1}, ..., \sum_{j=1}^{n} c_{\ell j} s^{j-1} \right)^t \tag{8.22}$$

From the previous proposition the following formula may be derived which is more convenient for numerical computations.

Proposition 8.3.5. *[Hin. & Pri. 2] If*

$$a(\underline{\alpha}, s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0, \ \underline{\alpha} = (1, \alpha_{n-1}, ..., \alpha_0) \tag{8.23}$$

is a Hurwitz polynomial written as, $a(\underline{\alpha}, s) = a_1(-s^2) + s a_2(-s^2)$ where $a_j(-s^2)$, $j = 1, 2$, are real polynomials in $-s^2$, then the real stability radius is given by

$$r_{\mathbb{R}}(\underline{\alpha}, C; \mathbb{C} \setminus \mathbb{C}_-) = \min \left\{ \alpha_0, \left(\max_{\omega^2 \in \mathbb{R}^+} f(\omega^2) \right)^{-1/2} \right\} \tag{8.24}$$

where

$$f(\omega^2) = \frac{1 + \omega^4 + \cdots + \omega^{2n-4}}{a_1^2(\omega^2) + a_2^2(\omega^2)}, \quad \text{if } n = 2k$$

or

$$f(\omega^2) = \frac{(1 + \omega^4 + \cdots + \omega^{2n-6})(1 + \omega^4 + \cdots \omega^{2n-2})}{a_1^2(\omega^2)(1 + \omega^4 + \cdots + \omega^{2n-6}) + a_2^2(\omega^2)(1 + \omega^4 + \cdots + \omega^{2n-2})}, \quad \text{if } n = 2k+1$$

From (8.24), we easily verify the following result.

Lemma 8.3.1. *Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$ where n is the degree of a stable polynomial $a(s)$. If $\hat{a}(s)$ is the perturbed polynomial with respect to the coefficients $\underline{\alpha}$ of a and $\|\underline{a}-\underline{\hat{a}}\| \leq r_{\underline{\alpha}}$, where $\underline{a}$, $\underline{\hat{a}}$ their coefficient-vectors respectively, then $\hat{a}(s)$ is also stable.*

The following criterion is now obtained, linking the decomposability of DAP to its stability.

Theorem 8.3.6. *Let $\underline{z} \in \wedge^2(\mathbb{R}^n)$ be a 2-vector and $\underline{\hat{z}}$ be its best decomposable approximation. If $\underline{a}$, $\underline{\hat{a}}$ are the coefficient-vectors of $a(s)$, $\underline{\hat{a}}(s)$ respectively, and $\|\underline{z}-\underline{\hat{z}}\| \leq r_{\underline{\alpha}}/\sigma_P$, where σ_P is the largest singular value of the Plücker matrix P and $a(s)$ is a stable polynomial, then $\hat{a}(s)$ is also stable.*

Proof. Let the spectral norm $\|A\|_2 = \max\{\lambda : \lambda \in \sigma(A^tA)\}$. From (8.24), the stability radius $r_{\underline{\alpha}}$ is computable and due to the forms of the initial and the approximate system, we immediately imply:

$$\|\underline{a}-\underline{\hat{a}}\| = \|(\underline{z}-\underline{\hat{z}})P\| \leq \|P\|_2 \frac{r_{\underline{\alpha}}}{\sigma_P} = r_{\underline{\alpha}}$$

The result now follows from Lemma 8.3.1. □

Theorem 8.3.6 does not only provide the means to test whether the perturbed solutions are at an acceptable distance from the original stable polynomial we had to assign, but also constitutes a criterion for the stability of the perturbed polynomial, *without* the calculation of its roots or their properties.

8.4 Optimization Algorithms

In this section we present a new algorithm for the best approximate solution of DAP, based on the minimization of $g_\wedge$. At first we use the Algebraic Geometry toolbox Macaulay 2 and then we present the generalization of Newton's algoritm for the minimization of the Rayleigh quotient on the Grassmann manifold, as this was presented in [Abs., etc. 1], [Abs., etc. 2], [Edel., etc. 1] and [Smi. 1]. We show that these two methodologies even though they may provide a number of useful results with regard to optimization over manifolds, they are not completely applicable for DAP. The former is usuful only when $\underline{z}$ is constant (not parametrized) since the program in the case of $\underline{z}(\underline{x})$ provides a large number of complicated solutions that are not easy to be verified, whereas Newton's algoritm may be used for the approximate DAP under several alternations and restrictions. Therefore, the determinatal assignment problem requires in general a different approach which is not met in these algorithms. At the last part of this section we present a new algorithm which is exclusively constructed for the approximate DAP, thus no special assumptions have to be made.

8.4.1 Decomposable Approximations via Macaulay 2

In this section we compare the results of the previous analysis with the ones that may be obtained by the Numerical Algebraic Geometry Toolbox *Macaulay 2*, [Eis., etc. 1] for the minimization problem

$$\min_{\underline{x}} \|\underline{z} - \underline{x}\| \tag{8.25}$$

when $\underline{z} \in \bigwedge^2(\mathbb{R}^n)$ and $\underline{x}$ is decomposable. The package originated by the need for numeric calculations within the context of invariant theory where the explicit description of polynomial functions, which remained invariant under the transformations of a given linear group, was examined [Sturm. 2]. Soon the program expanded for numerical applications to other fields, that used techniques from this area or related ones, such as optimal control, manifold optimization, etc., giving the opportunity to researchers to have an arithmetic point of view of their problems, to experiment with their ideas and to verify their results. In our case, the Macaulay 2 Toolbox is helpful for optimization oven the Grassmann variety, since it helps overcome the high complexity of the constraints, that constitute the problem impossible to solve numerically, via other toolboxes (a simple numerical substitution to problem (8.25) for the n=5 case in Matlab or Mathematica for instance, can verify this assertion).

In general, the system uses high level algebra objects, such as Galois fields, number fields, polynomial rings, exterior algebras, Weyl algebras, quotient rings, ideals, modules, homomorphisms of rings and modules, graded modules, maps between graded modules, chain complexes, maps between chain complexes, free resolutions, algebraic varieties, coherent sheaves and the Gröbner basis algorithm. The programming environment is similar to the one in Mathematica or Maple: the user enters mathematical expressions at the keyboard, and the program computes the value of the expression and displays the answer. The first input prompt offered to the user is of the form

$$i1 :$$

In response to the prompt, the user may enter, for example, a simple arithmetic expression, i.e.,

$$\begin{aligned} &i1 : 3/5 + 7/11 \\ &o1 = 68/55 \\ &o1 : QQ \end{aligned}$$

The answer itself is displayed to the right of the output label

$$o1 =$$

and its type (or class) is displayed to the right of the following label.

$$o1 :$$

The symbol QQ appearing in this example denotes the class of all rational numbers, and is meant to be reminiscent of the notation $\mathbb{Q}$.

In order to explain the Macaulay code for the $G_2(\mathbb{R}^5)$ case, we present the case of a plane curve

$$\ell = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\} \tag{8.26}$$

where f is a polynomial of degree d. We are given a point $(u, v) \in \mathbb{R}^2$ and we seek to minimize the distance from (u, v) to ℓ. For random f, this problem has d^2 complex critical points due to Bezout's theorem, i.e., the number of intersection points of n projective hyper-surfaces in a projective space of dimension n over an algebraic closed field, which are defined by n homogeneous polynomials in $n + 1$ variables, of degrees $d_1, ..., d_n$ is infinite, or the number of intersection points, counted with multiplicity, is equal to the product $d_1 \cdots d_n$, [Ful. 1]. The method of Lagrange multipliers says the gradient of $(u - x)^2 + (v - y)^2$ must be linearly dependent with the gradient of the constraint $f(x, y)$. Hence our equations to solve in this case are

$$f(x, y) = 0,\ (u - x) \cdot \frac{\partial f}{\partial y} = (v - y) \cdot \frac{\partial f}{\partial x}$$

where it is expected to have d^2 complex solutions, by Bezout's theorem. For the $G_2(\mathbb{R}^5)$ case, we present the following algorithm.

A Macaulay 2 Algorithm for the Approximate DAP

- R = $QQ[p_{12}, p_{13}, p_{23}, p_{14}, p_{24}, p_{34}, p_{15}, p_{25}, p_{35}, p_{45}]$;
- $\mathfrak{I}$ = Grassmannian$(1, 4, R)$.
- T = (transpose matrix apply(gens R, $p- > p$-random(-100,100)))—(jacobian($\mathfrak{I}$))
- $\mathfrak{J} = \mathfrak{I}$+minors(4,$T$);
- $\mathfrak{J} = \mathfrak{J}$: ideal($p_{12}p_{13}p_{23}p_{14}p_{24}p_{34}p_{15}p_{25}p_{35}p_{45}$);

Example 8.4.1. *Let* $R = QQ[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$ *and* $\mathfrak{I} = Grassmannian(1, 4, R)$. *These two functions generate the ideal of the Grassmannian defined by*

$$x_1x_8 - x_2x_6 + x_3x_5 = 0, x_1x_9 - x_2x_7 + x_4x_5 = 0, x_1x_{10} - x_3x_7 + x_4x_6 = 0,$$

$x_2x_{10} - x_3x_9 + x_4x_8 = 0, x_5x_{10} - x_6x_9 + x_7x_8 = 0$

for $x_1 := p_{12},\ x_2 := p_{13}, ..., x_{10} = p_{45}$ *since we will use lexicographical order. The third line builds the following* 10×6 *matrix*

$$o3 = \begin{pmatrix} (-1/p_{12} + 13) & 0 & 0 & p_{45} & p_{35} & p_{34} \\ (-1/p_{13} - 96) & 0 & p_{45} & 0 & -p_{25} & -p_{24} \\ (-1/p_{23} - 13) & p_{45} & 0 & 0 & p_{15} & p_{14} \\ (-1/p_{14} + 12) & 0 & -p_{35} & -p_{25} & 0 & p_{23} \\ (-1/p_{24} + 1) & -p_{35} & 0 & p_{15} & 0 & -p_{13} \\ (-1/p_{34} - 77) & p_{25} & p_{15} & 0 & 0 & p_{12} \\ (-1/p_{15} - 94) & 0 & p_{34} & p_{24} & p_{23} & 0 \\ (-1/p_{25} - 87) & p_{34} & 0 & -p_{14} & -p_{13} & 0 \\ (-1/p_{35} - 78) & -p_{24} & -p_{14} & 0 & p_{12} & 0 \\ (-1/p_{45} - 8) & p_{23} & p_{13} & p_{12} & 0 & 0 \end{pmatrix} \tag{8.27}$$

The first column is the gradient of our objective function which is the squared distance from some random point in $\mathbb{R}^{10}$ *and the last five columns are the Jacobian of the five constraints. Note that this* 10×5 *matrix has rank 3 on the Grassmannian, as it should, since 3 is the so-called codimension of* $G_2(\mathbb{R}^5)$*, i.e.,* $codim G_2(\mathbb{R}^5) = dim\mathbb{R}^5 - dim\mathbb{R}^2$*. Also, the forth command adds all the* 4×4 *minors of the* 6×10 *matrix to the ideal. This forces the matrix to have rank 3, so the first column is in the span of the last five. That ideal would have 63 complex solutions, counting multiplicities. However, we must remove extraneous solutions that have zero Plücker coordinates. This happens in line 5. The resulting ideal now has only one solution, resulting to 8.16558, which coincides with the solution of Example 5.3.1 when* $\underline{z}$ *was constant.*

In this code, the coordinates of the random point $\underline{z}$ in $\mathbb{R}^{10}$ were taken arbitrarily from -100 to 100. If one is interested in symbolic computations for $z_1, ..., z_{10}$ the algorithm provides a large number of equations which are difficult to be verified. This is the main reason that the approximate solution of DAP will be sought via a different procedure that takes into account the parameters of the linear variety $\mathcal{K}$.

8.4.2 Newton's method on the Grassmann Manifold

Newton's method is the simplest iterative method for computing a solution x_* of an equation of the form $f(x) = 0$, starting at an initial point x_0 where the iterations follow the formula

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)} \tag{8.28}$$

Graphically, x_{m+1} corresponds to the intersection of the tangent to the graph of f at x_m with the horizontal axis. Newton's method can be easily generalized to

functions $\underline{f} : \mathbb{R}^n \to \mathbb{R}^n$. Eqn. (8.28) takes the form

$$\underline{f}(\underline{x}_m) + D\underline{f}(\underline{x}_m)[\underline{x}_{m+1} - \underline{x}_m] = 0 \tag{8.29}$$

where $D\underline{f}(\underline{x})[\underline{y}]$ denotes the directional derivative of $\underline{f}$ along $\underline{y}$, i.e.,

$$D\underline{f}(\underline{x})[\underline{y}] = \lim_{h \to 0} \frac{\underline{f}(\underline{x} + h\underline{y}) - \underline{f}(\underline{x})}{h} \tag{8.30}$$

In other words, Newton's method simply updates a vector by subtracting the gradient vector premultiplied by the inverse of the Hessian. Now, if $\underline{f} = \text{grad}(\underline{f}) \equiv \nabla \underline{f}$, Newton's method becomes the problem of finding a critical point of a cost function on $\mathbb{R}^n$. In order to generalize this approach to manifolds and in particular to Grassmann manifolds, we must find geometric analogs to the various components of this formula. Tangent vectors on manifolds generalize the notion of a directional derivative.

Definition 8.4.1. *The set of all tangent vectors of $G_m(\mathbb{R}^n)$ at a point $\underline{x}$ is called the tangent space of the Grassmann manifold $G_m(\mathbb{R}^n)$ at $\underline{x}$ and is denoted by $T_{\underline{x}}G$.*

Now, the tangent space is easily computed by viewing the Grassmann manifold as a quotient space and its elements as matrices.

Definition 8.4.2. *[Edel., etc. 1] Let O_n be the orthogonal group consisting of $n \times n$ orthogonal matrices.*

i) The Stiefel manifold denoted as $V_{n,m}$, consisting of $n \times m$ orthonormal matrices is given by

$$V_{n,m} = O_n / O_{n-m} \tag{8.31}$$

where a point is the equivalence class

$$[Q] = \left\{ Q \begin{pmatrix} I_m & O \\ 0 & Q_{n-m} \end{pmatrix} : Q_{n-m} \in O_{n-m} \right\} \tag{8.32}$$

ii) The Grassmann manifold denoted as $G_{n,m}$, consisting those matrices of $V_{n,m}$ whose columns span the same subspace is given by

$$G_{n,m} = O_n / (O_n \times O_{n-m}) \tag{8.33}$$

where a point is the equivalence class

$$[Q] = \left\{ Q \begin{pmatrix} Q_m & O \\ 0 & Q_{n-m} \end{pmatrix} : Q_m \in O_m,\ Q_{n-m} \in O_{n-m} \right\} \tag{8.34}$$

Hence, $G_{n,m} = V_{n,m}/O_m$ and if we consider $Y \in V_{n,m}$ then tangent vectors take the form $X = YA + Y_\perp B$ where A is an $m \times m$ skew-symmetric matrix, B is $(n-m) \times m$ and $Y_\perp$ is any $n \times (n-m)$ matrix such that Y, $Y_\perp$ are orthogonal. But since $[Y] = \{YQ_m : Q_m \in O_m\}$, the vertical space at Y is the set of vectors of the form YA, therefore the horizontal space at Y is the set of vectors of the form

$$X = Y_\perp B \tag{8.35}$$

Because the horizontal space is equivalent to the tangent space of the quotient, [Edel., etc. 1], the tangent space of the Grassmann manifold at $[Y]$ is given by all $n \times m$ matrices A of the form in (8.35). Hence, the gradient of a function f from the Grassmann manifold to $\mathbb{R}$ at $[Y]$ is defined to be the tangent vector ∇f such that

$$\mathrm{tr} f_Y^t X = \mathrm{tr}(\nabla f)^t X \tag{8.36}$$

for all tangent vectors X at Y, where

$$(f_Y)_{ij} = \frac{\partial f}{\partial Y_{ij}} \tag{8.37}$$

Remark 8.4.1. *A different way to define the gradient of $\underline{f}$ in general at a point $\underline{x}$, without the use of the Grassmann manifold via matrix representation, is presented in [Abs., etc. 1], where the gradient of $\underline{f}$ is considered as the unique element of $T_{\underline{x}}G$ that satisfies*

$$< \mathrm{grad} f(\underline{x}), \underline{y} >_{\underline{x}} = Df(\underline{x})[\underline{y}],\ \forall \underline{y} \in T_{\underline{x}}G \tag{8.38}$$

where $<,>_{\underline{x}}$ is the endowed inner product of $T_{\underline{x}}G$. If $\underline{e}_i$ denotes the i-th coordinate of a vector field, i.e., a smooth function from $G_m(\mathbb{R}^n)$ that assigns to each point $\underline{x}$ a tangent vector $\underline{y} \in T_{\underline{x}}G$ then

$$\mathrm{grad} f(\underline{x}) = G_{\underline{x}} \begin{bmatrix} \partial_1 f(\underline{x}) \\ \vdots \\ \partial_d f(\underline{x}) \end{bmatrix} \tag{8.39}$$

where $G_{\underline{x}} :=< (\underline{e}_i, \underline{e}_j >_{\underline{x}}$ is the matrix whose ij element is $< (\underline{e}_i, \underline{e}_j >$ at $\underline{x}$.

Furthermore, the roots of $\mathrm{grad} f$ are still the critical points of f. The difference $\underline{x}_{m+1} - \underline{x}_m$, which is no longer defined since the iterates $\underline{x}_{m+1}$ and $\underline{x}_m$ belong to the Grassmann manifold, may be replaced either by

i) a tangent vector $\underline{\eta}_{\underline{x}_m}$ in the tangent space at $\underline{x}_m$, [Abs., etc. 1], [Abs., etc. 2] where the new iterate $\underline{x}_{m+1}$ is obtained from $\underline{\eta}_{\underline{x}_m}$ as $\underline{x}_{m+1} = R_{\underline{x}_m}(\underline{\eta}_{\underline{x}_m})$, for a retraction R, i.e., a mapping from $T_{\underline{x}}G$ to $G_m(\mathbb{R}^n)$ with a local rigidity condition that preserves gradients at a point $\underline{x}$ (in other words, a mapping that turns elements of $T_{\underline{x}}G$ into points of the manifold and transforms the cost functions defined in a neighborhood of $\underline{x}$ into cost functions defined on the vector space $T_{\underline{x}}G$) or

ii) by a suitable geodesic path, i.e.,

$$Y(t) = Y(0) \exp t \begin{bmatrix} 0 & -B^t \\ B & 0 \end{bmatrix}, \forall B \in \mathbb{R}^{(n-m)\times m} \tag{8.40}$$

which it may be computed via the following theorem.

Theorem 8.4.1. *[Edel., etc. 1] Let the Grassmannian geodesic*

$$Y(t) = Y(0) \exp t \begin{bmatrix} 0 & -B^t \\ B & 0 \end{bmatrix}, \forall B \in \mathbb{R}^{(n-m)\times m} \tag{8.41}$$

with $Y(0) = Y$ and $\dot{Y}(0) = H$. Then

$$Q(t) = [YVU] \begin{bmatrix} \cos St \\ \sin St \end{bmatrix} V^t \tag{8.42}$$

where USV^t is the compact singular value decomposition of H, i.e, $U \in \mathbb{R}^{n\times m}$ and both S and V are $m \times m$.

