Optimal Investment Choices Post Retirement in a Defined Contribution Pension Scheme

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Abstract

In defined contribution pension schemes, the financial risk is borne by the member. Financial risk occurs both during the accumulation phase (investment risk) and at retirement, when the annuity is bought (annuity risk). The annuity risk faced by the member can be reduced through the "income drawdown option": the retiree is allowed to choose when to convert the final capital into pension within a certain period of time after retirement. In some countries, there is a limiting age when annuitization becomes compulsory (in UK this age is 75). In the interim, the member can withdraw periodic amounts of money to provide for daily life, within certain limits imposed by the scheme's rules (or by law).

In this paper, we investigate the income drawdown option and define a stochastic optimal control problem, looking for optimal investment strategies to be adopted after retirement, when allowing for periodic fixed withdrawals from the fund. The risk attitude of the member is also considered, by changing a parameter in the disutility function chosen. We find that there is a natural target level of the fund, interpretable as a safety level, which can never be exceeded when optimal control is used.

Numerical examples are presented in order to analyse various indices — relevant to the pensioner — when the optimal investment allocation is adopted. These indices include, for example, the risk of outliving the assets before annuitization occurs (risk of ruin), the average time of ruin, the probability of reaching a certain pension target (that is greater than or equal to the pension that the member could buy immediately on retirement), the final outcome that can be reached (distribution of annuity that can be bought at limit age), and how the risk attitude of the member affects the key performance measures mentioned above.

Keywords: income drawdown option, stochastic optimal control, decumulation phase, immediate annuitization.

JEL classification: C61, G11, J26.

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1 Introduction

The income drawdown option in defined contribution (DC) pension schemes allows the member who retires not to convert the accumulated capital into an annuity immediately at retirement but to defer the purchase of the annuity until a certain point of time after retirement. During this period, the member can withdraw periodically a certain amount of money from the fund within prescribed limits. The period of time can also be limited: usually freedom is given for a fixed number of years after retirement and at a certain age the annuity must be bought.

In the UK, where the option was introduced in 1995, the periodic income drawn is bounded between 35% and 100% of the amount that the member would have received if she bought a level annuity at retirement. At age 75, the annuity must be bought with the remaining fund.

Comparing the drawdown option with the purchase of an annuity at retirement, we observe two important points: the member is given complete investment freedom (instead of locking the fund into bond-based assets, as is usual with annuities) and a bequest desire can be satisfied should the member die before buying the annuity (because, in the case of death, the fund remains as part of the individual's estate).

On the other hand, the drawdown option does not provide any hedge against longevity risk and financial risk: the retiree faces both the risk of outliving her own assets and the risk of buying, after the deferment period, a lower annuity than the one which was possible to buy at retirement.

In this paper, we consider the income drawdown option and investigate, by means of stochastic optimal control techniques, what should be the optimal investment allocation of the fund after retirement until the purchase of the annuity, given that the pensioner wishes to achieve a certain target when she buys the annuity. We assume here that the pensioner has no bequest motive and that the only reason for choosing the drawdown plan is the hope of being able to buy a better annuity in the future than the one which she could buy at retirement. It seems therefore a reasonable suggestion that the pensioner would have a certain "income target" in mind and attempt to pursue it when investing in the financial market. We deal with the bequest motive in a parallel paper (Gerrard, Haberman, Højgaard and Vigna, 2004). In this work, we do not consider the impact that some additional pre-existing annuity (either a public pension or a private one) would have on the decision whether to annuitize or take the income drawdown option. Also, the choice of the target does not take into account the possibility of future unexpected inflation.

The remainder of the paper is organized as follows. In section 2, we introduce the model. In section 3, we define the stochastic optimal control problem. In section 4, we solve the problem in the finite time horizon with two different forms for the target. In section 5, we solve the problem in the infinite time horizon. In section 6, we introduce the simulation part of the work, report the results and carry out a sensitivity analysis. In section 7, we draw conclusions and present some ideas for further research.

A number of authors have dealt with the problem of managing the financial resources of a pensioner after retirement, which arises from the fact that whole life annuities are felt by policyholders to be "poor value for money" (Orszag, 2000) and have investigated the other alternatives available to a retiree. Khorasanee (1996) compares the purchase of an index-linked annuity with two alternatives: level annuity and income withdrawal option. Milevsky (1998) proposes a strategy for the post-retirement period where the pensioner's consumption exactly matches what a level annuity purchased at retirement would pay and the pensioner invests the remaining part. Milevsky calculates the ruin time in a deterministic scenario and estimates the probability of being able to buy (after a certain number of years) at least a larger annuity than the one that can be purchased at retirement in the case when market returns are stochastic. Kapur and Orszag (1999) consider investment decisions in the decumulation phase of a DC plan by means of stochastic optimal control, choosing between equities and annuities, and find that complete annuitization eventually occurs. Mitchell, Poterba, Warshawsky and Brown (1999), in an attempt to answer the question why people do not buy annuities, compare the expected present value of payments from an annuity (calculated with a proper term structure of interest rates) with the amount of premium charged, and compare the expected utility from the annuity payments with the expected utility from an optimal consumption path, if there were no annuities in the market. Milevsky and Robinson (2000) consider the adoption of a drawdown option assuming a fixed amount withdrawn every year and investment of the remaining fund in one risky asset. They calculate exactly the eventual probability of ruin and then approximate the probability that ruin occurs before the random time of death, comparing their approximations with the frequency of ruin found via Monte Carlo simulations. Findlater (1999) and Wadsworth, Findlater and Boardman (2001) propose a product for the post- retirement period (the second paper being a more detailed exposition of the product) in which pensioners have flexibility in the way that they invest the fund and withdraw money, as in the drawdown option, with the difference that mortality credits are given to the survivors and there is no bequest at death. Blake, Cairns and Dowd (2003) use expected utility to compare immediate annuitization at retirement with two kinds of drawdown plan (one approach has survival bonuses but no bequest — as in the proposal by Wadsworth *et al.* — and the other is more traditional, with a bequest in the case of death and no mortality credits); they also investigate the optimal annuitization age. Albrecht and Maurer (2002) consider the probability that the pensioner outlives her own assets when taking the income drawdown option, withdrawing exactly the amount of money that an immediate level annuity bought at retirement would provide. Charupat and Milevsky (2002) find the optimal mix (constant over time) between a fixed immediate annuity and a variable immediate annuity, with different mortality assumptions, via the maximization of expected utility, and then compare it with the optimal mix found in the accumulation phase of a DC scheme. Lunnon (2002) lists nine alternatives to immediate and complete annuitization at retirement: three kinds of annuity, three kinds of income drawdown and three kinds of combinations of the two; he proposes criteria for the design of a post- retirement product and analyzes all of the choices with reference to these criteria.

2 The model

In our model, we consider the position of an individual who chooses the drawdown option at retirement, i.e. withdraws a certain income until she achieves the age at which the purchase of the annuity is compulsory.

The fund is invested in two assets, a riskless asset, with constant instantaneous rate of return, r, and a risky asset, whose price follows a geometric Brownian motion with drift λ and diffusion σ.

The pensioner withdraws an amount b_0 in the unit time. Therefore, the stochastic differential equation that describes the growth of the fund is the following (see, for instance, Merton, 1969):

$$\begin{aligned} &dX(t) = [X(t)(y(t)(\lambda - r) + r) - b_0]dt + X(t)y(t)\sigma dW(t) \\ &X(0) = x_0 \end{aligned} \tag{1}$$

where $X(t)$ is the fund at time t (with $X(0)$ being the fund at retirement), $y(t)$ is the proportion of the fund invested in the risky asset and $W(t)$ is the standard Brownian motion.

We introduce the following quadratic loss (or disutility) function:

$$L(t, X(t)) = (F(t) - X(t))^2 \tag{2}$$

The function of time $F(t)$ is a target that the individual wishes to achieve: deviations from this target are penalized so that a "cost", measured by the loss function, is paid when the fund is different from the target.
The use of a quadratic loss function is not new in the context of pension funds. Some examples are Boulier, Trussant and Florens (1995), Boulier, Michel and Wisnia (1996), and Cairns (2000). From a theoretical point of view, the quadratic loss function also penalizes any deviations above the target, and this can be considered as a drawback to the model. However, the choice of trying to achieve a target and no more than this has the effect of a natural limitation on the overall risk of the portfolio: once the target is reached, there is no reason for further exposure to risk and therefore the surplus becomes undesirable. The idea that people act by following subjective targets is accepted in the decision theory literature. For example, Kahneman and Tversky (1979) support the use of targets in the cost function, and, more recently, Bordley and Li Calzi (2000) investigate and support the target-based approach in decision making under uncertainty. Another example of the use of the targets in an insurance context is provided by Browne (1995), who derives optimal investment policies by minimizing the probability that the wealth hits a certain bottom level (ruin) before hitting a certain upper level (target).
In addition, as will be shown later, with a proper and not unreasonable choice of the target, the fund never exceeds the target. Hence, the choice of a quadratic loss function can be considered appropriate and has the advantage of leading to closed-form solutions.

