Optimal joint survival reinsurance: an efficient frontier approach

by

Dimitrina S. Dimitrova and Vladimir K. Kaishev*

Faculty of Actuarial Science and Insurance
Cass Business School, City University
106 Bunhill Row,
London EC1Y 8TZ
UK

*Corresponding author: V.Kaishev@city.ac.uk,
tel: +44 (0)2070408453

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Abstract

The problem of optimal excess of loss reinsurance with a limiting and a retention level is considered. It is demonstrated that this problem can be solved, combining specific risk and performance measures, under some relatively general assumptions for the risk model, under which the premium income is modelled by any non-negative, non-decreasing function, claim arrivals follow a Poisson process and claim amounts are modelled by any continuous joint distribution. As a performance measure, we define the expected profits at time x of the direct insurer and the reinsurer, given their joint survival up to x, and derive explicit expressions for their numerical evaluation. The probability of joint survival of the direct insurer and the reinsurer up to the finite time horizon x is employed as a risk measure. An efficient frontier type approach to setting the limiting and the retention levels, based on the probability of joint survival considered as a risk measure and on the expected profit given joint survival, considered as a performance measure is introduced. Several optimality problems are defined and their solutions are illustrated numerically on several examples of appropriate claim amount distributions, both for the case of dependent and independent claim severities.

Keywords: optimal excess of loss reinsurance, probability of ruin, Appell polynomials, joint survival of cedent and reinsurer, expected profit, efficient frontier, copula functions

1. Introduction

An upward trend in insurance and reinsurance claims frequency and severity has recently been observed, mostly due to catastrophic events, such as hurricane Katrina in the USA in 2005 and the winterstorm Kirill over northern Europe in 2007, causing enormous damage to households and infrastructure, measured in billions of dollars. As a result of this, both the insurance and reinsurance industry suffered severe losses, (see e.g. Zanetti, Schwarz and Lindemuth 2007 for an up-to-date account on world largest losses), and some companies became even insolvent. In order to cope with increasing future catastrophic risk, the industry faces the necessity of improving their internal risk models and especially, their implementation and use in the context of reinsurance. In particular, it becomes more clear that such models have to incorporate the interests of both insurance and reinsurance companies in order for them to maximize their chances of (joint) survival.

Coherent with these developments are the recent attempts in the actuarial literature to introduce joint risk and performance measures which can be used in determining the parameters of a reinsurance contract. Such reinsurance optimality criteria were first considered by Ignatov et al. (2004) and Kaishev and Dimitrova (2006). Along with these studies, extensive research on optimal reinsurance solely from the point of view of the direct insurer is carried on. More recent examples in this direction are the papers by Kaluszka (2004) and Verlaak and Beirlant (2003), who study mean-variance optimality criteria, Gajek and Zagrodny (2004a), Cao and Zhang (2007) and Balbás et al. (2009) who look at general risk measures, and Guerra and Centeno (2008), Liang and Guo (2007), Gajek and Zagrodny (2004b), and Schmidli (2004) where the risk is measured by the probability of ruin. A summary on the variety of research techniques used in setting optimal reinsurance arrangements and further references can be found in Centeno (2004), Aase (2002), Ignatov et al. (2004) and Balbás et al. (2009).

Recently, Ignatov et al. (2004) and Kaishev and Dimitrova (2006) considered a reinsurance optimality model, which combines the (contradicting) interests of both the cedent and the reinsurer under an excess of loss contract. Under this model, claims generated by a volume of risks arrive according to a Poisson process and the two parties share each individual claim and the total premium income in such a proportion that a certain joint optimality criterion is maximized (minimized). In their paper, Ignatov et al. (2004), assumed that claim severities have any discrete joint distribution and considered a simple excess of loss without a policy limit. As a joint risk measure they proposed to use the probability of joint survival of the cedent and the reinsurer up to a finite time horizon and derived explicit expressions for this probability. As a joint performance measure, the expected profit of each of the parties at a finite-time horizon, given their joint survival up to this instant has also been considered.

The model has been extended further in the paper by Kaishev and Dimitrova (2006), where it was assumed that claim amounts have any continuous (dependent) joint distribution and the excess of loss has a retention and a policy limit. Under these assumptions, closed form expressions for the probability of joint survival have been derived. Based on these expressions, it was demonstrated that retention and limiting levels could be optimally set by maximizing the probability of joint survival, given the premium income is split in a preassigned proportion or alternatively, an optimal split of the premium income between the two parties could be determined, given fixed retention and limiting levels.

In the present paper, we consider the model of Kaishev and Dimitrova (2006) and propose a Markowitz type efficient frontier solution to the problem of optimally setting the retention and limiting levels M and L, so that for a given level of the probability of joint survival the expected profits of the two parties are maximized. As an alternative, it is proposed to use an optimality criterion which provides for 'fair' distribution of the expected profits based on the agreed allocation of the premium income. In order to implement these ideas, we derive explicit expressions for the expected profit of the cedent and the reinsurer at some future moment in time, given their joint survival up to this instant.

The paper is organized as follows. In section 2, we briefly introduce the model and recall the formulae for the probability of joint survival of Kaishev and Dimitrova (2006). In section 3, explicit expressions for the expected profits of the direct insurer and the reinsurer are derived. The optimality problems, which incorporate these joint risk and performance measures, are formulated in section 4 and their efficient frontier solutions are illustrated. Section 5 concludes the paper with some comments on the results and possibilities of future research.

2. The excess of loss (XL) risk model of joint survival

2.1 The model

We consider an insurance portfolio, generating claims at some random moments of time. The claims inter-arrival times $\tau_1,\ \tau_2,\ ....$ are assumed identically, exponentially distributed r.v.s with parameter λ. Denote by $T_1 = \tau_1,\ T_2 = \tau_1 + \tau_2,\ ...$ the sequence of random variables representing the consecutive moments of occurrence of the claims. Let $N_t = \#\{i : T_i \le t\}$, where $\#$ is the number of elements of the set $\{.\}$. The claim severities are modeled by the continuous r.v.s. W_1, W_2, ..., W_k, ... with joint density function $\psi(w_1,\ ...,\ w_k)$. For convenience, we will introduce also the random variables $Y_1 = W_1,\ Y_2 = W_1 + W_2,\ ...$ representing the partial sums of consecutive claim amounts.

It is assumed that the r.v.s W_1, W_2, ... are independent of N_t. Then, the risk (surplus) process R_t, at time t, is given by $R_t = h(t) - Y_{N_t}$, where $h(t)$ is a nonnegative, non-decreasing, real function, defined on $\mathbb{R}_+$, representing the aggregate premium income up to time t. The function $h(t)$ may be continuous or not. If $h(t)$ is discontinuous, we define $h^{-1}(y) = \inf\{z : h(z) \ge y\}$. Note that the classical case $h(t) = u + c\,t$, with initial reserve u and premium rate c, is included in this rather general class of functions $h(t)$.

