Integrable derivations and stable equivalences of Morita type

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Abstract

Using that integrable derivations of symmetric algebras can be interpreted in terms of Bockstein homomorphisms in Hochschild cohomology, we show that integrable derivations are invariant under the transfer maps in Hochschild cohomology of symmetric algebras induced by stable equivalences of Morita type. With applications in block theory in mind, we allow complete discrete valuation rings of unequal characteristic. *Key words:* derivation, stable equivalence. *2010 Mathematics Subject classification:* 16E40, 16G30, 16D90.

1 Introduction

Throughout this paper, $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal $J(\mathcal{O}) = \pi\mathcal{O}$ for some nonzero element $\pi \in \mathcal{O}$, residue field $k = \mathcal{O}/J(\mathcal{O})$, and field K of fractions. Let A be an $\mathcal{O}$-algebra such that A is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer and let α be an $\mathcal{O}$-algebra automorphism of A such that α induces the identity map on $A/\pi^r A$; that is, for all $a \in A$ we have $\alpha(a) = a + \pi^r \mu(a)$ for some $\mathcal{O}$-linear endomorphism μ of A. It is well-known, and easy to verify, that the endomorphism of $A/\pi^r A$ induced by μ is a derivation on $A/\pi^r A$. Any derivation on $A/\pi^r A$ which arises in this way is called *integrable*; this concept goes back to work of Gerstenhaber [7] (see §3 for more details). The set of integrable derivations on $A/\pi^r A$ is an abelian group containing all inner derivations, hence determines a subgroup of $HH^1(A/\pi^r A)$, denoted $HH^1_A(A/\pi^r A)$. It is shown in [6, §4] that if $\mathcal{O} = k[[t]]$, $r = 1$, and if $A = \bar{A} \otimes_k k[[t]]$ for some finite-dimensional k-algebra $\bar{A}$, then $HH^1_A(\bar{A})$ is invariant under Morita equivalences between finite-dimensional k-algebras. We show that this invariance extends to stable equivalences of Morita type between symmetric algebras over an arbitrary complete discrete valuation ring $\mathcal{O}$. An $\mathcal{O}$-algebra A is called *symmetric* if A is $\mathcal{O}$-free of finite rank and if $A \cong A^\vee = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ as $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-modules. Following Broué [2, §5 A], given symmetric $\mathcal{O}$-algebras A and B, an A-B-bimodule M is said to induce a *stable equivalence of Morita type*, if M is finitely generated projective as a left A-module and as a right B-module, and if the bimodules $M \otimes_B M^\vee$ and $M^\vee \otimes_A M$ are isomorphic to A and B in the relatively $\mathcal{O}$-stable categories of $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-modules and $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-modules, respectively.

Theorem 1.1. *Let A and B be symmetric $\mathcal{O}$-algebras such that the semisimple quotients of A and B are separable. Let r be a positive integer. Suppose that the canonical maps $Z(A) \to Z(A/\pi^r A)$ and $Z(B) \to Z(B/\pi^r B)$ are surjective. Let M be an A-B-bimodule inducing a stable equivalence of Morita type. Then the functor $M \otimes_B - \otimes_B M^\vee$ induces an $\mathcal{O}$-linear isomorphism $HH^1(B/\pi^r B) \cong HH^1(A/\pi^r A)$ which restricts to an isomorphism of groups*

$$HH^1_B(B/\pi^r B) \cong HH^1_A(A/\pi^r A).$$

In particular, if A and B are derived or Morita equivalent, then $HH^1_B(B/\pi^r B) \cong HH^1_A(A/\pi^r A)$.

By a result of Rickard [12, 5.5], a derived equivalence between symmetric algebras induces a stable equivalence of Morita type, and hence the last statement in the theorem is an immediate consequece of the first. Theorem 1.1 will be proved, slightly more generally for relatively $\mathcal{O}$-selfinjective algebras, in Theorem 5.1.

Remark 1.2. The assumption that $Z(A) \to Z(A/\pi^r A)$ is surjective holds if A is a finite group algebra, or a block algebra, or a source algebra of a block. It also holds in the 'classic' case where $\mathcal{O} = k[[t]]$ and where A is of the form $\bar{A} \otimes_k k[[t]]$ for some finite-dimensional k-algebra $\bar{A}$. In that case, $HH^1_A(\bar{A})$ is a $Z(\bar{A})$-submodule, and a stable equivalence of Morita type between $\bar{A}$ and another finite-dimensional k-algebra $\bar{B}$ extends to a stable equivalence of Morita type between the $k[[t]]$-algebras $A = \bar{A} \otimes_k k[[t]]$ and $B = \bar{B} \otimes_k k[[t]]$. By contrast, in the more general situation of the above theorem, allowing k and K to have unequal characteristic, the subgroup $HH^1_A(A/\pi^r A)$ need not be an $\mathcal{O}/\pi^r\mathcal{O}$-submodule of $HH^1(A/\pi^r A)$, and not every stable equivalence of Morita type between $A/\pi^r A$ and $B/\pi^r B$ lifts necessarily to a stable equivalence of Morita type between A and B. See §6 for some examples.

Remark 1.3. One of the block theoretic applications of this material is the interpretation of the rank of $P/\mathfrak{foc}(\mathcal{F})$ in terms of invariants of the algebra structure of a block B of a finite group with a defect group P and fusion system $\mathcal{F}$ on P. See [9] for details.

Notation 1.4. In what follows, the use of algebra automorphisms as subscripts to modules is as in [8]. If α is an automorphism of an $\mathcal{O}$-algebra A and U an A-module, we denote by ${}_\alpha U$ the A-module which is equal to U as an $\mathcal{O}$-module, with $a \in A$ acting as $\alpha(a)$ on U. If α is inner, then ${}_\alpha U \cong U$; indeed, if $c \in A^\times$ such that $\alpha(a) = cac^{-1}$ for all $a \in A$, then the map sending $u \in U$ to cu is an A-module isomorphism $U \cong {}_\alpha U$. We use the analogous notation for right modules and bimodules. If U and V are A-A-bimodules and $\alpha \in \mathrm{Aut}(A)$, then we have an obvious isomorphism of A-A-bimodules $(U_\alpha) \otimes_A V \cong U \otimes_A ({}_{\alpha^{-1}}V)$.

2 Background material

We collect in this section some basic and well-known facts on Hochschild cohomology and stable categories. Let A be an $\mathcal{O}$-algebra such that A is free of finite rank as an $\mathcal{O}$-module. For any integer $n \geq 0$ and any $A \otimes_\mathcal{O} A^{\mathrm{op}}$-module M, the Hochschild cohomology in degree n of A with coefficients in M is $HH^n(A; M) = \mathrm{Ext}^n_{A\otimes_\mathcal{O} A^{\mathrm{op}}}(A; M)$; we set $HH^n(A) = HH^n(A; A)$. (We use here that A is $\mathcal{O}$-free, since in general, Hochschild cohomology is defined as a relative Ext-module.) We have $HH^0(A) \cong Z(A)$, and $HH^1(A)$ is the space of derivations on A modulo inner derivations. The graded $\mathcal{O}$-module $HH^*(A) = \oplus_{n\geq 0} HH^n(A)$ is a graded-commutative algebra, and the positive degree part is a graded Lie algebra of degree -1, with the Gerstenhaber bracket; in particular, $HH^1(A)$ is a Lie algebra. See e. g. [14, Ch. 9] for more material on Hochschild cohomology. For any $\mathcal{O}$-free A-modules U, V and any integers r, s such that $s \geq r > 0$ we have an obvious isomorphism $\mathrm{Hom}_A(U, V/\pi^r V) \cong \mathrm{Hom}_{A/\pi^s A}(U/\pi^s U, V/\pi^r V)$. If P is a projective resolution of U, then $P/\pi^s P$ is a projective resolution of the $A/\pi^s A$-module $U/\pi^s U$, and we have an isomorphism of cochain complexes $\mathrm{Hom}_A(P, V/\pi^r V) \cong \mathrm{Hom}_{A/\pi^s A}(P/\pi^s P, V/\pi^r V)$. Taking cohomology in degree $n \geq 0$ yields an isomorphism $\mathrm{Ext}^n_A(U, V/\pi^r V) \cong \mathrm{Ext}^n_{A/\pi^s A}(U/\pi^s U, V/\pi^r V)$. In particular, for any r, s such that $s \geq r > 0$ and any $n \geq 0$ we have

2.1.

$$HH^n(A; A/\pi^r A) \cong HH^n(A/\pi^s A; A/\pi^r A) \cong HH^n(A/\pi^r A)$$

In what follows the superscript π^r to an arrow means the map given by multiplication with π^r. We refer to [4] for background material on Bockstein homomorphisms and spectral sequences. Applying the functor $\mathrm{Hom}_{A\otimes_{\mathcal{O}} A^{\mathrm{op}}}(A, -)$ to the Bockstein short exact sequence of A-A-bimodules

$$0 \longrightarrow A \xrightarrow{\pi^r} A \longrightarrow A/\pi^r A \longrightarrow 0$$

and making use of the above identifications yields a long exact sequence of $\mathcal{O}$-modules which starts as follows:

2.2.

$$0 \longrightarrow Z(A) \xrightarrow{\pi^r} Z(A) \longrightarrow Z(A/\pi^r A) \longrightarrow HH^1(A) \xrightarrow{\pi^r} HH^1(A) \longrightarrow$$

$$HH^1(A/\pi^r A) \longrightarrow HH^2(A) \xrightarrow{\pi^r} HH^2(A) \longrightarrow \cdots$$

If r and n are positive integers such that that π^r annihilates $HH^n(A)$ and $HH^{n+1}(A)$, then it follows from the long exact sequence 2.2 that there is a short exact sequence

2.3.

$$0 \longrightarrow HH^n(A) \longrightarrow HH^n(A/\pi^r A) \longrightarrow HH^{n+1}(A) \longrightarrow 0$$

For future use, we briefly note some further immediate consequences of 2.2.