The terminal cost at the terminal time T, if the fund is different from the target is:

$$\varepsilon(F(T) - X(T))^2 = K(T, X(T))$$

$K(\cdot)$ has the same form of the loss function $L(\cdot)$, multiplied by a constant ε. The weighting factor ε can be useful if the cost experienced at final time T is to be considered more important than the running costs experienced before then.

Adopting the same approach as in Haberman and Vigna (2002), it can be shown that the target $F(t)$ is also a parameter that measures the risk attitude of the individual at time t: the higher its value, the lower the risk aversion of the individual. In fact, for a loss function $L(z)$, with z being the loss, the coefficient of absolute risk aversion is $\frac{L''(z)}{L'(z)}$, which in this case is:

$$\frac{L''(z(t))}{L'(z(t))} = \frac{1}{z(t)} = \frac{1}{F(t) - X(t)}$$

This relation between the target and the risk attitude is also intuitive: the less risk averse the individual is, the higher the target that she will pursue (and vice versa).

The targets are time dependent because, as time passes, the individual becomes older and her future life expectancy decreases: hence, the value of the annuity that would be purchased at the interruption of the income drawdown option decreases, ceteris paribus. We also observe that, as time passes, the fund on the one hand decreases due to the periodic income drawn, and on the other hand changes in value (and hopefully increases) due to the investment return from the two assets in which it is invested.

In a later section, we will produce results for different specification of the targets.

We now define the open set $U \subset \mathbf{R}_+ \times \mathbf{R}$, where the couples $(t, X(t))$ are allowed to range:

$$U = (0, T) \times (-\infty, +\infty), \tag{3}$$

where T is the time when purchase of the annuity becomes compulsory.

The first exit time of $(t, X(t))$ from the open set U is T.

A more realistic application would set finite bounds to the process $X(t)$.

In fact, retiring members of a DC scheme take the income drawdown option in the hope of doing better than buying an annuity at retirement. Therefore, it makes sense for them to have the wish of being able to buy a better annuity at a certain point of time after retirement than the annuity they would have purchased had they bought it at retirement. The option is thus taken with the final aim of buying a reasonably high pension and, if the size of the fund allows the purchase of the high pension before the compulsory age, the individual should stop investing the fund and lock it into an annuity, before the favourable conditions for the purchase of the desired level of pension vanish due either to increase in the annuity price or adverse performance of the asset returns leading to lower fund value. Therefore, the existence of a finite maximum bound for the fund process would be realistic.

Even more desirable than the existence of a maximum bound would be the existence of a minimum finite bound. A minimum limit would be intended to protect the retiree from outliving her assets and not being able to buy a minimum level pension at time T. Therefore, a minimum limit equal to at least 0 would be appropriate, as many other similar applications of HJB equation show (see, among others, the examples contained in Björk, 1998).

However, adding finite bounds to the state process means adding boundary conditions to the problem and this makes it very difficult to solve analytically, noting that a quadratic disutility function is here used (in the applications mentioned, the problem has been solved using a power utility function[1]). For this reason, we have left the state process unbounded, sacrificing an element of realism in order to obtain a solution in closed form.

We are thus assuming that the individual will use the income drawdown option until the maximum age allowed (time T), regardless of the size of the fund[2].

3 The stochastic optimal control problem

We are now ready to define the stochastic optimal control problem that we wish to solve.

The objective is to minimize the expected losses that can be experienced from retirement until the interruption of the income drawdown option, therefore the aim is to minimize the following expected value:

$$E_{0,x_0}\left[\int_0^T e^{-\rho s}L(s, X(s))ds + e^{-\rho T}K(T, X(T))\right] \tag{4}$$

where ρ is the (subjective) intertemporal discount factor and where the expectation is done at time $t = 0$, when the state of the system is x_0.

To solve this stochastic control problem, we define the performance criterion:

$$J(t, x; y(\cdot)) = E_{t,x}\left[\int_t^T e^{-\rho s}L(s, X(s))ds + e^{-\rho T}K(T, X(T))\right] \tag{5}$$

[1]However, the power utility function leads to a constant proportion of the portfolio being invested in the risky asset, and, as Boulier, Huang and Taillard (2001) notice, the fact that the investment allocation is independent of the size of the fund can be considered an undesirable feature in a pension fund context, where the income during retirement does depend on the amount of money in the fund, and, therefore, the power utility function can be regarded as inappropriate in modelling pension related problems.

[2]Actually, this is probably what a rich pensioner, unwilling to convert the capital into annuity, but willing to manage her money until the maximum age allowed by law, would do. Thus, the absence of limits to the wealth process can be considered to be less unrealistic for some classes of individuals.

where expectation is conditional on the state x at time t.

The value function is defined as:

$$H(t,x) := \inf_{y(\cdot)} J(t,x;y(\cdot)) = J(t,x;y^*(\cdot)) \qquad \forall(t,x) \in U \tag{6}$$

where $y^*(t,x)$ is the optimal control (if it exists).

We now want to determine the optimal control $y^*(t,x)$.

Let us consider, for any $v \in \mathbf{R}$ and any function $f \in C^2(\mathbf{R} \times \mathbf{R})$ the infinitesimal operator:

$$A^v f(t,x) := \frac{\partial}{\partial t} f(t,x) + b(t,x,v)\frac{\partial}{\partial x} f(t,x) + \frac{1}{2}\sigma^2(t,x,v)\frac{\partial^2}{\partial x^2} f(t,x) \tag{7}$$

where the functions $b(\cdot)$ and $\sigma(\cdot)$ are the drift and diffusion terms of the process $X(t)$ defined by (1).

In our case $A^v f$ becomes:

$$A^v f(t,x) := \frac{\partial f}{\partial t} + \{x[v(\lambda - r) + r] - b_0\}\frac{\partial f}{\partial x} + \frac{1}{2}x^2 v^2 \sigma^2 \frac{\partial^2 f}{\partial x^2} \tag{8}$$

Applying the HJB equation (see for example Øksendal, 1998) we get:

$$\begin{cases} \inf_{v \in \mathbf{R}}[e^{-\rho t}L(t,x) + A^v H(t,x)] = 0 & \forall(t,x) \in U \\ H(t,x) = e^{-\rho t}K(t,x) & \forall(t,x) \in \partial U \end{cases} \tag{9}$$

Applying (9) we obtain:

$$\inf_{y \in \mathbf{R}} \left\{ e^{-\rho t}(F(t) - x)^2 + \frac{\partial H}{\partial t} + \{x[y(\lambda - r) + r] - b_0\}\frac{\partial H}{\partial x} + \frac{1}{2}x^2 y^2 \sigma^2 \frac{\partial^2 H}{\partial x^2} \right\} = 0 \tag{10}$$

with the boundary condition:

$$H(T,x) = e^{-\rho T}K(T,x). \tag{11}$$

To have an easier notation, let us define:

$$\Phi(y,t,x) := e^{-\rho t}(F(t) - x)^2 + \frac{\partial H}{\partial t} + \{x[y(\lambda - r) + r] - b_0\}\frac{\partial H}{\partial x} + \frac{1}{2}x^2 y^2 \sigma^2 \frac{\partial^2 H}{\partial x^2}. \tag{12}$$

Equation (10) becomes:

$$\inf_y \Phi(y,t,x) = 0 \quad \Rightarrow \quad \Phi(y^*,t,x) = 0 \tag{13}$$

The first and second order conditions are:

$$\Phi'_y(y^*,t,x) = 0 \tag{14}$$
$$\Phi''_{yy}(y^*,t,x) > 0 \tag{15}$$

therefore:

$$x(\lambda - r)\frac{\partial H}{\partial x} + x^2 y^* \sigma^2 \frac{\partial^2 H}{\partial x^2} = 0$$

so that:

$$y^* = \frac{r - \lambda}{x\sigma^2}\frac{H'_x}{H''_{xx}}. \tag{16}$$

The sufficient condition is satisfied if and only if:

$$x^2\sigma^2\frac{\partial^2 H}{\partial x^2} > 0, \quad \text{which holds if and only if:} \quad \frac{\partial^2 H}{\partial x^2} > 0 \tag{17}$$

We will show later that this condition is actually satisfied, so that the solution is a minimum.