In this paper, we will be concerned with the case when the insurance company wants to reinsure its portfolio of risks by concluding an XL contract with a retention level $M \ge 0$ and a limiting level $L \ge M$. In other words, the cedent wants to reinsure the part of each claim which hits the layer $m = L - M$, i.e. each individual claim W_i is shared between the two parties so that

$W_i = W_i^c + W_i^r,\ i = 1, 2,\ ...,$ where W_i^c and W_i^r denote the parts covered respectively by the cedent and the reinsurer. Clearly, we can write

$$W_i^c = \min(W_i, M) + \max(0, W_i - L)$$

and

$$W_i^r = \min(L - M, \max(0, W_i - M)).$$

Denote by $Y_1^c = W_1^c$, $Y_2^c = W_1^c + W_2^c$, ... and by $Y_1^r = W_1^r$, $Y_2^r = W_1^r + W_2^r$, ... the consecutive partial sums of claims to the cedent and to the reinsurer, respectively. Under our XL reinsurance model, the total premium income $h(t)$ is also divided between the two parties so that $h(t) = h_c(t) + h_r(t)$, where $h_c(t)$, $h_r(t)$ are the premium incomes of the cedent and the reinsurer, assumed also non-negative, non-decreasing functions on $\mathbb{R}_+$. As a result, the risk process, R_t, can be represented as a superposition of two risk processes, that of the cedent

$$R_t^c = h_c(t) - Y_{N_t}^c \tag{1}$$

and of the reinsurer

$$R_t^r = h_r(t) - Y_{N_t}^r \tag{2}$$

i.e., $R_t = R_t^c + R_t^r$. Note that the two risk processes R_t^c and R_t^r are dependent through the common claim arrivals and the claim severities W_i, $i = 1, 2,\ ...$, as seen from (1) and (2).

Under this model, explicit formulae for the probability of joint survival, $P(T^c > x, T^r > x)$, of the cedent and the reinsurer within a finite time interval $[0, x]$, $x > 0$, were derived by Kaishev and Dimitrova (2006). The moments, T^c and T^r, of ruin of correspondingly the cedent and the reinsurer are defined as

$$T^c := \inf\ \{t : t > 0,\ R_t^c < 0\},$$

$$T^r := \inf\ \{t : t > 0,\ R_t^r < 0\}.$$

Clearly, the two events ($T^c > x$) and ($T^r > x$), of survival of the cedent and the reinsurer are dependent and hence, $P(T^c > x, T^r > x)$, is a meaningful measure of the risk the two parties share and jointly carry.

In section 2.3, we will define the expected profit for each of the two parties, given joint survival up to time x, and show how this performance measure can be used in combination with the risk measure $P(T^c > x, T^r > x)$ in finding the optimal set of parameters related to an XL reinsurance contract.

2.2 The probability of joint survival

There are two alternative optimization problems which have been stated in connection with the XL contract, considered here. The first is, given M and m are fixed, divide the premium income $h(t)$ between the two parties, so as to maximize the probability of joint survival, $P(T^c > x, T^r > x)$. And alternatively, if the total premium income, $h(t)$, is divided in an agreed way between the cedent and the reinsurer, i.e. $h_c(t)$ and $h_r(t) = h(t) - h_c(t)$ are fixed, set the parameters M and L of the XL contract so as to maximize $P(T^c > x, T^r > x)$. Obviously, both optimization problems are based solely on the joint risk measure $P(T^c > x, T^r > x)$. To address these problems, Kaishev and Dimitrova (2006) derived explicit expressions for $P(T^c > x, T^r > x)$ given by the following theorems.

Theorem 1. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$P(T^c > x, T^r > x) =$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{v}_1,\ \ldots,\ \tilde{v}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1\right) \tag{3}$$

where

$\tilde{\nu}_j = \min(\tilde{z}_j, x), \ \tilde{z}_j = \max\left(h_c^{-1}\left(y_j^c\right), h_r^{-1}\left(y_j^r\right)\right), \ y_j^c = \sum_{i=1}^{j} w_i^c, \ y_j^r = \sum_{i=1}^{j} w_i^r, \ j = 1, \ ..., k,$

$w_i^c = \min(w_i, M) + \max(0, w_i - L), \ w_i^r = \min(L - M, \max(0, w_i - M))$, *and*

$A_k(x; \tilde{\nu}_1, \ ..., \tilde{\nu}_k)$, $k = 1, 2, \ ...$ *are the classical Appell polynomials* $A_k(x)$ *of degree* k*, defined by*

$$A_0(x) = 1, \qquad A_k'(x) = A_{k-1}(x), \qquad A_k(\tilde{\nu}_k) = 0.$$

For further properties of Appell polynomials we refer to Kaz'min (2002). An alternative formula for $P(T^c > x, T^r > x)$ is provided by the following

Theorem 2. *The probability of joint survival of the cedent and the reinsurer up to a finite time* x *under an XL contract with a retention level* M *and a limiting level* L *is*

$$P(T^c > x, T^r > x) =$$

$$e^{-\lambda x}\left(\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1, \ ..., \tilde{z}_{l-1}, x)\, \psi(w_1, \ ..., w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1\right) \tag{4}$$

where

$B_l(\tilde{z}_1, \ ..., \tilde{z}_{l-1}, x) = \sum_{j=0}^{l-1} (-\lambda)^j b_j(\tilde{z}_1, \ ..., \tilde{z}_j)\left(\sum_{m=0}^{l-j-1} \frac{(x\lambda)^m}{m!}\right)$, with $B_0(\cdot) \equiv 0, \ B_1(\cdot) = 1$, l *is such that* $\tilde{z}_1 \leq \ ... \leq \tilde{z}_{l-1} \leq x < \tilde{z}_l$,

$b_j(\tilde{z}_1, \ ..., \tilde{z}_j) = \sum_{i=1}^{j} (-1)^{j+i} \frac{\tilde{z}_j^{j-i+1}}{(j-i+1)!} b_{i-1}(\tilde{z}_1, \ ..., \tilde{z}_{i-1})$, *with* $b_0 \equiv 1$,

$\tilde{z}_j$ *are defined as in Theorem 1.*

As noted in Kaishev and Dimitrova (2006), the above two expressions can be used interchangeably and depending on the specified parameters and the software used for implementation either (3) or (4) can be faster and less computationally involved.

In the next section, we will supplement the risk measure $P(T^c > x, T^r > x)$ by a performance measure and in section 3 we will demonstrate how the two measures can be combined into a single optimization problem, which incorporates the contradictory goals of maximizing the profit and minimizing the risk of the cedent and the reinsurer.

3. The expected profit given joint survival

Under the general model of an XL contract with a retention level M and a limiting level L, and assuming claims have any continuous joint distribution, we will be concerned here with the profit at time x, each of the parties are expected to make, given they both survive up to x. Considering a joint optimality criterion, based on expected profit given joint survival, is reasonable since with the eventual ruin of either of the parties the XL reinsurance contract will cease and this will affect the risk and profitability of the surviving party. So, obviously the two parties have mutually dependent performance with respect not only to the risk they carry but also with respect to their expected profits. Expected profit assuming joint survival was first considered by Ignatov et al. (2004) in the case of a simple XL contract with one retention level and discrete integer-valued claims.

In what follows, we will present some explicit expressions for these quantities and a result establishing the existence of values of M and L such that the expected profits of the two parties are in the same proportion as their premium incomes. First, we will introduce some useful definitions and notation. Following Ignatov et al. (2004), we will define the profits at time x of the cedent and the reinsurer, correspondingly as the values, R_x^c and R_x^r, of their risk processes, given by (1) and (2), at time x. Denote by I_A and I_B the indicator random variables of the events $A = \{T^c > x\}$ and $B = \{T^r > x\}$. There exists a suitable function $\phi(u, v)$ such that the conditional expectation $E(R_x^c \mid I_A, I_B) \overset{a.s.}{=} \phi(I_A, I_B)$. When $I_A \equiv 1$ and $I_B \equiv 1$, we obtain $\phi(1, 1) = E[R_x^c \mid (T^c > x, T^r > x)]$ which we will call the expected profit of the cedent at time x, given the two parties' joint survival up to time x. Similarly, $E[R_x^r \mid (T^c > x,\ T^r > x)]$ denotes the reinsurer's expected profit at time x, given its and the insurer's joint survival up to time x.