Lemma 2.4. *Let r be a positive integer.*

(i) The canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective if and only if $HH^1(A)$ is $\mathcal{O}$-free.

(ii) The connecting homomorphism $HH^1(A/\pi^r A) \to HH^2(A)$ is injective if and only if $HH^1(A) = \{0\}$.

Proof. The above long exact sequence implies that the map $Z(A) \to Z(A/\pi^r A)$ is surjective if and only if the map $Z(A/\pi^r A) \to HH^1(A)$ is zero, hence if and only if the map $HH^1(A) \to HH^1(A)$ induced by multiplication with π^r is injective. This is the case if and only if $HH^1(A)$ is $\mathcal{O}$-free, proving (i). Similarly, the connecting homomorphism $HH^1(A/\pi^r A) \to HH^2(A)$ is injective if and only if the map $HH^1(A) \to HH^1(A/\pi^r A)$ is zero, hence if and only if the map $HH^1(A) \to HH^1(A)$ induced by multiplication by π^r is surjective. By Nakayama's Lemma, this is equivalent to $HH^1(A)$ being zero, which proves (ii). $\square$

The property of $HH^1(A)$ being $\mathcal{O}$-free or zero does not depend on the integer r in the above lemma. It follows therefore from the first statement of the lemma that if $Z(A) \to Z(A/\pi^r A)$ is surjective for some positive integer r, then it is surjective for any positive integer r. Similarly, the second statement of this lemma implies that if $HH^1(A/\pi^r A) \to HH^2(A)$ is injective for some positive integer r, then it is injective for any positive integer r. The following observation is well-known.

Lemma 2.5. *Suppose that $K \otimes_{\mathcal{O}} A$ is separable. Then $HH^n(A)$ is a torsion $\mathcal{O}$-module for all positive integers n.*

Proof. Since $K \otimes_{\mathcal{O}} A$ is separable it follows that $HH^n(K \otimes_{\mathcal{O}} A)$ is zero for all positive integers n. The exactness of the functor $K \otimes_{\mathcal{O}} -$ implies that $\{0\} = HH^n(K \otimes_{\mathcal{O}} A) \cong K \otimes_{\mathcal{O}} HH^n(A)$. Since $K \otimes_{\mathcal{O}} -$ annihilates exactly the torsion $\mathcal{O}$-modules, the result follows. $\square$

Lemma 2.6. *Suppose that $K \otimes_{\mathcal{O}} A$ is separable. Let r be a positive integer. The following are equivalent.*

(i) The canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective.

(ii) We have $HH^1(A) = \{0\}$.

(iii) The connecting homomorphism $HH^1(A/\pi^r A) \to HH^2(A)$ is injective.

Proof. If (i) holds, then the two previous lemmas imply that $HH^1(A)$ is both $\mathcal{O}$-torsion and torsion free, hence zero. The rest follows from 2.4. $\square$

As before, if (i) or (iii) in this lemma holds for some positive integer r, it holds for all positive integers r, since (ii) does not depend on r. This implies further that if the equivalent statements in the lemma hold, then all canonical maps $Z(A/\pi^r A) \to Z(A/\pi^{r+1} A)$ are surjective.

Lemma 2.7. *Suppose that $K \otimes_{\mathcal{O}} A$ is separable and that the canonical map $Z(A) \to Z(A/\pi A)$ is surjective. Let r be a positive integer. The bimodule homomorphism $A/\pi^r A \to A/\pi^{r+1} A$ given by multiplication with π on A induces an injective map $HH^1(A/\pi^r A) \to HH^1(A/\pi^{r+1} A)$ making the diagram*

$$\begin{tikzcd} HH^1(A/\pi^r A) \arrow[r] \arrow[d] & HH^1(A/\pi^{r+1} A) \arrow[d] \\ HH^2(A) \arrow[r, equal] & HH^2(A) \end{tikzcd}$$

commutative, where the vertical arrows are the connecting homomorphisms. Moreover, if π^r annihilates $HH^2(A)$, then all maps in this diagram are isomorphisms. In particular, $HH^2(A)$ has an ascending finite filtration of subspaces isomorphic to $HH^1(A/\pi^i A)$, with $i \geq 1$.

Proof. The commutativity of the diagram follows from comparing the long exact sequences from 2.2 for r and $r+1$. Since the canonical map $Z(A) \to Z(A/\pi A)$ is surjective and since $K \otimes_{\mathcal{O}} A$ is separable, it follows from 2.6 that the connecting homomorphisms in the statement are injective, and hence that the map $HH^1(A/\pi^r A) \to HH^1(A/\pi^{r+1} A)$ is injective. If π^r annihilates $HH^2(A)$, then the long exact sequence 2.2 implies that the connecting homomorphism $HH^1(A/\pi^r A) \to HH^2(A)$ is also surjective, whence the result. $\square$

We conclude this section with a very brief review of relatively $\mathcal{O}$-stable categories and stable equivalences; more details can be found for instance in [3]. Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module.

An A-module U is called *relatively $\mathcal{O}$-projective* if U is isomorphic to a direct summand of $A \otimes_{\mathcal{O}} V$ for some $\mathcal{O}$-module V. If U is relatively $\mathcal{O}$-projective, then the canonical map $A \otimes_{\mathcal{O}} U \to U$ sending $a \otimes u$ to au splits (this is a standard fact on the splitting of adjunction units and counits).

Dually, U is called *relatively $\mathcal{O}$-injective* if U is isomorphic to a direct summand of $\mathrm{Hom}_{\mathcal{O}}(A, V)$ for some $\mathcal{O}$-module V, where the A-module structure on $\mathrm{Hom}_{\mathcal{O}}(A, V)$ is given by $(a \cdot \varphi)(b) = \varphi(ba)$, for a, $b \in A$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}}(A, V)$. As before, if U is relatively $\mathcal{O}$-injective, then the canonical map $U \to \mathrm{Hom}_{\mathcal{O}}(A, U)$ sending $u \in U$ to the map $a \mapsto au$ is split.

Suppose that $k \otimes_{\mathcal{O}} A$ is selfinjective. Then $k \otimes_{\mathcal{O}} A$ is injective as a left and right $k \otimes_{\mathcal{O}} A$-module, hence its k-dual is a progenerator as a left and right $k \otimes_{\mathcal{O}} A$-module. Since finitely generated projective $k \otimes_{\mathcal{O}} A$-modules lift uniquely, up to isomorphism, to finitely generated projective A-modules, it follows that the $\mathcal{O}$-dual $A^\vee = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ is a progenerator of A as a left and right A-module. Thus U is relatively $\mathcal{O}$-projective if and only if U is isomorphic to a direct summand of finitely many copies of $A^\vee \otimes_{\mathcal{O}} U = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}) \otimes_{\mathcal{O}} U \cong \mathrm{Hom}_{\mathcal{O}}(A, U)$, where the last map sends $\alpha \otimes u$ to the map $a \mapsto \varphi(a)u$, for $a \in A$, $u \in U$, $\alpha \in \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$. Thus U is relatively $\mathcal{O}$-projective if and only if U is relatively $\mathcal{O}$-injective. Identifying the relatively $\mathcal{O}$-projective modules to zero yields therefore a triangulated category, denoted $\mathrm{stmod}(A)$, and called the *relatively $\mathcal{O}$-stable category of finitely generated A-modules.* The objects of $\mathrm{stmod}(A)$ are the finitely generated A-modules. For any two finitely generated A-modules U, U', denote by $\mathrm{Hom}_A^{\mathrm{pr}}(U, U')$ the set of A-homomorphisms from U to U' which factor through a relatively $\mathcal{O}$-projective A-module. This is an $\mathcal{O}$-submodule of $\mathrm{Hom}_A(U, U')$. The space of morphisms in $\mathrm{stmod}(A)$ from U to U' is the $\mathcal{O}$-module $\underline{\mathrm{Hom}}_A(U, U') = \mathrm{Hom}_A(U, U')/\mathrm{Hom}_A^{\mathrm{pr}}(U, U')$, with composition of morphisms induced by that in $\mathrm{mod}(A)$. The triangulated structure in $\mathrm{stmod}(A)$ is induced by $\mathcal{O}$-split short exact sequences of finitely generated A-modules.

Suppose that A and B are two $\mathcal{O}$-algebras which are $\mathcal{O}$-free of finite rank such that $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$ are selfinjective. Then $k \otimes_{\mathcal{O}} (A \otimes_{\mathcal{O}} B)$ and $k \otimes_{\mathcal{O}} A^{\mathrm{op}}$ are selfinjective as well. Let M be an A-B-bimodule such that M is finitely generated projective as a left A-module and as a right B-module. The functor $M \otimes_B -$ from $\mathrm{mod}(B)$ to $\mathrm{mod}(A)$ is exact and preserves the classes of relatively $\mathcal{O}$-projective modules, hence induces a functor from $\mathrm{stmod}(B)$ to $\mathrm{stmod}(A)$ as triangulated categories. Let N be a B-A-bimodule which is finitely generated projective as a left B-module and as a right A-module. Following Broué [2] we say that *M and N induce a stable equivalence of Morita type between A and B* if we have isomorphisms $M \otimes_B N \cong B \oplus Y$ and $N \otimes_A M \cong A \oplus X$ as $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-modules and $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-modules, respectively, such that Y is a projective $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module and X is a projective $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-module. In that case, the functors $M \otimes_B -$ and $N \otimes_A -$ induces equivalences between $\mathrm{stmod}(B)$ and $\mathrm{stmod}(A)$ which are inverse to each other. Moreover, for any positive integer r the bimodules $M/\pi^r M$ and $N/\pi^r N$ induce a stable equivalence of Morita type between the $\mathcal{O}/\pi^r\mathcal{O}$-algebras $A/\pi^r A$ and $B/\pi^r B$. If A is symmetric, then $k \otimes_{\mathcal{O}} A$ is selfinjective. If also B is symmetric and if as before M and N induce a stable equivalence of Morita type between A and B, then M and its $\mathcal{O}$-dual $M^\vee$ induce a stable equivalence of Morita type. The following observation is an immediate consequence of the fact that if $k \otimes_{\mathcal{O}} A$ is selfinjective, then projective modules are relatively $\mathcal{O}$-injective. For the convenience of the reader we give a direct proof (using some of the above comments).