By substituting (16) into (13) we obtain:

$$0 = e^{-\rho t}(F(t)-x)^2 + \frac{\partial H}{\partial t} + (rx - b_0)\frac{\partial H}{\partial x} - \frac{1}{2}\left(\frac{r-\lambda}{\sigma}\right)^2\frac{(H'_x)^2}{H''_{xx}}. \tag{18}$$

We try a solution of the form:

$$H(t,x) = e^{-\rho t}[A(t)x^2 + B(t)x + C(t)] \tag{19}$$

The boundary condition (11) becomes:

$$\varepsilon e^{-\rho T}(F(T)-x)^2 = e^{-\rho T}[A(T)x^2 + B(T)x + C(T)]$$

so that

$$\begin{cases} A(T) &= \varepsilon \\ B(T) &= -2\varepsilon F(T) \\ C(T) &= \varepsilon F(T)^2 \end{cases} \tag{20}$$

The partial derivatives of H are:

$$\begin{cases} H'_t &= -\rho e^{-\rho t}[A(t)x^2 + B(t)x + C(t)] + e^{-\rho t}[A'(t)x^2 + B'(t)x + C'(t)] \\ H'_x &= e^{-\rho t}[2A(t)x + B(t)] \\ H''_{xx} &= 2e^{-\rho t}A(t) \end{cases} \tag{21}$$

From (16) we derive the optimal investment strategy at time t:

$$y^*(t,x) = \frac{r-\lambda}{\sigma^2}\left(1 + \frac{B(t)}{2A(t)x}\right) \tag{22}$$

Substituting the partial derivatives of H in (18) we have:

$$\begin{aligned} 0 &= \{1 - \rho A(t) + A'(t) - \beta^2 A(t) + 2rA(t)\}x^2 + \\ &\quad + \{B'(t) - 2F(t) - \rho B(t) + rB(t) - 2b_0 A(t) - \beta^2 B(t)\}x + \\ &\quad + \left\{F(t)^2 - \rho C(t) + C'(t) - b_0 B(t) - \beta^2\frac{B(t)^2}{4A(t)}\right\} \end{aligned} \tag{23}$$

by defining: $\beta = \frac{\lambda - r}{\sigma}$, which is the Sharpe ratio of the risky asset (see later discussion, in section 4.1).

Since (23) must hold $\forall (t,x)$, we obtain the following system of ordinary differential equations:

$$\begin{cases} A'(t) &= [\rho + \beta^2 - 2r]A(t) - 1 = aA(t) - 1 \\ B'(t) &= [\rho + \beta^2 - r]B(t) + 2F(t) + 2b_0 A(t) = (a+r)B(t) + 2F(t) + 2b_0 A(t) \\ C'(t) &= \rho C(t) - F(t)^2 + b_0 B(t) + \beta^2 B(t)^2 (4A(t))^{-1} \end{cases} \tag{24}$$

by defining $a := [\rho + \beta^2 - 2r]$ and with the boundary conditions (20).

4 The behaviour of $X(t)$ under optimal control

From (1), the evolution of the fund size $X(t)$ is described by the equation

$$dX(t) = -b_0\,dt + y(t)X(t)(\lambda\,dt + \sigma\,dW(t)) + (1-y(t))X(t)r\,dt.$$

If we define

$$G(t) = -\frac{B(t)}{2A(t)},$$

then the optimal control may be written as

$$y^*(t,x) = \frac{\lambda - r}{\sigma^2}\left(\frac{G(t)-x}{x}\right),$$

and the stochastic differential equation satisfied by the fund when optimal control is applied, $X^*(t)$, becomes:

$$dX^*(t) = [rG(t) - b_0 + (\beta^2 - r)(G(t) - X^*(t))]\,dt + \beta(G(t) - X^*(t))\,dW(t).$$

The form of this equation suggests that it will be profitable to consider the evolution of the process

$$S(t) = G(t) - X^*(t)$$

rather than that of $X(t)$ itself. We have

$$dS(t) = G'(t)\,dt - dX^*(t)$$

and

$$\begin{aligned} G'(t) &= -\frac{B'(t)}{2A(t)} + \frac{B(t)A'(t)}{2A(t)^2} \\ &= -b_0 + rG(t) + \frac{1}{A(t)}[G(t) - F(t)]. \end{aligned}$$

Therefore

$$dS(t) = \frac{1}{A(t)}[G(t) - F(t)]\,dt + (r - \beta^2)S(t)\,dt - \beta S(t)\,dW(t).$$

This equation is soluble in general, but the solution has a particularly simple form — that of a geometric Brownian motion — in the case that

$$F(t) = G(t) \text{ for all } 0 \le t \le T. \tag{25}$$

This is only an implicit equation for $F(t)$ since from (24) we see that $B(t)$, and hence $G(t)$, includes an integral involving $F(t)$. Define $\tilde{F}(t)$ as the function which solves (25), which will be derived below. If $F(t) = \tilde{F}(t)$ then

$$dS(t) = (r - \beta^2)S(t)\,dt - \beta S(t)\,dW(t),$$

which has solution

$$S(t) = S(0)\exp\left\{\left(r - \tfrac{3}{2}\beta^2\right)t - \beta W(t)\right\},$$

where $S(0) = G(0) - x_0$. An important property of the geometric Brownian motion is that it is always positive if the starting point is positive. As a consequence we can state that, if $F(t) = \tilde{F}(t)$ for all t and if $x_0 < G(0)$, then

$$X(t) < G(t) \text{ for all } 0 \le t \le T.$$

In other words, the event that the fund exceeds the target, with consequent penalization of the surplus, never occurs with such a choice of the sequence of targets.

It is now appropriate to derive the solution $\tilde{F}$ to (25). We can rewrite the definition of $\tilde{F(t)}$ as:

$$\begin{aligned} \frac{d}{dt}\left(e^{(a+r)(T-t)}A(t)\tilde{F}(t)\right) &= \frac{d}{dt}\left(-\frac{1}{2}\left(e^{(a+r)(T-t)}B(t)\right)\right) & (26) \\ -(a+r)A(t)\tilde{F}(t)+A'(t)\tilde{F}(t)+A(t)\tilde{F}'(t) &= -\frac{1}{2}\left(B'(t)-(a+r)B(t)\right), & (27) \end{aligned}$$

which reduces to

$$\tilde{F}'(t) - r\tilde{F}(t) = -b_0,$$

an equation whose solution is

$$\tilde{F}(t) = \frac{b_0}{r} + \left(\tilde{F}(T) - \frac{b_0}{r}\right)e^{-r(T-t)}. \tag{28}$$

The value $\tilde{F}(T)$ may be chosen arbitrarily.

This definition for the targets allows an easy interpretation, writing (28) in the following form:

$$\tilde{F}(t) = \tilde{F}(T)e^{-r(T-t)} + b_0\left(\frac{1-e^{-r(T-t)}}{r}\right).$$

Should the fund reach the value of the target at time t, the pensioner could immediately invest in the riskless asset what would be necessary to reach the final target at time T and still consume an amount b_0 for the remaining $T-t$ years. Therefore, she would achieve the final target with certainty, meanwhile consuming the amount b_0 that immediate annuitization at retirement would have provided. However, we notice that, due to the way in which they are constructed, the targets can never be reached.

A similar result can be found in Browne (1997). In the same setting as ours (the same financial market and fixed consumption), he aims to maximize the probability of hitting a certain upper boundary before ruin. He finds that, subject to optimal control, the difference between the "safety level" (the minimum level of the fund that guarantees the fixed consumption by investing entirely in the riskless asset) and the fund level is a geometric Brownian motion, so never decreases to zero, implying that the safety level can never be reached. We notice that also in our case the target $\tilde{F}$ represents a safety level and is never achieved.

4.1 Risky assets with the same Sharpe ratio

The evolution of the fund under optimal control is:

$$\begin{aligned} &dX^*(t) = [X^*(t)(y^*(t)(\lambda - r) + r) - b_0]dt + X^*(t)y^*(t)\sigma dW(t) \\ &X^*(0) = x_0 \end{aligned} \tag{29}$$

If we consider the Sharpe ratio of the risky asset, i.e. the dimensionless quantity:

$$\beta = \frac{\lambda - r}{\sigma}$$

we can easily show that, if the riskless rate of return r does not vary, the evolution of the fund under optimal control is invariant for assets with the same Sharpe ratio — changing in an appropriate way the optimal allocation in the risky asset. In fact, given a certain Sharpe ratio, it is possible to obtain the same ratio by multiplying by the same constant k both the expected excess rate of return (over the riskless asset) and the volatility of the risky asset. Thus, if the optimal control for the asset with expected excess rate of return $\lambda - r$ and volatility σ is $y^*(t)$, then the optimal control for the asset with expected excess rate of return $k(\lambda - r)$ and volatility $k\sigma$ is $\frac{y^*(t)}{k}$, and it is easy to see, by looking at (29), that the two assets with the same Sharpe ratio give rise to the same evolution of the fund (if the return of the riskless asset remains unchanged).

This simple result does in effect imply that the distribution of the final annuity will be the same if one invests optimally in two different assets with the same Sharpe ratio, and therefore, that the conclusions found for a particular risky asset hold for a wide class of assets, namely the assets that have the same Sharpe ratio. This will allow us to find more general conclusions than expected.

5 Solution of the problem

5.1 Finite time horizon

We have solved the problem with two definitions for the targets.