The following two theorems give explicit expressions for $E[R_x^c \mid (T^c > x,\ T^r > x)]$ and $E[R_x^r \mid (T^c > x,\ T^r > x)]$ correspondingly.

Theorem 3. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E[R_x^c \mid (T^c > x,\ T^r > x)] =$$

$$h_c(x) - \left\{ \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^c\, A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1 \right\} \Bigg/ \tag{5}$$

$$\left\{ 1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1 \right\}$$

where $y_k^c,\ \tilde{\nu}_j,\ j = 1,\ \ldots,\ k$ and $A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)$ *are defined as in Theorem 1.*

Proof. In view of the definitions (1) and (2) of the risk processes R_t^c and R_t^r, and expression (3) for the probability of joint survival, we can express the unconditional expectation $E(R_x^c \cdot I_A \cdot I_B)$ as

$$E(R_x^c \cdot I_A \cdot I_B) =$$

$$e^{-x\lambda} \left\{ h_c(x) + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} \left(h_c(x) - \sum_{i=1}^{k} w_i^c \right) A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1 \right\} \tag{6}$$

Note that in equality (6), if k claims have occurred up to time x, where $k = 1, 2, \ldots$, the profit of the cedent at the end of the time horizon $[0, x]$ is equal to $h_c(x) - \sum_{i=1}^{k} w_i^c$, and if no claims have occurred, i.e. $k = 0$, the profit is equal to the premium income at time x, i.e. $h_c(x)$, which is accounted for by the first term of the sum in (6). The unconditional expectation (6) can be rewritten as

$$E(R_x^c \cdot I_A \cdot I_B) = e^{-x\lambda}\, h_c(x) \left\{1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,; \tilde{v}_1, \ldots, \tilde{v}_k)\ \psi(w_1, \ldots, w_k)\, d w_k \ldots d w_2\, d w_1 \right\} - \tag{7}$$

$$e^{-x\lambda} \left\{\sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} \left(\sum_{i=1}^{k} w_i^c\right) A_k(x\,; \tilde{v}_1, \ldots, \tilde{v}_k)\ \psi(w_1, \ldots, w_k)\, d w_k \ldots d w_2\, d w_1 \right\}$$

For the conditional expectation $E[R_x^c \mid (T^c > x,\ T^r > x)]$ we have

$$E[R_x^c \mid (T^c > x,\ T^r > x)] = \frac{E\,(R_x^c \cdot I_A \cdot I_B)}{P(T^c > x,\ T^r > x)} \tag{8}$$

Substituting (7) and (3) in (8), and after cancelling appropriate terms, recalling the notation $\sum_{i=1}^{k} w_i^c = y_k^c$, we obtain the assertion of the theorem.□

Similarly, for the expected profit of the reinsurer we have

Theorem 4. *The expected profit of the reinsurer at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$$E[R_x^r \mid (T^c > x,\, T^r > x)] =$$

$$h_r(x) - \left\{ \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^r\, A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1 \right\} \Big/ \qquad (9)$$

$$\left\{ 1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots dw_2\, dw_1 \right\}$$

where $w_i^r,\ \tilde{\nu}_j,\ j = 1,\ \ldots,\ k$ and $A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)$ *are defined as in Theorem 1.*

Proof. The proof follows the same lines of reasoning as in Theorem 3, replacing the premium income and the claims to the cedent with the ones to the reinsurer.□

Alternative formulae for $E[R_x^c \mid (T^c > x,\, T^r > x)]$ and $E[R_x^r \mid (T^c > x,\, T^r > x)]$ can be derived using expression (4) for $P(T^c > x,\, T^r > x)$ and its derivation. They are given in the next two theorems.

Theorem 5. *The expected profit of the cedent at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$E[R_x^c \mid (T^c > x,\ T^r > x)] =$

$$h_c(x) - \Big\{\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} y_k^c\, B_l(\tilde{z}_1,\ \ldots,\ \tilde{z}_{l-1},\ x)$$

$$\psi(w_1,\ \ldots,\ w_k)\, dw_k\, dw_{k-1}\ldots dw_2\, dw_1\Big\} \Big/ \qquad (10)$$

$$\Big\{\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1,\ \ldots,\ \tilde{z}_{l-1},\ x)$$

$$\psi(w_1,\ \ldots,\ w_k)\, dw_k\, dw_{k-1}\ldots dw_2\, dw_1\Big\}$$

where $\tilde{z}_j$ and $B_l(\tilde{z}_1,\ \ldots,\ \tilde{z}_{l-1},\ x)$ *are defined as in Theorem 2.*

Theorem 6. *The expected profit of the reinsurer at time x, under an XL contract with a retention level M and a limiting level L, given the joint survival of the cedent and the reinsurer up to time x, is*

$E[R_x^r \mid (T^c > x,\ T^r > x)] =$

$$h_r(x) - \Big\{\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} y_k^r\, B_l(\tilde{z}_1,\ \ldots,\ \tilde{z}_{l-1},\ x)$$

$$\psi(w_1,\ \ldots,\ w_k)\, dw_k\, dw_{k-1}\ldots dw_2\, dw_1\Big\} \Big/ \qquad (11)$$

$$\Big\{\sum_{k=1}^{\infty}\int_0^{h(x)}\int_0^{h(x)-w_1}\cdots\int_0^{h(x)-w_1-\ldots-w_{k-2}}\int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} B_l(\tilde{z}_1,\ \ldots,\ \tilde{z}_{l-1},\ x)$$

$$\psi(w_1,\ \ldots,\ w_k)\, dw_k\, dw_{k-1}\ldots dw_2\, dw_1\Big\}$$

where $\tilde{z}_j$ and $B_l(\tilde{z}_1,\ \ldots,\ \tilde{z}_{l-1},\ x)$ *are defined as in Theorem 2.*

As with (3) and (4) for $P(T^c > x,\ T^r > x)$, the expressions (5), (9) and (10), (11) can be used interchangeably and depending on the specified parameters and the software used for implementation either of them can converge faster and be less computationally involved.

4. Combining the risk and performance measures in setting an optimal XL contract

In this section, we will illustrate how the probability of joint survival up to time x and the expected profits at time x, given joint survival of the cedent and the reinsurer up to x, can be used in combination, correspondingly as risk and performance measures, in order to set (optimally) the parameters of an XL reinsurance contract. Our approach is motivated by the mean-variance, portfolio optimization model of Markowitz (1952), in which an efficient frontier is found where the expected return from an investment portfolio over the investment horizon x is maximized for a given level of risk, measured by the variance of the portfolio return.

We outline and discuss several alternative approaches of solving the optimal XL reinsurance problem. The solution under any of them is obtained as a reasonable compromise between the contradictory risk and performance optimality criteria. On one hand, it is in the interest of the direct insurance company to possibly maximize the risk and minimize the premium income it transfers to the reinsurer. On the other hand, the reinsurance company aims at minimizing the risk and maximizing the portion of the premium it charges. In this way, both companies are aiming at optimizing their individual risk and performance measures. At the same time, it is reasonable to assume that the two parties are rational investors and hence, are interested in decreasing their joint probability of ruin and increasing their expected profits, given joint survival. Here, we state three problems which illustrate different approaches for determining the values of the retention and the limiting levels, M and L, given a split of the premium income $h(t) = h_c(t) + h_r(t)$, which balances the conflicting goals of the cedent and the reinsurer. The complexity of the expressions derived in Theorems 1 to 6 precludes the possibility of solving the stated problems analytically but as we will see, finding the numerical solutions is straightforward. For convenience, throughout this section we will use the notation $m = L - M$ for the layer covered by the reinsurer.