Finally, the Hessian is obtained by twice differentiating the function along a geodesic, i.e.,

$$\text{Hess} f(X, X) = \frac{d^2}{dt^2}|_{t=0} f(Y(t)) \tag{8.43}$$

or

$$\text{Hess} f(X_1, X_2) = f_{YY}(X_1, X_2) - \text{tr}(X_1^t X_2 Y^t f_Y), \ X_1 \neq X_2 \tag{8.44}$$

where $Y(t)$ is a geodesic with tangent $X = \dot{Y}(0)$ and

$$(\underline{f}_{YY})_{ij,k\ell} = \frac{\partial^2 f}{\partial Y_{ij} \partial Y_{k\ell}} \tag{8.45}$$

For Newton's method, one must determine $X = -\text{Hess}^{-1}(\nabla f)$, which for the Grassmann manifold is expressed as the linear problem

$$f_{YY}(X) - X(Y^t f_Y) = -\nabla f \tag{8.46}$$

with $Y^t X = 0$. For the algorithm one needs a smooth function $f(Y)$ with $Y \in \mathbb{R}^{n\times m}$ and $Y^t Y = I_m$ on the Grassmann manifold, i.e., $f(Y) = f(YQ)$ for all orthogonal matrices $Q \in \mathbb{R}^{m\times m}$. Then, the algorithm is as follows, [Edel., etc. 1].

Newton's Method on the Grassmann Manifold

- Given Y such that $Y^tY = I_m$, compute $G = f_Y - YY^t f_Y$ and $X = -\text{Hess}^{-1}G$ such that $Y^tX = 0$ and $f_{YY}(X) - X(Y^t f_Y) = -G$.

- Move from Y in direction X to $Y(1)$ using the geodesic formula

$$Y(t) = YV\cos(St)V^t + U\sin(St)V^t$$

where USV^t is the compact SVD of X.

- Repeat.

Clearly, Newton's method is constructed for functions defined on the Grassmann manifold, whereas DAP is looking for the gap between a "point" in the projective space and the Grassmann variety. However, from the analysis of the approximate DAP via the gap g and $g_\wedge$ in the previous sections, we see that Newton's method is ideal for approximate decomposable solutions when the following conditions hold all together:

(i) $Y = T_{\underline{z}}$

(ii) the gap g (or the gap $g_\wedge$ for the Grassmann variety $G_2(\mathbb{R}^5)$) can be written as a Rayleigh quotient or trace-style function, e.g., $\text{tr}(Y^tAY)$, for a symmetric matrix A, something that is feasible by using the formulae in Remark 5.4.6.

(iii) the dimensions $\dim\mathcal{V} = n > m$ of the Grassmann variety $G_m(\mathcal{V})$ satisfy the condition $m = n - 1$ or $\mathcal{V} \cong \mathcal{V}^*$, where $\mathcal{V}^*$ is the dual space of $\mathcal{V}$ or when the Grassmann variety is *almost* the entire projective space, i.e., its biggest subset that covers most of its area.

In [Mah. 1] there is an analytical approach to all cases that involve optimization on the projective space, where it becomes even more clear that for the approximate DAP we need a more suitable algorithm that calculates gaps between multivectors in the projective space and the Grassmann variety. In Section 8.4.3 we will build such an algorithm for the computation of the gap as this was proved in Theorem 6.3.1.

8.4.3 An Approximate DAP algorithm

As we have seen in the previous two sections, the algorithms concerning optimization on the Grassmann manifold may be applied for DAP only for special cases, whereas the numerical computations on Macaulay 2 provide solid solutions

for numerical data. Now we will present an algorithm which is explicitly constructed for the approximate DAP that covers all related frequency assignment problems, i.e., solve the system $\det(H \cdot M(s)) = a(s)$ in terms of H when $a(s)$ is Hurwitz and if this is not possible, derive the best possible approximate solution.

Let $s_1, s_2, ..., s_n,\ n > 6$ be the roots of $a(s)$. If $\underline{z} := C_m(H)$ and V is the matrix representation of $\mathcal{N}_\ell(P)$, then DAP is transformed into $\underline{z}^t = \lambda \underline{h}_0^t + \underline{\kappa}^t V = [\lambda, \underline{\kappa}^t] \begin{bmatrix} \underline{h}_0^t \\ V \end{bmatrix}$. If this is not possible, one has to find $\underline{z}$ such that the gap between $\underline{z}$ and $G_m(\mathbb{R}^n)$ the least possible. Therefore, we have:

An Algorithm for the Approximate DAP

- Select a polynomial $a(s)$ whose roots s_i satisfy the property $\mathrm{Re}(s_i) < 0$.

- Calculate an orthonormal matrix-basis $[\underline{v}_1, ..., \underline{v}_r]^t$ for the linear problem $\underline{h}^t P = \underline{a}^t$. Then the linear variety $\mathcal{K}$ is described by $\underline{h} = [\underline{v}_1, ..., \underline{v}_r] \cdot \underline{x} \equiv \underline{z}(\underline{x})$, where $\underline{x} \in \mathbb{R}^{r \times 1}$ is a free vector.

- Solve in terms of $\underline{x}$ the optimization problem $\min_{\underline{x}} g([\underline{v}_1, ..., \underline{v}_r] \cdot \underline{x}, G_m(\mathbb{R}^n))$ s.t. $||\underline{x}|| = 1$ and let $\underline{z}_0 = [\underline{v}_1, ..., \underline{v}_r] \cdot \underline{x}_0$ be its minimizer, i.e., the vector of the linear variety closest to $G_m(\mathbb{R}^n)$.

- If $\underline{z}_0$ is decomposable then obtain H by $\underline{z}_0 := C_2(H)$. Then DAP has an exact solution. Else, decompose $\underline{z}_0$ via the prime decomposition. A candidate solution is the decomposable vector $\sigma_k \cdot \underline{e}_{2k} \wedge \underline{e}_{2k-1}$.

Example 8.4.2. *Let*

$$M(s) = \begin{pmatrix} (1+s)^4 & 0 \\ -2+s^2 & s^3 \\ 1+s^3 & s^2 \\ 2s & -2+s \\ 1 & 1 \end{pmatrix}$$

We select the stable polynomial $a(s) = 9.80179 + 50.0464s + 109.122s^2 + 131.717s^3 + 95.06s^4 + 41.02s^5 + 9.8s^6 + s^7$ *whose roots are*

$$s_1 = -1.7, s_2 = -1.6, s_3 = -1.5, s_4 = -1.4, s_5 = -1.3, s_6 = -1.2, s_7 = -1.1$$

If P is the Plücker matrix, then the matrix representation of an orthonormal basis

of $\underline{h}^t P = \underline{a}^t$ *is*

$$[\underline{v}_1, \underline{v}_2, \underline{v}_3] = \begin{bmatrix} 0.0212483 & 0.0031971 & 0.0198759 \\ 0.14123 & 0.0008332 & 0.107242 \\ 0.179195 & 0.108632 & 0.267088 \\ 0.686415 & 0.311853 & 0.54408 \\ 0.017989 & -0.0177103 & -0.00803 \\ 0.058825 & 0.247088 & -0.202414 \\ -0.480937 & 0.505633 & 0.293198 \\ 0.142346 & 0.268904 & -0.308593 \\ -0.127068 & 0.700893 & -0.219786 \\ -0.452585 & -0.101711 & 0.591784 \end{bmatrix}$$

The substitution of $\underline{z}(x_1, x_2, x_3) := [\underline{v}_1, \underline{v}_2, \underline{v}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *to the gap* g *implies the 4th order homogeneous polynomial*

$$\begin{aligned} F(x_1, x_2, x_3) = \; & 0.0276749x_1^4 - 0.0445024x_1^3x_2 + 0.050691x_1^2x_2^2 + 0.0127932x_1x_2^3 + \\ & + 0.0018031x_2^4 - 0.0348223x_1^3x_3 - 0.0986078x_1^2x_2x_3 - 0.0735073x_1x_2^2x_3 + \\ & + 0.013395x_2^3x_3 + 0.0052206x_1^2x_3^2 + 0.0119637x_1x_2x_3^2 + 0.041775x_2^2x_3^2 - \\ & - 0.0414718x_3^3 + 0.0414718x_2x_3^3 + 0.054624x_3^4 \end{aligned}$$

Hence, we have

$$\min F(x_1, x_2, x_3) \text{ subject to } x_1^2 + x_2^2 + x_3^2 = 1$$

The least gap is achieved at

$$(x_1 = -0.711111, x_2 = -0.348945, x_3 = 0.610376)$$

where we get

$$\begin{aligned} \underline{z}_0 = (&-0.0283575, -0.166179, -0.328359, -0.92903, -0.00117062, \\ &- 0.00450339, -0.013399, -0.00669876, -0.0200615, -0.0038811) \end{aligned}$$

Vector $\underline{z}_0$ *is not decomposable since*

$$\begin{aligned} (\underline{z}_0 \wedge \underline{z}_0)/2 = (&- 0.000174025, -0.000570197, \\ &- 0.00010584, 0.000280932, 0) \neq \underline{0} \end{aligned}$$

Therefore, we proceed to the calculation of its best decomposable approximation. The spectral analysis of

$$T_{\underline{z}_0} = \begin{pmatrix} 0 & -0.0283575 & -0.166179 & -0.328359 & -0.92903 \\ 0.0283575 & 0 & -0.00117062 & -0.00450339 & -0.013399 \\ 0.166179 & 0.00117062 & 0 & -0.00669876 & -0.0200615 \\ 0.328359 & 0.00450339 & 0.00669876 & 0 & -0.0038811 \\ 0.92903 & 0.013399 & 0.0200615 & 0.0038811 & 0 \end{pmatrix}$$

implies

$$\underline{z}_0 = \underline{z}_1 + \underline{z}_2$$

where

$$\underline{z}_1 = \sigma_2\underline{e}_2 \wedge \underline{e}_3 = (-0.0283592, -0.166177, -0.328363, -0.929028, -0.00177318, \\ -0.00460241, -0.0133529, -0.00643763, -0.0201559, -0.00383742)$$

and

$$\underline{z}_2 = \sigma_1\underline{e}_4 \wedge \underline{e}_5 = (1.86224 \cdot 10^{-6}, -2.04133 \cdot 10^{-6}, 4.24203 \cdot 10^{-6}, -1.19104 \cdot 10^{-6}, \\ -0.0000669825, 0.0000990208, -0.0000469799, -0.000261124, 0.0000943383, \\ -0.0000436849)$$

Hence, the closest decomposable vector to $\underline{z}_0$ *is* $\underline{z}_1$*, which can be re-written dy division by the first coordinate as:*

$$\underline{v} = (1, 5.85971, 11.5787, 32.7594, 0.0625258, \\ 0.16229, 0.470849, 0.227004, 0.710735, 0.135315)$$

This corresponds to the 2×5 *matrix* $\hat{H}$

$$\hat{H} = \begin{pmatrix} 1 & 0 & -0.0625258 & -0.16229 & -0.470849 \\ 0 & 1 & 5.85971 & 11.5787 & 32.7594 \end{pmatrix}$$

The approximate matrix $\hat{H}$ *implies the perturbed polynomial*

$$\hat{a}(s) = \det(\hat{H} \cdot M(s)) = -9.83676 - 28.2452s - -22.4508s^2 + \\ + 13.8753s^3 + 35.4504s^4 + 24.9495s^5 + 7.99053s^6 + s^7$$

whose roots are

$$(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (-2.27804 - 0.576901i, -2.27804 + 0.576901i, -1.34073 - \\ - 0.809347i, -1.34073 + 0.809347i, -0.903325, -0.82816 - \\ - 0.343749i, -0.82816 + 0.343749i)$$

Hence, $\hat{a}(s)$ *is stable. Furthermore,*

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 \\ 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ -2 & -7 & -8 & -2 & 2 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 & -1 & 0 \\ 4 & -2 & -2 & 1 & -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -4 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\sigma_P = 14.0414$. *The stability radius for* $a(s) = (s+1.7)(s+1.6)(s+1.5)(s+1.4)(s+1.3)(s+1.2)(s+1.1)$ *from* (8.24) *is* $r_{\underline{\alpha}} = 7.3246$. *Therefore,*

$$\|\underline{z} - \underline{\hat{z}}\| = 0.000673828 < 0.522663 = \frac{r_{\underline{\alpha}}}{\sigma_P}$$

Hence the approximate polynomial $\hat{a}(s)$ *is stable, which verifies Theorem 8.3.6.*

8.5 A Zero Assignment by "Squaring Down" Application

The problem of zero assignment by "squaring down" has been presented in the second chapter as a special case of the Determinantal Assignment Problem (DAP). In this section we present an example to show how the above approximate DAP analysis may be implemented to the zero assignment by "squaring down" problem.

Example 8.5.1. *Let*

$$N(s) = \begin{bmatrix} s-1 & 0 & 0 & -1 & -2 \\ 0 & s-2 & 0 & 0 & -1 \\ 0 & 0 & s-3 & 0 & -1 \end{bmatrix}$$

We will find a matrix $K \in \mathbb{R}^{2\times 3}$ *such that,* $\det(K \cdot N(s)) = a(s) = s^3 + 6s^2 + 11s + 6$ *where the zeros of* $a(s)$ *are -1, -2, -3. Moreover,*

$$a(s) = s^3 + 6s^2 + 11s + 6 = [1, s, s^2, s^3] \begin{bmatrix} 6 \\ 11 \\ 6 \\ 1 \end{bmatrix} \equiv \underline{e}_3(s) \cdot \underline{\alpha}$$

For the calculation of the Plücker matrix P *we have that*

$$\begin{aligned} C_3(N(s)) = [&-6 + 11s - 6s^2 + s^3, 0, 2 - 3s + s^2, 0, -3 + 4s - s^2, 0, \\ &6 - 5s + s^2, 12 - 10s + 2s^2, 2 - s, s - 3] = \\ &= [1, s, s^2, s^3] \begin{bmatrix} -6 & 0 & 2 & 0 & -3 & 0 & 6 & 12 & 2 & -3 \\ 11 & 0 & -3 & 0 & 4 & 0 & -5 & -10 & -1 & 1 \\ -6 & 0 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence

$$P = \begin{bmatrix} -6 & 0 & 2 & 0 & -3 & 0 & 6 & 12 & 2 & -3 \\ 11 & 0 & -3 & 0 & 4 & 0 & -5 & -10 & -1 & 1 \\ -6 & 0 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, the linear system $P\underline{z} = \underline{\alpha}$ *has the solution* $\underline{z}(\underline{x}) = (1, x_2, x_3, x_4, x_5, x_6, 12 + x_5 - x_3 - 2x_8, x_8, -120 + 2x_3, -60 + x_5)$. *The QPRs for* $G_3(\mathbb{R}^5)$ *are given by the set*

$$\begin{aligned} x_4x_8 - x_5x_7 + x_1x_{10} &= 0 \\ x_4x_9 - x_6x_7 + x_2x_{10} &= 0 \\ x_5x_9 - x_6x_8 + x_3x_{10} &= 0 \end{aligned}$$

The substitution of these equations to the minimization of $\sum QPR^2/\|\underline{z}\|^2$ *s.t.* $\|\underline{x}\| = 1$ *implies the zero solution, i.e.,* $\underline{z}$ *is decomposable. Hence, if* $\underline{z} = C_3(H)$ *for* $H \in \mathbb{R}^{5\times 3}$, *then as proved in [Gia. 1]*

$$H = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \\ x_7 & -x_4 & x_2 \\ x_8 & -x_5 & x_3 \end{bmatrix}$$

and the requested matrix K *is given by the last* $k(=2)$ *lines of* H, *i.e.,*

$$K = \begin{bmatrix} x_7 & -x_4 & x_2 \\ x_8 & -x_5 & x_3 \end{bmatrix}$$

The substitution of $\underline{z}(\underline{x})$ *to the QPRs implies* $x_2 = (30x_4 + x_6)/20$, $x_3 = 30$, $x_5 = 20$ *and* $x_8 = 0$. *Hence,*

$$K = \begin{bmatrix} 2 & -x_4 & (30x_4 + x_6)/20 \\ 0 & -20 & 30 \end{bmatrix},\ x_4, x_6 \in \mathbb{R}$$

8.6 Conclusions

In this chapter, we have seen how the approximate determinantal assignment problem may be solved as a distance problem between the Grassmann variety and a linear variety defined by the properties of a desirable polynomial. The study of the problem was split to three basic problems: (i) The distance problem of a point of the projective space from the Grassmann variety; (ii) The extension of the above to the case where we have the distance of a linear variety from the Grassmann variety; (iii) The characterization of acceptability of the optimal distance solutions as far as the nature of the resulting assigned polynomial. The key minimization problem implied by the second problem (formulation of the (8.2) type), was addressed and a closed form solution was derived, which is similar in nature to the first optimization problem for a constant point in $\wedge^2(\mathbb{R}^n)$. The results of this approach have been demonstrated via a new approximate DAP algorithm, since the existing methodologies such as Newton's method on the

Grassmann manifold provide results for special cases of DAP. The results were specialized to different types of frequency assignment, such as the state feedback and the design of an asymptotic observer. Furthermore, the polynomials corresponding to the approximate solutions are at some distance from the nominal polynomial and for solutions to be acceptable we need the resulting polynomials to be stable. We have used the stability radius results [Hin. & Pri. 2] to derive a condition that can be used to check stability without root calculations.

The above results are based on the prime decomposition (5.26) of 2-vectors which has implied significant simplifications, such as the formulation of DAP into a 4th order polynomial minimization problem, constrained to the unit sphere for the $G_2(\mathbb{R}^5)$ case. This approach, which is usually met in tensor decomposition problems, [Yok. 1] or expansions of the standard matrix SVD, [Dela., etc. 1] has not been used before for general frequency assignment problems and it may be implemented as a new pole placement method, that uses no generic or exact solvability conditions.

The results obtained in this chapter, along with the degenerate cases studied in the previous chapter, now clearly provide the *complete* solution of the Approximate DAP in the 2-dimensional case and constitute the core of the thesis. In the next chapter, we examine the generalization into $G_3(\mathbb{R}^n)$. The difficulty in m-decompositions, $m \geq 3$, lies in the fact that the matrices which in our case provide the representation of the points of the projective space, become m-tensors, thus the approximate DAP should be naturally studied via tensor decomposition algorithms.

Chapter 9

Decomposable Approximations via 3-Tensor Decompositions

9.1 Introduction

As we have already seen, the difficulty in solving the approximate DAP is the solution of problem (8.2). For the $G_2(\mathbb{R}^n)$ case, the problem has been thoroughly examined in the previous chapters. In this chapter we will refer to the $G_3(\mathbb{R}^n)$ case, that may be solved numerically by the so called *Higher-Order Singular Value Decompositions*. These methods aim to the approximation a 3-tensor $\mathbf{A} \in \mathbb{R}^{n\times m\times k}$ by a second tensor $\mathbf{B}$ of equal dimensions but of lower rank, i.e.,

$$\min_{\mathbf{B}} \|\mathbf{A} - \mathbf{B}\| \tag{9.1}$$

In our case, $\mathbf{A} \equiv T_{\underline{z}}$, i.e., $\mathbf{A}$ is a skew-symmetric tensor and $\mathbf{B}$ a skew-symmetric tensor of lower rank.

Problem (9.1) has a similar nature as the approximation problem we examined in the previous chapters; if a tensor can not be written as a tensor product of the form $\mathbf{A} = \underline{a}_1 \otimes \cdots \otimes \underline{a}_n$, then try to decompose $\mathbf{A}$ as a sum $\mathbf{A} = \sum_{i=1}^{r} \mathbf{A}_i$ where $\mathbf{A}_i$ are rank one tensors for the least possible r, which is referred to as the rank of the tensor. The most well-known non-algorithmic result is Segre's Theorem, who showed that a general $2 \times 2 \times 2$ tensor has a unique decomposition as a sum of 2 decomposable tensors/multivectors. Segre's theorem as well as expansions to other cases, e.g., $3 \times 2 \times 2$ are examined in [Land. 1]. All other results concern the study of the problem via numerical techniques, where it was shown [Kol. 1], [Kol. 2] that low-rank approximation for tensors via Higher-Order SVD methods do not guarantee an optimal approximation but at least one low-rank approximate solution (or one approximate decomposable 3-tensor in our case). The same result stands for the case examined in [Zha. & Gil. 1], where the decomposition satisfied some special orthogonal properties. The core of the problem is traced to

the non-uniqueness of the rank of a tensor; while the rank of an ordinary matrix is unique, the rank of a tensor may vary. Hence, the application of higher-order tensor decompositions to the approximate DAP may not imply the optimal best decomposable approximation, as in the 2-dimensional case, but at least *one* best approximation.