5.1.1 Case 1: targets $\tilde{F}(t)$.

We assume that $F(t) = \tilde{F}(t)$, that is:

$$F(t) = \frac{b_0}{r} + \left(F(T) - \frac{b_0}{r}\right) e^{-r(T-t)} = \frac{b_0}{r} + \left(F - \frac{b_0}{r}\right) e^{-r(T-t)}$$

where F is the final target at time T. F can be, for example, the price of the desired annuity at the age of compulsory annuitization, or could even be linked to the benefit provided by a defined benefit pension scheme.

The solution of (24) is:

$$\begin{cases} A(t) &= (\varepsilon - \frac{1}{a})e^{-a(T-t)} + \frac{1}{a} \\ B(t) &= -2F\varepsilon e^{-(a+r)(T-t)} + \frac{2F}{a}[e^{-(a+r)(T-t)} - e^{-r(T-t)}] + \frac{2b_0\varepsilon}{r}[e^{-(a+r)(T-t)} - e^{-a(T-t)}] + \\ & \quad + \frac{2b_0}{ar}[e^{-r(T-t)} + e^{-a(T-t)} - 1 - e^{-(a+r)(T-t)}] \\ C(t) &= \varepsilon F^2 e^{-\rho(T-t)} + \left(\frac{b_0}{r}\right)^2 \frac{1}{\rho}(1 - e^{-\rho(T-t)}) + \frac{2b_0}{r(\rho - r)}\left(F - \frac{b_0}{r}\right)[e^{-r(T-t)} - e^{-\rho(T-t)}] + \\ & \quad + \frac{1}{\rho - 2r}\left(F - \frac{b_0}{r}\right)^2 [e^{-2r(T-t)} - e^{-\rho(T-t)}] - e^{-\rho(T-t)} \int_t^T e^{\rho(T-s)} \left[b_0 B(s) + \beta^2 \frac{B(s)^2}{4A(s)}\right] ds \end{cases} \tag{30}$$

The condition $H''_{xx} > 0$ is also satisfied. In fact:

$$H''_{xx} = 2e^{-\rho t}A(t) = 2e^{-\rho t}\left(\varepsilon e^{-a(T-t)} + a^{-1}(1 - e^{-a(T-t)})\right)$$

If $a > 0$, then $\left(\varepsilon e^{-a(T-t)} + a^{-1}(1 - e^{-a(T-t)})\right) > 0$, obviously.

If $a < 0$, then $\left(\varepsilon e^{-a(T-t)} + a^{-1}(1 - e^{-a(T-t)})\right) > 0$, because $a^{-1} < 0$ and also $1 - e^{-a(T-t)} < 0$.

We observe that, with the quadratic loss function chosen, a high risk aversion is associated with a low desired level of final target F and vice versa. The parameter F is thus a measure of the risk attitude of the individual.

The condition $x_0 < G(0)$, that ensures the positivity of the shortfall under optimal control, is fulfilled if the final target F is such that:

$$F > \frac{b_0}{r} + \left(x_0 - \frac{b_0}{r}\right) e^{rT}. \tag{31}$$

5.1.2 Case 2: exponential targets.

Another sensible choice for the target can be the price of a level annuity. The price of a life annuity can be approximated by the present value of an annuity paid with certainty for $\Omega - t$ years, by choosing carefully the value of Ω. Therefore, we choose the target at time t as the present value at the rate of return r of the annuity which pays an amount b_1 per unit time for $\Omega - t$ years:

$$F(t) = b_1 \int_0^{\Omega - t} e^{-rs}\, ds = \frac{b_1}{r}(1 - e^{-r(\Omega - t)}) \tag{32}$$

This definition of the targets has the advantage of leading to a closed-form solution. If the true price of the annuity had to be considered, the solution would have to be found numerically.

The interest rate used to price the annuity could be considered as a stochastic process, but this would complicate further the model. Furthermore, Milevsky (1998), who assumes that the interest rate used to price the annuity follows the Cox-Ingersoll-Ross dynamics and makes a sensitivity analysis on the parameters of the process, finds that the uncertainty surrounding the future interest rates has little effect on the probability of being better off when deferring annuitization.

As mentioned above, the value of Ω should be properly chosen, so that the resulting sequence of targets approximates well the price of a lifetime annuity at any time t. Ω can be the expected remaining lifetime at retirement or could be chosen such that $\Omega - T$ is the expected remaining lifetime at the age of compulsory annuitization. In the latter case, all the resulting targets would be higher than the true prices of the annuities at the different ages. In fact, following the actuarial equivalence principle, the price of a continuously paid life annuity is the expectation of a concave function of the remaining lifetime T_x of the individual aged x. The target at time T would be the same concave function evaluated in the expected value of the random variable T_x, and, by applying Jensen's inequality[3], the target at time T would be greater than the actuarially fair price. It is then easy to see (considering that $E(T_{x-1}) < E(T_x) + 1$ for any x) that also all the other targets between retirement and time T would be higher than the price of the corresponding annuity.

Since the main aim is to try to reach the final target (due to the fact that the drawdown plan does not stop until compulsory annuitization), in the simulation part we have chosen Ω so that the final target in this formulation is approximately equal to the true price of a lifetime annuity issued at the age of compulsory annuitization. This choice also leads to a sequence of slightly higher targets than the fair prices of the annuities at the different ages, except for the last target.

A reasonable choice for b_1 would be $b_1 \geq b_0$. Here, the risk aversion is measured by the level of the desired pension income b_1. A high risk aversion is associated with a low desired level of pension b_1 and vice versa.

[3] $E[f(T_x)] \leq f[E(T_x)]$ for any concave function f. In this case, $f(T_x) = \frac{1-e^{-rT_x}}{r}$.

The solution of (24) is now:

$$\begin{cases} A(t) &= (\varepsilon - \frac{1}{a})e^{-a(T-t)} + \frac{1}{a} \\ B(t) &= 2\varepsilon \frac{b_1}{r} e^{-(a+r)(T-t)}(e^{-r(\Omega-T)} - 1) - \frac{2\frac{b_1}{r}+2b_0a^{-1}}{(a+r)}(1 - e^{-(a+r)(T-t)}) + \\ & \quad - \frac{2b_0(a\varepsilon-1)}{ar}(e^{-a(T-t)} - e^{-(a+r)(T-t)}) + \frac{2b_1}{ar}e^{r(t-\Omega)}(1 - e^{-a(T-t)}) \\ C(t) &= \varepsilon(F(T)^2)e^{-a(T-t)} - \rho^{-1}(F(T)^2)(e^{-\rho(T-t)} - 1) + \\ & \quad -e^{-\rho(T-t)}\left[b_0 \int_t^T e^{\rho(T-s)}B(s)ds + \beta^2 \int_t^T e^{\rho(T-s)}\frac{B(s)^2}{4A(s)}\,ds\right] \end{cases} \tag{33}$$

We observe that the sufficient condition for the minimum is satisfied, as $A(t)$ does not change when the targets change.

5.2 Infinite time horizon

As an extreme and special case, we consider the case where the time horizon is equal to infinity. Therefore, the objective is to minimize the following expectation:

$$E_{t,x}\left[\int_t^\infty e^{-\rho s}L(s, X(s))ds\right]. \tag{34}$$

Targets are chosen to be fixed over time ($F(t) = F$ for any t) and the bequest function has been chosen equal to 0.

This problem is easier to solve than before, because in the trial solution we can separate time and wealth, which was not possible in the finite time horizon case.

The trial solution for $H(t,x)$ is now:

$$H(t,x) = e^{-\rho t}[Ax^2 + Bx + C] = e^{-\rho t}H(0,x). \tag{35}$$

The "transversality" condition is now:

$$\lim_{t\to+\infty} E\left(H(t, X^*(t))\right) \quad = \quad 0 \tag{36}$$

and it can be shown that it is satisfied if:

$$2r < \rho + \beta^2. \tag{37}$$

By calculating the partial derivatives H'_t, H'_x and H''_{xx} and substituting them into equation (18) and setting equal to 0 the coefficients of x^2, x and the constant term, we obtain:

$$\begin{cases} A &= (\rho + \beta^2 - 2r)^{-1} \\ B &= \frac{2F+2b_0A}{r-\rho-\beta^2} \\ C &= [F^2 - b_0B - \beta^2B^2/(4A)]/\rho \end{cases} \tag{38}$$

The optimal investment strategy is:

$$y^*(x) = \frac{r-\lambda}{\sigma^2}\left(1 + \frac{B}{2Ax}\right) \tag{39}$$

which is independent of time.

The sufficient condition for $y^*(x)$ to be a minimum is $A > 0$, which holds if and only if:

$$2r < \rho + \beta^2,$$

which is exactly the condition (37) above.