In order to exemplify these approaches, formulae (3), (4), (5), (9), (10) and (11), given by Theorems 1 to 6, were implemented in *Mathematica* under two sets of model assumptions: one with independent exponentially distributed claim amounts and one with dependent claim severities, modelled by a Rotated Clayton Copula, $C^{\mathrm{RCl}}(F(w_1), ..., F(w_k); \theta)$, with $F \equiv$ Weibull(α, β) marginals and dependence parameter θ. In this way, we are able to study also the effect of dependence on the choice of the parameters of an XL contract. In both cases, we have assumed linear premium income function $h(t) = u + c\,t$, where u is the total initial reserve and c is the total premium rate per unit of time.

A random sample of 500 simulated data points from a bivariate Rotated Clayton copula, with dependence parameter $\theta = 1$ and Weibull(2.12, 1.14) marginals is presented in Fig.1. One of the properties of this particular type of copula is that it has an upper tail dependence and therefore, in our context it models positive dependence between large claim amounts. We refer the reader to Kaishev and Dimitrova (2006), where the expressions for a multidimensional Rotated Clayton copula and its density, together with some further applications in modelling dependence among claims severities, can be found.

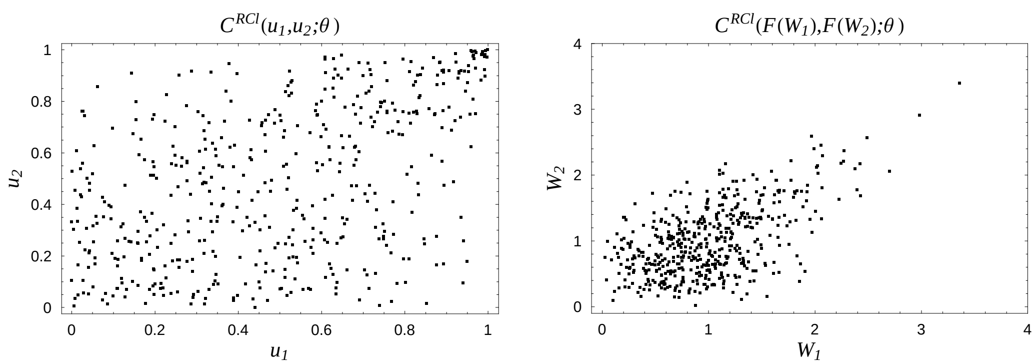

Fig. 1. A random sample of 500 simulations from a bivariate Rotated Clayton copula, with dependence parameter $\theta = 1$, marginals $F \equiv$ Weibull(2.12, 1.14).

Being able to calculate $P(T^c > x, T^r > x)$, $E[R^c_x \mid (T^c > x, T^r > x)]$ and $E[R^r_x \mid (T^c > x, T^r > x)]$, the 'individual' approach of the cedent and the reinsurer for finding optimal values of M and m, given $h(t) = h_c(t) + h_r(t)$, can be formulated as follows.

Problem 1. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$, find

$$\max_{M,m} \; E[.\,|\,(T^c > x,\, T^r > x)] \qquad \text{subject to} \quad P(T^c > x,\, T^r > x) = p\,. \tag{12}$$

The expectation $E[.\,|\,(T^c > x,\, T^r > x)]$ in (12) is taken with respect to either R_x^c or R_x^r.

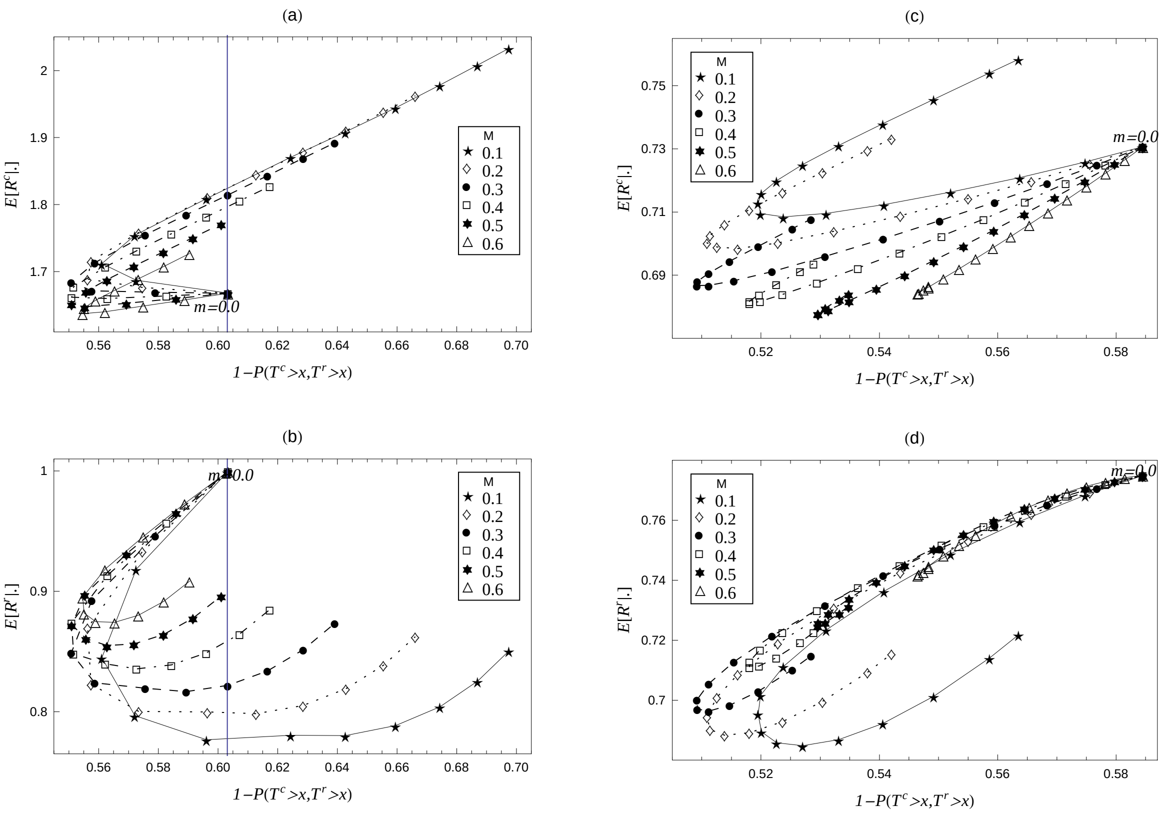

Fig. 2. $E[R_x^c\,|\,(T^c > x,\, T^r > x)]$ and $E[R_x^r\,|\,(T^c > x,\, T^r > x)]$ respectively plotted against $1 - P(T^c > x,\, T^r > x)$ in the case of: (a) and (b) - independent claim severities, Exp(1) distributed, $m = 0.0, 0.1, 0.2, \ldots, 1.0$, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.5$; (c) and (d) - dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), \ldots, F(w_k); \theta)$ distributed with $F \equiv \text{Weibull}(2.12, 1.14)$ marginals and $\theta = 1$, $m = 0.0, 0.05, 0.1, \ldots, 0.8$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$.

Solving Problem 1 simply means that the cedent and the reinsurer would choose points (M^c, m^c) and (M^r, m^r) respectively from their 'individual' efficient frontiers. The efficient frontier in our context is the set of dominant pairs of retention and limiting levels, (M, L), in the sense that the latter provide the highest return, measured by $E[.\,|\,(T^c > x,\, T^r > x)]$, for a chosen level of risk, measured by $1 - P(T^c > x,\, T^r > x)$.