Higher Order Tensor decompositions originated with Hitchcock in 1927, [Hit. 1] and the work of Cattell in [Cat. 1]. These concepts started receiving more attention during the 1960s with the work of Tucker in [Tuc. 1] and Carroll and Chang [Car. & Cha. 1]. The latter two, derived one of the most popular numerical tensor decompositions, the so called *CANDECOMP* (canonical decomposition) until Harshman [Hars. 1] in the 1970s came up with an upgraded version called *the CANDECOMP/PARAFAC decomposition* (parallel factors), which has given better approximate results.

In this chapter we present for the first time a symbolic tensor decomposition based on the CANDECOMP/PARAFAC decomposition, where the entries of the initial tensor which is to be approximated depend on some free parameters, for the $\bigwedge^3(\mathbb{R}^6)$ case. This is essential for the approximate DAP, since the linear sub-problem yields a number of free parameters $x_1, x_2, ...$ as we have seen in the previous chapter, whereas the higher-order SVD algorithms are designed for numerical data, rather than symbolic.

This chapter is organized as follows: In Section 9.3 we introduce some basic definitions and results with regards to tensors when these are considered as multi-dimensional arrays, and not as multi-vectors as we saw in Chapter 3. This numerical analysis aspect of tensors is critical for the understanding of the approximate DAP in higher dimensions, as well as its solution.

In Section 9.4 we introduce the notions of symmetry and skew-symmetry for tensors as the natural generalizations of symmetric and skew-symmetric matrices. Furthermore, we present a new algorithm for the construction of a 3-rd order skew-symmetric tensor, since in the respective literature, one may find algorithms only for the symmetric case. Furthermore, in 9.5 we explain the matricization techniques and procedures of a tensor. Since we aim to approximate a tensor by another one of lower rank, we must follow a similar method as we did with matrices. We present the main ways which are mostly used to "transform" a tensor into a matrix.

In Section 9.6 we introduce the different ways that two tensors may be multiplied; in 9.6.1 we explain the p-mode product which is based on the idea of multiplying a tensor by a matrix (or a vector) (in mode p) and in 9.6.2 we define the Kronecker, Khatri-Rao, and Hadamard products for the matrices obtained

by the matricization of a tensor. Moreover, in 9.7, we study the notions of the rank and rank-1 of a tensor and the problem of its uniqueness (contrary to the case of matrices, the rank of a tensor *is not* uniquely defined) and we explain the various ways several researchers have managed to calculate the rank of a tensor - for special dimensions.

In Section 9.8, we discuss the CANDECOMP/PARAFAC (CP) decomposition of a tensor, the most common approximation technique for tensor problems. We also briefly refer to the second most useful method, the so-called Tucker decomposition and a few others, which may be considered as variations of these two. Specifically, in 9.8.1 we elaborate on the mathematical framework of the CP decomposition and in 9.8.2 we present the algorithm. Finally, in Section 9.9 we present the main result of this chapter and one of the most significant results of this thesis; we built a new algorithm, based on the CP decomposition, for a symbolic tensor for DAP applications. Our algorithm works theoretically for any number of parameters in the entries of the tensor. We present an example with one free parameter x to test the results.

9.2 Tensor Basics

Definition 9.2.1. *[Gal. 1] A tensor product of $k \geq 2$ vector spaces $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_k$ is a vector space $\mathcal{T}$, together with a k-linear map $\underline{\varphi} : \mathcal{V}^k \to \mathcal{T}$, such that, for every vector space $\mathcal{W}$ and for every k-linear map $\underline{f} : \mathcal{V}^k \to \mathcal{W}$ there is a unique linear map $\underline{f}_{\otimes} : \mathcal{T} \to \mathcal{W}$ with*

$$\underline{f}(\underline{x}_1, \underline{x}_2, ..., \underline{x}_k) = \underline{f}_{\otimes}\left(\underline{\varphi}(\underline{x}_1, \underline{x}_2, ..., \underline{x}_k)\right) \tag{9.2}$$

Proposition 9.2.1. *[Gal. 1] Any two tensor products $(\mathcal{T}_1, \underline{\varphi}_1)$, $(\mathcal{T}_2, \underline{\varphi}_2)$ are isomorphic, i.e., there is an isomorphism $\underline{h} : \mathcal{T}_1 \to \mathcal{T}_2$ such that*

$$\underline{\varphi}_2 = \underline{h} \circ \underline{\varphi}_1 \tag{9.3}$$

Since tensor products are unique up to isomorphism, we can obtain a construction that produces one. If we denote $\underline{\varphi}(\underline{x}_1, \underline{x}_2, ..., \underline{x}_k)$ as $\underline{x}_1 \otimes \underline{x}_2 \otimes \cdots \otimes \underline{x}_k$ then we can construct the tensor product $(\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_k, \underline{\varphi}) \equiv \mathcal{V}^{\otimes k}$ which is generated by $\underline{x}_1 \otimes \underline{x}_2 \otimes \cdots \otimes \underline{x}_k$. Furthermore, for every multilinear map $\underline{f} : \mathcal{V}^k \to \mathcal{W}$, the unique linear map $\underline{f}_{\otimes} : \mathcal{V}^{\otimes k} \to \mathcal{W}$ is defined by

$$\underline{f}_{\otimes}(\underline{x}_1 \otimes \underline{x}_2 \otimes \cdots \otimes \underline{x}_k) = \underline{f}(\underline{x}_1, \underline{x}_2, ..., \underline{x}_k) \tag{9.4}$$

This construction does not only provide the derivation of a tensor product, but produces a tensor product with the universal mapping property with respect to multilinear maps, i.e.,

$$\mathrm{Hom}(\mathcal{V}^{\otimes k}, \mathcal{W}) \cong \mathrm{Hom}(\mathcal{V}_1, ..., \mathcal{V}_k; \mathcal{W}) \tag{9.5}$$

In other words, we constructed the new vector space $\mathcal{V}^{\otimes k}$ such that the multilinear map $\underline{f} : \mathcal{V}^k \to \mathcal{W}$ is turned into a *linear* map $\underline{f}_{\otimes} : \mathcal{V}^{\otimes k} \to \mathcal{W}$, which is equivalent to $\underline{f}$ in the strong sense of isomorphism.

However, the vectors $\underline{x}_1 \otimes \underline{x}_2 \otimes \cdots \otimes \underline{x}_k$ that generate $\mathcal{V}^{\otimes k}$ are not selected to be linearly independent. Thus, we need a basis for $\mathcal{V}^{\otimes k}$. Next example, demonstrates this important procedure as well as the notation which will be used for tensors in the rest of the thesis.

Example 9.2.1. *Let three vector spaces with* $dim\mathcal{X}_1 = n_1$, $dim\mathcal{X}_2 = n_2$, $dim\mathcal{X}_3 = n_3$ *and*

$$\underline{x}_1 = (x_1^1, ..., x_1^{n_1}) \in \mathcal{X}_1,\ \underline{x}_2 = (x_2^1, ..., x_2^{n_2}) \in \mathcal{X}_2,\ \underline{x}_3 = (x_3^1, ..., x_3^{n_3}) \in \mathcal{X}_3$$

If $\{\underline{e}_{1,i_1}\}$, $\{\underline{e}_{2,i_2}\}$, $\{\underline{e}_{3,i_3}\}$ *are their respective bases with* $1 \leq i_1 \leq n_1,\ 1 \leq i_2 \leq n_2,\ 1 \leq i_3 \leq n_3$, *then*

$$\underline{x}_1 = (\underline{e}_{1,1}, ..., \underline{e}_{1,n_1}) \begin{pmatrix} x_1^1 \\ \vdots \\ x_1^{n_1} \end{pmatrix}, \underline{x}_2 = (\underline{e}_{2,1}, ..., \underline{e}_{2,n_2}) \begin{pmatrix} x_2^1 \\ \vdots \\ x_2^{n_2} \end{pmatrix}, \underline{x}_3 = (\underline{e}_{3,1}, ..., \underline{e}_{3,n_3}) \begin{pmatrix} x_3^1 \\ \vdots \\ x_3^{n_3} \end{pmatrix}$$

Then it is obvious that

$$\{\underline{e}_{1,i_1} \otimes \underline{e}_{2,i_2} \otimes \underline{e}_{3,i_3}\}$$

for $1 \leq i \leq n_1,\ 1 \leq j \leq n_2,\ 1 \leq p \leq n_3$ *is a basis of* $X^{\otimes 3}$ *and any vector in* $X^{\otimes 3}$ *is written as*

$$\mathbf{T} = \sum_{\substack{1 \leq i_1 \leq n_1 \\ 1 \leq i_2 \leq n_2 \\ 1 \leq i_3 \leq n_3}} x_1^i x_2^j x_3^k \underline{e}_{1,i_1} \otimes \underline{e}_{2,i_2} \otimes \underline{e}_{3,i_3} \equiv t^{ijk} \underline{e}_{1,i_1} \otimes \underline{e}_{2,i_2} \otimes \underline{e}_{3,i_3} \tag{9.6}$$

where $t^{ijk} \equiv x_1^i x_2^j x_3^k$ *and the summation symbol* $\sum$ *along with its ranges can be assumed by the superindices and subindices with the same letter and their positions (Einstein's summation convention).*

It is evident now, that $\mathbf{T}$ in (9.6) is a *tensor* and in this example specifically, a 3rd order tensor.

For the generalization in random-order tensors, one more notice has to be taken with respect to dual basis and the indices in (9.6); in many cases, tensors are easier to compute with respect to their dual basis. Therefore, the upper indices, called *contravariant coordinates* (where their main property is that under a change of basis, the components of the vectors $\underline{x}_i, i = 1, 2, 3$, transform with the inverse of the respective transformation matrix) and the lower indices, called *covariant coordinates* (similarly, components transform via the same matrix),

should somehow indicate whether a basis is dual or not. Hence, if we had the tensor product $\mathcal{X}_1 \otimes \mathcal{X}_2^* \otimes \mathcal{X}_3$ in the previous example, where $\mathcal{X}_2^*$ is the dual of $\mathcal{X}_2$, then we have to express this duality by writing

$$\mathbf{T} = x_1^i x_{2,j}^* x_3^k \underline{e}_{1,i_1} \otimes \underline{e}_2^{*i_2} \otimes \underline{e}_{3,i_3} \equiv t_{o\ j\ o}^{i\ o\ k} \underline{e}_{1,i} \otimes \underline{e}_2^{*j} \otimes \underline{e}_{3,p} \tag{9.7}$$

where (o) denotes the dual position from the contravariant and the covariant perspective, respectively.

Proposition 9.2.2. *[Tol. & Cast. 1] Given $k \geq 2$ vector spaces $\mathcal{V}_1, ..., \mathcal{V}_k$ with $dim\mathcal{V}_i = n_i,\ i = 1, ..., k$ and $\{\underline{e}_i^j\}_{i \in F_j},\ F_j = \{1, 2, ..., j\}$ there respective bases for $1 \leq j \leq k$, then the family of vectors*

$$\{\underline{e}_{1,i_1} \otimes \underline{e}_{2,i_2} \otimes \cdots \otimes \underline{e}_{k,i_k}\}_{(i_1,...,i_k) \in F_1 \times F_2 \times \cdots \times F_k} \tag{9.8}$$

is a basis of the tensor product $\mathcal{V}^{\otimes k}$. Thus, every tensor $\mathbf{T} \in V^{\otimes k}$ is written as

$$\mathbf{T} = t^{i_1 i_2 ... i_k} \underline{e}_{1,i_1} \otimes \underline{e}_{2,i_2} \otimes \cdots \otimes \underline{e}_{k,i_k} \tag{9.9}$$

Moreover, if some linear spaces are given with respect to their dual basis instead of the fundamental initial bases then

$$\mathbf{T} = t_{o\ i_2\ o\ \cdots}^{i_1\ o\ i_3\ \cdots} \underline{e}_{1,i_1} \otimes \underline{e}_2^{*i_2} \otimes \cdots \otimes \underline{e}_{k,i_k} \tag{9.10}$$

Elements of the form (9.9) or (9.10), are called *(homogeneous) tensors*. If a tensor is written as $\underline{x}_1 \otimes \underline{x}_2 \otimes \cdots \otimes \underline{x}_k$ then it is called *indecomposable* and the tensors that are not indecomposable are called *compound tensors*. If it is easy to distinguish the respective basis and the dual-position, we can omit the numbers $1, 2, ..., k$ from the lower indices and the symbol (o) and we can gather the contravariant coordinates in the first r arguments and the covariant coordinates in the last s arguments. Then we have

$$\mathbf{T} = a_{j_1,...,j_s}^{i_1,...,i_r} \underline{e}_{i_1} \otimes \cdots \otimes \underline{e}_{i_r} \otimes \underline{e}^{*j_1} \otimes \cdots \otimes \underline{e}^{*j_s} \tag{9.11}$$

In this case, tensors are considered as the multidimensional arrays of the coefficients

$$\left(a_{j_1,...,j_s}^{i_1,...,i_r}\right) \tag{9.12}$$

The following definitions will help us obtain the respective vector space, i.e., algebra of homogeneous tensors, which will be very helpful for the next section when we will define a special group of tensors, the skew-symmetric tensors.

Definition 9.2.2. *[Gal. 1] Let*

$$\mathcal{V}_m^{\otimes} := \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{m\ times} \tag{9.13}$$

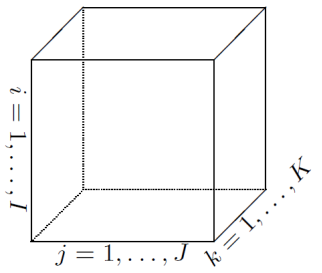


Figure 9.1: 3-rd Order Tensor

i) The vector space

$$\mathcal{T}(\mathcal{V}) := \bigoplus_{m \geq 0} \mathcal{V}_m^{\otimes},\ \mathcal{V}_0^{\otimes} \equiv \mathcal{F},\ \mathcal{V}_1^{\otimes} \equiv \mathcal{V} \tag{9.14}$$

is a tensor algebra.

ii) The tensor space $\mathcal{T}_{r,s}(\mathcal{V})$ is called a tensor product of type (r, s) if

$$\mathcal{T}_{r,s}(\mathcal{V}) = \mathcal{V}_r^{\otimes} \otimes (\mathcal{V}^*)_s^{\otimes} \tag{9.15}$$

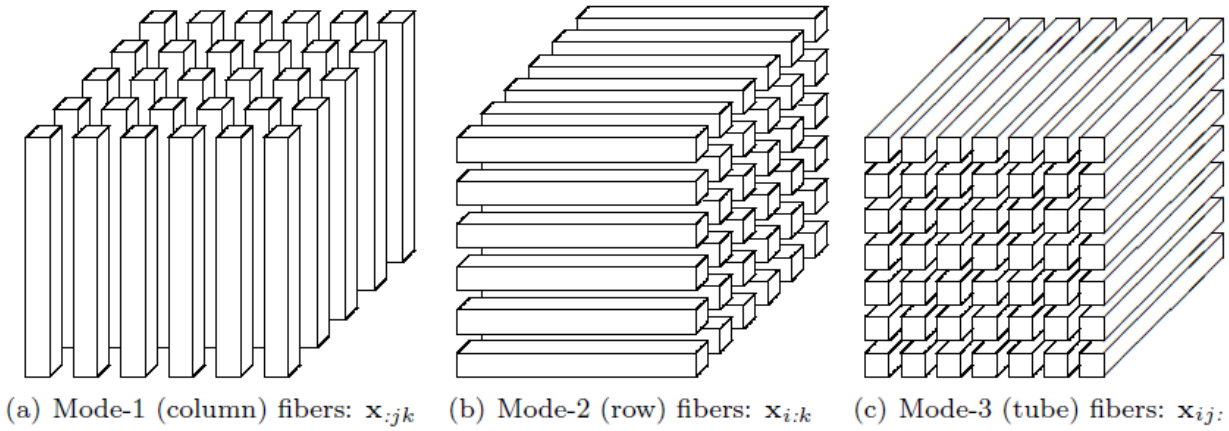
Tensors in $\mathcal{T}_{r,s}(\mathcal{V})$ are usually called *homogeneous* of degree (r, s) and it is easily proved that

$$\mathcal{T}^{\bullet}(\mathcal{V}) := \bigoplus_{r,s \geq 0} \mathcal{T}_{(r,s)}^{\otimes} \tag{9.16}$$

is also a tensor algebra. It also shown, [Gal. 1], that all tensors in $\mathcal{T}_{r,s}(\mathcal{V})$ may take the form (9.11) or very similar ones. Thus, the study of tensors in most cases reduces to the study of homogeneous tensors. All tensors in this thesis will be considered homogeneous, unless stated otherwise.

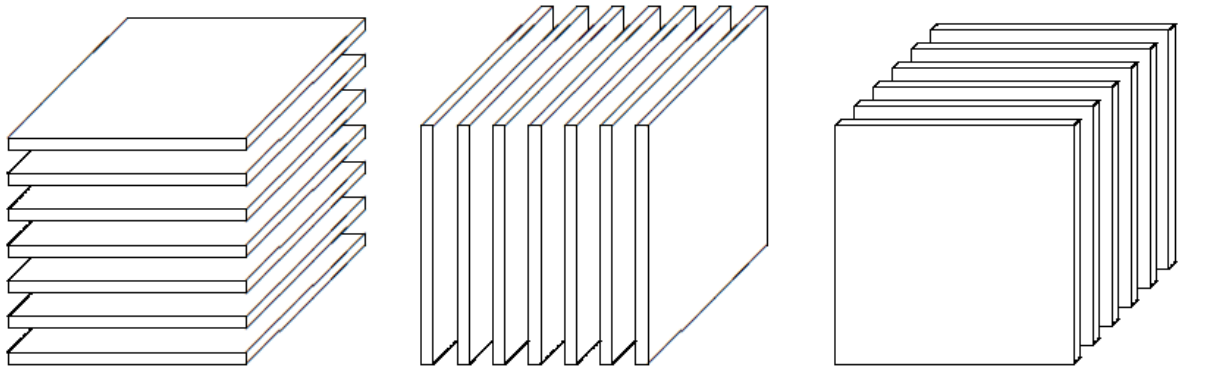
9.3 Preliminary Framework

As we saw in the previous section, a tensor may be considered as the multidimensional array or the multidimensional matrix representation of a multivector , as in Figure 9.1, which is usually referred to as an n-way or an n-order tensor and it is an element of the tensor product of n vector spaces, each of which has its own coordinate system. In this chapter, the i-th entry of a vector $\underline{z}$ will be denoted by z_i, the ij entry of a matrix A by denoted by a_{ij}, and the ijk element of a third-order tensor $\mathbf{X}$ will be denoted by x_{ijk}. Indices typically range from 1 to their respective final capital index, e.g., $i = 1, 2..., \iota_F$, where ι_F denotes the final index. The i-th element in a sequence is denoted by a superscript in parentheses, e.g., $A^{(i)}$ denotes the i-th matrix in a sequence.



(a) Mode-1 (column) fibers: $\mathbf{x}_{:jk}$ (b) Mode-2 (row) fibers: $\mathbf{x}_{i:k}$ (c) Mode-3 (tube) fibers: $\mathbf{x}_{ij:}$

Figure 9.2: Fibers of a 3-Tensor



(a) Horizontal slices: $\mathbf{X}_{i::}$ (b) Lateral slices: $\mathbf{X}_{:j:}$ (c) Frontal slices: $\mathbf{X}_{::k}$ (or $\mathbf{X}_k$)

Figure 9.3: Slices of a 3-Tensor

Subarrays, [Kol. & Bad. 3], are formed when a subset of the indices is fixed, i.e., the j-th column of a matrix A is denoted by $a_{:j}$ and the i-th row of a matrix A is denoted by $a_{i:}$.