6 Simulations

We have carried out some simulations in order to investigate the behaviour of the optimal investment strategy and its appropriateness in terms of:

1. the risk of outliving the assets: ie risk of ruin;
2. the average time of ruin, given that ruin occurs;
3. the probability of reaching the target (e.g. the desired level of annuity) at time T;
4. the distribution of the annuity that can be bought at time T, compared to the target pursued;
5. how the risk attitude of the individual can affect optimal choices and final results.

In the simulations we have considered both the formulations of the targets, $\tilde{F}(t)$ and exponential. For a consistent comparison, the final target at time T in the two formulations coincide, i.e.:

$$F = \frac{b_1}{r}\left(1 - e^{-r(\Omega - T)}\right). \tag{40}$$

This choice of the final target F ensures the positivity of the shortfall under optimal control, observing that the condition (31) holds with the chosen values of the parameters.

Apart from the targets at time T, the targets in the exponential formulation are higher than the targets in the other formulation (as figure 1 in the next section shows).

6.1 Assumptions

The assumptions made are the following:

- the member retires at the age of 60;
- the compulsory age for annuitization is 75; therefore, $T = 15$;
- the initial fund is $X(0) = 100$;
- the constant amount b_0 withdrawn every year is equal to the amount that an annuity purchased at retirement would provide[4]; the mortality table used in computing b_0 is the Italian projected mortality table (RG48);
- the level of desired pension is $b_1 = \frac{3}{2} b_0$[5];
- the value of Ω is 25, chosen such that the final target coincide with the expected present value of a whole life annuity of b_1 per annum issued to a male aged 75, that is: $F(15) \approx b_1 a_{75}$; curiously, 25 almost coincides with the expected remaining lifetime at retirement (age 60);

[4] This assumption is not new in the current actuarial literature on income drawdown option: see, among others, Khorasanee (1996), Milevsky (1998), Albrecht and Maurer (2002). This choice allows a consistent comparison of immediate annuitization with the consumption path over whole life when choosing drawdown.

[5] We choose the target at age 75 to be 1.5 times the income purchasable at retirement because we think that 15 years is a sufficiently long period over which to increase the amount of pension rate by 50%. If the time horizon were shorter (e.g. 10 years), the choice of the coefficient for the target would have been different and probably lower than 1.5.

- the assumptions on investment returns parameters are the following: $r = 5\%$; $\lambda = 10\%$; $\sigma = 20\%$;
- the intertemporal discount factor is $\rho = 5\%$ and the weight given to the loss at the time horizon T is $\varepsilon = 1$;
- different risk profiles were considered, by choosing different values of b_1, eg $b_1 = 2b_0$ and $b_1 = 1.25b_0$, whereas a higher target indicates a low risk aversion and vice versa.

The targets in the two formulation with $b_1 = 1.5b_0$ are plotted in Figure 1. The upper line shows the exponential targets and the lower one the $\tilde{F}(t)$.

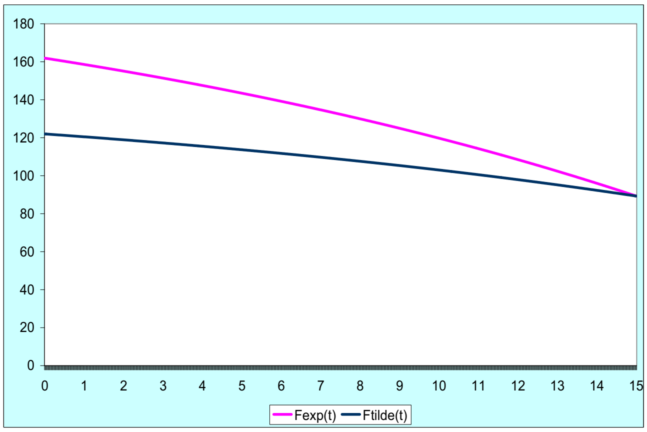

Figure 1: $\tilde{F}(t)$ and $F_{exp}(t)$ over time.

In discretizing the process, we have chosen the time interval h equal to 1 week: this simplification leads to 780 time points in which the pensioner has to decide about the investment strategy and, therefore, the aim is to find the values of $y^*(t)$ for $t = 0, 1, \ldots, 779$.

We have carried out 1000 Monte Carlo simulations for each value of b_1, using the same 1000 streams of pseudo-random numbers for each value of b_1. In each simulation, we have simulated the Brownian motion (with the discretization chosen) and hence the behaviour over time (15 years) of the risky asset. In each scenario of market returns, the optimal value $y^*(t)$ has been calculated (for $t = 0, 1, \ldots, 779$) and then adopted for the growth of the fund.

In the simulations, when ruin occurs the drawdown plan stops and the fund remains zero until time T, and so does the optimal allocation in the risky asset, $y^*(t)$.

6.2 Simulation results

The results from the simulations provide the following information:

- the optimal investment strategy is analysed by looking at certain percentiles of the distribution of $y^*(t)$;

- the risk of outliving the assets (ruin probability) is analysed by looking at the frequency over the 1000 simulations of the event $X(t) \leq 0$ for some $t \leq 780$ and the average time of ruin (when ruin occurs) is also calculated;
- the probability of reaching the final target is analysed by looking at the frequency over the 1000 simulations of the event $X(T) \geq F(T)$; this gives positive answers only in the exponential target formulation (we recall that the probability of reaching the target in the $\tilde{F}$ formulation is zero by construction); similarly, also the probability of not being able at time T to buy the annuity that was purchasable at retirement is calculated;
- the final outcome of the income drawdown option is considered by looking at the distribution of the annuity which can be bought at age 75 with the remaining fund (by means of a histogram);
- the effect of risk aversion is considered by comparing the results for different values of b_1.

6.3 Optimal investment strategy

The following graphs in Figure 2 show the 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentiles of the distribution of the optimal investment strategy $y^*(t)$ $(t = 0, 1, \ldots, 779)$ for the different formulations of the targets and for the different degrees of risk aversion. In other words, the graphs capture the behaviour over time of 90% of the trajectories of $y^*(t)$ obtained by applying the optimal strategy.

On the left there are the three graphs for the $\tilde{F}$ targets, on the right the exponential targets. Going down the page, there are the three different levels of risk attitude: $b_1 = 1.25b_0$ on the top, $b_1 = 1.5b_0$ in the middle and $b_1 = 2b_0$ in the bottom.