The solution of Problem 1 is illustrated in Fig. 2, where it is assumed that the risk for each of the two parties of the XL reinsurance contract is measured by the complement of the probability of their joint survival up to time x. The probability of joint survival up to x in (12) should be fixed by the cedent and the reinsurer to an acceptable value p according to their 'joint' level of risk aversion. It is obvious that, given $h(t) = h_c(t) + h_r(t)$ and fixed level p, such an 'individual' approach may not lead to one and the same optimal solution (M, m), since the interests of the two parties are contradictory. As can be seen from Fig. 2, if $p = p^* = \max_{M,m} P(T^c > x,\, T^r > x) = \min_{M,m}(1 - P(T^c > x,\, T^r > x))$ the solution to Problem 1 will be one and the same for the two parties and will coincide with the solution of Problem 1 of Kaishev and Dimitrova (2006). However, as seen from Fig. 2 (a) and (b), in the case of i.i.d. $\mathrm{Exp}(1)$ distributed claim amounts for instance, if $p = 0.603$ (the vertical blue line in Fig. 2 (a) and (b)), the reinsurer's solution is any pair $(M, 0)$, a solution which is unacceptable for the direct insurer and indeed, leads to its lowest expected profit for this level of risk. It should be noted that in the Exp(1) iid case for example, the values of $E[.\,|\,(T^c > x, T^r > x)]$ and $P(T^c > x,\, T^r > x)$ for the pairs $(M, 1.0)$, $(M, 1.1)$, $(M, 1.2)$, … coincide, so the curves in Fig. 2 (a) and (b) are complete and do not extend further in any direction.

Table 1. Optimal values of M and m, maximizing $E[R_x^c\,|\,(T^c > x, T^r > x)]$ or $E[R_x^r\,|\,(T^c > x, T^r > x)]$ respectively subject to $P(T^c > x,\, T^r > x) = 1 - p$, in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.5$.

$max_{M,m}\ E[.\,|\,(T^c > x, T^r > x)]$	$p^* = 0.551$	$p = 0.585$	$p = 0.603$	$p = 0.70$
(M^c, m^c)	(0.3, 0.3)	(0.2, 0.4)	(0.1, 0.4)	(0.1, 1.5)
(M^r, m^r)	(0.3, 0.3)	(0.8, 0.2)	$(M, 0)$	(0.1, 1.5)

Tables 1 and 2 provide a list of solutions (M^c, m^c) and (M^r, m^r) of optimality problem (12) for different levels p, in the cases of independent and dependent claims severities, illustrated in Fig.

2 (a), (b) and (c), (d) respectively.The optimal values are given for the grid considered. However, they do not change significantly if the grid is refined, as our numerical tests confirm.

Table 2. Optimal values of M and m, maximizing $E[R_x^c \mid (T^c > x, T^r > x)]$ or $E[R_x^r \mid (T^c > x, T^r > x)]$ respectively subject to $P(T^c > x, T^r > x) = 1 - p$, in the case of dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), ..., F(w_k); \theta)$ distributed with $F \equiv$ Weibull(2.12, 1.14) marginals and $\theta = 1$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.775$.

$max_{M,m}\ E[. \mid (T^c > x, T^r > x)]$	$p^* = 0.509$	$p = 0.515$	$p = 0.54$	$p = 0.56$
(M^c, m^c)	(0.3, 0.5)	(0.2, 0.4)	(0.1, 0.6)	(0.1, 0.8)
(M^r, m^r)	(0.3, 0.5)	(0.3, 0.4)	(0.4, 0.3)	(0.5, 0.2)

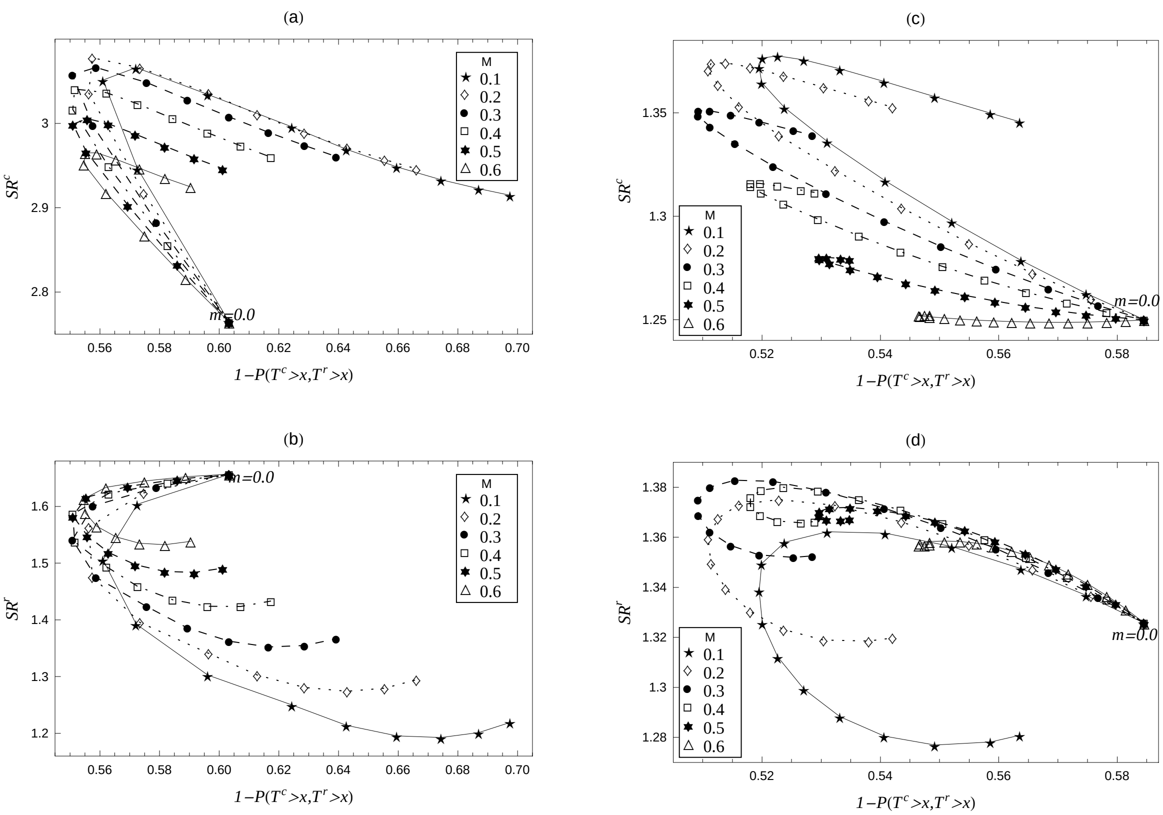

Fig. 3. SR^c and SR^r respectively plotted against $1 - P(T^c > x, T^r > x)$ in the case of: (a) and (b) - independent claim severities, Exp(1) distributed, $m = 0.0, 0.1, 0.2, ..., 1.0$, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.5$; (c) and (d) - dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), ..., F(w_k); \theta)$ distributed with $F \equiv$ Weibull(2.12, 1.14) marginals and $\theta = 1$, $m = 0.0, 0.05, 0.1, ..., 0.8$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.775$.

It has to be noted that, instead of solving (12), an alternative 'individual' approach for each of the two parties could be to try and find their set of values (M', m') which gives the highest 'return per unit of risk taken'. The latter means that (M', m') would provide the highest Sharpe ratio, defined as $\mathrm{SR}^c = E[R_x^c \mid (T^c > x,\, T^r > x)] / (1 - P(T^c > x,\, T^r > x))$ and $\mathrm{SR}^r = E[R_x^r \mid (T^c > x,\, T^r > x)] / (1 - P(T^c > x,\, T^r > x))$ respectively. However, this would again lead to possibly two different optimal solutions, ($M^{c}{'}$, $m^{c}{'}$) and ($M^{r}{'}$, $m^{r}{'}$), for the direct insurer and the reinsurer respectively and therefore, it suffers the same drawback as Problem 1. For instance, in Fig. 3 (c) and (d) we see that the combination (0.1, 0.5) gives the maximum value of SR^c, whereas $\max \mathrm{SR}^r$ is achieved for (0.3, 0.4).