Fibers, [Kol. & Bad. 3], are the higher order analogue of matrix rows and columns. A fiber is defined by fixing every index but one. A matrix column is a mode-1 fiber and a matrix row is a mode-2 fiber. Third-order tensors have column, row, and tube fibers which we denote by $x_{:jk}$, $x_{i:k}$, $x_{ij:}$, respectively, as in Figure 9.3. Note that when extracted from the tensor, fibers are always assumed to be oriented as column vectors.

Slices, [Kol. & Bad. 3], are two-dimensional sections of a tensor, defined by fixing

all but two indices. The figure above shows the horizontal, lateral, and frontal slides of a third-order tensor $\mathbf{X}$, denoted by $X_{i::}$, $X_{:j:}$, $X_{::k}$, respectively. Alternatively, the k-th frontal slice of a third-order tensor may be denoted as X_k.

The *inner product* of two same-sized tensors $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \cdots \times \iota_n}$, is the sum of the products of their entries, i.e.,

$$< \mathbf{A}, \mathbf{B} >= \sum_{i_1=1}^{\iota_1} \sum_{i_2=1}^{\iota_2} \cdots \sum_{i_n=1}^{\iota_n} a_{i_1 i_2 \cdots i_n} b_{i_1 i_2 \cdots i_n} \tag{9.17}$$

and the *norm* of a tensor $\mathbf{A} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \cdots \times \iota_n}$ is the square root of the sum of the squares of all its components, i.e.,

$$\|\mathbf{A}\| = \left\{ \sum_{i_1=1}^{\iota_1} \sum_{i_2=1}^{\iota_2} \cdots \sum_{i_n=1}^{\iota_n} a^2_{i_1 i_2 \cdots i_n} \right\}^{1/2} \tag{9.18}$$

It readily follows that $< \mathbf{A}, \mathbf{A} >= \|\mathbf{A}\|^2$.

9.4 Symmetry of Tensors

In order to talk about symmetry and tensors, we must assume that every mode is of the same size, i.e., $\mathbf{A} \in \mathbb{R}^{\iota_F \times \iota_F \times \cdots \times \iota_F}$.

Definition 9.4.1. *[Kol. & Bad. 3]*

i) A tensor is called cubical if every mode has the same size.

ii) A cubical tensor is called symmetric (or supersymmetric) if its elements remain constant under any permutation of the indices, i.e.,

$$x_{ijk\ldots} = x_{ikj\ldots} = x_{jik\ldots} = x_{jki\ldots} = x_{kij\ldots} = x_{kji\ldots} = \cdots, \ \forall i, j, k = 1, ..., \iota_F. \tag{9.19}$$

iii) A cubical tensor is called skew-symmetric (or antisymmetric) if it alternates sign when any two indices are interchanged, i.e.,

$$x_{ijk\ldots} = -x_{ikj\ldots} = x_{jik\ldots} = -x_{jki\ldots} = x_{kij\ldots} = -x_{kji\ldots} = \cdots, \ \forall i, j, k = 1, ..., \iota_F. \tag{9.20}$$

Furthermore, tensors can be *partially symmetric*, [Kol. & Bad. 3] in two or more modes as well. For example, a three-way tensor $\mathbf{A} \in \mathbb{R}^{\iota_F \times \iota_F \times \kappa_F}$ is symmetric in modes one and two if all its frontal slices are symmetric, i.e.,

$$X_k = X_k^t, \ \forall k = 1, ..., \kappa_F \tag{9.21}$$

On the other hand, if a skew-symmetric tensor changes sign under *any* pair of its indices then the tensor is called *totally skew-symmetric*.

Remark 9.4.1. *Immediately from Chapter 3, one may imply that a completely skew-symmetric contravariant tensor is actually a k-vector (multivector), whereas a completely skew-symmetric covariant tensor is a k-form.*

In the respective literature, one may find many examples and algorithms on how to construct symmetric tensors, e.g., [Dela., etc. 1], [Kol. & Bad. 3], [Sav. & Li.1]. Since, there is no specific algorithm for the construction of a skew-symmetric tensor, which is necessary for our problem, we present for the first time a Matlab algorithm for the 3-tensor case that yields the 3-rd order skew-symmetric tensor $T_{\underline{z}} \in G_3(\mathbb{R}^6)$ as in Figure 9.4.

A 3-Skew-Symmetric Tensor Matlab Algorithm

- axis([-.1, 1.1, -.1, 1.1, -.1, 1.1]);
- Define custom matrix: A=zeros(n, n, n);
- Set: q = [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20];
- $iq = 1$;

for $i = 1 : n$
  for $j = 1 : i - 1$
    for $k = 1 : j - 1$
    x = input([Give element (num2str(i) , num2str(j) , num2str(k)):]);
    $x = q(iq)$;
    $iq = iq + 1$;
    $a(i, j, k) = x$;
    $a(i, k, j) = -x$;
    $a(j, i, k) = -x$;
    $a(k, j, i) = -x$;
    $a(j, k, i) = +x$;
    $a(k, i, j) = +x$;
    end
  end
end

hold on; for $i = 1 : n$
  for $j = 1 : n$
    for $k = 1 : n$
$pr = num2str(a(i, j, k))$;
    if $j < i$, $k < j$
    color = black;
    text((i-1)/(n-1), (j-1)/(n-1), (k-1)/(n-1), ... pr, Color, color, FontWeight, bold, FontSize, 9);
    else
    color = red;
    text((i-1)/(n-1), (j-1)/(n-1), (k-1)/(n-1), ... pr, Color, color, FontSize, 8);
    end
  end
end

- Draw planes:

tfill = $[(i-1)/(n-1), 0, 0; (i-1)/(n-1), 1, 0; (i-1)/(n-1), 1, 1; (i-1)/(n-1), 0, 1]$;
fill3(tfill(:,1), tfill(:,2), tfill(:,3), blue, FaceAlpha, .2, EdgeColor, none);
end

hold off

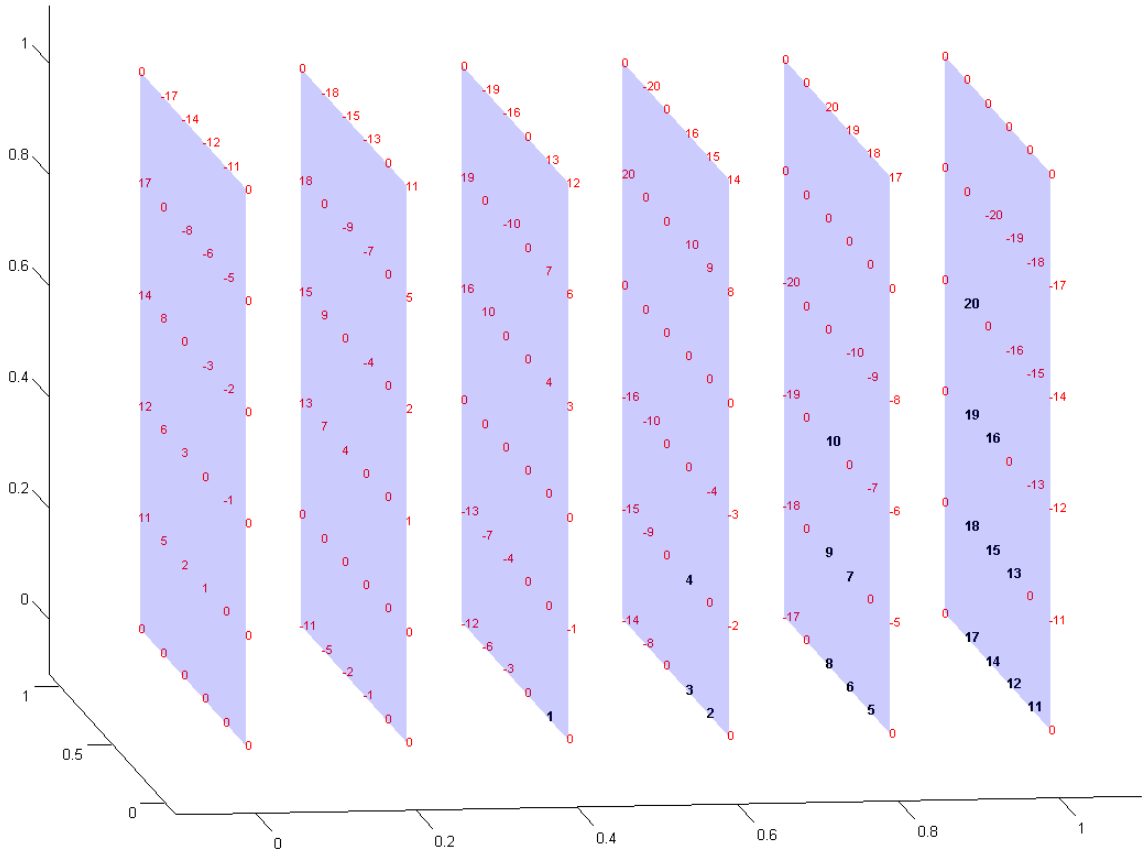


Figure 9.4: A $6 \times 6 \times 6$ Skew-Symmetric Tensor

9.5 Matricization Techniques of a Tensor

The process of reordering the elements of an n- way array into a matrix is known as matricization (or unfolding or flattening) of the n-tensor. For example a $3 \times 6 \times 8$ tensor can be rearranged as a 18×8 or as a 6×24 matrix, and so on. This n-mode unfolding may be achieved in different ways, e.g., [Dela., etc. 1], [Kol. & Bad. 3]. Here we follow the methodology proposed in [Kol. & Bad. 3], where the mode- n matricization of a tensor $\mathbf{A} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \cdots \times \iota_n}$ is denoted by $A_{(n)}$ and arranges the mode-n fibers to be the columns of the resulting matrix. Strictly speaking, the authors mapped the $(i_1, i_2, ..., i_n)$ element of the tensor to

the matrix element (i_n, j) where

$$j = 1 + \sum_{\substack{m=1 \\ m \neq n}}^{n} \left[(i_m - 1) \prod_{\substack{\ell=1 \\ \ell \neq n}}^{m-1} I_\ell \right] \tag{9.22}$$

Example 9.5.1. *Let*

$$A_1 = \begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{bmatrix}, \; A_2 = \begin{bmatrix} m & p & s & v \\ n & q & t & w \\ o & r & u & x \end{bmatrix}$$

be the frontal slices of a $3 \times 4 \times 2$ *tensor* $\mathbf{A}$*. Then, the three matricizations are*

$$A_{(1)} = \begin{bmatrix} a & d & g & j & m & p & s & v \\ b & e & h & k & n & q & t & w \\ c & f & i & l & o & r & u & x \end{bmatrix},$$

$$A_{(2)} = \begin{bmatrix} a & b & c & d & e & f & g & h & i & j & k & l \\ m & n & o & p & q & r & s & t & u & v & w & x \end{bmatrix},$$

$$A_{(3)} = [A_1^t | A_2^t]$$

Remark 9.5.1. *Note that it is possible not only to "matricize" a tensor, but to "vectorize" it also. For the previous example, that vector would be:*

$$\text{vec}(\mathbf{A}) = (a, b, ..., w, x)^t \tag{9.23}$$

9.6 Tensor Multiplication

There is a number of different ways to calculate the product between to tensors. In this section we study the p-mode product, i.e., multiplying a tensor by a matrix (or a vector) in mode p and the matrix Kronecker, Khatri-Rao, and Hadamard products as these were adopted in [Kol. & Bad. 3]. A detailed treatment of tensor multiplication may be found in [Bad. & Kol. 1].

9.6.1 The p- mode tensor product method

The p-mode matrix product of a tensor $\mathbf{A} \in \mathbb{R}^{\iota_1 \times \cdots \iota_n}$ with a matrix $B \in \mathbb{R}^{j \times \iota_p}$, $p = 1, ..., n$ is the multiplication of each mode-p fiber by the matrix B, i.e.,

$$\mathbf{A} \times_p B \in \mathbb{R}^{\iota_1 \times \cdots \times \iota_{p-1} \times j \times \iota_{p+1} \times \cdots \times \iota_n} \tag{9.24}$$

or equivalently in element form

$$\mathbf{A} \times_p B = \sum_{i_p=1}^{\iota_p} a_{i_1 i_2 \cdots i_n} b_{j i_p} \tag{9.25}$$

For distinct modes in a series of multiplications, the order of the multiplication is irrelevant, i.e.,

$$\mathbf{A} \times_p B \times_m C = \mathbf{A} \times_m C \times_n B \tag{9.26}$$

and when the modes are the same, then

$$\mathbf{A} \times_p B \times_n C = \mathbf{A} \times_p (CB) \tag{9.27}$$

Example 9.6.1. *Let*

$$A_1 = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \; A_2 = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & 17 & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}$$

be the frontal slices of a $3 \times 4 \times 2$ *tensor* $\mathbf{A}$ *and let*

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Then the $\mathbf{Y} = \mathbf{A} \times_1 B$ *product is given by*

$$Y_1 = \begin{bmatrix} 22 & 49 & 76 & 103 \\ 28 & 64 & 100 & 136 \end{bmatrix}, \; Y_2 = \begin{bmatrix} 130 & 157 & 184 & 211 \\ 172 & 208 & 244 & 280 \end{bmatrix},$$

Remark 9.6.1. *It is easy to see that this kind of multiplication, is similar to a change of basis in the case when a tensor defines a multilinear operator, as we have seen in Chapter 3.*

Similarly, the p-mode vector product of a tensor $\mathbf{A} \in \mathbb{R}^{\iota_1 \times \cdots \times \iota_n}$ with a vector $\underline{b} \in \mathbb{R}^{\iota_p}$ rises as the idea to compute the inner product of each mode-p fiber with the vector, i.e.,

$$\mathbf{A} \underline{\times}_p \underline{b} = \sum_{i_p=1}^{\iota_n} a_{i_1 i_2 \cdots i_n} b_{i_p} \tag{9.28}$$

Example 9.6.2. *If* $\mathbf{A}$ *is given as in the previous example and* $\underline{b} = (1, 2, 3, 4)^t$ *then*

$$\mathbf{A} \underline{\times}_2 \underline{b} = \begin{bmatrix} 70 & 190 \\ 80 & 200 \\ 90 & 210 \end{bmatrix}$$

When it comes to mode-n vector multiplication, precedence matters because the order of the intermediate results change, i.e.,

$$\mathbf{A} \underline{\times}_p \underline{b} \underline{\times}_m \underline{c} = (\mathbf{A} \underline{\times}_p \underline{b}) \underline{\times}_{m-1} \underline{c} = (\mathbf{A} \underline{\times}_m \underline{c}) \times_p \underline{b} \tag{9.29}$$

9.6.2 The matrix Kronecker, Khatri-Rao, and Hadamard products

In this section we present the matrix-products we will use for the approximation of a symbolic skew-symmetric 3-tensor by another one of lower rank.

Definition 9.6.1. *[Kol. & Bad. 3](The Kronecker product of matrices) Let* $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{k\times \ell}$. *The Kronecker product denoted by* $A \otimes B$ *is given by*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix} = \\ = [\underline{a}_1 \otimes \underline{b}_1, \underline{a}_1 \otimes \underline{b}_2, ..., \underline{a}_m \otimes \underline{b}_{\ell-1}, \underline{a}_m \otimes \underline{b}_\ell] \tag{9.30}$$

where $\underline{a}_i \otimes \underline{b}_j$ *is the tensor product defined in Chapter 3.*

The Hadamard product is the element-wise matrix product.

Definition 9.6.2. *[Tol. & Cast. 1] (The Hadamard product of matrices) Let* $A, B \in \mathbb{R}^{n\times m}$. *The Hadamard product denoted by* $A * B$ *is given by*

$$A * B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1m}b_{1m} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2m}b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{nm}b_{nm} \end{bmatrix} \tag{9.31}$$

The Khatri-Rao product is considered as the "matching column-wise" Kronecker product.

Definition 9.6.3. *[Smil., etc. 1](The Khatri-Rao matrix product)Let* $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{m\times k}$. *The Khatri-Rao matrix denoted by* $A \odot B$ *is given by*

$$A \odot B = [\underline{a}_1 \otimes \underline{b}_1, \underline{a}_1 \otimes \underline{b}_2, ..., \underline{a}_k \otimes \underline{b}_k] \tag{9.32}$$

Next theorem provides some useful properties of these matrices that we will use in our algorithm. $A^\dagger$ denotes the pseudoinverse matrix (Moore-Penrose matrix) of A.

Theorem 9.6.1. *[Smil., etc. 1](Properties of the matrix products)*

i) $(A \otimes B)(C \otimes D) = AC \otimes BD$.

ii) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

iii) $A \odot B \odot C = (A \odot B) \odot C = A \odot (B \odot C)$.

iv) $(A \odot B)^t(A \odot B) = A^tA * B^tB.$

v) $(A \odot B)^\dagger = ((A^tA) * (B^tB))^\dagger (A \odot B)^t$

Some very interesting examples regarding these matrix products and their properties may be found in [Kol. 2].

9.7 Tensor Rank

The rank of a tensor may be defined as the generalization of the rank of a matrix, if the latter is viewed in the following way: since, the rank of any real matrix A is the minimum number of column vectors needed to span the range of the matrix, we may say that A has rank one if there exist vectors $\underline{a}$, $\underline{b}$ such that

$$A = \underline{a} \times \underline{b}^t \tag{9.33}$$

Then, the rank of A is the length of the smallest decomposition of a matrix A into a sum of such rank-1 outer products:

$$A = \underline{a}_1 \times \underline{b}_1^t + \cdots + \underline{a}_n \times \underline{b}_n^t \tag{9.34}$$

The definition of the rank of a tensor in the form (9.11) now readily follows.

Definition 9.7.1. *A tensor* $\mathbf{A}$ *has rank one, if there exist vectors* $\underline{x}_1$, $\underline{x}_2, ..., \underline{x}_n$ *in a vector space* V *such that*

$$\mathbf{A} = \underline{x}_1 \otimes \underline{x}_2 \otimes \cdots \otimes \underline{x}_n \tag{9.35}$$

Then the rank of $\mathbf{A}$ *is defined to be the minimum number of rank one tensors with which* $\mathcal{A}$ *is expressed as a sum.*

The above definition of tensor rank was first proposed by Hitchcock, [Hit. 1] in 1927, and Kruskal [Kru. 1] did so independently 50 years later. But even though the definition of tensor rank is an exact analogue to the definition of matrix rank, the properties of matrix and tensor ranks are quite different. One difference is that the rank of a real-valued tensor may actually be different over $\mathbb{R}$ and $\mathbb{C}$. In [Kru. 1], Kruskal gave an example to verify his allegement; if

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{9.36}$$

are the frontal slices of a tensor $\mathbf{A}$, then $\mathbf{A}$ is written as a sum of the matrices:

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \tag{9.37}$$

whereas over $\mathbb{C}$ these matrices are:

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \; Y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \tag{9.38}$$

Another major difference between matrix and tensor rank is that (except in special cases such as the example above), there is no straightforward algorithm to determine the rank of a specific given tensor. In fact, in [Kru. 1] a specific example was cited of a particular $9 \times 9 \times 9$ tensor whose rank can not be computed, except its boundaries, that were calculated between 18 and 23. This has led to the definition of the *maximum* and *typical* rank of a tensor. The *maximum rank* is defined as the largest attainable rank, whereas the typical rank is any rank that occurs with probability greater than zero (i.e., on a set with positive Lebesgue measure). For the collection of $n \times m$ matrices, the maximum and typical ranks are identical and equal to $\min\{n, m\}$. For tensors, the two ranks may be different. Furthermore, over $\mathbb{R}$, there may be more than one typical ranks, whereas over $\mathbb{C}$, as shown in [Kol. & Bad. 3] there is always only one typical rank. In fact, Monte Carlo experiments (which randomly draw each entry of the tensor from a normal distribution with mean zero and standard deviation one) reveal that the set of $2 \times 2 \times 2$ tensors of rank two fills about 0.79 of the space while those of rank three fill 0.21. Rank-one tensors are possible but occur with zero probability.