Lemma 2.8. *Suppose that $k \otimes_{\mathcal{O}} A$ is selfinjective. Let U be an $\mathcal{O}$-free A-module, let Q be a finitely generated projective A-module, and let r be a positive integer. The map $\mathrm{Hom}_A(U, Q) \to \mathrm{Hom}_A(U, Q/\pi^r Q)$ induced by the canonical surjection $Q \to Q/\pi^r Q$ is surjective.*

Proof. Since $k \otimes_{\mathcal{O}} A$ is selfinjective, it follows that every projective indecomposable $k \otimes_{\mathcal{O}} A$-module is the k-dual of a projective indecomposable right $k \otimes_{\mathcal{O}} A$-module. Since finitely generated projective $k \otimes_{\mathcal{O}} A$-modules lift uniquely, up to isomorphism, to finitely generated projective A-modules, it follows that $Q \cong \mathrm{Hom}_{\mathcal{O}}(R, \mathcal{O})$ for some finitely generated projective right A-module R, and we have $Q/\pi^r Q \cong \mathrm{Hom}_{\mathcal{O}}(R, \mathcal{O}/\pi^r\mathcal{O})$. The tensor-Hom adjunction yields isomorphisms

$$\mathrm{Hom}_A(U, Q) \cong \mathrm{Hom}_A(U, \mathrm{Hom}_{\mathcal{O}}(R, \mathcal{O})) \cong \mathrm{Hom}_{\mathcal{O}}(R \otimes_A U, \mathcal{O})$$

and

$$\mathrm{Hom}_A(U, Q/\pi^r Q) \cong \mathrm{Hom}_A(U, \mathrm{Hom}_{\mathcal{O}}(R, \mathcal{O}/\pi^r\mathcal{O})) \cong \mathrm{Hom}_{\mathcal{O}}(R \otimes_A U, \mathcal{O}/\pi^r\mathcal{O})$$

Since R is finitely generated projective as a right A-module and U is $\mathcal{O}$-free, it follows that $R \otimes_A U$ is $\mathcal{O}$-free. This implies that the canonical map $\mathrm{Hom}_{\mathcal{O}}(R \otimes_A U, \mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}(R \otimes_A U, \mathcal{O}/\pi^r\mathcal{O})$ is surjective, and hence so is the canonical map $\mathrm{Hom}_A(U, Q) \to \mathrm{Hom}_A(U, Q/\pi^r Q)$ as claimed. $\square$

3 On derivations and algebra automorphisms

For background material on integrable derivations in the context of deformations of algebras, see for instance Gerstenhaber [7] and Matsumura [10]. We first review some basic facts on integrable derivations, adapted to arbitrary complete discrete valuation rings rather than rings of power series.

Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. We denote by $\mathrm{Aut}(A)$ the group of $\mathcal{O}$-algebra automorphisms of A, by $\mathrm{Inn}(A)$ the normal subgroup of inner automorphisms, and by $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ the group of outer $\mathcal{O}$-algebra automorphisms of A. For any positive integer r we denote by $\mathrm{Aut}_r(A)$ the subgroup of $\mathrm{Aut}(A)$ consisting of all $\mathcal{O}$-algebra automorphisms of A which induce the identity on $A/\pi^r A$, with the convention $\mathrm{Aut}_0(A) = \mathrm{Aut}(A)$. That is, $\mathrm{Aut}_r(A)$ is the kernel of the canonical map $\mathrm{Aut}(A) \to \mathrm{Aut}(A/\pi^r A)$. We denote by $\mathrm{Out}_r(A)$ the image of $\mathrm{Aut}_r(A)$ in $\mathrm{Out}(A)$.

Lemma 3.1. *Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module, and let r be a positive integer. The group $\mathrm{Out}_r(A)$ is equal to the kernel of the canonical map $\mathrm{Out}(A) \to \mathrm{Out}(A/\pi^r A)$.*

Proof. The image of $\mathrm{Aut}_r(A)$ in $\mathrm{Out}(A)$ belongs trivially to the kernel of the canonical map $\mathrm{Out}(A) \to \mathrm{Out}(A/\pi^r A)$. We need to show that any element of this kernel is represented by an automorphism in $\mathrm{Aut}_r(A)$. Let $\alpha \in \mathrm{Aut}(A)$ such that α induces inner automorphism of $A/\pi^r A$, given by conjugation with an invertible element $\bar{u}$ in $A/\pi^r A$. Since the canonical map $A^\times \to (A/\pi^r A)^\times$ is surjective, it follows that $\bar{u}$ lifts to an invertible element u in A. The composition of α with conjugation by u^{-1} yields an automorphism α' which belongs to the same class as α in $\mathrm{Out}(A)$ and which induces the identity on $A/\pi^r A$, hence which belongs to $\mathrm{Aut}_r(A)$. This shows that every class in the kernel of $\mathrm{Out}(A) \to \mathrm{Out}(A/\pi^r A)$ has a representative in $\mathrm{Aut}_r(A)$, as required. $\square$

As mentioned before, the extra hypothesis on $Z(A) \to Z(A/\pi^r A)$ being surjective in many of the statements below is satisfied automatically if $A = \bar{A} \otimes_k k[[t]]$ for some k-algebra $\bar{A}$.

Proposition 3.2. *Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer. Suppose that the canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective. Let $\alpha \in \mathrm{Aut}_r(A)$, and let $\mu : A \to A$ be the unique linear map satisfying $\alpha(a) = a + \pi^r \mu(a)$ for all $a \in A$.*
(i) The map $\bar{\mu} : A/\pi^r A \to A/\pi^r A$ induced by μ is a derivation. The class of $\bar{\mu}$ in $HH^1(A/\pi^r A)$ depends only on the class of α in $\mathrm{Out}(A)$.

(ii) If α is an inner automorphism of A, then α is induced by conjugation with an element of the form $c = 1 + \pi^r d$ for some $d \in A$, and we have $\bar{\mu} = [\bar{d}, -]$; in particular, $\bar{\mu}$ is an inner derivation in that case.

Proof. For $a, b \in A$, comparing the expressions $\alpha(ab) = ab + \pi^r \mu(ab)$ and $\alpha(a)\alpha(b) = ab + \pi^r a\mu(b) + \pi^r \mu(a)b + \pi^{2r}\mu(a)\mu(b)$ yields the equality

$$\mu(ab) = a\mu(b) + \mu(a)b + \pi^r \mu(a)\mu(b) \ .$$

Reducing this modulo $\pi^r A$ shows that $\bar{\mu}$ is a derivation on $A/\pi^r A$. In order to show that the class of $\bar{\mu}$ in $HH^1(A)$ depends only on the image of α in $\mathrm{Out}(A)$ it suffices to prove (ii). Suppose that α is inner, induced by conjugation with an element $c \in A^\times$. Since α induces the identity on $A/\pi^r A$, it follows that $\bar{c} \in Z(A/\pi^r A)^\times$. Since the map $Z(A) \to Z(A/\pi^r A)$ is assumed to be surjective, there is an element $z \in Z(A)^\times$ such that $\bar{z} = \bar{c}$, hence such that $cz^{-1} \in 1 + \pi^r A$. Thus, after replacing c by cz^{-1}, we may assume that $c = 1 + \pi^r d$ for some $d \in A$. For $a \in A$ we have $cac^{-1} = \alpha(a) = a + \pi^r \mu(a)$, hence $ca = ac + \pi^r \mu(a)c$, or equivalently, $[c, a] = \pi^r \mu(a)c$. Replacing c by $1 + \pi^r d$ in this equation and dividing by π^r yields $[d, a] = \mu(a) + \pi^r \mu(a)d$, whence $[\bar{d}, \bar{a}] = \bar{\mu}(\bar{a})$ as claimed in (ii). □

Definition 3.3. Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module, and let r be a positive integer. A derivation δ on $A/\pi^r A$ is called *A-integrable* if there is an algebra automorphism α of A and an $\mathcal{O}$-linear endomorphism μ of A such that $\alpha(a) = a + \pi^r \mu(a)$ for all $a \in A$ and such that δ is equal to the map induced by μ on $A/\pi^r A$. A class $\eta \in HH^1(A/\pi^r A)$ is called *A-integrable* if it can be represented by an A-integrable derivation. We denote by $HH^1_A(A/\pi^r A)$ the set of A-integrable classes in $HH^1(A/\pi^r A)$.

Remark 3.4. One can extend the notation from Definition 3.3 in a way which is analogous to Hasse-Schmidt derivations of positive length considered for commutative algebras in [11]. Let $s \geq 2r$ and let α be an automorphism of $A/\pi^s A$ which induces the identity on $A/\pi^r A$. Then, for $\bar{a} = a + \pi^s A \in A/\pi^s A$ we have $\alpha(\bar{a}) = \bar{a} + \pi^r \nu(\bar{a})$ for some element $\nu(a)$ of $A/\pi^s A$. Unlike in the preceding definition, multiplication by π^r on $A/\pi^s A$ is not injective, and hence the map ν need not be a linear endomorphism of $A/\pi^s A$. The annihilator of π^r in $A/\pi^s A$ is $\pi^{s-r} A/\pi^s A$. Since $r \leq s - r$ it follows that ν induces a linear endomorphism of $A/\pi^r A$, and as before, this is a derivation. The derivations and corresponding classes in $HH^1(A/\pi^r A)$ which arise in this way are called *$A/\pi^s A$-integrable.* We denote by $HH^1_{A/\pi^s A}(A/\pi^r A)$ the set of $A/\pi^s A$-integrable classes. We have $HH^1_{A/\pi^{2r} A}(A/\pi^r A) = HH^1(A/\pi^r A)$ because if μ is a linear endomorphism of A inducing a derivation on $A/\pi^r A$, then α defined by $\alpha(a + \pi^{2r} A) = a + \pi^r \mu(a) + \pi^{2r} A$ for any $a \in A$ is an automorphism of $A/\pi^{2r} A$. We have obvious inclusions $HH^1_A(A/\pi^r A) \subseteq HH^1_{A/\pi^{s+1} A}(A/\pi^r A) \subseteq HH^1_{A/\pi^s A}(A/\pi^r A)$. These inclusions need not be equalities since an automorphism of $A/\pi^s A$ which induces the identity on $A/\pi^r A$ does not necessarily lift to an automorphism of $A/\pi^{s+1} A$ or of A. Another variation of this definition arises from replacing π by a suitable element in $Z(A)$ which is not a zero divisor. This point of view appears in Roggenkamp and Scott [13, §5].