Another approach to the optimal reinsurance problem, which gives a common solution (M', m') for the two parties involved in an XL reinsurance arrangement, could be to use the total expected profit of the cedent and the reinsurer as an optimization criterion for finding values of M and m, given $h(t) = h_c(t) + h_r(t)$. Namely, the optimality problem could be to find

$$\max_{M,\,m} \{E[R_x^c \mid (T^c > x,\, T^r > x)] + E[R_x^r \mid (T^c > x,\, T^r > x)]\} \qquad (13)$$
$$\text{subject to } P(T^c > x,\, T^r > x) = p\,.$$

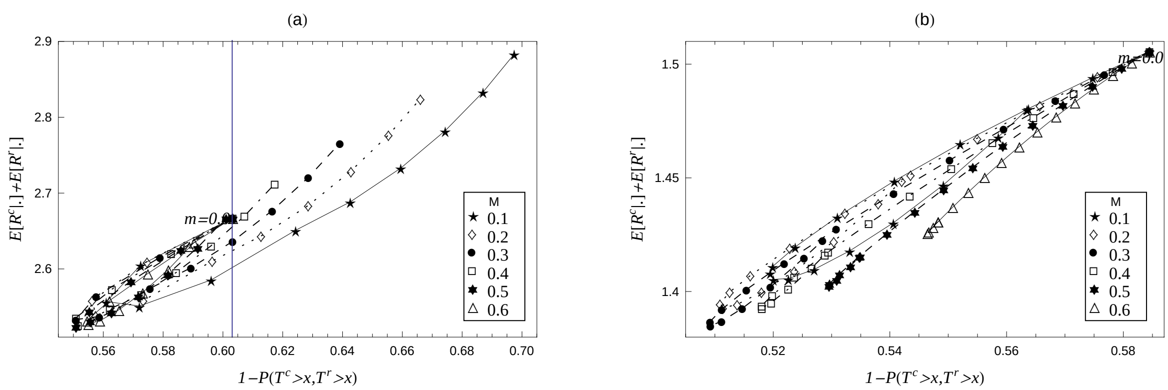

Fig. 4. $E[R_x^c \mid (T^c > x,\, T^r > x)] + E[R_x^r \mid (T^c > x,\, T^r > x)]$ plotted against $1 - P(T^c > x,\, T^r > x)$ in the case of: (a) - independent claim severities, Exp(1) distributed, $m = 0.0, 0.1, 0.2, \ldots, 1.0$, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.5$; (b) - dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), \ldots, F(w_k); \theta)$ distributed with $F \equiv$ Weibull(2.12, 1.14) marginals and $\theta = 1$, $m = 0.0, 0.05, 0.1, \ldots, 0.8$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$.

However, such a criterion seems not to be 'fair' with respect to both the cedent and the reinsurer, since as can be seen form Fig. 4, depending on the level p, (13) could be maximized due to maximizing the expected profit of only one of the two parties at the expense of the other. For example, when $p = 0.603$ (the vertical blue line in Fig. 4 (a)) a solution of (13) is any point $(M, 0)$, which is not adequate for the cedent, as has been already mentioned with respect to Problem 1, since it pays a non-zero reinsurance premium against zero reinsurance coverage. In Fig. 4 (a) and (b), the contradictory goals of maximizing $P(T^c > x, T^r > x)$ and maximizing $E[R_x^c \mid (T^c > x, T^r > x)] + E[R_x^r \mid (T^c > x, T^r > x)]$, as functions of M and m, are also illustrated.

In fact, optimality problem (13) does not explicitly take into account the information of how the premium income $h(t)$ is split between the two parties. The conditional on joint survival up to x, expected profits of the cedent, $E[R_x^c \mid (T^c > x, T^r > x)]$, and of the reinsurer, $E[R_x^r \mid (T^c > x, T^r > x)]$, can be used in defining the following criterion for optimally setting the XL levels M and L, which takes into account the way in which $h(t)$ is split and transfers it into the ratio of the expected profits at time x.

Problem 2. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$ with $h_c(t) = \alpha\, h(t)$, $h_r(t) = (1-\alpha)\, h(t)$, $0 \le \alpha \le 1$, i.e. given that at any $t \ge 0$ the cedent retains $100\,\alpha\,\%$ of $h(t)$ and the rest $100\,(1-\alpha)\,\%$ is taken by the reinsurer, find values of M and m such that

$$\frac{E[R_x^c \mid (T^c > x, T^r > x)]}{E[R_x^r \mid (T^c > x, T^r > x)]} = q \tag{14}$$

where

$$q = \frac{h_c(t)}{h_r(t)} = \frac{\alpha\, h(t)}{(1-\alpha)\, h(t)} = \frac{\alpha}{1-\alpha}\,. \tag{15}$$

In order to be able to address this optimality problem, we will use the explicit formulae for the corresponding expected profits given in Theorems 3 to 6. First, we will prove the following theorem, which states the existence of a solution to Problem 3.

Theorem 7. *If the total premium income, $h(t) = h_c(t) + h_r(t)$, is shared between the cedent and the reinsurer in such a way that $h_c(t)/h_r(t) = q$, for any $t \geq 0$, where $q \geq 0$, then there always exist $M \geq 0$ and $L \geq M$, such that*

$$E[R_x^c \mid (T^c > x,\, T^r > x)] / E[R_x^r \mid (T^c > x,\, T^r > x)] = q. \tag{16}$$

Proof. Varying $0 \leq \alpha \leq 1$ in (15) one can see that $0 \leq q \leq \infty$. Applying equations (5) and (9), established by Theorems 3 and 4 respectively, to express the numerator and the denominator of the ratio in (16), it is easy to verify that, given $h_c(t)/h_r(t) = q$ for any $t \geq 0$, the expected profits of the two parties will be in the same proportion, q, if and only if

$$\left(\sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^c\, A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots\right.$$
$$\left. dw_2\, dw_1\right) \Big/$$
$$\left(\sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} y_k^r\, A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)\ \psi(w_1,\ \ldots,\ w_k)\, dw_k \ldots\right.$$
$$\left. dw_2\, dw_1\right) = q \tag{17}$$

Note that the numerator and the denominator in (17) depend on M and L through y_k^c, y_k^r and $A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)$. From their definitions, given in Theorem 1, it can be seen that y_k^c, y_k^r and $A_k(x\,;\tilde{\nu}_1,\ \ldots,\ \tilde{\nu}_k)$ are continuous functions of M and L, and hence both the numerator and the denominator in (17) are also continuous functions of M and L.

Varying $M \geq 0$ and $L \geq M$, the left-hand side of (17) takes the whole range of values from 0 to ∞, e.g. when $M = 0$, $L = \infty$ we have $y_k^c = 0$, $0 < y_k^r < \infty$ for every $k = 1, 2, \ldots$ and hence the left-hand side of (17) is zero. On the other extreme when $M = L$, we have $y_k^r = 0$, $0 < y_k^c < \infty$ for every $k = 1, 2, \ldots$ and hence the left-hand side of (17) is infinity. Therefore, there should exist a pair M and L, for which the left-hand side of (17) will be equal to q and so, the ratio of the cedent's and the reinsurer's expected profits will be equal to q. This completes the proof of the theorem.□

In summary, Theorem 7 states that there always exists a solution to Problem 2, however the following remarks should be made.