For a general third-order tensor $\mathbf{A} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \iota_3}$, only the following weak upper bound on its maximum rank is known, [Kru. 1]

$$\text{rank}\mathbf{A} \leq \min\{\iota_1\iota_2, \iota_2\iota_3, \iota_3\iota_1\} \tag{9.39}$$

In [Kol. & Bad. 3] there may be found more results on maximum and typical ranks for tensors but only of specific sizes.

9.8 Tensor Algorithmic Decompositions

The difficulty in defining and therefore calculating the rank of a tensor in a specific framework has led to a number of different algorithms that decompose a random tensor into a sum of tensors of lower rank. In this section we will refer to the (numerical) CANDECOMP/PARAFAC (CP) decomposition that we will expand for symbolic calculations in the next section. Other tensor decompositions that may be found in the respective literature, is the Tucker decomposition, [Tuc. 1], a higher-order form of principal component analysis, which is the second most used method for tensor decompositions and it is based on the concept that any tensor $\mathbf{X} \in \mathbb{R}^{i \times j \times k}$ may be written as

$$\mathbf{X} \approx \mathbf{G} \times_1 A \times_2 B \times_3 C \tag{9.40}$$

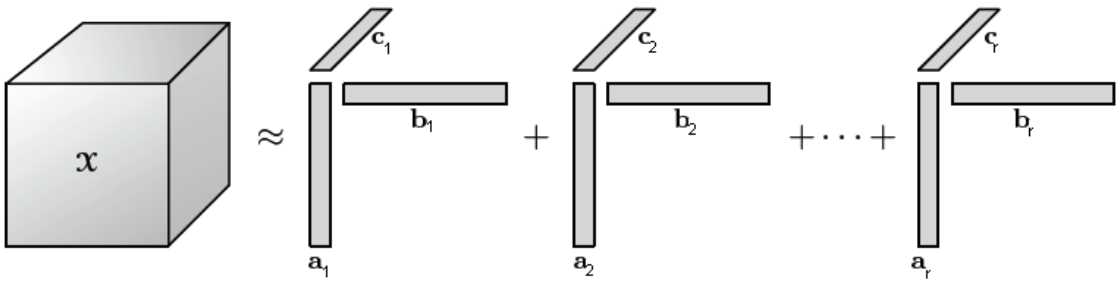


Figure 9.5: The CP decomposition of a 3-rd order tensor

where $A \in \mathbb{R}^{i \times p}$, $B \in \mathbb{R}^{j \times q}$, $C \in \mathbb{R}^{k \times r}$ and $\mathbf{G} \in \mathbb{R}^{p \times q \times r}$ is the so-called *core tensor* whose entries show the level of interaction between the different components. The algorithms created for the CP and the Tucker decompositions follow the same philosophy more or less. However, for the approximate DAP, the CP decomposition is more suitable since its formulation directly generalizes the prime decomposition, as we will see later in this chapter. Other variations of these two decompositions also met and used are the INDSCAL, PARAFAC2, CANDELINC, DEDICOM, and PARATUCK2 decompositions. More details on them may be found in [Dela., etc. 1], [Kol. & Bad. 3] and the references therein.

9.8.1 The CANDECOMP/PARAFAC Decomposition

The CP decomposition factorizes a tensor into a sum of component rank-one tensors. For example, given a third-order tensor $\mathbf{X} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \iota_3}$, we want to write $\mathbf{A}$ as

$$\mathbf{X} \approx \sum_{i=1}^{r} \underline{a}_i \circ \underline{b}_i \circ \underline{c}_i \tag{9.41}$$

where $r \in \mathbb{N}$ and $\underline{a}_i \in \mathbb{R}^{\iota_1}$, $\underline{b}_i \in \mathbb{R}^{\iota_2}$, $\underline{c}_i \in \mathbb{R}^{\iota_3}$ for $i = 1, ..., r$, as illustrated in the diagram above. Now, if $A = [\underline{a}_1, \underline{a}_2, ..., \underline{a}_r]$ and likewise for B and C, then (9.41) may be matricized as

$$\begin{aligned} X_{(1)} &\approx A(C \odot B)^t, \\ X_{(2)} &\approx B(C \odot A)^t, \\ X_{(3)} &\approx C(B \odot A)^t \end{aligned}$$

where $\odot$ denotes the the Khatri-Rao product we introduced earlier. Analogous equations can be written for the horizontal and lateral slices. In general, though, slice-wise expressions do not easily extend beyond three dimensions. Now, fol-

lowing [Kol. & Bad. 3] the CP model can be concisely expressed as

$$\mathbf{X} \approx [[A, B, C]] := \sum_{i=1}^{r} \underline{a}_i \circ \underline{b}_i \circ \underline{c}_i \tag{9.42}$$

It is often useful to assume that the columns of A, B and C are normalized to length one with the weights absorbed into a vector $\underline{\lambda} \in \mathbb{R}^r$ so that

$$\mathbf{X} \approx [[\underline{\lambda}; A, B, C]] := \sum_{i=1}^{r} \underline{\lambda}_i \underline{a}_i \circ \underline{b}_i \circ \underline{c}_i \tag{9.43}$$

Remark 9.8.1. *The above analysis concerns the 3-tensor case, because we will deal with approximations over the Grassmann variety $G_3(\mathbb{R}^6)$. The n-order tensor CP is respectively*

$$\mathbf{X} \approx [[\underline{\lambda}; A^{(1)}, A^{(2)}, ..., A^{(n)}]] := \sum_{i=1}^{r} \underline{\lambda}_i \underline{a}_i^{(1)} \circ \underline{a}_i^{(s)} \circ \cdots \circ \underline{a}_i^{(n)} \tag{9.44}$$

9.8.2 The CANDECOMP/PARAFAC Algorithm

As mentioned previously, there is no finite algorithm for determining the rank of a tensor; consequently, the first issue that arises in computing a CP decomposition is how to choose the number of rank-one components. Most procedures fit multiple CP decompositions with different numbers of components until one is "good". Ideally, if the data are noise-free and we have a procedure for calculating the CPD with a given number of components, then we can do that computation for $r = 1, 2, 3, ..$ number of components and stop at the first value of r that gives a fit of 100%. However, there are many problems with this procedure. In [Kol. & Bad. 3] there is an analytical discussion with respect to these obstacles. Nevertheless, there are two algorithms that seem to work quite well for the CP decomposition, the so-called "workhorse" algorithm for CPD and the alternating least squares (ALS) method proposed in the original papers by Carroll and Chang, [Car. & Cha. 1] and Harshman, [Hars. 1]. We will present the 3-rd order case only, which is needed for our problem, i.e., if $\mathbf{X} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \iota_3}$ we will calculate a CP decomposition with r components that best approximates $\mathbf{X}$. In other words, we will solve the minimization problem

$$\min_{\hat{\mathbf{X}}} \|\mathbf{X} - \hat{\mathbf{X}}\|, \ \hat{\mathbf{X}} = [[\underline{\lambda}; A, B, C]] := \sum_{i=1}^{r} \underline{\lambda}_i \underline{a}_i \circ \underline{b}_i \circ \underline{c}_i \tag{9.45}$$

The ALS method fixes B and C to solve for A, then fixes A and C to solve for B, fixes A and B to solve for C, and continues to repeat the entire procedure until some convergence criterion is satisfied. Having fixed all but one matrix, the

problem reduces to a linear least squares problem. If for example, B and C are fixed problem (9.45) is written as

$$\min_{\hat{A}} \|X_{(1)} - \hat{A}(C \odot B)^t\|_F, \ \hat{A} = A \cdot \text{diag}(\underline{\lambda}) \tag{9.46}$$

Hence, the optimal solution is given by

$$\hat{A} = X_{(1)} \left((C \odot B)^t\right)^\dagger \tag{9.47}$$

or equivalently, due to Theorem 9.6.1,

$$\hat{A} = X_{(1)} \left((C \odot B)\right) (C^t C * B^t B)^\dagger \tag{9.48}$$

The advantage of (9.48) is that we need only to calculate the pseudoinverse of an $r \times r$ matrix rather than a $mk \times r$ matrix, as in (9.47). Finally, we normalize the columns of $\hat{A}$ to get A, i.e., $\lambda_i = \|\hat{\underline{a}}_i\|$, $\underline{a}_i = \hat{\underline{a}}_i/\lambda_i$, for $i = 1, ..., r$. Thus we obtain the following function:

A CPD-ALS Algorithm

- Function CP-ALS($\mathbf{X}, r$);
- Give a positive integer n.
- for $i = 1, ..., n : A^{(i)} = \mathbb{O}$; (initial values)
- end for
- for $i = 1, ..., n$ set

  $V := A^{(1)t}A^{(1)} * \cdots * A^{(i-1)t}A^{(i-1)} * A^{(i+1)t}A^{(i+1)} * \cdots * A^{(n-1)t}A^{(n-1)}$;
  $A^{(i)} := X^{(i)}(A^{(n)} \odot A^{(i-1)} \odot A^{(i+1)} \odot \cdots \odot A^{(1)})V^\dagger$;
  $\lambda_i := \|\underline{a}_i\|$; (Normalize columns of $A^{(1)}$)
- end for
- repeat until maximum iterations exhausted.
- Print $\lambda_i, A^{(i)}, \ i = 1, ..., n$.
- end function

Although the ALS method is simple to implement, it may take many iterations to converge and it is not guaranteed to converge to a global minimum nor even a stationary point of (9.45), (only to a solution where the objective function of (9.45) ceases to decrease). The final solution can be heavily dependent on the starting guess as well. Some techniques for improving the efficiency of ALS include line searches, [Raj. & Com. 1], where a line search is added after each major iteration in order to update all component matrices simultaneously based on the standard ALS search directions. A different improvement of the CP was presented in [Dela., etc. 1] based on simultaneous matrix diagonalization but only in the case where $\mathbf{A} \in \mathbb{R}^{\iota_1 \times \iota_2 \times \iota_3}$ and $\text{rank}(\mathbf{A}) \leq \max\{\iota_1, \iota_2, \iota_3\}$ for a 3-rd order and the respective restrictions for the n-tensor case. Nevertheless, the algorithm is considered the most reliable for higher-order approximations. Next we apply the above methodology for the Determinantal Assignment Problem, for the $G_3(\mathbb{R}^6)$ case.

9.9 Parametric Tensor Decompositions with Applications to DAP

As we have already seen, if the linear problem of DAP is solvable then one expects more than one solution, i.e., a solution depending on some free parameters. Let $\underline{z}$ be the parametric solution of the linear subproblem $\underline{z}^t P = \underline{a}^t$. We want to calculate the best decomposable approximation of $\underline{z} \in \bigwedge^3(\mathbb{R}^r)$, i.e., find which $\hat{\underline{z}}$ in the Grassmann variety $G_3(\mathbb{R}^6)$ attains the least distance from $\underline{z}$. If $T_{\underline{z}}$ is the corresponding 3rd order skew-symmetric tensor, then we may apply the previous low-rank approximation techniques and algorithms, where the entries of the tensor in our case depend on some parameters $\underline{x}$ implied by the solution of the linear system $\underline{z}^t P = \underline{a}^t$. Then the approximate solution will also depend on $\underline{x}$. In order to test whether our approximation is acceptable (decomposable) or not (since the parameterized approximation is very complicated and large-scaled, contrary to the non-parametric data case where it is easy to verify what kind of approximation the algorithm gave), we substitute the parameterized approximation to the QPR. If this set of equations is zero then our parameterized solution is acceptable. Otherwise, we look for those $x_1, x_2, ...$ that satisfy the QPR. Then our approximation is decomposable, even thought it may not be the best one, as analytically explained in [Kol. & Bad. 3].

In this section, we start by the construction of a new CPD-ALS Algorithm that works for the parametric case as well. First, we must calculate the Khatri-Rao product for symbolic mathematics.

A Matlab algorithm for the computation of the Symbolic Khatri-Rao Product

- function p = khatrirao-sym(varargin).

if nargin == 2 and iscell(varargin(1)) (input is a single cell array)
A = varargin(1);
else
A = (varargin(1:end-1));
end

- matorder = length(A):-1:1;
else
if nargin == 1 and iscell(varargin(1))
A = varargin(1); (input is a single cell array) else
A = varargin; (input is a sequence of matrices)
end

- matorder = 1:length(A);
end.
- N = size(A1,2);
- M = 1;

for i = matorder
if ndims(A) = 2
error(Each argument must be a matrix);
end

if (N = size(A(i),2))
error(All matrices must have the same number of columns.)
end

M = M * size(Ai,1);
end
- p = [ ];
for n = 1 : N
ab = A(matorder(1))(:,n); (Loop through all the matrices)
for i = matorder(2:end) ab = Ai(:,n) * ab(:).; (Compute outer product of n-th columns)
end
- p = [p, ab(:)]; (Fill nth column of p with reshaped result)
end

Next we present the matricized tensor times Khatri-Rao product for a multi-dimensional array of symbolic objects.

A Matlab algorithm for the computation of the symbolic matrix product of the n-mode matricization of a tensor with the symbolic Khatri-Rao product

- function V = mttkrp-sym(X,U,n).
- N = ndims(X);
- if ($N < 2$)
  error('MTTKRP-SYM is invalid for tensors lowe than 2 dimensions');
end
if (length(U) = N)
  error(Cell array is the wrong length);
end
- if n == 1
  R = size(U(2),2);
else
  R = size(U(1),2);
end

- for i = 1:N
  quad if i == n, continue;
end
  if (size(U(i),1) = size(X,i)) or (size(U(i),2) = R)
error(Entry d of cell array is wrong size, i);
end
end

- Xn = permute(X,[n 1:n-1,n+1:N]);
- Xn = reshape(Xn, size(X,n), prod(size(X))/size(X,n));
- Z = khatrirao-sym(U[1:n-1,n+1:N],r);
- V = Xn*Z;

We now give the CPD-ALS algorithm for the symbolic case. The main idea is that the approximate solution will be implied after a given number of iterations. In other words, due to the fact that the entries depend on an unknown parameter, the recursive procedures will not stop automatically when a specific property is satisfied (ending property), but we select every time the number of

iterations according to the complexity of the tensor.

A Symbolic CPD-ALS algorithm

- function U = cp-als-sym(X, R).
- Give maxiters.
- N = ndims(X);
- Uinit = cell(N,1);
- for i=1:N

Uiniti = sym(fix(10*rand(size(X,3), R)));
end

- U = Uinit;
- P = [ ];
- for iter = 1:maxiters

for $n = 1 : N$
Unew = simplify(mttkrp-sym(X,U,n));
$Y = ones(R, R)$;
for $i = [1 : n - 1, n + 1 : N]$
$Y = Y. * (U(i)^t * U(i))$;
end

- Y= simplify(Y);
- Unew = $(Y\ Unew^t)^t$;
- Un = simplify(Unew);

end
end

It is not difficult to show that stability criterion in Theorem 8.3.6 holds for this case also, i.e., if $\underline{z} \in \wedge^3(\mathbb{R}^n)$ is a 3-vector and $\hat{\underline{z}}$ is one decomposable approximation with $\|\underline{z} - \hat{\underline{z}}\| \leq r_{\underline{\alpha}}/\sigma_P$, where σ_P is the largest singular value of the Plücker matrix P and $\underline{a}$, $\hat{\underline{a}}$ are the coefficient-vectors of $a(s)$, $\hat{\underline{a}}(s)$ respectively, with $a(s)$ being a stable polynomial then $\hat{a}(s)$ is also stable. We now present an example to show how the previous algorithm works.

Example 9.9.1. *Consider the linear subproblem of a DAP,* $\underline{z}^t P = \underline{a}^t$ *where*

$\underline{a} = (-1, -1, ..., -1) \in \mathbb{R}^{20}$ *and the Plücker matrix* P *is given by*

$$\left[\begin{array}{cccccccccccccccccccc}
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 2 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

The solution of the linear subproblem is given by $\underline{z} = (1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, x, x)^t$ *where* $x \in \mathbb{R}$. *It is not difficult to show that* $\underline{z}$ *is not decomposable; if* $z_1 := p_{012},\ z_2 := p_{013}, ..., z_{20} = p_{345}$, *then by using the functions, [Sturm. 2]*

$v =$sdpvar((nchoosek $(6, 3), 1)$

$[pol, v_{string}] =$ grassmannian$(v, 3, 6)$

$pol = [pol; \text{sum}(v.^2) - 1]$;

$\text{sdisplay}_{\text{vec}}(pol,\ 'v\,', v_{string})$

in Macaulay 2, [Eis., etc. 1] we obtain the 35 QPRs:

$$\begin{array}{c}
p_{014}p_{023} - p_{013}p_{024} + p_{012}p_{034} = 0,\ p_{014}p_{123} - p_{013}p_{124} + p_{012}p_{134} = 0, \\
p_{015}p_{023} - p_{013}p_{025} + p_{012}p_{035} = 0,\ p_{015}p_{024} - p_{014}p_{025} + p_{012}p_{045} = 0, \\
p_{015}p_{034} - p_{014}p_{035} + p_{013}p_{045} = 0,\ p_{015}p_{123} - p_{013}p_{125} + p_{012}p_{135} = 0, \\
\vdots
\end{array}$$

where we see that the first term of the third equation $(p_{015}p_{024})$ *is equal to 1 and the rest of the products are zero. Hence, we calculate a best decomposable of* $\underline{z}$.