Target: $\tilde{F}(t)$ Target: $F_{exp}(t)$

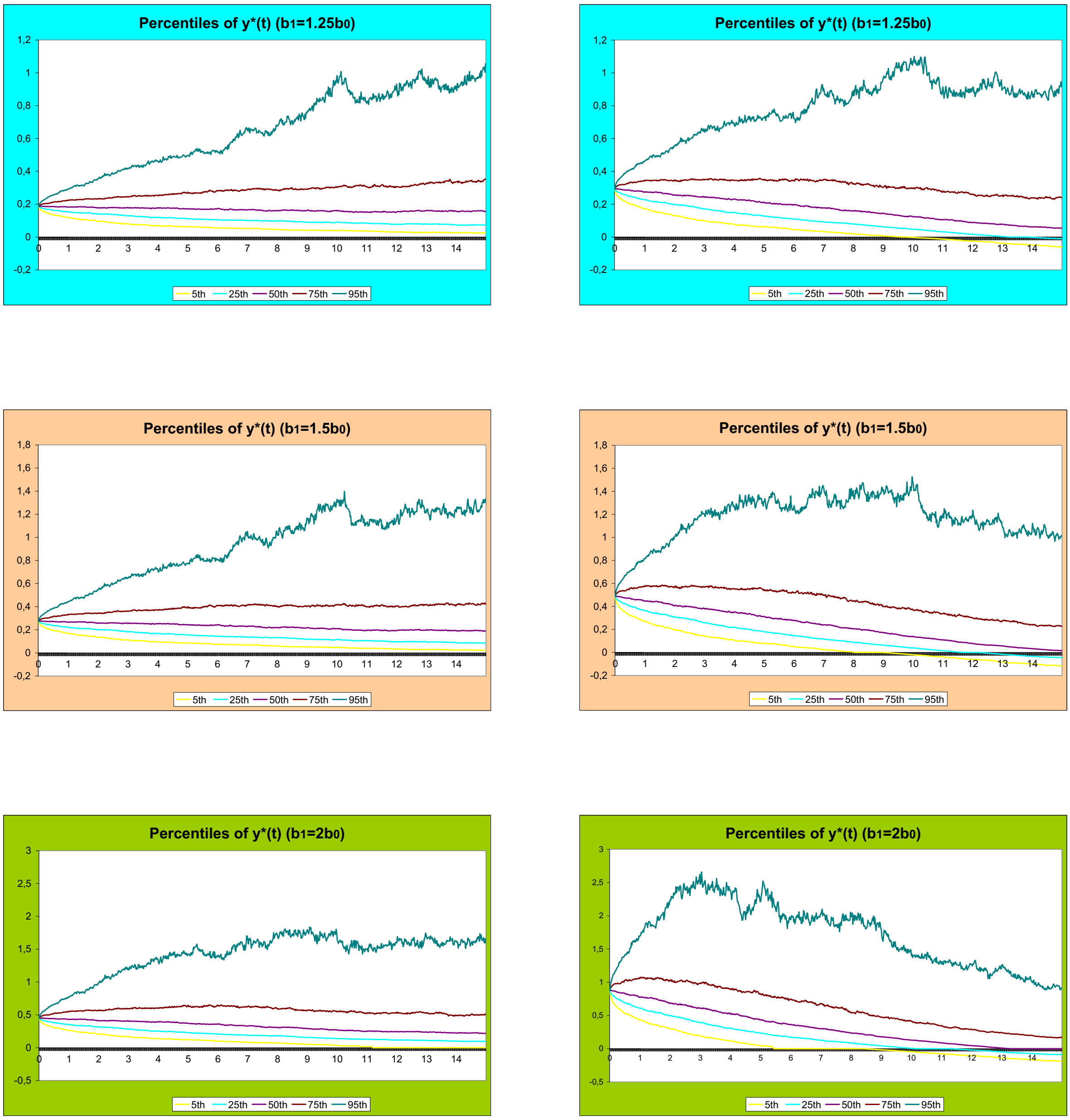

Figure 2: Behaviour of the optimal investment strategy over time

We notice that the trend of $y^*(t)$ depends on the formulation of the targets. In fact, with $\tilde{F}$ the optimal allocation in the risky asset $y^*(t)$ is almost stable over time (it is even slightly increasing on average). However, with exponential targets, the optimal allocation in the risky asset has a decreasing trend, which means that it seems to decrease as time T approaches, in a sort of post-retirement "lifestyle strategy". Due to a higher initial target in the exponential case, the starting points $y^*(0)$ are lower for the $\tilde{F}$ targets than for the exponential targets; the optimal investment in the risky asset at time T, on the other hand, is on average higher for $\tilde{F}$ than for the exponential case. This indicates that there is a cross-over time t at which the average investment in the risky asset is the same in both cases.

Looking at how the percentiles are spread around the median, we observe that the patterns of the strategies are more stable with the $\tilde{F}$ targets than with the exponential ones. This is confirmed also by considering the level of the standard deviation of $y^*(t)$ over time, which turns to be in general much higher in the case of exponential targets.

The values of $y^*(t)$ are higher with higher values of b_1, which is intuitive: the lower is the risk aversion, the higher is the proportion of the portfolio invested in the risky asset. This can be also proved mathematically, by looking at the expression for $y^*(t)$ (equations (22), (30), (33) and (40)): the function $B(t)$ is decreasing in b_1, and $y^*(t)$ is decreasing in $B(t)$, and therefore $y^*(t)$ is increasing in b_1. Going down the page, the volatility of the optimal investment allocation also increases, with the percentiles of the strategies being more spread.

We notice that the value of $y^*(t)$ is always positive in the $\tilde{F}$ formulation, because the shortfall is always positive (noting that negative values of the fund are not permitted in the simulations).

6.4 Probability of ruin, probability of failing the target and distribution of the final annuity

Table 1 reports, for the two formulation of the targets and the three different risk profiles: the estimated ruin probability, which is the probability of outliving the assets (calculated as the frequency in the simulations that at some time t before T the pensioner runs out of money); the average time of ruin when ruin occurs (in years); the estimated probability of failing to reach the target (b_1); the estimated probability of failing to reach the annuity that it was possible to buy at retirement (b_0); the mean shortfall from the target, when the target is missed; and key percentiles and moments of the distribution of the annuity that can be bought at time T with the remaining fund (noting that, with the assumptions made, the annuity purchasable at retirement is $b_0 = 7.56$ and the targeted annuity — depending on the risk aversion — is 9.45, 11.35 or 15.13). In some cases, the 5^{th} percentile of the final annuity is 0 because in the simulations when ruin occurs there is no more investment and the fund remains 0 until time T. This happens more frequently with a low risk aversion (hence, more aggressive strategies) and the exponential targets.

	$b_1 = 1.25b_0 = 9.45$		$b_1 = 1.5b_0 = 11.35$		$b_1 = 2b_0 = 15.13$	
	$\tilde{F}(t)$	$F_{exp}(t)$	$\tilde{F}(t)$	$F_{exp}(t)$	$\tilde{F}(t)$	$F_{exp}(t)$
probability of ruin	1.8%	2.6%	3.4%	5.2%	6.7%	11.5%
mean ruin time when ruin occurs (years)	12.2	10.1	10.4	8.8	9	6.7
probability final annuity $< b_1$	100%	74.5%	100%	63.2%	100%	53.7%
probability final annuity $< b_0$	30.9%	23.3%	19.7%	14.4%	15.2%	15.1%
mean shortfall from b_1	1.7	1.8	2.5	2.8	4.1	5.7
final annuity: 5th perc.	4.4	4	3.4	0	0	0
final annuity: 25th perc.	7.2	7.7	8.1	9	9.7	11.3
final annuity: 50th perc.	8.3	8.9	9.7	10.9	12.4	14.8
final annuity: 75th perc.	8.8	9.5	10.5	11.7	13.7	16.1
final annuity: 95th perc.	9.2	9.9	10.9	12.4	14.5	17.7
final annuity: mean	7.7	8.2	8.8	9.8	11	12.7
final annuity: standard deviation	1.7	2	2.4	3	3.9	5.4

Table 1: Risk Measures for Different Targets and Risk Profiles

The probability of ruin is always lower with the $\tilde{F}(t)$ formulation of the targets than with the exponential one, and this is mainly due to the lower aggressiveness of the strategies which generally apply in the former case. Furthermore, ruin occurs on average later (two-three years later) in the $\tilde{F}(t)$ formulation of the targets. This is due to the fact that, with exponential targets, there is a consistent investment in the risky asset at the beginning of the drawdown phase and in the case of adverse performance of the risky asset, ruin occurs relatively early. The ruin probability significantly increases when the value of the targeted b_1 increases: it roughly doubles when passing from $b_1 = 1.25b_0$ to $b_1 = 1.5b_0$ and when passing from $b_1 = 1.5b_0$ to $b_1 = 2b_0$ (in this last case, it more than doubles with exponential targets).

The probability of failing to reach the target is obviously 100% in all cases for the $\tilde{F}(t)$ formulation of the targets and is decreasing as the target increases with the exponential formulation. However, the mean shortfall from b_1 is lower with the $\tilde{F}(t)$ formulation of the targets than with the exponential one, apart from low values of b_1, when it is almost equal (with a very low value of b_1 it is easier to approach the desired level of pension).

With low values of b_1, the probability of failing to reach b_0 is lower for the exponential formulation than for the $\tilde{F}(t)$ one, whereas it is almost the same for $b_1 = 2b_0$. The probability of having a final annuity lower than the one which it was possible to buy at retirement is low when the target pursued is not too low: in the range 14–20% with $b_1 = 1.5b_0$ and $b_1 = 2b_0$, depending on the risk profile (generally decreasing when b_1 increases, ie when riskier strategies are adopted). On the other hand, it is relatively high when the target is low: in the range 23–31% with $b_1 = 1.25b_0$. Unsurprisingly, this seems to suggest that retirees with a high risk aversion, or who try to pursue a target not too much higher than the annuity they could have purchased on immediate annuitization at retirement should consider the relatively high chances of being worse off by choosing the income drawdown option (almost one case out of three or one out of four, depending on the formulation of the interim targets). On the other hand, if the risk aversion is not too high, and the target pursued is not too low, they should probably opt for the income drawdown option, even if they were not allowed to annuitize before age 75. This simple conclusion is even stronger if one recalls that, in the case of death before age 75, the fund would remain in the member's estate. In order to check the robustness of our assertions, we have undertaken a further sensitivity analysis in the next section.

The percentiles of the table show that the distribution of the final annuity purchasable at time T is more spread out with the exponential formulation of the target than with the $\tilde{F}(t)$ one. This is also confirmed by the mean and the standard deviation, which are both higher in the exponential case. This results are evident also by looking at the graphs reported in Figure 3, which report the empirical distribution from the 1000 simulations of the final annuity, with both formulations of the targets and with the three levels of b_1. In each histogram, the cases of ruin are collected in a rectangle with base located in the interval $[-1, 0]$ on the x-axis and height equal to the frequency of ruin.

In the $\tilde{F}(t)$ formulation it is possible to derive analytically the density function of the final annuity (observing from section 4 that $X(T) = F - S(T)$ and that the shortfall at time T, $S(T)$, is lognormal). In Figure 3, for the $\tilde{F}(t)$ targets, we have plotted together the density function and the empirical distribution of the final annuity derived from the simulations. Although it is not necessary to show also the empirical distribution when the density function is available, this allows a better comparison with the results of the exponential formulation, considering that the scenarios generated are the same.