Remark 1. The solution to Problem 2 may not be unique. There may exist a whole curve of combinations of M and m, for which the ratio of the expected profits of the cedent and the reinsurer is equal to q. We will refer to it as the 'fair' curve. For an illustration of this phenomenon see the right panels in Fig. 5, 6 and 7, where the 'fair' curve is the intersection between the plane $q = h_c(t)/h_r(t) = \text{const}$ and the surface $E[R_x^c \mid (T^c > x, T^r > x)]/E[R_x^r \mid (T^c > x, T^r > x)]$ as a function of M and m.

Remark 2. The numerator and the denominator in (17) coincide with the unconditional expectations $E\left[Y_{N_x}^c \cdot I_A \cdot I_B\right]$ and $E\left[Y_{N_x}^r \cdot I_A \cdot I_B\right]$ which in fact are the unconditional expected aggregate claim amounts at time x of the cedent and the reinsurer respectively, assuming they both survive up to x. So, as is natural to expect, in order for the expected profits to be in proportion q, it is necessary for the expected aggregate claim amounts to be in proportion q, since the premium income, $h(t)$, has been shared in the same proportion.

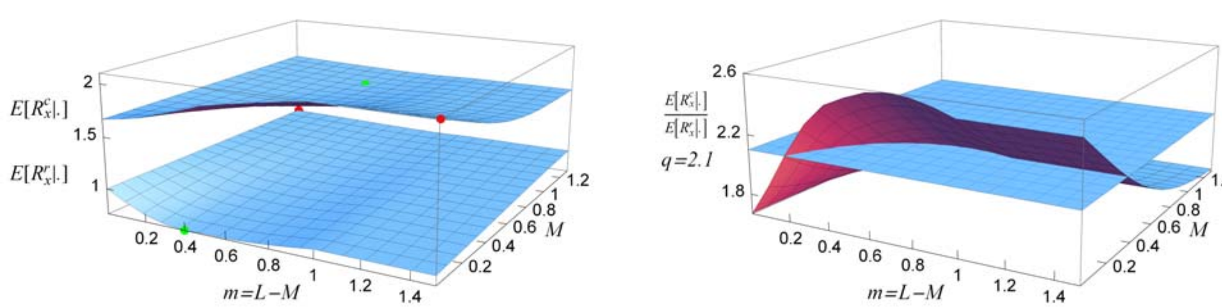

Fig. 5. Solutions to the optimality Problem 2, in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.5$, $q = 2.1$.

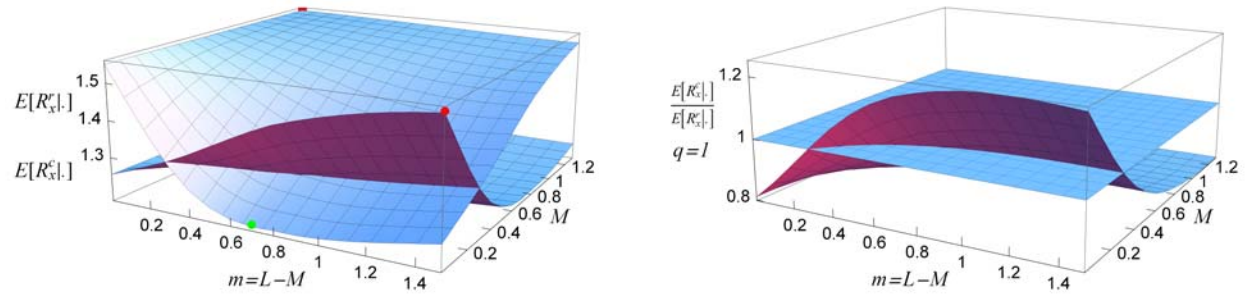

Fig. 6. Solutions to the optimality Problem 2, in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.775$, $q = 1$.

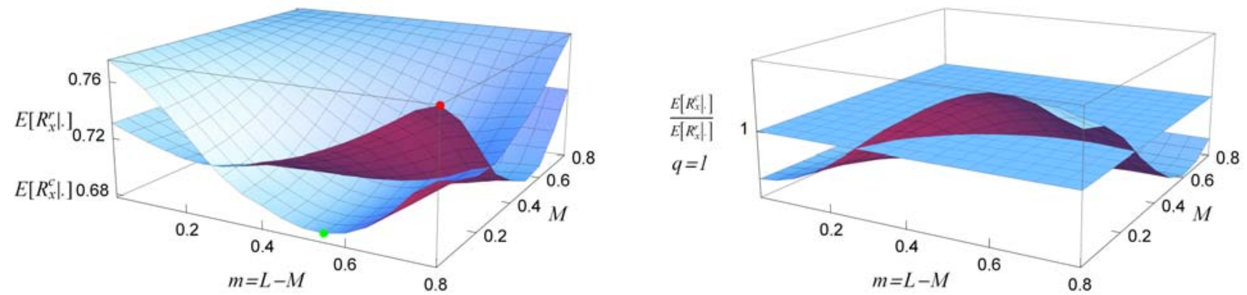

Fig. 7. Solutions to the optimality Problem 2, in the case of dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), \ldots, F(w_k); \theta)$ distributed with $F \equiv$ Weibull(2.12, 1.14) marginals and $\theta = 1$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.775$, $q = 1$.

As can be seen from the right panels of Fig. 6 and 7, given a fixed split of the premium income $q = h_c(t)/h_r(t) = 1$ for all $t \geq 0$, the value of m which lies on the 'fair' curve in the case of dependence between the claim amounts (Fig. 7) may be either smaller or larger, compared to the value of m in the independent case (Fig. 6), depending on the retention level M. For example, for $M = 0.2$, in the case of i.i.d. claim severities $m = 0.5$, whereas in the case of dependent claim sizes $m = 0.4$, which means that the size of the layer covered by the reinsurer is smaller for the same fixed split of $h(t)$. Our experience shows that the effect of dependence modelled through a copula function is complex and may be different for different choices of copulas, marginals and values of the dependence parameter (for further comments see Kaishev and Dimitrova 2006).

Having a whole curve of solutions which provide for a 'fair' distribution of the expected profit at x, given joint survival up to x, the cedent and the reinsurer face the necessity of choosing one particular pair (M', m') from the 'fair' line. In such a situation, the most natural choice would be the pair of values of the parameters (M, m) with the highest probability of joint survival, i.e. the solution of the following problem.

Problem 3. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$ with $h_c(t) = \alpha\, h(t)$, $h_r(t) = (1-\alpha)\, h(t)$, $0 \le \alpha \le 1$, so that $h_c(t)/h_r(t) = q$, find (M, m) for which

$$\min_{M,m} [1 - P(T^c > x,\ T^r > x)]$$

$$\text{subject to} \quad \frac{E[R_x^c \mid (T^c > x,\ T^r > x)]}{E[R_x^r \mid (T^c > x,\ T^r > x)]} = q\,. \qquad (18)$$

It is clear that there always exists a unique solution to Problem 3. As illustrated in Fig. 8 (a) and (b), it is (0.2, 0.3) in the case of i.i.d. claim sizes and $q = h_c(t)/h_r(t) = 1.05\,t/0.5\,t = 2.1$, and (0.25, 0.5) in the dependent case with $q = h_c(t)/h_r(t) = 0.775\,t/0.775\,t = 1$.