Proposition 3.5. *Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer. Suppose that the canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective. Let $\alpha \in \mathrm{Aut}_r(A)$, and let $\mu : A \to A$ be the unique linear map satisfying $\alpha(a) = a + \pi^r \mu(a)$ for all $a \in A$. Denote by $\bar{\mu}$ the derivation on $A/\pi^r A$ induced by μ.*

(i) The image of $\bar{\mu}$ in $HH^1(A/\pi^r A)$ is zero if and only if α induces an inner automorphism on $A/\pi^{2r}A$.

(ii) The map sending α to the class of $\bar{\mu}$ induces a group homomorphism $\mathrm{Out}_r(A) \to HH^1(A/\pi^r A)$, with kernel equal to $\mathrm{Out}_{2r}(A)$. In particular, $HH^1_A(A/\pi^r A)$ is a subgroup of $HH^1(A/\pi^r A)$, and we have a short exact sequence of groups

$$1 \longrightarrow \mathrm{Out}_{2r}(A) \longrightarrow \mathrm{Out}_r(A) \longrightarrow HH^1_A(A/\pi^r A) \longrightarrow 1$$

Proof. In order to show (i), suppose first that $\bar{\mu} = [\bar{d}, -]$ for some $d \in A$. Set $c = 1 + \pi^r d$. Write $[d, a] = \mu(a) - \pi^r \tau(a)$ for some $\tau(a) \in A$. Then $\pi^r \mu(a) = [c, a] + \pi^{2r}\tau(a) = ca - ac + \pi^{2r}\tau(a)$. Multiplying this by c^{-1} on the right implies that $cac^{-1} = a + \pi^r\mu(a)c^{-1} - \pi^{2r}\tau(a)c^{-1}$. Using $\alpha(a) = a + \pi^r\mu(a)$, it follows that $\alpha(a) - cac^{-1} = \pi^r\mu(a)(1-c^{-1}) + \pi^{2r}\tau(a)c^{-1}$. Since $c \in 1 + \pi^r A$, we have $c^{-1} \in 1 + \pi^r A$, hence $1 - c^{-1} \in \pi^r A$. This shows that $\alpha(a) - cac^{-1} \in \pi^{2r}A$, hence α induces an inner automorphism on $A/\pi^{2r}A$. Suppose conversely that α acts as inner automorphism on $A/\pi^{2r}A$. That is, the class of α belongs to the kernel of the canonical map $\mathrm{Out}(A) \to \mathrm{Out}(A/\pi^{2r}A)$. By 3.1, the class of α has a representative in $\mathrm{Aut}_{2r}(A)$. Thus, after replacing α by a suitable representative in the class of α in $\mathrm{Out}(A)$ we may assume that $\alpha \in \mathrm{Aut}_{2r}(A)$. Thus $\alpha(a) = a + \pi^{2r}\mu'(a)$ for some $\mu'(a) \in A$, or equivalently, $\mu(a) = \pi^r\mu'(a)$. This shows that μ induces the zero map on $A/\pi^r A$, proving (i). Let $\beta \in \mathrm{Aut}_r(A)$ and ν such that $\beta(a) = a + \pi^r\nu(a)$ for all $a \in A$. A short calculation shows that $\beta(\alpha(a)) = a + \pi^r\mu(a) + \pi^r\nu(a) + \pi^{2r}\nu(\mu(a)) = \pi^r(\mu(a) + \nu(a) + \pi^r\nu(\mu(a)))$. This shows that the class determined by $\beta \circ \alpha$ in $HH^1(A/\pi^r A)$ is the class determined by $\bar{\mu} + \bar{\nu}$. Statement (ii) follows from (i). □

The propositions above imply that $HH^1_A(A/\pi^r A)$ is a subgroup of $HH^1(A/\pi^r A)$, and that any inner derivation of $A/\pi^r A$ is A-integrable. Thus if a class $\eta \in HH^1(A/\pi^r A)$ is A-integrable, then any derivation representing η is A-integrable. The group $HH^1_A(A/\pi^r A)$ depends in general on A, not just on $A/\pi^r A$; see the examples in §6 below. If A is clear from the context, we simply say that a class η, or a derivation representing η, is integrable instead of A-integrable. The arguments in the proof of the previous proposition extend in the obvious way to $A/\pi^{2r}A$-integrable derivations, and yield the following statement. The set $HH^1_{A/\pi^{2r}A}(A/\pi^r A)$ is a subgroup of $HH^1(A/\pi^r A)$. Denote by $\mathrm{Out}_r(A/\pi^{2r}A)$ the group modulo inner automorphisms of algebra automorphisms of $A/\pi^{2r}A$ which induce the identity on $A/\pi^r A$. Proposition 3.5 (ii) and the fact that the image of $\mathrm{Out}_{2r}(A)$ in $\mathrm{Out}(A/\pi^{2r}A)$ is trivial imply that we have a group isomorphism

$$\mathrm{Out}_r(A/\pi^{2r}A) \cong HH^1_{A/\pi^{2r}A}(A/\pi^r A) .$$

4 Integrable derivations and Bockstein homomorphisms

We interpret integrable derivations as images under certain Bockstein homomorphisms. Let A be an $\mathcal{O}$-algebra. Recall that if

$$0 \longrightarrow X \xrightarrow{\tau} Y \xrightarrow{\sigma} Z \longrightarrow 0$$

is a short exact sequence of cochain complexes X, Y, Z of A-modules, with differentials denoted δ, ϵ, ζ, respectively, and if n is an integer, then the connecting homomorphism $H^n(Z) \to H^{n+1}(X)$

is constructed as follows. Let $\bar{z} = z + \mathrm{Im}(\zeta^{n-1}) \in H^n(Z)$ for some $z \in \ker(\zeta^n) \subseteq Z^n$. Let $y \in Y^n$ such that $\sigma^n(y) = z$. Then $\epsilon^n(y) \in Y^{n+1}$ satisfies $\sigma^{n+1}(\epsilon^n(y)) = \zeta^n(\sigma^n(y)) = \zeta^n(z) = 0$, hence $y \in \ker(\sigma^{n+1}) = \mathrm{Im}(\tau^{n+1})$. Thus there is $x \in X^{n+1}$ such that $\tau^{n+1}(x) = \epsilon^n(y)$. An easy verification shows that $x \in \ker(\delta^{n+1})$ and that the class $\bar{x} = x + \mathrm{Im}(\delta^n) \in H^{n+1}(X)$ depends only on the class $\bar{z}$ of z in $H^n(Z)$. The connecting homomorphism sends $\bar{z}$ to $\bar{x}$ as defined above. We use algebra automorphisms as subscripts to modules and bimodules as described in 1.4 above. Write $A^e = A \otimes_{\mathcal{O}} A^{\mathrm{op}}$. If A is free of finite rank as an $\mathcal{O}$-module, and if $\alpha \in \mathrm{Aut}_r(A)$ for some positive integer r, then ${}_\alpha(A/\pi^r A) \cong A/\pi^r A \cong (A/\pi^r A)_\alpha$ as A^e-modules, but the A^e-modules $A/\pi^{2r}A$ and $(A/\pi^{2r}A)_\alpha$ need not be isomorphic.

Proposition 4.1. *Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer. Suppose that the canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective. Let $\alpha \in \mathrm{Aut}_r(A)$, and let $\mu : A \to A$ be the unique linear map satisfying $\alpha(a) = a + \pi^r\mu(a)$ for all $a \in A$. Let P be a projective resolution of A as an A^e-module. Applying the functor $\mathrm{Hom}_{A^e}(P, -)$ to the the canonical short exact sequence of A^e-modules*

$$0 \longrightarrow A/\pi^r A \longrightarrow (A/\pi^{2r}A)_\alpha \longrightarrow A/\pi^r A \longrightarrow 0$$

yields a short exact sequence of cochain complexes of $\mathcal{O}$-modules

$$0 \longrightarrow \mathrm{Hom}_{A^e}(P, A/\pi^r A) \longrightarrow \mathrm{Hom}_{A^e}(P, (A/\pi^{2r}A)_\alpha) \longrightarrow \mathrm{Hom}_{A^e}(P, A/\pi^r A) \longrightarrow 0$$

The first nontrivial connecting homomorphism of the induced long exact sequence in cohomology can be canonically identified with a map

$$\mathrm{End}_{A^e}(A/\pi^r A) \longrightarrow HH^1(A/\pi^r A)$$

The image of $\mathrm{Id}_{A/\pi^r A}$ under this map is the class of the derivation on $A/\pi^r A$ induced by μ.