Target: $\tilde{F}(t)$ Target: $F_{exp}(t)$

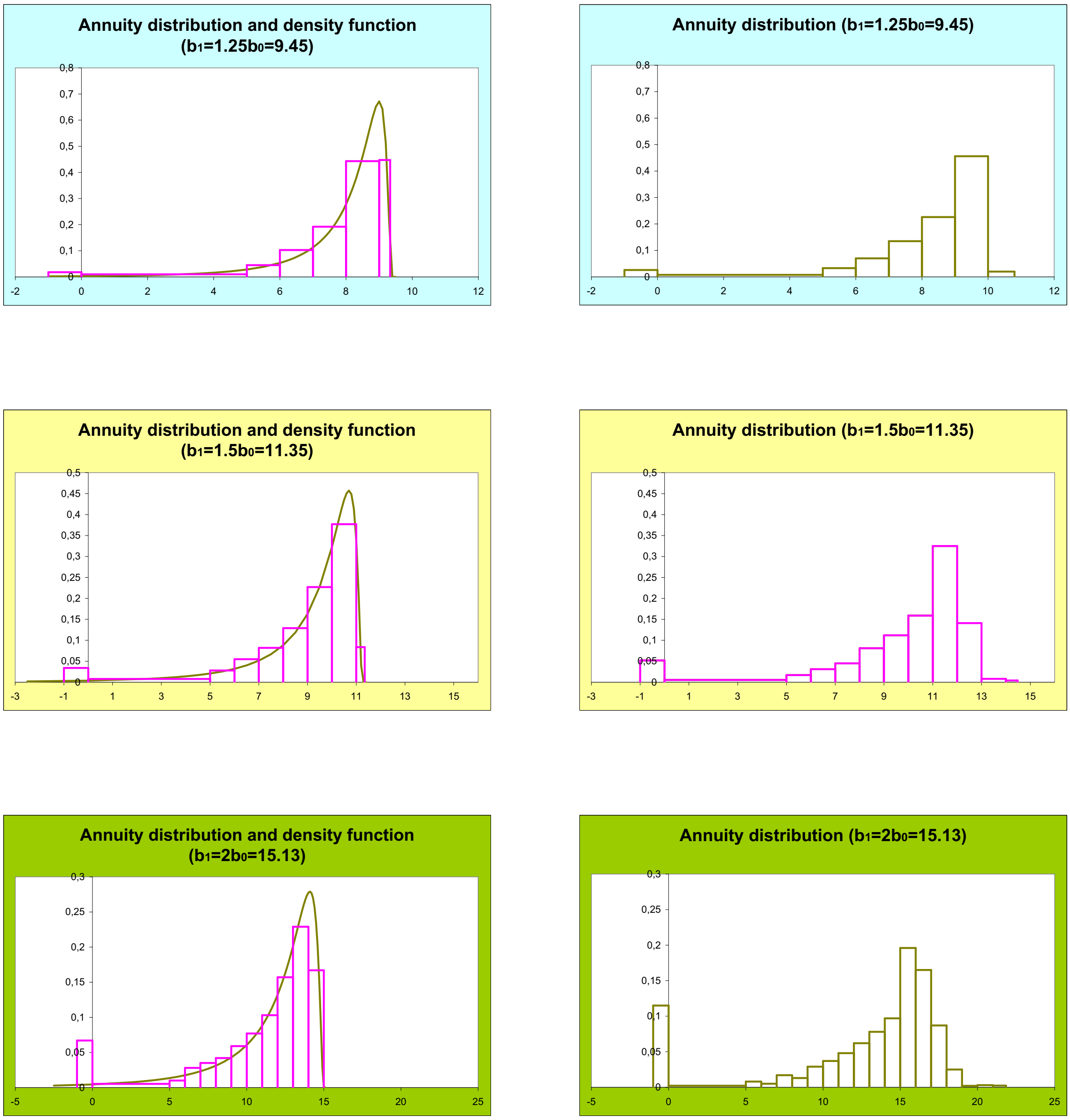

Figure 3: Density function and empirical distribution of the final annuity

As one may expect, the distribution of the final annuity for the $\tilde{F}(t)$ formulation is highly concentrated in the area on the immediate left hand side of b_1: the final target can never be exceeded but the chances of getting very close to it are significantly high. Furthermore, the strategies are more stable than with the exponential targets and the frequency of ruin is lower. On the other hand, with the exponential formulation of the targets, the right tail of the distribution is longer in all cases, leading to better results when the performance of the financial market is positive. In particular, a direct comparison between the two final annuities achieved with the two different formulations in each scenario of market returns shows that with the exponential targets the final annuity purchased is higher than the one provided with the $\tilde{F}(t)$ formulation in 85–90% of the cases (888 cases out of 1000 when $b_1 = 1.25b_0$, 887 when $b_1 = 1.5b_0$, 861 when $b_1 = 2b_0$).

With both formulations of the targets, the distribution moves towards the right when the risk aversion decreases and b_1 increases. By increasing the level of desired income, one has a better chance of being better off than with a lower target. However, the distribution has also a slightly thicker left tail, and this leads to worse results when the financial market performs poorly, and a higher probability of running out of money.

6.5 Sensitivity analysis

We have undertaken a sensitivity analysis with respect to changing the Sharpe ratio, i.e. varying[6]:

$$SR = \frac{\lambda - r}{\sigma}$$

In the base scenario, the assumptions for the parameters of the market returns are: $r = 5\%$; $\lambda = 10\%$; $\sigma = 20\%$. Therefore, we have $SR = 0.25$. We have considered the following values for the Sharpe ratio, by changing either the excess return on the risky asset or the volatility: 0.20, 0.33 and 0.38. As we have shown in section 4.1, when looking at the final annuity distribution it is not important whether the excess return or the volatility is changed (as long as the riskless rate r remains unchanged), because two different assets with the same Sharpe ratio give rise to the same dynamics of the fund and the same distribution of the final annuity.

In what follows, we show results for the $\tilde{F}(t)$ formulation of the targets and $b_1 = 1.5b_0$. The results for the exponential formulation of the targets and for different choices of b_1, available from the authors upon request, show similar trends.

Table 2 reports the probability of ruin, the mean ruin time when ruin occurs, the probability of failing to achieve the initial level of pension b_0, the mean shortfall from b_1 (the shortfall occurs with certainty) and some percentiles and mean and standard deviation of the final annuity for the different values of the Sharpe ratio.

[6] We note that in earlier sections of the paper we have called this ratio β, we now change notation in order to avoid confusion with the known "beta" of an asset (see, for example, Elton & Gruber, 1987), which has a different meaning from the one intended here.

	$SR = 0.2$	$SR = 0.25$	$SR = 0.33$	$SR = 0.38$
probability of ruin	3.6%	3.4%	3.0%	2.4%
mean ruin time when ruin occurs (years)	11.6	10.4	9.5	8.4
probability final annuity $< b_0$	34.1%	19.7%	6.8%	3.7%
mean shortfall from b_1	3.5	2.5	1.4	0.9
final annuity: 5th perc.	2.0	3.4	6.5	8.1
final annuity: 25th perc.	6.8	8.1	9.9	10.5
final annuity: 50th perc.	8.6	9.7	10.7	11.0
final annuity: 75th perc.	9.8	10.5	11.0	11.2
final annuity: 95th perc.	10.5	10.9	11.2	11.2
final annuity: mean	7.9	8.8	10.0	10.4
final annuity: standard deviation	2.6	2.4	2.1	1.9

Table 2: Risk Measures for Different Sharpe Ratio: $\tilde{F}$ and $b_1 = 1.5b_0$

Main comments on the table are:

- the probability of outliving the assets decreases as SR increases, but not so remarkably, suggesting that it is not heavily affected by the value of SR: ruin occurs in 2–4% of the cases for all values of the Sharpe ratio considered;

- the average time of ruin, given that ruin occurs, decreases as SR increases, as we expect; when SR is low, the exposure in the risky asset is very low at the beginning and increases when time passes if the return on the risky asset is low (as it is necessary to invest more in it in order to reach the target). Therefore, if ruin occurs, it tends to occur later after retirement. On the other hand, when SR is high, the portfolio is heavily invested in the risky asset at the beginning of the drawdown plan, and if returns turn out to be adverse immediately after retirement, the high exposure to risk may lead to rapid ruin;

- the probability of having a final annuity lower than the one which could have been purchased at retirement decreases sharply as SR increases. This result is intuitive. This probability is sensitive to the value of SR: it roughly halves when passing from $SR = 0.2$ to $SR = 0.25$, reduces to one third when passing from $SR = 0.25$ to $SR = 0.33$ and halves again when passing from $SR = 0.33$ to $SR = 0.38$. The results of other simulations not shown here report that this probability goes down to 1.7% with a Sharpe ratio of 0.5;

- with a very low value of SR ($SR = 0.2$) the probability of ending up with a lower annuity than the one that could be purchased at retirement is quite high (34.1%). This suggests that the investor should choose the risky asset carefully when adopting optimal strategies (although a Sharpe index of 0.2 is low for a standard equity portfolio). However, we note that this probability sharply decreases to 19.7% when slightly increasing the value of SR ($SR = 0.25$);

- the mean of the final annuity increases and the standard deviation decreases as SR increases, as we would expect. This feature is also clear from inspection of the density function of the final annuity in Figure 4.