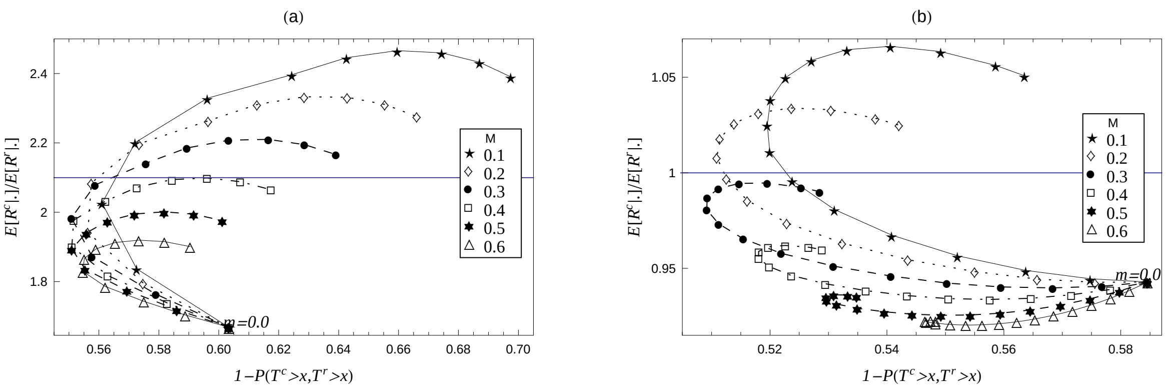

Fig. 8. Solutions to the optimality Problem 3, in the case of: (a) - independent claim severities, Exp(1) distributed, $m = 0.0, 0.1, 0.2, \ldots, 1.0$, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.5$, $q = 2.1$; (b) - dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), \ldots, F(w_k); \theta)$ distributed with $F \equiv$ Weibull(2.12, 1.14) marginals and $\theta = 1$, $m = 0.0, 0.05, 0.1, \ldots, 0.8$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$, $q = 1$.

Finally, in Fig 9, we give a plot of the reinsurer's versus cedent's expected profits, for fixed levels of the non-survival probability $p = 1 - P(T^c > x,\ T^r > x)$, which provides a different point of view to the possible selection of values for (M, m), such that the two profits are shared in an appropriate proportion, different from the proportion, q, in which the premium income is shared.

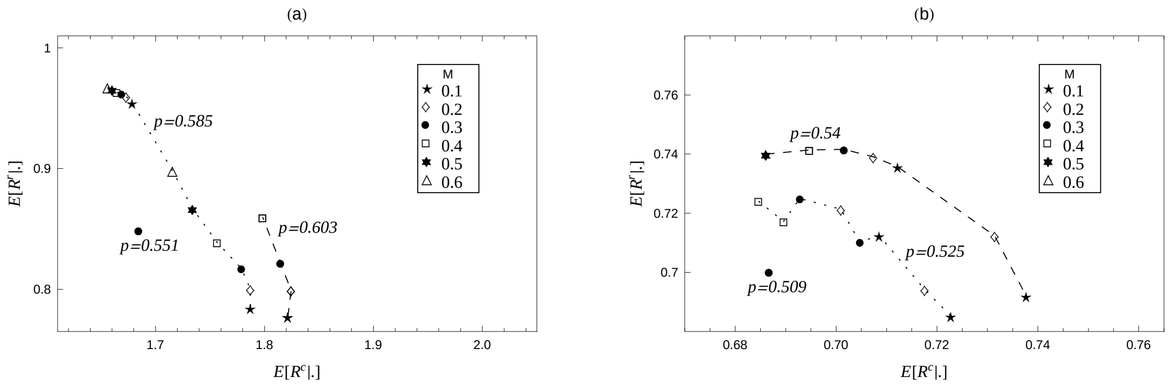

Fig. 9. Reinsurer's versus cedent's expected profits for different levels of the non-survival probability $p = 1 - P(T^c > x, T^r > x)$, in the case of: (a) - independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.5$; (b) - dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), \ldots, F(w_k); \theta)$ distributed with $F \equiv$ Weibull(2.12, 1.14) marginals and $\theta = 1$, with $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $c_r = 0.775$.

5. Comments and conclusions

In the present paper, we have shown how the problem of optimal XL reinsurance can be solved, combining specific risk and performance measures, under a relatively general assumptions for the risk model. As a performance measure, we have defined the expected profits at time x of the direct insurer and the reinsurer given their joint survival up to x, and derived explicit expressions for their numerical evaluation. The results of Kaishev and Dimitrova (2006) for the probability of joint survival of the direct insurer and the reinsurer up to time x have been recalled and employed as a risk measure. Three optimality problems have been defined and their solutions have been numerically illustrated and discussed under the assumption of both dependent and independent claim severities. It is interesting to mention that the effect of dependence of the claim severities is rather complex and difficult to predict based on purely intuitive reasoning. Henceforth, the model presented here provides a very promising framework for future exploration of the effect of dependence on the optimal choice of the parameters of reinsurance contracts. It should also be noted that inverse optimality problems in which the two parties set the retention and the limiting levels and seek for an optimal sharing of the total premium income between them can also be formulated and solved using the techniques and the formulae described in sections 3 and 4.

References

Aase, K. (2002). Perspectives of Risk Sharing. *Scand. Actuarial J.*, 2, 73-128.

Balbás, A., Balbás, B. and Heras, A. (2009). Optimal reinsurance with general risk measures. *Insurance: Mathematics and Economics,* 44, 374-384.

Cao, Y. and Zhang, Y. (2007). Optimal reinsurance under the general mixture risk measures. *Applied Mathematics and Computation*, 185 (1), 229-239.

Centeno, M. L. (2004). Retention and Reinsurance Programmes. in *Encyclopedia of Actuarial Science*, Teugels, J., Sundt, B. edt.-in-chief , John Wiley and Sons Ltd.

Gajek, L. and Zagrodny, D. (2004a). Optimal reinsurance under general risk measure. *Insurance:Mathematics and Economics*, 34, 227-240.

Gajek, L. and Zagrodny, D. (2004b). Reinsurance arrangements maximizing insurer's survival probability. *The Journal of Risk and Insurance*, 71 (3), 421-435.

Guerra, M. and Centeno, M.L. (2008). Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria. *Insurance: Mathematics and Economics*, 42, 529-539.

Ignatov, Z. G. and Kaishev, V. K. (2004). A finite time ruin probability formula for continuous claim severities. *Journal of Applied Probability,* 41, 570-578.

Ignatov, Z. G., Kaishev, V. K. and Krachunov, R. S. (2004). Optimal retention levels, given the joint survival of cedent and reinsurer. *Scand. Actuarial J.,* 6, 401-430.

Kaishev, V. K. and Dimitrova, D. S. (2006). Excess of loss reinsurance under joint survival optimality. *Insurance:Mathematics and Economics*, 39, 376-389.

Kaluszka, M. (2004). Mean-variance optimal reinsurance arrangements. *Scand. Actuarial J.*, 1, 28-41.

Kaz'min Yu. A.(2002). Appell polynomials. *Encyclopaedia of Mathematics*, Edt. by Michiel Hazewinkel. Springer Verlag, Berlin.

Liang, Z. and Guo, J. (2007). Optimal Proportional Reinsurance and Ruin Probability. *Stochastic Models*, 23 (2), 333-350.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance, 7, 77-91*.

Schmidli, H. (2004). On Cramér-Lundberg approximations for ruin probabilities under optimal excess of loss reinsurance. *Working paper 193,* Laboratory of Actuarial Mathematics, University of Copenhagen.

Verlaak, R. and Beirlant, J. (2003). Optimal reinsurance programs: An optimal combination of several reinsurance protections on a heterogeneous insurance portfolio. *Insurance:Mathematics and Economics*, 33, 381-403.

Zanetti, A., Schwarz, S. and Lindemuth, A. (2007). Natural catastrophes and man-made disasters in 2006: low insured losses. *Sigma* Insurance Research, No 2, Swiss Re Publications.