Proof. We choose for P the canonical projective resolution of A, which in degree $n \geq 0$ is equal to $A^{\otimes n+2}$, and which is zero in negative degree. The canonical quasi-isomorphism $P \to A$ is given by the map $A \otimes_{\mathcal{O}} A \to A$ induced by multiplication in A. The last nonzero differential of P is the map $\delta_1 : A^{\otimes 3} \to A^{\otimes 2}$ sending $a \otimes b \otimes c$ to $ab \otimes c - a \otimes bc$, where a, b, $c \in A$. Using a standard sign convention, the first nonzero differential of the cochain complex $\mathrm{Hom}_{A^e}(P, (A/\pi^{2r})_\alpha)$ is given by precomposing with $-\delta_1$. By the remarks from the beginning of the previous section, we have canonical identifications

$$H^0(\mathrm{Hom}_{A^e}(P, A/\pi^r A)) = HH^0(A, A/\pi^r A) \cong HH^0(A/\pi^r A) = \mathrm{End}_{A^e}(A/\pi^r A)$$

The element on the left side corresponding to the identity map $\mathrm{Id}_{A/\pi^r A}$ on the right side is represented by the A^e-homomorphism

$$\zeta : A \otimes_{\mathcal{O}} A \to A/\pi^r A$$

given by $\zeta(a \otimes b) = ab + \pi^r A$ for all a, $b \in A$. This map lifts to an A^e-homomorphism

$$\hat{\zeta} : A \otimes_{\mathcal{O}} A \to (A/\pi^{2r}A)_\alpha$$

defined by $\hat{\zeta}(a\otimes b) = a\alpha(b)+\pi^{2r}A$ for all a, $b \in A$, thanks to the fact that α induces the identity on $A/\pi^r A$. By the construction of connecting homomorphisms in a long exact cohomology sequence of a short exact sequence of cochain complexes, as reviewed above, we need to apply to the 0-cochain $\hat{\zeta}$ the first nonzero differential of the complex $\mathrm{Hom}_{A^e}(P, (A/\pi^{2r}A)_\alpha)$. This differential is given by precomposing with $-\delta_1$, hence yields the 1-coboundary

$$-\hat{\zeta}\circ\delta_1 : A^{\otimes 3} \to (A/\pi^{2r}A)_\alpha$$

For all a, b, $c \in A$ we have

$$(-\hat{\zeta}\circ\delta_1)(a\otimes b\otimes c) = -\hat{\zeta}(ab\otimes c + a\otimes bc) = -ab\alpha(c) + a\alpha(bc) = a(\alpha(b)-b)\alpha(c) \ = \pi^r a\mu(b)\alpha(c)$$

Thus $-\hat{\zeta}\circ\delta_1$ is the image of the map induced by μ under the homomorphism $\mathrm{Hom}_{A^e}(P, A/\pi^r A) \to \mathrm{Hom}_{A^e}(P, (A/\pi^{2r}A)_\alpha)$ from the short exact sequence in the statement. The interpretation of $HH^1(A/\pi^r A)$ in terms of derivations comes from restricting an A^e-homomorphism $A\otimes_{\mathcal{O}} A\otimes_{\mathcal{O}} A \to A/\pi^r A$ to $1\otimes A\otimes 1$. This means that the element in $HH^1(A/\pi^r A)$ determined by μ is the image of $\mathrm{Id}_{A/\pi^r A}$ as claimed. □

We write as before $A^e = A\otimes_{\mathcal{O}} A^{\mathrm{op}}$ and use the analogous notation for the quotients $A/\pi^r A$ of A. By well-known standard facts relating Ext and extensions, the representations of an element in $HH^1(A)$ by a short exact sequence and by a 1-cycle $\varphi : A^{\otimes 3} \to A$ are related via a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & A\otimes_{\mathcal{O}} A\otimes_{\mathcal{O}} A \arrow[r, "\delta_1"] \arrow[d, "\varphi"'] & A\otimes_{\mathcal{O}} A \arrow[r, "\delta_0"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"'] & X \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

The derivation μ representing this element is obtained from restricting φ to $1\otimes A\otimes 1$. Using $\delta_1(1\otimes a\otimes 1) = a\otimes 1 - 1\otimes a$ we get that $\sigma(a\otimes 1 - 1\otimes a) = \iota(\mu(a))$. We have a decomposition $X = \iota(A)\oplus\sigma(A\otimes 1)$ as a direct sum of left A-modules, both of which are isomorphic to A as left A-modules. The first copy, $\iota(A)$, is isomorphic to A as an A^e-module, but the second is not a bimodule: the right action on $\sigma(A\otimes 1)$ is determined for $a\in A$, by

$$\sigma(1\otimes 1)\cdot a = \sigma(1\otimes a) = \sigma(a\otimes 1) + \sigma(1\otimes a) - \sigma(a\otimes 1) = \sigma(a\otimes 1) - \iota(\mu(a))$$

In other words, after identifying $X = A\oplus A$ as left A-modules, the first copy of A is an A-A-subbimodule, and the right action on the second copy is given by $(0,1)a = (-\mu(a), a)$. These considersations remain true for arbitrary commutative base rings, so long as the algebra is projective over its base ring (that is, we can apply these considerations to the $\mathcal{O}/\pi^r\mathcal{O}$-algebras $A/\pi^r A$, for r any positive integer). The following result interprets the class of an integrable derivation as the image of an element in the appropriate Ext-module determined by the first short exact sequence in the previous proposition.

Proposition 4.2. *Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer. Suppose that the canonical map $Z(A) \to Z(A/\pi^r A)$ is surjective. Let $\alpha \in$*

$\mathrm{Aut}_r(A)$*, and let* $\mu : A \to A$ *be the unique linear map satisfying* $\alpha(a) = a + \pi^r\mu(a)$ *for all* $a \in A$*. The exact sequence of* $(A/\pi^{2r}A)^e$*-modules*

$$0 \longrightarrow A/\pi^r A \longrightarrow (A/\pi^{2r}A)_\alpha \longrightarrow A/\pi^r A \longrightarrow 0$$

determines an element $\eta(\alpha)$ *in* $\mathrm{Ext}^1_{(A/\pi^{2r}A)^e}(A/\pi^r A, A/\pi^r A)$*. The canonical surjective homomorphism* $A/\pi^{2r}A \to A/\pi^r A$ *induces a homomorphism*

$$\mathrm{Ext}^1_{(A/\pi^{2r}A)^e}(A/\pi^r A, A/\pi^r A) \longrightarrow \mathrm{Ext}^1_{(A/\pi^{2r}A)^e}(A/\pi^{2r}A, A/\pi^r A) \cong HH^1(A/\pi^r A)$$

which sends $\eta(\alpha)$ *to the class of the derivation induced by* $-\mu$*.*

Proof. We consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A/\pi^r A \arrow[r] & (A/\pi^{2r}A)_\alpha \arrow[r] & A/\pi^r A \arrow[r] & 0 \\
0 \arrow[r] & A/\pi^r A \arrow[r] \arrow[u, equal] \arrow[d, equal] & X \arrow[r] \arrow[u] \arrow[d] & A/\pi^{2r}A \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & A/\pi^r A \arrow[r] & X/\pi^r X \arrow[r] & A/\pi^r A \arrow[r] & 0
\end{tikzcd}$$

The first row in this diagram is the canonical exact sequence representing the element $\eta(\alpha)$ in $\mathrm{Ext}^1_{(A/\pi^{2r}A)^e}(A/\pi^r A, A/\pi^r A)$ as in the statement. The second row is obtained from the first by taking for X the pullback of the two canonical maps from $(A/\pi^{2r}A)_\alpha$ and $A/\pi^{2r}A$ to $A/\pi^r A$. Thus the second row represents the image of $\eta(\alpha)$ in $\mathrm{Ext}^1_{(A/\pi^{2r}A)^e}(A/\pi^{2r}A, A/\pi^r A)$ as in the statement. The third row represents the image of $\eta(\alpha)$ in $HH^1(A/\pi^r A)$ under the isomorphism $\mathrm{Ext}^1_{(A/\pi^{2r}A)^e}(A/\pi^{2r}A, A/\pi^r A) \cong HH^1(A/\pi^r A)$ from 2.1, applied with $s = 2r$ and $n = 1$. We have

$$X = \{(u + \pi^{2r}A, v + \pi^{2r}A) \mid u, v \in A,\ u - v \in \pi^r A\}$$

and the maps from X to $(A/\pi^{2r}A)_\alpha$ and to $A/\pi^{2r}A$ are induced by the canonical projections. The map $A/\pi^r A \to X$ in the diagram sends $a + \pi^r A$ to $(\pi^r a + \pi^{2r}A, 0 + \pi^{2r}A)$. An easy verification shows that

$$\pi^r X = \{(u + \pi^{2r}A, u + \pi^{2r}A) \mid u \in \pi^r A\}$$

The vertical maps from the second to the third row are the canonical surjections. Denote by Y the image of the map $A/\pi^r A \to X$ in the diagram. Set

$$Z = \{(u + \pi^{2r}A, u + \pi^{2r}A) \mid u \in A\}$$

Note that $\pi^r Z = \pi^r X$. Thus the images of Y and Z in $X/\pi^r X$ yield a direct sum decomposition of $X/\pi^r X$ as a left $A/\pi^r A$-module of two copies isomorphic to $A/\pi^r A$. This is the decomposition as described in the paragraph before the statement: the first of these two copies is a bimodule, but the second is not. Moreover, the right action by A on the second copy detects the negative of the derivation corresponding to the element of $HH^1(A/\pi^r A)$ determined by the third row of the above diagram. This right action by $a \in A$ on the image in $X/\pi^r X$ of the element $(u+\pi^{2r}A, u+\pi^{2r}A) \in Z$ is represented by the element in X of the form

$$(u\alpha(a) + \pi^{2r}A, ua + \pi^{2r}A) = (ua + \pi^{2r}A, ua + \pi^{2r}A) + (\pi^r u\mu(a) + \pi^{2r}A, 0 + \pi^{2r}A)$$

The image in $X/\pi^r X$ of the second element on the right side is the image of $(u\mu(a)+\pi^r A)$ under the map $A/\pi^r A \to X/\pi^r X$. The result follows from the identification between derivations and short exact sequences representing elements in $HH^1(A/\pi^r A)$ as described in the paragraph preceding this proposition. $\square$

The fact that we obtain the class represented by μ in 4.1 and the class represented by $-\mu$ in 4.2 is essentially a matter of sign conventions. For the sake of completeness, we mention that Bockstein homomorphisms in Hochschild cohomology are graded derivations, just as in the case of group cohomology (cf. [1, Lemma 4.3.3]), for instance. We sketch a proof for the convenience of the reader. As before, we will use without further mention the canonical isomorphisms $HH^n(A; A/\pi^r A) \cong HH^n(A/\pi^r A)$ for any integers $n \geq 0$, $r > 0$, and any $\mathcal{O}$-algebra A which is free of finite rank as an $\mathcal{O}$-module.