The respective skew-symmetric tensor $\mathbf{T}_{\underline{z}}$ *is* $\mathbf{T}_{\underline{z}} = (T_{\underline{z}}^{(1)}, ..., T_{\underline{z}}^{(6)})$, *where*

$$T_{\underline{z}}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_{\underline{z}}^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{\underline{z}}^{(3)} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & x & 0 \end{bmatrix}, \; T_{\underline{z}}^{(4)} = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & x & 0 \end{bmatrix}$$

$$T_{\underline{z}}^{(5)} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & -x & 0 & 0 \end{bmatrix}, \; T_{\underline{z}}^{(6)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we apply our algorithm, the approximate solution implied is too complicated and unmanageable. We then substitute its components to the above set of QPR, where for $x = 0.001$ *it implies the following decomposable approximation (all terms of the QPR aggregate to the approximate zero):*

$$\hat{T}_{\underline{z}}^{(1)} = \begin{bmatrix} 0.0002 & 0 & -0.0002 & 0 & 0.003 & 0.0002 \\ 0 & -0.0001 & -1 & -0.9998 & -1.0003 & 0 \\ 0.0001 & 0.9998 & 0.0005 & -0.9998 & -0.0006 & -0.0004 \\ 0.0005 & 0.9997 & 1.0003 & 0.0004 & -0.0010 & -0.0002 \\ -0.0004 & 1.0003 & -0.0003 & -0.0005 & 0.0010 & 0 \\ 0.0001 & 0.0000 & 0.0002 & -0.0001 & -0.0001 & 0 \end{bmatrix}$$

$$\hat{T}_{\underline{z}}^{(2)} = \begin{bmatrix} -0.0001 & -0.0001 & 1.0002 & 1.0000 & 0.9996 & 0.0000 \\ -0.0001 & 0.0001 & -0.0000 & -0.0002 & 0.0004 & -0.0000 \\ -1.0000 & 0.0002 & -0.0005 & -1.0002 & 0.0006 & -0.0000 \\ -1.0005 & 0.0003 & 0.9997 & -0.0004 & 0.0010 & 0.0001 \\ -0.9997 & -0.0003 & 0.0003 & 0.0005 & -0.0010 & 0.0000 \\ 0.0000 & -0.0002 & -0.0000 & 0.0001 & 0.0000 & -0.0000 \end{bmatrix}$$

$$\hat{T}_{\underline{z}}^{(3)} = \begin{bmatrix} -0.0001 & -1.0001 & 0.0000 & 1.0001 & -0.0001 & -0.0005 \\ 0.9999 & 0.0001 & 0. & 0.9999 & 0.0002 & 0.0001 \\ 0.0001 & -0.0001 & -0.0001 & -0.0000 & 0.0001 & 0.0004 \\ -1.0002 & -0.9999 & 0.0000 & -0.0001 & 0.0001 & -0.0001 \\ 0.0002 & -0.0001 & -0.0001 & 0.0001 & -0.0002 & -0.0005 \\ 0.0000 & 0.0002 & -0.0000 & -0.0000 & 0.0009 & 0.0000 \end{bmatrix}$$

$$\hat{T}^{(4)}_{\underline{z}} = \begin{bmatrix} -0.0002 & -0.9999 & -0.9999 & 0.0000 & -0.0001 & -0.0000 \\ 1.0001 & -0.0000 & -1.0000 & -0.0001 & 0.0001 & -0.0001 \\ 0.9998 & 1.0002 & -0.0004 & -0.0001 & 0.0006 & -0.0001 \\ -0.0004 & 0.0003 & -0.0003 & -0.0003 & 0.0008 & -0.0004 \\ 0.0003 & -0.0003 & 0.0002 & 0.0003 & -0.0008 & -0.0005 \\ 0.0001 & -0.0001 & 0.0002 & -0.0002 & 0.0007 & -0.0000 \end{bmatrix}$$

$$\hat{T}^{(5)}_{\underline{z}} = \begin{bmatrix} 0.0001 & -0.9999 & -0.0001 & -0.0001 & 0.0003 & -0.0000 \\ 1.0001 & -0.0001 & 0.0000 & 0.0002 & -0.0003 & 0.0001 \\ -0.0001 & -0.0000 & 0.0003 & 0.0002 & -0.0003 & 0.0008 \\ 0.0003 & -0.0002 & 0.0001 & 0.0003 & -0.0006 & 0.0008 \\ -0.0002 & 0.0002 & -0.0001 & -0.0004 & 0.0006 & -0.0002 \\ 0.0001 & -0.0002 & -0.0011 & -0.0009 & 0.0001 & 0.0000 \end{bmatrix}$$

$$\hat{T}^{(6)}_{\underline{z}} = \begin{bmatrix} 0.0478 & -0.0560 & -0.0794 & 0.0850 & 0.1091 & -0.0009 \\ -0.0556 & 0.1690 & -0.1947 & -0.2985 & 0.0642 & -0.0015 \\ -0.2714 & -0.0038 & -0.0465 & 0.3286 & -0.4642 & -0.0005 \\ -0.0146 & -0.1982 & -0.2285 & -0.0832 & -0.4928 & -0.0007 \\ -0.1555 & 0.3312 & 0.9044 & 0.9390 & -0.1802 & 0.0009 \\ 0.0009 & 0.0009 & 0.0005 & -0.0010 & 0.0002 & 0.0000 \end{bmatrix}$$

Hence we obtain the approximate stable polynomial

$$\begin{aligned} \hat{a}(s) = (&-1.001, -0.998, -1., -0.506, -0., -0.998, -1.013, -1.4, \\ &-1.842, -0.998, -1.001, -1.001, -1., -0.998, -1., \\ &-1., -0.998, -1.001, -1.001, -1.001) \end{aligned}$$

9.10 Conclusions

In this chapter we have presented an algorithm based on the CANDECOMP/ PARAFAC Decomposition for the approximation of a 3-rd order skew-symmetric tensor by another of lower-rank, i.e., a tensor in the Grassmann variety $G_3(\mathbb{R}^6)$. We thoroughly defined every notion and preliminary result regarding tensor and multidimensional array theory in the first sections so that the algorithm is as easy-to-understand as possible, even for the reader who is coming in touch with tensor approximation techniques for the first time.

The main idea behind our algorithm was to generalize into symbolic language the tensor Khatri-Rao product and the p-mode product that only worked for numerical data in [Kol. & Bad. 3]. Then we had to do same for the main algorithm CPD-ALS and find a way to overcome the problem of infinite iterations that comes up when we work with parameters. We gave a small number of iterations at first and based on the fact that the algorithms in [Kol. & Bad. 3] give reliable approximations (at least for numerical data) we verified that our approximation

is an element of the Grassmann variety $G_3(\mathbb{R}^6)$ with the help of the corresponding QPR. Of course, our solution may not be the *optimal solution*, since lower-rank approximation problems via these techniques may imply do not guarantee the so-called *best approximation*.

In general, the above problem is referred to as one of *degeneracy*, where a tensor is *degenerate* if it may be approximated *arbitrarily well* by a factorization of lower rank. If we could not reach a suitable approximation, we could resort to the so called border rank, [Bin. 1] which is defined as the minimum number of rank-one tensors that are sufficient to approximate the given tensor with arbitrarily small nonzero error. This concept was introduced in 1979 and developed within the algebraic complexity community through the 1980s. Much of the work on border rank has been done in the context of bilinear forms and matrix multiplication.

To sum up, the CP decomposition may always calculate an approximation of a tensor and it is possible to find an approximation even if the CP decomposition fails, but there is no guarantee that the approximation implied is the "nearest" to the original tensor. From this aspect, our algorithm provides one decomposable approximation for the approximate DAP in the case of the $\bigwedge^3(\mathbb{R}^6)$ space, which is an important new result for the study and behavior of determinantal-type assignment problems in higher dimensions.

Chapter 10

Conclusions and Further work

The main objective of this thesis was to develop a new approach regarding the computational construction of the solutions of the exact and approximate Determinantal Assignment Problem (DAP). Previous work, [Gia. 1], [Kar. & Gia. 5], [Kar. & Lev. 9], [Lev. 1] was focused on the examination of the exterior algebra/algebraic geometry nature of the problem along with the solvability and assignability conditions that follow. A key result for the construction of solutions was presented in [Lev. 1] along with several new solvability results for the general DAP as well as an algorithm for constructing real feedbacks of generic systems that concerned a wider family of systems (such as decentralized, dynamic, etc.,) than those described in [Bro. 1], [Wil. & Hes. 1]. The work in [Lev. 1] regarded the compensators as elements of the Grassmann variety or as elements of a subvariety of the Grassmann variety, covering in this way the decentralized DAP case as well, i.e., controllers operating on local information to accomplice global goals rather than under the influence of a central controller (centralized DAP), which was initiated in [Lai. 1].

The starting points of this thesis were: (i) to enrich the existing DAP framework by formulating a concept that may provide solutions for DAP without the use of generic or special solvability/assignabilty conditions, therefore to define approximate solutions when exact solutions do not exist and (ii) to enhance the computational concept of DAP with new tools, ideas and algorithms that could imply solid, practical solutions. The need to introduce a new method based on approximation theories and manifold optimization techniques, was motivated by the difficulty of deriving real solutions for DAP when solvability conditions fail, i.e., when the number of controller free parameters is less than the number of constraints, or equivalently, when the respective varieties do not intersect or when it is difficult to verify their real intersections. Note that this a case often met when dealing with real solutions of determinantal-type assignment problems, since the natural field for intersection theory of varieties is the field of complex numbers $\mathbb{C}$ (which is algebraically closed or in other words every polynomial equation in one

variable is solvable) whereas $\mathbb{R}$ is not algebraically closed.

The work in this thesis has three main directions: the presentation of all existing methodologies via theoretic results or specific examples that highlight the need for a new framework regarding determinantal-type assignment problems, the complete investigation of the problem in 2-dimensions for the derivation of approximate solutions in all cases (unique solutions, degenerate issues, alternative-equivalent solutions, stability criteria, closed-form and algorithmic results) and finally the derivation of an algorithm for approximation in higher dimensions. The mathematical tools used stem form the areas of algebraic geometry, tensor algebra, optimization-approximation theory and numerical algebraic/differential geometry and one of our main goals was to present them as simple and complete as possible. The majority of the mathematical tools used here, have been adjusted to the specific context of the thesis and many of them have been further developed, such as the generalization of the standard Grassmann variety in Chapter 7 and the parametric decomposition/low-rank approximation of 3rd order tensors. Moreover, several of our results can be used almost directly to other areas and have their own interest, such as the minimization of the gap between a 2-vector and the corresponding Grassmann variety which is closely related to the open problem of the minimization of the maximum eigenvalue of a matrix, met in linear algebra and matrix-optimization theory. However, further examination of such issues has been outside the scope of this thesis and constitutes an area of future research.

Specifically, the work in Chapter 2, aimed at introducing DAP and the results obtained by previous authors with respect to its relation with other frequency assignment problems and its solvability conditions. To present an overall view of the problem and in order to be as complete and independent as possible, we started by reviewing the most important notions and terminology of dynamical systems and control theory, which are met in different parts of the thesis. Similarly, in Chapter 3, our purpose was to review the important mathematical tools which are used in the next chapters, based on Tensor-Exterior Algebra and Algebraic Geometry techniques. We gave a gradually and overall presentation of the notions mentioned, so that every definition or notion is fully clarified before naturally leading to the next, instead of a partial apposition of results. The material covered in this chapter may be also developed with respect to other problems in algebraic systems theory, such as implicit systems theory, i.e., mathematical models whose variables of interest satisfy dynamic and static relations among them, [Ba. & Kar. 1].

Moreover, in Chapter 4 we have provided some of the most well-known techniques that are usually used to solve assignment of frequencies in the affine space, before introducing the projective techniques, originated in [Kar. & Gia. 5]. These meth-

ods vary from simple applications for solving systems of polynomial equations, i.e., Gröbner bases to geometric techniques and Schubert calculus. Our purpose was to examine what the existing methodologies provide for determinantal-type assignment problems, before executing approximate techniques. In all cases, we show that the derivation of a suitable controller concerns either special cases (generic systems, algorithms with convergence uncertainty, etc.) or informations with respect to the solvability of the problem (assignability conditions, number of intersections, etc.).

The main part of our work starts in Chapter 5, where the approximate DAP is introduced, as a distance problem form the corresponding Grassmann variety for the 2-dimensional and its Hodge-dual case. The least distance problem from the simplest Grassmann variety $G_2(\mathbb{R}^4)$ which is described by one QPR only, is easily solved via the Lagrange method but unfortunately as the dimensions increase this method becomes inefficient. The observation that the least distance coincides with the smallest singular value of the corresponding Grassmann matrix, has given us a useful alternative criterion to calculate least distances, via matrix decompositions. Thus, for higher dimensions we defined the 2-vector via its equivalent skew-symmetric matrix and calculated the so-called prime decomposition where the 2-vector is written as a sum of decomposable vectors, one of which is its "best" approximation, i.e., the one in the Grassmann variety $G_2(\mathbb{R}^n)$ that achieves the least distance. The fact that the least singular value of the Grassmann matrix is equal to the minimum distance from the Grassmann variety in this general case also, as well as the fact that the skew-symmetric matrix of the 2-vector may be written as a Grassmann matrix, are among the most important new results of our thesis.

In Chapter 6, we have examined the case of degeneracy, i.e., repeated eigenvalues, in the prime decomposition, for the complete study of the problem. This is very common among tensor decompositions theorists, since equal or special-structured eigenvalues/singular values yield non-uniqueness issues for the decomposition and therefore the approximate solution. The connection of the uniqueness of the prime decomposition with the uniqueness of matrix least squares distance functions problems, has helped us derive solid uniqueness criteria for the prime decomposition and to connect the approximation problem with the original one. The main contribution however, was the investigation of the non-uniqueness case (which has not been thoroughly examined in the respective literature so far, except for special applications in isotropic matrix theory, [Sal. & Cr. 1]) via a completely new approach, the use of *Extremal Varieties*. We showed that when we have degenerate eigenvalues, the approximation implied by the prime decomposition is the worst and we calculated the respective gap. The new varieties have been defined in terms of path-wise connectivity, polynomial sums of squares and congugacy-duality properties. These results were applied for the derivation of

the best decomposable approximation via some new alternative formulae. With the help of these formulae, we managed to solve a number of problems related to manifold optimization, such as the computation of the Lagrange multipliers for the first time in closed-form formulae.

The basic aim in Chapter 7 was to investigate how the problem of deriving the best decomposable/rank-1 approximation of a multivector for the 2-dimensional case, may be generalized on sets that expand the notion of decomposability of the standard Grassmann variety. Tensor decompositions, especially in the case of 2-vectors, offer a fertile ground for experimenting on generalizations with various applications, [Fri. & Tor. 1], [Kol. 2], [Lu-S., etc. 1]. These lower-rank generalizations are met for the first time in [Gol. & Van. 2], where a best-low rank matrix approximation was achieved for a matrix whose specified columns remained fixed. Our approach lied within the concept of expanding the standard exterior algebra/tensor theories, [Hod. & Ped. 1], [Mar. 1] by using similar generalization tools as those in the so called *generalized Grassmann algebras*, where the properties of the classic exterior (Grassmann) algebra have been equipped with multi-linear structures instead of bilinear ones, [Ohn. & Kam. 1], [Kwa. 1]. Note that our approach may also view the new general Grassmann variety as a closed subscheme of a related projective subspace. This kind of generalization via scheme-theory is met in [Rav., etc. 1] for applications on dynamic output feedback problems. The key-result of this chapter was the derivation of a new Cauchy-Schwartz type inequality that may solve these general distance/ best-approximation problems based on the eigenvalues of the 2-vector. This inequality may cover all classic 2-dimensional decompositions, including degenerate issues (equal or similar structured eigenvalues) and it is one of the main results of this thesis, since this is the first time a spectral-type inequality is directly applied to manifold constrained optimization/ best rank-r approximations when $r \geq 1$.

In Chapter 8 we implemented the new results, techniques and formulae shown in the previous chapters to the solution of the approximate DAP, i.e., construction of the approximate controller and we computed the stability properties of the approximate solution. We saw that the approximate 2-vector implies a new polynomial that lies in the stability area of the original stable polynomial, using stability radius results, [Hin. & Pri. 2]. This was the main objective of the entire thesis and the result was established by expanding the prime decomposition, the gap metric and the other formulae that were examined in Chapters 5, 6 and 7, into parameterized 2-vectors, since this was the main idea from the beginning in order to comply with the solutions extracted from the linear subproblem of DAP. Other important results of this chapter, concerned the computational construction of the approximate controller, where we showed that the problem of gap-minimization is equivalent to the minimization of a 4-th order polynomial constrained to the unit sphere. This is a very important result not only

for deriving the approximate controller in practical applications of determinantal assignment problems without any solvability restrictions, but it may be seen as a new technique for optimization in the projective space, [Mah. 1]. Note, also that this approach may be also suitable for higher order Grassmann varieties, where the approximation derived may be considered as a sub-optimal decomposable approximation a case often met in approximation theory via tensor decompositions, [Kol. & Bad. 3]. In the same chapter we also examined and implemented for the first time a new algorithm which works ideally for DAP approximations.

Our purpose in Chapter 9, was to solve the approximate DAP in 3-dimensional Grassmann varieties. In this case the prime decomposition was transformed into the CANDECOMP/PARAFAC (CP) decomposition where the construction of the approximate solution followed the numeric laws and properties of these higher-order tensor SVD-like techniques. We transformed the algorithms presented in [Kol. & Bad. 3] that worked only for numerical data, i.e., constant tensors, to algorithms which allow parameters at the entries of the tensor. This allowed us to apply a parametric alternating least squares/ CP decomposition (parametric ALS/CPD algorithm) where we managed to imply the parametric decomposable approximation. We tested the acceptability of the solution (since in the parametric case, contrary to the constant case, the comparison of the implied approximation with the original tensor is not straight-forward), via the use of the QPR set. With the help of this set we obtained at least one decomposable approximation of the initial controller, since the method does not guarantee in general the optimal solution, [Kol. & Bad. 3]. This is the first result regarding the construction of an approximate controller in determinantal assignment problems in higher dimensions, ever presented within the algebraic systems theory content.

The range of results presented in this thesis by no means exhaust the application of approximate theory techniques to determinantal assignment problems, but they may be considered as a new approach to the longstanding issue of deriving optimization methods for applications to dynamic systems problems, [Helm. & Mo. 1]. The 2-dimensional case may be considered completely solved from all aspects. For 3-rd order Grassmann varieties, we focused on the original definition of DAP, eqn.(2.24). Stabilization issues have been examined, with respect to Theorem 8.3.6 which holds in higher dimensions, since its proof did not depend on any dimensions. Our approach was based on the calculation of the approximate solution at a first step and its stability test afterwards.

Thus, the contributions of this thesis for the Approximate DAP may be summarized as follows:

(a) The interpretation of the approximate DAP as a minimization problem

between a parameterized multivector (defining a linear variety) and the Grassmann variety and closed form solutions in the $\bigwedge^2(\mathbb{R}^n)$, $\bigwedge^3(\mathbb{R}^6)$ dimensions.

(b) New stability criteria for the solution of the approximate DAP and its approximate polynomial implied by the approximate multivector of the previous optimization problem.

(c) Solution of the problem in all cases of degeneracy and pathologies for the $\bigwedge^2(\mathbb{R}^n)$ dimensions.

(d) Generalization and solution of the problem for larger sets of decomposable vectors than the Grassmann variety.

(e) Construction of new algorithms specifically designed to solve DAP problems and connection with other approaches such as Newton's algorithm.

(f) A new Candecomp/Parafac decomposition for parametric tensors which is used for the first time in tensor related problems.

Further Work

This thesis is the starting point for the examination of a variety of problems related to DAP and its exact and approximate solutions. The transformation of the exact synthesis to design tools requires development of further areas which involve:

(i) Extending the frequency assignment framework described by the Approximate DAP to one requiring stabilization rather than frequency assignment. Such a version of the exact and approximate DAP is closer in spirit to design and would involve constrained optimization where we seek for the best approximation when $a(s)$ is stable. This involves solving the optimization problem

$$\min_{\underline{z}} g\left(\underline{z}, G_{n,m}\right) \text{ s.t. } P\underline{z} \text{ is stable}$$

which guarantees the desirable stability of the approximation. The main difficulty in this case is that the set of stable polynomials is not given via a closed formula or a specific property that could be transformed into a second constraint in our DAP or the parametric CPD-ALS algorithms. The solution to such an optimization problem has to involve constraints, such as the stability radius, which can guarantee that the final solution is close to the roots of the desirable polynomial and thus achieve stability.

(ii) The methodology for the constant exact DAP has been already extended in [Lev. & Kar. 4] to dynamic problems as well as decentralised versions of

DAP Extension of the above results to dynamic and decentralized schemes [Kar. etc. 8], [Lev. & Kar. 8], [Kar. & Wil. 15] etc. Note that such problems are reduced to structured forms of higher dimension DAP formulations and thus the current approach may be transferred. Such extensions of the current optimization approach would involve:

(a) the study of properties of the resulting Grassmann varieties (for the case of the dynamic problems) and the corresponding sub-varieties of the Grassmann variety for the decentralized problems.

(b) Develop the study of such varieties as functions of the McMillan degree of the compensation, as well as the structure of decentralization of the control scheme.

(c) Apply the current optimization approach to the above families of approximate DAP under different values of the compensator degree and alternative forms of decentralization.

The dynamic and decentralised feedback cases benefit from their equivalent constant DAP formulation, but they are considerably much more difficult and usually lead to non-convex optimization problems. The examination of these issues via our methodologies and algorithms is a significant challenge.