Figure 4 reports the density function of the final annuity with the different values of the Sharpe ratio.

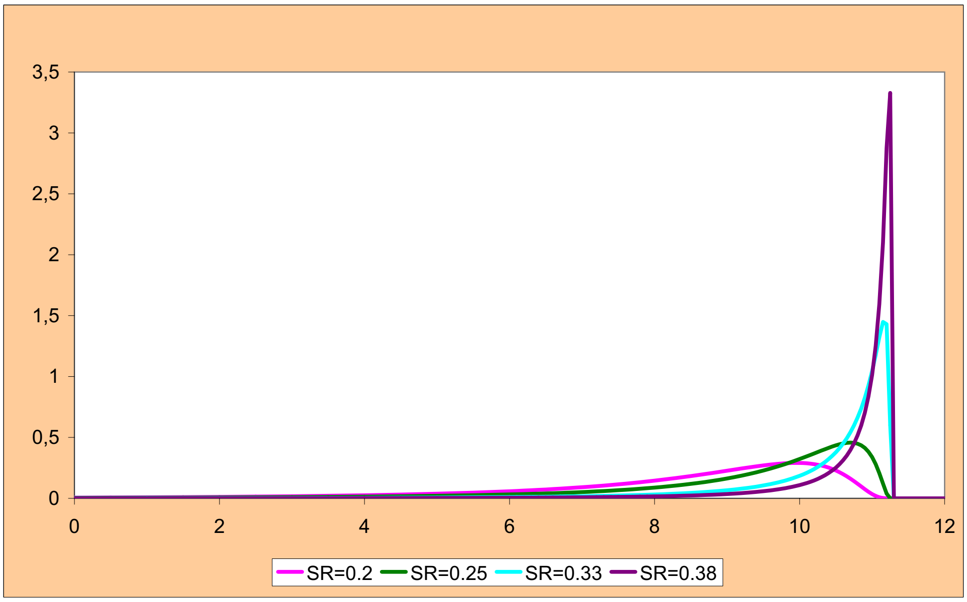

Figure 4: Density function of the final annuity changing the Sharpe ratio.

We note the following features:

- the density function, unsurprisingly, moves to the right when SR increases: substantial improvements in the final annuity distribution are to be expected when SR passes from 0.25 (base scenario) to 0.33 and much better results when it passes from 0.33 to 0.38. On the other hand, the distribution of annuity amounts looks worse, but not remarkably so, when SR reduces to 0.2. This suggests that improvements in the upward direction have a much more significant effect on the distribution of annuity amount than reductions in the downward direction;

- looking at more extreme cases, other simulation results not shown here (but available upon request) show that, with very low values of SR (namely, $SR = 0.125$ and 0.08), the distribution concentrates in the left tail in the range [0, 10], with roughly 70% of the pension incomes being in the range [5, 10] and 25% in the range [0, 5]. Further, the probability of having a final annuity lower than the one that could have been purchased at retirement is greater than 50% (precisely, it is 63% and 86% with $S = 0.125$ and 0.08 respectively). This suggests that immediate annuitization would probably be preferred to income drawdown option in this case; this conclusion is consistent with the results of Yaari (1965) and Mitchell *et al.* (1999). Yaari shows that, in the absence of a bequest motive, one should invest one's own wealth in actuarial notes (a more general form of annuities) rather than in regular notes (a more general form of riskless assets). Mitchell *et al.* (1999) show that the money's worth of annuities traded in the market is high enough to make them more convenient (in terms of expected utility) to the investor in comparison with other investment vehicles. Yaari's regular notes and the assets used by Mitchell *et al.* are riskless assets, and, therefore, the conclusions drawn in these studies can be regarded as extreme cases of the situation examined here. The choice for immediate annuitization could be considered rational and justified also with certain classes of risky assets (securities whose Sharpe ratio is very low);

- the density functions shown indicate that for high values of SR ($SR = 0.33$, $S = 0.38$) the distribution of the final annuity is principally concentrated in the range [10, 11.35]; this is supported by the results of the simulations run, that show that this happens with a frequency of 73% with $SR = 0.33$ and of 85% with SR=0.38. We find that this is a satisfactory result in terms of the appropriateness of the strategy, considering that the desired level of annuity is 11.35 and that values of the Sharpe ratio for risky assets greater or equal than 0.33 are not atypical in the financial market.

7 Conclusions and further research

In this paper, we consider the position of a retiree member of a DC pension scheme who takes the income drawdown option and defers annuitization of the fund until the latest possible age permitted by the scheme's rules or by legislation. We assume, as is common in defining this kind of problem, that the retiree periodically withdraws from the fund the exact amount of money that an annuity bought at retirement would provide, invests the remaining money in the financial market and, when the age of compulsory annuitization comes, purchases the annuity with the remaining fund (if any). Comparisons with immediate annuitization are then possible, since the individual has withdrawn in the deferment period the exact amount that an immediate annuity would have provided.

We assume that there are no bequest motives and that the only reason that leads the pensioner to defer annuitization is the desire of being able to buy a better annuity later in life than the one purchasable at retirement. With this aim in mind, the pensioner sets a certain pension income target, which depends on the level of the income that could have been purchased at retirement, and is equal to or higher than this. A stochastic optimal control problem is defined with a quadratic loss function: the pensioner invests the money in a typical Merton (1969) financial market with a view to minimizing the square of the difference of the fund from the amount needed to provide the desired income. The risk aversion of the individual is also considered, by changing the level of the target in the disutility function. We find the solution of the problem and find a closed form for the optimal investment in the risky asset at any time between retirement and compulsory age, with two different specifications for the target.

We then carry out some Monte Carlo simulations in order to investigate the appropriateness of the optimal investment strategy found in terms of a number of measures of the downside risk borne by the member, including the probability of outliving the assets, the probability of ending up with a worse annuity than the original one, the probability of not being able to buy after the deferment period the desired annuity and also in terms of the distribution of the final annuity which can be purchased at the compulsory age. The simulations have been carried out with three different levels of risk aversion, and a sensitivity analysis has been performed with respect to changes in the Sharpe ratio of the risky asset.

The main results of our investigation are:

- the probability of ruin seems to be more sensitive to the risk profile of the individual rather than to the Sharpe ratio of the risky asset: given a fixed formulation of the target function, the ruin frequency varies more significantly when changing the desired level of pension income b_1 than when changing the Sharpe ratio of the risky asset; this probability varies between 2% and 11%, depending on the risk aversion and the formulation of the interim targets;

- retirees with a high degree of risk aversion or who aim to a target pension which is not too much higher than the one they could receive with immediate annuitization should carefully consider the relatively high probability of being worse off when adopting income drawdown option (almost 25–33%), even if the ruin probability is very small;

- the probability of not being able to buy a better annuity than the one that could have been purchased at retirement is very low for not particularly high values of the Sharpe ratio: it is about 7% for a value of 0.33 and less than 4% for a value of 0.38. However, with lower values such as 0.25 and 0.2 this probability increases to significant values, like 20% and 34%, respectively. This indicates that the income drawdown option should be preferred to immediate annuitization if sufficiently good (in terms of their risk-reward characteristics) risky assets are to be found in the financial market, but should maybe be avoided if the assets available are not good enough. In fact, with very low values of the index this probability is quite high (63% for a value of the index of 0.125 up to 86% when the index is 0.08), suggesting that the risky asset should be properly selected when choosing the portfolio composition. At the extreme, if the risky asset were coincident with the riskless one, the conclusion seems to indicate that immediate annuitization would be optimal, which coincides with the results of Yaari (1965) and Mitchell *et al* (1999);

- with high levels of the Sharpe ratio, the distribution of the final annuity is significantly concentrated in the area at the immediate left hand side of the desired level of the pension income (the distribution is spread out also in the area in the right hand side with the exponential definition of the target);

- the main conclusion seems to be that for a pensioner with a not too high risk aversion, the income drawdown option should be preferred to immediate annuitization, adopting optimal investment strategies with a sufficiently good risky asset.

In the present work we have considered only the portfolio allocation as a control variable; a more extended work would include also the consumption policy as control variable. We have not considered how incorporating mortality in the model can affect the results. We have also solved the problem without constraints on the control variables. In a parallel paper (Gerrard *et al.* 2004), we deal with the enlarged consumption-investment problem, consider a model allowing for mortality and mention how the solution would change when introducing constraints on the control variables. The exact solution of the problem with constraints is the subject of ongoing research.

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