Proposition 4.3. *Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. Let r be a positive integer, and set $\bar{A} = A/\pi^r A$. Let P be a projective resolution of A as an A^e-module. For any nonnegative integer n denote by $\beta_n : HH^n(\bar{A}) \to HH^{n+1}(\bar{A})$ the connecting homomorphism in the long exact cohomology sequence of the canonical short exact sequence*

$$0 \longrightarrow \mathrm{Hom}_{A^e}(P, \bar{A}) \longrightarrow \mathrm{Hom}_{A^e}(P, A/\pi^{2r}A) \longrightarrow \mathrm{Hom}_{A^e}(P, \bar{A}) \longrightarrow 0$$

of cochain complexes. Then for any nonnegative integers m, n, any $\zeta \in HH^m(\bar{A})$ and any $\eta \in HH^n(\bar{A})$ we have

$$\beta_{m+n}(\zeta\eta) = \beta_m(\zeta)\eta + (-1)^m \zeta\beta_n(\eta) \ .$$

Proof. Denote by $\epsilon_n : P_n \to P_{n-1}$ the differential of P for $n > 0$, and by $\epsilon_0 : P_0 \to A$ a map with kernel $\mathrm{Im}(\epsilon_1)$ inducing a quasi-isomorphism $P \to A$. We have $A \otimes_A A \cong A$, and hence $P \otimes_A P$ is also a projective resolution of A. Thus there is a homotopy equivalence $h : P \simeq P \otimes_A P$ which lifts the identity on A. The homotopy class of h with this property is unique. The term in degree $n \geq 0$ of $P \otimes_A P$ is the direct sum of the terms $P_i \otimes_A P_j$, with i, j running over all nonnegative integers such that $i + j = n$. The differential $\hat{\epsilon}$ of $P \otimes_A P$ is obtained by taking the sum of the maps $(-1)^i \mathrm{Id} \otimes \epsilon_j : P_i \otimes_A P_j \to P_i \otimes_A P_{j-1}$ and $\epsilon_i \otimes \mathrm{Id} : P_i \otimes_A P_j \to P_{i-1} \otimes_A P_j$.

Let m, n be nonnegative integers, $\zeta \in HH^m(\bar{A})$ and $\eta \in HH^n(\bar{A})$. These classes are represented by cocycles, abusively denoted by the same letters, $\zeta : P_m \to \bar{A}$ and $\eta : P_n \to \bar{A}$. The product in $HH^*(\bar{A})$ is induced by the map sending ζ and η to the cocycle $(\zeta \otimes \eta) \circ h_{m+n}$. Via the canonical surjection $A/\pi^{2r}A \to \bar{A}$, we lift the cocycles ζ and η to cocycles $\hat{\zeta} : P_m \to A/\pi^{2r}A$ and $\hat{\eta} : P_n \to A/\pi^{2r}A$. The cocycle

$$\widehat{\zeta\eta} = (\hat{\zeta} \otimes \hat{\eta}) \circ h_{m+n} : P_{m+n} \to A/\pi^{2r}A$$

lifts then a cocycle $P_{m+n} \to \bar{A}$ which represents the product $\zeta\eta \in HH^{m+n}(\bar{A})$.

Using these cocycles, we calculate the images under the Bockstein homomorphisms of ζ, η, and $\zeta\eta$. The connecting homomorphism β_m sends ζ to the class of a cocycle $\tilde{\zeta} : P_{m+1} \to \bar{A}$ which upon identifying $\bar{A}$ with its image $\pi^r A/\pi^{2r} A$ in A/π^{2r} is equal to

$$\pi^r \hat{\zeta} \circ \epsilon_{m+1} : P_{m+1} \to \bar{A} \ .$$

Similarly, β_n sends η to the class of a cocycle $\tilde{\eta} : P_{n+1} \to \bar{A}$ equal to $\pi^r \hat{\eta} \circ \epsilon_{n+1}$, with the analogous identification. The connecting homomorphism β_{m+n} sends $\zeta\eta$ to a class in $HH^{m+n+1}(A)$ represented by a cocycle $P_{m+n+1} \to \bar{A}$ which upon identifying $\bar{A}$ with its image $\pi^r A/\pi^{2r} A$ in $A/\pi^{2r} A$ is equal to the cocycle

$$\pi^r (\hat{\zeta} \otimes \hat{\eta}) \circ \hat{\epsilon}_{m+n+1} \circ h_{m+n+1} \ .$$

In view of the explicit description of the differential of $P \otimes_A P$ above, this expression is equal to

$$\pi^r (\hat{\zeta} \circ \epsilon_{m+1} \otimes \hat{\eta} + (-1)^m \hat{\zeta} \otimes \hat{\eta} \circ \epsilon_{n+1}) \circ h_{m+n+1} = (\pi^r \hat{\zeta} \circ \epsilon_{m+1} \otimes \hat{\eta} + (-1)^m \hat{\zeta} \otimes \pi^r \hat{\eta} \circ \epsilon_{n+1}) \circ h_{m+n+1} \ .$$

By the above description of the images of ζ and η under Bockstein homomorphisms, this is equal to

$$(\tilde{\zeta} \otimes \hat{\eta} + (-1)^m \hat{\zeta} \otimes \tilde{\eta}) \circ h_{m+n+1}$$

This represents $\beta(\zeta)\eta + (-1)^m \zeta\beta(\eta)$ as required. □

5 Integrable derivations and stable equivalences

Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. The group $\mathrm{Out}(A)$ need not be invariant under Morita equivalences, but the subgroups $\mathrm{Out}_r(A)$ are invariant under Morita equivalences. Indeed, if $\alpha \in \mathrm{Out}_r(A)$ for some $r > 0$, then α induces in particular the identity on $A/\pi A$, hence stabilises the isomorphism class of any simple $A/\pi A$-module, hence of any projective indecomposable A-module. The subgroup of $\mathrm{Out}(A)$ represented by those automorphisms which stabilise all projective indecomposable modules is well-known to be invariant under Morita equivalences, and thus so are the groups $\mathrm{Out}_r(A)$. More explicitly, if A, B are two $\mathcal{O}$-free algebras of finite $\mathcal{O}$-rank and if M is an A-B-bimodule inducing a Morita equivalence, then for any $\alpha \in \mathrm{Aut}_r(A)$ there is $\beta \in \mathrm{Aut}_r(B)$ such that ${}_{\alpha^{-1}}M \cong M_\beta$ as A-B-bimodules, and the correspondence $\alpha \mapsto \beta$ yields a group isomorphism $\mathrm{Out}_r(A) \cong \mathrm{Out}_r(B)$. As before, the use of automorphisms as subscripts to modules and bimodules is as in 1.4 above. We use without further reference some standard facts relating bimodules and automorphisms (see e. g. [5, §55A]): if $\alpha \in \mathrm{Aut}(A)$, then the A-A-bimodule A_α induces a Morita equivalence on A, we have $A_\alpha \cong A$ as A-A-bimodules if and only if α is inner, the map sending $a \in A$ to $\alpha(a)$ is an A-A-bimodule isomorphism ${}_{\alpha^{-1}}A \cong A_\alpha$, and the map sending α to A_α induces an injective group homomophism from $\mathrm{Out}(A)$ to the Picard group $\mathrm{Pic}(A)$ of isomorphism classes of bimodules inducing Morita equivalences on A. We will show next that if the involved algebras are relatively $\mathcal{O}$-selfinjective with separable semisimple quotients, then the groups $\mathrm{Out}_r(A)$ are in fact invariant under stable equivalences of Morita type in a way which is compatible with the associated groups of integrable derivations. Symmetric $\mathcal{O}$-algebras remain symmetric over k, hence selfinjective, and thus Theorem 1.1 is a special case of the following result.

Theorem 5.1. *Let A, B be $\mathcal{O}$-algebras which are free of finite rank as $\mathcal{O}$-modules, such that the k-algebras $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$ are indecomposable nonsimple selfinjective with separable semisimple quotients. Let r be a positive integer. Suppose that the canonical maps $Z(A) \to Z(A/\pi^r A)$ and $Z(B) \to Z(B/\pi^r B)$ are surjective. Let M be an A-B-bimodule and N a B-A-bimodule inducing a stable equivalence of Morita type between A and B. For any $\alpha \in \mathrm{Aut}_r(A)$ there is $\beta \in \mathrm{Aut}_r(B)$ such that ${}_{\alpha^{-1}}M \cong M_\beta$ as A-B-bimodules, and the correspondence $\alpha \mapsto \beta$ induces a group isomorphism $\mathrm{Out}_r(A) \cong \mathrm{Out}_r(B)$ making the following diagram of groups commutative:*

$$\begin{array}{ccc} \mathrm{Out}_r(A) & \xrightarrow{\cong} & \mathrm{Out}_r(B) \\ \downarrow & & \downarrow \\ HH^1_A(A/\pi^r A) & \xrightarrow{\cong} & HH^1_B(B/\pi^r B) \end{array}$$

where the vertical maps are the group homomorphisms from Proposition 3.5 (ii) and where the lower horizontal isomorphism is induced by the functor $N \otimes_A - \otimes_A M$.

If M and N induce a Morita equivalence, then the hypothesis on $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$ being selfinjective with separable semisimple quotients is not needed; see the Remark 5.4 below for the necessary adjustments. This yields an alternative proof of a result due to Farkas, Geiss, and Marcos in [6]. The first part of the statement of 5.1 is the following variation of [8, Theorem 4.2].