(iii) Extension of the above results to problems of stabilization using semi-algebraic sets or Kharitonov's Theorem is also a line for future work. In this area there is scope for development of both exact and approximate DAP. A similar approach reducing DAP to a linear sub-problem described by the linear variety K, the multi-linear sub-problem described by the QPRs of the related Grassmann variety and the semi-algebraic variety implied by the four polynomials in Kharitonov's Theorem may now be adopted whose stability is described by the non-linear inequalities that the coefficients of the original polynomial a(s) must satisfy.

(iv) Research is also needed for the construction of spectral sets that generalize the algebro-geometric and topological properties and the structure of the Grassmannians and the derivation of suitable spectral-tensor inequalities in higher dimensions, in order to obtain a connection between the eigenvalues of a tensor and the Grassmann varieties. Hence, the Grassmannians may be connected with spectral properties, or other sets, where the problems of multi-linearity associated with them are easily tackled, as it was already shown in this thesis proved in the 2-dimensional case, [Lev., etc. 9].

(v) There are strong indications that the properties of Grassmann matrices [Kar. & Gia. 5] may be further expanded by introducing some notions of duality (based on Hodge-Pedoe duality). A combination of properties

of Grassmann and their duals may provide alternative tools for tackling optimization problem.

The development of algorithmic procedures for computing approximate solutions for DAP will provide the basis for new tools for Control Design and may also be applied to areas beyond Control, where Multi-linear Algebra, Applied Algebraic Geometry, Manifold Optimization and Tensor Approximations are involved. Many problems in the analysis of large data sets emerging from signal processing may benefit from such tools, [Dra. etc. 1].

References

CONTROL THEORY:

[Apl. 1] J. D. Aplevich. *On the squaring problem in feedback design.* IEEE Conference on Decision and Control, pages 29-30, 1979.

[Bar. 1] S. Barnett. *Matrices in Control Theory.* Van Nostrad Reinhold, London, 1972.

[Ba. & Kar. 1] U. Baser and N. Karcanias. *Exterior algebra and invariant spaces of implicit systems: The Grassmann representative approach.* Kybernetika, Vol. 30, No. 1, pages 1-22, 1994.

[Bra. & Pear. 1] F. M. Brasch and J. B. Pearson. *Pole placement using dynamic compensators.* IEEE, Trans. AC 15, pages 33-43.

[Bro. 1] R. W. Brockett. *Nonlinear systems and differential geometry.* Proceedings of the IEEE, Vol. 64, No. 1, pages 61-72, 1976.

[Bro. & Byr. 2] R. W. Brockett and C. I. Byrnes. *Multivariable Nyquist criterion, root loci and pole placement: A geometric viewpoint.* SIAM, IEEE Trans. Aut. Control, AC-26, pages 271-283, 1981.

[Byr. 1] C. I. Byrnes. *Algebraic and geometric aspects of the analysis of feedback systems.* Geometrical Methods for the Theory of Linear Systems, pages 85-124, 1980.

[Byr. 2] C. I. Byrnes. *On compactifications of spaces of systems and dynamic compensation.* Proc. IEEE Conference on Decision and Control, San Antonio, TX, pages 889-894, 1983.

[Byr. 3] C. I. Byrnes. *Stabilizability of multivariable systems and the Ljusternic-Snirelmann category of real Grassmannians.* Systems Control Lett., Vol. 3, pages 255-262, 1983.

[Byr. & Stev. 3] C. I. Byrnes and P. K. Stevens. *Global properties of the root-locus map.* Feedback Control of Linear and Nonlinear Systems, Vol. 39, pages 9-29, 1982.

[Caro., etc. 1] L. Carotenuto, G. Franze and P. Muraca. *Some results on the genericity of the pole assignment problem*. System and Control Letters, Vol. 42, pages 291-298, 2001.

[Chen. 1] C. T. Chen. *Linear system theory and design*. H. R. W, N. Y., 1984.

[Das. 1] S. Dasgupta. *Kharitonov's theorem revisited*. Systems and Control Letters, Vol. 11, pages 381-384, 1988.

[Dav. & Wan. 1] E. J. Davison and S. H. Wang. *On pole assignment in linear multivariable systems using output feedback*. IEEE Trans. Automat. Cont., Vol. AC-20, pages 516-518, 1975.

[Sak. 1] A. K. El-Sakkary. *The gap metric: Robustness of stabilization of feedback systems.* IEEE Transactions on Automatic Control, Vol. AC-30, No.3, pages 240-247.

[Fal. 1] P. Falb. *Methods of algebraic geometry in control theory II: Multivariate linear systems and projective algebraic geometry*. Birkhäuser, 1999.

[For. 1] G. D. Forney. *Minimal Bases of rational vector spaces, with applications to multivariable control*. SIAM J. Control, Vol. 13, No. 3, pages 493-520, 1975.

[Gia. 1] C. A. Giannakopoulos. *Frequency assignment problems of linear multivariable systems: An exterior algebra and algebraic geometry approach*. Control Engineering Centre, City University, PhD Thesis, 1984.

[Gia. & Kar. 2] C. Giannakopoulos and N. Karcanias. *Necessary and sufficient conditions for zero assignment by constant squaring down*. Linear Algebra and its applications, 123/124, pages 415-446, 1989.

[Gia. & Kar. 3] C. Giannakopoulos and N. Karcanias. *Pole assignment of strictly proper and proper linear system by constant output feedback*. Int. J. Control, Vol. 42, pages 543-565, 1985.

[Gr. & Lan. 1] S. Graillat and P. Langlois. *Pseudozero set decides on polynomial stability*. Proc. Symp. on Math. Theory of Networks and Systems, Leuven, Belgium, pages 1-11, 2004.

[Helm. & Mo. 1] U. Helmke and J. B. Moore. *Optimization and dynamical systems*. Springer, 1996.

[Her. & Mar. 1] R. Herman and C. F. Martin. *Applications of algebraic geometry to systems theory, part I* . IEEE Trans. Autom. Contr., Vol. AC-22, pages 19-25, 1977.

[Herm., etc. 1] A. S. Herman, W. A. Wolovich and B. D. O. Anderson. *Generic pole assignment preliminary results*. Proc. 20th IEEE Conf. on Decision and Control, San Diego, 1981.

[Hin. & Pri. 1] D. Hinrichsen and A. J. Pritchard. *Mathematical systems theory I*. Springer-Verlag, Berlin, 2005.

[Hin. & Pri. 2] D. Hinrichsen and A. J. Pritchard. *On the robustness of root locations of polynomials under complex and real perturbations*. Proceedings of the 27th Conference on Decision and Control, Austin, Texas, December, 1988.

[Hub. & Ver. 1] B. Huber and J. Verschelde. *Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control*. SIAM J. Control Optim., Vol. 38, No.4, pages 1265-1287, 2000.

[Isi. 1] A. Isidori. *Nonlinear control systems*. Communication and Control Engineering Series, Springer-Verlag, New York, 2nd edition, 1989.

[Kai. 1] T. Kailath. *Linear systems*. Prentice Hall Inc., 1980.

[Kalm. 1] R. E. Kalman. *Mathematical description of linear dynamical systems*. SIAM J. Cont., Vol. 1, pages 152-192, 1963.

[Kal. 1] G. E. Kalogeropoulos. *Matrix pencils and linear system theory*. Control Engineering Centre, City University, PhD Thesis, 1985.

[Kar. 1] N. Karcanias. *Linear systems: Kronecker canonical forms and invariants*. Concise Encyclopedia of Modelling and Simulation, Perganon Press, Eds. D. P Atherton, P. Borne, pages 248-253, 1993.

[Kar. 2] N. Karcanias. *Matrix pencil approach to geometric system theory*. Proc. lEE., Vol. 126, pages 585-590, 1979.

[Kar. 3] N. Karcanias. *Minimal bases of matrix pencils: Algebraic, Toeplitz structure and geometric properties*. Linear Algebra and its Applications, Vol. 205-206, pages 831-868, 1994.

[Kar. 4] N. Karcanias. *Multivariable Poles and Zeros*. Control Systems, Robotics and Automation, Encyclopedia of Life Support Systems (EOLSS), Developed under the Auspices of the UNESCO, Eolss Publishers, Oxford, UK, 2005.

[Kar. 4] N. Karcanias. *Degenerate Solutions of the Decentralised Dynamic Assignment Problem*. Control Eng.Centre, City Univ. Res. Report, 1999.

[Kar. & Gia. 5] N. Karcanias and C. Giannakopoulos. *Grassmann invariants, almost zeros and the determinantal zero, pole assignment problems of linear multivariable systems.* International Journal of Control, Vol. 40, pages 673-698, 1984.

[Kar. & Gia. 6] N. Karcanias and C. Giannakopoulos. *Grassmann matrices, decomposability of multivectors and the determinantal assignment problem.* Linear Circuit Systems and Signal Processing, North Holland, pages 307-312, 1988.

[Kar. & Gia. 7] N. Karcanias and C. Giannakopoulos. *Necessary and sufficient conditions for zero assignment by constant squaring down.* Lin. Alg. and Appl., pages 415-446, 1989.

[Kar. etc. 8] N. Karkanias, B. Laios and C. Giannakopoulos. *Decentralized determinantal assignment problem: fixed and almost fixed modes and zeros.* Int. J. Cont., Vol. 48, No. 1, pages 129-147, 1988.

[Kar. & Lev. 9] N. Karcanias and J. Leventides. *Grassmann matrices, determinantal assignment problem and approximate decomposability.* Proceedings of 3rd IFAC Symposium on Systems Structure and Control Symposium (SSSC 07), 17-19 October, Foz do Iguacu, Brazil, 2007.

[Kar. & Lev. 10] N. Karcanias and J. Leventides. *Grassmann invariants, matrix pencils and linear systems properties.* Linear Algebra and its Applications, Vol. 241-243, pages 705-731, 1996.

[Kar. & Lev. 11] N. Karcanias and J. Leventides. *Design of decentralised control schemes: An algebraic approach.* IFAC Proc. Vol. 16, pages 922-927.

[Kar. & Mil. 12] N. Karcanias and E. Milonidis. *Structural Methods for Linear Systems: An introduction.* Mathematical Methods for Robust and Nonlinear Control, Springer London, Vol. 367, pages 47-98, 2007.

[Kar.& Vaf. 13] N. Karcanias and D. Vafiadis. *Canonical Forms for State Space Descriptions.* Control Systems, Robotics and Automation, Encyclopedia of Life Support Systems (EOLSS), Developed under the Auspices of the UNESCO, Eolss Publishers, Oxford ,UK, 2005.

[Kar. etc. 14] N. Karcanias, D. Vafiadis and J. Leventides. *Symbolic Methods in Control System Analysis and Design: Algebraic, algebrogeometric methods and symbolic computations in linear control problems.* The Institution of Electrical Engineers, Control Engineering Series, Vol. 56, 1999.

[Kar. & Wil. 15] N. Karcanias and D. Wilson. *Decentralised Diagonal Dynamic Stabilization of Linear Multi-variable Systems.* Lin. Alg. Appl., Vol. 121, pages 455-474.

[Ki. 2] S.-W. Kim. *Mismatching problem between generic pole-assignabilities by static output feedback and dynamic output feedback in linear systems.* Int. J. Control, Automation and System, Vol. 3, pages 56-69, 2005.

[Ki. 1] S.-W. Kim. *Construction algorithm of Grassmann space parameter linear output feedback systems.* Int. J. Control, Automation and System, Vol. 3, pp. 430-443, 2005.

[Ki., etc. 2] S.-W. Kim, C.-J. Boo, S. Kim and H.-C. Kim. *Stable controller design of MIMO systems in real Grassmann space.* Int. J. Contr., Autom, Syst., Vol. 10, No. 2, pages 213-226, 2012.

[Kim. 1] H. Kimura. *Pole assignment by gain output feedback.* IEEE Trans. Autom. Cont., Vol. AC-20, pages 509-516, 1975.

[Kim. 2] H. Kimura. *A further result on the problem of pole assignment by output feedback.* IEEE Trans. Autom. Cont., Vol. AC-22, pages 458-462, 1977.

[Kim. 3] H. Kimura. *Pole assignment by output feedback : A longstanding open problem.* IEEE Conference Publications, Proceedings of the 33rd IEEE Conference on Decision and Control, Vol. 3, pages 2101-2105, 1994.

[Ko. & Le. 1] K. Ko and J. W. Lee. *Multiuser MIMO user selection based on chordal distance.* IEEE Transactions on Communications, Vol. 60, No. 3, pages 649-654, 2012.

[Kou. & Mac. 1] B. Kouvaritakis and A. G. J. MacFarlane. *Geometric approach to analysis and synthesis of system zeros, part I: Square systems, part II: Non -square systems.* Int. J. Control, Vol. 23, pages 149-166, 167-181, 1976.

[Kuc. 1] V. Kucera. *Discrete linear control, the polynomial equation approach.* Wiley, New York, 1979.

[Lai. 1] B. Laios. *A unified approach to decentralised control.* Phd. Thesis, Control Eng. Centre, City University, 1990.

[Lev. 1] J. Leventides. *Algebrogeometric and topological methods in control theory.* Control Engineering Centre, City University, PhD Thesis, 1993.

[Lev. & Kar. 2] J. Leventides and N. Karcanias. *A new sufficient condition for arbitrary pole placement by real constant output feedback.* Systems and Control Letters, Vol. 18, No. 3, pages 191-199, 1992.

[Lev. & Kar. 3] J. Leventides and N. Karcanias. *Global asymtotic linearisation of the pole placement map: A closed-form solution of the Output Feedback Problem.* Automatica, pages 1303 - 1309, 1995.

[Lev. & Kar. 4] J. Leventides and N. Karcanias. *Dynamic pole assignment using global blow up linearisation: Low complexity solution.* Journal of Optimization Theory and Applications, 1996.

[Lev. & Kar. 5] J. Leventides and N. Karcanias. *A perturbation method for the solution of the output feedback pole placement problem.* Proceedings of IEEE CDC Conference, New Orleans, 13-15, pages 2176-2717, Dec. 1995.

[Lev. & Kar. 6] J. Leventides and N. Karcanias. *Insensitive solutions to the global linearisation of the constant pole assignment problem.* Proceedings of IFAC Conf Syst. Struct. and Control, pages 178-182, Nantes, July 1995.

[Lev. & Kar. 7] J. Leventides and N. Karcanias. *The Pole Placement Map, its Properties and Relationships to System invariants.* IEEE Trans. on Auto Control, Vol. 38, No. 8, pages 1266-1270.

[Lev. & Kar. 8] J. Leventides and N. Karcanias. *The decentralized Markov parameters and the selection of control structures.* Int. J. Cont., Vol. 70, No. 5, pages 815-830, 1998.

[Lev., etc. 9] J. Leventides, G. Petroulakis and N. Karcanias. *The Approximate Determinantal Assignment Problem.* Linear Algebra and its Applications, Vol. 461, pages 139-162, 2014.

[Lew. 1] F. L. Lewis. *A survey of linear singular systems.* Circuit Systems and Signal Processing, Vol. 5, pages 3-36, 1986.

[Lin. 1] Z. Lin. *Grobner bases and applications in control and systems.* 7th International Conference on Control, Automation, Robotics and Vision, Conference Publications ICARCV, Vol. 2, pages 1077 -1082, 2002.

[MacFar. & Kar. 1] A. J. MacFarlane and N. Karcanias. *Poles and Zeros of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory.* Int. J. Control, No. 1, vol. 24, pages 33-74, 1976.

[Mart. & Her. 1] C. F. Martin and R. Herman. *Applications of algebraic geometry to systems theory, Part II: The McMillan degree and Kronecker indices as topological and morphic invariants.* SIAM J. Contr., Vol. 16, No. 5, pages 743-755, 1978.

[Mos. 1] R. G. Mosier. *Root neighborhoods of a polynomial.* Math. Comp., Vol. 47, No. 175, pages 265-273, 1986.

[Mun. 1] N. Munro. *Pole Assignment: A review of methods.* Systems and Control Encyclopedia, M. G. Singh, Pergamon Press, pages 3710-3717, 1987.

[Pa. 1] R. V. Patel. *Pole assignment by means of unrestricted-rank output feedback*. Proceedings of IEEE, Vol. 121, No. 8, pages 874-878, 1974.

[Rav., etc. 1] M. S. Ravi, J. Rosenthal and X. Wang. *Degree of the generalized Plücker embedding of a quot scheme and quantum cohomology*. Mathematische Annalen, Vol. 311, No. 1, pages 11-26, 1998.

[Rav., etc. 2] M. S. Ravi, J. Rosenthal and X. Wang. *Dynamic pole assignment and Schubert calculus*. SIAM J. Contr. Optim., Vol. 34, pages 813-832, 1996.

[Ros. 1] H. H. Rosenbrock. *State space and multivariable theory*. Nelson, London, 1970.

[Ro. 1] J. Rosenthal. *New results in pole assignment by real output feedback*. SIAM J. Control and Optimization, Vol. 30, No. 1, pages 203-211, January 1992.

[Ro. 2] J. Rosenthal. *Tuning natural frequences by output feedback*. Computation and Control, K. Bowers and J. Lund Eds., Boston, 1989.

[Rot. & Lall. 1] M. Rotkowitz and S. Lall. *A characterization of convex problems in decentralized control*. Proc. IEEE Conference on Decision and Control, pages 569-575, 2002.

[Sal. & Cr. 1] J. K. Salisbury and J. J. Craig. *Articulated hands: Force control and kinematic issues*. Int. J. Robot. Res., Vol. 1, No. 1, pp. 4-12, 1982.

[Sand. 1] I. W. Sandberg. *On the theory of linear multi-loop feedback systems*. Bell Syst. Tech. J., No. 42, pages 355-382, 1963.

[Ser. 1] M. Seraji. *Pole placement using output feedback*. IEEE, Vol. 14: 23, pages 739-741, 1978.

[So. & Mun. 1] M. T. Söylemez and N. Munro. *A new technique for partial pole placement using constant output-feedback*. Conf. Decision Control, Tampa, FL, Vol. 12, pages 1722-1727, 1998.

[So. & Mun. 2] M. T. Söylemez and N. Munro. *A parametric solution to the pole assignment problem using dynamic output-feedback*. IEEE Transactions on Automatic Control, Vol. 46, No. 5, 2001.

[Syr., etc. 1] V. L. Syrmos, C. T. Abdallah, P. Dorato and K. Grigoriadis. *Static output feedback-a survey*. Auromatica, Vol. 33, No. 2, pages 125-137, 1997.

[Var. 1] A. I. G Vardulakis. *Zero placement and the squaring down problem: A polynomial matrix approach*. Cambridge University, Engineering Department, Technical Report, CUED/F-CAMS/TR 193, 1979.

[Ver. 1] G. Verghese. *Infinite frequency behaviour in generilized dynamical systems*. Stanford University, Ph.D thesis, U.S.A., 1978.

[Vid. 1] M. Vidyasagar. *Control system sythesis-A factorization approach*. MIT Press, 1987.

[Wan. 1] X. Wang. *Grassmannian, central projection and output feedback pole-assignment of linear systems*. IEEE Trans. Automat. Control, Vol. 41, pages 786-794, 1996.

[Wan. & Ros. 2] X. Wang and J. Rosenthal. *A cell structure for the set of autoregressive systems*. Linear Algebra and its Applications, Vol. 205-206, pages 1203-1225, 1994.

[Wil. & Hes. 1] J. C. Willens and W. H. Hesselink. *Generic properties of the pole assignment problem*. Proc. 7th IFAC World Congress, Helsinki, 1978.

[Won. 1] W. M. Wonham. *On pole assignment in multi-input controllable linear systems*. IEEE Trans. Autom. Contr., Vol. AC-12, pages 660-665, 1967.

[Yan. & Ti. 1] Y. Yang and A. L. Tits. *Generic pole assignment may produce very fragile designs*. Proc. of the 37rd conf. on Decision and Control, Tampa, FL, USA, pages 1745-1746, 1998.

LINEAR/MULTILINEAR ALGEBRA-ALGEBRAIC GEOMETRY:

[Ad. & Lou.1] W. Adams and P. Loustaunau. *An introduction to Gröbner bases*, 1st ed. American Mathematical Society, Rhode Island, 1996.

[Bel. 1] R. Bellmann. *Introduction to matrix analysis*, 2nd Ed. SIAM, 1996.

[Bir. & Macl. 1] G. Birkhoff and S. MacLane. *A survey of modern algebra*, 4th Ed. MacMillan, New York, 1977.

[Bro. 1] R. W. Brockett. *Least squares matching problems*. Lin. Alg. Appl., Vol. 122-124, pp. 761-777, 1989. ArXiv, Math A.G., 2014.

[Cox. 1] D. Cox, J. Little and D. O'Shea. *Ideals, varieties and algorithms*, 2nd Ed. Springer-Verlag, N.Y., 1996.

[Dra. etc. 1] J. Draisma, E. Horobet, G. Ottaviani, B. Sturmfels and R. R. Thomas. *The Euclidean Distance Degree of an Algebraic Variety*. ArXiv, Math A.G., 2014.

[Det. 1] R. De Traubenberg. *Clifford algebras of polynomials, generalized grassmann algebras and q deformed Heisenberg algebras*. Advances in Applied Clifford Algebras, Vol. 4, pages, 131-144, 1994.

[Die. 1] J. Dieudonne. *The Weil conjectures*. The Mathematical Intelligencer, Vol. 10, pages 7-21, 1975.

[Eck. & You. 1] C. Eckart and G. Young. *The approximation of one matrix by another of lower rank*. Psychometrika, Vol. 1, pages 211-218, 1936.

[Fri. & Tor. 1] S. Friedland and A. Torokhti. *Generalized rank-constrained matrix approximations*. SIAM J. on Matr. Anal. Appl., Vol. 29, No. 2, pp. 656-659, 2007.

[Ful. 1] W. Fulton. *Intersection theory*. Springer-Verlag, 1984.

[Gan. 1] F. R. Gantmacher. *The theory of matrices*, Vol. 2. Chelsea Publishing Co., New York, 1971.

[Gee. 1] G. van der Geer. *Exterior algebra*. U.V.A Publications, 2009.

[Gold. 1] R. Goldman and R. Krasauskas (eds). Topics in algebraic geometry and geometric modelling. *AMS Contemporary Mathematics*, Vol. 334, Proc. of Vilnius Workshop, Lithuania, 2003.

[Gol., etc. 1] G. H. Golub, A. Hoffmann and G. W. Stewart. *A generalization of the Eckart-Young-Mirsky matrix approximation theorem*. Lin. Alg. and its Appl., Vols. 88/89, pages 317-327, 1987.

[Gol. & Van. 2] G.H. Golub and C.F. Van Loan. *Matrix computations*, 3rd ed. John Hopkins University Press, Baltimore, MD, 1996.

[Got. 1] L. Göttsche. *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*. Math. Ann., 286, pages 193-207, 1990.

[Hel. & Shay. 1] U. Helmke and M. A. Shayman. *Critical points of matrix least squares distance functions*. Lin. Alg. Appl., Vol. 215, pages 1-19, 1995.

[Hig. 1] N. J. Higham. *Computing the polar decomposition with applications*. SIAM J. Sci. Stat. Comput., Vol. 7, No. 4, 1986.

[Hod. & Ped. 1] W. Hodge and D. Pedoe. *Methods of algebraic geometry*, Vols.1, 2, 3. Cambridge University Press, 1952.

[Horn. & Joh. 1] R. Horn and C. Johnson. *Matrix analysis*. Cambridge University Press, 1999.

[Jia. 1] X. Jiang L.-H. Lim, Y. Yao and Y. Ye. Statistical ranking and combinatorial hodge theory. *Mathematical Programming*, Vol. 1, No. 203-244, 2011.

[Kl. & Lak. 1] S. L. Kleiman and D. Laksov. *Schubert calculus*. The American Mathematical Monthly, Vol. 79, No. 10, pages 1061-1082, 1972.

[Kolh. 1] R. Kolhatkar. *Grassmann varieties*. Department of Mathematics and Statistics, McGill University, PhD Thesis, 2004.

[Kwa. 1] A. K. Kwaśniewski. *On generalized clifford algebras - a Survey of Applications*. Clifford Analysis and Its Applications, Vol. 25, pages 163-171, 2001.

[Land. 1] J. M. Landsberg. *Tensors: geometry and applications*. AMS, 2012.

[Lev., etc. 9] J. Leventides, G. Petroulakis and N. Karcanias. *Distance Optimization and the extremal variety of the Grassmann variety* $G_2(\mathbb{R}^n)$. Submitted.

[Lu-S., etc. 1] W. -S. Lu, S. -C. Pei and P. -H. Wang. *Weighted low-rank approximation of general complex matrices and its application in the design of 2-D digital filters*. IEEE Trans. on Circuits and Systems- I: Fundamental Theory and Appl., Vol. 44, No. 7, pp. 650-655, 1997.

[Mar. 1] M. Marcus. *Finite dimensional multilinear algebra, parts 1 and 2*. Marcel Deker, New York, 1973.

[Mir. 1] L. Mirsky. A trace inequality of John von Neumann. *Monatsh. Math.*, Vol. 79, No. 4, pp. 303-306, 1975.

[Mor. 1] F. Morgan. *The exterior algebra* $\wedge^k\mathbb{R}^n$ *and area minimization*. Lin. Alg. and its Appl., Vol. 66, pages 1-28, 1985.

[Morr. 1] A. O. Morris. *On generalized Clifford algebra*. Quarterly Journal of Mathematics, Vol. 18, pages 7-12, 1967.

[Nit. 1] N. Nitsure. *Construction of Hilbert and quot schemes*. AMS, Fundamental Algebraic Geometry, Math. Surveys Monogr., Vol. 23, pages 105-137.

[Ohn. & Kam. 1] Y. Ohnuki and S. Kamefuchi. *Generalized Grassmann algebras with applications to Fermi systems*. J. Math. Phys., Vol. 21, No. 4, pp. 601-608, 1980.

[O'Neil. 1] K. A. O'Neil. Critical points of the singular value decomposition. *SIAM J. Matrix Anal. Appl.*, Vol. 27, pp. 459-473, 2005.

[Rei. 1] K. J. Reinschke. *Multivariable control: A graph theoretical approach*. Lecture Notes in Cont. Infor. Sci., Vol. 108, Springer-Verlag, Berlin, 1988.

[Sot. 1] F. Sottile. *Enumerative geometry for the real grassmannian of lines in projective space.* Duke Math. J., Vol. 87, pages 59-85, 1997.

[Sot. 2] F. Sottile. *Real rational curves in Grassmannians.* J. Amer. Math. Soc., Vol. 13, pages 333-341, 2000.

[Stew. 1] G. W. Stewart. *Error and pertubation bounds for subspaces associated with certain eigenvalue problems.* SIAM Rev., 15, pages 727-769, 1973.

[Stew. 2] G. W. Stewart. *Rank degeneracy.* SIAM J. Sci. Stat. Comput., Vol. 5, pages 401-413, 1984.

[Tho. 1] J. Thomassen, P. Johansen and T. Dokken. Closest points, moving surfaces and algebraic geometry. *Mod. Methods Math.*, Nashboro Press, Brentwood, TN, 2005.

[Tol. & Cast. 1] J. R. R. Tolosa and E. Castillo. *From vectors to tensors.* Springer-Verlag, 2005.

[Tre. 1] W. F. Trench. *Inverse eigenproblems and associated approximation problems for matrices with generalized symmetry or skew symmetry.* Lin. Alg. Appl., Vol. 380, pages 199-211, 2004.

[Wey. 1] H. Weyl. *On unitary metrics in projective space.* Annals of Mathematics, 2nd Series, Vol. 40, No. 1, pages 141-148, 1939.

[Wilk. 1] J. H. Wilkinson. *The algebraic eigenvalue problem.* Clarendon Press, Oxford, 1968.

[Xia. 1] H. Xiang. *A note on the minimax representation for the subspace distance and singular values.* Linear Algebra and its Applications, Vol. 414, pages 470-473, 2006.

[Yok. 1] T. Yokonuma. *Tensor spaces and exterior algebra.* Translations of Mathematical Monographs, AMS, 1977.

DIFFERENTIAL GEOMETRY:

[Avr. & Her. 1] L. L. Avramov and J. Herzog. *Jacobian criteria for complete intersections. The graded case.* Inventiones Mathematicae, Vol. 117, No. 1, pages 75-88, 1994.

[Bur. 1] R. B. Burckel. *An Introduction to classical complex Analysis.* Vol. 64 of Mathematische Reihe. Birkhäuser, Basel, 1979.

[Bus. 1] H. Busemann. *Convexity on Grassmann manifolds.* Enseign. Math., Vol. 7, pages 139- 152, 1961.

[Chee. 2] J. Cheeger. *Critical points of distance functions and applications to geometry.* Geometric Topology: Recent Developments, Springer-Verlag, 1991.

[Chee. 1] J. Cheeger. *Finiteness theorems for Riemannian manifolds.* Amer. J. Math., Vol. 92, pages 61-74, 1970.

[Gal. 1] J. Gallier. *Notes on differential geometry and Lie groups.* University of Pennsylvania, Texts in Mathematics, 2012.

[Good. & Pol. 1] J. E. Goodman and R. Pollack. *Foundations of a theory of convexity on affine Grassmann manifolds.* Mathematika, Vol. 42, pages 305-328, 1995.

[Har. & Law. 1] R. Harvey and H. B. Lawson, Jr. *Calibrated geometries.* Acta Mathematica, 148, pages 47-157, 1982.

[Jost. 1] J. Jost. *Riemannian geometry and geometric analysis*, 4th Ed. Universitext, Springer-Verlag, 2005.

[Kob. & Mom. 1] S. Kobayashi and K. Nomizu. *Foundations of differential geometry*, vols. 1 and 2. Wiley, New York, 1969.

[Kre. & Kras. 1] M. G. Kreĭn and M. A. Krasnosellskii. *Fundamental theorems concerning the extension of Hermitian operators and some of their applications to the theory of orthogonal polynomials and the moment problem.* Uspekhi Mat. Nauk., Vol. 2, pages 60-106, 1947.

[Lu. 1] Q. K. Lu. *The Elliptic geometry of extended spaces.* Acta Math. Sinica, Vol. 4, pages 54-69, 1963.

[San., etc. 1] R. Sanyal, F. Sottile and B. Sturmfels. *Orbitopes.* Mathematika, Vol. 57- 02, pages 275-314, 2011.

[Tur., etc. 1] P. Turaga, A. Veeraraghavan and R. Chellappa. *Statistical analysis on Stiefel and Grassmann manifolds with applications in computer vision.* IEEE Conference on Computer Vision and Pattern Recognition, pages 1-8, 2008.

[Wong. 1] Y. C. Wong. *Differential geometry of Grassmannian manifolds.* Proc. Nat. Acad. Sci. pages 589-594, 1967.

MATRIX SPECTRAL INEQUALITIES:

[And. 1] T. Ando. *Majorizations and inequalities in matrix theory.* Lin. Alg. Appl., Vol. 199, pages 17-67, 1994.

[Boy., etc. 1] S. Boyd, L. E. Ghauoui and V. Balakrishnan. *Linear matrix inequalities in system and control thoery*. SIAM, 1994.

[Has. 1] M. A. Hasan. *Generalized Wielandt and Cauchy-Schwarz inequalities*. IEEE Proceedings of the 2004 American Control Conference, Boston, Massachusetts June 30- July 2, pages 2142-2147, 2004.

[Hor. & Mat. 2] R. A. Horn and R. Mathias. *Cauchy-Schwartz inequalities associated with positive semidefinite matrices*. Linear Algebra and its Applications, Vol. 142, pages 63-82, 1990.

[Mar. & Min. 2] M. Marcus and H. Minc. *A survey of matrix theory and matrix inequalities*. Allyn and Bacon, Boston, 1964.

[Mars., etc. 1] A. W. Marshall, I. Olkin and B. C. Arnold. *inequalities: Theory of majorization and its applications*, 2nd Ed. Springer New York, 2011.

[Thom. 1] R. C. Thompson. *The Schubert calculus and matrix spectral inequalities*. Lin. Alg. Appl., Vol. 117, pages 176-179, 1989.

[Wey. 2] H. Weyl. *Inequalities between two kinds of eigenvalues of a linear transformation*. Proc. N. A. S., Vol. 35, No. 7, 1949.

OPTIMIZATION AND ALGORITHMS:

[Abs., etc. 1] P.-A. Absil, R. Mahony, and R. Sepulchre. *Optimization algorithms on matrix manifolds*. Princeton University Press, 2008.

[Abs., etc. 2] P.-A. Absil, R. Mahony, and R. Sepulchre. *Riemannian geometry of Grassmann manifolds with a view on algorithmic computation*. Acta Applicandae Mathematicae, 80(2), pages 199 -220, 2004.

[Bas., etc. 1] S. Basu, R. Pollack and M.-F. Roy. *Algorithms in real algebraic geometry*, 2nd ed. Algorithms and Computation in Mathematics, Vol. 10, Springer-Verlag, Berlin, 2006.

[Chow., etc. 1] S. Chow, J. Mallet-Paret and J. A. Yorke. *Finding zeros of maps: Homotopy methods that are constructive with probability one*. Mathematics of Computation, Vol. 32, No. 143, pages 887-899, 1978.

[Edel., etc. 1] A. Edelman, T. A. Arias and S. T. Smith. *The geometry of algorithms with orthogonality constraints*. SIAM J. Matrix Anal. Appl., Vol. 20, No. 2, pages 303 -353, 1998.

[Eis., etc. 1] D. Eisenbud, D. R. Grayson, M. Stillman and B. Sturmfels. *Computations in algebraic geometry with Macaulay 2*. Springer-Verlag, 2001.

[Le. & Over. 1] A. S. Lewis and M. L. Overton. *Eigenvalue optimization*. Acta Numerica, Vol. 5, pages 149-190, 1996.

[Mah. 1] R. Mahony. *Optimization algorithms on homogeneous spaces with applications in linear systems theory*. PhD Thesis, Australian National University, 1994.

[Mit., etc. 1] M. Mitrouli, N. Karcanias and C. Giannakopoulos. *The computational framework of the Determinantal Assignment Problem*. Proc. of 1st European Control Conf., Vol. 1, pages 98-103, Grenoble, 1991.

[Mit., etc. 2] M. Mitrouli, N. Karcanias and C. Koukouvinos. *Numerical aspects for nongeneric computations in control problems and related applications*. Congr. Numer., Vol. 126, pages 5-19, 1997.

[Rid. 1] R. C. Riddell. *Minimax problems on Grassmann manifolds. Sums of eigenvalues*. Advances in Mathematics. Vol. 54, pages 107-199, 1984.

[Roc. 1] R. T. Rockafellar. *Conjugate duality and optimization*. SIAM, 1974.

[Sha. & Fa. 1] A. Shapiro and M. K. H. Fan. *On eigenvalue optimization*. SIAM J. Opt., Vol. 5, No. 3, pages 552-569, 1995.

[Smi. 1] S. T. Smith. *Optimization techniques on Riemannian manifolds*. Fields Institute Communications, Vol. 3, AMS, Providence, RI, pages 113 -146, 1994.

[Sturm. 2] B. Sturmfels. *Algorithms in invariant theory*, 2nd Edition. Springer, 2008.

HIGH ORDER TENSOR DECOMPOSITIONS:

[Bad. & Kol. 1] B. W. Bader and T. G. Kolda. *Algorithm 862: MATLAB tensor classes for fast algorithm prototyping*. ACM Transactions on Mathematical Software, Vol. 32, pages 635-653, 2006.

[Bin. 1] D. Bini. *The role of tensor rank in the complexity analysis of bilinear forms*. Presentation at ICIAM07, Zürich, Switzerland, July 2007.

[Car. & Cha. 1] J. D. Carroll and J. J. Chang. *Analysis of individual differences in multidimensional scaling via an N-way generalization of "Eckart-Young" decomposition*. Psychometrika, Vol. 35, pages 283-319, 1970

[Cat. 1] R. B. Cattell. *The three basic factor-analytic research designs-their interrelations and derivatives*. Psychological Bulletin, Vol. 49, pages 499-452, 1952.

[Dela., etc. 1] L. De Lathauwer, B. De Moor, and J. Vandewalle. *A multilinear singular value decomposition.* SIAM J. Matr. Anal. App., Vol. 21, pages 1253-1278, 2000.

[Del. 1] W. F. De La Vega, M. Karpinski, R. Kannan, and S. Vempala. *Tensor decomposition and approximation schemes for constraint satisfaction problems.* ACM Press, Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing, pages 747-754, 2005.

[Eid. & Sav. 1] L. Eidén and B. Savas. *A Newton-Grassmann method for computing the best multilinear rank-*(R_1, R_2, R_3) *approximation of a tensor.* SIAM Journal on Matrix Analysis and Applications, Vol. 31, Issue 2, March 2009.

[Hars. 1] R. A. Harshman. *Foundations of the PARAFAC procedure: Models and conditions for an "explanatory" multi-modal factor analysis.* UCLA working papers in phonetics, Vol. 16, pages 1-84, 1970.

[Hit. 1] F. L. Hitchcock. *The expression of a tensor or a polyadic as a sum of products.* Journal of Mathematics and Physics, Vol. 6, pages 164-189, 1927.

[Kol. 1] T. Kolda. *A counterexample to the possibility of an extension of the Eckart-Young low-rank approximation theorem for the orthogonal rank tensor approximation.* SIAM J. Matrix Anal.Appl., Vol. 24, No. 3, pp. 762-767, 2003.

[Kol. 2] T. Kolda. *Multilinear operators for higher-order decompositions.* Tech. Report SAND2006-2081, Sandia National Laboratories, Albuquerque, New Mexico and Livermore, California, Apr., 2006.

[Kol. & Bad. 3] T. Kolda and B. Bader. *Tensor decompositions and applications.* SIAM Reviews, June 2008.

[Kru. 1] J. B. Kruskal. *Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics.* Linear Algebra and its Applications, Vol. 18, pages 95-138, 1977.

[Lat.. etc. 1] L. De Lathauwer, B. De Moor and J. Vaulewalle. *A multilinear singular value decomposition.* SIAM Journal of Matrix Analysis and Applications, Vol. 21, No. 4, pages 1253-1278.

[Raj. & Com. 1] M. Rajih and P. Comon. *Enhanced line search: A novel method to accelerate PARAFAC.* EUSIPCO-05: Proceedings of the 13th European Signal Processing Conference, 2005.

[Sav. & Li.1] B. Savas and L.-H. Lim. *Quasi-Newton methods on Grassmannians and multilinear approximations of tensors.* SIAM J. Sci. Comput., Vol.32, No. 6, pages 3352-3393.

[Smil., etc. 1] A. Smilde, R. Bro, and P. Geladi. *Multiway analysis: applications in the chemical sciences*. Wiley, West Sussex, England, 2004.

[Ten. & Sid. 1] J. M. F. Ten Berge and N. D. Sidiropoulos. *On uniqueness in CANDECOMP/PARAFAC*. Psychometrika, Vol. 67, pages 399-409, 2002.

[Tuc. 1] L. R. Tucker. *Some mathematical notes on three-mode factor analysis*. Psychometrika, Vol. 31, pp. 279-311, 1966.

[Vaf. 1] C. Vafa. *Topological mirrors and quantum rings*. Essays on Mirror Manifolds, International Press, 1992.

[Zha. & Gil. 1] T. Zhang and G. H. Golub. *Rank-one approximation to high order tensors*. SIAM. J. Matrix Analysis and Applications, Vol. 23, No. 2, pages 534-550, 2001.