Lemma 5.2. *With the notation and hypotheses of 5.1, for any $\alpha \in \mathrm{Aut}_r(A)$ there is $\beta \in \mathrm{Aut}_r(B)$ such that ${}_{\alpha^{-1}}M \cong M_\beta$ as A-B-bimodules, and the correspondence $\alpha \mapsto \beta$ induces a group isomorphism $\mathrm{Out}_r(A) \cong \mathrm{Out}_r(B)$.*

Proof. Let $\alpha \in \mathrm{Aut}_r(A)$. Then α induces the identity on $A/\pi A \cong k \otimes_{\mathcal{O}} A$, hence stabilises the isomorphism classes of all $A/\pi A$-modules, hence in particular of all simple $A/\pi A$-modules and of all finitely generated projective A-modules. By [8, Theorem 4.2] there exists an automorphism β of B, unique up to inner automorphisms, such that we have an isomorphism ${}_{\alpha^{-1}}M \cong M_\beta$ of A-B-bimodules. We sketch the argument to prove this. The self-stable equivalence induced by $N \otimes_A {}_{\alpha^{-1}}M$ reduces to the identity functor on the stable module category of $k \otimes_{\mathcal{O}} B$-modules because α induces the identity on $k \otimes_{\mathcal{O}} A$. By [8, Theorem 2.1], a stable equivalence of Morita type preserving simple modules is induced by a Morita equivalence. Since this Morita equivalence fixes the isomorphism class of any simple B-module, it is given by a bimodule of the form B_β, for some $\beta \in \mathrm{Aut}(B)$. Thus the B-B-bimodule B_β is, up to isomorphism, the unique nonprojective direct summand of $N \otimes_A {}_{\alpha^{-1}}M$. Tensoring by $M \otimes_B -$ shows that $M_\beta \cong {}_{\alpha^{-1}}M$. We observe next that β belongs to $\mathrm{Out}_r(B)$. We have an isomorphism of $A/\pi^r A$-$B/\pi^r B$-bimodules

$$M_\beta/\pi^r M_\beta \cong {}_{\alpha^{-1}}M/\pi^r({}_{\alpha^{-1}}M)\ .$$

Since α induces the identity on $A/\pi^r A$, it follows that this bimodule is isomorphic to $M/\pi^r M$. Tensoring with N on the left implies that the nonprojective summands of $N \otimes_A M/\pi^r(N \otimes_B M)$ and $N \otimes_A M_\beta/\pi^r(N \otimes_B M_\beta)$ are isomorphic, hence that $B/\pi^r B$ and $B_\beta/\pi^r B_\beta$ are isomorphic as B-B-bimodules. This implies that the automorphism on $B/\pi^r B$ induced by β is inner. A straightforward verification shows that the correspondence $\alpha \mapsto \beta$ induces a group homomorphism $\mathrm{Out}_r(A) \to \mathrm{Out}_r(B)$. Exchanging the roles of A and B yields the inverse of this group homomorphism. $\square$

It follows immediately from 5.2, applied with r and $2r$, together with 3.2 and 3.5 that in the situation of Theorem 5.1 there is a group isomorphism $HH^1_A(A/\pi^r A) \cong HH^1_B(B/\pi^r B)$ which makes the diagram in 5.1 commutative. In order to show that this isomorphism is induced by the functor $N \otimes_A - \otimes_A M$, we will need the following technical observation.

Lemma 5.3. *With the notation and hypotheses of 5.1, let $\alpha \in \mathrm{Aut}_r(A)$ and $\beta \in \mathrm{Aut}_r(B)$ such that ${}_{\alpha^{-1}}M \cong M_\beta$ as A-B-bimodules. Then there is an A-B-bimodule isomorphism $\varphi : {}_{\alpha^{-1}}M \cong M_\beta$ which induces the identity map on $M/\pi^r M$.*

Proof. Let $\psi : {}_{\alpha^{-1}}M \cong M_\beta$ be an isomorphism of A-B-bimodules. That is, ψ is an $\mathcal{O}$-linear automorphism of M satisfying

$$\psi(\alpha^{-1}(a)mb) = a\psi(m)\beta(b)$$

for all $a \in A$, $b \in B$, and $m \in M$. Set $\bar{\mathcal{O}} = \mathcal{O}/\pi^r\mathcal{O}$. Then $\bar{A} = A/\pi^r A$ and $\bar{B} = B/\pi^r B$ are $\bar{\mathcal{O}}$-algebras, and the $\bar{A}$-$\bar{B}$-bimodule $\bar{M} = M/\pi^r M$ and the $\bar{B}$-$\bar{A}$-bimodule $\bar{N} = N/\pi^r N$ induce a stable equivalence of Morita type between $\bar{A}$ and $\bar{B}$. Multiplication by elements in $Z(\bar{A})$ induces an $\bar{\mathcal{O}}$-algebra isomorphism $Z(\bar{A}) \cong \mathrm{End}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{A}^{\mathrm{op}}}(\bar{A})$. We denote by $Z^{\mathrm{pr}}(\bar{A})$ the ideal in $Z(\bar{A})$ corresponding to those endomorphisms of $\bar{A}$ which factor through a projective $\bar{A} \otimes_{\bar{\mathcal{O}}} \bar{A}^{\mathrm{op}}$-module. Following [2, §5] we set $Z_{\mathrm{st}}(\bar{A}) = Z(\bar{A})/Z^{\mathrm{pr}}(\bar{A})$. We use the analogous notation for $\bar{B}$. We denote by $\underline{\mathrm{End}}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M})$ the quotient of $\mathrm{End}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M})$ by the ideal of endomorphisms which factor through a projective $\bar{A} \otimes_{\bar{\mathcal{O}}} \bar{B}^{\mathrm{op}}$-module. By the proof of [2, 5.4] we have $\bar{\mathcal{O}}$-algebra isomorphisms

$$Z_{\mathrm{st}}(\bar{A}) \cong \underline{\mathrm{End}}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}) \cong Z_{\mathrm{st}}(\bar{B})$$

induced by left and right mulitplication with elements of $Z(\bar{A})$ and $Z(\bar{B})$ on $\bar{M}$, respectively. The map $\bar{\psi}$ on $\bar{M}$ induced by ψ is in fact an endormorphism of $\bar{M}$ as an $\bar{A}$-$\bar{B}$-bimodule, because α and β induce the identity on $\bar{A}$ and $\bar{B}$, respectively. Since the map $Z(A) \to Z(\bar{A})$ is assumed to be surjective, there is an element $z \in Z(A)$ such that the A-B-endomorphism λ of M_β given by left multiplication with z induces an $\bar{A}$-$\bar{B}$-endomorphism $\bar{\lambda}$ on $\bar{M}$ belonging to the same class as that of $\bar{\psi}$ in $\underline{\mathrm{End}}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M})$. Thus, after replacing ψ by $z^{-1} \cdot \psi$, we may assume that the $\bar{A}$-$\bar{B}$-endomorphism $\bar{\psi} - \mathrm{Id}_{\bar{M}}$ of $\bar{M}$ factors through a projective cover $\bar{Q}$ of $\bar{M}$; say $\bar{\psi} - \mathrm{Id}_{\bar{M}} = \bar{\epsilon} \circ \bar{\delta}$ for some bimodule homomorphisms $\bar{\delta} : \bar{M} \to \bar{Q}$ and $\bar{\epsilon} : \bar{Q} \to \bar{M}$. Let Q be a projective cover of M_β. Since β induces the identity on $\bar{B}$, it follows that $Q/\pi^r Q$ is a projective cover of $\bar{M}$, hence isomorphic to $\bar{Q}$. Choose notation such that $Q/\pi^r Q = \bar{Q}$. The bimodule $\bar{\epsilon}$ lifts to an A-B-bimodule homomorphism $\epsilon : Q \to M_\beta$ because Q is projective. By the assumptions, the algebra $k \otimes_{\mathcal{O}} (A \otimes_{\mathcal{O}} B\mathrm{op})$ is selfinjective. Since the involved modules M and Q are $\mathcal{O}$-free, it follows from 2.8 that the bimodule homomorphism $\bar{\delta}$ lifts to an A-B-bimodule homomorphism $\delta : {}_{\alpha^{-1}}M \to Q$. Since M is indecomposable nonprojective, it follows that $\psi - \epsilon \circ \delta : {}_{\alpha^{-1}}M \cong M_\beta$ is still an isomorphism. By construction, $\psi - \epsilon \circ \delta$ induces the identity on $\bar{M}$, whence the result. □

Proof of Theorem 5.1. The first statement holds by 5.2. It remains to verify the commutativity of the diagram in the statement. Let $\alpha \in \mathrm{Aut}_r(A)$ and $\beta \in \mathrm{Aut}_r(B)$ such that ${}_{\alpha^{-1}}M \cong M_\beta$. By 5.3, there is an isomorphism $\varphi : {}_{\alpha^{-1}}M \cong M_\beta$ which induces the identity on $M/\pi^r M$. Let μ and ν be the $\mathcal{O}$-linear endomorphisms of A and B, respectively, satisfying $\alpha(a) = a + \pi^r\mu(a)$ for all $a \in A$ and $\beta(b) = b + \pi^r\nu(b)$ for all $b \in B$. Set $\bar{A} = A/\pi^r A$ and $\bar{B} = B/\pi^r B$. Denote by $\bar{\mu}$ and $\bar{\nu}$ the classes in $HH^1_A(\bar{A})$ and $HH^1_B(\bar{B})$ determined by μ and ν; that is, $\bar{\mu}$ and $\bar{\nu}$ are the images of the classes of α and β under the canonical group homomorphisms $\mathrm{Out}_r(A) \to HH^1(\bar{A})$ and

$\mathrm{Out}_r(B) \to HH^1(\bar{B})$, respectively. Set $\bar{M} = M/\pi^r M$ and $\bar{\mathcal{O}} = \mathcal{O}/\pi^r\mathcal{O}$. The assumptions imply that tensoring by $\bar{M}$ yields a stable equivalence of Morita type between $\bar{A}$ and $\bar{B}$. In particular, we have isomorphisms

$$HH^1(\bar{A}) \cong \mathrm{Ext}^1_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}, \bar{M}) \cong HH^1(\bar{B})$$

induced by the functors $-\otimes_A M$ and $M \otimes_B -$. Since $N \otimes_A M$ is isomorphic to B in the relatively $\mathcal{O}$-stable category of $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-modules, it follows that the composition

$$HH^1(\bar{A}) \cong HH^1(\bar{B})$$

of the two previous isomorphisms is induced by the functor $N \otimes_A - \otimes_A M$. Thus we need to show that $M \otimes_B -$ and $- \otimes_A M$ send $\bar{\nu}$ and $\bar{\mu}$, respectively, to the the same class in $\mathrm{Ext}^1_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}, \bar{M})$. Note that the functors $M \otimes_B -$ and $- \otimes_A M$ induce obvious algebra homomorphisms

$$\mathrm{End}_{\bar{A}^e}(\bar{A}) \longrightarrow \mathrm{End}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}) \longleftarrow \mathrm{End}_{\bar{B}^e}(\bar{B}) \quad ,$$

where $\bar{A}^e = \bar{A} \otimes_{\bar{\mathcal{O}}} \bar{A}^{\mathrm{op}}$ and $\bar{B}^e = \bar{B} \otimes_{\bar{\mathcal{O}}} \bar{B}^{\mathrm{op}}$. Tensoring the short exact sequences

$$0 \longrightarrow \bar{A} \longrightarrow (A/\pi^{2r}A)_\alpha \longrightarrow \bar{A} \longrightarrow 0$$

$$0 \longrightarrow \bar{B} \longrightarrow (B/\pi^{2r}B)_\beta \longrightarrow \bar{B} \longrightarrow 0$$

by $- \otimes_A M$ and $M \otimes_B -$, respectively, yields short exact sequences of the form

$$0 \longrightarrow \bar{M} \longrightarrow {}_{\alpha^{-1}}(M/\pi^{2r}M) \longrightarrow \bar{M} \longrightarrow 0$$

$$0 \longrightarrow \bar{M} \longrightarrow (M/\pi^{2r}M)_\beta \longrightarrow \bar{M} \longrightarrow 0$$

These two exact sequences are equivalent, because the isomorphism φ above induces an isomorphism ${}_{\alpha^{-1}}(M/\pi^{2r}M) \cong (M/\pi^{2r}M)_\beta$ which induces in turn the identity map on $\bar{M}$. Applying the functor $\mathrm{Hom}_{A\otimes_{\mathcal{O}}B^{\mathrm{op}}}(M, -)$ to these two short exact sequences, with identifications analogous to those in 4.1, yields therefore the same connecting homomorphism

$$\mathrm{End}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(M) \to \mathrm{Ext}^1_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}, \bar{M})$$

Denote by $\bar{\mu} \otimes \mathrm{Id}_{\bar{M}}$ and $\mathrm{Id}_{\bar{M}} \otimes \bar{\nu}$ the images in $\mathrm{Ext}^1_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}, \bar{M})$ of $\bar{\mu}$ and $\bar{\nu}$ under the functors $- \otimes_A M$ and $M \otimes_B -$, respectively. By the naturality properties of connecting homomorphisms, it follows from the description of $\bar{\mu}$ and $\bar{\nu}$ in 4.1, that $\bar{\mu} \otimes \mathrm{Id}_{\bar{M}}$ and $\mathrm{Id}_{\bar{M}} \otimes \bar{\nu}$ are both equal to the image of $\mathrm{Id}_{\bar{M}}$ under the above connecting homomorphism. This shows that the group isomorphism $HH^1_A(\bar{A}) \cong HH^1_B(\bar{B})$ induced by the group isomorphism $\mathrm{Out}_r(A) \cong \mathrm{Out}_r(B)$ is equal to the group isomorphism determined by the functor $N \otimes_A - \otimes_A M$, completing the proof of Theorem 5.1. □

Remark 5.4. With the notation of 5.1, if M and N induce a Morita equivalence, then the conclusions of the theorem and the lemmas in this section hold without the assumption that A and B are relatively $\mathcal{O}$-injective with separable semisimple quotients. This hypothesis is needed in 5.2 via there reference [8, Theorem 4.2], but it is not needed if M and N induce a Morita equivalence, by the remarks at the beginning of this section. This hypothesis is also needed in the proof of

Lemma 5.3, but again, if M and N induce a Morita equivalence, then left and right multiplication by elements in $Z(\bar{A})$ and $Z(\bar{B})$, respectively, induce algebra isomorphisms

$$Z(\bar{A}) \cong \mathrm{End}_{\bar{A}\otimes_{\bar{\mathcal{O}}}\bar{B}^{\mathrm{op}}}(\bar{M}) \cong Z(\bar{B}) \ ,$$

where the notation is as in the proof of 5.3. Thus $\bar{\psi}$ is induced by multiplication with an element $\bar{z} \in Z(\bar{A})$, hence after replacing ψ by $z^{-1}\cdot\psi$ we get that $\bar{\psi} = \mathrm{Id}_{\bar{M}}$. As mentioned above, this shows the invariance of the groups of integrable derivations under Morita equivalences, a result which is due to Farkas, Geiss, and Marcos [6] in the case $\mathcal{O} = k[[t]]$, here slightly generalised to arbitrary complete discrete valuation rings.

6 Examples

The following examples illustrate some of the basic connections between integrable elements in $HH^1(A/\pi^m A)$, the algebra A, and the ramification of the ring $\mathcal{O}$.

Example 6.1. Suppose that k has prime characteristic p and that $\mathcal{O}$ has characteristic zero. Let P be a nontrivial finite abelian p-group. Suppose that K contains a primitive $|P|$-th root of unity, denoted τ. Let r be the positive integer such that $\pi^r\mathcal{O} = (\tau-1)\mathcal{O}$.

(a) Any $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ which induces the identity on kP is of the form $\alpha(u) = \zeta(u)u$ for all $u \in P$, where $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$. Thus $\mathrm{Aut}_1(\mathcal{O}P) \cong \mathrm{Hom}(P, \mathcal{O}^\times)$, which is (noncanonically) isomorphic to P, and the elements in $HH^1_{\mathcal{O}P}(kP)$ are induced by linear endomorphisms of $\mathcal{O}P$ sending $u \in P$ to $\frac{\zeta(u)-1}{\pi}u$ for all $u \in P$. If $r \geq 2$, then π divides the coefficients $\frac{\zeta(u)-1}{\pi}$, and hence the induced linear endomorphisms of kP are zero. Thus if $r \geq 2$, then $HH^1_{\mathcal{O}P}(kP) = \{0\}$.

(b) Set $\bar{\mathcal{O}} = \mathcal{O}/(\tau-1)\mathcal{O} = \mathcal{O}/\pi^r\mathcal{O}$. Any $\mathcal{O}$-algebra automorphism of $\mathcal{O}P$ which induces the identity on kP induces by the above description the identity on $\bar{\mathcal{O}}P$, hence is as before of the form $\alpha(u) = \zeta(u)u$ for all $u \in P$, where $\zeta : P \to \mathcal{O}^\times$ is a group homomorphism. Thus, as before, we have $\mathrm{Aut}_r(\mathcal{O}P) \cong \mathrm{Hom}(P, \mathcal{O}^\times)$. However, $HH^1_{\mathcal{O}P}(\bar{\mathcal{O}}P)$ is a nontrivial finite subgroup of $HH^1(\bar{\mathcal{O}}P)$ consisting of all derivations d_ζ of $\bar{\mathcal{O}}P$ induced by the linear maps on $\mathcal{O}P$ sending $u \in P$ to $\frac{\zeta(u)-1}{\tau-1}u$, where $\zeta \in \mathrm{Hom}(P, \mathcal{O}^\times)$.

(c) It is shown in [6] that if $\mathcal{O} = k[[t]]$, then any integrable derivation on A/tA preserves the Jacobson radical of A/tA. The result does not carry over to general $\mathcal{O}$ and the Artinian quotients $A/\pi^r A$ of A. Choose P cyclic of order p with a generator denoted y. The automorphism α of $\mathcal{O}P$ sending y to τy determines a derivation of $\bar{\mathcal{O}}P$ induced by the linear endomorphism μ of $\mathcal{O}P$ sending y^i to $\frac{\tau^i-1}{\tau-1}y^i$. This endomorphism sends $y-1$ to y, and hence the derivation on $\bar{\mathcal{O}}P$ induced by μ does not preserve the Jacobson radical.

Example 6.2. Different lifts of a symmetric k-algebra yield in general different groups of integrable derivations. Suppose that k has prime characteristic p. Let P be a finite cyclic p-group of order p^s for some positive integer s; let y be a generator of P. There is a k-algebra isomorphism $kP \cong k[x]/(x^{p^s})$ sending $y-1$ to $x+(x^{p^s})$. We have $\dim_k(HH^1(kP)) = p^s$; more precisely, for any i such that $0 \leq i \leq p^s-1$ there is a unique derivation d_i on kP sending x to x^i, and the set $\{d_i\}_{0\leq i\leq p^s-1}$ is a k-basis of $HH^1(kP)$. If $A = \mathcal{O}P$, then by the previous example, $HH^1_{\mathcal{O}P}(kP)$ is a finite (possibly trivial) subgroup of $HH^1(kP)$. By contrast, if $A = \mathcal{O}[x]/(x^{p^s})$, then $HH^1_A(kP)$

is the $(p^s - 1)$-dimensional k-subspace spanned by the set $\{d_i\}_{1 \leq i \leq p^s - 1}$. Indeed, in that case, for $1 \leq i \leq p^s - 1$ and $\lambda \in \mathcal{O}$ there is an $\mathcal{O}$-algebra automorphism $\alpha_{i,\lambda}$ in $\mathrm{Aut}_1(A)$ sending x to $x + \pi\lambda x^i$, and this automorphism gives rise to the derivation $\bar{\lambda} d_i$ on kP, where $\bar{\lambda}$ is the image of λ in k.

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